Physics 5150
Homework Set # 5
Due 5 pm Thursday 02/22/2018

SOLUTIONS

Problem 1: **Gradient and Curvature drifts.**
An infinite straight wire carries a constant current I in the $+z$ direction. At $t = 0$, an electron of small gyroradius is at $z = 0$ and $r = r_0$ with $v_{\perp,0} = v_{\parallel,0}$ ($\perp$ and $\parallel$ refer to the direction relative to the magnetic field generated by the current in the wire). Calculate the magnitude and direction of the resulting guiding center drift velocity due to the sum of the gradient and curvature drifts.

Solution:
The electron will experience two drifts: the curvature drift and the gradient drift. Since the only current in the problem is that carried by the wire and the electron in question is located in the vacuum region outside of it, the magnetic fields strength drops off with radius r as $1/r$ and so the gradient length $L_B \equiv B/|\nabla B|$ is just equal to r, i.e., the same as the radius of field-line curvature $R_c = r$. Then we can combine the expressions for the two drifts as (here I am using cgs units)

$$\mathbf{v}_{\rm d} = \mathbf{v}_R + \mathbf{v}_{\nabla B} = -\frac{m_e c}{eB}\,\frac{\hat{\mathbf{R}}_c \times \hat{\mathbf{B}}}{R_c}\left(v_\parallel^2 + v_\perp^2/2\right). \tag{1}$$

In our case, $\mathbf{R}_c = r_0\,\hat{r}$ and the magnetic field is that created by the infinite straight wire with current I: $\mathbf{B} = (2I/cr_0)\,\hat{\theta}$. Thus,

$$\mathbf{v}_{\rm d} = -\frac{m_e c}{eB}\,\frac{[\hat{r}\times\hat{\theta}]}{r_0}\,\frac{3}{2}\,v_\perp^2 = -\frac{3}{4}\,\frac{m_e c^2 v_\perp^2}{eI}\,\hat{z}\,. \tag{2}$$

Problem 2:
Consider a particle moving along a stationary but spatially-varying magnetic field into a region of stronger field. As you know, the magnetic moment μ of the particle is preserved during this motion, and so the magnitude of its perpendicular velocity increases. From the point of view of the particle (i.e., in the frame of reference associated with the particle guiding center's parallel motion), what force is responsible for this perpendicular acceleration?

Solution:
When a particle moves along a stationary but non-uniform magnetic field, it effectively feels

a time-varying magnetic field: $d\mathbf{B}/dt = v_\parallel \nabla_\parallel \mathbf{B}$. This time-changing magnetic field induces an electric field according to Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\,\dot{\mathbf{B}} = -\,[v_\parallel \nabla_\parallel \mathbf{B}]/c\,. \tag{3}$$

It is this electric field that accelerates (or decelerates) the particle's perpendicular motion. The change in the perpendicular kinetic energy over one gyro-period is

$$\delta\Big(\frac{mv_\perp^2}{2}\Big) = \int\limits_0^{2\pi/\omega_c} q\mathbf{E}\cdot\mathbf{v}_\perp\, dt = \int q\mathbf{E}\cdot d\mathbf{l}\,. \tag{4}$$

Using Stokes' theorem, this contour integral can be written as

$$\delta\Big(\frac{mv_\perp^2}{2}\Big) = \int q\mathbf{E}\cdot d\mathbf{l} = q\int\limits_S [\nabla\times\mathbf{E}]\cdot d\mathbf{S} = -\,\frac{q}{c}\int\limits_S \dot{\mathbf{B}}\cdot d\mathbf{S} = \pm\,\frac{q}{c}\,\dot{B}\pi r_L^2 = \frac{mv_\perp^2}{2B}\,\frac{2\pi\dot{B}}{\Omega_c} = \mu\,\delta B\,, \tag{5}$$

where $\delta B = 2\pi\dot{B}/\Omega_c$ is the change of the magnetic field over one cyclotron period. Since $mv_\perp^2/2 = \mu B$, we see that $\delta\mu = 0$, i.e., μ is conserved. Furthermore, $-\mu v_\parallel \nabla_\parallel \mathbf{B}$ is equal to the work $v_\parallel F_\parallel$ done by the mirror force that decelerates the parallel motion of the particle. Thus, if we are going back to the laboratory frame, the total energy of the particle is unchanged: the increase in the perpendicular kinetic energy of the particle comes at the expense of its parallel kinetic energy.

