# Physics 5150
## Homework Set # 8
### Due 5 pm Thursday 3/22/2018

## SOLUTIONS

**Problem 1: Diamagnetic Drift**
*A cylindrical column of hydrogen plasma with uniform* $T_e = T_i = \text{const}$ *and with* $n_e = n_i = n(r)$*, where* $r$ *is the cylindrical radius, is immersed in a uniform magnetic field along the* $z$*-axis, of strength* $B_0$*. There is no electric field. The plasma density profile has the form:*

$$n(r) = n_0 \left[1 - (r/a)^2\right], \qquad 0 \le r \le a\,. \tag{1}$$

*a) Calculate the ion and electron diamagnetic flows for* $0 \le r \le a$*.*
*b) Calculate the associated diamagnetic current in this region.*
*c) Determine magnetic field resulting from the diamagnetic current calculated in (b).*

**Solution:**
**(a)** The diamagnetic drift velocity is given by

$$\mathbf{v}_D = -\frac{\nabla p \times \mathbf{B}}{qnB^2}\,. \tag{2}$$

In this problem the pressure gradient is equal to

$$\nabla p = kT\,\frac{\partial n}{\partial r}\,\hat{r} = -\,2n_0kT\,\frac{r}{a^2}\,\hat{r}\,, \tag{3}$$

where $\hat{r}$ is a unit vector in the radial direction. Substituting this expression into the above formula for $\mathbf{v}_D$, we get

$$\mathbf{v}_D = -\,\frac{2r}{a^2 - r^2}\,\frac{kT}{qB_0}\,\hat{\theta}\,. \tag{4}$$

Thus we have

$$\mathbf{v}_{Di} = -\,\frac{2r}{a^2 - r^2}\,\frac{kT}{eB_0}\,\hat{\theta} = -\,\mathbf{v}_{De}\,. \tag{5}$$

**(b)** The diamagnetic current density is

$$\mathbf{j}_D = en(r)\,(\mathbf{v}_{Di} - \mathbf{v}_{De}) = -\,4\,r\,\frac{n_0kT}{a^2B_0}\,\hat{\theta}\,. \tag{6}$$

**(c)** From Ampere's law we have:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_D \quad \Rightarrow \quad -\frac{\partial B_z}{\partial r} = -\,4\,r\,\frac{\mu_0 n_0 kT}{a^2 B_0}\,. \tag{7}$$

Integrating, we obtain

$$B_z(r) = C + 2\,\frac{r^2}{a^2}\,\frac{\mu_0 n_0 kT}{B_0}\,. \tag{8}$$

Next, in this geometry the diamagnetic current only affects the magnetic field inside the plasma, it produces no magnetic field for $r \geq a$. Thus, we need to impose a boundary condition $B(r = a) = B_0$, which enables us to determine the integration constant $C$, and hence write

$$B_z(r) = B_0 - 2\,\frac{\mu_0 n_0 kT}{B_0}\left(1 - \frac{r^2}{a^2}\right). \tag{9}$$

Thus, the additional magnetic field produced by the diamagnetic current is

$$\Delta B_{z,D} = -\,\frac{2\mu_0 n_0 kT}{B_0}\left(1 - \frac{r^2}{a^2}\right) = -\,B_0\,\frac{n_0 kT}{B_0^2/2\mu_0}\left(1 - \frac{r^2}{a^2}\right). \tag{10}$$

As can be seen from the above expression, the magnetic field due to the diamagnetic current is negligible compared with the externally imposed field $B_0$ if $n_0 kT \ll B_0^2/2\mu_0$, that is, if the plasma pressure is small compared with the magnetic pressure.

