# Householder transformation

In linear algebra, a **Householder transformation** (also known as **Householder reflection** or **elementary reflector**) is a linear transformation that describes a reflection about a plane or hyperplane containing the origin. The Householder transformation was introduced in 1958 by Alston Scott Householder.[1]

Its analogue over general inner product spaces is the Householder operator.

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# Definition

## Transformation

The reflection hyperplane can be defined by a unit vector $v$ (a vector with length $\mathbf{1}$) which is orthogonal to the hyperplane. The reflection of a point $x$ about this hyperplane is the linear transformation:

$$x - 2\langle x, v \rangle v = x - 2v(v^{\mathrm{H}}x),$$

where $v$ is given as a column unit vector with Hermitian transpose $v^{\mathrm{H}}$.

## Householder matrix

The matrix constructed from this transformation as:

$$P = I - 2vv^{\mathrm{H}}, \text{ where } I \text{ is the identity matrix}$$

is known as the **Householder matrix**.

## Properties

The Householder matrix has the following properties:

- it is Hermitian: $P = P^H$,
- it is unitary: $P^{-1} = P^H$,
- hence it is involutory: $P^2 = I$.
- A Householder matrix has eigenvalues $\pm 1$. To see this, notice that if $u$ is orthogonal to the vector $v$ which was used to create the reflector, then $Pu = u$, i.e., $1$ is an eigenvalue of multiplicity $n - 1$, since there are $n - 1$ independent vectors orthogonal to $v$. Also, notice $Pv = -v$, and so $-1$ is an eigenvalue with multiplicity $1$.
- The determinant of a Householder reflector is $-1$, since the determinant of a matrix is the product of its eigenvalues, in this case one of which is $-1$ with the remainder being $1$ (as in the previous point).

# Applications

## Geometric optics

In geometric optics, specular reflection can be expressed in terms of the Householder matrix.

## Numerical linear algebra

Householder transformations are widely used in numerical linear algebra, to perform QR decompositions and is the first step of the QR algorithm. They are also widely used for tridiagonalization of symmetric matrices and for transforming non-symmetric matrices to a Hessenberg form.

### QR decomposition

Householder reflections can be used to calculate QR decompositions by reflecting first one column of a matrix onto a multiple of a standard basis vector, calculating the transformation matrix, multiplying it with the original matrix and then recursing down the $(i, i)$ minors of that product.

### Tridiagonalization

This procedure is taken from the book: Numerical Analysis, Burden and Faires, 8th Edition. In the first step, to form the Householder matrix in each step we need to determine $\alpha$ and $r$, which are:

$$\alpha = -\operatorname{sgn}(a_{21}) \sqrt{\sum_{j=2}^{n} a_{j1}^2} \, ;$$

$$r = \sqrt{\frac{1}{2}(\alpha^2 - a_{21}\alpha)} \, ;$$

From $\alpha$ and $r$, construct vector $v$:

$$v^{(1)} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix},$$

where $v_1 = 0$, $v_2 = \dfrac{a_{21} - \alpha}{2r}$, and

$$v_k = \frac{a_{k1}}{2r} \text{ for each } k = 3, 4 \ldots n$$

Then compute:

$$P^1 = I - 2v^{(1)}(v^{(1)})^{\mathrm{T}}$$

$$A^{(2)} = P^1 A P^1$$

Having found $P^1$ and computed $A^{(2)}$ the process is repeated for $k = 2, 3, \ldots, n-1$ as follows:

$$\alpha = -\operatorname{sgn}(a^k_{k+1,k}) \sqrt{\sum_{j=k+1}^{n} (a^k_{jk})^2} \,;$$

$$r = \sqrt{\frac{1}{2}(\alpha^2 - a^k_{k+1,k}\alpha)}\,;$$

$$v^k_1 = v^k_2 = \cdots = v^k_k = 0;$$

$$v^k_{k+1} = \frac{a^k_{k+1,k} - \alpha}{2r}$$

$$v^k_j = \frac{a^k_{jk}}{2r} \text{ for } j = k+2; k+3, \ldots, n$$

$$P^k = I - 2v^{(k)}(v^{(k)})^{\mathrm{T}}$$

$$A^{(k+1)} = P^k A^{(k)} P^k$$

Continuing in this manner, the tridiagonal and symmetric matrix is formed.

## Examples

This example is taken from the book "Numerical Analysis" by Richard L. Burden (Author), J. Douglas Faires. In this example, the given matrix is transformed to the similar tridiagonal matrix $A_2$ by using the Householder method.

