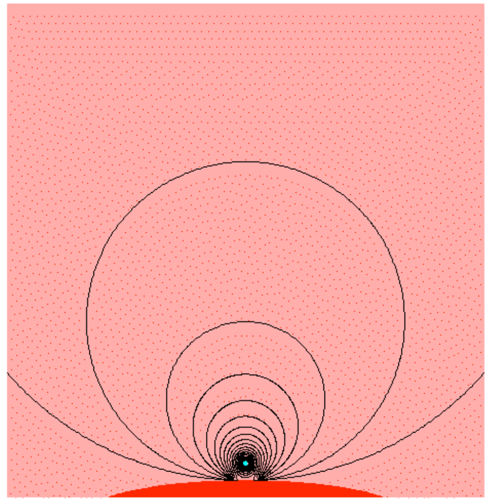
Lecture 7 - Magnetohydrodynamics II

- Topics in today's lecture:
    - Equation of Motion
    - Lorentz Forces
    - Magnetic Pressure and Tension
    - MHD Equilibria



Magnetic Forces

- First consider, momentum equation: $\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \rho \mathbf{g}$
- Moving plasma will experience the Lorentz force ($\mathbf{j}\times\mathbf{B}$) =>

$$\boxed{\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}}$$

- How do we interpret the Lorentz Force? Using Ampere's Law:

$$\mathbf{j} \times \mathbf{B} = -\frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B})$$

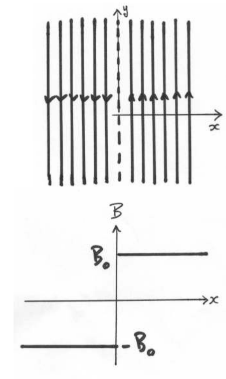
but using the vector identity $\nabla\left(\frac{1}{2}\mathbf{B}\cdot\mathbf{B}\right) = \mathbf{B} \times (\nabla \times \mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{B}$

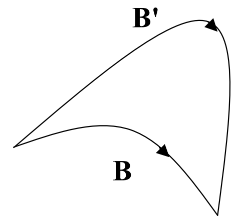
$$\Rightarrow \mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla\left(\frac{B^2}{2\mu_0}\right)$$

- 1^{st} term is magnetic tension parallel to B - important for curved fields. 2^{nd} term is magnetic pressure - important when |B| changes along field.

Magnetic Pressure and Tension

- Lorentz force can be resolved into two components:
  - Pressure term ($B^2/2\mu_0$): Isotropic. Gradient of a scalar. Perpendicular to B in example.
  - Tension term (B/μ_0) : Directed towards centre of curvature of B. e.g., B' has larger radius of curvature than B, therefore larger restoring force due to tension.
- Magnetic pressure and tension represent two kinds of restoring forces => associated with distinct wave modes: *Alfven waves and magnetoacoustic waves*.





MHD Equation of Motion

- Motion in of plasma can be described by:

$$\boxed{\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j}\times\mathbf{B} + \rho\mathbf{g}}$$

(1) *(2)* *(3)* *(4)*

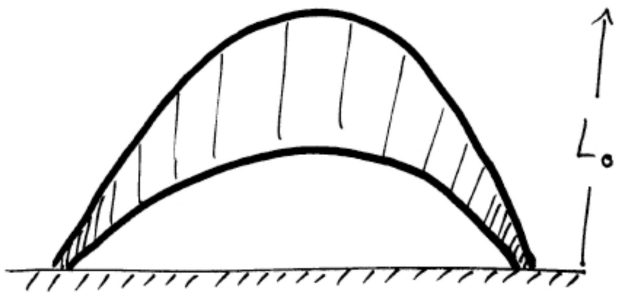
- In corona, term *(3)* dominates.
- Along **B**, **j** || **B** => *(3)* = 0, so *(2)* and *(4)* important.

$$\frac{(2)}{(4)} = \frac{p_0 / L_0}{\rho_0\, g} >> 1$$

$$=> L_0 << \frac{p_0}{\rho_0 g} = H \qquad H = \text{Scale height}$$

i.e., for length-scales smalled than the scale height.

MHD Equation of Motion



- $L_0 < H \Rightarrow p \sim$ costant.
- $L_0 > H \Rightarrow p$ falls

Pressure isobars

- Pressure gradient (∇p) - acts from high to low p.
- Is normal to isobars.

MHD Equation of Motion

$$\boxed{\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}}$$

(1) *(2)* *(3)* *(4)*

(i) $$\frac{(2)}{(3)} = \beta = \frac{p}{B^2/(2\mu)}$$

- When $\beta \ll 1$, $\mathbf{j} \times \mathbf{B}$ dominates. Condition applies in corona.

(ii) $$(1) \approx (3) \rightarrow v \approx v_A = \frac{B}{\sqrt{\mu\rho}}$$

- Perturbations or waves propagate at a a characteristic velocity, called the Alfven speed.