**Problem 2: The Eddington limit**

*Consider the hydrogen plasma (consisting of electrons and protons only) in the upper atmosphere of a bright massive star [mass $M_*$, radius $R_*$, and luminosity (total radiative power emitted by the star) $L_*$]. The radiation pressure of the star exerts an outward radial force on the charged particles in the star's atmosphere: $f_{\rm rad} = \sigma_T F_{\rm rad}/c$. Here, $c$ is the speed of light, $F_{\rm rad} = L_*/4\pi R^2$ is the radiative energy flux per unit area, and $\sigma_T$ is the Thomson scattering cross-section: $\sigma_T = (8\pi/3) r_j^2$, where $r_j$ is called the classical radius of the particles (index $j$ stands for electrons or protons), defined as $r_j = e^2/(4\pi\epsilon_0 m_j c^2)$ (equal to $2.82 \times 10^{-13}$ cm for electrons and 1836 times smaller for protons). Because the cross-section for protons is by a factor of $(m_p/m_e)^2$ times smaller than for electrons, the radiation pressure force on the protons can be completely neglected compared with the radiation pressure on the electrons.*

*(a) Calculate the maximum luminosity $L_*$ that the star can have before the net outward radiation pressure force on the plasma starts to overwhelm the gravitational force on the plasma. Take into account that the radiation pressure acts mostly on the electrons, whereas the gravity acts mostly on the ions. Can the radiation pressure blow the electrons away leaving the heavier ions behind? Why?*

*(b) This maximum luminosity is known as the Eddington limit, or the Eddington luminosity, $L_{\rm Edd}$. It plays a very important role in astrophysics: if the luminosity of a star starts to approach the Eddington limit, the star starts to blow out a powerful radiation-driven wind*

*that causes significant mass loss and thus affects the star's evolution. Estimate this maximum luminosity for the Sun ($M_* = M_\odot = 2 \times 10^{33}\,$g), and compare it with the actual solar luminosity ($L_* = L_\odot = 4 \times 10^{33}\,$erg/s). Do you think the solar wind is driven by the Sun's radiation pressure or by some other force?*

**Solution:**

**(a)** The Eddington limit is achieved when the total radiation pressure force on a plasma is equal to the total gravitational force. The key idea here is that, even though the radiation pressure acts mainly on the electrons whereas the gravity acts on the ions, the two species are tightly coupled by an electric field that ensures that they always move together. Because of this, the electrons cannot be blown away while leaving ions behind.

The radial *electron* equation of motion is basically a balance between the radiation pressure force, $f_e$, and the electric force:

$$f_{e,\mathrm{rad}} - enE_r = 0\,,$$

whereas for ions we have a balance between the gravitational and electric forces:

$$f_{i,\mathrm{grav}} + enE_r = 0\,.$$

(here we make use of the quasineutrality condition: $n_e = n_i = n$.)

Thus, when we consider the total force on the plasma — the sum of the above two equations — the electric force cancels out and the only two forces that remain are $f_{e,\mathrm{rad}}$ and $f_{i,\mathrm{grav}}$, so that $f_{e,\mathrm{rad}} = -\,f_{i,\mathrm{grav}}$.

The (outward) radiation pressure force on the electrons (per unit volume) is

$$f_{e,\mathrm{rad}} = n\sigma_{T,e} L_*/(4\pi R^2 c)\,.$$

The (inward) gravitational force on the ions (per unit volume) is

$$f_{i,\mathrm{grav}} = -\,nGM_* m_p/R^2\,.$$

Notice that both forces have the same $R$-dependence: they are both inversely proportional to $R^2$. Therefore, the condition that the two forces are in balance can be cast as the statement that the star's luminosity has a certain critical value (the Eddington limit $L_{\mathrm{Edd}}$) that is independent of the radius and depends only on the star's mass:

$$L_{\mathrm{Edd}} = \frac{4\pi G M_* m_p c}{\sigma_{T,e}} \simeq 1.3 \times 10^{38} \left(\frac{M_*}{M_\odot}\right)\,\mathrm{erg/s}\,.$$

**(b)** In astrophysics, it is often convenient to recast the Eddington luminosity by normalizing the luminosity and the mass of the star by the Sun's mass, $M_\odot = 2.0 \times 10^{33}\,$g, and the Sun's luminosity $L_\odot = 3.85 \times 10^{33}\,$erg/s. The resulting expression is:

$$L_{\mathrm{Edd}} = 3.3 \times 10^4 \left(\frac{M_*}{M_\odot}\right)\,L_\odot\,.$$

From this we see that the actual solar luminosity is much less than the Eddington limit corresponding to 1 solar mass. It is thus clear that the solar wind is NOT driven by the solar radiation pressure (it is driven by the coronal gas pressure instead).

