PHYS/ASTR-5150 Spring 2018

Physics 5150

Homework Set # 7

Due 4 pm Friday 3/16/2018!

SOLUTIONS

Problem 1: Pressure tensor
Show that if the distribution function is isotropic (i.e., spherically symmetric), then the pressure tensor is diagonal.

Solution:
A tensor is diagonal if its off-diagonal components are equal to zero. Consider, for example, the xy component of the pressure tensor:

$$\Pi_{xy} = m \int\int\int w_x w_y \, f(\mathbf{v}) \, d^3\mathbf{v} = m \int\limits_{-\infty}^{+\infty} dv_x \int\limits_{-\infty}^{+\infty} dv_y \int\limits_{-\infty}^{+\infty} dv_z \, w_x w_y \, f(\mathbf{v}) \,. \tag{1}$$

First, since $f(\mathbf{v})$ is spherically symmetric, the fluid velocity is zero, $\mathbf{u} \equiv \bar{\mathbf{v}} = n^{-1} \int \mathbf{v} f(\mathbf{v}) \, d^3\mathbf{v} = 0$, and so $\mathbf{w} \equiv \mathbf{v} - \mathbf{u} = \mathbf{v}$. Thus,

$$\Pi_{xy} = m \int\int\int v_x v_y \, f(\mathbf{v}) \, d^3\mathbf{v} = m \int\limits_{-\infty}^{+\infty} dv_x \int\limits_{-\infty}^{+\infty} dv_y \int\limits_{-\infty}^{+\infty} dv_z \, v_x v_y \, f(\mathbf{v}) \,. \tag{2}$$

Since the distribution function $f(\mathbf{v})$ is spherically symmetric, i.e., depends only on the magnitude, but not on the direction of the velocity vector, this function is even in each of the velocity components, e.g., $f(v_x, v_y, v_z) = f(-v_x, v_y, v_z) = f(v_x, -v_y, v_z)$, etc. Therefore, in the above expression, the v_x integral, for example, automatically gives zero, i.e., $\Pi_{xy} = 0$. One can show, in a similar manner, that all the other off-diagonal components are also identically zero.

Problem 2: Energy Distribution Function
Consider a gas with a given isotropic velocity distribution function, $f(\mathbf{v}) = f(|v|)$. What is the energy distribution function, $F_\epsilon(\epsilon)$ of such a gas? Here, $\epsilon = mv^2/2$ is the kinetic energy of a particle and F_ϵ is normalized according to

$$\int\limits_0^{\infty} F_\epsilon(\epsilon) \, d\epsilon = n \,. \tag{3}$$

Solution:

Let us consider particles in a certain small energy bin $(\epsilon, \epsilon + d\epsilon)$. The number density of such particles is $F_\epsilon(\epsilon)d\epsilon$. On the other hand, we can also look at these same particles in the 3D velocity space, where their number density can be written as $f(\mathbf{v})d^3\mathbf{v}$. Because these are the same particles, the two densities are equal:

$$F_\epsilon(\epsilon)\, d\epsilon = f(\mathbf{v})\, d^3\mathbf{v}\,. \tag{4}$$

We just need to figure out the 3D velocity-space volume element $d^3\mathbf{v}$. In this velocity space, the particles under consideration occupy an infinitesimally thin spherical shell of radius $v = (2\epsilon/m)^{1/2}$ and of thickness dv related to $d\epsilon$ via $d\epsilon = d(mv^2/2) = mvdv$. The volume of this shell in the velocity space is $4\pi v^2 dv$ and thus we obtain:

$$F_\epsilon(\epsilon)d\epsilon = F_\epsilon(\epsilon)\, mvdv = f(v)\, 4\pi v^2 dv\,. \tag{5}$$

That is,

$$F_\epsilon(\epsilon) = f(v)\,\frac{4\pi v}{m} = f\left(\sqrt{\frac{2\epsilon}{m}}\right)\frac{4\pi}{m}\sqrt{\frac{2\epsilon}{m}}\,. \tag{6}$$

Problem 3: Maxwellian distribution function

The general form of a drifting Maxwellian distribution is

$$f(\mathbf{v}) = n_0\left(\frac{m}{2\pi kT}\right)^{3/2}\exp[-\,m(\mathbf{v}-\mathbf{U}_0)^2/2kT]\,. \tag{7}$$

