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CLASS GROUPS AND RIEMMAN-ROCH THEOREM: AN APPROACH OF ALGEBRAIC GEOMETRY

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Abstract. Class groups and Riemann-Roch theorem are basic notion and theorem in arthmetics and algebra. I will introduce them in a geometric or topological way by using the language of schemes. First, we may turn arithmetical and algebraic objects (rings and modules) into geometric and topological objects (schemes and quasicoherent sheaves), then we use the methods of algebraic topology ((co)homology) to study them. We will see some theories in algebraic topology have an algebraic geometry version, which are very powerful.

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Introduction

In the first section, I will give a brief introduction to the language of schemes, which builds a bridge from arithmetics to topology, so that we can apply methods of algebraic topology to develop arithmetics. The language of schemes is conventioned by Alexander Grothendieck. The motivation for him to develop scheme is

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to solve Weil conjecture totally. Weil conjecture is conjecture about the connection between the number of finite field points and topology of complex points on a kind of variety, where the former one is about arithmetic and the latter one is about topology. Finally, Grothendieck's student, Deligne solved Weil conjecture by using the language of schemes and étale cohomology.

In the second section, I will introduce quasicoherent sheaves and how we define a generalized notion of class groups by quasicoherent sheaves. In particular, This survey focus on desribe the relations among divisors, line bundles and algebraic cycles.

In topology, bundles of a topological space character the space effectively; in ring theory, modules and ideals of a ring character the ring in an analogous way; in algebraic geometry, an analogous pattern is the study of quasicoherent sheaves on schemes, which draws on the experience the study of bundles and the study of modules. The second section will be presented in this spirit.

In the third section, I will show the construction of cohomology machinery for quasicoherent sheaves on schemes and an important algebraic invariant, arithmetic genus via Euler characteristic. Then I give a statement and proof of Riemann-Roch theorem on regular projective curves. There should have had some application of Riemann-Roch theorem in this survey, but time is limit, so there is no application of Riemann-Roch theorem.

1. Geometric viewpoint: Language of schemes

1.1. Notion of schemes.

Definition 1.1 (Scheme)**.** A locally ringed space $(X, \mathscr{O}_X)$ is an **affine scheme** if there exists a ring A such that $(X, \mathscr{O}_X) \cong \operatorname{Spec} A$ as locally ringed space. Generally, $(X, \mathscr{O}_X)$ is a **scheme** if for any point $x \in X$, there exists an open neighbourhood U of x such that $(U, \mathscr{O}_X|U)$ is isomorphic to an affine scheme. For convenience, we may simply write X instead of $(X, \mathscr{O}_X)$.

Notation: Given a point $p \in X$, we denote the stalk, a local ring, at p by $\mathscr{O}_{X,p}$ and the residue field $k(p)$.

Remark 1.2. This notion is similar to the notion of manifolds: a real topological manifold is "glued" by copies of $\mathbb{R}^n$s vs a scheme is "glued" by $\{\operatorname{Spec} A_i\}$. The word "glued" is strictly defined by using cocycle conditions. See [Wed16] Proposition 4.11 for details.

Since schemes are locally ringed spaces, we just take them as a subcategory of the category of locally ringed spaces i.e. morphisms between schemes are morphisms between locally ringed space. Here are some examples: n-dimensional affine space $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$. We may view the a closed set in $\mathbb{A}^n_k$ as a set of solutions to an algebraic equations (consist of polynomials). This example is a trivial but intuitive one. Note that it is not true that every scheme is an affine scheme and we will see the non-trivial construction in 1.3.

Although the concept of schemes is similar to manifolds, there are some subtle phenomenon that only happens on schemes.

Definition 1.3 (Generic point)**.** X is topological space and K is a closed subset of X. We call k is a generic point of K if $\overline{\{k\}} = K$.

Definition 1.4 (Closed point)**.** A point of topological space is said to be a closed point if

$$\overline{\{p\}} = \{p\}$$

We will see that in Hausdorff space, every point is a closed point, so this notion is trivial for manifolds. However, in schemes, this notion makes sense.

Proposition 1.1. *Every closed point in* $\operatorname{Spec} A$ *corresponds to a maximal ideal in* A*.*

Now to give the concept about **generalization** and **specialization**. Suppose X is a topological space, given a pair of points x, y in X, then we call x is a specialization of y and y is generalization of x if $x \in \overline{\{y\}}$.

Example 1.5 (An interesting but not formal example)**.** This idea comes from Mumford's treasure maps, see [Mum99] P72 Example C for more details. Let us image $\operatorname{Spec} \mathbb{Z}$ as a tape measure in the picture below. Its closed points (or special points) $[(2)], [(3)]$, etc. are marked along the strip, while the generic point $[(0)]$ is always hidden in the case. We can then preform a specialization of the generic point to special point $[(p)]$ by pulling out the strip as far as the mark for $[(p)]$ indicates. The reverse line segment, with endpoints$[(p)]$ and $[(0)]$, is a good picture for $\operatorname{Spec} \mathbb{Z}_{(p)}$. The reverse process of the strip returning to the case is of course a generization. Our mental picture is that the generic point should always be surrounded by all the special points, as in the case, so that each neighborhood of each special point contains the generic point. Equivalently, we may image that the generic point permeates the entire spectrum so it is everywhere and shapeless.

1.2. **Some properties of schemes and morphisms I.** Actually, properties on morphisms (relative version) are more essential than properties on schemes diretly (absolute version). This idea comes from Grothendieck.

Before we introduce properties, we need to classify types of properties.

Definition 1.6. Let P be a property on schemes, we say P is an **affine-local property** the following assertion is ture: X has property P if and only if there is an affine open cover $\{U_I\}$ of X and each open subscheme has property P. In other words, one can check whether X has property P by check on any affine open cover.

Let P be a property of morphisms, we say P is **affine local on the targets (resp sources)** if we can check whether morphism $\pi\colon X \to Y$ has property P by checking any affine open over on Y (resp. X).

Most of important properties of schemes and morphisms are affine-local, which means that we can verify properties handily. However, checking whether a propert is affine-local or not still need some tools:

Lemma 1.7 (Affine Communication Lemma)**.** *Let P be some property enjoyed by some affine open subsets of a scheme X, such that:*

1. *if an affine open subset* $\operatorname{Spec} A \hookrightarrow X$ *has property P, then for any* $f \in A$, $\operatorname{Spec} A_f \hookrightarrow X$ *has P, too*
2. *if* $(f_1, \ldots, f_n) = A$, *and* $\operatorname{Spec} A_{f_i} \hookrightarrow X$ *has property P, then so does* $\operatorname{Spec} A \hookrightarrow X$

Suppose that $X = \cup_{i\in I} \operatorname{Spec} A_i$ *where* $\operatorname{Spec} A_i$ *has property P. Then every open subset of X has P, too.*

To prove this lemma, we still need a proposition:

Proposition 1.2. *Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* X*. Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distinguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$*.*

Proof. Given any point $p \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we let $\operatorname{Spec} A_f \subset \operatorname{Spec} A \cap \operatorname{Spec} B$ be an distinguished open set of $\operatorname{Spec} A$ that contains p. Let $\operatorname{Spec} B_g \subset \operatorname{Spec} A_f$ be a distinguished open subsecheme of $\operatorname{Spec} B$ that contains p. Then $g \in \Gamma(\operatorname{Spec} B, \mathscr{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathscr{O}_X)$.The points of $\operatorname{Spec} A_f$ where g vanishes aare precisely the points if $\operatorname{Spec} A_f$ that g' vanishes. Hence $\operatorname{Spec} B_g = \operatorname{Spec} A_F \backslash \{[\mathfrak{p}] \mid g' \in \mathfrak{p}\} = \operatorname{Spec}(A_f)_{g'}$. We may write $g' = g''/f^n$ where $g'' \in A$, then $\operatorname{Spec} B_g = \operatorname{Spec} A_{fg''}$, and for each point we can find such distinguished open subscheme and it is done. $\square$

Proof of Lemma 6.1. Let $\operatorname{Spec} A$ be an affine subscheme of X, we may assume $\operatorname{Spec} A$ is covered by finite number of affine open subscheme of $\operatorname{Spec} A_f$, each of which is also distinguished in some $\operatorname{Spec} A_i$. Then by (1), each $\operatorname{Spec} A_f$ has property P, then by assumption (2), $\operatorname{Spec} A$ has property P. $\square$

Definition 1.8 (Some topological properties of schemes)**.** Let X be scheme, X is **quasicompact** if any open cover of X admits a finite subcover. X is **quasiseparated** if the intersection of any two quasicompact subsets X is quasicompact.

One can check that any affine scheme is quasicompact and quasiseparated. We next show the relative version:

Definition 1.9. A morphism $\pi : X \to Y$ is **quasicompact** if for every open affine subset U of Y, $\pi^{-1}(U)$ is quasicompact. π is **quasiseparated** if $\pi^{-1}(U)$ is a quasiseparated scheme.

Remark 1.10. Actually, quasicompactnees and quasiseparatedness of morphisms are affine local on the target.

Remark 1.11. Note that the terminal object in the category of schemes is $\operatorname{Spec} \mathbb{Z}$ i.e. for any scheme X, there is a unique structure map $X \to \operatorname{Spec} \mathbb{Z}$, then the relative version coincide with the absolute version in the following way: the $X \to \operatorname{Spec} \mathbb{Z}$ is quasicompact (resp. quasiseparated), then X is quasicompact (resp. quasiseparated). The next lemma will show how these two properties of morphism help us. Conversely, a morphism from a quasiseparated scheme is quasiseparated. However, in general, a morphism from a quasicompact scheme may not be quasicompact, but the result is true when considering Noetherian scheme, see Proposition 1.6.

Lemma 1.12 (Qcqs Lemma)**.** *If* X *is a quasicompact and quasiseparated scheme and* $s \in \Gamma(X, \mathscr{O}_X)$*, then the natural map* $\Gamma(X, \mathscr{O}_X)_s \to \Gamma(X_s, \mathscr{O}_X)$ *is an isomorphism.*

Here X_s means the set of points that s does not vanish and one can check that it is indeed an open set. The proof of this lemma is in [Vak17], 7.3.5. This lemma is significant, because it plays an important role in the proof of the assertion that given a quasicompact and quasiseparated morphism $X \to Y$, the pushforward of a quasicoherent sheaf on X is a quasicoherent sheaf on Y.

Definition 1.13 (Affine morphism)**.** Let $f\colon X \to Y$ be a morphism, then we say f is an **affine morphism** if for any affine open subset $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ for some ring A.

Proposition 1.3. *Affineness of morphisms is affine local on the targets,*

Proof. By using the Qcqs lemma, we can prove this proposition, see [Vark17], 7.3.7. □

Next part is about some basic algebraic properties of schemes (some of them will be related to topological properties).

