Physics 5150
Homework Set # 2
Due 5 pm Thursday 2/1/2018

SOLUTIONS

Problem 1: Debye Shielding:
Consider a positive point test charge immersed in a plasma. Calculate the electrostatic potential and charge density distributions around the test charge and then find the total plasma charge of the Debye shielding cloud. How big is it compared with the test charge? For simplicity, assume that the ions are fixed and that $e\phi \ll k_B T_e$.

Solution:
From Poissons law in spherical coordinates, we have:

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = -4\pi\rho = -4\pi e(n_i - n_e)\,,$$

where we took into account that the problem is spherically symmetric and hence the potential should depend only on radius but not on the angle variables.

Next, the electron density is related to the potential via

$$n_e(r) = n_0\exp(-q\phi/k_BT) = n_0\exp(+e\phi/k_BT),$$

whereas the ions are fixed in this problem: $n_i = n_0$. Thus, the total charge density in the plasma is

$$\rho(r) = e[n_i(r)n_e(r)] = en_0\left[1-\exp(+e\phi(r)/k_BT)\right] \approx -n_0e^2\phi(r)/k_BT,$$

where we used $|e\phi| \ll k_BT$ to do Taylor expansion of the exponential.

Substituting this into the Poisson equation above, we get:

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{1}{\lambda_D^2}\,\phi\,,$$

where $\lambda_D^2 \equiv k_BT/4\pi n_0e^2$.

To solve this equation, let us define new variables: $x \equiv r/\lambda_D$, and $u(x) \equiv r\phi(r)$. Then the above equation can be rewritten as $u''(x) = u$, and the general solution is $u(x) = A\exp(x) + B\exp(-x)$. Because the $\exp(x)$ solution is exponentially growing and is thus clearly unphysical for our problem, we have to set $A = 0$, and hence we are left with the exponentially-decaying solution:

$$\phi(r) = (B/r)\exp(-r/\lambda_D).$$

Matching this to the well-known vacuum potential of a point-like charge at small radius, we have:

$$\phi(r) = \frac{Q_0}{r}\exp(-r/\lambda_D).$$

The charge density distribution is then given by:

$$\rho(r) \approx -\,n_0 e^2\phi(r)/k_B T = -\,\phi(r)/4\pi\lambda_D^2 \approx -\,Q_0/(4\pi r\lambda_D^2)\exp(-r/\lambda_D).$$

Next, in order to calculate the total plasma charge within the Debye shielding cloud, Q_D, one can integrate this charge density from zero out to infinity. However, we can actually show that this charge is precisely equal to the test charge Q_0 in a much more elegant and simpler way. To do this, we make use of Gausss law, according to which the net charge enclosed within an arbitrary closed surface $\mathcal{S}$ is proportional to the total flux of the electric field across this surface:

$$Q_{\rm net} = Q_0 + Q_D = (4\pi)^{-1}\int_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{A}.$$

Taking the surface to be a sphere of some very large radius $R \gg \lambda_D$, and taking into account the spherical symmetry of the problem, we see that the surface integral on the right-hand-side becomes simply $R^2 E_r(R) = -R^2\phi'(R) = Q_0[1 + R/\lambda_D]\exp(-R/\lambda_D)$. Because of the decaying exponential factor, this expression rapidly goes to zero as $R/\lambda_D \to \infty$. Thus, the total electric charge enclosed within a sphere of radius R goes to zero, which means that the charge of the Debye shielding cloud exactly cancels the test charge.

<u>Problem 2:</u> Heliospheric Termination Shock:

The Solar Wind is a supersonic flow of hydrogen plasma streaming outward from the Sun with a constant radial velocity of 400 km/sec*, and a density that is equal to about* $8\,\mathrm{cm}^{-3}$ *at* 1 AU *(Astronomical Unit, the distance between the Sun and the Earth) and that falls off with distance as* r^{-2}*. The solar wind is stopped by the pressure of the interstellar medium (ISM) at the so-called Heliospheric Termination Shock, where the ram pressure of the wind* (ρv^2) *equals to the total pressure of the ISM,* $P_{\rm ISM} = 4\times 10^{-13}\,\mathrm{Pa}$*. Approximately, how far from the Sun is the termination shock?*

<u>Solution:</u>

According to the pressure balance condition at the termination shock (TS), the ram pressure of the solar wind is equal to the external pressure of the ISM; therefore, the mass density of the plasma is equal to

$$\rho_{\rm TS} = \frac{P_{\rm ISM}}{v^2} = 2.5\times 10^{-24}\,\mathrm{kg/m^3}\,.$$

Since the solar wind plasma consist mostly of protons and electrons, the corresponding plasma number density ($n = n_i = n_e$) is

$$n_{\rm TS} = \frac{\rho_{\rm TS}}{m_p} = \frac{P_{\rm ISM}}{m_p v^2} = 1.5 \times 10^3\,{\rm m}^{-3}\,.$$