Show by explicit calculation that:

(a) *the plasma density corresponding to this distribution is equal to n_0;*

(b) *the average particle velocity corresponding to this distribution is equal to $\mathbf{U}_0$;*

(c) *the average particle kinetic energy is $\bar{\mathcal{E}} = (3/2)kT + mU_0^2/2$;*

(d) *the pressure tensor corresponding to this distribution is diagonal and isotropic — i.e., that $\mathbf{\Pi} = \mathrm{diag}\{P,P,P\}$ or, equivalently, $\Pi_{ij} = P\delta_{ij}$ — and that the scalar pressure is given by $P = nkT$.*

Solution:

(a) Plasma density:

Using the substitution $\mathbf{w} = \mathbf{v} - \mathbf{U_0}$, we get:

$$\int_{-\infty}^{+\infty} f(\mathbf{v})\, d^3v \;=\; \int_{-\infty}^{+\infty} f(\mathbf{w}+\mathbf{U_0})\, d^3w = n_0\left(\frac{m}{2\pi kT}\right)^{3/2}\int_{-\infty}^{+\infty}\exp(-\,mw^2/2kT)\, d^3w \tag{8}$$

$$=\; n_0\left(\frac{m}{2\pi kT}\right)^{3/2}\left[\int_{-\infty}^{+\infty}\exp(-\,mw_x^2/2kT)\, dw_x\right]^3 \tag{9}$$

$$= \quad n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{2kT}{m}\right)^{3/2} \left[\int\limits_{-\infty}^{+\infty} \exp(-\,\xi^2)\, d\xi\right]^3 = n_0\,. \tag{10}$$

(b) *Average velocity:*

$$\bar{\mathbf{v}} \;=\; \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} \mathbf{v}\, f(\mathbf{v})\, d^3v = \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} (\mathbf{w} + \mathbf{U_0})\, f(\mathbf{w} + \mathbf{U_0})\, d^3w \tag{11}$$

$$=\; \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} \mathbf{w}\, f(\mathbf{w} + \mathbf{U_0})\, d^3w + \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} \mathbf{U_0}\, f(\mathbf{w} + \mathbf{U_0})\, d^3w\,. \tag{12}$$

The first integral is zero because it's an integral of an antisymmetric function of $\mathbf{w}$ *over a symmetric domain, e.g.,:*

$$\int\limits_{-\infty}^{+\infty} w_x\, f(\mathbf{w} + \mathbf{U_0})\, dw_x \sim \int\limits_{-\infty}^{+\infty} w_x\, \exp(-\,mw_x^2/2kT)\, dw_x = 0\,. \tag{13}$$

In the second integral, we can just pull $\mathbf{U_0}$ *out of the integral and thus get*

$$\bar{\mathbf{v}} = \frac{\mathbf{U_0}}{n_0} \int\limits_{-\infty}^{+\infty} f(\mathbf{w} + \mathbf{U_0})\, d^3w = \frac{\mathbf{U_0}}{n_0}\, n_0 = \mathbf{U_0}\,. \tag{14}$$

(c) *Average particle energy:*

$$\bar{\mathcal{E}} \;=\; \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} \frac{mv^2}{2}\, f(\mathbf{v})\, d^3v = \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} \frac{m(\mathbf{w} + \mathbf{U_0})^2}{2}\, f(\mathbf{w} + \mathbf{U_0})\, d^3w \tag{15}$$

$$=\; \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} \frac{mw^2}{2}\, f(\mathbf{w} + \mathbf{U_0})\, d^3w + \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} m(\mathbf{w} \cdot \mathbf{U_0})\, f(\mathbf{w} + \mathbf{U_0})\, d^3w + \frac{1}{n_0} \int\limits_{-\infty}^{+\infty} \frac{mU_0^2}{2}\, f(\mathbf{w} + \mathbf{U_0})\, d^3w \tag{16}$$