Definition 1.14 (Reducedness and integrality)**.** Let X be a scheme, X is a **reduced (resp. integral)** if $\mathscr{O}_X(U)$ is a reduced ring (resp. integral domain) for any open subset U of X.

Definition 1.15 (Function field)**.** Let X be an integral scheme so that X has a unique generic point η. The stalk $\mathscr{O}_{X,\eta}$ is called **function field** of X. We denote it by $K(X)$. In particular, when A is an integral domain, $K(\operatorname{Spec} A) = K(A)$, the fraction field of A.

Remark 1.16*.* Reducedness is stalk local i.e. one can check whether a scheme is reudced by checking all the stalks. However, integrallity is not stalk local, because it shows the irreduciblity and the data of stalks cannot tell us whether a scheme is irreducible, see next proposition.

Proposition 1.4. *A scheme X is integral if and only if X is irreducible and reduced.*

Proof. Suppose X is integral, then $\mathscr{O}_X(U)$ is integral and in particular reduced for any open subset U. It remains to show X is irreducible. Suppose X is not irreducible, then we have two disjoint non-empty open subset U, V of X. Consider $\mathscr{O}_X(U \cup V)$, by sheaf axiom, $\mathscr{O}_X(U \cup V) \cong \mathscr{O}_X(U) \times \mathscr{O}_X(V)$, contradiction to $\mathscr{O}_X(U \cup V)$ is integral.

Conversely, if X is irreducible and reduced. For an affine open subset $\operatorname{Spec} A$ of X, let ideals I, J with $IJ = 0$ in A, then $V(I) \cup V(J) = V(IJ) = \operatorname{Spec} A$. If both of $V(I), V(J)$ are proper closed subsets, then there exists two proper closed subset Z, W of X such that $Z \cap \operatorname{Spec} A = V(I)$ and $W \cap \operatorname{Spec} A = V(J)$. Then $X = Z \cup W \cup (\operatorname{Spec} A)^c$ a union of proper closed subset, contradiction. Hence either $V(I) = \operatorname{Spec} A$ or $V(J) = \operatorname{Spec} A$. We may assume $V(I) = \operatorname{Spec} A$, then I is contained in the radical of A, then $I = 0$ because A is reduced (the radical is 0). Hence A is an integral domain. □

Remark 1.17*.* We see that in algebraic geometry, topological properties and algebraic properties are related.

Definition 1.18 ((locally) Noetherian scheme)**.** Let X be a scheme, we say X is a **locally Noetherian scheme** if X can be cover by affine open subsets $\operatorname{Spec} A$ where A is Noetherian. Further, X is a **Noetherian scheme** if X is a quasicompact locally Noetherian scheme.

Proposition 1.5. *Locally Noetherian schemes are quasiseparated.*

Proof. Let X be a locally Noetherian scheme covered by $\{\operatorname{Spec} A_i\}$ where A_i is Noetherian ring. We first claim that when A is a Noetherian ring, then any open set of $\operatorname{Spec} A$ is quasicompact. Suppose $U = \operatorname{Spec} A \setminus V(I)$ for some ideal I, and $I = (f_1, \dots, f_n)$ because A is Noetherian, then $U = \operatorname{Spec} A \setminus (\cup_{i=1}^n V(f_i)) = \cap_i^n D(f_i) = D(f_1 \dots f_n) = \operatorname{Spec} A_{f_1 \dots f_n}$, which is quasicompact because it is affine. Further, $A_{f_1 \dots f_n}$ is a Noetherian ring.

Given any two quasicompact open set U, W of X, since U is quasicompact, suppose U is covered by $\{\operatorname{Spec} A_i\}_i^n$. Then $U \cap W = \cup_{i=1}^n (\operatorname{Spec} A_i \cap (U \cap V))$. By previous claim, $U \cap W$ is a finite union of quasicompact open sets, hence $U \cap W$ is compact. □

According to the proof of the first claim, we have a lemma:

Lemma 1.19. *Let A be a Noetherian ring, then any non-empty open set of* $\operatorname{Spec} A$ *is affine, in particular, quasicompact.*

Use the lemma, we can prove the next proposition easily:

Proposition 1.6. *Any morphism from a Noetherina scheme is quasicompact.*

Definition 1.20 (Normality and factoriality)**.** We say a scheme X is **normal (resp. factorial)** if all of its stalks are **integrall closed domain i.e. normal (resp. UFD)**

Remark 1.21. Normality can imply reducedness, becaus they are all stalk local, while normality cannot imply integrality. Note that a UFD is normal, so factoriality can imply normality.

1.3. **Important construction: projective schemes.** Let $S = \bigoplus_{n=0}^{\infty} S_n$ be a $\mathbb{N}$-graded ring and $S_+ = \bigoplus_{n=1}^{\infty} S_n$ be the irrelevant ideals. Now to construct $\mathrm{Proj} S$, the projective scheme: First, points of $\mathrm{Proj} S$ are the set of homogeneous prime ideals of S that do not contain S_+. Second, to define the topology on $\mathrm{Proj} S$, we set the closed subset of $\mathrm{Proj} S$ is of the form $V(I) = \{[\mathfrak{p}] \in \mathrm{Proj} S \mid I \subset \mathfrak{p}\}$ where I is a homogeneous ideal of S. It is clear that this set-up satifies the axioms of topology. Third, we need to construct the structure sheaf of $\mathrm{Proj} S$. Recall the precedure that we define structure on $\operatorname{Spec} A$: we define sheaves on the base of $\operatorname{Spec} A$ (the distinguished open subsets of $\operatorname{Spec} A$) then glue the sheaves on base together to get a sheaf on $\operatorname{Spec} A$ because it satisfies cocycle condition. We now get our goal in an analogous way. Let $D(f) := \mathrm{Proj} S \setminus V(f)$ be the projective distinguished open set for a homogeneous element $f \in S_+$. Now it suffices to show a projective distinguished open set is an affine open set. According to the definition of localization, the set of $D(f)$ is the set of homogeneous prime ideals of S_f.

Proposition 1.7. *There is a bijection between the set of homogeneous prime ideals of S_f and the prime ideals of $(S_f)_0$. In this way, $D(f) = \operatorname{Spec}(S_f)_0$.*

Proof. Let $(S_f) = A$ and we have a natural inclusion $A_0 \hookrightarrow A$. For homogeneous prime ideal $\mathfrak{p}$ of A, $\mathfrak{p} \cap A_0$ is a prime ideal of A_0. This gives a map from the set of homogeneous prime ideals of A to $\operatorname{Spec} A$. Conversely, let $\mathfrak{p}_0$ be a prime ideal in A_0 and define $\mathfrak{p} \subset A$ as $\mathfrak{p} = \oplus P_i$ where $P_i := \{a_i \in A_i \mid a_i^{\deg f}/f^i \in p_0$ for $i > 0$ and $P_0 = \mathfrak{p}_0$.

First, $\mathfrak{p} \cap A_0 = \mathfrak{p}_0$ clearly.

Second, it is obvious that $a \in P_i$ if and only if $a^2 \in P_{2i}$.

Third, claim that if $a, b \in P_i$, then $a + b \in P_i$: $a^{\deg f}/f^i, b^{\deg f}/f^i \in P_0$ implies $(ab)^{\deg f}/f^2 i \in P_0$, then $ab \in P_{2i}$. Consider $(a+b)^{2\deg f}/f^{2i} = \sum c_n a^n b^{2\deg f - n}/f^{2i}$, we have either $n \geqslant \deg f$ or $2\deg f - n \geqslant \deg f$, then $a^n b^{2\deg f - n}/f^{2i} \in P_0$ for all n because either $a^{\deg f}/f^i$ or $b^{\deg f}/f^i$ divides it. Then $(a+b)^2 \in P_{2i}$. By the claim in the second step, $a+b \in P_i$. Hence $\mathfrak{p}$ is a homogeneous ideal clearly. Fourth to show $\mathfrak{p}$ is prime, it suffices to show for any $g_i \in A_i, g_j \in A_j$ such that $g_i g_j \in \mathfrak{p}$, we have either $g_i \in \mathfrak{p}$ or $g_j \in \mathfrak{p}$ (the general case can be reduced to this case by writing elements into a sum of homogenuous items). Since $g_i g_j \in P_{i+j}$, we have $(g_i g_j)^{\deg f}/f^{i+j} = (g_i^{\deg f}/f^i)(g_j^{\deg f}/f^j) \in \mathfrak{p}_0$, then either $g_i^{\deg f}/f^i \in \mathfrak{p}_0$ or $g_j^{\deg f}/f^j \in \mathfrak{p}_0$. Hence either $g_i \in \mathfrak{p}$ or $g_j \in \mathfrak{p}$. Hence $\mathfrak{p} \mapsto \mathfrak{p} \cap A_0$ is surjective. It is injective, if there is homogeneous element $x \in \mathfrak{p}$ but $x \notin \mathfrak{p}'$, then $x^{\deg^f}/f^{\deg x} \notin \mathfrak{p}' \cap A_0$. Thus we have a bijection. □

After above construction and arguement, Proj$S$ is indeed a scheme. If $S = A[x_0, \dots, x_n]$, we denote Proj$S$ by $\mathbb{P}^n_A$ and call it **n-dimensional A-projective space** or simply projective space. In particular, if $n = 1$, we call it **projective line**; if $n = 2$, we call it **projective space**.

If $S = k[x_0, \dots, x_n]$, then we will see that S_+ is finitely generated by $x_0, \dots, x_n$ as $S_0 = k$-algebra, in otherwords, $D(x_i)$ will be an affine open cover of $\mathbb{P}^n_k$. Specifically,

$$(S_{x_i})_0 = \{\sum_{k_0+\dots+k_n=0} a_k x_0^{k_0} \dots x_n^{k_n} \mid k_j \geqslant 0 \text{ for } j \neq i;\ k_i \in \mathbb{Z}\} \tag{1}$$

Further

$$(S_{x_i})_0 \cong k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$$

Example 1.22 (k-projective is not affine)**.** If $\mathbb{P}^1_k$ is an affine scheme, then it is isomorphic to Spec$(\Gamma(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}))$. Now claim that

$$\Gamma(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}) = k$$

First, let $D(x_0)$ and $D(x_1)$ covers $\mathbb{P}^1_k$ and by sheaf axiom, we have the exact sequence:

$$0 \to \Gamma(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}) \to \Gamma(D(x_0), \mathscr{O}_{\mathbb{P}^1_k}) \times \Gamma(D(x_1), \mathscr{O}_{\mathbb{P}^1_k}) \to \Gamma(D(x_0x_1), \mathscr{O}_{\mathbb{P}^1_k})$$

Hence in the form of formula 1

$$\Gamma(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}) = \{\sum_{k_0+k_1=0} a_k x_0^{k_0} x_1^{k_1} \mid k_1 \geqslant 0\ \ k_0 \in \mathbb{Z}\} \cap \{\sum_{k_0+k_1=0} a_k x_0^{k_0} x_1^{k_1} \mid k_0 \geqslant 0\ \ k_1 \in \mathbb{Z}\}$$

Then

$$\Gamma(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}) = \{\sum_{k_0+k_1=0} a_k x_0^{k_0} x_1^{k_1} \mid k_1, k_0 \geqslant 0\} = k$$

However, $\mathbb{P}^1_k$ is not a single point, so $\mathbb{P}^1_k \neq \operatorname{Spec} k$, which means that it is not affine.

