Physics 5150

Homework Set # 12

Due 5 pm Thursday 4/26/2018

SOLUTIONS

Problem 1: Energy Partitioning in Alfvén Waves

Consider an incompressible Alfvén wave propagating parallel to a background magnetic field $\mathbf{B_0} = B_0\,\hat{z}$. *What is the greater: the plasma kinetic energy density due to the perturbed velocity,* $E_{\rm kin} = \rho_0|u_1|^2/2$, *or the magnetic energy density of the perturbed magnetic field,* $E_{\rm magn} = |B_1|^2/(8\pi)$, *and by what factor ?*

Solution:

Since the Alfvén wave is an ideal MHD phenomenon, the fluid velocity is that corresponding to the $\mathbf{E} \times \mathbf{B}$ drift, i.e., $|u_1| = c|E_1|/B_0$. On the other hand, according to Faraday's law, $\nabla \times \mathbf{E}_1 = -\dot{\mathbf{B}}/c$, the perturbed electric and magnetic field magnitudes are related via $|E_1| = |B_1|\,|\omega/(kc)|$, and so $|u_1|^2 = \omega^2/k^2\,|B_1|^2$. Thus we have

$$\frac{E_{\rm kin}}{E_{\rm magn}} = \frac{\rho_0|u_1|^2/2}{|B_1|^2/8\pi} = \frac{4\pi\rho_0}{B_0^2}\,\frac{k^2}{\omega^2} = \frac{k^2V_A^2}{\omega^2} \tag{1}$$

Since the dispersion relation for this parallel wave is $\omega^2 = k^2V_A^2$, we see that the two energies are equal to each other.

Problem 2: Alfvén Speed

Calculate the Alfvén speed in the following environments, assuming a fully ionized pure hydrogen plasma, $m_i = m_p$, $n_i = n_e$ *(please pay attention to units):*
(a) Tokamak: $B_0 = 4$ *Tesla;* $n_e = 10^{14}\,\mathrm{cm}^{-3}$*;*
(b) Earth Magnetosphere: $B_0 = 20$ *nT (nano-Tesla);* $n_e = 0.1\,\mathrm{cm}^{-3}$*;*
(c) Solar Corona: $B_0 = 100$ *G;* $n_e = 10^9\,\mathrm{cm}^{-3}$.

Solution:

We find:
(a) Tokamak ($B_0 = 4$ Tesla; $n_e = 10^{14}\,\mathrm{cm}^{-3}$): $\Rightarrow\ V_A \simeq 9 \times 10^8$ cm/s $= 9 \times 10^3$ km/s.
(b) Magnetosphere ($B_0 = 20$ nT; $n_e = 0.1\,\mathrm{cm}^{-3}$): $\Rightarrow\ V_A \simeq 1.4 \times 10^8$ cm/s $= 1.4 \times 10^3$ km/s.
(c) Solar Corona ($B_0 = 100$ G; $n_e = 10^9\,\mathrm{cm}^{-3}$): $\Rightarrow\ V_A \simeq 7 \times 10^8$ cm/s $= 7 \times 10^3$ km/s.

It is quite remarkable that these Alfvén speeds are all in the same ball park, even though the underlying plasma parameters (B_0 and n_e) vary greatly.

Problem 3: Relativistic Alfvén Wave

Derive the dispersion relation, and also the phase and group velocities, of an Alfvén wave propagating parallel to the background magnetic field ($[\mathbf{k} \times \mathbf{B}_0] = 0$) in the case when the formal expression for the Alfvén velocity that we derived in class, $V_A = B_0/\sqrt{4\pi\rho_0}$, exceeds the speed of light (this is equivalent to saying that the magnetic field is so strong that the magnetic energy density, $B_0^2/8\pi$, is greater than one half of the plasma rest-mass energy density, $\rho_0 c^2$).

(Hint: Since we are considering linear waves, the wave's amplitude is treated as being as infinitesimally small. In particular, this means that the perturbed velocity, $\mathbf{u}_1$, is much smaller than the speed of light, i.e., the actual plasma motions are non-relativistic. Then, only one equation in our derivation needs to be modified: namely, the displacement current needs to be kept in in Ampere's law.)

