Topological Data Analysis and Neuroscience

Chapter 1: Basics

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Section 1:
Topological Spaces

Topological Spaces

- A topological space is a pair $\langle X, \mathcal{T} \rangle$
  - a set X endowed with a topological structure $\mathcal{T}$ (i.e. a collection of subsets of X), such that:
    - both the empty set and X are elements of $\mathcal{T}$
    - the union of arbitrary many elements of $\mathcal{T}$ is also an element of $\mathcal{T}$
    - the intersection of finitely many elements of $\mathcal{T}$ is an element of $\mathcal{T}$

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- Open/closed sets
  - A subset $A \subseteq X$ is open (w.r.t. $\langle X, \mathcal{T} \rangle$) is A is an element in $\mathcal{T}$
  - A subset $A \subseteq X$ is closed if its complement is open

Examples

- The discrete topology on a set X is the power set of X.
- The indiscrete topology on X is the set $\{\emptyset, X\}$.
- The Euclidean topology on $\mathbb{R}$ is the set $\{\cup_{i\in I}\,(a_i, b_i) | a_i < b_i\}$ of unions of open intervals.

Metric Spaces

- A metric space (X, d_X) is a set X equipped with a function $d_X: X \times X \to R$ such that
  - $d_X(x, y) \geq 0$ for all $x, y \in X$
  - $d_X(x, x) = 0$ if and only if $x = y$
  - $d_X(x, y) = d_X(y, x)$ for all $x, y \in X$
  - $d_X(x, y) + d_X(y, z) \geq d_X(x, z)$, for any $x, y, z \in X$

- Examples:
  - Euclidean space $(R^d, ||\cdot||_2)$

Metric Space Topology

- "Open ball" topology

- Given (X, d), a natural topology $\langle X, \mathcal{T} \rangle$
  - open ϵ-ball $B(x, \epsilon) \coloneqq \{ y \in X \mid d(x, y) < \epsilon \}$
  - $\mathcal{B} \coloneqq \{ B(x, \epsilon) \mid x \in X, \epsilon > 0 \}$
  - $\mathcal{T}$ =collection of unions of open balls from $\mathcal{B}$

- Example: the real line R

- Remark:
  - $\mathcal{B}$ above also forms the basis of the resulting topology $\langle X, \mathcal{T} \rangle$

Subspace Topology

- Given a topological space $\langle X, \mathcal{T} \rangle$, and a subset $Y \subset X$
  - $\langle Y, \mathcal{T}_Y \rangle$: the subspace topology on Y induced from $\langle X, \mathcal{T} \rangle$ where
    - open sets in $\mathcal{T}_Y$ are the intersection of open sets from $\mathcal{T}$ with Y

- Examples:
  - a line segment
  - A surface in R^3

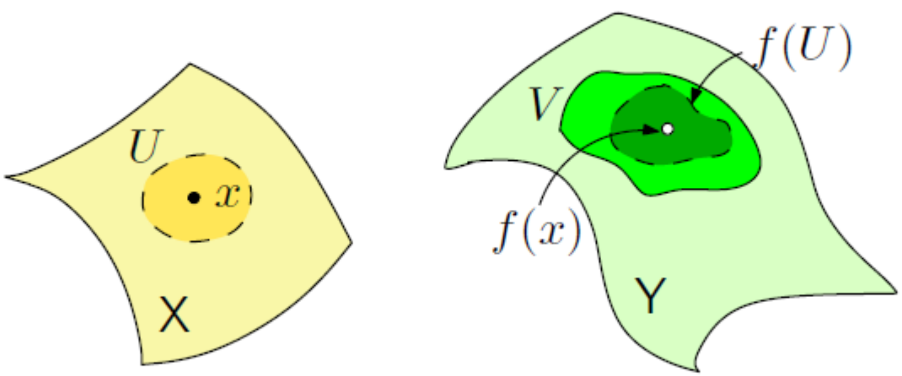
Closure/interior

- Given a subset A of a topological space X
  - the **closure of A**, denoted by $\bar{A}$, is the intersection of all closed sets containing A
  - the **interior of A**, denoted by $int\ A$, is the union of all open sets contained in A
  - the **boundary of A** is $bd\ A := \bar{A} \cap \overline{X - A}$

Section 2:
Maps, homeomorphism, and homotopy

Continuous Functions

- An important concept to describe relations between (the connectivity of) two topological spaces
- A neighborhood of a point $x \in X$ is simply an open set of X containing x
- A function $f: X \to Y$
  - is continuous at $x \in X$ if for any neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.
  - is continuous if it is continuous at all points in X



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  - is continuous if it is continuous at all points in X
- Equivalently, a function $f: X \to Y$
  - is continuous if for any open set V in Y, its preimage $f^{-1}(V)$ is also open

Remarks

- The perhaps more familiar (ϵ, δ)-definition of a continuous real-valued function $f: R \to R$