Proposition 1.8. *If S is a finitely generated graded ring over S_0 such that S_+ is generated by S_1, then $\Gamma(\mathrm{Proj}S, \mathscr{O}_{\mathrm{Proj}S}) = S_0$.*

Proof. We can use the method in previous example to prove this proposition. □

Definition 1.23. A **quasiprojective** A-scheme is a quasicompact open subscheme of a projective A-scheme.

Definition 1.24. A **rational map** from $X \to Y$ is a morphism on a dense open subset U of X. A rational map is **dominant** if its image is dense in Y.

we may use (U, f) to represent a rational map f, where $U \subset X$ is an open dense subset i.e. the generic points of irreducible components of X are in U.

Remark 1.25. If X is an irreducible scheme, then U is an open dense set of X if and only if U contains the generic point of X.

Remark 1.26. We may identify

$$\Gamma(X, \mathscr{O}_X) \cong \mathrm{Mor}_{\mathbf{Sch}}(X, \mathbb{A}^1_{\mathbb{Z}})$$

further if X is a k-scheme, then

$$\Gamma(X, \mathscr{O}_X) \cong \mathrm{Mor}_{\mathbf{Sch}}(X, \mathbb{A}^1_k)$$

If X is an integral scheme, we may identify the function field $K(X)$ with the field of rational maps from X to $\mathbb{A}^1$, which are called **rational functions**. In other words, a rational function is a section on a open dense subset of X.

Example 1.27. The rational function field of $\mathbb{P}^n_k$ is

$$\{f/g \in k(x_0, \dots, x_n) \mid f, g \text{ are homongenous and of the same degree}\})$$

.

1.4. Some properties of schemes and morphisms II.

1.4.1. *Closed embedding.* Let X be a scheme, U be an open subset of X, the subscheme structure of U is easily to be defined by $(U, \mathscr{O}_X|_U)$, but for closed subsets, the situation is more subtle: how to defined the structure sheaf on the closed subset so that the definition makes sense. For affine case, any closed subset of $\operatorname{Spec} A$ is of the form $V(I)$ for some ideal I and we have the natural correspondence $V(I) \sim \operatorname{Spec} A/I$. Follow this observation, we have the following definition:

Definition 1.28 (Closed embedding)**.** A morphism $\pi\colon X \to Y$ is a **closed embedding** if it is an affine morphism and for any affine subset $\operatorname{Spec} B$ of Y, the restriction $\pi|_{\operatorname{Spec} A} : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to a surjective ring homomorphism $\pi^{\#} : B \to A$. If X is an closed subset of Y, then we say X is a **closed subscheme** of Y.

Remark 1.29. The data of a closed embedding gives a sheaf morphism on Y

$$\mathscr{O}_Y \to \pi_* \mathscr{O}_X$$

Note that $\pi_* \mathscr{O}_X$ is an $\mathscr{O}_Y$-module and this sheaf morphism has the data of all the induced surjective ring homomorphisms in the definition:

$$\mathscr{O}_Y(\operatorname{Spec} B) = B \to \pi_* \mathscr{O}_X(\operatorname{Spec} A) = A \cong B/I$$

This shows some connection between closed subscheme of Y and some kind of $\mathscr{O}_Y$-module on Y. We will show this connection in the section of quasicoherent sheaves.

Example 1.30. A closed subscheme of $\mathbb{P}^n_k$ cut out by a singele nonzero homogenuous equation is called a **hyersurface.** The **degree of the hypersurface** is defined to be the degree of the homogenuous polynomial.

Definition 1.31 (Locally closed embedding)**.** A morphism $\pi_! : X \to Y$ is a **locally closed embedding**, if there is an open subscheme U of Y such that $\pi^{-1}(U) = X$ and $\pi : X \to U$ is a closed embedding. If X is a subscheme of Y, then X is a **locally closed subscheme** of Y.

Theorem 1.32. *Suppose $f\colon X \to Z$ and $g\colon Y \to Z$ are arbitrary two morphisms between schemes, then the fiber product (or pull-back) exists (in the category of schemes).*

Remark 1.33. Note that for any point p in a scheme Y, we may identify p with a morphism $\operatorname{Spec} k(p) \to Y$ ($k(p)$ is the residue field of p), then given a morphism $\pi\colon X \to Y$, $\pi^{-1}(p) = \operatorname{Spec} k(p) \times_Y X$, which is called **the fiber of π at** p.

Remark 1.34. Let U, V be two open subscheme of X, the the pullback of the inclusions $U \times_X V$ can be identified with the open subscheme $U \cap V$.

Proposition 1.9. *Let $\pi : X \to Y$ be a morphism, then the diagonal morphisms $\delta\colon X \to X \times_Y X$ is a locally closed embedding.*

Sketch proof. We just reduce it to affine case $\operatorname{Spec} A \to \operatorname{Spec} B$ and clearly $A \otimes_B A \to A$ is surjective. ☐

Remark 1.35. Suppose X is an A-scheme, Δ be the locally closed subscheme of the diagnal map $\delta\colon X \to X \times_Y X$.

Remark 1.36. Let X be an A-scheme, U, V be two open subsets of X, then we have

$$\Delta \cap (U \times_A V) \cong U \cap V$$

this is because

$$X \times_{X \times_A X} (U \times_A V) \cong U \times_X V$$

then recall Remark 1.34.

1.4.2. *Separatedness.*

Definition 1.37 (Separatedness)**.** A morphism $\pi : X \to Y$ is **separated** if the diagonal map $X \to X \times_Y X$ is a closed embedding. X is a **separated scheme** if the identity map is separated.

Remark 1.38. Separatedness is an affine local property.

Definition 1.39 (Variety)**.** A **variety** over a field k is a reduced, separated scheme of finite type over k. We also call them k-varieties.

Proposition 1.10. *Let X be a separated scheme, then the intersection of any two affine open subsets of X is affine open.*

Theorem 1.40. *Let $\pi\colon X \to Y$ and $\pi'\colon X \to Y$ be two morphisms from a reduced scheme X to a separated scheme Y, if they agree on a dense open subset of X, then they are the same.*

1.4.3. *Dimension.*

Definition 1.41 (Dimension)**.** Let $(X, \mathscr{O}_X)$ be a scheme and X is the underlying topological space, then the dimension of the scheme X is the the dimension of the topological space X: the supremum of lengths of chains of closed irreducible sets (the index starts with 0). We say a topological space is **equidimensional** or **pure dimensional**. An equidimensional topological space of dimension 1 (resp. 2, n) is saied to be a **curve** (resp. **surface**, **n-fold**). In this notes, we take k-varieties as the topological spaces when we mention curves, surfaces and n-folds.

Remark 1.42. Let $X = \operatorname{Spec} A$, then the **Krull dimension** of A is the same as the dimension of $\operatorname{Spec} A$.

Definition 1.43 (Codimension)**.** The **codimension of an irreducible subset** $Y \subset X$ of a topological space is the supremum of lengths of **increasing** chains of ireducible closed subsets starting with $\bar{Y}$ (the index starts with 0), which is denoted by $\operatorname{codim}_X Y$.

Remark 1.44. Let $X = \operatorname{Spec} A$ and an irreducible closed set of X corresponds to a prime ideal $\mathfrak{p}$ of A. Then the **height** of $\mathfrak{p}$ is the same as $\operatorname{codim}_X V(\mathfrak{p})$. Further, the height of a prime ideal $\mathfrak{p}$ is the same as the dimension of $A_{\mathfrak{p}}$.

Proposition 1.11. *Suppose Y is an irreducible closed subset of a scheme X and η is the generic point of Y. Then the codimension of Y is the dimension of the local ring $\mathscr{O}_{X,\eta}$.*

Sketch proof. According to the definition of codimension, we have $\operatorname{codim}_X Y = \operatorname{codim}_X \eta$. Then we may reduce the case to affine case by assuming X is affine (otherwise pick an affine open neighbourhood X' of η and $Y' = Y \cap X'$. It is clearly that $\operatorname{codim}_{X'} Y' = \operatorname{codim}_X Y$), then the result follows by Remark 1.44. □

Remark 1.45. We will see that a generic point η of an irreducible closed subset Y of a scheme X contains a lot of data of Y and sometimes we can use the generic point η to represents Y, especially when we talk about divisors.

Theorem 1.46. *Suppose X is an equidimensional k-scheme locally of finite type, Y is an irreducible closed subset and η is the generic point of Y. Then* $\dim Y + \dim \mathscr{O}_{X,\eta} = \dim X$. *By Proposition 1.11,* $\dim Y + \operatorname{codim}_X Y = \dim X$

Theorem 1.47 (Krull's Principal Ideal Theorem: geometric version)**.** *Suppose X is a locally Noetherian scheme and f is a function on X. The irreducible components of $V(f)$ are of codimension 0 or 1.*

Sketch proof. This result follows the Krull's principal ideal theorem in commutative algebra. □

1.4.4. *Regularity.* This property is very important when we discuss divisors.

Theorem 1.48. *Suppose $(A, \mathfrak{m}, k)$ is a Noetherian local ring, then* $\dim A \leqslant \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Definition 1.49. A local ring $(A, \mathfrak{m}, k)$ is **regular** if $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. For a scheme X, a point $x \in X$ is a **regular point** if $\mathscr{O}_{X,x}$ is a regular local ring. A scheme is **regular if all the points are regular**. If a point of a scheme is not regular, then we call this point **singular point**.

We will see that regularity is a very good property for Noetherian local rings.

Theorem 1.50. *Suppose $(A, \mathfrak{m})$ is a Noetherian local ring of dimension 1. Then the following are equivalent.*

(1) *$(A, \mathfrak{m})$ is regular.*
(2) *$\mathfrak{m}$ is principal.*
(3) *all ideals are of the form $\mathfrak{m}^n$ for $n \geqslant 0$ or (0).*
(4) *A is a principal domain.*
(5) *A is a discrete valuation ring.*
(6) *A is a normal domain i.e. integrally closed.*

Proof. See [Vak17] 12.5 or any textbook about commutative algebra. □

Example 1.51. Let A be a Dedekind domain, $\operatorname{Spec} A$ is a regular scheme because $A_{\mathfrak{p}}$ is a discrete valuation ring for each prime ideal $\mathfrak{p}$ in A.

