# Physics 5150
# Homework Set # 13 (OPTIONAL)
## Due: N/A

## SOLUTIONS

**Problem 1: Electrostatic Electron Waves**
*Consider a one-dimensional electron-ion plasma with ions providing a uniform stationary neutralizing background and with the electron unperturbed velocity distribution function that has a triangular shape:*

$$f(v_x) = A\,(1 - \frac{|v_x|}{v_0}), \quad |v_x| \le v_0\,, \tag{1}$$

*and*

$$f(v_x) = 0 \quad |v_x| > v_0\,. \tag{2}$$

*Consider electrostatic electron plasma waves in this plasma. Use the Vlasov and Poisson equations to derive the real part of the dispersion relation $\omega(k)$ for electrostatic electron plasma assuming that the phase velocity is large: $v_\phi \gg v_0$ [remember to keep the first two terms when Taylor-expanding $(\omega - kv_x)^{-2}$ where needed]. Calculate the phase and group velocities of the wave. Ignore Landau damping.*

**Solution:**
From the linearized Vlasov equation in the Fourier representation,

$$-\,i\omega\,\tilde{f}_1 + ikv_x\,\tilde{f}_1 + ik\frac{e}{m_e}\,\tilde{\phi}_1\,\partial_{v_x} f_0 = 0\,, \tag{3}$$

we get

$$\tilde{f}_1 = -\,\frac{1}{v_x - \omega/k}\,\frac{e}{m_e}\,\tilde{\phi}_1\,\partial_{v_x} f_0\,. \tag{4}$$

Substituting this into the Fourier form of the Poisson equation:

$$-k^2\,\tilde{\phi}_1 = \frac{e}{\epsilon_0}\int\limits_{-\infty}^{+\infty}\tilde{f}_1(v_x)\,dv_x\,, \tag{5}$$

we get the general electrostatic dispersion relation

$$1 = \frac{\omega_p^2}{k^2}\int\limits_{-\infty}^{+\infty}\frac{\partial\hat{f}_0/\partial v_x}{v_x - v_\phi}\,dv_x\,, \tag{6}$$

where $\hat{f}_0$ is the equilibrium distribution function normalized to 1: $\hat{f}_0 = f/n = f/Av_0$.

Integrating this by parts and using the fact that $\hat{f}_0(\pm\infty) = 0$, we get:

$$1 = \frac{\omega_p^2}{k^2} \int_{-\infty}^{+\infty} \frac{\hat{f}_0\, dv_x}{(v_x - v_\phi)^2} = \frac{\omega_p^2}{k^2} \int_{-v_0}^{+v_0} \frac{\hat{f}_0\, dv_x}{(v_x - v_\phi)^2} \, . \tag{7}$$

Since $v_0 \ll v_\phi$, we can Taylor-expand the denominator and get

$$1 = \frac{\omega_p^2}{k^2} \frac{1}{(\omega/k)^2} \int_{-v_0}^{+v_0} \hat{f}_0 \left(1 + \frac{2v_x}{v_\phi} + \frac{3v_x^2}{v_\phi^2} + ...\right) dv_x \, . \tag{8}$$

Since the distribution function $\hat{f}_0(v_x)$ is symmetric with respect to $v_x = 0$, only terms that are even in $v_x$ contribute to the integral, so that

$$1 = \frac{\omega_p^2}{\omega^2} \int_{-v_0}^{+v_0} \hat{f}_0 \left(1 + \frac{3v_x^2}{v_\phi^2} + ...\right) dv_x = \frac{\omega_p^2}{\omega^2} \left(1 + \frac{3}{v_\phi^2} < v_x^2 > + ...\right). \tag{9}$$

As one can easily show, $< v_x^2 >= v_0^2/6$ for this equilibrium distribution (see Homework # 6, Problem 1), and we thus get

$$1 = \frac{\omega_p^2}{\omega^2} \left(1 + \frac{v_0^2}{2v_\phi^2} + ...\right) \tag{10}$$

and hence

$$\omega^2 \simeq \omega_p^2 + \frac{k^2 v_0^2}{2} \quad \Rightarrow \quad \omega \simeq \omega_p + \frac{k^2 v_0^2}{4\omega_p} \, . \tag{11}$$

Then the phase and the group velocities are

$$v_\phi = \frac{\omega}{k} \simeq \frac{\omega_p}{k} + \frac{k v_0^2}{4\omega_p} \, , \tag{12}$$

$$v_{\rm gr} = \frac{\partial \omega}{\partial k} \simeq \frac{k v_0^2}{2\omega_p} \, . \tag{13}$$

