Taylor Series of Arcsine Function

From ProofWiki

It has been suggested that this article or section be renamed:

Taylor Series of Real Arcsine Function

One may discuss this suggestion on the talk page.

Theorem

The (real) arcsine function has a Taylor series expansion:

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{2n+1}$$

which converges for $-1 \le x \le 1$.

Proof

From the General Binomial Theorem:

$$\begin{aligned} \left(1 - x^2\right)^{-1/2} &= 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots \\ (1): \quad &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}x^{2n} \end{aligned}$$

for $-1 < x < 1$.

From Power Series is Termwise Integrable within Radius of Convergence, (1) can be integrated term by term:

$$\begin{aligned} \int_0^x \frac{1}{\sqrt{1-t^2}}\mathrm{d}t &= \sum_{n=0}^{\infty} \int_0^x \frac{(2n)!}{2^{2n}(n!)^2}t^{2n}\,\mathrm{d}t \\ \implies \quad \arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{2n+1} \qquad \text{Derivative of Arcsine Function} \end{aligned}$$

We will now prove that the series converges for $-1 \le x \le 1$.

By Stirling's Formula:

$$\begin{aligned} \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{2n+1} &\sim \frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{2^{2n}n^{2n}e^{-2n}2\pi n} \frac{x^{2n+1}}{2n+1} \\ &= \frac{1}{\sqrt{\pi n}} \frac{x^{2n+1}}{2n+1} \end{aligned}$$

Then:

$$\begin{aligned} \left\lvert \frac{1}{\sqrt{\pi n}} \frac{x^{2n+1}}{2n+1} \right\rvert &< \left\lvert \frac{x^{2n+1}}{n^{3/2}} \right\rvert \\ &\le \frac{1}{n^{3/2}} \end{aligned}$$

By P-Series Converges Absolutely:

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is convergent.

So by the Comparison Test, the Taylor series is convergent for $-1 \le x \le 1$.

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