Proposition 1.12. *Suppose X is a Noetherian normal scheme, then it is regular in codimension 1 i.e. every point of codimension at most 1 is regular.*

Proof. See the Definition 1.20 of normal scheme and Theorem 1.50, then the result is straightforward. □

Geometric interpretation of regularity: Recall the cotangent space of a smooth manifold M, and let $\mathscr{O}_M$ be the sheaf of smooth functions on M. Suppose $x \in M$ and the cotangent space is $\mathfrak{m}_x/\mathfrak{m}_x^2$ where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathscr{O}_{M,x}$, and it is canonically isomorphic to the dual space of the tangent space. We will see that the dimension of (co)tangent of any point on a smooth manifold is the same as the dimension of the manifold. For a singular point, consider a cusp of a curve, where there are two linear independent tangent vector on the cusp point, the dimension of the "tangent space" of a singular point of a node is larger than 1, the dimension of a curve.

In algebraic geometry, for a scheme, we can define **Zariski cotangent space** in an analogous way and define **Zariski tangent space** as the dual space of the Zariski cotangent space.

For a special case, let X be an integral Noetherian normal scheme, for any regular point p of codimension 1, given $f \in K(\mathscr{O}_{X,p}) \subset K(X)$ and let v_p be the valuation of $\mathscr{O}_{X,p}$, if $v_p(f) > 0$, we say f **has a zero of order** $v_p(f)$ **at** p; if $v_p(f) < 0$, we say f **has a pole of order** $-v_p(f)$ **at** p.

From this view point, normality is a good property. Given a reduced scheme, we may consider its **normalization** instead of itself sometimes.(About normalization see [Vak17], 9.7)

In arithemtic case, given a number field K, the set of valuations on K forms a scheme, more specifically, an algebraic curve, where each stalk is a DVR (discrete valuation ring) corresponding to a valuation on K. (Let $\mathcal{O}_K$ be the ring of integers of K, $\operatorname{Spec}\mathcal{O}_K$ is the algebraic curve.)

2. Quasicoherent sheaves on schemes

Definition 2.1 ($\mathscr{O}_X$-modules)**.** Let $(X, \mathscr{O}_X)$ be a ringed space, **a $\mathscr{O}_X$-module $\mathscr{F}$ on X is a sheaf of modules** such that $\Gamma(U, \mathscr{F})$ is an $\mathscr{O}_X(U)$-module for each open subset $U \subset X$.

Definition 2.2. Let $\mathscr{F}$ be a $\mathscr{O}_X$-module on a scheme X, then we define **the fiber of $\mathscr{F}$ at point** p by

$$\mathscr{F}|_p := \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} k(p)$$

Remark 2.3. Let X be a scheme, the category of $\mathscr{O}_X$-modules is an abelian category.

Example 2.4 ($\widetilde{\ }$ construction)**.** Let M be an A-module, we define $\widetilde{M}$ be an $\mathscr{O}_{\operatorname{Spec} A}$-module by setting

$$\Gamma(\widetilde{M}, D(f)) = M_f$$

Remark 2.5. Actually, $\widetilde{\ }$ is an equivalence from the category of A-modules to the category of quasicoherent sheaves over $\operatorname{Spec} A$.

Definition 2.6 (Locally free sheaves)**.** Let $(X, \mathscr{O}_X)$ be a ringed space, an $\mathscr{O}_X$-module $\mathscr{F}$ is a **locally free sheaf of rank n** if there exists an open cover $\{U_i\}_{i\in I}$ of X such that $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus n}$ for each $i \in I$.

Example 2.7 (Vector bundles)**.** Let M be a smooth real manifold, a rank n vector bundle $E \to M$ is a locally free sheaf of rank n on M by viewing E as an Etalé space of a sheaf. Actually, the notion of vector bundles is equivalent to the notion of locally free sheaves.

Definition 2.8. Let X be a scheme, then an $\mathscr{O}_X$-module $\mathscr{F}$ is a **quasicoherent sheaf** if for every affine open subset $\operatorname{Spec} A \subset X$, $\mathscr{F}_{\operatorname{Spec} A} \cong \widetilde{M}$ for some A-module M. Further $\mathscr{F}$ is a **finite type quasicoherent sheaf** if M is a finitely generated A-module for each $\operatorname{Spec} A$; $\mathscr{F}$ is a **coherent sheaf** if M is a coherent A-module for each $\operatorname{Spec} A$.

Remark 2.9. We can character a quasicoherent sheaf in this way: for every affine open subscheme, consider the diagram:

$$\begin{tikzcd}
\Gamma(\operatorname{Spec} A, \mathscr{F}) \arrow[rr, "\phi"] \arrow[dr, "\otimes_A A_f"'] & & \Gamma(\operatorname{Spec} A_f, \mathscr{F}) \\
& \Gamma(\operatorname{Spec} A, \mathscr{F})_f \arrow[ur, "\alpha"'] &
\end{tikzcd} \tag{2}$$

where ϕ is the restriction map and α is induced by the universal property of localization. $\mathscr{F}$ **is a quasicoherent sheaf if for every affine open subscheme** $\operatorname{Spec} A$**,** α **is an isomorphism.**

Proposition 2.1 (Quasicoherentness is an affine local property)**.** *Let X be a scheme, then an $\mathscr{O}_X$-module $\mathscr{F}$ is a* ***quasicoherent sheaf*** *if there exists an affine open cover $\{\operatorname{Spec} A_i\}$ such that $\mathscr{F}_{\operatorname{Spec} A_i} \cong \widetilde{M_i}$.*

Sketch proof. Let P be the property of affine open subschemes $\operatorname{Spec} A$ of that $\mathscr{F}_{\operatorname{Spec} A} \cong \widetilde{M}$ for some A-module. Then check P satisfies two hypothesis in Affine communication lemma 1.7. □

Example 2.10. Locally free sheaves are quasicoherent sheaves.

Example 2.11 (Sheaf of ideals is a quasicoherent sheaf)**.** Let $i : X \to Y$ be a closed embedding, then we have a surjection on sheaves

$$\mathscr{O}_Y \to i_*\mathscr{O}_X$$

Now consider the exact sequence of sheaves

$$0 \longrightarrow \mathscr{I}_{X/Y} \longrightarrow \mathscr{O}_Y \longrightarrow i_*\mathscr{O}_X \longrightarrow 0$$

where $\mathscr{I}_{X/Y}$ is a sheaf of ideals: for each affine open subscheme $\operatorname{Spec} A$ of Y, we have

$$X \cap \operatorname{Spec} A = V(\mathscr{I}_{X/Y}(\operatorname{Spec} A))$$

Moreover,

$$X \cap \operatorname{Spec} A_f = V(\mathscr{I}_{X/Y}(\operatorname{Spec} A_f)) = \operatorname{Spec} A_f \cap V(\mathscr{I}_{X/Y}(\operatorname{Spec} A))$$

thus we have

$$\mathscr{I}_{X/Y}(\operatorname{Spec} A_f) = \mathscr{I}_{X/Y}(\operatorname{Spec} A)_f$$

which shows that $\mathscr{I}_{X/Y}$ is a quasicoherent sheaf. We now define **sheaf on ideals on** Y **is a quasicoherent sheaf** $\mathscr{I}$ **such that** $\mathscr{I}(\operatorname{Spec} A)$ **is an ideal of** A **for each affine open subscheme** $\operatorname{Spec} A \subset Y$**.**

Actually, closed subschemes of Y one-one correspond to sheaf of ideals on Y.

In the begining of this section, I mention that quasicoherent sheaf is a kind of generalization of modules over a ring, so we ecept there will be enough module-like contruction on the category of quasicoherent sheaf. (Notation: Qch_X means the category of quasicoherent sheaves on a scheme X). Then we claim that Qch_X has the following structures:

(1) Qch_X is an abelian category,
(2) Qch_X is a tensor category.

Definition 2.12 (Direct sum and tensor product)**.** Suppose X is a scheme, $\mathscr{F}$ and sG are in Qch_X, define $\mathscr{F} \oplus \mathscr{G}$ by

$$\Gamma(U, \mathscr{F} \oplus \mathscr{G}) := \Gamma(U, \mathscr{F}) \oplus \Gamma(U, \mathscr{G})$$

for open subset $U \subset X$; similarly, define $\mathscr{F} \otimes \mathscr{G}$ by

$$\Gamma(U, \mathscr{F} \otimes \mathscr{G}) := \Gamma(U, \mathscr{F}) \otimes \Gamma(U, \mathscr{G})$$

(Note that $(M \otimes_A N)_f \cong M_f \otimes_{A_f} N_f$).

Remark 2.13. In vector bundles, the direct sum is Whitney sum.

Definition 2.14. Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$-modules on X, the hom sheaf $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is defined by

$$\Gamma(U, \mathscr{H}om(\mathscr{F}, \mathscr{G})) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}|_U, \mathscr{G}|_U)$$

for any open subset U of X. Clearly, $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is sheaf of abelian groups. In particular, **the dual of** $\mathscr{F}$ is $\mathscr{H}om(\mathscr{F}, \mathscr{O}_X)$, denoted by $\mathscr{F}^\vee$.

Remark 2.15. If $\mathscr{F}$ is locally free of rank n and $\mathscr{G}$ is locally free of rank m, then $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is locally free of rank nm.

Proposition 2.2 (Geometric Nakayama)**.** *Suppose X is a scheme and $\mathscr{F}$ is a finite type quasicoherent sheaf. If $U \subset X$ is an open neighborhood of $p \in X$ and $a_1, \ldots, a_n \in \mathscr{F}(U)$ so that their images $a_1|_p, \ldots, a_n|_p$ generat the fiber $\mathscr{F}|_p = \mathscr{F}_p \otimes k(p)$, then there is an affine open neighbirhood $p \in \operatorname{Spec} A \subset U$ such that:*

(1) $a_1|_{\operatorname{Spec} A}, \ldots, a_n|_{\operatorname{Spec} A}$ generate $\mathscr{F}(\operatorname{Spec} A)$ as an A-module.
(2) For each $q \in \operatorname{Spec} A$, $a_1, \ldots, a_n$ generate the stalk $\mathscr{F}_q$ as an $\mathscr{O}_{X,q}$-module.

