Binomial Theorem/General Binomial Theorem

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Theorem

Let $\alpha \in \mathbb{R}$ be a real number.

Let $x \in \mathbb{R}$ be a real number such that $|x| < 1$.

Then:

$$(1 + x)^\alpha = \sum_{n = 0}^\infty \frac{\alpha^{\underline n}}{n!} x^n = \sum_{n = 0}^\infty \binom \alpha n x^n = \sum_{n = 0}^\infty \frac 1 {n!} \left({\prod_{k = 0}^{n - 1} \left({\alpha - k}\right)}\right) x^n$$

where:

$\alpha^{\underline n}$ denotes the falling factorial
$\dbinom \alpha n$ denotes a binomial coefficient.

That is:

$$(1 + x)^\alpha = 1 + \alpha x + \frac {\alpha \left({\alpha - 1}\right)} {2!} x^2 + \frac {\alpha \left({\alpha - 1}\right) \left({\alpha - 2}\right)} {3!} x^3 + \cdots$$

Proof

Let R be the radius of convergence of the power series:

$$f \left({x}\right) = \sum_{n = 0}^\infty \frac {\prod \limits_{k = 0}^{n - 1} \left({\alpha - k}\right)} {n!} x^n$$

By Radius of Convergence from Limit of Sequence:

$$\frac 1 R = \lim_{n \to \infty} \frac {\left\vert{\alpha \left({\alpha - 1}\right) \cdots \left({\alpha - n}\right)}\right\vert} {\left({n + 1}\right)!} \frac {n!} {\left\vert{\alpha \left({\alpha - 1}\right) \cdots \left({\alpha - n + 1}\right)}\right\vert}$$

$$\begin{aligned} \frac 1 R &= \lim_{n \to \infty} \frac {\left\vert{\alpha \left({\alpha - 1}\right) \cdots \left({\alpha - n}\right)}\right\vert} {\left({n + 1}\right)!} \frac {n!} {\left\vert{\alpha \left({\alpha - 1}\right) \cdots \left({\alpha - n + 1}\right)}\right\vert} \\ &= \lim_{n \to \infty} \frac {\left\vert{\alpha - n}\right\vert} {n + 1} \\ &= 1 \end{aligned}$$

Thus for $|x| < 1$, Power Series Differentiable on Interval of Convergence applies:

$$D_x f \left({x}\right) = \sum_{n = 1}^\infty \frac {\prod \limits_{k = 0}^{n - 1} \left({\alpha - k}\right)} {n!} n x^{n - 1}$$

This leads to:

$$
\begin{aligned}
(1+x) D_x f(x) &= \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1} (\alpha - k)}{(n-1)!} x^{n-1} + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1} (\alpha - k)}{(n-1)!} x^n \\
&= \alpha + \sum_{n=1}^{\infty} \left(\frac{\prod_{k=0}^{n} (\alpha - k)}{n!} + \frac{\prod_{k=0}^{n-1} (\alpha - k)}{(n-1)!} \right) x^n \\
&= \alpha + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n} (\alpha - k)}{(n-1)!} \left(\frac{1}{n} + \frac{1}{\alpha - n} \right) x^n \\
&= \alpha + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n} (\alpha - k)}{(n-1)!} \frac{\alpha}{n(\alpha - n)} x^n \\
&= \alpha \left(1 + \sum_{n=1}^{\infty} \frac{\prod_{k=0}^{n-1} (\alpha - k)}{n!} x^n \right) \\
&= \alpha f(x)
\end{aligned}
$$

Gathering up:

$$(1+x) D_x f(x) = \alpha f(x)$$

Thus:

$$D_x \left(\frac{f(x)}{(1+x)^\alpha} \right) = -\alpha (1+x)^{-\alpha-1} f(x) + (1+x)^{-\alpha} D_x f(x) = 0$$

So $f(x) = c(1+x)^\alpha$ when $|x| < 1$ for some constant c.

But $f(0) = 1$ and hence $c = 1$.

$\blacksquare$

Historical Note

The General Binomial Theorem was first conceived by Isaac Newton during the years 1665 to 1667 when he was living in his home in Woolsthorpe.

He announced the result formally, in letters to Henry Oldenburg (https://en.wikipedia.org/wiki/Henry_Oldenburg) on 13th June 1676 and 24th October 1676 but did not provide a proper proof (at that time the need for the appropriate level of rigor had not been recognised).

Leonhard Paul Euler made an incomplete attempt in 1774, but the full proof had to wait for Carl Friedrich Gauss to provide it in 1812.

This was, in fact, the first time anything about infinite summations was proved adequately.

Sources

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