Adjugate matrix

In linear algebra, the **adjugate**, **classical adjoint**, or **adjunct** of a square matrix is the *transpose of its cofactor matrix*.[1]

The adjugate[2] has sometimes been called the "adjoint",[3] but today the "adjoint" of a matrix normally refers to its corresponding adjoint operator, which is its conjugate transpose.

Contents

Definition

The adjugate of **A** is the *transpose of the cofactor matrix* **C** of **A**,

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^\mathsf{T} \, .$$

In more detail, suppose R is a commutative ring and **A** is an $n \times n$ matrix with entries from R.

- The (i,j) *minor* of **A**, denoted $\mathbf{M}_{ij}$, is the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of **A**.
- The cofactor matrix of **A** is the $n \times n$ matrix **C** whose (i,j) entry is the (i,j) *cofactor* of **A**,

$$\mathbf{C}_{ij} = (-1)^{i+j}\mathbf{M}_{ij} \, .$$

- The adjugate of **A** is the *transpose* of **C**, that is, the $n \times n$ matrix whose (i,j) entry is the (j,i) cofactor of **A**,

$$\operatorname{adj}(\mathbf{A})_{ij} = \mathbf{C}_{ji} = (-1)^{i+j}\mathbf{M}_{ji} \, .$$

The adjugate is defined as it is so that the product of **A** with its adjugate yields a diagonal matrix whose diagonal entries are $\det(\mathbf{A})$,

$$\boxed{\mathbf{A}\operatorname{adj}(\mathbf{A}) = \det(\mathbf{A})\,\mathbf{I} \, .}$$

A is invertible if and only if det(**A**) is an invertible element of R, and in that case the equation above yields

$$\begin{aligned}\operatorname{adj}(\mathbf{A}) &= \det(\mathbf{A})\mathbf{A}^{-1} \, , \\ \mathbf{A}^{-1} &= \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A}) \, .\end{aligned}$$

Examples

1 × 1 generic matrix

The adjugate of any 1×1 matrix is $\mathbf{I} = (1)$.

2 × 2 generic matrix

The adjugate of the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $\operatorname{adj}(\mathbf{A}) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. It is seen that det(adj(**A**)) = det(**A**) and hence that adj(adj(**A**)) = **A**.

3 × 3 generic matrix

Consider a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

so its cofactor matrix is

$$\mathbf{C} = \begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

where

$$\begin{vmatrix} a_{im} & a_{in} \\ a_{jm} & a_{jn} \end{vmatrix} = \det \begin{pmatrix} a_{im} & a_{in} \\ a_{jm} & a_{jn} \end{pmatrix}.$$

Its adjugate is the transpose of its cofactor matrix,

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}} = \begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}.$$

3 × 3 numeric matrix

As a specific example, we have

$$\operatorname{adj}\begin{pmatrix} -3 & 2 & -5 \\ -1 & 0 & -2 \\ 3 & -4 & 1 \end{pmatrix} = \begin{pmatrix} -8 & 18 & -4 \\ -5 & 12 & -1 \\ 4 & -6 & 2 \end{pmatrix}.$$

The −6 in the third row, second column of the adjugate was computed as follows:

$$(-1)^{2+3}\;\det\begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix} = -((-3)(-4) - (3)(2)) = -6.$$

Again, the (3,2) entry of the adjugate is the (2,3) cofactor of A. Thus, the submatrix

$$\begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}$$

was obtained by deleting the second row and third column of the original matrix **A**.

It is easy to check the adjugate is the inverse times the determinant, −6.

Properties

The adjugate has the properties

$$\begin{aligned} &\operatorname{adj}(\mathbf{I}) = \mathbf{I}, \\ &\operatorname{adj}(\mathbf{AB}) = \operatorname{adj}(\mathbf{B})\operatorname{adj}(\mathbf{A}), \\ &\operatorname{adj}(c\mathbf{A}) = c^{n-1}\operatorname{adj}(\mathbf{A})\,, \end{aligned}$$

for $n\times n$ matrices **A** and **B**. The second line follows from equations adj(**B**)adj(**A**) = det(**B**)$\mathbf{B}^{-1}$ det(**A**)$\mathbf{A}^{-1}$ = det(**AB**)$(\mathbf{AB})^{-1}$.