Proof. According to the local version of Nakayama's lemma, $a_{1,p}, \ldots, a_{n,p}$ generates $\mathscr{F}_p$. Then we can find an affine open neighborhood $\operatorname{Spec} B$ of p such that $a_{i,p}$ can be represented by a section a_i on $\operatorname{Spec} B$ for each i, i.e. $a_1, \ldots, a_n \in \mathscr{F}(\operatorname{Spec} B) = M$. Let $N = M/(a_1, \ldots, a_n)$ a finite generated B-module, then the support of N, $\mathrm{Supp}N$ is a closed set and $p \notin \mathrm{Supp}N$. Then we can find an $f \in B$ such that $p \in \operatorname{Spec} B_f \subset \operatorname{Spec} B \setminus \mathrm{Supp}N$. Let $A = B_f$, then $\operatorname{Spec} A$ is what we need: for each point $q \in \operatorname{Spec} A$, $N_q = M_q/(a_{1,q}, \ldots, a_{n,q}) = 0$ and $N_f = 0$. □

(We use the fact that the support of a finitely generated module is closed)

Proposition 2.3. *Suppose $\mathscr{F}$ is a finitely presented quasicoherent sheaf on a scheme X, then if $\mathscr{F}_p$ is a free $\mathscr{O}_{X,p}$-module, there exists a neighborhod U of p such that $\mathscr{F}_U$ is a free sheaf on U.*

Proof. We reduce the case to the affine case. We may assume $X = \operatorname{Spec} A$ and $\mathscr{F} = \widetilde{M}$ for an A-module M, then M_p is a free A_p for some $p \in X$, then there exists $a_1, \ldots, a_n \in M$ such that they generate M_p as an A_p-module. Then we have a linear map

$$\varphi \colon A^n \to M$$

determined by $a_1, \dots, a_n$. Then we have two finitely generated A-modules $\ker\varphi$ and cokerφ. Note that Supp($\ker\varphi$) and Supp(cokerφ) are two closed subsets and p is in neither one. Hence we can find an $f \in A$ such that

$$p \subset \operatorname{Spec} A_f \subset \operatorname{Spec} A \setminus (\operatorname{Supp}(\ker\varphi) \cup \operatorname{Supp}(\operatorname{coker}\varphi))$$

Then for each point q in $\operatorname{Spec} A_f$, M_q is a free sheaf and M_f is a free sheaf. □

2.1. **Invertible sheaves and line bundles.**

Definition 2.16 (Line bundle or invertible sheaf)**.** A locally free sheaf of rank 1 on X is called **line bundle** or **invertibld sheaf**.

Proposition 2.4. *Let $\mathscr{F}$ be an invertible sheaf on scheme X, then*

$$\mathscr{F} \otimes \mathscr{F}^\vee \cong \mathscr{O}_X$$

Definition 2.17 (Picard group)**.** Let X be a scheme, the isomorphic classes of invertible sheaves forms an abelian group under tensor product. This abelian group is **Picard group on** X, denoted by $\operatorname{Pic}(X)$.

Example 2.18 (Class groups and Picard groups)**.** Let A be a Dedekind domain, ClA be its class group. Recall that a **fractional ideal** $\mathfrak{a}$ is an A-module in $K(A)$, the fraction field of A, such that $a\mathfrak{a} \subset A$ for some element $a \in A$. The set of non-zero fractional ideals of A form a semi-abelian group under multiplication. We get class group from such semi-abelian group by moduling the subgroup of principal fractional ideals (ideals of the form rA for some $r \in K(A)^\times$). **We claim that the**

$$\operatorname{Pic}(\operatorname{Spec} A) \cong \operatorname{Cl}(A)$$

To show this claim, we need to show the following assertions:

Assertion 1: A frational ideal $\mathfrak{a}$ of A yields an invertible sheaf on $\operatorname{Spec} A$. Note that $\mathfrak{a}$ is an A-module, we just need to consider $\widetilde{\mathfrak{a}}$ and show it is an invertible sheaf clearly. We check it stalk by stalk: suppose $\mathfrak{a} = \frac{1}{a} I$ for some ideal I of A, then for a prime $\mathfrak{p}$ of A, consider $\mathfrak{a}_\mathfrak{p} = \frac{1}{a} I_\mathfrak{p}$. Since A is a Dedekind domain, $A_\mathfrak{p}$ is a dvr, in particular a PID, thus $I_\mathfrak{p} = a_\mathfrak{p} A_\mathfrak{p}$ for some $a_\mathfrak{p} \in A_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p} = \frac{a_p}{a} A_\mathfrak{p}$, which is a free $A_\mathfrak{p}$-module of rank 1 (ref to Proposition 2.3).

Assertion 2: The invertible sheaf yielded by a principal fractional ideal is isomorphic to $\mathscr{O}_{\operatorname{Spec} A}$. This assertion is straightforword and it can deduce that two fractional ideals that differ by a principal fractional ideal yield the same invertible sheaf up to isomorphism.

Assertion 3: Two fractional ideals that yield the same invertible sheaf up to isomorphism differ by a principal fractional ideal. Let $\frac{1}{a} I$ and $\frac{1}{b} J$ be two fractional ideals that yield the same invertible sheaf up to isomorphism, then it means that

$$\frac{1}{a} I \cong \frac{1}{b} J$$

then we have

$$I \cong \frac{a}{b} J$$

we conclude that I and J differ by a principal fractional ideal and the assertion follows.

Remark 2.19. This example shows how we study a arithmetic problem in the language of algebraic geometry. We translate arithmetic objects into a topological objects.

2.1.1. *Line bundles on projective spaces.*

Definition 2.20 (Twisting line bundle)**.** Let $\mathbb{P}_k^n$ be a k-projective space, given an integer d, we define the **twisting line bundle** $\mathscr{O}_{\mathbb{P}_k^n}(d)$ by

$$\Gamma(D(f), \mathscr{O}_{\mathbb{P}_k^n}(d)) = \operatorname{Spec}(k[x_0, \dots, x_n]_f)_d$$

for any $f \in k[x_0, \dots, x_n]$. More specifically

$$\begin{aligned}\operatorname{Spec}(k[x_0, \dots, x_n]_f)_d =& \{\sum g_i/f^{k_i} \mid g_i \text{ is homogenuous and} \deg g_i - k_i \deg f = d\} \\ =& \operatorname{Spec}(k[x_0, \dots, x_n]_f(d))_0\end{aligned}$$

(recall that given a graded ring S, the d-shift graded ring $S(n)$ of S is defined by $S(n)_i := S_{i+n}$)

Remark 2.21. We now show that a twisting line bundle is indeed a line bundle: first it is a quasicoherent sheaf according to the definition clearly, then it is locally free of rank 1 by considering an affine open cover $\cup_{j=0}^n D(x_j) = \mathbb{P}_k^n$,

$$\begin{aligned}\Gamma(D(x_j), \mathscr{O}_{\mathbb{P}_k^n}(d)) =& \{\sum g_i/x_j^{k_i} \mid g_i \text{ is homogenuous and} \deg g_i - k_i = d\} \\ =& x_j^d k[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}]\end{aligned}$$

which is free of rank 1 on $D(x_j)$.

Proposition 2.5. *The dimension of the global section*

$$\dim_k \Gamma(\mathbb{P}_k^n, \mathscr{O}_{\mathbb{P}_k^n}(d)) = \binom{n+d}{n}$$

Proof. According to the definition,

$$\Gamma(\mathbb{P}_k^n, \mathscr{O}_{\mathbb{P}_k^n}(d)) = \{\sum a_{i_0 \dots i_k} x_{i_0}^{d_0} \dots x_{i_k}^{d_k} \mid \sum d_i = d, d_i \in \mathbb{N}^*\}$$

and a k-vector space basis is

$$\{x_{i_0}^{d_0} \dots x_{i_k}^{d_k} \mid \sum d_i = d, d_i \in \mathbb{N}^*\}$$

Then according to some combination calculation, the cardinary of the basis is $\binom{n+d}{n}$ □

Corollary 2.1. *The group homomorphism $\mathbb{Z} \to \operatorname{Pic}\mathbb{P}_k^n$ given by $n \mapsto \mathscr{O}(n)$ is injective.*

Proof. First, it is indeed a group homomorphism: for given two integer m, n, $\mathscr{O}(m) \otimes \mathscr{O}(n) \cong \mathscr{O}(m+n)$ according to Remark 2.21. Then Proposition 2.5 shows when one of m, n is not less than 0, if $m \neq n$, then $\mathscr{O}(m) \ncong \mathscr{O}(n)$, because their global sections are not isomorphic. For general case, if $m \neq n$, $\mathscr{O}(m) \cong \mathscr{O}(n)$, then twist a large N such that $m + N > 0$ and $n + N > 0$, then $\mathscr{O}(m+N) \cong \mathscr{O}(m) \otimes \mathscr{O}(N) \cong \mathscr{O}(n) \otimes \mathscr{O}(N) \cong \mathscr{O}(n+N)$, contradiction. □

Proposition 2.6. *Every invertible sheaf on $\mathbb{P}_n^1$ is of the form $\mathscr{O}(n)$ for some integer n.*

Proof. Note that $\mathbb{P}^1_k = \text{Proj}k[x,y]$ and $\mathbb{P}^1_k = \operatorname{Spec} k[x/y] \cup \operatorname{Spec} k[y/x]$. Any invertible sheaf on $\mathbb{A}^1_k$ corresponds to $\widetilde{M}$ on $\mathbb{A}^1_k$ and $\widetilde{M}$ is torsion free of rank 1. According to the classification of finitely generated modules over a PID, M is a free module. Hence let $\mathscr{F}$ be an invertible sheaf on $\mathbb{P}^1_k$, $\mathscr{F}|_{\operatorname{Spec} k[x/y]}$ and $\mathscr{F}|_{\operatorname{Spec} k[y/x]}$ are trivial line bundles i.e. $\mathscr{F}|_{\operatorname{Spec} k[x/y]} \cong \mathscr{O}_{\operatorname{Spec} k[x/y]}$ and $\mathscr{F}|_{\operatorname{Spec} k[y/x]} \cong \mathscr{O}_{\operatorname{Spec} k[y/x]}$. Then we have the exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{O}_{\operatorname{Spec} k[x/y]} \oplus \mathscr{O}_{\operatorname{Spec} k[y/x]} \longrightarrow \mathscr{O}_{\operatorname{Spec} k[x/y,y/x]}$$

and the transition function on $\operatorname{Spec} k[x/y, y/x]$ determines the cocycle condition. The cocycle condition determines the invertible sheaf. Hence we just need to classify transition functions.

According to the definition of cocycle condition, the transition function corresponds an invertible element of $k[x/y, y/x]$, i.e. $(y/x)^n$ for some integer n, up to a scalar mutiplication of k.

