Physics 5150
Homework Set # 11
Due 5 pm Thursday 4/19/2018

SOLUTIONS

Problem 1: **Sound waves**
Derive the dispersion relation for sound waves in a neutral gas (e.g., air) with an adiabatic equation of state ($P \sim \rho^\gamma$), using the gas dynamics (fluid) equations. What are the phase and group velocities of these waves?

Solution:
By linearizing and taking the Fourier transform of the continuity and momentum equations, we find:

$$-i\omega\,\tilde{\rho}_1 = -i\,\mathbf{k}\cdot\tilde{\mathbf{u}}_1\,\rho_0\,, \tag{1}$$

and

$$-i\omega\,\rho_0\tilde{\mathbf{u}}_1 = -i\,\mathbf{k}\,\tilde{P}_1 = -i\,\mathbf{k}\,\tilde{\rho}_1\gamma P_0/\rho_0\,, \tag{2}$$

where we used the equation of state to establish the last equality.

By combining these two equations we get

$$\omega^2\,\tilde{\rho}_1 = \omega\,\mathbf{k}\cdot\tilde{\mathbf{u}}_1\rho_0 = \mathbf{k}\cdot\mathbf{k}\,\tilde{\rho}_1\gamma P_0/\rho_0\,, \tag{3}$$

from which we immediately get the dispersion relation

$$\omega^2 = k^2 c_s^2\,, \tag{4}$$

where we defined the sound speed $c_s \equiv \sqrt{\gamma P_0/\rho_0}$.

The phase and group velocities for this wave are the same and are equal to

$$\mathbf{v}_\phi = \mathbf{v}_g = \pm\,c_s\hat{k}\,. \tag{5}$$

Problem 2: **Energy Partitioning in Plasma Oscillations**
Consider Langmuir's electron plasma oscillations in an unmagnetized cold plasma (treat ions as a stationary uniform neutralizing background, $n_i = \mathrm{const}$, $u_i = 0$). What is the greater: the electron kinetic energy density due to the perturbed electron velocity, $\mathcal{E}_{\rm kin} = n_0 m_e\,|u_{e,1}|^2/2$, or the energy density of the perturbed electric field, $\mathcal{E}_{\rm el} = |E_1|^2/(8\pi)$, and by what factor? For definiteness, just consider the oscillation amplitudes for both energy density components.

Solution:
The simplest way to address this question is to look at the linearized cold electron fluid equation of motion in the Fourier form:

$$-\,i\omega n_0 m_e\,\tilde{\mathbf{u}}_{e1} = -\,en_0\,\tilde{\mathbf{E}}_1 \quad \Rightarrow \quad \tilde{\mathbf{u}}_{e1} = \frac{e}{i\,m_e\omega}\,\tilde{\mathbf{E}}_1\,. \tag{6}$$

Thus,

$$\frac{\mathcal{E}_{\rm kin}}{\mathcal{E}_{\rm el}} = \frac{n_0 m_e\,|\tilde{\mathbf{u}}_{e,1}|^2/2}{|\tilde{\mathbf{E}}_1|^2/(8\pi)} = \frac{4\pi n_0 e^2}{m_e\omega^2}\,. \tag{7}$$

Since for electron plasma oscillations $\omega = \omega_{pe} = (4\pi n_0 e^2/m_e)^{1/2}$, we see that this ratio is equal to 1, i.e., there is an equipartition between the wave's electron kinetic energy and the wave's electric field energy.

Problem 3: **Radio Blackout**
A space capsule making a reentry into the Earth's atmosphere suffers a communication blackout because a plasma is generated by the shock wave in front of the capsule. If the radio operates at a frequency of 300 MHz, what is the minimum plasma density during the blackout?

Solution:
The radio blackout happens happens because the electron plasma frequency $\omega_{pe}(n_e)$ of the shocked plasma exceeds the communication frequency $\omega = 2\pi f$. By setting these two frequencies equal to each other, we immediately estimate the minimum plasma density:

$$n_{\rm min} = m_e(2\pi f)^2/4\pi e^2 \simeq 1.1\times 10^9\ {\rm cm}^{-3}\,.$$