Problem 3:

Consider a charged particle moving in a stationary but inhomogeneous magnetic field (varying in space on a scale much longer than the particle's Larmor radius). How does a magnetic flux enclosed by the particle's gyro-orbit vary as the particle moves in the direction of increasing field strength?

Solution:

The magnetic flux enclosed by the particle's gyro-orbit of radius r_L is

$$\Phi \;=\; AB = \pi r_L^2 B = \pi\,\frac{v_\perp^2}{\Omega_c^2}\,B = \pi\,\frac{v_\perp^2}{q^2B^2/m^2c^2}\,B \tag{6}$$

$$\;=\; 2\pi\,\frac{mv_\perp^2}{2B}\,\frac{mc^2}{q^2} \equiv 2\pi\,\frac{mc^2}{q^2}\,\mu\,, \tag{7}$$

where $\mu \equiv mv_\perp^2/2B$ is the magnetic moment of the particle. Since under the conditions of this problem μ is conserved during the particle's motion, it then follows that the enclosed magnetic flux Φ stays constant. [By the way, incidentally, you may recognize the combination mc^2/q^2 as the inverse classical radius of the particle.]

Problem 4:
A plasma with an isotropic velocity distribution is placed in a magnetic mirror trap with mirror ratio $R_m = B_{\rm max}/B_{\rm min} = 4$. There are no collisions, so the particles in the loss cone simply escape, and the rest remain trapped. What fraction is trapped?

Solution:
The loss cone angle, θ_m, is given by:

$$\sin^2\theta_m = 1/R_m = 1/4 \;\Rightarrow\; \theta_m = \pi/6\,. \tag{8}$$

The solid angle within each of the two oppositely directed loss cones is then:

$$\Omega_{\rm loss} = \int\limits_0^{\theta_m} 2\pi\sin\theta\, d\theta = 2\pi\,(1-\cos\theta_m) = 2\pi\,(1-\sqrt{3}/2)\,. \tag{9}$$

Because the particle velocity distribution is isotropic, meaning that all directions are equally likely, the fraction of the escaping particles — i.e., particles with velocities inside one of the two loss cones — is simply equal to the solid angle subtended by the loss cones divided by the entire solid angle corresponding to a sphere (4π):

$$f_{\rm escape} = \frac{2\,\Omega_{\rm loss}}{4\pi} = 1 - \frac{\sqrt{3}}{2}\,. \tag{10}$$

Correspondingly, the fraction of the particles that are trapped in the magnetic mirror machine (i.e., not escaping) is

$$f_{\rm trap} = 1 - f_{\rm escape} = \frac{\sqrt{3}}{2} \simeq 0.866\,. \tag{11}$$

Problem 5:
A cosmic ray proton is trapped and bounces between two moving $R_m = B_{\rm max}/B_{\rm min} = 5$ magnetic mirrors, initially separated by a distance $L = 0.1$ parsec. The proton initially has energy $\mathcal{E} = 10\,\mathrm{keV}$ and $v_\perp = v_\parallel$ at the midplane between the two mirrors. Each mirror moves toward the midplane with a velocity $v_m = 10\,\mathrm{km/s}$.

(a) Find the energy to which the proton will be accelerated before it escapes the mirrors.
(b) Estimate roughly (ignoring factors of order unity) how long it will take the proton to reach this energy.