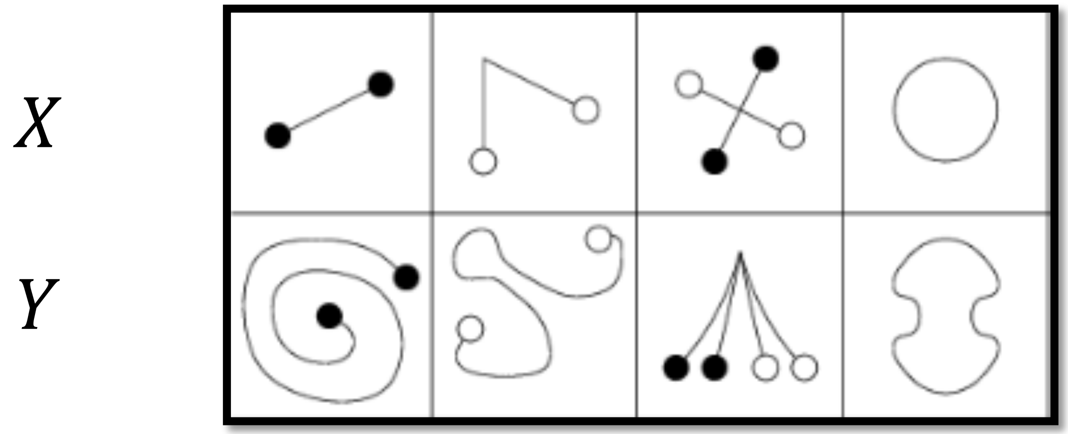
- If we know a basis generating the topology of Y,
  - then to check whether a function $f: X \to Y$ is continuous, we only need to verify that for each basis element of Y, its pre-image is open in X.

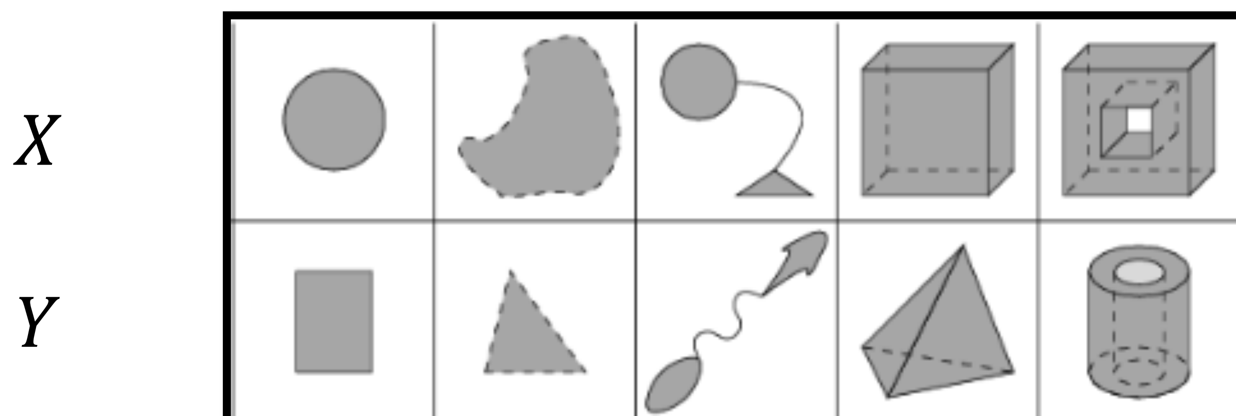
Homeomorphism

- Intuitively, two topological spaces are homeomorphic if they are equivalent as topological spaces.

- Given two topological spaces X and Y
  - a homeomorphism between them is a continuous function $h: X \to Y$ such that h is bijection and the inverse of h is also continuous.
  - In this case, X is homeomorphic to Y, denoted by $X \cong Y$

- Examples: $X \cong Y$ in each pair below





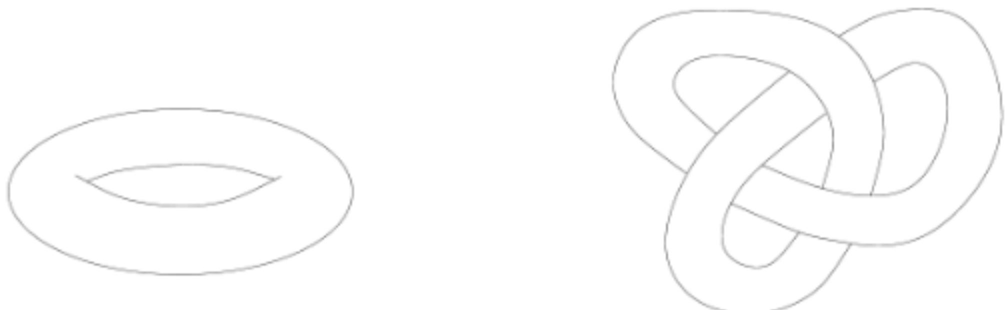
More Examples

- Constructing homeomorphisms
  - Open d-ball and R^d
  - Sphere and the boundary of a tetrahedron
  - Sphere with north-pole removed and R^2

Embeddings

- Given two topological spaces X and Y

  ϕ is an embedding of X into Y if it induces a homeomorphism between X and $\phi(X)$



Two embeddings of the torus in R^3

Homotopy

- Two continuous functions $f, g: X \to Y$ are homotopic if
  - there is a continuous map $H: X \times [0,1] \to Y$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. We denote this by $X \simeq Y$.
  - the map H is called a homotopy connecting f and g

- Example:
  - Two maps from $\mathrm{f}, \mathrm{g}: \mathbb{R} \to \mathbb{R}$ given by $\mathrm{g}(\mathrm{x}) = \mathrm{x}$ and $\mathrm{f}(\mathrm{x}) = 0$.
  - A homotopy between them is given by $\mathrm{H}(\mathrm{x}, \mathrm{t}) = \mathrm{t} * \mathrm{x}$.

