TDA+Neuro Exercise set # 1

Due Date: 5:00pm EST on Monday, February 6th. You may submit your solutions via email to clause.15@osu.edu, or you may submit them in person to Nate Clause's office, 549 in the OSU Math Tower (feel free to leave under the door if Nate's not in). Feel free to work with classmates, and to ask the instructors questions via email/slack/office hours. Note: email/slack are not a great modes of communication for rigorous mathematical statements and proofs, so if your question is of such a nature we may request a meeting instead of fully answering via email.

Linear Algebra Exercises

Exercise 1. Suppose U, V are linear subspaces of a vector space W. Is $U \cap V$ a linear subspace of W? Is $U \cup V$ a linear subspace of W? Justify each with a proof or counterexample.

Exercise 2. For each of the following, prove that the function f stated is a linear transformation, or demonstrate why it is not:

(i) Let $V = \mathcal{P}_n(\mathbb{R})$ be polynomials with degree less than or equal to n, with coefficients in $\mathbb{R}$, and let $\mathbb{F} = \mathbb{R}$. Addition and scalar multiplication are defined as usual. Let $c \in \mathbb{R}$, with $c \neq 0$. Define $f : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ by $f(p(x)) = c \cdot p(x)$.

(ii) Let $\ell(x) = ax + b$ be a linear function. Define $f : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_{n+1}(\mathbb{R})$ by $f(p(x)) = \ell(x) \cdot p(x)$.

(iii) Define $f : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_{n-1}(\mathbb{R})$ by $f(p(x)) = \frac{d}{dx}p(x)$, the usual derivative of $p(x)$.

Exercise 3. Let $f : V \to W$ be a linear transformation. Prove that $\ker(f)$ is a linear subspace of V and that $\operatorname{im}(f)$ is a linear subspace of W.

Exercise 4. Let $f : U \to V$ and $g : V \to W$ be linear transformations. Prove that $\operatorname{im}(f) \subseteq \ker(g)$ if and only if $g \circ f = 0$.

Exercise 5. Let T be the 4x4 matrix:

$$T = \begin{bmatrix} 4 & 1 & 5 & 2 \\ 2 & 2 & -1 & 3 \\ 3 & 1 & 4 & 1 \\ 3 & 1 & 0 & 5 \end{bmatrix}$$

T gives a linear transformation $f : \mathbb{R}^4 \to \mathbb{R}^4$. By hand, compute $\ker(f)$ and $\operatorname{im}(f)$. Then compute these two using MATLAB, stating the results. The bases MATLAB give may be different than the ones you computed by hand. If so, demonstrate that the bases are equivalent.

Metric Spaces Exercises

Exercise 6. Let (X, d_X) and (Y, d_Y) be metric spaces. Define (Z, d_Z), where $Z = X \times Y$ and $d_Z((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$. Prove that (Z, d_Z) is a metric space.

Exercise 7. We say two metrics $d, d' : X \times X \to \mathbb{R}$ on a set X are (strongly) *equivalent metrics* if there exist constants c_1, c_2 such that for all $x, x' \in X$, $d(x, x') \leq c_2 \cdot d'(x, x')$ and $d'(x, x') \leq c_1 \cdot d(x, x')$. Let $n \in \mathbb{Z}$, $n \geq 1$, and let d_2, d_∞ be the usual p-metrics on $\mathbb{R}^n$. Prove that d_2 and d_∞ are equivalent metrics.

As a remark, two metrics on a set being equivalent does not mean that the two metric spaces are isometric, but it does yield that the topologies on the two metric spaces generated by open balls have the same open sets.

Exercise 8. As discussed in the second recitation video for "Week 1", we can convert a weighted graph into a metric space using the shortest path metric. One assumption needed here is that edges have strictly positive weight. If we allow edges to have weight 0, will the shortest path approach necessarily yield a metric space? If not, is there some other terminology we have for the resulting space?

Exercise 9. Recall, we say (X, d_X) isometrically embeds into (Y, d_Y) if there exists $f : X \to Y$ such that for all $x, x' \in X$, $d_X(x, x') = d_Y(f(x), f(x'))$. Construct an example of a finite metric space (X, d_X), which *does not* isometrically embed into $(\mathbb{R}^2, d_2)$ (the plane with the usual Euclidean metric), and prove why it doesn't. Hint: you will need X to have at least 4 points, but examples with 4 points do exist.

Exercise 10. (challenge, not required to submit) Let (X, d_X) be a finite metric space with n points $\{x_1, x_2, \ldots, x_n\}$ for some $n \geq 1$. Show that (X, d_X) isometrically embeds into $(\mathbb{R}^n, d_\infty)$.

Hint: try $f : X \to \mathbb{R}^n$ given by $f(x_i) = (d_X(x_i, x_1), d_X(x_i, x_2), \ldots, d_X(x_i, x_n))$.