Finally, because the density in the solar wind decreases with the distance r from the Sun as

$$n(r) = n(1\,{\rm AU})\left(\frac{r}{1\,{\rm AU}}\right)^{-2},$$

the distance to the termination shock can be estimated as

$$r_{\rm TS} = \sqrt{\frac{n(1\,{\rm AU})}{n_{\rm TS}}}\,{\rm AU} \simeq 73\,{\rm AU} \simeq 11\,{\rm bln\ km} = 1.1 \times 10^{13}\,{\rm m}\,.$$

<u>Problem 3:</u> Pulsars:

A pulsar (a rotating magnetized neutron star) has a radius $R = 10$ km and a mass $M = 1.4\,M_\odot$, where $M_\odot = 2 \times 10^{30}\,{\rm kg}$ is the mass of the Sun. The pulsar spins around its rotation axis with a period $P = 1\,{\rm sec}$. Multi-year observations show that the pulsar is slowly spinning down, so that its rotation period increases at a rate of $\dot{P} = 10^{-8}\,{\rm sec/year}$. It is believed that the corresponding decay of the pulsar's rotational energy ($E_{\rm rot} = I\Omega^2/2$, where I is the pulsar moment of inertia and $\Omega = 2\pi/P$ is its angular rotation rate) powers the pulsar wind by creating and ejecting highly relativistic electrons and positrons. These relativistic particles then fill the pulsar wind nebula (like the Crab Nebula) and gradually radiate away all their energy via synchrotron radiation. In the following, assume that the system is in a steady state and take the pulsar moment of inertia to be $I = (2/5)MR^2$. Also assume that each electron and positron has initial energy $\epsilon = \gamma\, m_e c^2$, with a Lorentz factor $\gamma = 10^4$.

(a) *What is the total luminosity (i.e., power) of the nebula?*
(b) *How many electrons and positrons are created by the pulsar per second?*

<u>Solution:</u>

(a) The luminosity L of the nebula is equal to the rate of loss of rotational energy of the pulsar:

$$L = -\frac{d}{dt}\,E_{\rm rot} = -\,I\Omega\dot{\Omega} = I\,\frac{4\pi^2}{P^3}\,\dot{P}\,. \tag{1}$$

Taking $I = (2/5)MR^2 = 1.12 \times 10^{38}\,{\rm kg\,m^2}$, and $\dot{P} = 10^{-8}\,{\rm sec/year} \simeq 3.2 \times 10^{-16}$, we get

$$L \simeq 1.4 \times 10^{24}\,{\rm W}\,. \tag{2}$$

(b) The energy of each electron or positron ejected by the pulsar is $\epsilon = \gamma m_e c^2 \simeq 8.2 \times 10^{-10}$ J. Thus the number of particles of each species created each second is

$$\dot{N}_- = \dot{N}_+ = \frac{1}{2}\,\dot{N} = \frac{1}{2}\,\frac{L}{\epsilon} \simeq 8.5 \times 10^{32}\,{\rm s}^{-1}\,. \tag{3}$$

Problem 4: Quasar Accretion Power:

A galaxy consisting of 10^{12} *Sun-like stars, each star shining at* $L_\odot = 4 \times 10^{26}\,\mathrm{W}$*, is located 300 million light years away from us. Right at the center of the galaxy there is a supermassive black hole with a mass* $M_{\rm BH} = 10^8\, M_\odot$*. The black hole has an accretion disk around it and accretes gas at a rate of* $\dot{M} = 1\, M_\odot/\mathrm{year}$*. A fraction* $\eta = 10\%$ *of the rest-mass energy (*$E = mc^2$*) of the accreted material is converted to radiation that escapes the system. What is the total radiative luminosity of the accreting black hole? How does it compare with the total star light coming from the galaxy?*

Solution:

The total radiative luminosity of the black hole is

$$L_{\rm BH} = \eta\, \dot{M} c^2 \simeq 5.7 \times 10^{38}\,\mathrm{W}\,. \tag{4}$$

This is comparable, and in fact somewhat higher than the total luminosity of all the trillion stars in the galaxy: $L_{\rm stars} = 4 \times 10^{38}\,\mathrm{W}$.

That's why this accreting supermassive black hole appears as a bright point-like source right at the center of the galaxy and often outshining it. This is what is called an Active Galactic Nucleus (AGN). The brightest Active Galactic Nuclei — corresponding to largest black holes and highest accretion rates, like the one in the example above — are called Quasars. There is also a supermassive black hole at the center of our Galaxy, but it is relatively light (with a mass of about $4 \times 10^6\, M_\odot$) and its accretion rate is very small, so it is not considered to be an AGN.