Homotopy Equivalence

- Two topological spaces X and Y are homotopy equivalent if
  - there is a pair of continuous maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g$ is homotopic to id_Y, the identity in Y, and $g \circ f$ is homotopic to id_X, the identity in X.
  - denoted by $X \simeq Y$

- Intuition and examples

- Theorem:
  - Two homeomorphic spaces are also homotopy equivalent, but not vice versa.

Deformation Retraction

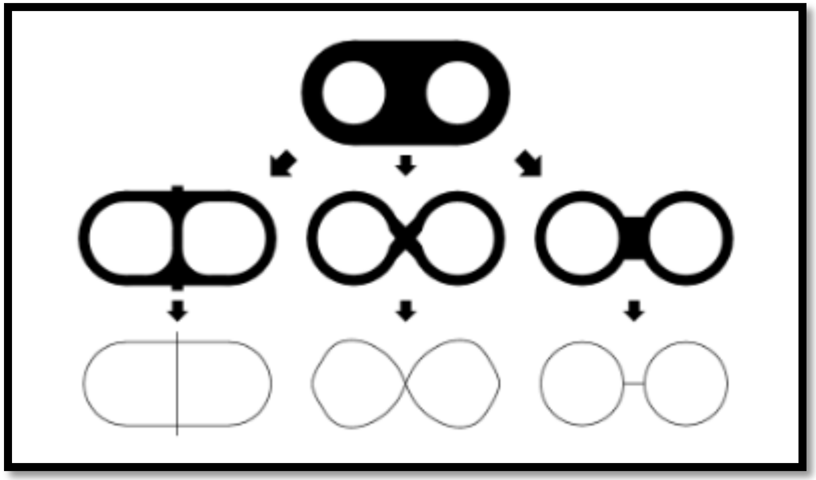
- A special type of homotopy equivalence, which is more intuitive and often easier to construct

- Retraction:
  - Given $A \subseteq X$, a retraction map is a continuous function $r: X \to A$ such that $r(x) = x$ for any $x \in A$
- Deformation retraction:
  - a retraction map $r: X \to A$ is a deformation retraction if $r \simeq id_X$ (i.e, r is homotopic to the identity map in X). A is called a deformation retract of X in this case.

- Theorem:
  - If $A \subseteq X$ is a deformation retract of X, then $X \simeq A$, i.e, they are homotopy equivalent.

- Theorem:
  - If $X \cong Y, Y \cong Z$, then $X \cong Z$.
  - If $X \simeq Y, Y \simeq Z$, then $X \simeq Z$.

Examples

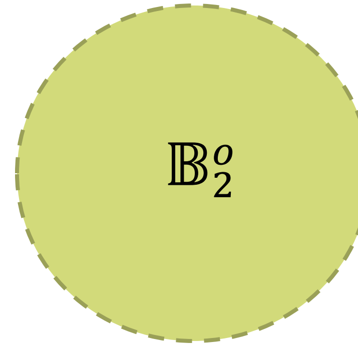


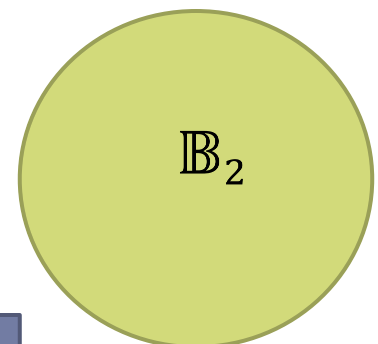
Section 3: Clustering

Section 3: Manifolds

Some Notations

- Open d-ball:
  - $\mathbb{B}_d^o := \{ x \in R^d \mid \|x\| < 1\}$
- Closed d-ball:
  - $\mathbb{B}_d := \{ x \in R^d \mid \|x\| \leq 1\}$
- d-sphere:
  - $\mathbb{S}_d = \{ x \in R^{d+1} \mid \|x\| = 1\}$
  - $\mathbb{S}_d = bd\ \mathbb{B}_{d+1}$





What is $\mathbb{S}_1$? $\mathbb{S}_2$? $\mathbb{S}_0$?

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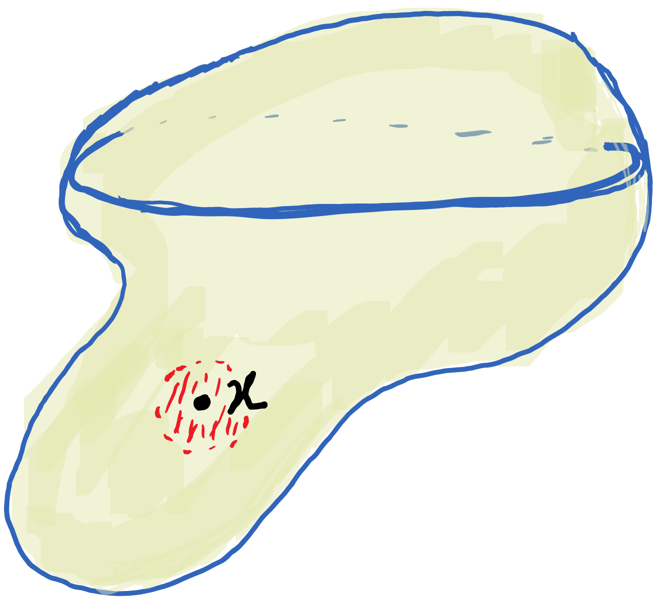
- Relation between
  - $\mathbb{B}_d^o$ and R^d?
  - $\mathbb{B}_2$ and a triangle?
  - $\mathbb{B}_3$ and a cube?
  - $\mathbb{B}_d$ and a point?
  - What is the space $\mathbb{S}_d - (0,0, \dots, 1)$ (i.e, d-sphere with north pole removed) homeomorphic to?

