Physics 5150

Homework Set # 4

Due 5 pm Thursday 02/15/2018

SOLUTIONS

Problem 1: **Larmor radius.**

Compute the Larmor radius r_L for the following cases (neglecting $v_\|$):
(a) *10-keV electron in the Earth's magnetic field of 0.5 G.*
(b) *A solar wind proton with streaming velocity 300 km/s, $B = 5$ nT (nano-Tesla).*
(c) *A 1-keV He^+ ion in the solar atmosphere near a sunspot, where $B = 0.5$ kG.*
(d) *A 3.5 MeV He^{++} ash alpha-particle in an 8 Tesla DT fusion reactor.*

Solution:
Using formulae obtained in class, we obtain:

(a) $v_\perp = 5.9 \times 10^4\,\mathrm{km/s}$, $r_L = 6.75\,\mathrm{m}$;
(b) $v_\perp = 300\,\mathrm{km/s}$, $r_L = 630\,\mathrm{km}$;
(c) $v_\perp = 220\,\mathrm{km/s}$, $r_L = 18\,\mathrm{cm}$;
(d) $v_\perp = 1.3 \times 10^4\,\mathrm{km/s}$, $r_L = 3.4\,\mathrm{cm}$.

Problem 2: **Larmor radius.**

In the TFTR (Tokamak Fusion Test Reactor in Princeton, NJ), the plasma was heated by injection of 200-keV neutral deuterium (D) atoms, which, after entering the magnetic field, are converted into 200-keV D ions (with atomic weight $A = 2$) by charge exchange. These ions are confined within the tokamak only if their Larmor radius r_L is less than the minor radius $a = 0.6$ m of the toroidal plasma inside the tokamak. Compute the maximum (i.e., corresponding to particles with $v_\| = 0$) Larmor radius of these D ions in a 5 Tesla magnetic field to see whether this confinement condition is satisfied.

Solution:
Since all the newly produced deuterium ions have the same total kinetic energy of $E = 200\,\mathrm{keV} \simeq 3.2 \times 10^{-14}\,\mathrm{J}$, the maximum Larmor radius will belong to those particles that have no parallel velocity, i.e., for which $E = m_D v_\perp^2/2$, where $m_D = 2m_p$. Their Larmor radius is

$$r_{L,\max} = \frac{v_{\perp,\max}}{\omega_{c,D}} = \sqrt{\frac{2E}{m_D}}\,\frac{m_D c}{eB} = c\,\frac{\sqrt{2E\,m_D}}{eB} \simeq 1.8\,\mathrm{cm}\,. \tag{1}$$

Thus, these ions are well confined in a tokamak of minor radius $a = 0.6$ m.

Problem 3: $E \times B$ **drift.**

An unneutralized electron beam has a density $n_e = 10^{14}\,\mathrm{m}^{-3}$ and radius $a = 1$ cm and flows along a $B = 2$ T uniform and straight magnetic field in the z-direction. Calculate the magnitude and direction of the $\mathbf{E} \times \mathbf{B}$ drift at $r = a$ caused by the electric field due to the beam's charge.

Solution:

The electric field produced by the electron beam is directed radially inward, $\mathbf{E} = E_r(r)\,\hat{r}$, where $E_r < 0$ and where $\hat{r}$ is the unit vector in the radial direction. Using Gauss's law, the magnitude of the electric field at any radius r outside of the beam times the circumference of the circle of radius r is equal to the enclosed line charge density times 4π:

$$2\pi r E_r(r) = -\,4\pi^2 a^2\, e n_e\,, \tag{2}$$

that is, $E_r(r) = -\,2\pi a^2 e n_e/r$. At $r = a$ we have $E_r(a) = -\,2\pi a e n_e \simeq 0.3\,\mathrm{cgs} = -9.0$ kV/m.

The $\mathbf{E} \times \mathbf{B}$ drift velocity then is

$$\mathbf{v}_E(r) = c\frac{\mathbf{E} \times \mathbf{B}}{B^2} = c\frac{E_r(r)}{B}\,[\hat{r} \times \hat{z}] = -\,2\pi c\frac{a^2 e n_e}{r\,B}\,(-\,\hat{\theta}) = 2\pi c\frac{a^2 e n_e}{r\,B}\,\hat{\theta}\,. \tag{3}$$

In particular, for $r = a$ we have

$$\mathbf{v}_E(r) = 2\pi c\frac{a e n_e}{B}\,\hat{\theta} \simeq 4.5\,\mathrm{km/sec}\,. \tag{4}$$

Problem 4: **Cyclotron and Plasma Frequency.**

There two important fundamental frequency scales in plasma physics (it is more convenient to work in the cgs units in this problem): (i) the (electron) cyclotron frequency, $\Omega_{ce} \equiv eB/m_e c$ (which we have covered in class), and (ii) the electron plasma frequency, $\omega_{pe} \equiv (4\pi n_e e^2/m_e)^{1/2}$ (which we will study later). Let us say we want to determine which one of these two frequencies is higher under given conditions. Express the ratio of the two frequencies in terms of the ratio of two energy densities: the magnetic field energy density, $B^2/8\pi$, and the electron rest-mass energy density, $n_e m_e c^2$.