The first integral yields:

$$\frac{1}{n_0} \int\limits_{-\infty}^{+\infty} \frac{mw^2}{2}\, f(\mathbf{w} + \mathbf{U_0})\, d^3w \tag{17}$$

$$=\; \frac{1}{n_0} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \frac{m(w_x^2 + w_y^2 + w_z^2)}{2}\, n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\,mw^2/2kT}\, dw_x dw_y dw_z \tag{18}$$

$$=\; 3 \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \frac{mw_x^2}{2} \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\,mw^2/2kT}\, dw_x dw_y dw_z \tag{19}$$

$$=\; 3\pi^{-3/2}\, kT \left[\int\limits_{-\infty}^{+\infty} \xi^2 e^{-\xi^2}\, d\xi\right] \left[\int\limits_{-\infty}^{+\infty} e^{-\xi^2}\, d\xi\right]^2 = 3\pi^{-3/2}\, kT\, \frac{\sqrt{\pi}}{2}\, (\sqrt{\pi})^2 = \frac{3}{2}\, kT\,. \tag{20}$$

The second integral is zero because it's an integral of an antisymmetric function of $\mathbf{w} \cdot \mathbf{U_0}$ *over a symmetric domain, e.g.,:*

$$mU_{0x} \int_{-\infty}^{+\infty} w_x \, f(\mathbf{w} + \mathbf{U_0}) \, dw_x \sim mU_{0x} \int_{-\infty}^{+\infty} w_x \, \exp(-\, m w_x^2 / 2kT) \, dw_x = 0 \, . \tag{21}$$

In the third integral, we can just pull $mU_0^2/2$ *out of the integral and get*

$$\frac{1}{n_0} \frac{mU_0^2}{2} \int_{-\infty}^{+\infty} f(\mathbf{w} + \mathbf{U_0}) \, d^3w = \frac{1}{n_0} \frac{mU_0^2}{2} \, n_0 = \frac{mU_0^2}{2} \, . \tag{22}$$

Putting all this together, we get

$$\bar{\mathcal{E}} = \frac{3}{2} \, kT + \frac{mU_0^2}{2} \, . \tag{23}$$

(d) <u>*Pressure tensor:*</u>

$$\mathbf{\Pi} \;=\; m \int_{-\infty}^{+\infty} \mathbf{ww} \, f(\mathbf{v}) \, d^3v = m \int_{-\infty}^{+\infty} \mathbf{ww} \, f(\mathbf{w} + \mathbf{U_0}) \, d^3w \tag{24}$$

$$\;=\; m \, n_0 \left(\frac{m}{2\pi kT} \right)^{3/2} \int_{-\infty}^{+\infty} \mathbf{ww} \, \exp[-\, m(w_x^2 + w_y^2 + w_z^2)/2kT] \, dw_x dw_y dw_z \, . \tag{25}$$

It is clear from symmetry considerations that all the off-diagonal components of this tensor vanish because they involve integrating an odd function over a symmetric domain. It is also clear from the above expression that all the diagonal components are equal to each other, $\Pi_{xx} = \Pi_{yy} = \Pi_{zz}$*, and are equal to*

$$P \;\equiv\; m \, n_0 \left(\frac{m}{2\pi kT} \right)^{3/2} \int_{-\infty}^{+\infty} w_x^2 \, \exp[-\, m(w_x^2 + w_y^2 + w_z^2)/2kT] \, dw_x dw_y dw_z \tag{26}$$

$$\;=\; 2kT \, n_0 \, \pi^{-3/2} \int_{-\infty}^{+\infty} \xi_x^2 \, \exp[-\, (\xi_x^2 + \xi_y^2 + \xi_z^2)] \, d\xi_x \, d\xi_y \, d\xi_z = n_0 kT \, . \tag{27}$$

<u>**Problem 4:**</u>

Consider air (80% N_2 and 20% O_2) at normal pressure and at a room pressure (20 C). Assuming (non-drifting) Maxwellian distribution function for the air molecules, how many nitrogen molecules in 1 m^3 *have velocities in the range between 1000 m/sec and 1001 m/sec? How many have velocities between 2000 m/sec and 2001 m/sec?*

<u>**Solution:**</u>

For any isotropic distribution function f (such as Maxwellian) the number density of particles *in an infinitesimal velocity interval* $(v, v + dv)$ *is*

$$f(\mathbf{v})d^3\mathbf{v} = 4\pi v^2\, f(v)\, dv\,. \tag{28}$$

The Maxwellian distribution function is

$$f_M(\mathbf{v}) = n\,(\sqrt{\pi} v_{\rm th})^{-3}\, \exp\,(-v^2/v_{\rm th}^2)\,, \tag{29}$$

where $v_{\rm th} \equiv (2kT/m)^{1/2}$ *is the thermal velocity of particles of mass* m *at temperature* T*. For molecular nitrogen at room temperature the thermal velocity is*

$$v_{N_2,{\rm th}} = \sqrt{\frac{2kT}{m_{N_2}}} \simeq \sqrt{\frac{2kT}{28m_p}} \simeq 416\,{\rm m/sec}\,. \tag{30}$$