Manifolds

- A d-manifold without boundary is a topological space M such that each point $x \in M$ has a neighborhood homeomorphic to R^d (i.e, to $\mathbb{B}_d^o$).



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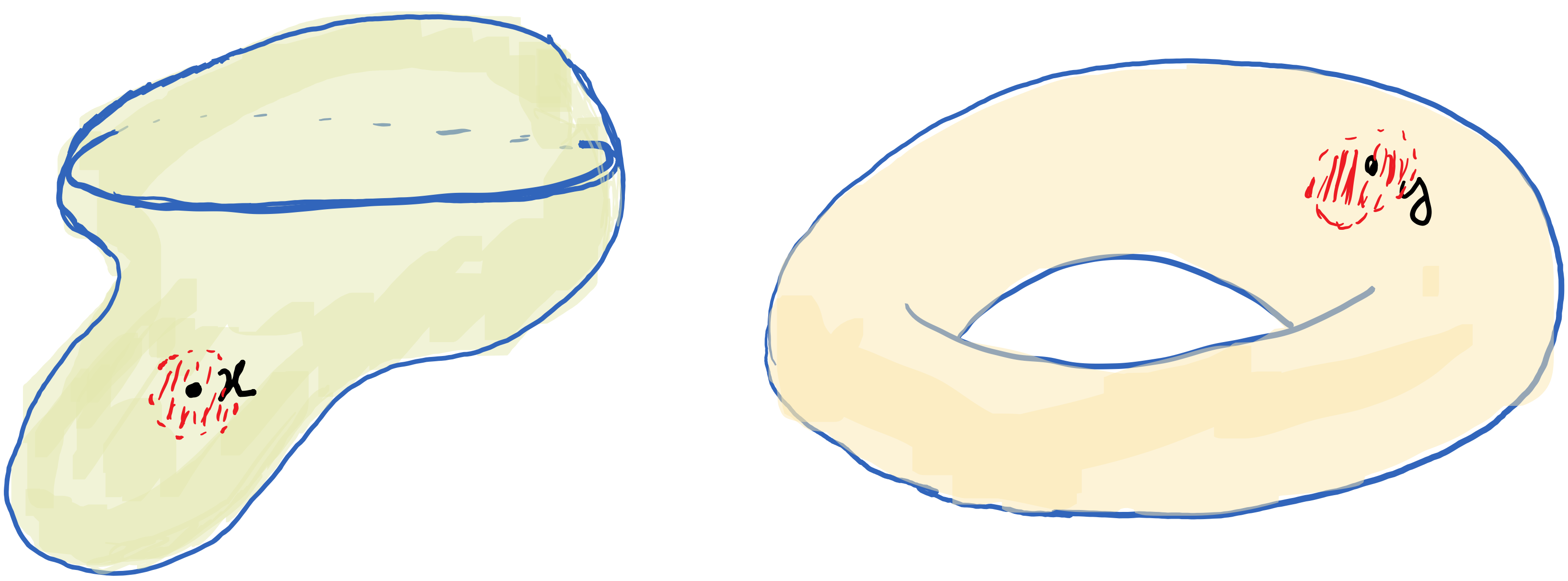

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x is an interior point.

x is on the boundary.

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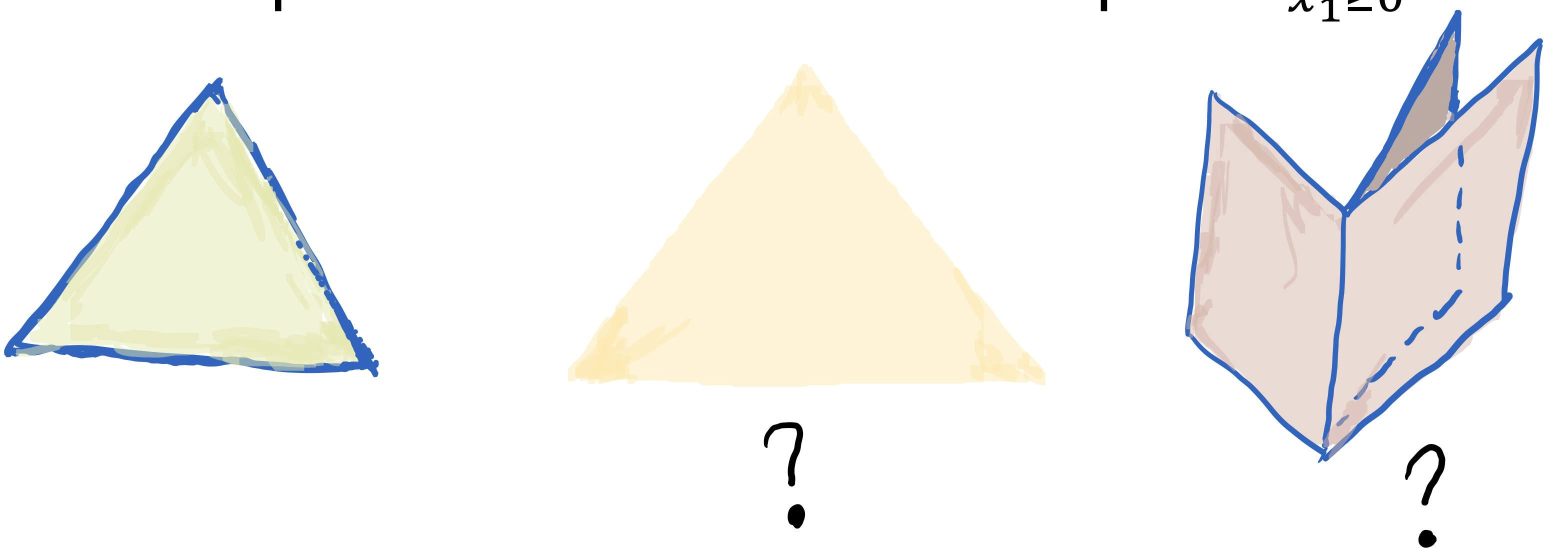

