HW 12 of Plasma

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1. When $\vec{k} \parallel \vec{B}$, $\quad B_1 = -B_0 \cdot \frac{u_1}{V_A}$

so $\frac{E_m}{E_k} = \frac{\frac{1}{8\pi} \cdot B_0^2 \cdot \frac{u_1^2}{V_A^2}}{\frac{1}{2} \rho_0 \cdot u_1^2} = \frac{B_0^2}{4\pi \rho_0 \cdot V_A^2}$, apply. $V_A = \frac{B_0}{\sqrt{4\pi\rho_0}}$

$E_m / E_k = 1$, the disturbed magnetic & kinetic energy are same.

2. Applying $V_A = B_0 / \sqrt{\mu_0 \cdot n_i \cdot m_p}$

(a) $V_A = 2.725 \times 10^6 \ m/s$

(b) $V_A = 1.380 \times 10^6 \ m/s$

(c) $V_A = 6.898 \times 10^6 \ m/s$

3. Taking displacement current into consideration.

$$\nabla\times B = \mu_0 J + \frac{1}{c^2}\partial_t E$$

(6.5.2) becomes: $\rho_0 \partial_t \vec{U} = \frac{1}{\mu_0}(\nabla\times B - \frac{1}{c^2}\partial_t E)\times B_0 - \nabla P$

then (6.5.7) becomes: $-i\omega\rho_0\cdot\vec{U} = \frac{1}{\mu_0}(i\vec{k}\times\vec{B} + \frac{i\omega}{c}E)\times B_0 - \nabla P$

and (6.5.12) becomes: $\omega^2\vec{U} = \frac{1}{\mu_0\rho_0}\left(\vec{k}\times[\vec{k}\times(\vec{U}\times\vec{B}_0) - \frac{\omega^2}{c^2}E\right)\times B_0 + V_s^2\,\vec{k}(\vec{k}\cdot\vec{U})$

Now apply $\vec{B}_0 = B_0\hat{k}$, $\vec{k} = k\hat{k}$, $\theta = 0$ $(\hat{i},\hat{j},\hat{k}$ coordinate)

Solving extra E term, we need help from Faraday Law:

$$\nabla\times E = -\partial_t B \quad , \text{ linearized as}$$

$$\vec{k}\times\vec{E} = \omega\vec{B}$$

so $-E\times B = \vec{B}\times\vec{E} = \frac{B_0}{k}\cdot\vec{k}\times\vec{E} = \frac{\omega B_0}{k}\cdot\vec{B}$, then apply

$\omega\vec{B} = -\vec{k}\times(\vec{U}\times\vec{B}_0)$, we get:

$$-\frac{\omega^2}{c^2}E\times B = -\frac{\omega^2}{c^2}\cdot\frac{B_0}{k}\cdot\vec{k}\times(\vec{U}\times\vec{B}_0) = \frac{\omega^2 B_0^2}{c^2}\begin{pmatrix}U_x\\U_y\\0\end{pmatrix}$$ add it into

(6.5.13) we get: $\left(\frac{\omega}{k}\right)^2\cdot\begin{pmatrix}U_x\\U_y\\U_z\end{pmatrix} = V_A^2\cdot\left(1+\frac{\omega^2}{k^2c^2}\right)\cdot\begin{pmatrix}U_x\\U_y\\0\end{pmatrix} + V_s^2\cdot\begin{pmatrix}0\\0\\U_z\end{pmatrix}$

so the despersion relation should be: $\boxed{\left(\frac{\omega}{k}\right)^2 = V_A^2\left(1+\left(\frac{\omega}{k}\right)^2/c^2\right)}$

or: $\frac{\omega}{k} = V_A/\sqrt{1-(V_A/c)^2} = \gamma_A\cdot V_A$

so $U_p = U_g = \gamma_A\cdot V_A$; $(\gamma_A = 1/\sqrt{1-(V_A/c)^2})$

4. When propagation direction is perpendicular to $B_0 \hat{z}$,

$\vec{v} = (v_x, v_y)$, then (6.5.13) becomes

$$v_p^2 \begin{pmatrix} v_x \\ v_y \end{pmatrix} = V_A^2 \begin{pmatrix} v_x \\ v_y \cdot \cos^2\theta \end{pmatrix} + V_s^2 \begin{pmatrix} v_x \cdot \sin^2\theta \\ 0 \end{pmatrix}$$

namely:

$$\begin{pmatrix} v_p^2 - v_A^2 - v_s^2 \cdot \sin^2\theta , & 0 \\ 0 , & v_p^2 - v_A^2 \cdot \cos^2\theta \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0$$

the eigen-roots are:

$$v_p^2 = V_A^2 + V_s^2 \sin^2\theta \qquad (a)$$

$$v_p^2 = V_A^2 \cdot \cos^2\theta \qquad (b$$

the fast magnetosonic mode is (a), and dispersion relation is:

$$\frac{\omega}{k} = \sqrt{V_A^2 + V_s^2 \cdot \sin^2\theta}$$

the $v_p = v_g = \sqrt{V_A^2 + V_s^2 \cdot \sin^2\theta} = \sqrt{\frac{B_0^2}{\mu_0 \rho_0} + \frac{\gamma P_0}{\rho_0} \sin^2\theta}$