Solution:

By combining the linearized and Fourier-transformed Maxwell equations (including the displacement current)

$$\partial_t \mathbf{B} = -c\nabla \times \mathbf{E} \Rightarrow -i\omega\tilde{\mathbf{B}} = -ic\mathbf{k} \times \tilde{\mathbf{E}} \tag{2}$$

and

$$\nabla \times \mathbf{B} = \frac{1}{c}\,\partial_t \mathbf{E} + \frac{4\pi}{c}\,\mathbf{j} \Rightarrow i\mathbf{k} \times \tilde{\mathbf{B}} = -\frac{i\omega}{c}\,\tilde{\mathbf{E}} + \frac{4\pi}{c}\,\tilde{\mathbf{j}}, \tag{3}$$

we find $\tilde{\mathbf{B}} = (c\mathbf{k}/\omega) \times \tilde{\mathbf{E}}$ and then

$$\frac{c\mathbf{k}}{\omega} \times [\frac{c\mathbf{k}}{\omega} \times \tilde{\mathbf{E}}] = -\tilde{\mathbf{E}} - \frac{4\pi i}{\omega}\,\tilde{\mathbf{j}}\,. \tag{4}$$

Since an Alfvén wave propagating in the parallel propagation, $\mathbf{k} \parallel \mathbf{B}_0$, has transverse polarization, $\mathbf{k} \cdot \tilde{\mathbf{E}}$, the left hand side of this equation becomes simply $-(ck/\omega)^2\,\tilde{\mathbf{E}}$, and thus this equation can be written in a compact form,

$$\left(\frac{c^2k^2}{\omega^2} - 1\right)\tilde{\mathbf{E}} = \frac{4\pi i}{\omega}\,\tilde{\mathbf{j}}\,. \tag{5}$$

Next, we need to consider the MHD equation of motion:

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \frac{1}{c}\,[\mathbf{j} \times \mathbf{B}],\,. \tag{6}$$

Just as in the non-relativistic case, we here consider a wave with an incompressible polarization, $\nabla \cdot \mathbf{u} = 0$, and so, according to the continuity equation $\partial_t \rho = -\nabla \cdot (\rho\mathbf{u})$, the plasma density, and hence the pressure $p = p(\rho)$, are unperturbed. Then, dropping the pressure term and linearizing and Fourier-transforming the equation of motion we obtain:

$$-\,i\omega\rho_0\tilde{\mathbf{u}} = \frac{1}{c}\,[\tilde{\mathbf{j}} \times \mathbf{B}_0] \Rightarrow \tilde{\mathbf{u}} = \frac{i}{c\omega\rho_0}\,[\tilde{\mathbf{j}} \times \mathbf{B}_0]. \tag{7}$$

Finally, we use the linearized ideal-MHD Ohm's law, $c\tilde{\mathbf{E}} = -\,\tilde{\mathbf{u}}\times\mathbf{B}_0$. By combining it with equations (5) and (7), we obtain:

$$c\tilde{\mathbf{E}} = -\,\tilde{\mathbf{u}}\times\mathbf{B}_0 = -\,\frac{i}{c\omega\rho_0}\,[\tilde{\mathbf{j}}\times\mathbf{B}_0]\times\mathbf{B}_0 = -\,\frac{i}{c\omega\rho_0}\left(\frac{c^2k^2}{\omega^2}-1\right)\frac{\omega}{4\pi i}\,[\tilde{\mathbf{E}}\times\mathbf{B}_0]\times\mathbf{B}_0\,, \tag{8}$$

or, once again using $\tilde{\mathbf{E}}\cdot\mathbf{B}_0 = 0$,

$$\tilde{\mathbf{E}} = \frac{B_0^2}{4\pi\rho_0 c^2}\left(\frac{c^2k^2}{\omega^2}-1\right)\tilde{\mathbf{E}}\,. \tag{9}$$

Canceling $\tilde{\mathbf{E}}$ on both sides, we arrive at the dispersion relation

$$\omega^2 = k^2 V_{\rm A}^2\,\frac{c^2}{c^2+V_{\rm A}^2}\,. \tag{10}$$

where for convenience we have introduced the nonrelativstic Alfvén speed $V_{\rm A}^2 \equiv B_0^2/4\pi\rho_0$.

As one can see, this dispersion relation becomes the usual non-relativistic Alfvén wave dispersion relation $\omega^2 = k^2V_{\rm A}^2$ in the non-relativistic limiting case $V_{\rm A}\ll c$, and takes the form of the usual vacuum electromagnetic wave dispersion relation $\omega^2 = k^2c^2$ in the ultra-relativistic limit $V_{\rm A}\gg c$.

The phase and group velocities of this wave are

$$v_\phi = v_{\rm gr} = \pm\frac{V_{\rm A}c}{\sqrt{c^2+V_{\rm A}^2}}\,. \tag{11}$$

<u>Problem 4:</u> **Fast Magnetosonic Wave**

Using compressible ideal MHD equations with an adiabatic equation of state $P\propto\rho^\gamma$, derive the dispersion relation for a fast magnetosonic wave propagating perpendicular to a uniform background magnetic field $\mathbf{B_0} = B_0\,\hat{z}$. Find the phase and group velocities of the wave.

<u>Solution:</u>

In a perpendicular fast magnetosonic wave the motions are compressible and the restoring force is provided by both magnetic pressure and gas pressure. To derive the dispersion relation we need to consider a linearized set of MHD equations in Fourier representation.