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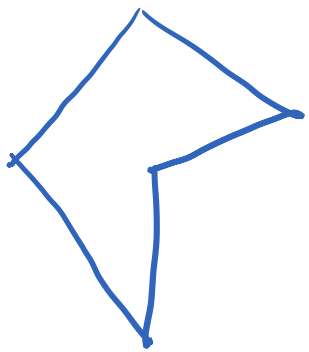
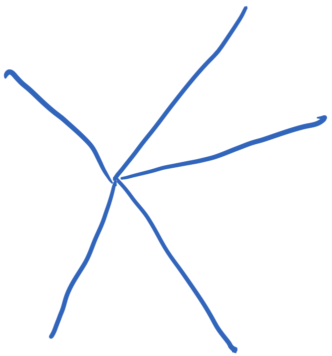
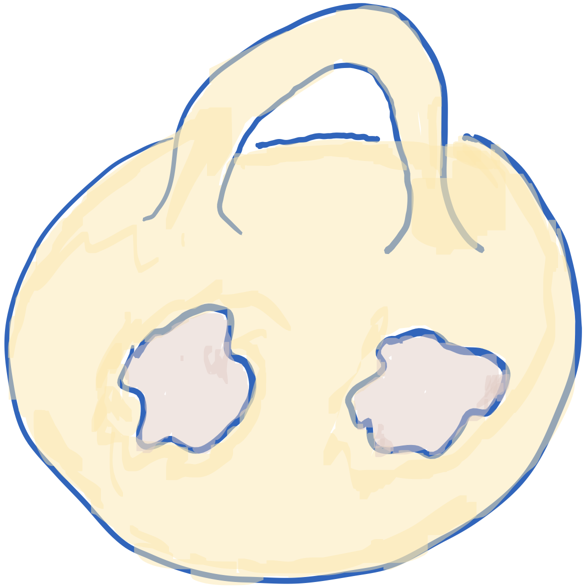
Theorem

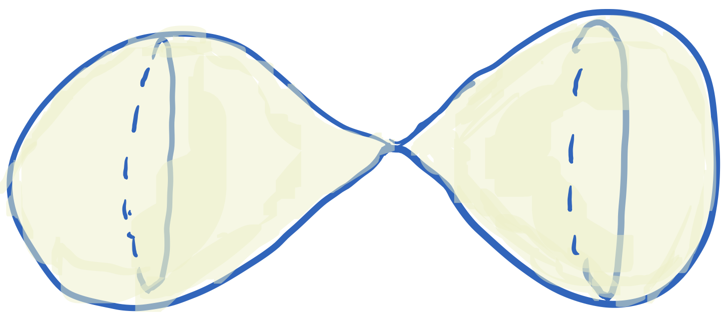
The boundary of a d-manifold is a $(d-1)$-manifold without boundary (potentially with multiple connected components).

More Examples









Intrinsic / ambient dimension

- Given a m-dimensional manifold M embedded in R^d
  - intrinsic dimension: m
  - ambient dimension: d

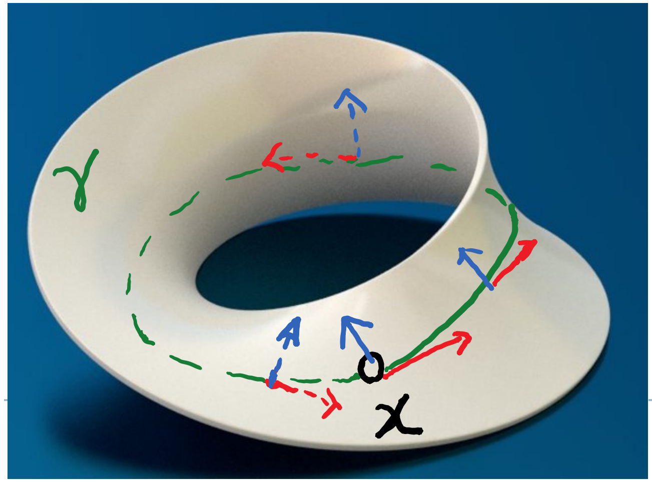
- Often in practice, $m \ll d$
  - It is desirable to have time complexity of an algorithm depending mostly on m instead of on d

Section 4:
2-Manifolds and their classification

Orientable / non-orientable surfaces

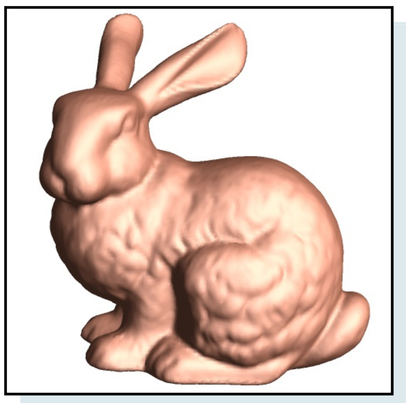
- A closed curve $\gamma \subseteq M$ is
  - orientation-reversing: if the hand-ness of a local frame changes as we traverse the curve once.
  - orientation-preserving: otherwise.

