Physics 5150

Homework Set # 9

Due 5 pm Thursday 4/5/2018

SOLUTIONS

Problem 1:
In our derivation of the MHD equations, we used Ampere's law in the form $\nabla \times \mathbf{B} = 4\pi\mathbf{j}/c$, *thereby neglecting the displacement current* $\partial_t\mathbf{E}$. *Within the non-relativistic ideal MHD framework (i.e.,* $\mathbf{E} = -[\mathbf{u}\times\mathbf{B}]/c$ *with* $|\mathbf{u}| \ll c$*) is this a good approximation? Does it remain to be valid in relativistic MHD (when* $|\mathbf{u}| \sim c$*)?*

Solution:
In ideal MHD, $\mathbf{E} = -[\mathbf{u}\times\mathbf{B}]/c$, we can estimate roughly the typical strength of the electric field, $E \sim (u/c)B$. Then, the ratio of the displacement current to the $\nabla \times \mathbf{B}$ term in Ampere's law can be estimated as

$$\frac{|c^{-1}\,\partial_t\mathbf{E}|}{|\nabla\times\mathbf{B}|} \sim \frac{E/cT}{B/L} \sim \frac{L}{cT}\,\frac{u}{c}\,,$$

where L and T are the characteristic length- and time-scales, respectively. Since in MHD any time evolution is usually associated with (or caused by) the fluid motions, L and T are related via the typical fluid velocity, $L \sim uT$. Thus, we see that

$$\frac{|\partial_t\mathbf{E}|}{|\nabla\times\mathbf{B}|} \sim \frac{u^2}{c^2}\,,$$

i.e., the displacement current is smaller than the $\nabla\times\mathbf{B}$ term by something like a factor of u^2/c^2, which has to be small in order for the system to be in the nonrelativistic MHD regime. In the case of relativistic MHD, where we deal with relativistic fluid motions, $u \sim c$ and the displacement current is in general not negligible.

Problem 2:
A star with a radius $R_* = 1$ million km *and a uniform magnetic field* $B_* = 100\,\mathrm{Gauss}$ *on its surface ends its life by suddenly collapsing into a very compact and dense neutron star of radius* $R_{ns} = 10\,\mathrm{km}$. *The collapse is so rapid that the magnetic field doesn't have time to diffuse away and hence the perfect flux-freezing assumption holds.*
(a) *What is the strength of the resulting magnetic field on the surface of the neutron star?*
(b) *What is the electron cyclotron frequency in this magnetic field, and to which part of the electromagnetic spectrum does the frequency correspond?*

Solution:
This problem illustrates the practical usefulness of the concept of flux freezing. The magnetic flux through the star before the collapse is equal to $\Phi_* = B_* \pi R_*^2$, and the magnetic flux through the neutron star right after the collapse is $\Phi_{\rm ns} = B_{\rm ns} \pi R_{\rm ns}^2$. Since the flux is preserved, $\Phi_{\rm ns} = \Phi_*$, we have

$$B_{\rm ns} = B_* \frac{R_*^2}{R_{\rm ns}^2} = 10^{10}\, B_* = 10^{12}\,\text{Gauss} = 10^8\,\text{T}\,.$$

This is indeed a typical magnetic field strength observationally inferred for neutron stars!

The electron cyclotron frequency corresponding to such a tremendously high magnetic field is $\nu_c = eB/2\pi m_e c = 2.8 \cdot 10^{18}\,\text{Hz}$, and the corresponding photon energy is $h\nu_c \simeq 12\,\text{keV}$, which falls into the hard X-ray range.

Problem 3:
Consider an infinitely-long straight wire carrying a current I, surrounded by vacuum. Calculate the magnetic tension and magnetic pressure gradient forces at a distance R from the wire. Which force is stronger? Where is the net force (the sum of the two forces) pointing?

