Physics 5150

Homework Set # 10

Due 5 pm Thursday 4/12/2018

SOLUTIONS

Problem 1: **Z-pinch Equilibrium**

A plasma device called the Z-pinch is a cylindrical plasma column with the plasma current flowing along the axis (in the z-direction) and with no external magnetic field. The only magnetic field present is that produced by the plasma current itself and is in the azimuthal direction. The plasma in the Z-pinch is confined radially by the Lorentz force associated with the axial current and the azimthal magnetic field (it is simply a manifestation of the idea that parallel currents attract each other).

Consider an infinitely long Z-pinch of radius a with a uniform axial current density: $j_z(r) =$ *const. Find the equilibrium radial gas pressure profile* $p(r)$ *satisfying the boundary condition* $p(a) = 0$.

Solution:

First, let us find the azimuthal magnetic field profile; using Ampere's law and the fact that $j_z(r) = \text{const}$, we get

$$2\pi r B_\theta(r) = \frac{4\pi}{c}\, I(r) = \frac{4\pi}{c}\, \pi r^2\, j_z \quad \Rightarrow \quad B_\theta(r) = \frac{2\pi}{c}\, r j_z\,. \tag{1}$$

The pressure profile is then determined from the magnetostatic equilibrium condition, i.e., by balancing the pressure's gradient with the radial Lorentz force (also called the *"pinch force"*):

$$\nabla p = \frac{1}{c}\mathbf{j}\times\mathbf{B} \quad \Rightarrow \quad \frac{dp(r)}{dr} = c^{-1} j_z(r) B_\theta(r) [\hat{\mathbf{z}}\times\hat{\theta}]_r = -\frac{2\pi}{c^2}\, r j_z^2\,. \tag{2}$$

Since $j_z(r) = \text{const}$, this equation can be trivially integrated:

$$p(r) = p(0) + \int_0^r \frac{dp}{dr}\, dr = p(0) - \frac{\pi}{c^2}\, j_z^2 r^2\,. \tag{3}$$

Applying the boundary condition $p(a) = 0$, we finally get

$$p(r) = \frac{\pi}{c^2}\, j_z^2\, (a^2 - r^2)\,. \tag{4}$$

Problem 2: **Plasma beta**

The plasma β is an important parameter in plasma physics; it is defined as the ratio of the thermal plasma pressure, $P = 2nk_BT$, to the magnetic pressure, $P_{\rm magn} = B^2/8\pi$. Estimate the plasma β parameter for the following environments (please pay attention to units!):

1) Tokamak plasma: $B = 5$ Tesla; $n = 10^{20}\ \mathrm{m}^{-3}$; $kT_i = kT_e = 10$ keV
2) Solar Corona: $B = 100$ Gauss; $n = 10^{10}\ \mathrm{cm}^{-3}$; $T_i = T_e = 2\times10^6$ K
3) Solar Photosphere: $B = 100$ Gauss; $n = 10^{17}\ \mathrm{cm}^{-3}$; $T_i = T_e = 6,000$ K.

Which of these environments are magnetically-dominated ($\beta \ll 1$) ?

Solution:

The general expression for the plasma β parameter is:

$$\beta = \frac{P}{B^2/8\pi} = 8\pi\,\frac{P_e + P_i}{B^2} = 16\pi\,\frac{nkT}{B^2}\,. \tag{5}$$

Substituting the relevant parameter values into this expression, we get:
1) Tokamak: $\beta \simeq 0.01 \ll 1$;
2) Solar Corona: $\beta \simeq 1.3\times10^{-2} \ll 1$;
3) Solar Photosphere: $\beta \simeq 400 \gg 1$.

Thus, the tokamak and the solar corona are magnetically-dominated, whereas the solar photosphere is not.

Problem 3: **Screw-pinch (force-free) equilibrium**

If a low-β plasma is in a magnetostatic equilibrium, such an equilibrium is called force-free, basically because the magnetic forces have to balance themselves to zeroth order in β (the pressure-gradient force is small). Neglecting the inertial and pressure forces, the MHD equation of motion then simply becomes $\mathbf{j}\times\mathbf{B} = 0$; that is, the current density vector is parallel to the magnetic field and hence can be written as $\mathbf{j}(\mathbf{r}) = (c/4\pi)\alpha(\mathbf{r})\,\mathbf{B}(\mathbf{r})$, where $\alpha(\mathbf{r})$ is a scalar function. It is customary to rewrite this equation using nonrelativistic Ampere's law, $\mathbf{j} = (c/4\pi)[\nabla\times\mathbf{B}]$, as

$$\nabla\times\mathbf{B} = \alpha(\mathbf{r})\,\mathbf{B}(\mathbf{r})\,. \tag{6}$$

Furthermore, by taking the divergence of this equation, it is easy to show that $\alpha(\mathbf{r})$ has to be constant along magnetic field lines, $\mathbf{B}\cdot\nabla\alpha = 0$.