- A 2-manifold is
  - non-orientable: if it contains any orientation-reversing closed curve
  - orientable: otherwise.



Remarks and Examples

- Any compact surface embedded in R^3 is orientable.





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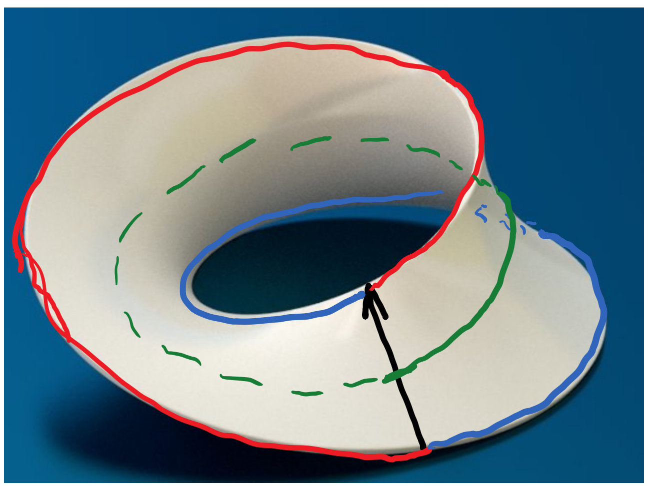
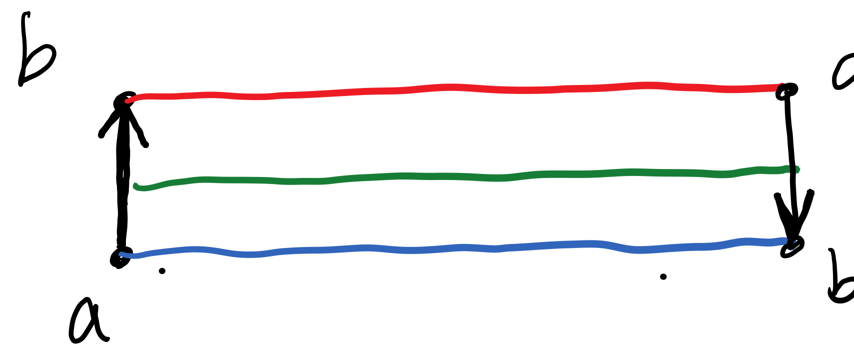
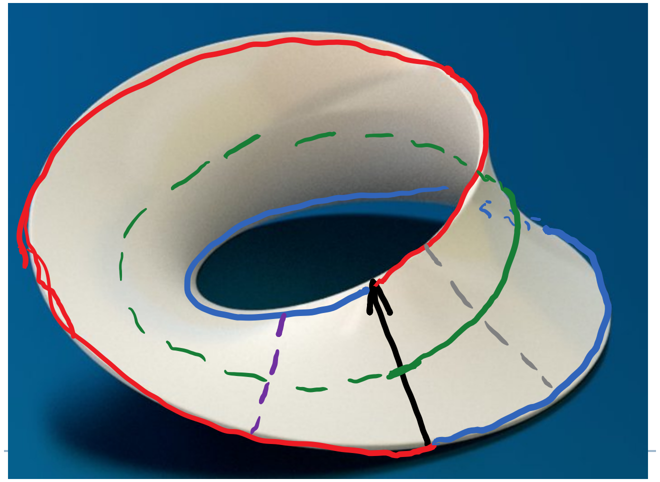
- Non-orientable surface
  - Möbius strip (simplest non-orientable surface with boundary)

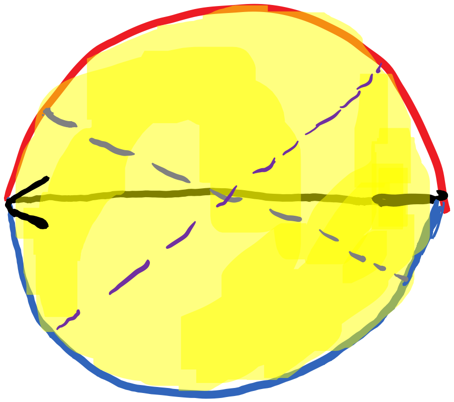




Remarks and Examples

- Any compact surface embedded in R^3 is orientable.
- Non-orientable surface
  - Möbius strip (simplest non-orientable surface with boundary)
  - Projective plane (simplest non-orientable surface without boundary)
    - obtained by gluing the boundary of a Möbius strip to that of a disk
    - or: think of a disk, then glue the pairs of antipodal points of it





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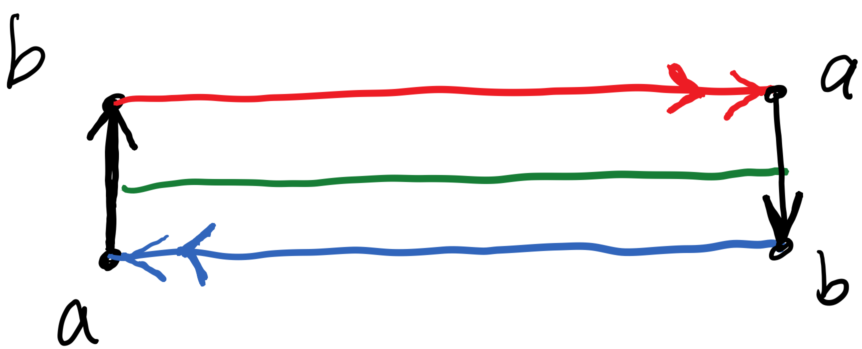