Substituting in the second line $\mathbf{B} = \mathbf{A}^{m-1}$ and performing the recursion, one finds, for all integer m,

$$\operatorname{adj}(\mathbf{A}^m) = \operatorname{adj}(\mathbf{A})^m\,.$$

The adjugate preserves transposition,

$$\operatorname{adj}(\mathbf{A}^\mathsf{T}) = (\operatorname{adj}\mathbf{A})^\mathsf{T}\,.$$

Furthermore,

> If **A** is a $n\times n$ matrix with $n \geq 2$, then $\det(\operatorname{adj}(\mathbf{A})) = \det(\mathbf{A})^{n-1}$, and
> If **A** is an invertible $n\times n$ matrix, then $\operatorname{adj}(\operatorname{adj}(\mathbf{A})) = \det(\mathbf{A})^{n-2}\mathbf{A}\,,$

so that, if $n = 2$ and **A** is invertible, then det(adj(**A**)) = det(**A**) and adj(adj(**A**)) = **A**.

Taking the adjugate of an invertible matrix **A** k times yields

$$\begin{aligned} &\operatorname{adj}_k(\mathbf{A}) = \det(\mathbf{A})^{\frac{(n-1)^k-(-1)^k}{n}}\mathbf{A}^{(-1)^k}\,, \\ &\det(\operatorname{adj}_k(\mathbf{A})) = \det(\mathbf{A})^{(n-1)^k}\,. \end{aligned}$$

Inverses

In consequence of Laplace's formula for the determinant of an $n\times n$ matrix **A**,

$$\mathbf{A}\,\operatorname{adj}(\mathbf{A}) = \operatorname{adj}(\mathbf{A})\,\mathbf{A} = \det(\mathbf{A})\,\mathbf{I}_n \qquad (*)$$

where $\mathbf{I}_n$ is the $n{\times}n$ identity matrix. Indeed, the (i,i) entry of the product **A** adj(**A**) is the scalar product of row i of **A** with row i of the cofactor matrix **C**, which is simply the Laplace formula for det(**A**) expanded by row i.

Moreover, for $i \neq j$ the (i,j) entry of the product is the scalar product of row i of **A** with row j of **C**, which is the Laplace formula for the determinant of a matrix whose i and j rows are equal, and therefore vanishes.

From this formula follows one of the central results in matrix algebra: A matrix **A** over a commutative ring R is invertible if and only if det(**A**) is invertible in R.

For, if **A** is an invertible matrix, then

$$1 = \det(\mathbf{I}_n) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A})\det(\mathbf{A}^{-1})\,,$$

and equation (*) above implies

$$\mathbf{A}^{-1} = \det(\mathbf{A})^{-1}\operatorname{adj}(\mathbf{A})\,.$$

Similarly, the **resolvent** of **A** is

$$R(t;\mathbf{A}) \equiv \frac{\mathbf{I}}{t\mathbf{I} - \mathbf{A}} = \frac{\operatorname{adj}(t\mathbf{I} - \mathbf{A})}{p(t)}\,,$$

where $p(t)$ is the characteristic polynomial of **A**.

Characteristic polynomial

If

$$p(t) \stackrel{\text{def}}{=} \det(t\mathbf{I} - \mathbf{A}) = \sum_{i=0}^{n} p_i t^i \in R[t],$$

is the characteristic polynomial of the matrix n-by-n matrix **A** with coefficients in the ring R, then

$$\operatorname{adj}(s\mathbf{I} - \mathbf{A}) = \Delta p(s, \mathbf{A})\,,$$

where

$$\Delta p(s,t) \stackrel{\text{def}}{=} \frac{p(s) - p(t)}{s - t} = \sum_{j=0}^{n-1}\sum_{i=0}^{n-j-1} p_{i+j+1} s^i t^j \in R[s,t]$$

is the first divided difference of p, a symmetric polynomial of degree $n - 1$.