Solution:

As one can straightforwardly see,

$$\frac{\omega_{pe}}{\Omega_{ce}} = \sqrt{\frac{1}{2}\,\frac{n_e m_e c^2}{(B^2/8\pi)}}\,.$$

That is, the ratio of the plasma and cyclotron frequency is equal to $2^{-1/2}$ times the square root of the ratio of the electron plasma rest-mass energy density to the magnetic energy density.

Problem 5: Ultra-relativistic gyro-motion.
Calculate the gyro-radius and the gyro-frequency of an ultra-relativistic particle of energy $\epsilon \equiv \gamma mc^2$, $\gamma \gg 1$, in a uniform constant magnetic field $\mathbf{B}$, for an arbitrary pitch angle α (angle between the particle's velocity and $\mathbf{B}$).

Solution:
Let us consider this problem in the cgs units. The equation of motion for a relativistic particle in a magnetic field reads:

$$\frac{d}{dt}\,\mathbf{p} = \frac{d}{dt}\,(m\gamma\mathbf{v}) = q\,\frac{\mathbf{v}\times\mathbf{B}}{c}\,. \tag{5}$$

An important thing to note is that since the Lorentz force on the right-hand side is always perpendicular to the particle's velocity, it does not perform any work on the particle and hence does not change its energy ϵ. Consequently, the Lorentz factor $\gamma = \epsilon/mc^2 = (1 - v^2/c^2)^{-1/2}$ remains constant during the particle's motion. We can then take it out of the derivative on the left-hand side of the above equation (along with the mass m) and obtain:

$$m\gamma\frac{d}{dt}\,\mathbf{v} = q\,\frac{\mathbf{v}\times\mathbf{B}}{c}\,. \tag{6}$$

As we see, this equation is exactly the same as the one we dealt with when we considered the gyro-motion of a non-relativistic particle, except that the particle's mass m is replaced by the product $m\gamma = \text{const}$. Therefore, the particle motion of a relativistic particle is basically the same as that of a non-relativistic particle — free streaming along the magnetic field and gyro-motion across the field, with the cyclotron frequency

$$\Omega_c = \frac{|q|B}{\gamma mc}\,, \tag{7}$$

and a gyro-radius

$$\rho_c = \frac{v_\perp}{\Omega_c} = \frac{v}{c}\,\frac{\gamma mc^2}{|q|B}\,\sin\alpha\,, \tag{8}$$

where we made use of $v_\perp = v\sin\alpha$. For an ultra-relativistic particle, $\gamma \gg 1$, $v \approx c$, we then simply get:

$$\rho_c = \frac{\gamma mc^2}{|q|B}\,\sin\alpha = \frac{\epsilon}{|q|B}\,\sin\alpha\,. \tag{9}$$

Problem 6: Rotating Plasma Column.
A cylindrical column of plasma rotates around its central axis (as though it were a rigid solid) at an angular velocity ω_0. A constant uniform magnetic field $\mathbf{B}_0$ is present parallel to the axis of rotation.

(a) *Assuming that the rigid rotation can be described by $v_E = \omega_\mathbf{0} \times \mathbf{r}$, where $v_E = c\,\mathbf{E} \times \mathbf{B}/B^2$, compute the electric field $\mathbf{E}$ in the plasma column. Use cylindrical (r, ϕ, z) coordinates.*

(b) *Is there a polarization charge $\rho_q = (4\pi)^{-1}\nabla \cdot \mathbf{E}$ associated with this electric field? If so, how does ρ_q depend on the distance from the central axis?*

(c) *Find the electrostatic potential φ such that $\mathbf{E} = -\nabla\varphi$.*

Solution:
First, a note. If, instead of a uniform magnetic field, we had a dipole magnetic field, the problem would describe the co-rotating inner magnetosphere of a magnetized planet (the Earth) or a star (e.g., a neutron star, a pulsar). In the latter context, this problem would correspond to the famous Goldreich-Julian (1969) model of aligned pulsar magnetosphere.

(a) The electric field is clearly radial and is given by:

$$\mathbf{E} = -\,[\mathbf{v} \times \mathbf{B_0}]/c = -\,[[\omega_\mathbf{0} \times \mathbf{r}] \times \mathbf{B_0}]/c = -\,\mathbf{r}\,(\omega_\mathbf{0} \cdot \mathbf{B_0})/c = \mp\,\mathbf{r}\,\omega_0 B_0/c\,, \tag{10}$$

where we have "-" sign if the magnetic field is in the same direction as the angular velocity vector and "+" if it is opposite.

(b) There is a polarization charge density (the so-called co-rotation, or Goldreich-Julian charge density):

$$\rho_q = (4\pi)^{-1}\nabla \cdot E = (4\pi)^{-1} r^{-1}\,\partial_r(rE_r) = -\,(\omega_\mathbf{0} \cdot \mathbf{B_0})/(2\pi c)\,. \tag{11}$$

Interestingly, the charge density in this case is uniform, independent of r.

(c) Since the magnetic field in this problem is time-independent, the electric field is electrostatic, $\nabla \times E \sim \dot{B} = 0$, and hence can be expressed in terms of a potential: $E = -\nabla\varphi$. A suitable choice of the potential is given by

$$\varphi = \frac{r^2}{2c}\,(\omega_\mathbf{0} \cdot \mathbf{B_0})\,. \tag{12}$$