Claim that if the transition function is

$$\begin{tikzcd} k[x/y] \arrow[r, bend left, "\times(y/x)^n"] & k[y/x] \arrow[l, bend left, "\times(x/y)^n"] \end{tikzcd}$$

then $\mathscr{F} \cong \mathscr{O}(n)$. It is clear if we give a trivializations of $\mathscr{O}(n)$, its transition function is the same as the above diagram. (see Remark 2.21 and consider the trivalization by sending x_j^d to 1.) ☐

Remark 2.22. Actually, this proposition is true for $\mathbb{P}^m_k$, but we need to use the language of divisor to show it.

2.2. **Divisors. In the this subsection, we just consider Noetherian scheme (All the schemes I mention are Noetherian and normal).**

Definition 2.23. A **Weil divisor** on a scheme X is a formal $\mathbb{Z}$ linear combination of codimension 1 irreducible closed subsets of X. We can write it as the form

$$\sum_{Y \subset X,\, \operatorname{codim}_X Y = 1} n_Y[Y]$$

where n_Y are integers. Or we may consider $[Y]$ as the generic point of Y. The abelian group of Weil divisors on X is denoted by Weil$X$.

We say $[Y]$ is an **irreducible divisor**. We say a Weil divisor $D = \sum n_Y[Y]$ is **effective** if all $n_Y \geqslant 0$, and we denote it $D \geqslant 0$. For Weil divisors D_1 and D_2, we say $D_1 \geqslant D_2$ if $D_1 - D_2 \geqslant 0$. The **support of a Divisor** D is $\cup_{n_Y \neq} Y$. If $U \subset X$ is an open set, then we define the **restriction map** Weil$X \to$ Weil$U$ by $\sum n_Y[Y] \mapsto \sum_{Y \cap U \neq \emptyset} n_Y[Y \cap U]$.

Remark 2.24. If X is a regular curve, then a Weil divisor on X is a $\mathbb{Z}$ linear combination of closed points of X.

Now we suppose X is regular in codimension 1 (or a Noetherian normal scheme, see Proposition 1.12), then the stalks on dim 1 generic points are DVR i.e. **every codimension 1 irreducible closed subset** Y **corresponds to a valuation** val$_Y$ **of the function field.**

Definition 2.25 (Principal divisor)**.** Let X be a Noetherian normal scheme, f be a rational function (see Remark 1.26) on X i.e an element in the function field $K(X)$, then a **principal divisor** is

$$\mathrm{div} f = \sum_Y \mathrm{val}_Y(f)[Y]$$

where val_Y is the valuation corresponding to Y. We also call $\mathrm{div} f$ the **divisor of zeros and poles** of a rational function f. (see Geometric interpretation of regularity 1.4.4.)

In other words, **a Weil divisor D is principal if there exists a rational function f such that $D = \mathrm{div} f$.** We say D is **locally principal** if there exists an open cover $\{U_i\}$ of X such that $D|_{U_i}$ is a principal divisor.

Remark 2.26. $\mathrm{div} f$ is indeed a Weil divisor i.e. a finite sum of non-zero items. To show this, we reduce it to affine case that $X = \operatorname{Spec} A$ and $f = g/h$ for $g/h \in A$, then there are only finite prime ideals of A containing f or g.

Remark 2.27. If f is a regular function (an element in $\mathscr{O}_X(X)$), then $\mathrm{div} f \geqslant 0$; if f is an invertible regular function, then $\mathrm{div} f = 0$.

Example 2.28. Let A be a Dedkind domain, $f \in K(A)$ and (f) is a fractional ideal, then we have a unique decomoposition of prime ideals

$$(f) = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_n^{e_n}$$

where $\mathfrak{p}_i$ is a prime ideal and e_i is an integer for each i. Then

$$\mathrm{div} f = \sum e_i[\mathfrak{p}_i]$$

Example 2.29. On $\mathbb{A}_k^1$, $\mathrm{div}(x^3/(x-1)) = 3[(x)] - [(x+1)]$, which means it has a simple pole at $[(x+1)]$ and an order 3 zero at $[(x)]$.

If $k = \mathbb{C}$, $\mathbb{A}_{\mathbb{C}}^1 \sim \mathbb{C}$ and we view it from the view point of complex analysis, the notions of rational functions, zeros and poles make sense in complex analysis.

According to the property of valuation, we have

$$\mathrm{div} fg = \mathrm{div} f + \mathrm{div} g$$

then we deduce that principal divisors form a subgroup of $\mathrm{Weil} X$. **We denote the subgroup of principal divisors by $\mathrm{Prin}(X)$.** Hence we have a group homomorphism

$$\mathrm{div} \colon K(X)^* \to \mathrm{Prin}(X)$$

Similarly, we denote the subgroup of locally principal Weil divisors by $\mathrm{LocPrin}(X)$.

Definition 2.30. We define the **Weil divisor class group** $\mathrm{Cl}(X)$ on a Noetherian normal scheme X by $\mathrm{Cl}(X) = \mathrm{Weil}(X)/\mathrm{Prin}(X)$.

Remark 2.31. We have an exact sequence

$$0 \longrightarrow \mathscr{O}_X(X)^* \hookrightarrow K(X)^* \xrightarrow{\mathrm{div}} \mathrm{Prin}(X) \longrightarrow \mathrm{Weil}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

In Example 2.18, we see that the class group of a Dedkind domain A is the same as the Picard group of $\operatorname{Spec} A$. By Example 2.28 and the exact sequence in Remark 2.31, we easily find that the Weil divisor class group of $\operatorname{Spec} A$ is the same as the class group of A. Next we will show the connection between divisors and invertible sheaves to find the connection between the Weil divisor class group and the Picard group of invertible sheaves.

Definition 2.32. Let X be a Noetherian normal scheme, D is a Weil divisor on X, then we define a sheaf $\mathscr{O}_X(D)$ by

$$\Gamma(U, \mathscr{O}_X(D)) := \{t \in K(X)^* \mid \operatorname{div}|_U(t) + D|_U \geqslant 0\} \cup 0$$

where $\operatorname{div}|U$ means we regard t as a rational function on U and take the valuations on U.

Remark 2.33. $\mathscr{O}_X(D_1) \otimes \mathscr{O}(D_2) \cong \mathscr{O}_X(D_1 + D_2)$

Proposition 2.7. *Assume the assumption in Definition 2.32, $\mathscr{O}(D)$ is a quasicoherent sheaf.*

Proof. We just need to show for any affine open subset $\operatorname{Spec} A \subset X$, we have

$$\Gamma(\operatorname{Spec} A, \mathscr{O}_X(D))_f \cong \Gamma(\operatorname{Spec} A_f, \mathscr{O}_X(D)).$$

Note that for any $n \subset \mathbb{Z}$, $\operatorname{div}|_{\operatorname{Spec} A_f}(f^n) = 0$, thus $\operatorname{div}|_{\operatorname{Spec} A_f}(t) = \operatorname{div}|_{\operatorname{Spec} A_f}(t/f^n)$. Then $\Gamma(\operatorname{Spec} A, \mathscr{O}_X(D))_f \cong \Gamma(\operatorname{Spec} A_f, \mathscr{O}_X(D))$ naturally. $\square$

Proposition 2.8. *Suppose D is a locally principal divisor, then $\mathscr{O}(D)$ is an invertibld sheaf.*

Proof. It is obvious. We just need to check Definition 2.25 and process it locally. $\square$

Example 2.34. Let $D = [(x)] - 2[(x-1)]$ be a Weil divisor on $\mathbb{A}^1$, then the global section is

$$\Gamma(\mathbb{A}^1_k, \mathscr{O}_{\mathbb{A}^1_k}(D)) = \frac{x}{(x-1)^2} k[x]$$

Next we use this notion to show the relation between invertible sheaves and Weil divisors.

Proposition 2.9. *Let $\mathscr{L}$ be an invertible sheaf on a Noetherian normal scheme X, s be a non-zero rational section of $\mathscr{L}$ i.e. $s \in \Gamma(U, \mathscr{L})$ for an open dense subset $U \subset X$. Applying trivialization on s, $\operatorname{div} s$ makes sense and is a Weil divisor, then we have*

$$\mathscr{O}(\operatorname{div} s) \cong \mathscr{L}$$

Proof. Let $\{U_i\}$ be an affine open cover of X such that $\mathscr{L}|_{U_i} \cong \mathscr{O}_{U_i}$. We just need to check that $\mathscr{O}(\operatorname{div} s)|_{U_i} \cong \mathscr{O}_{U_i}$ for each U_i. Further, suppose $U_i = \operatorname{Spec} A_i$, since they are all quasicoherent sheaves, it remains to show

$$\Gamma(U_i, \mathscr{O}_{U_i}(\operatorname{div} s)) \cong A_i$$

where we may identify s with a rational function on U_i i.e. $s = f/g \in K(A)$. Obviously, $\Gamma(U_i, \mathscr{O}_{U_i}(\operatorname{div} f/g)) \cong f/gA_i$ and $f/gA_i \cong A_i$ as A_i-modules. $\square$

In this proof, we can extend div from $K(X)^*$ to $\{(\mathscr{L}, s)\}/\text{iso.}$ where $\mathscr{L}$ is an invertible sheaf and s is a rational section of $\mathscr{L}$. Moreover $\operatorname{div}\colon \{(\mathscr{L}, s)\}/\text{iso.} \to \operatorname{Weil} X$ is an injective. Because if $\operatorname{div}(\mathscr{L}, s) = 0$, then s is a global section and $\times s\colon \mathscr{O}_X \to \mathscr{L}$ is an isomorphism.

Proposition 2.10. *Suppose X is a Noetherian normal scheme, the map*

$$\operatorname{div}\colon \{(\mathscr{L}, s)\}/iso. \to \operatorname{LocPrin}(X)$$

is an isomorphism.

Proof. We have shown div is injective. It suffices to show it is surjective.

Note that an invertible sheaf is the same notion as the line bundle. Gluing all the trivalizations and charts of the invertible sheaf together, we have a map

$$\sigma\colon K(X) \to \{\text{rational sections of } \mathscr{L}\}.$$

In detials, let $\{(U_i = \operatorname{Spec} A_i, \phi_i)\}$ be an atlas of the bundle charts , where $\mathscr{L}|_{U_i} \cong \mathscr{O}_{U_i}$ and $\phi_i\colon A_i \to \mathscr{L}(U_i)$ is an isomorphism, then for a rational function f on X, we just consider $\phi_i(f)$ for each bundle chart that are contained in the domain of f and glue $\phi_i(f)$ to get a unique rational section of $\mathscr{L}$ by sheaf axioms.

