Every skew-symmetric matrix has a non-negative determinant

> Let A be a skew-symmetric $n \times n$-matrix over the real numbers. Show that $\det A$ is nonnegative.

I'm breaking this up into the even case and odd case (if A is an $n \times n$ skew-symmetric matrix).

So when n is odd, we have:

$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) \Rightarrow \det(A) = -\det(A) \Rightarrow \det(A) = 0$

So $\det(A)$ is non-negative when n is odd.

When n is even, we have:

$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) \Rightarrow \det(A) = \det(A)$

But why can't $\det(A)$ be negative in this case?

(linear-algebra) (determinant)

edited Sep 19 '15 at 12:58
darij grinberg
8,420 ● 3 ■ 26 ▲ 55

asked Nov 26 '14 at 0:30
Caleb
51 ■ 1 ▲ 2

1 en.wikipedia.org/wiki/Pfaffian – Andrea Mori Nov 26 '14 at 1:02

2 Answers

Since A is skew-symmetric, if λ is an eigenvalue of A then $-\lambda$ is also an eigenvalue. Thus, in even dimension, the eigenvalues of A come in pairs $\pm\lambda$. Moreover, it is known that all the eigenvalues are imaginary. That is, the set of eigenvalues of A is of the form $\{\pm\lambda_1 i, \cdots, \pm\lambda_n i\}$ Thus,

$$\det(A) = \lambda_1 i \cdot (-\lambda_1 i) \cdots \lambda_n i \cdot (-\lambda_n i) = \lambda_1^2 \cdots \lambda_n^2 \geq 0.$$

Without the use of eigenvalues we can proceed as follows. The result is clear if the matrix is of order 2. If it has order 4 is of the form:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

If the first row is 0 then $\det = 0$ and we are done. So, there is $a_{1i} \neq 0$. Assume it is $a_{12} \neq 0$. In other case change row 2 with row i and column 2 with column i. Now, we make zeros in the first two rows and columns and get the matrix

$$\begin{pmatrix} 0 & a_{12} & 0 & 0 \\ -a_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & a'_{34} \\ 0 & 0 & -a'_{34} & 0 \end{pmatrix}.$$

All changes we have done doesn't change the determinant of the matrix. Now we get that the determinant is

$$a_{12}^2 (a'_{34})^2 \geq 0.$$

We can do this argument with any symmetric matrix of order n. We change columns and rows, if necessary, to get

$$\begin{pmatrix} 0 & a_{12} & \cdots \\ -a_{12} & 0 & \cdots \\ \vdots & \vdots & A \end{pmatrix}$$

with $a_{12} \neq 0$. Now, we make zeros in the two first rows and columns to get

$$\begin{pmatrix} 0 & a_{12} & 0 \\ -a_{12} & 0 & 0 \\ 0 & 0 & A' \end{pmatrix}.$$

We can show that A' is skew-symmetric and that the determinant is $a_{12}^2 \det(A')$. Now, A' is a skew-symmetric matrix of order $n-2$ and we repeat the process.

edited Sep 19 '15 at 13:18
amd
17.6k ● 2 ■ 6 ▲ 36

answered Nov 26 '14 at 0:58
mfl
22.6k ● 1 ■ 17 ▲ 36

Is there a solution that doesn't use eignevalues? We haven't covered them yet. – Caleb Nov 26 '14 at 1:01

See Lemma 5.6 (and the following discussion) in Keith Conrad's *Bilinear Forms*.

answered Sep 19 '15 at 12:57

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