

Math Homework Week 1

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1 Measure Spaces

1.1 Problem 1.3

- $\mathcal{G}_1 = \{A, A \subset \mathbb{R}, A \text{ open}\}$
 - if A is an open set on $(-\infty, 0)$ then its complement is $[0, \infty)$
 - According to the definition of a algebra, A is an \mathcal{A} or \mathcal{S} if it's complement is a closed set. And since A^c is only half closed, \mathcal{G}_1 cannot be an algebra
- $\mathcal{G}_2 = \{A, A \text{ is a finite union of intervals of the form } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$
 - if $A = \cup_{i=1}^n (a_i, b_i]$ where $-\infty \leq a_i \leq b_i \leq \infty$ then
 - then $A^c = (-\infty, a_i] \cup \bigcup_{i=1}^{n-1} (b_i, a_{i+1}] \cup (b_n, \infty)$
 - given the complement, the function \mathcal{G}_2 is an algebra since it goes on indefinitely on both ends and is closed
- $\mathcal{G}_3 = \{A, A \text{ is a countable union of } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$
 - \mathcal{G}_3 is a σ -algebra

1.2 Exercise 1.7

Explain why these are the 'largest' and 'smallest' possible σ -algebras, respectively, in the following sense: if \mathcal{A} is any σ -algebra, then $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$.

- $\{\emptyset, X\}$: this is the smallest algebra σ -algebra since it contains an empty set and a non-empty set (the empty set's complement)
- $\mathcal{P}(X)$: the Power Set is all subsets of X including the empty set making $\mathcal{A} \subset \mathcal{P}(X)$ mechanically larger than $\{\emptyset, X\}$

1.3 Exercise 1.10

Prove the following Proposition:

Let $\{\mathcal{S}_\sigma\}$ be a family of σ -algebras on X . Then $\cap_\sigma \mathcal{S}_\sigma$ is also a σ -algebra.

Since each set is a σ -algebra, there must be a set of empty sets, which satisfies the first condition of a σ -algebra.

1.4 Exercise 1.17

Let (X, \mathcal{S}, μ) be a measure space. Prove the following:

- μ is monotone: if $A, B \in \mathcal{S}$ then $\mu(A) \leq \mu(B)$
- μ is countably subadditive: if $\{A_i\}_{i=1}^\infty$ then $\mu(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu(A_i)$

1.5 Exercise 1.18

Let (X, \mathcal{S}, μ) be a measure space. Let $B \in \mathcal{S}$. Show that $\lambda: \mathcal{S} \rightarrow [0, \infty]$ defined by $\lambda(A) = \mu(A \cap B)$ is also a measure of (X, \mathcal{S})

1.6 Exercise 1.20

Prove (ii)

2 Construction of Lebesgue Measure

2.1 Exercise 2.10

2.2 Exercise 2.14

3 Measureable Functions

3.1 Exercise 3.1

3.2 Exercise 3.4

3.3 Exercise 3.7

3.4 Exercise 3.14

4 Lebesgue Integration

4.1 Exercise 4.14

4.2 Exercise 4.15

4.3 Exercise 4.17

4.4 Exercise 4.21