Physics 5150

Homework Set # 6

Due 5 pm Thursday 3/1/2018

SOLUTIONS

Problem 1: **Gradient and Curvature drifts.**
The equation for a dipole magnetic field in spherical coordinates (r, θ, ϕ) is given by (in SI units)

$$\vec{B} = \frac{\mu_0 M}{4\pi} \frac{1}{r^3} \left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right), \tag{1}$$

where M is the magnetic moment, and $\hat{r}$ and $\hat{\theta}$ are the unit vectors in the r and θ directions, respectively.

(a) *Show that the equation for a magnetic field line is $r = R\sin^2\theta$, where R is the radius of the magnetic field line at the equator ($\theta = \pi/2$).*

Solution:
The shape of a field line is given by

$$\frac{dr}{rd\theta} = \frac{B_r}{B_\theta} = \frac{2\cos\theta}{\sin\theta} \tag{2}$$

so that

$$d\ln r = 2\,\frac{\cos\theta d\theta}{\sin\theta} = 2\,\frac{d\sin\theta}{\sin\theta} = 2\,d\ln(\sin\theta) = d\ln\sin^2\theta \tag{3}$$

integrating which we obtain:

$$r(\theta) = \text{const} \times \sin^2\theta, \tag{4}$$

or

$$r(\theta) = R\sin^2\theta, \tag{5}$$

where $R \equiv r(\theta = \pi/2)$.

(b) *Show that the radius of curvature of a magnetic field line at the equator is $R_c = R/3$. [Hint: in general, the radius of curvature is given by $R_c = |(\hat{b}\cdot\nabla)\hat{b}|^{-1}$, where $\hat{b} \equiv \vec{B}/B$ is a unit vector in the direction of the magnetic field.]*

Solution:
Let us calculate the vector $(\hat{b}\cdot\nabla)\hat{b}$ at $\theta = \pi/2$.

The spherical components of the unit vector $\hat{b}$ are

$$\hat{b}_r = \frac{2\cos\theta}{\sqrt{4\cos^2\theta + \sin^2\theta}} = \frac{2\cos\theta}{\sqrt{1+3\cos^2\theta}}\,, \tag{6}$$

and

$$\hat{b}_\theta = \frac{\sin\theta}{\sqrt{1+3\cos^2\theta}}\,, \tag{7}$$

while $\hat{b}_\phi = 0$ everywhere.

The θ component of the vector $(\hat{b}\cdot\nabla)\hat{b}$, equal to

$$\hat{b}_r\partial_r\hat{b}_\theta + \frac{\hat{b}_\theta}{r}\partial_\theta\hat{b}_\theta + \frac{\hat{b}_r\hat{b}_\theta}{r}\,, \tag{8}$$

vanishes at the equator because $\hat{b}_r = 0$ and $\partial_\theta\hat{b}_\theta = 0$ here (since $\hat{b}_\theta$ is an even function of θ with respect to $\theta = \pi/2$). The ϕ (azimuthal) component of $(\hat{b}\cdot\nabla)\hat{b}$ is also obviously zero because $\hat{b}_\phi = 0$ everywhere. Thus, only the radial component remains:

$$[(\hat{b}\cdot\nabla)\hat{b}]_r = \hat{b}_r\partial_r\hat{b}_r + \frac{\hat{b}_\theta}{r}\partial_\theta\hat{b}_r - \frac{\hat{b}_\theta^2}{r}\,. \tag{9}$$

The first term in this expression vanishes at the equator because $\hat{b}$ here is in the θ direction. The second term, evaluated at the equator, is

$$\frac{\hat{b}_\theta}{r}\partial_\theta\hat{b}_r|_{\theta=\pi/2} = \frac{1}{R}\,\partial_\theta\hat{b}_r|_{\theta=\pi/2} = -\,\frac{2}{R}\,, \tag{10}$$

where $R = r(\pi/2)$. Finally, the third term is

$$-\frac{\hat{b}_\theta^2}{r} = -\frac{|\hat{b}|^2}{r} - \frac{1}{R}\,. \tag{11}$$

Thus, putting all together

$$|(\hat{b}\cdot\nabla)\hat{b}|_{\pi/2} = |[(\hat{b}\cdot\nabla)\hat{b}]_r|_{\pi/2} = \frac{3}{R}\,, \tag{12}$$

and thus the radius of curvature is equal to $R_c = R/3$.

(c) *Compute the curvature drift of a particle with a positive charge q and parallel velocity $v_\parallel$ at a radial distance R at the equator.*

Solution:
The curvature drift is given by

$$\vec{v}_c = \frac{mv_\parallel^2}{qB^2}\,\frac{[(R_c\hat{r})\times\vec{B}]}{R_c^2} = \frac{mv_\parallel^2}{qBR_c}\,[(\hat{r})\times\hat{b}]\,. \tag{13}$$

At the equator $\theta = \pi/2$, $\hat{b} = \hat{\theta}$, and so $[(\hat{r}) \times \hat{b}] = \hat{\phi}$. Thus, the curvature drift is in the azimuthal direction and is given by

$$\vec{v}_c = \frac{mv_\parallel^2}{qBR_c}\,\hat{\phi} = \frac{3mv_\parallel^2}{qBR}\,\hat{\phi}\,. \tag{14}$$

where we used the results of part (b) to get the last expression.

