HW7 of Plasma

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1. When the distribution function is spherically symmetric, average velocity

$$V_{sz} = \frac{1}{n_s}\int v\cdot\cos\theta \cdot f_s \cdot dv \cdot v\sin\theta\, d\theta \cdot v\, d\varphi \qquad \text{(in spherical coordinate)}$$

$$= \frac{1}{n_s} 2\pi \int f_s \cdot v^2 dv \int_0^{\pi} \sin\theta\cdot\cos\theta\, d\theta$$

$\int_0^{\pi} \sin\theta\cdot\cos\theta\, d\theta = 0$, so $V_{sz} = 0$, and $\vec{V}_s = 0$ under isotropy

and $$P_{s\,ij} = m_s \int (v_i - V_{si})(v_j - V_{sj}) \cdot f_s\, d^3v$$

$$= m_s \int v_i \cdot v_j \cdot \sin\theta\, d\theta \cdot d\varphi \cdot f_s v^2 dv$$

choose any two orthogonal $i \neq j$, such as $i=x$, $j=y$. $v_i = v\cdot\sin\theta\cos\varphi$; $v_j = v\cdot\sin\theta\sin\varphi$

$$P_{s\,xy} = m_s \cdot \int_0^{\pi} \sin^3\theta\, d\theta \int_0^{2\pi} \sin\varphi\cos\varphi\, d\varphi \cdot \int_{\pm\infty} f_s \cdot v^4 dv$$

because $\int_0^{2\pi} \sin\varphi\cos\varphi\, d\varphi = 0$, $P_{s\,xy} = 0$, for any $i \neq j$. $P_{sij} = 0$

so $\overleftrightarrow{P}_s$ is diagonal.

2. normalization requires: $\int_0^{+\infty} f(v)\cdot 4\pi v^2 dv = n$

and $$\int_0^{+\infty} f\left(v = \sqrt{\frac{2\varepsilon}{m}}\right) \cdot 4\pi \cdot \frac{2\varepsilon}{m} \cdot \frac{dv}{d\varepsilon} \cdot d\varepsilon = n$$

so $$F_\varepsilon(\varepsilon) = f\left(\sqrt{\frac{2\varepsilon}{m}}\right) \cdot 4\pi \cdot \frac{2\varepsilon}{m} \cdot \left(\frac{d\varepsilon}{dv}\right)^{-1} = f\left(\sqrt{\frac{2\varepsilon}{m}}\right) \cdot 8\pi\varepsilon \cdot \sqrt{\frac{2\varepsilon}{m}}$$

3. (a) $\rho = \int f(\vec{v})\, d^3\vec{v} = n_0 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \int_{\pm\infty} dv_x \cdot e^{-\frac{m}{2kT}\cdot(v_x - v_{0x})^2} \int dv_y \sim \cdot \int dv_z \sim$

$\int_{\pm\infty} dv_x \cdot e^{\sim} = \int_{\pm\infty} d(v_x - v_{0x})\, e^{\sim} = \sqrt{\pi \cdot \frac{2kT}{m}}$, similar to v_y, v_z

so $\rho = n_0 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \left(\pi \cdot \frac{2kT}{m}\right)^{\frac{3}{2}} = n_0$

(b) $f(v_i) = \sqrt{\frac{m}{2\pi kT}} \cdot \exp\left[-m(v_i - v_{0i})^2/2kT\right]$ defined so,

$f(\vec{v}) = n_0 f(v_x) f(v_y) f(v_z)$

as in $\hat{x}$ $\langle v_x \rangle = \frac{1}{n_0}\int_{\pm\infty} n_0 v_i f(v_i)\, dv_i \cdot \underbrace{\int f(v_y) dv_y}_{=1} \underbrace{\int f(v_z)\, dv_z}_{=1}$

$= \sqrt{\frac{m}{2\pi kT}} \cdot \int_{\pm\infty} v_x \cdot e^{-\frac{m}{2kT}(v_x - v_{0x})^2} \cdot dv_x$, define $v_x - v_{0x} = x$

$= \sqrt{\frac{m}{2\pi kT}} \cdot \int_{\pm\infty} (x + v_{0x})\, e^{-\frac{m}{2kT}x^2}\, dx$; $x \cdot e^{-x^2}$ is odd, so $\int_{\pm\infty} x e^{-\frac{m}{2kT}x^2} dx = 0$.