**Problem 3: Quasineutrality**

*When deriving the (nonrelativistic) MHD equations from the two-fluid theory, one uses the assumption of quasineutrality,* $|\rho_e| \equiv e|\Delta n| \equiv e|n_i - n_e| \ll en \simeq en_e$. *Using Poisson's law (*$\nabla \cdot \mathbf{E} = 4\pi\rho_e$*), nonrelativistic Ampere's law (*$\nabla \times \mathbf{B} = 4\pi\mathbf{j}/c$*), and ideal-MHD Ohm's law* $\mathbf{E} = -[\mathbf{u} \times \mathbf{B}]/c$, *show that quasineutrality is indeed a good approximation in the nonrelativistic limit, i.e., when the fluid velocity is much less than the speed of light,* $|\mathbf{u}| \ll c$.

**Solution:**

We can estimate the charge density using Poisson's equation, $|\rho_e| = |\nabla \cdot \mathbf{E}|/4\pi \sim E/4\pi L$, where $L$ is the characteristic length scale of the problem. Estimating the electric field in the ideal-MHD regime as $E \sim uB/c$, we get $|\rho_e| \sim (u/c)\, B/4\pi L$. From Ampere's law (neglecting the displacement current), we have $B/4\pi L \sim j/c \sim en\, u_{\rm dr}/c$, where $j$ is the typical magnitude of the current density and $u_{\rm dr}$ is the typical magnitude of the drift velocity of the charged particles (usually, electrons) responsible for carrying the current. Thus we find $\rho_e \sim en\,(u/c)\,(u_{\rm dr}/c) \ll 1$ in a nonrelativistic plasma, $u \ll c$.

**Problem 4 (optional): Entropy**

*Using the variational principle, show that the non-drifting Maxwellian velocity distribution is the result of maximizing the entropy density*

$$S = -\int f \ln f d^3\mathbf{v}\,, \tag{11}$$

*subject to the constraints of a fixed particle number density,* $n = \int f d^3\mathbf{v} = \text{const}$, *and a fixed particle energy density,* $E = \int (mv^2/2)\, f d^3\mathbf{v} = \text{const}$.

**Solution:**

According to the variational principle we need to maximize the functional

$$I[f(\mathbf{v})] \equiv S[f] - \lambda_1\Big(\int f d^3\mathbf{v} - n\Big) - \lambda_2\Big(\int (mv^2/2)\, f d^3\mathbf{v} - E\Big), \tag{12}$$

where $\lambda_1$ and $\lambda_2$ are the Lagrange multipliers. Varying it with respect to $f(\mathbf{v})$ we get

$$\delta I[f] = -\int (\ln f + 1)\delta f d^3\mathbf{v} - \lambda_1 \int \delta f d^3\mathbf{v} - \lambda_2 \int (mv^2/2)\, \delta f d^3\mathbf{v} = -\int [\ln f + 1 + \lambda_1 + \lambda_2(mv^2/2)]\delta f d^3\mathbf{v}\,. \tag{13}$$

The variation of $I[f]$ vanishes for an arbitrary function $\delta f(\mathbf{v})$ only if

$$\ln f + 1 + \lambda_1 + \lambda_2(mv^2/2) = 0\,. \tag{14}$$

which means

$$f(\mathbf{v}) = C\, \exp[-\lambda_2(mv^2/2)] \tag{15}$$

where $C = \exp(-\lambda_1 - 1) = \text{const}$, and where the second Lagrange multiplier, $\lambda_2$, can be identified with $1/kT$.