HW 6 of Plasma

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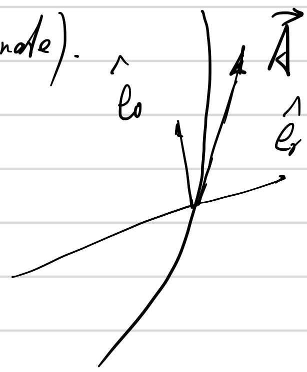
1. (a) for any tangent vector on line $r = R\cdot\sin^2\theta$ (spherical coordinate).



$$\frac{A_r}{A_\theta} = \frac{1}{r}\cdot\frac{dr}{d\theta}$$, apply $\vec{B}$ into $\vec{A}$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{2\cos\theta}{\sin\theta} \Rightarrow \ln r = 2\ln\sin\theta + C.$$

define $r(\theta=\pi/2) = R$, so $C = R$, $r = R\cdot\sin^2\theta$ #

(b) $$\vec{b} = \vec{B}/B = \frac{1}{\sqrt{1+3\cos^2\theta}}\left(2\cos\theta\cdot\hat{e}_r + \sin\theta\cdot\hat{e}_\theta\right)$$

given $\nabla(\vec{f}\cdot\vec{g}) = (\vec{g}\cdot\nabla)\vec{f} + (\vec{f}\cdot\nabla)\vec{g} + \vec{g}\times(\nabla\times\vec{f}) + \vec{f}\times(\nabla\times\vec{g})$

replace $\vec{g}, \vec{f}$ by $\vec{b}$, and apply $(\vec{b})^2 \equiv 1$

$$(\vec{b}\cdot\nabla)\vec{b} = \frac{1}{2}\nabla(b^2) - \vec{b}\times(\nabla\times\vec{b}) = -\vec{b}\times(\nabla\times\vec{b})$$

$b_\varphi \equiv 0$, so $\nabla\times\vec{b} = \frac{1}{r}\left(\partial_r(r b_\theta) - \partial_\theta b_r\right)\hat{e}_\varphi$, $\nabla\times\vec{b} \perp \vec{b}$

so $|(b\cdot\nabla)b| = |\vec{b}|\cdot|\nabla\times\vec{b}| = |\nabla\times\vec{b}|$, now calculate $\nabla\times\vec{b}$

$$\partial_r(r b_\theta) = b_\theta, \quad \partial_\theta b_r = \frac{2\sin\theta}{1+3\cos^2\theta}\cdot\frac{1}{\sqrt{1+3\cos^2\theta}}$$

$$|\nabla\times\vec{b}| = \frac{1}{r}\left|\left(\sin\theta + \frac{2\sin\theta}{1+3\cos^2\theta}\right)\cdot\frac{1}{\sqrt{1+3\cos^2\theta}}\right| = \left|\frac{\sin\theta}{r}\cdot\frac{3(1+\cos^2\theta)}{1+3\cos^2\theta}\right|$$

at equator, $\theta = \frac{\pi}{2}$, $\sin\theta = 1$, $\cos\theta = 0$; $r = R\sin^2\theta = R$

$$R_c = 1/|\nabla\times\vec{b}| = R/3.$$ #

(c). Curvature drift $\vec{v}_R = \frac{m v_\parallel^2}{qB^2} \cdot \frac{\vec{R}_c \times \vec{B}}{R_c^2}$, at equator

$$\vec{R}_c = \frac{1}{3} R \cdot \hat{e}_r , \quad \vec{B} = \frac{\mu_0 M}{4\pi} \cdot \frac{1}{R^3} \cdot \hat{e}_\theta$$

so $\vec{v}_R = \frac{3 m v_\parallel^2 R^2}{q} \cdot \frac{4\pi}{\mu_0 M} \cdot \hat{e}_\varphi$

(d) ∇B drift $\vec{v}_{\nabla B} = \frac{m v_\perp^2}{2qB^2} \cdot \frac{\vec{R}_c \times \vec{B}}{R_c^2}$, applying (c)

$$\vec{v}_{\nabla B} = \frac{3 m v_\perp^2 R^2}{2q} \cdot \frac{4\pi}{\mu_0 M} \cdot \hat{e}_\varphi$$

(e) $\vec{v}_R$ and $\vec{v}_{\nabla B}$ have the same direction, while $|v_R| = 2|v_{\nabla B}|$

2. Constancy of magnetic momentum indicates the vertical kinetic energy $E_\perp$ has form of:

$$\frac{E_\perp}{2B_z} = \left.\frac{E_\perp}{2B_z}\right|_{z=0}$$

so $v_\perp^2 = v_{\perp,0}^2 \left(1 + \left(\frac{z}{L}\right)^2\right)$, and $E_\parallel$ will reduce to

$$E_\parallel = E_{\parallel,0} - (v_\perp^2 - v_{\perp,0}^2)$$

$\Rightarrow v_\parallel^2 = v_{\parallel,0}^2 - v_{\perp,0}^2 \cdot \left(\frac{z}{L}\right)^2$, having harmonic oscillation form

on z direction : $\frac{1}{2} m \cdot v_\parallel^2 + \frac{1}{2} m \cdot \frac{v_{\perp,0}^2}{L^2} \cdot z^2 = \frac{1}{2} m \cdot v_{\parallel,0}^2$ (Const)

so bounce frequency $f = \frac{1}{2\pi} \cdot \sqrt{\frac{k}{m}}$ $\left(k = m \cdot \frac{v_{\perp,0}^2}{L^2} \right)$

as $k = \frac{2E \cdot \sin^2\alpha_0}{L^2}$, $f = \frac{\sin\alpha_0}{2\pi L} \cdot \sqrt{\frac{2E}{m}}$

3. (a) normalization parameter A satisfies $\int_{-\infty}^{+\infty} f(v_x) \cdot dv_x = 1$

so $v_0 \cdot A = 1$ $A = v_0^{-1}$

(b) 1> $\langle v_x \rangle = \int_{-\infty}^{+\infty} v_x f(v_x) \cdot dv_x$, since f is even function, and v_x is odd

$v_x \cdot f$ is odd, so $\langle v_x \rangle = 0$

2> $\langle |v_x| \rangle = 2\int_0^{+\infty} v_x \cdot f(v_x) \cdot dv_x = \frac{2}{v_0^2} \int_0^1 x(v_0 - x)\, dx = \frac{1}{3} v_0$

3> $\langle \frac{1}{2} m v_x^2 \rangle = \frac{1}{2} m \cdot 2 \int_0^{+\infty} v_x^2 \cdot f(v_x) \cdot dv_x = \frac{m}{v_0^2} \int_0^1 x^2 (v_0 - x)\, dx = \frac{1}{12} m \cdot v_0^2$