(d) *Compute the ∇B drift of a particle with a positive charge q and perpendicular velocity $v_\perp$ at a radial distance R at the equator.*

Solution:
The gradient drift is given by

$$\vec{v}_{\nabla B} = \frac{mv_\perp^2}{2qB}\,\frac{[\vec{B} \times \nabla B]}{B^2} = \frac{mv_\perp^2}{2qB}\,\frac{[\hat{b} \times \nabla B]}{B}\,. \tag{15}$$

At the equator, $\hat{b} = \hat{\theta}$ and $B \sim r^{-3}$, so that $\nabla B = -3(B/R)\hat{r}$. Then, $[\hat{b} \times \nabla B] = -3(B/R)\,[\hat{\theta} \times \hat{r}] = 3(B/R)\hat{\phi}$, and so

$$\vec{v}_{\nabla B} = \frac{3mv_\perp^2}{2qBR}\,\hat{\phi}\,. \tag{16}$$

(e) *Compare the directions and magnitudes of the curvature and ∇B drifts at the equator.*

Solution:
We see that the two drifts are both in the same direction (positive azimuthal direction) and their ratio is

$$\frac{|\vec{v}_c|}{|\vec{v}_{\nabla B}|} = \frac{2v_\parallel^2}{v_\perp^2}\,. \tag{17}$$

Problem 2:

A particle is trapped in a magnetic mirror field given by

$$B_z = B_0\left[1 + \left(\frac{z}{L}\right)^2\right] \tag{18}$$

and has a total kinetic energy $E = mv^2/2$ and pitch angle α_0 at $z = 0$. Find the oscillation (bounce) frequency in terms of L, E, and α_0.

Solution:
The parallel (z) equation of motion for this particle, governed by the mirror force, is

$$\frac{d^2 z}{dt^2} = \frac{dv_\parallel}{dt} = \frac{F_z}{m} = -\,\frac{\mu}{m}\,\frac{\partial B}{\partial z} = -\,\frac{\mu}{m}\,B_0\frac{2z}{L^2} = -\,\frac{2\mu B_0}{mL^2}\,z\,, \tag{19}$$

where $\mu = E_\perp / B = (E/B_0)\sin\alpha_0^2$ is the magnetic moment of the particle.

Thus, we can see that the equation of parallel motion of the particle in this magnetic mirror is that of a simple harmonic oscillator with an oscillation frequency of

$$\omega_b = \sqrt{\frac{2\mu B_0}{mL^2}} = \sqrt{\frac{2E\sin\alpha_0^2}{mL^2}} = \sqrt{\frac{2E}{m}}\frac{\sin\alpha_0}{L}\,. \tag{20}$$

Problem 3:

Consider a one-dimensional gas of particles with a velocity distribution function that has a triangular shape:

$$f(v_x) = A\,(1 - \frac{|v_x|}{v_0}), \quad |v_x| \leq v_0\,, \tag{21}$$

and

$$f(v_x) = 0 \quad |v_x| > v_0\,. \tag{22}$$

(a) *Express the constant parameter A in terms of the particle density n and v_0.*

(b) *Calculate, in terms of v_0, the following quantities:*
(i) the average velocity, $< v_x >$;
(ii) the average magnitude of velocity, $< |v_x| >$;
(iii) the average kinetic energy, $< mv_x^2/2 >$.

Solution:
(a) The normalization condition for $f(v_x)$ gives:

$$n = \int_{-\infty}^{+\infty} f(v_x)\,dv_x = A \int_{-v_0}^{+v_0} (1 - |v_x|/v_0)\,dv_x = Av_0\,. \tag{23}$$

Therefore, the normalization constant is

$$A = \frac{n}{v_0}\,. \tag{24}$$

(b) (i) The average velocity, $< v_x >$, is obviously equal to zero, since the distribution function is even:

$$< v_x >= \frac{1}{n}\int_{-\infty}^{+\infty} v_x f(v_x)\,dv_x = 0 \tag{25}$$

(ii) The average magnitude of velocity, $< |v_x| >$ is

$$< |v_x| >= \frac{1}{n}\int_{-\infty}^{+\infty} |v_x| f(v_x)\,dv_x = \frac{2}{n}\int_{0}^{v_0} v_x f(v_x)\,dv_x = 2v_0 \int_0^1 u\,(1-u)\,du = \frac{v_0}{3}\,. \tag{26}$$

(iii) the average kinetic energy, $< mv_x^2/2 >$ is:

$$< mv_x^2/2 > \;=\; \frac{m}{2n} \int\limits_{-\infty}^{+\infty} v_x^2\, f(v_x)\, dv_x = \frac{m}{n} \int\limits_{0}^{+v_0} v_x^2\, f(v_x)\, dv_x \tag{27}$$

$$= mv_0^2 \int\limits_0^1 u^2\, (1-u)\, du = \frac{mv_0^2}{12}\,. \tag{28}$$