**<u>Problem 2:</u> Landau Damping and the "bump-on-tail" instability**

*An infinite, uniform plasma with fixed ions has an electron distribution function composed of (1) a Maxwellian distribution of "plasma" electrons with density $n_p$ and temperature $T_p$ at rest in the laboratory frame, and (2) a drifting Maxwellian distribution of "beam" electrons with density $n_b$ and temperature $T_b \ll T_p$ centered at $\mathbf{v} = V\hat{x}$. If $n_b$ is infinitesimally small, plasma oscillations in the $x$ direction are Landau-damped. If $n_b$ is relatively large, there will*

*be an electrostatic two-stream instability. Find the critical value of the density ratio $n_b/n_p$ at which the instability sets in, assuming that the beam velocity $V$ is much greater than the thermal velocity of the beam electrons, $V \gg v_{\rm th,b} \equiv (2k_B T_b/m_e)^{1/2}$, but, at the same time, $V v_{\rm th,b} \ll v_{\rm th,p}^2 \equiv 2k_B T_p/m_e$.*
*Hint: the condition for the instability onset can be found by setting the slope of the combined distribution function to zero.*

**Solution:**
The combined distribution function (integrated over $v_y$ and $v_z$), describing the two populations, is

$$f(v_x) = f_p(v_x) + f_b(v_x) = n_p \, \frac{1}{\sqrt{\pi}\, v_{\rm th,p}}\, e^{-v_x^2/v_{\rm th,p}^2} + n_b \, \frac{1}{\sqrt{\pi}\, v_{\rm th,b}}\, e^{-(v_x - V)^2/v_{\rm th,b}^2} \, . \tag{14}$$

Instability is possible if there is a range of positive values of $v_x$ where $f'(v_x) > 0$. For this to be possible, the function $f(v_x)$ must go through a minimum at some critical velocity $v_c$, which we expect to be close to, but somewhat below, the beam velocity $V$. Let us investigate the conditions when such a critical velocity exist. If it does exist, then $f'(v_c)$ has to be zero, i.e.:

$$f'(v_c) = -\, 2 v_c \, \frac{n_p}{\sqrt{\pi}\, v_{\rm th,p}^3}\, e^{-v_c^2/v_{\rm th,p}^2} - 2(v_c - V)\, \frac{n_b}{\sqrt{\pi}\, v_{\rm th,b}^3}\, e^{-(v_c - V)^2/v_{\rm th,b}^2} = 0 \, , \tag{15}$$

from which we get

$$(v_c - V)\, e^{-(v_c - V)^2/v_{\rm th,b}^2} = -\, \frac{n_p}{n_b} \left(\frac{T_b}{T_p}\right)^{3/2} v_c \, e^{-v_c^2/v_{\rm th,p}^2} \simeq -\, \frac{n_p}{n_b} \left(\frac{T_b}{T_p}\right)^{3/2} V \, e^{-V^2/v_{\rm th,p}^2} \, . \tag{16}$$

Introducing a dimensionless variable $w \equiv (V - v_c)/v_{\rm th,b} > 0$, we can rewrite this as

$$F(w) \equiv w \, e^{-w^2} = \frac{n_p}{n_b} \left(\frac{T_b}{T_p}\right)^{3/2} \frac{V}{v_{\rm th,b}}\, e^{-V^2/v_{\rm th,p}^2} = \frac{n_p}{n_b} \left(\frac{T_b}{T_p}\right) \frac{V}{v_{\rm th,p}}\, e^{-V^2/v_{\rm th,p}^2} \tag{17}$$

The critical density ratio $n_b/n_p$ above which instability is possible is then determined as the minimum density ratio for which the above equation has a positive solution, i.e., for which the right hand side equals to the maximum possible value of the function on the left hand side, $F_{\max} = F(w = 1/\sqrt{2}) = 1/\sqrt{2e}$. Thus, we find

$$\left(\frac{n_b}{n_p}\right)_c = F_{\max}^{-1}\, \frac{T_b}{T_p}\, \frac{V}{v_{\rm th,p}}\, e^{-V^2/v_{\rm th,p}^2} = \sqrt{2e}\, \frac{T_b}{T_p}\, \frac{V}{v_{\rm th,p}}\, e^{-V^2/v_{\rm th,p}^2} \, . \tag{18}$$