

Binomial Theorem/General Binomial Theorem

From ProofWiki
< Binomial Theorem

KubeCon + CloudNativeCon

Join leading technologists for Kubernetes, Docker, and Cloud Native architecture events.linuxfoundation.org

Contents

- 1 Theorem
- 2 Proof
- 3 Historical Note
- 4 Sources

Theorem

Let $\alpha \in \mathbb{R}$ be a real number.

Let $x \in \mathbb{R}$ be a real number such that $|x| < 1$.

Then:

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} x^n = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{k=0}^{n-1} (\alpha - k) \right) x^n$$

where:

α^n denotes the falling factorial
 $\binom{\alpha}{n}$ denotes a binomial coefficient.

That is:

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots$$

Proof

Let R be the radius of convergence of the power series:

$$f(x) = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1} (\alpha - k)}{n!} x^n$$

By Radius of Convergence from Limit of Sequence:

$$\begin{aligned} \frac{1}{R} &= \lim_{n \rightarrow \infty} \frac{|\alpha(\alpha-1) \cdots (\alpha-n)|}{(n+1)!} \frac{n!}{|\alpha(\alpha-1) \cdots (\alpha-n+1)|} \\ \frac{1}{R} &= \lim_{n \rightarrow \infty} \frac{|\alpha(\alpha-1) \cdots (\alpha-n)|}{(n+1)!} \frac{n!}{|\alpha(\alpha-1) \cdots (\alpha-n+1)|} \\ &= \lim_{n \rightarrow \infty} \frac{|\alpha-n|}{n+1} \\ &= 1 \end{aligned}$$

Thus for $|x| < 1$, Power Series Differentiable on Interval of Convergence applies:

$$D_x f(x) = \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1} (\alpha - k)}{n!} n x^{n-1}$$

This leads to:

$$\begin{aligned}
(1+x)D_x f(x) &= \sum_{n=1}^\infty \frac{\prod_{k=0}^{n-1} (\alpha - k)}{(n-1)!} x^{n-1} + \sum_{n=1}^\infty \frac{\prod_{k=0}^{n-1} (\alpha - k)}{(n-1)!} x^n \\
&= \alpha + \sum_{n=1}^\infty \left(\frac{\prod_{k=0}^n (\alpha - k)}{n!} + \frac{\prod_{k=0}^{n-1} (\alpha - k)}{(n-1)!} \right) x^n \\
&= \alpha + \sum_{n=1}^\infty \frac{\prod_{k=0}^n (\alpha - k)}{(n-1)!} \left(\frac{1}{n} + \frac{1}{\alpha - n} \right) x^n \\
&= \alpha + \sum_{n=1}^\infty \frac{\prod_{k=0}^n (\alpha - k)}{(n-1)!} \frac{\alpha}{n(\alpha - n)} x^n \\
&= \alpha \left(1 + \sum_{n=1}^\infty \frac{\prod_{k=0}^{n-1} (\alpha - k)}{n!} x^n \right) \\
&= \alpha f(x)
\end{aligned}$$

Gathering up:

$$(1+x)D_x f(x) = \alpha f(x)$$

Thus:

$$D_x \left(\frac{f(x)}{(1+x)^\alpha} \right) = -\alpha(1+x)^{-\alpha-1}f(x) + (1+x)^{-\alpha}D_x f(x) = 0$$

So $f(x) = c(1+x)^\alpha$ when $|x| < 1$ for some constant c .

But $f(0) = 1$ and hence $c = 1$.

■

Historical Note

The General Binomial Theorem was first conceived by Isaac Newton during the years 1665 to 1667 when he was living in his home in Woolsthorpe.

He announced the result formally, in letters to Henry Oldenburg (https://en.wikipedia.org/wiki/Henry_Oldenburg) on 13th June 1676 and 24th October 1676 but did not provide a proper proof (at that time the need for the appropriate level of rigor had not been recognised).

Leonhard Paul Euler made an incomplete attempt in 1774, but the full proof had to wait for Carl Friedrich Gauss to provide it in 1812.

This was, in fact, the first time anything about infinite summations was proved adequately.

Sources

- 1937: Eric Temple Bell: *Men of Mathematics* ... (previous) ... (next): Chapter VI: On the Seashore
- 1968: Murray R. Spiegel: *Mathematical Handbook of Formulas and Tables*: 20.4
- 1977: K.G. Binmore: *Mathematical Analysis: A Straightforward Approach* ... (previous) ... (next): §15.10 (1)
- 1997: Donald E. Knuth: *The Art of Computer Programming: Volume 1: Fundamental Algorithms* (3rd ed.) ... (previous) ... (next): §1.2.6: Binomial Coefficients: F (15)

KubeCon 2017 in Austin

Join leading technologists for Kubernetes, Docker, and Cloud Native architecture events.linuxfoundation.org

Retrieved from "https://proofwiki.org/w/index.php?title=Binomial_Theorem/General_Binomial_Theorem&oldid=280958"

Categories: Proven Results | Binomial Theorem

- This page was last modified on 29 December 2016, at 17:58 and is 4,952 bytes
- Content is available under Creative Commons Attribution-ShareAlike License unless otherwise noted.