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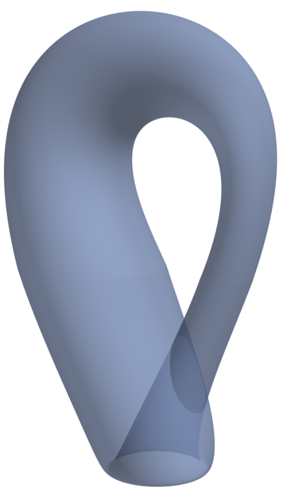
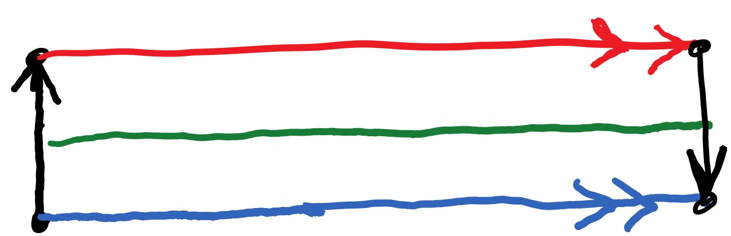
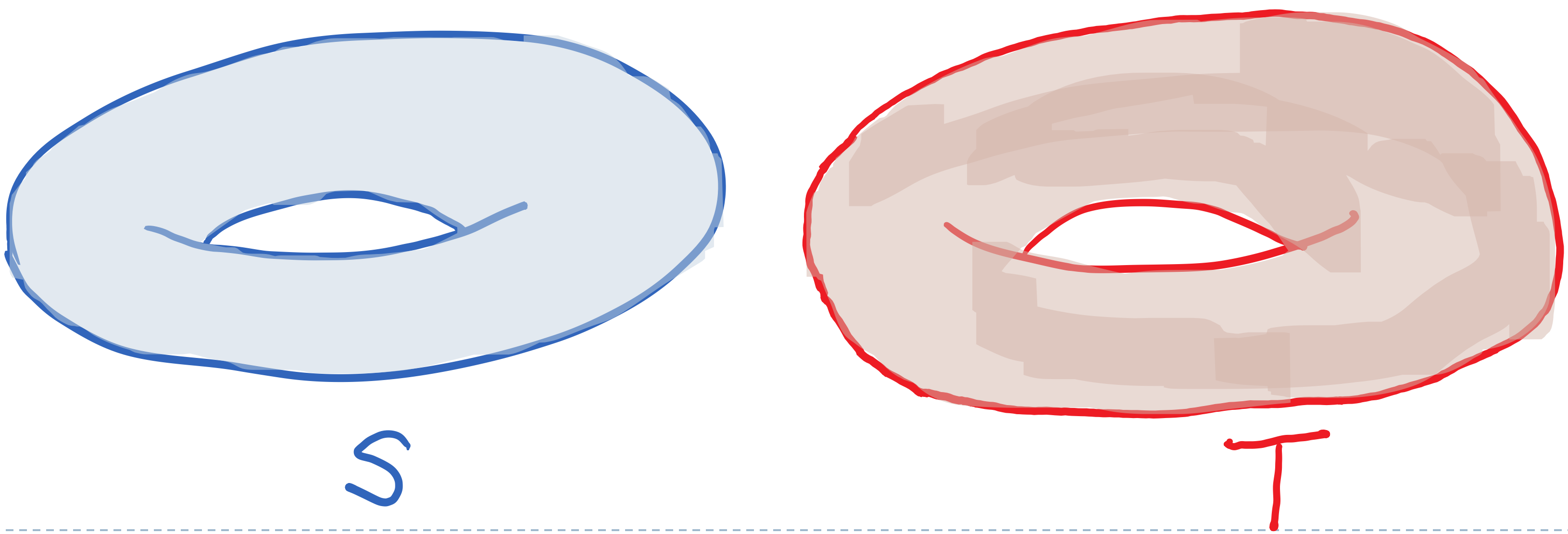
- Non-orientable surface
  - Möbius strip (simplest non-orientable surface with boundary)
  - Projective plane (simplest non-orientable surface without boundary)
  - Klein bottle
    - Obtained by gluing two Möbius strips along their boundaries





Connected Sum

- A way to construct ``more complicated'' surfaces
  - Given two surfaces S and T, their connected sum $S \# T$ is constructed by:
    - first, removing a disk from both,
    - then, gluing them along the boundary of the disks