Jacobi's formula

The adjugate also appears in Jacobi's formula for the derivative of the determinant,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\det(A) = \operatorname{tr}\left(\operatorname{adj}(A)\frac{\mathrm{d}A}{\mathrm{d}\alpha}\right).$$

Cayley–Hamilton formula

The Cayley–Hamilton theorem allows the adjugate of **A** to be represented in terms of traces and powers of **A**,

$$\mathrm{adj}(\mathbf{A}) = \sum_{s=0}^{n-1} \mathbf{A}^s \sum_{k_1,k_2,\ldots,k_{n-1}} \prod_{l=1}^{n-1} \frac{(-1)^{k_l+1}}{l^{k_l} k_l!} \mathrm{tr}(\mathbf{A}^l)^{k_l},$$

where n is the dimension of $\mathbf{A}$, and the sum is taken over s and all sequences of $k_l \geq 0$ satisfying the linear Diophantine equation

$$s + \sum_{l=1}^{n-1} l k_l = n - 1 \,.$$

For the 2×2 case, this gives

$$\mathrm{adj}(\mathbf{A}) = \mathbf{I}_2 \,\mathrm{tr}\,\mathbf{A} - \mathbf{A} \,.$$

For the 3×3 case, this gives

$$\mathrm{adj}(\mathbf{A}) = \frac{1}{2}\left((\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}\,\mathbf{A}^2\right)\mathbf{I}_3 - \mathbf{A}\,\mathrm{tr}\,\mathbf{A} + \mathbf{A}^2.$$

For the 4×4 case, this gives

$$\mathrm{adj}(\mathbf{A}) = \frac{1}{6}\left((\mathrm{tr}\,\mathbf{A})^3 - 3\,\mathrm{tr}\,\mathbf{A}\,\mathrm{tr}\,\mathbf{A}^2 + 2\,\mathrm{tr}\,\mathbf{A}^3\right)\mathbf{I}_4 - \frac{1}{2}\mathbf{A}\left((\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}\,\mathbf{A}^2\right) + \mathbf{A}^2\,\mathrm{tr}\,\mathbf{A} - \mathbf{A}^3 \,..$$

The same $\mathrm{adj}(\mathbf{A}) = (-)^{n-1}\sum_{s=0}^{n-1} c_{s+1}\mathbf{A}^s$ follows directly from the terminating step of the fast Faddeev–LeVerrier algorithm, where the coefficients determined above are those of the characteristic polynomial of $\mathbf{A}$, namely, $p(\lambda) = \sum_{k=0}^{n} c_k \lambda^k$.

See also

- Cayley–Hamilton theorem
- Cramer's rule
- Trace diagram
- Jacobi's formula
- Faddeev–LeVerrier algorithm

References

1. Gantmacher, F. R. (1960). *The Theory of Matrices* (https://books.google.com/books?id=ePFtMw9v92sC&pg=PA76). **1**. New York: Chelsea. pp. 76–89. ISBN 0-8218-1376-5.
2. Strang, Gilbert (1988). "Section 4.4: Applications of determinants". *Linear Algebra and its Applications* (3rd ed.). Harcourt Brace Jovanovich. pp. 231–232. ISBN 0-15-551005-3.
3. Householder, Alston S. (2006). *The Theory of Matrices in Numerical Analysis*. Dover Books on Mathematics. pp. 166–168. ISBN 0-486-44972-6.

- Roger A. Horn and Charles R. Johnson (1991), *Topics in Matrix Analysis*. Cambridge University Press, ISBN 978-0-521-46713-1

External links

- Matrix Reference Manual (http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/property.html#adjoint)
- Online matrix calculator (determinant, track, inverse, adjoint, transpose) (http://www.elektro-energetika.cz/calculations/matreg.php?language=english) Compute Adjugate matrix up to order 8
- "adjugate of { { a, b, c }, { d, e, f }, { g, h, i } }" (http://www.wolframalpha.com/input/?i=adjugate+of+{+{+a%2C+b%2C+c+}%2C+{+d%2C+e%2C+f+}%2C+{+g%2C+h%2C+i+}+}). *Wolfram Alpha*.