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -2 & 2 \\ 1 & 2 & 0 & 1 \\ -2 & 0 & 3 & -2 \\ 2 & 1 & -2 & -1 \end{bmatrix},$$

Following those steps in the Householder method, we have:

The first Householder matrix:

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/3 & 2/3 & -2/3 \\ 0 & 2/3 & 2/3 & 1/3 \\ 0 & -2/3 & 1/3 & 2/3 \end{bmatrix},$$

$$A_1 = Q_1 A Q_1 = \begin{bmatrix} 4 & -3 & 0 & 0 \\ -3 & 10/3 & 1 & 4/3 \\ 0 & 1 & 5/3 & -4/3 \\ 0 & 4/3 & -4/3 & -1 \end{bmatrix},$$

Used $A_1$ to form $Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3/5 & -4/5 \\ 0 & 0 & -4/5 & 3/5 \end{bmatrix},$

$$A_2 = Q_2 A_1 Q_2 = \begin{bmatrix} 4 & -3 & 0 & 0 \\ -3 & 10/3 & -5/3 & 0 \\ 0 & -5/3 & -33/25 & -68/75 \\ 0 & 0 & -68/75 & 149/75 \end{bmatrix},$$

As we can see, the final result is a tridiagonal symmetric matrix which is similar to the original one. The process is finished after two steps.

# Computational and theoretical relationship to other unitary transformations

The Householder transformation is a reflection about a certain hyperplane, namely, the one with unit normal vector $v$, as stated earlier. An $N$-by-$N$ unitary transformation $U$ satisfies $UU^H = I$. Taking the determinant ($N$-th power of the geometric mean) and trace (proportional to arithmetic mean) of a unitary matrix reveals that its eigenvalues $\lambda_i$ have unit modulus. This can be seen directly and swiftly:

$$\frac{\text{Trace}(UU^H)}{N} = \frac{\sum_{j=2}^{N} |\lambda_j|^2}{N} = 1, \det(UU^H) = \prod_{j=1}^{N} |\lambda_j|^2 = 1.$$

Since arithmetic and geometric means are equal if the variables are constant (see inequality of arithmetic and geometric means), we establish the claim of unit modulus.

For the case of real valued unitary matrixes we obtain orthogonal matrices, $UU^T = I$. It follows rather readily (see orthogonal matrix) that any orthogonal matrix can be decomposed into a product of 2 by 2 rotations, called Givens Rotations, and Householder reflections. This is appealing intuitively since multiplication of a vector by an orthogonal matrix preserves the length of that vector, and rotations and reflections exhaust the set of (real valued) geometric operations that render invariant a vector's length.

The Householder transformation was shown to have a one-to-one relationship with the canonical coset decomposition of unitary matrices defined in group theory, which can be used to parametrize unitary operators in a very efficient manner.[2]

Finally we note that a single Householder transform, unlike a solitary Givens transform, can act on all columns of a matrix, and as such exhibits the lowest computational cost for QR decomposition and tridiagonalization. The penalty for this "computational optimality" is, of course, that Householder operations cannot be as deeply or efficiently parallelized. As such Householder is preferred for dense matrices on sequential machines, whilst Givens is preferred on sparse matrices, and/or parallel machines.

# References

1. Householder, A. S. (1958). "Unitary Triangularization of a Nonsymmetric Matrix". *Journal of the ACM*. **5** (4): 339–342. doi:10.1145/320941.320947 (https://doi.org/10.1145%2F320941.320947). MR 0111128 (https://www.ams.org/mathscinet-getitem?mr=0111128).

2. Renan Cabrera; Traci Strohecker; Herschel Rabitz (2010). "The canonical coset decomposition of unitary matrices through Householder transformations". *Journal of Mathematical Physics*. **51** (8). doi:10.1063/1.3466798 (https://doi.org/10.1063%2F1.3466798).

- LaBudde, C.D. (1963). "The reduction of an arbitrary real square matrix to tridiagonal form using similarity transformations". *Mathematics of Computation*. American Mathematical Society. **17** (84): 433–437. doi:10.2307/2004005 (https://doi.org/10.2307%2F2004005). JSTOR 2004005 (https://www.jstor.org/stable/2004005). MR 0156455 (https://www.ams.org/mathscinet-getitem?mr=0156455).
- Morrison, D.D. (1960). "Remarks on the Unitary Triangularization of a Nonsymmetric Matrix". *Journal of the ACM*. **7** (2): 185–186. doi:10.1145/321021.321030 (https://doi.org/10.1145%2F321021.321030). MR 0114291 (https://www.ams.org/mathscinet-getitem?mr=0114291).
- Cipra, Barry A. (2000). "The Best of the 20th Century: Editors Name Top 10 Algorithms" (http://www.siam.org/pdf/news/637.pdf) (PDF). **33** (4): 1. (Herein Householder Transformation is cited as a top 10 algorithm of this century)
- Press, WH; Teukolsky, SA; Vetterling, WT; Flannery, BP (2007). "Section 11.3.2. Householder Method" (http://apps.nrbook.com/empanel/index.html#pg=578). *Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press. ISBN 978-0-521-88068-8