Equation (6) above is called the force-free equation; it is the main partial differential equation (PDE) governing the structure of a force-free equilibrium. Any magnetic field satisfying this equation is called a force-free field. An important special case of the force-free equilibrium is the case when $\alpha(\mathbf{r})$ is not just constant along field lines, but is constant everywhere in space: $\alpha_0(\mathbf{r}) = \alpha_0 = \mathrm{const}$. This is called a linear force-free field, because the force-free equation (6) becomes linear in this case. A particularly important example of a linear force-free equilibrium is the so-called screw-pinch, an axisymmetric (and also translationally symmetric in z) cylindrical configuration with both an axial (z) and an azimuthal (θ) magnetic field components, but with no radial field; it is basically a combination of a θ-pinch and a Z-pinch.

Use the θ and z components of equation (6) with constant $\alpha(\mathbf{r}) = \alpha_0$, to derive the structure of the screw pinch (i.e., the radial profiles of B_z and B_θ). Use the boundary conditions $B_\theta(r = 0) = 0$, and $dB_z/dr(r = 0) = 0$. [Hint: you should get the so-called Bessel differential equation as a result, and the solutions most relevant to this problem are the special functions known as the zeroth and first order Bessel functions of the first kind. You can read about them on Wikipedia or in Wolfram MathWorld or in any mathematical handbook, if you need to get more information.]

Solution:

First, although I am not asking you to prove that α is constant along field lines, I still would like to show you how this is done. To do this, one just needs to take divergence of equation (6):

$$\nabla \cdot [\nabla \times \mathbf{B}] = \nabla \cdot (\alpha \mathbf{B}) = \alpha_0 (\nabla \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla)\alpha \,. \tag{7}$$

Using the mathematical identity $\nabla \cdot [\nabla \times \mathbf{B}] \equiv 0$ and the fact that $\nabla \cdot \mathbf{B} = 0$, we thus get

$$(\mathbf{B} \cdot \nabla)\alpha = 0 \,, \tag{8}$$

which means that α is constant along the magnetic field.

Now, let us consider the structure of the force-free screw pinch. Here we will work in cylindrical polar coordinates and take into account both the axial symmetry ($\partial_\theta = 0$) and the translational symmetry along the z axis ($\partial_z = 0$). Then all the quantities are functions of radius r only. We also take into account that $B_r = 0$.

The z-component of the force-free equation (6) yields

$$\frac{1}{r} \frac{d}{dr} (r B_\theta) = \frac{dB_\theta}{dr} + \frac{B_\theta}{r} = \alpha_0 \, B_z \,, \tag{9}$$

and the θ-component of equation (6), using $B_r = 0$, yields

$$-\frac{dB_z}{dr} = \alpha_0 \, B_\theta \,. \tag{10}$$

To find $B_z(r)$, we can substitute B_θ from eq. (10) into the left-hand side of eq. (9); we then get

$$\frac{d^2 B_z}{dr^2} + \frac{1}{r} \frac{dB_z}{dr} + \alpha_0^2 B_z = 0 \,. \tag{11}$$

By defining a new dimensionless coordinate $x \equiv \alpha_0 r$, this equation can be written as

$$x^2 B_z''(x) + x B_z'(x) + x^2 B_z(x) = 0 \,, \tag{12}$$

where prime denotes differentiation with respect to x. This equation is a standard linear ordinary differential equation called the Bessel equation of zeroth order. Its general solution that is regular at the origin $x = 0$ is

$$B_z = B_0 \, J_0(x) = B_0 \, J_0(\alpha_0 r) \,, \tag{13}$$

where B_0 is an arbitrary normalization constant and $J_0(x)$ is the zeroth order Bessel function of the first kind.

To find $B_\theta(r)$, one can use equation (10) and the following mathematical property of Bessel's functions: $J_0'(x) = -\,J_1(x)$, where $J_1(x)$ is the first order Bessel function of the first kind. we then have

$$B_\theta(r) = -\,\frac{1}{\alpha_0}\,\frac{dB_z}{dr} = -\,B_0\,\frac{dJ_0}{dx} = B_0 J_1(\alpha_0 r)\,. \tag{14}$$