First, the linearized MHD equation of continuity gives:

$$\dot{\rho_1} = -\,\nabla\cdot(\rho_0\mathbf{u}_1) \quad\Rightarrow\quad \tilde{\rho}_1 = \frac{\mathbf{k}\cdot\tilde{\mathbf{u}}_1}{\omega}\,\rho_0\,. \tag{12}$$

The linearized equation of motion gives:

$$\rho_0\,\dot{\mathbf{u}}_1 = -\,\nabla p_1 + (\mathbf{B}_0\cdot\nabla)\mathbf{B}_1/4\pi - \nabla(2\mathbf{B}_0\cdot\mathbf{B}_1/8\pi)\,. \tag{13}$$

The first term on the right-hand side can be transformed using the adiabatic equation of state: $p_1/p_0 = \gamma\rho_1/\rho_0$, i.e., $p_1 = \rho_1 c_s^2$, where $c_s^2 \equiv \gamma p_0/\rho_0$. Thus, $-\nabla p_1 = -c_s^2\,\nabla\rho_1$.

The second term on the right-hand side — the magnetic tension force — does not contribute to this particular wave motion because we consider the case of the wave vector being perpendicular to the background magnetic field, and hence there is no variation along $\mathbf{B}_0$.

Finally, the third term on the right-hand side — the magnetic pressure force — does contribute; it is equal to $-(1/4\pi)\,\nabla(\mathbf{B}_0\cdot\mathbf{B}_1)$. Putting all the terms together and taking Fourier transform, we get

$$-i\omega\tilde{\mathbf{u}}_1 = -c_s^2 i\mathbf{k}\rho_1/\rho_0 - V_A^2\, i\mathbf{k}\,(\hat{z}\cdot\tilde{\mathbf{B}}_1)/B_0\,. \tag{14}$$

where we defined the Alfvén speed:

$$V_A \equiv \frac{B_0}{\sqrt{4\pi\rho_0}}\,. \tag{15}$$

Using equation (12) we can write this equation of motion as

$$\tilde{\mathbf{u}}_1 = c_s^2\mathbf{k}\frac{\mathbf{k}\cdot\tilde{\mathbf{u}}_1}{\omega^2} + V_A^2\,\frac{\mathbf{k}}{\omega}\,(\hat{z}\cdot\tilde{\mathbf{B}}_1)/B_0\,. \tag{16}$$

Finally, let us consider the linearized ideal MHD magnetic induction equation:

$$\partial_t\mathbf{B}_1 = -\nabla\times[\mathbf{u}_1\times\mathbf{B}_0] = \mathbf{u}_1(\nabla\cdot\mathbf{B}_0) - \mathbf{B}_0(\nabla\cdot\mathbf{u}_1) + (\mathbf{B}_0\cdot\nabla)\mathbf{u}_1 - (\mathbf{u}_1\cdot\nabla)\mathbf{B}_0\,. \tag{17}$$

The first term on the right-hand side vanishes because $\nabla\cdot\mathbf{B} = 0$; the third term vanishes because we chose a wave with $\mathbf{k}$ perpendicular to $\mathbf{B}_0$, and so there is no variation of $\mathbf{u}_1$ along $\mathbf{B}_0$; and the last term vanishes because $\mathbf{B}_0$ is uniform. Thus we have only the second term left:

$$\partial_t\mathbf{B}_1 = -\mathbf{B}_0(\nabla\cdot\mathbf{u}_1)\,, \tag{18}$$

Fourier-transforming which we get

$$-i\omega\tilde{\mathbf{B}}_1 = -i\,\mathbf{B}_0(\mathbf{k}\cdot\tilde{\mathbf{u}}_1) \quad\Rightarrow\quad \tilde{\mathbf{B}}_1 = \mathbf{B_0}\,\frac{\mathbf{k}\cdot\tilde{\mathbf{u}}_1}{\omega} = B_0\,\hat{z}\,\frac{\mathbf{k}\cdot\tilde{\mathbf{u}}_1}{\omega}\,. \tag{19}$$

Combining this result with equation (16), we get

$$\tilde{\mathbf{u}}_1 = (V_A^2 + c_s^2)\,\mathbf{k}\,\frac{\mathbf{k}\cdot\tilde{\mathbf{u}}_1}{\omega^2}\,, \tag{20}$$

that is, $\tilde{\mathbf{u}}_1$ is in the same direction as $\mathbf{k}$, and hence we get the dispersion relation

$$\omega^2 = k^2\,(V_A^2 + c_s^2)\,. \tag{21}$$

The phase and group velocities are both equal to

$$\mathbf{v}_\phi = \mathbf{v}_g = \hat{\mathbf{k}}\,\sqrt{V_A^2 + c_s^2}\,. \tag{22}$$