Typical values on Sun

	Photosphere	Chromosphere	Corona
N (m^{-3})	10^{23}	10^{20}	10^{15}
T (K)	6000	10^{4}	10^{6}
B (G)	5 - 10^{3}	100	10
β	10^{6} - 1	10^{-1}	10^{-3}
v_A (km/s)	0.05 - 10	10	10^{3}

- *1 Tesla* = 10^4 *Gauss (G)*
- $\beta = 3.5 \times 10^{-21}\, N\, T/B^2$, $v_A = 2 \times 10^9\, B/N^{1/2}$

MHD Equation of Motion - Equilibria

$$\boxed{\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j}\times\mathbf{B} + \rho\,\mathbf{g}}$$

(1) *(2)* *(3)* *(4)*

- If $v \ll v_A$, then (1) << (3) and so

$$\boxed{0 = -\nabla p + \mathbf{j}\times\mathbf{B} + \rho\,\mathbf{g},}$$

- Called *equation of magnetohydrostatic equilibrium.*
- Note: $\mathbf{j} = \nabla\times\mathbf{B}/\mu$, $\nabla.\mathbf{B} = \mathbf{0}$, $\rho = p/(\mathcal{R}\,T)$

MHD Equation of Motion - Equilibria

$$\underset{}{0} = \underset{(2)}{-\nabla p} + \underset{(3)}{\mathbf{j}\times\mathbf{B}} + \underset{(4)}{\rho\mathbf{g}}$$

- If $L_o << H$, then (4) << (2) and

$$\boxed{0 = -\nabla p + \mathbf{j}\times\mathbf{B}}$$

- Called *equation of magnetostatic equilibrium.*

- If $L_o << 2H/\beta$, then (4) << (3) and

$$\boxed{0 = \mathbf{j}\times\mathbf{B}}$$

- Called *force-free configuration.*

Potential fields

- A simple solution to the force-free equation is $\mathbf{j} = 0$, which is called a *potental field configuration.*

- From Ampere's Law, $\nabla \times \mathbf{B} = 0$

  which has a general solution $\mathbf{B} = \nabla\varphi$,

  where φ is the scalar magnetic potential.

- Substituting the general solution into the solenoid condition ($\nabla\cdot\mathbf{B}=0$) gives,

$$\nabla^2\varphi = 0 \qquad => \varphi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a_{lm}r^l + b_{lm}r^{-(l+1)})P_l^m(Cos\theta)e^{im\phi}$$

- Potential fields therefore satisfy Laplace's equation. General solution can be given in terms of associated Legendre polynomials.

Force-free fields

- Assume that $L_0 << H$ and $\beta << 1$, we have the force-free field equation. If the magnetic field is not potential then the general solution is that the current must be parallel to the magnetic field. Thus,

$$\mu \mathbf{j} \propto \mathbf{B} \quad \text{or} \quad \mu \mathbf{j} = \alpha \mathbf{B}$$

  where α is a scalar function of position (i.e., $\alpha = \alpha(\mathbf{r})$)

- From Ampere's Law => $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

- Now since $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot (\nabla \times \mathbf{B}) = 0$, we obtain $\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\alpha \mathbf{B})$

$$= \alpha \nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla \alpha$$

- Hence, $\mathbf{B} \cdot \nabla \alpha = 0$

- So that α is constant along each field line, although it may vary from field line to field line. If $\alpha = 0$, then the magnetic field reduces to the potential case.

MHD Equation of Motion - Equilibria

- If $\alpha = const$, then $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ => $\nabla \times (\nabla \times \mathbf{B}) = \nabla \times (\alpha \mathbf{B}) = \alpha \nabla \times \mathbf{B} = \alpha^2 \mathbf{B}$

- However, $\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$ and so,

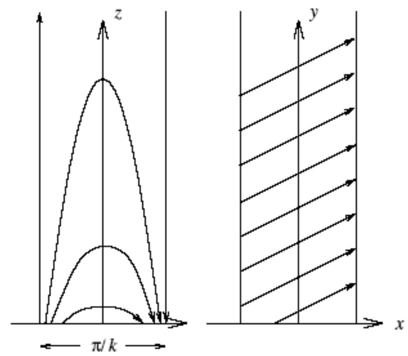
$$\nabla^2 \mathbf{B} + \alpha^2 \mathbf{B} = 0$$

- Which is the *Helmholtz Equation*. A field that fulfils this condition is force-free. When boundary conditions specified, can then use standard mathematical methods to solve.

- E.g., consider a coronal arcade with an arc (e.g, sinusoidal) in the *xz* plane and uniform in *y*. Also require field to vanish in at high altitude (e.g., exponential).

$$\begin{aligned} B_x &= B_{x,0} \sin(kx) e^{-lz} \\ B_y &= B_{y,0} \sin(kx) e^{-lz} \\ B_z &= B_0 \cos(kx) e^{-lz} \,. \end{aligned}$$

Linear force-free model



- Now from $\nabla \times \mathbf{B} = \alpha \mathbf{B}$;,

$$
\begin{aligned}
lB_{y,0} &= \alpha B_{x,0} \\
-lB_{x,0} + kB_0 &= \alpha B_{y,0} \\
kB_{y,0} &= \alpha B_0 \,.
\end{aligned}
$$

- Therefore,
$$
\begin{aligned}
B_x &= (l/k) B_0 \sin(kx) e^{-lz} \\
B_y &= (\alpha/k) B_0 \sin(kx) e^{-lz} \\
B_z &= B_0 \cos(kx) e^{-lz} \,,
\end{aligned}
$$
  where $l^2 = k^2 - \alpha^2$.

- The projection of the field lines onto the *xy*-plane are parallel straight lines:

$$B_y = \frac{\alpha}{(k^2 - \alpha^2)^{1/2}} B_x$$

  whereas the projection onto the *xz*-plane are arcs (see figure above).