$\langle v_x \rangle = \sqrt{\frac{m}{2\pi kT}} \cdot v_{0x} \cdot \int e^{-\frac{m}{2kT}x^2} dx = v_{0x}$, so $\langle \vec{v} \rangle = \vec{v}_0$

(c) $\bar{\varepsilon} = \frac{1}{n_0}\int \frac{1}{2} m |\vec{v}|^2 \cdot f(\vec{v})\, d^3\vec{v} = \frac{m}{2} \int (v_x^2 + v_y^2 + v_z^2)\, f(v_x) f(v_y) f(v_z)\, dv_x dv_y dv_z$

$= \frac{m}{2} \int f_y\, dv_y \int f_z\, dv_z \left[\int v_x^2 f_x\, dv_x + (v_y^2 + v_z^2)\int f_x\, dv_x\right]$

* $\int v_x^2 f_x\, dv_x = \sqrt{\frac{a}{\pi}} \int (x + v_{0x})^2 \exp(-ax^2)\, dx$ (with $x = v_x - v_{0x}$; $a = \frac{m}{2kT}$)

$= \sqrt{\frac{a}{\pi}} \int x^2 e^{-ax^2} dx + \sqrt{\frac{a}{\pi}} v_{0x}^2 \int e^{-ax^2} dx$ ($x \cdot v_{0x} e^{-ax^2}$ is odd, vanishes under integration)

$= \sqrt{\frac{a}{\pi}} \cdot \sqrt{\pi} \cdot \frac{a^{-\frac{3}{2}}}{4} + v_{0x}^2 = \frac{kT}{m} + v_{0x}^2$

so $\bar{\varepsilon} = \frac{m}{2} \int f_y\, dv_y \cdot \int \left(\frac{kT}{m} + v_{0x}^2 + v_y^2 + v_z^2\right) f_z\, dv_z$, similarly

$= \frac{m}{2} \int f_y \cdot \left(\frac{kT}{m} + v_{0x}^2 + \frac{kT}{m} + v_{0z}^2 + v_y^2\right) f_y\, dv_y$

$= \frac{m}{2}\left(\frac{kT}{m} + v_{0x}^2 + \frac{kT}{m} + v_{0z}^2 + \frac{kT}{m} + v_{0y}^2\right) = \frac{3kT}{2} + \frac{1}{2} m (v_{0x}^2 + v_{0y}^2 + v_{0z}^2)$

3. (d) $\vec{\vec{P}}_s = m\int (\vec{v}-\vec{v}_0)(\vec{v}-\vec{v}_0)\cdot f(\vec{v})\, d^3\vec{v}$

$= n_0 m\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\cdot\int \vec{u}\,\vec{u}\cdot \exp\left[-\frac{m}{2kT}\cdot u^2\right] d^3\vec{u}$

similarly to Question 1, with $\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\cdot e^{-\frac{m}{2kT}u^2}$ as modified isotropic distribution

the $\vec{\vec{P}}_s$ must be diagonal and isotropic

$P_{s,xx} = n_0\sqrt{\frac{a}{\pi}}\int u_x u_x\cdot e^{-au_x^2}\, du_x = n_0 m\cdot\frac{kT}{m}$ (according to (c))

so $P_{s,xx} = n_0 kT = p$, $\vec{\vec{P}}_s = \mathrm{diag}\{p, p, p\}$

4. $f(\vec{v}) = n_0\cdot\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\cdot \exp\left[-\frac{m}{2kT}\cdot|\vec{v}|^2\right]$, scalar speed has distribution in form of:

$$f(v) = n_0\cdot 4\pi v^2\cdot\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\cdot \exp\left(-\frac{m}{2kT}\cdot v^2\right)$$

$m_{N_2} = 14.0067\,u = 2.326\times10^{-26}\,kg$, $T = 293.15\,K$

and $n_0 = 80\%\cdot P/kT = 0.8\cdot\frac{1\,atm}{kT} = 2.0029\times10^{25}/m^3$

$N(1000\sim1001) \doteq f(v=1000)\cdot\Delta v = 1.008\times10^{22}$

$N(2000\sim2001) \doteq f(v=2000)\cdot\Delta v = 3.871\times10^{18}$

5. $\partial_t f + \vec{v} \cdot \nabla f + \frac{q}{m} \left(\vec{E} + \vec{v} \times \vec{B} \right) \cdot \nabla_v f = 0$, vlasov equation

when $B=0$, it becomes: $\partial_t f + \vec{v} \cdot \nabla f + \frac{q\vec{E}}{m} \cdot \nabla_v f = 0$.

given $f(\vec{v}, \vec{x}, t) = n_s \left(\frac{m_s}{2\pi k T_s} \right)^{\frac{3}{2}} \cdot \exp\left(-\frac{m_s}{2kT_s} \cdot v^2 - \frac{q_s}{kT_s} \cdot \varphi \right)$

$= A \cdot \exp(-a v^2) \cdot \exp(-b\varphi)$,

$\partial_t f = 0$,

$\nabla f = \frac{\partial f}{\partial \varphi} \cdot \nabla \varphi = bf \cdot \vec{E}$

$\nabla_v f = \frac{\partial f}{\partial v^2} \cdot \nabla_v v^2 = -af \cdot 2\vec{v}$

$\vec{v} \cdot \nabla f + \frac{q_s}{m_s} \cdot \vec{E} \cdot \nabla_v f$

$= f \cdot \left(\frac{q_s}{kT_s} \cdot \vec{v} \cdot \vec{E} - \frac{q_s}{m_s} \cdot \frac{m_s}{2kT_s} \cdot 2\vec{v} \cdot \vec{E} \right) = 0$, satisfied .