We now construct a map from $\operatorname{LocPrin}(X)$ to $\{(\mathscr{L}, s)\}/\text{iso.}$ by

$$D \mapsto (\mathscr{O}(D), \sigma(1))$$

where the target of σ is the set of rational sections of $\mathscr{O}(D)$. By checking definition, we claim $\operatorname{div}\sigma(1) = D$. To show the claim, we just consider it affine-locally and we may assume D is a principal divisor on an affine scheme $\operatorname{Spec} A$. Let $D = \operatorname{div} f$, then $\sigma\colon A \to fA$ (recall Example 2.34) and $\sigma(1) = f$ in particular. □

Corollary 2.2. *Let X be a Noetherian normal scheme, then $\operatorname{Pic}(X) \subset \operatorname{Cl}(X)$.*

We just consider the following diagram:

$$\begin{tikzcd}
\{(\mathscr{L}, s)\}/\text{iso.} \arrow[r, "\sim"] \arrow[d] & \operatorname{LocPrin}(X) \arrow[r, hook] \arrow[d, two heads, "/\operatorname{Prin}(X)"] & \operatorname{Weil}(X) \arrow[d, two heads, "/\operatorname{Prin}(X)"] \\
\operatorname{Pic}(X) = \{\mathscr{L}\}/\text{iso.} \arrow[r, "\sim"] & \operatorname{LocPrin}(X)/\operatorname{Prin}(X) \arrow[r, hook] & \operatorname{Cl}(X)
\end{tikzcd} \tag{3}$$

Now the question is: when do $\operatorname{Pic}(X) \cong \operatorname{Cl}(X)$?

Proposition 2.11. *If X is Noetherian and factorial, then for any Weil divisor D, $\mathscr{O}_X(D)$ is invertible and hence the map $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ is an isomorphism.*

If X is Noetherian and factorial, in particular smooth(though I do not give the definition of smoothness in this survey, claim that if X is a regular scheme over a perfect field k, then X is a smooth scheme over k), then $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ and we define:

Definition 2.35. For each $\mathscr{L} \in \operatorname{Pic}(X)$, we define the **first Chern class** of $\mathscr{L}$ to be

$$c_1(\mathscr{L}) = [D]$$

where $[D] \in \operatorname{Cl}(X)$ such that $\mathscr{O}_X(D) \cong \mathscr{L}$. In other words, $c_1\colon \operatorname{Pic}(X) \to \operatorname{Cl}(X)$ is the inverse homormophism of $\operatorname{Cl}(X) \to \operatorname{Pic}(X)$ by $D \mapsto \mathscr{O}_X(D)$.

Excision sequence for class groups: Let X be a Noetherian normal scheme, let Z be a codimension 1 irreducible closed set of X and $i\colon Z \hookrightarrow X$ be the closed embedding. Then we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Weil}(X) \xrightarrow{|_{X-Z}} \operatorname{Weil}(X - Z) \longrightarrow 0 \tag{4}$$

Then we quotient then by the subgroups of principal divisors to get the exicision sequence for class groups

$$\mathbb{Z} \xrightarrow{1\mapsto[Z]} \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X-Z) \longrightarrow 0 \tag{5}$$

We can use the excision sequence of divisor to calculate some examples.

Example 2.36. First, $\mathrm{Cl}(\mathbb{A}_1^n) = 0$ because $k[x_1, \dots, x_n]$ is a UFD and the class group is trivial clearly. Then we let $X = \mathbb{P}_n^k$, $Z = V(x_0)$ the hyperplane cut by x_0 and $X - Z \cong \mathbb{A}_k^n$. Now consider the excision sequence of class groups:

$$\mathbb{Z} \xrightarrow{1\mapsto[Z]} \mathrm{Cl}(\mathbb{P}_k^n) \longrightarrow \mathrm{Cl}(\mathbb{A}_k^n) \longrightarrow 0$$

Then we get a surjection $Z \to \mathrm{Cl}(\mathbb{P}_k^n)$ i.e. $\mathrm{Cl}(X)$ is generated by $[Z]$. Note that $\mathscr{O}([Z]) \cong \mathscr{O}(1)$ according to Remark 2.21. Thus $\mathrm{Pic}(\mathbb{P}_k^n) \cong \mathrm{Cl}(\mathbb{P}_k^n) \cong \mathbb{Z}$.

Hence for each invertible sheaf $\mathscr{L}$ on $\mathbb{P}_k^n$, $\mathscr{L}) \cong \mathscr{O}(d)$ for a unique integer d, **we say the degree of $\mathscr{L}$ is d.** In particular, if $\mathscr{L} = \mathscr{O}(D)$ for some divisor D on $\mathbb{P}_k^n$, we say the degree of D is d.

Example 2.37. Suppose that Y is a closed subscheme of $\mathbb{P}_n^k$ cut by a homogenuous irreducible polynomial of degree d, then accirding to the excision sequence of class groups

$$\mathbb{Z} \xrightarrow{1\mapsto[Y]} \mathrm{Cl}(\mathbb{P}_k^n) \longrightarrow \mathrm{Cl}(\mathbb{P}_k^n - Y) \longrightarrow 0$$

and note that $\mathscr{O}([Y]) = \mathscr{O}(d)$, hence $\mathrm{Cl}(\mathbb{P}_k^n - Y) = Z/(d)$.

2.3. **Algebraic cycles and Chow groups.** We say a closed subscheme X of a scheme Y is **locally principal** if there is an affine open cover $\{U_i\} = \operatorname{Spec} A_i$ of Y such that $X \cap U_i = V(s_i)$ for some $s_i \in A_i$. If for each i, s_i is a not a zerodivisor, then we called the locally principal closed subsecheme X **efferctive Cartier divisor.** Now we consider the sheaf of ideals $\mathscr{I}_{X/Y}$ of the effective Cartier divisor:

$$0 \longrightarrow \mathscr{I}_{X/Y} \longrightarrow \mathscr{O}_Y \longrightarrow i_*\mathscr{O}_X \longrightarrow 0$$

when restrict on the $\operatorname{Spec} A_i$, we get

$$0 \longrightarrow A_i \xrightarrow{\times s_i} A_i \longrightarrow A_i/(s_i) \longrightarrow 0$$

Since s_i is not a zero divisor, the map $\times s_i$ is injective. Then **we see the sheaf of ideals associated to the effective Cartier divisor is an invertible sheaf**. Conversely, an invertible sheaf of ideal will determine an effective Cartier divisor. Thus actually, **the concept of effective Cartier divisors is the same as the concept of invertible sheaves of ideals.** More specifically, according to Krull's principal ideal theorem, an effective Cartier divisor X is of codimension 1 and it is indeed a Weil divisor on X. By this, we claim that we have an exact sequence

$$0 \longrightarrow \mathscr{O}_Y(-X) \to \mathscr{O}_Y \to i_*\mathscr{O}_X \longrightarrow 0$$

and so the invertible sheaf of ideals correspoinding to X is $\mathscr{O}_Y(-X)$ actually.

In this section, we show that we can descirbe a phenomenon from two perspectives, where one is from closed subschemes and the other is from quasicoherent sheaves and these two viewpoints give the same result. Informally speaking, **the geometric data of quasicoherent sheaves on Y is almost the same as the data of closed subschemes of Y.** In the case of algebraic topology, de Rham theorem and Ṕoincare duality show that the relation should be a kind of duality.

We believe that there will be analogous way of such duality in the world of algebraic geometry. Further, we may consider the algebraic structures on good quasicoherent sheaves (such as invertible sheaves) and good closed subschemes (such as effective Cartier divisors or closed subvarieties), then we need the notions of Grothendieck K-ring and Chow ring, and Grothendieck-Riemann-Roch theorem shows the connection between them. Next we show that how to use algebraic cycles and Chow group to define class group.

Definition 2.38. Let X be a Noetherian normal scheme. The group of n-cycles, denoted by $Z_n(X)$, is the free abelian group generated by the n-dimensional irreducible subvarieties of X. For each n-dimensional subvariety $V \subset X$, we denote by $[V]$ the corresponding element of $Z_n(X)$. A n-cycle α is an element in $Z_n(X)$, and an algebraic cycle α on X is an element of abelian group $\oplus_n Z_n(X)$.

Definition 2.39. An algebraic cycle α on X is **rationally equivalent to zero**, written as $\alpha \sim 0$, if there are irreducible subvarieties $V_1, \dots, V_m$ of X and a rational function f_i on each V_i such that

$$\alpha = \sum_{i=1}^{m} \operatorname{div}(f_i)$$

where $\operatorname{div}(f_i)$ is a Weil divisor on X and is a $\mathbb{Z}$-linear combination of codimension 1 irreducible closed subsets of V_i (at the same time, they are irreducible subvarities of X as well).

Remark 2.40. Algebraic cycles that are rationally equivalent 0 forms a group clearly and we denote the subgroup of k-cycles that are rationally equivalent to 0 by $B_k(X)$.

Definition 2.41 (Chow group)**.** The **Chow group** of k-cycles on X, denoted by $A_n(X)$, is

$$A_n(X) := Z_n(X)/B_n(X)$$

The direct sum

$$A_*(X) = \bigoplus_n A_n(X)$$

is called the **Chow group of** X.

Next we show the relation between Chow groups and class groups:

Example 2.42. If X is a Noetherian normal k-variety and $\dim(X) = n$, then $A_{n-1} \cong \mathrm{Cl}(X)$.

Actually, Chow groups is an analogy to singular homology groups in algebraic topology. Informally, we may compare a k-cycle on a scheme X with a k-simplex in a manifold (or generally, topological space) M. However, the given Definition 2.39 of rational equivalence is quit strange if we consider it in algebraic topology. Now we given an equivalent definition of rational equivalence, which makes us think about more algebraic topology.

Definition 2.43. A cycle $\alpha \in Z_k(X)$ is rationally equivalent to zero iff there exists $k+1$ dimensional irreducible varieties $W_1, \dots, W_m$ of $X \times \mathbb{P}^1$ such that the projective maps $p_i : W_i \to \mathbb{P}^1$ are dominant and

$$\alpha = \sum_{i=1}^{m} (p_i^{-1}(0) - p_i^{-1}(\infty))$$

where $p_i^{-1}(0)$ and $p_i^{-1}(\infty)$ are scheme-theoretic fiber; $0 = [0:1] \in \mathbb{P}^1$ and $\infty = [1:0] \in \mathbb{P}^1$.

The proof of the equivalence between Definition 2.39 and Definition 2.43 is in [Ful98],1.6.

Remark 2.44. Informally speaking, we may view it as a kind of homotopy parametrized by the projective line (in classical algebraic geometry, homotopy is parametrized by the real line $\mathbb{R}$). Specifically, given two k-dimensional irreducible subvarieties V_0, V_1 of X, we say $[V_0]$ is rational equivalent to $[V_1]$, denoted by $V_0 \sim V_1$ if $[V_0] - [V_1] \sim 0$, and according to Definition 2.43, $V_0 \sim V_1$ if and only if there is a $k+1$-irreducible subvariety W of $X \times \mathbb{P}^1$ such that V_0 is the fiber of 0 under the projection to $\mathbb{P}^1$ and V_1 is the fiber of ∞. In this setup, we may regard W as a kind of "homotopy cylinder" between V_0 and V_1.