Solution:
Of course, it is clear from the start, without doing any calculations, that the two forces exactly cancel each other, so the sum is zero and has no direction. This is because, as we have discussed in class, the magnetic tension force and the magnetic pressure gradient force are just a convenient way to split the total Lorentz force, $\mathbf{j} \times \mathbf{B}$, into two components. Now, the total $\mathbf{j} \times \mathbf{B}$ force in this problem is clearly zero, because here we are talking about a vacuum region surrounding the wire, and there are no charge carriers and hence no current flowing in vacuum, i.e., $\mathbf{j} = 0$.

At the same, one can also show that the two magnetic forces are equal to each other and are in opposite direction by a direct explicit calculation, and this is in fact what you are asked to do in this problem. To perform this calculation, we start with the expression for the magnetic field produced by a straight long wire carrying current I. This field is purely in the toroidal (azimuthal) direction and is equal to $B = 2I/(cR)$. Then, the magnetic pressure is

$$P_{\rm magn} = \frac{B^2}{8\pi} = \frac{1}{2\pi}\left(\frac{I}{cR}\right)^2,$$

and the associated force is

$$-\nabla P_{\rm magn} = +\frac{I^2}{\pi c^2 R^3}\,\hat{\mathbf{r}} = \frac{B^2}{4\pi R}\,\hat{\mathbf{r}}\,.$$

As for the magnetic tension force, it points radially inward (toward the wire) and is equal to

$$\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} = \frac{B_\phi}{4\pi}\frac{\partial}{R\partial\phi}(B_\phi\hat{\phi}) = \frac{B_\phi^2}{4\pi R}\frac{\partial\hat{\phi}}{\partial\phi} = -\frac{B_\phi^2}{4\pi R}\,\hat{\mathbf{r}}\,.$$

This is exactly equal to the magnetic pressure gradient force with a minus sign. Hence the two forces cancel each other and the total magnetic force is zero.

Problem 4:

Our planet's magnetosphere protects life on Earth from the harmful flow of charged particles streaming from the Sun, known as the solar wind. Estimate roughly how far from the Earth (in centimeters and in Earth's radii) is the pressure of the magnetosphere's dipole magnetic field able to stop the incoming solar wind? Assume that the magnetic field is $0.3\,\mathrm{G}$ *at the Earth's equator and drops off inversely proportionally to* R^3 *and that the solar wind is a fully ionized hydrogen plasma with a velocity of* $400\,\mathrm{km/sec}$ *and a density of* $10\,\mathrm{cm}^{-3}$*. Neglect the thermal and magnetic pressure of the solar wind itself and only take into account its ram pressure.*

Solution:

The goal of this problem is to give you an example of a situation where a quick and simple rough calculation gives one an approximate but nevertheless very useful estimate, in this case, an estimate for the location of the day-side boundary of the Earth's magnetosphere (called the *magnetopause*). As in many space- and astrophysics problems, you don't need to be very precise in this problem, a factor of 2 accuracy will suffice.

The Earth magnetic field stops the incoming solar wind at a distance r_0 where its magnetic pressure is equal to the ram pressure of the solar wind:

$$P_{\mathrm{magn}} = \frac{B^2(r_0)}{8\pi} = P_{\mathrm{ram}} = \rho v^2 \,.$$

Because the Earth's magnetic field in this problem is approximated as a dipole, $B(r) = B_E\,(R_E/r)^3$, this critical distance can be estimated as

$$r_c = R_E \left[\frac{B_E^2/8\pi}{\rho v^2}\right]^{1/6} \simeq 7\,R_E \simeq 45,000\,\mathrm{km}\,.$$

In reality this distance varies over time within a factor of 2 or so because of the fluctuations in the solar wind. Also, because the solar wind's ram pressure compresses the Earth's magnetosphere, the actual magnetic field strength just inside the magnetopause is somewhat higher than that given by the dipole formula. Nevertheless, the above estimate is actually very useful. It tells us, for example, that the Moon is well outside the protection of the Earth's magnetosphere and hence is exposed to the solar wind; therefore, astronauts on the Moon will have to deal with a much higher level of radiation than those on a low-Earth orbit.