Next, we need the number density of nitrogen molecules. The total number density of all molecules in the atmosphere is

$$n = \frac{P}{kT} \simeq 2.5 \times 10^{25}\,{\rm m}^{-3}\,, \tag{31}$$

where $P = 101.3\,{\rm kPa}$ *is the normal atmospheric pressure at sea level. Since nitrogen constitutes approximately 80% of air molecules, its particle number density is then about*

$$n_{N_2} \simeq 2 \times 10^{25}\,{\rm m}^{-3}\,. \tag{32}$$

Thus, the number of nitrogen molecules in 1 ${\rm m}^3$ *with velocities from* $v_1 = 1000$ *m/sec and* $v_1 + dv = 1001$ *m/sec is*

$$n_1 = 4\pi v_1^2\, n_{N_2}\,(\sqrt{\pi} v_{N_2,{\rm th}})^{-3}\, \exp\,(-v_1^2/v_{N_2,{\rm th}}^2)\, dv \simeq 1.9 \times 10^{21}\,{\rm m}^{-3}\,, \tag{33}$$

and similarly, the number of nitrogen molecules in 1 ${\rm m}^3$ *with velocities from* $v_2 = 2000$ *m/sec and* $v_2 + dv = 2001$ *m/sec is*

$$n_2 = 4\pi v_2^2\, n_{N_2}\,(\sqrt{\pi} v_{N_2,{\rm th}})^{-3}\, \exp\,(-v_2^2/v_{N_2,{\rm th}}^2)\, dv \simeq 2.3 \times 10^{14}\,{\rm m}^{-3}\,. \tag{34}$$

Problem 5:

A particle species s *has a distribution function of the form*

$$f_s(\mathbf{x}, \mathbf{v}, t) = n_s \left(\frac{m_s}{2\pi k T_s}\right)^{3/2} \exp\Big[-\frac{m_s v^2/2 + q_s\phi}{kT_s}\Big]\,, \tag{35}$$

where $\phi = \phi(\mathbf{x})$ *is an electrostatic potential (constant in time),* $\mathbf{E} = -\nabla\phi$*, and the particle number density* $n_s = n_{0s}$ *and temperature* $T_s = T_{0s}$ *are both uniform in space and constant in time. Show that this distribution function satisfies the Vlasov equation with* $\mathbf{B} = 0$*,*

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q_s\mathbf{E}}{m_s}\cdot\nabla_{\mathbf{v}} f = 0\,. \tag{36}$$

Solution:

Here one assumes that the parameters n_s and T_s are constant in space and time and hence the spatial variation of the distribution function arises only due to the spatial non-uniformity of the electrostatic potential $\phi(\mathbf{r})$. Since the potential is taken to be stationary in time, the distribution function is also stationary, and hence the first term in the Vlasov equation is zero:

$$\frac{\partial f}{\partial t} = 0\,. \tag{37}$$

The second term in the Vlasov equation is, using chain rule,

$$\mathbf{v}\cdot\nabla f = \frac{\partial f}{\partial \phi}\,\mathbf{v}\cdot\nabla\phi(\mathbf{r}) = -\frac{q_s}{kT_s}\,f\,(-\,\mathbf{v}\cdot\mathbf{E}) = \frac{q_s}{kT_s}\,f\,(\mathbf{v}\cdot\mathbf{E})\,. \tag{38}$$

The third term is:

$$\frac{q\mathbf{E}}{m_s}\cdot\nabla_{\mathbf{v}} f = \frac{q\mathbf{E}}{m_s}\cdot\left(-\frac{m_s\mathbf{v}}{kT_s}\right) f = -\frac{q_s}{kT_s}\,f(\mathbf{E}\cdot\mathbf{v})\,. \tag{39}$$

Thus we see that the second and third terms cancel and hence the entire left-hand side of the Vlasov equation yields zero.