Solution:
(a) First, let us try to understand the physics behind this problem. The easiest way to understand what is going on here is to utilize the first two conserved adiabatic invariants:

the magnetic moment μ (the 1st adiabatic invariant) and the second adiabatic invariant related to the bounce motion, $J_2 = 2mv_\parallel L$. The proton is initially trapped between the two magnetic mirrors because its initial pitch angle at the midplane (45 degrees) is outside the mirror's loss cone. However, as time progresses, the proton gains energy by Fermi acceleration and, importantly, it is the energy of the parallel motion that increases, according to the conservation of the 2nd adiabatic invariant. The energy of the perpendicular motion in fact does not change over one bounce cycle, because of the conservation of the 1st adiabatic invariant: $mv_\perp^2/2 = \mu B = \text{const}$. Thus, we see that $v_\parallel$ increases from one cycle to the next, whereas $v_\perp$ stays constant, and hence the particle's pitch angle $\alpha = \text{atan} v_\perp/v_\parallel$ decreases. At some point it will drop below the critical loss-cone pitch angle $\alpha_c = \text{asin} R_m^{-1/2} = \text{asin}(1/\sqrt{5})$ and then the particle will escape.

To get an actual quantitative estimate, we can express the midplane pitch angle in terms of the particle's energy as

$$\sin^2\alpha = \frac{v_\perp^2}{v^2} = \frac{v_{\perp,\text{initial}}^2}{v^2} = \frac{\mathcal{E}_{\perp,\text{initial}}}{\mathcal{E}} = \frac{\mathcal{E}_{\text{initial}}/2}{\mathcal{E}}\,, \tag{12}$$

and thus we see that this pitch angle becomes equal to α_c when the particle's energy reaches

$$\mathcal{E} = \frac{1}{2}\,\mathcal{E}_{\text{initial}}\,\sin^{-2}\alpha_c = \frac{R_m}{2}\,\mathcal{E}_{\text{initial}} = 2.5\,\mathcal{E}_{\text{initial}} = 25\,\text{keV}\,. \tag{13}$$

Another, equivalent way to obtain the same result to argue then the particle will no longer be confined between the magnetic mirrors when the magnetic field at its reflection point, B', will start to exceed the maximum magnetic field of the mirrors, $B_{\text{max}} = R_m B_{\text{midplane}}$. Since at the reflection point all of the particle's kinetic energy is that of the perpendicular motion, one can write this energy as $\mathcal{E} = \mathcal{E}_\perp = \mu B' = \mu B_{\text{max}} = R_m \mu B_{\text{midplane}}$. If one now recalls that neither μ nor B_{midplane} change over time, one can evaluate the product μB_{midplane} in terms of the initial perpendicular kinetic energy at the midplane: $\mu B_{\text{midpane}} = mv_{\perp,\text{initial}}^2/2 = \mathcal{E}_{\perp,\text{initial}} = 0.5\,\mathcal{E}_{\text{initial}} = 5\,\text{keV}$ and hence $\mathcal{E} = R_m \mathcal{E}_{\perp,\text{initial}} = 25\,\text{keV}$.

(b) The second part of the problem is a straight-forward application of the conservation of the 2nd adiabatic invariant, $J_2 \sim v_\parallel L = \text{const}$. The initial midplane parallel kinetic energy is $\mathcal{E}_{\parallel,\text{init}} = 0.5\,\mathcal{E}_{\text{init}} = 5\,\text{keV}$; the final midplane parallel kinetic energy at escape is $\mathcal{E}_{\parallel,\text{fin}} = \mathcal{E}_{\text{fin}} - \mathcal{E}_{\perp,\text{fin}} = \mathcal{E}_{\text{fin}} - \mathcal{E}_{\perp,\text{init}} = 25\,\text{keV} - 5\,\text{keV} = 20\,\text{keV}$, i.e., 4 times higher than $\mathcal{E}_{\parallel,\text{init}}$. Therefore, the final parallel velocity is 2 times higher than the initial one, $v_{\parallel,\text{fin}} = 2\,v_{\parallel,\text{init}}$, and, consequently, the final length between the two mirrors is one half of the original length: $L_{\text{fin}} = 0.5\,L_{\text{init}}$. Thus, the particle escapes when the two approaching mirrors have travelled to 1/2 of their original separation and the time it takes is $t_{\text{esc}} = (L_{\text{init}}/2)/(2v_m) = L_{\text{init}}/(4v_m) \simeq 7.8 \times 10^{10}\,\text{s} \simeq 2500\,\text{years}$.