Math Homework Week 1

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1 Measure Spaces

1.1 Problem 1.3

- $\mathcal{G}_1 = \{A, A \subset \mathbb{R}, A\ open\}$
  - if A is an open set on $(-\infty, 0)$ than its complement is $[0, \infty)$
  - According to the definition of a alegebra, A is an $\mathcal{A}$ or $\mathcal{S}$ if it's complement is a closed set. And since A^c is only half closed, $\mathcal{G}_1$ cannot be an algebra
- $\mathcal{G}_2 = \{$ A, A is a finite union of intervals of the form $(a, b], (-\infty, b], and (a, \infty)\}$
  - if $A = \cup_{i=1}^{n}(a_i, b_i]$ where $-\infty \leq a_i \leq b_i \leq \infty$ then
  - then $A^c = (-\infty, a_i] \cup \bigcup_{i=1}^{n-1}(b_o, a_{i+1}] \cup (b_n, \infty))$
  - given the complement, the function $\mathcal{G}_2$ is an algebra since it goes on indefinitely on both ends and is closed
- $\mathcal{G}_3 = \{$ A, A is a countable union of $(a, b], (-\infty, b], and (a, \infty)\}$
  - $\mathcal{G}_3$ is a σ-algebra

1.2 Exercise 1.7

Explain why these are the 'largest' and 'smallest' possible σ-algebras, respectively, in the following sense: if A is any σ-algebra, then $\{\emptyset, A\} \subset \mathcal{A} \subset \mathcal{P}(X)$.

- $\{\emptyset, X\}$: this is the smallest algebra σ-algebra since it contains an empty set and a non-empty set (the empty set's complement
- $\mathcal{P}(X)$: the Power Set is all subsets of $\mathcal{X}$ including the empty set making $\mathcal{A} \subset \mathcal{P}(X)$ mechanically larger than $\{\emptyset, X\}$

1.3 Exercise 1.10

Prove the following Proposition:

Let $\{\mathcal{S}_\sigma\}$ be a family of σ-algebras on X. Then $\cap_\sigma \mathcal{S}_\sigma$ is also a σ-algebra.

Since each set is a σ-algebra, there must be a set of empty sets, which satisifies the first condition of a an algebra.

1.4 Exercise 1.17

Let $(X, \mathcal{S}, \mu)$ be a measure space. Prove the following:

- μ is monotone: if $A, B \in \mathcal{S}$ then $\mu(A) \leq \mu(B)$
- μ is countably subadditive: if $\{A_i\}_{i=1}^{\infty}$ then $\mu(\cup_{i=1}^{\infty} A_i) \leq \Sigma_{i=1}^{\infty} \mu(A)$

1.5 Exercise 1.18

Let $(X, \mathcal{S}\mu)$ be a measure space. Let $B \in \mathcal{S}$. Show that $\lambda\, \mathcal{S} \to [0, \infty]$ defined by $\lambda(A) = \mu(A \cap B)$ is also a measure of $(X, \mathcal{S})$

1.6 Exercise 1.20

Prove (ii)

2 Construction of Lebesgue Measure

2.1 Exercise 2.10

2.2 Exercise 2.14

3 Measureable Functions

3.1 Exercise 3.1

3.2 Exercise 3.4

3.3 Exercise 3.7

3.4 Exercise 3.14

4 Lebesque Integration

4.1 Exercise 4.14

4.2 Exercise 4.15

4.3 Exercise 4.17

4.4 Exercise 4.21