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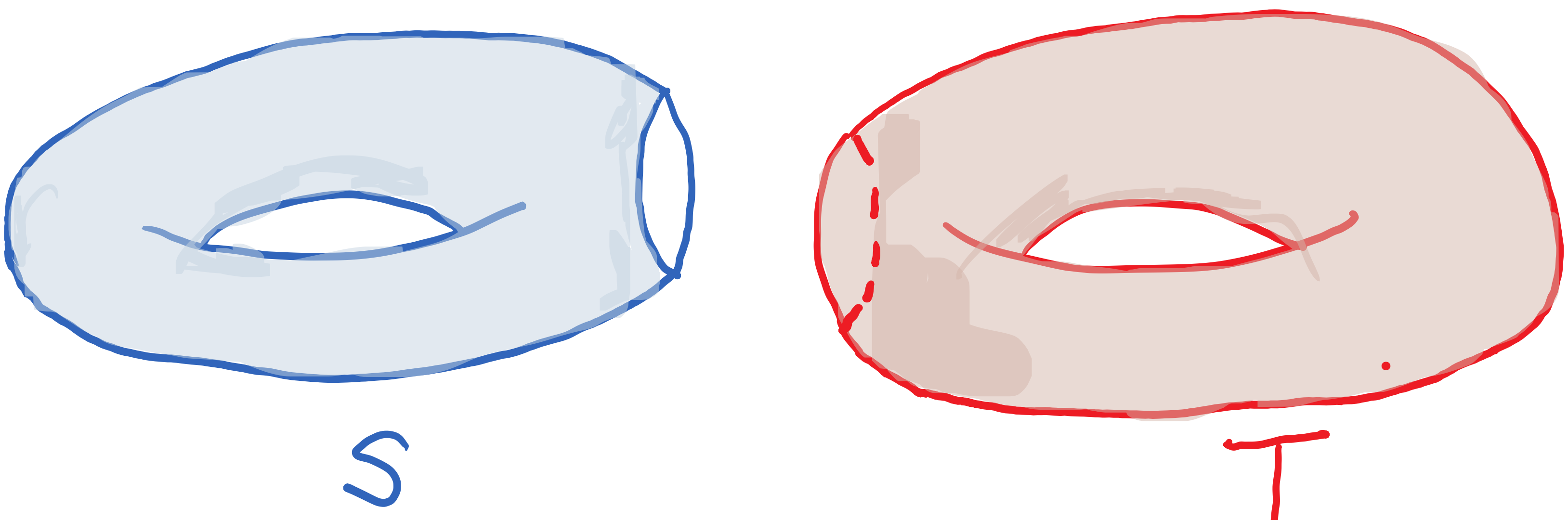

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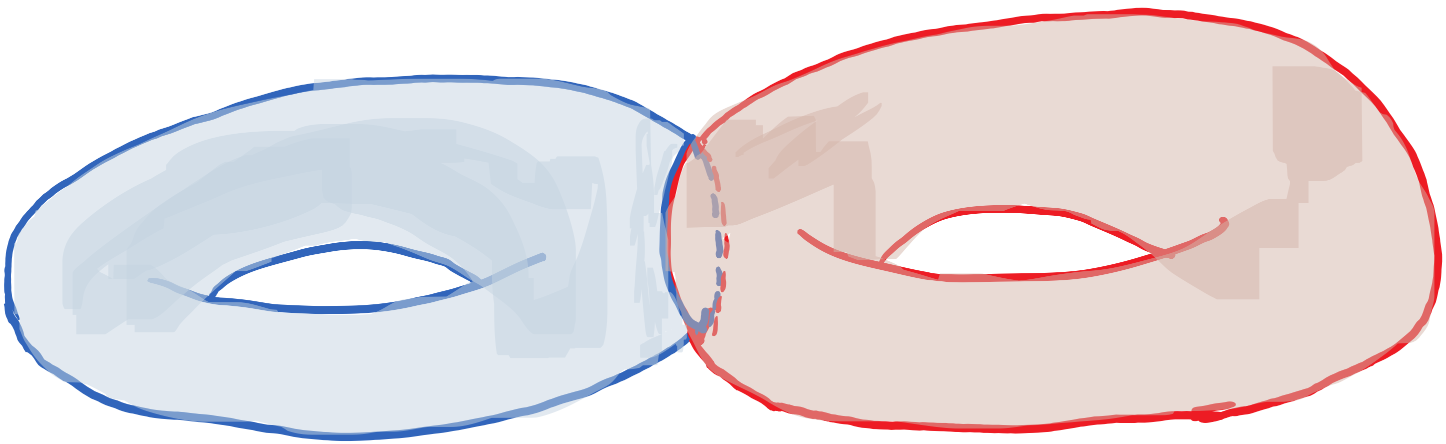

$S \# T$

Connected Sum

- A way to construct ``more complicated'' surfaces
  - Given two surfaces S and T, their connected sum $S \# T$ is constructed by:
    - first, removing a disk from both,
    - then, gluing them along the boundary of the disks
  - More examples: $\mathbb{S}$: 2-sphere, $\mathbb{T}$: torus, P: projective plane
    - $\mathbb{S} \# \mathbb{S}$, $\mathbb{S} \# \mathbb{T}$, $\mathbb{T} \# \mathbb{T}$, $\mathbb{T} \# \mathbb{T} \# \mathbb{T}$
    - What is $P \# P$?$\mathbb{S} \# P$?

Surface Classification

- Complete classification of compact 2-manifolds

Theorem 4.2 (Classification Theorem) *The two infinite families* $\mathbb{S}$, $\mathbb{T}$, $\mathbb{T}\#\mathbb{T}$, $\ldots$, *and* $\mathbb{P}$, $\mathbb{P}\#\mathbb{P}$, $\ldots$, *exhaust the family of compact 2-manifold without boundary (upto homeomorphism). The first family of surfaces are all orientable; while the second family are all non-orientable. Furthermore, no two surfaces in these sequences are homeomorphic.*

- $M \# \mathbb{T}$: adding a handle
  - All orientable compact surfaces can be obtained by adding handles to a sphere
- $M \# P$: adding a cross-cap
  - All non-orientable compact surfaces can be obtained by adding cross-caps to a sphere