2.4. **Pushforward and Pullback.**

In this subsection, we just show some facts about pullback and pushforward.

2.4.1. *Pushforward.*

Let $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes and suppose M is an A-module and $\widetilde{M}$ is a quasicoherent sheaf on A. Note that M_B is a B-module via ring morphism $\pi^{\#}\colon B \to A$. Then the pushforward $\widetilde{M}$ via π is

$$\pi_* \widetilde{M} \cong \widetilde{M_B}$$

In particular, $\pi_*\widetilde{M}$ is quasicoherent. Moreover, suppose $\pi\colon X \to Y$ is an affine morphism, $\mathscr{F}$ is a quasicoherent sheaf on X, then $\pi_*\mathscr{F}$ is a quasicoherent sheaf on Y that is defined affine-locally.

Proposition 2.12. *If $\pi: X \to Y$ is an affine morphism, π_* is an exact functor $Qch_X \to Qch_Y$.*

Theorem 2.45. *Suppose $\pi: X \to Y$ is a quasicompact quasiseparated morphism and $\mathscr{F}$ is a quasicoherent sheaf on X, then $\pi_*\mathscr{F}$ is a quasicoherent sheaf on Y.*

Combine Qcqs lemma 1.12 and Remark 2.9, then the proof follows.

2.4.2. *Pullback.*

Suppose $\phi\colon X \to Y$ is a morphism of scheme, and $\mathscr{G}$ is a quasicoherent sheaf on Y, then **the pullback of $\mathscr{G}$ under π is**

$$\pi^*\mathscr{G} := \pi^{-1}\mathscr{G} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

Note that a morphism a scheme contains the data $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. Specifically, let $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset Y$ be two affine open subschemes such that $\pi(\operatorname{Spec} A) \subset \operatorname{Spec} B$, and $\mathscr{G}|_{\operatorname{Spec} B} = \widetilde{N}$ for a B-module N, then

$$\Gamma(\operatorname{Spec} A, \pi^*\mathscr{G}) = N \otimes_B A$$

The pullback of a quasicoherent sheaf is still a quasicoherent sheaf. Further, ϕ^* is a covariant functor $Qch_Y \to Qch_X$.

Example 2.46. Let X be a scheme, $p \in X$ and $\mathscr{F}$ be a quasicoherent sheaf on X. We may identify p with an inclusion $i\colon \operatorname{Spec} k(p) \to X$ by sending the single point $\operatorname{Spec} k(p)$ to p. Then $i^*\mathscr{F} = \mathscr{F}|_p$ (recall Definition 2.2) and this is why we call $\mathscr{F}|_p$ the fiber of $\mathscr{F}$ at point p.

Proposition 2.13. *Let $\pi : X \to Y$ be a quasicompact quasiseparated morphism, (π^*, π_*) is an adjoint pair: there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathscr{O}_X}(\pi^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, \pi_*\mathscr{F})$$

Corollary 2.3. *Suppose $i\colon U \to X$ is an open embedding, $\mathscr{F}$ is an $\mathscr{O}_X$-module and $\mathscr{E}$, then there is bijection*

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{E}) \leftrightarrow \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, i_*\mathscr{E})$$

There are some other properties of pullback: suppose $\pi\colon X \to Y$ is a morphism of schemes and $\mathscr{G}$ is a quasicoherent sheaf on Y.

(1) If $\pi(p) = q$, then the pullbucak induced an isomorphism

$$\pi^*\mathscr{G}_p \cong \mathscr{G}_q \otimes_{\mathscr{O}_{Y,q}} \mathscr{O}_{X,p}$$

(2) Let $\mathscr{G}'$ be another quasicoherent sheaf on Y

$$\pi^*(\mathscr{G} \otimes_{\mathscr{O}_Y} \mathscr{G}') \cong \pi^*\mathscr{G} \otimes_{\mathscr{O}_X} \pi^*\mathscr{G}'$$

(3) If $\pi(p) = q$, then

$$(\pi^*\mathscr{G})|_p \cong \mathscr{G}|_q \otimes_{k(q)} k(p)$$

2.5. **Line bundles and maps to projective schemes.**

Definition 2.47. Let X be a scheme and $\mathscr{F}$ be an $\mathscr{O}_X$-module, then we say $\mathscr{F}$ is **globally generated** if there exists a surjection

$$\bigoplus_I \mathscr{O}_X \to \mathscr{F}$$

where I is an index set. If I is finite, then we say $\mathscr{F}$ is finitely globally generated.

An equivalent definition is that there exists a family of global sections $\{s_i\}_{i\in I}$ in $\mathscr{F}(X)$ such that $\mathscr{F}_p$ is generated by $\{s_{i,p}\}_{i\in I}$ for each point $p \in X$.

In particular, we say $\mathscr{F}$ is globally generated at point $p \in X$, when $\mathscr{F}_p$ is generated by global sections of $\mathscr{F}$.

Note that in general case, quasicoherent sheaf may not be globally generated, for example $\mathscr{O}(-1)$ on $\mathbb{P}^n_k$, due to its trivial global section.

Proposition 2.14. *Suppose $\mathscr{F}$ is a finite-type quasicoherent sheaf, then $\mathscr{F}$ is globally generated at p if and only the fiber of $\mathscr{F}$ is generated by global sections at p i.e. the map from global sections yo the fiber $\mathscr{F}_p/\mathfrak{m}\mathscr{F}_p$ is surjective, where $\mathfrak{m}$ is the maximal ideal of $\mathscr{O}_{X,p}$.*

Applying geometric Nakayama'lemma, the result follows.

Corollary 2.4. *An invertible sheaf $\mathscr{L}$ on X is globally generated if and only if for any point $p \in X$, there is a global section of $\mathscr{L}$ not vanishing at p.*

The "only if" part is obvious. Conversely, if there is a global section of $\mathscr{F}$ does not vanish at p, then the pullback this global section does not vanish at the fiber of $\mathscr{F}$ at p. Since the fiber of $\mathscr{F}$ is a vector space of dimension 1, then the fiber of $\mathscr{F}$ at each point p is generated by global sections and by previous proposition, the result follows.

Definition 2.48. If $\mathscr{L}$ is an invertible sheaf on a scheme, then those points where $\mathscr{L}$ vanishes are called **base points** of $\mathscr{L}$. Note that the set of base points is a closed subset and we called it **base point locus**.

The main reason we need this definition is due to the following proposition:

Proposition 2.15. *Suppose X is a k-scheme, and $s_0, \dots, s_n$ are global sections of a line bundle $\mathscr{L}$ on X without common zeros, then these global sections determine a morphism $X \to \mathbb{P}^n_k$.*

Proof. Sketch proof If U is an open bundle chart of $\mathscr{L}$, then apply trivialization on $s_0, \dots, s_n$, s_i are functions on U i.e. $s_i \in \mathrm{Hom}(U, \mathbb{A}^1_k)$, then we have $\phi|_U : U \to \mathbb{P}^n_k$ defined by

$$x \mapsto [s_0(x), \dots, s_n(x)]$$

Since the trivialization and transition functions are compatible, for any two open bundle charts, previous definition of the morphisms agree on the intersection. Then we can glue these morphisms into a morphism $X \to \mathbb{P}^n_k$. □

Definition 2.49 (Linear series)**.** Suppose X is a k-scheme and V is a finite dimensional k dimensional with a linear map $\lambda : V \to \Gamma(X, \mathscr{F})$, then we usually simply denote it by V and call it linear series on X. Similarly, we can define base point for linear series.

If $U \subset X$ is base point free for an $n+1$ linear series, then we have a morphism $U \to \mathbb{P}^n_k$.

Theorem 2.50. *For a fixed scheme X, morphisms $X \to \mathbb{P}^n_k$ are in bijection with the data $(\mathscr{L}, s_0, \dots, s_n)$, where $\mathscr{L}$ is a line bundle on X and $s_0, \dots, s_n$ are global sections of $\mathscr{L}$ with no common zeros, up to isomorphism of these data. More specifically, given a morphism $\pi : X \to \mathbb{P}^n_k$, the data $(\pi^*\mathscr{O}(1), \pi^* x_0, \dots, \pi^* x_n)$ is the corresponding line bundle.*

Remark 2.51. This theorem shows that $\mathbb{P}^n_k$ is the moduli space of line bundle $\mathscr{L}$ with $n+1$ sections without common zeros.

Next we show some application of this theorem.

Example 2.52. The automorphism group $\mathrm{Aut}(\mathbb{P}^n_k)$ is $\mathrm{PgL}(n+1, k)$ called projective general linear group, where elements are $(n+1) \times (n+1)$ invertible matrix modulo scalar.

Proof. For an element in $\mathrm{PgL}(n, k)$, the matrix determines an automorphism on $\mathbb{P}^n_k$ clearly. We just need to show given any automorphism ϕ, it is an element in $\mathrm{PgL}(n, k)$. First, ϕ^* is an automorphism on $\mathrm{Pic}(\mathbb{P}^n_k)$ and according to Example 2.36, $\mathrm{Pic}(\mathbb{P}^n_k)$ is generated by $\mathscr{O}(1)$ or $\mathscr{O}(-1)$, thus $\phi^*\mathscr{O}(1) = \mathscr{O}(1)$ or $\mathscr{O}(-1)$. Since $\mathscr{O}(-1)$ has no non-trivial global sections, we must have $\phi^*\mathscr{O}(1) = \mathscr{O}(1)$. Let $s_i = \phi^*(x_i)$ and $s_i = a_{ij} x_j$ and (a_{ij}) forms an invertible $(n+1) \times (n+1)$ matrix A. Then A determines an isomorphism between graded rings

$$k[x_0, \dots, x_n] \to k[s_0, \dots, s_n]$$

and this induced

$$\mathrm{Proj}(k[s_0, \dots, s_n]) \to \mathrm{Proj}(k[x_0, \dots, x_n])$$

which is ϕ exactly due to the previous theorem. □

Definition 2.53. Suppose $\pi : X \to \mathrm{Spec}\, A$ is a proper morphism and $\mathscr{L}$ is an invertible sheaf on X. We say $\mathscr{L}$ is **very ample over** A if there exists a finite generated graded ring S_* over A generated in degree 1 such that $X \cong \mathrm{Proj} S_*$ and $\mathscr{L} \cong \mathscr{O}_{\mathrm{Proj} S_*}(1)$.

Proposition 2.16. *Suppose $\pi\colon X \to \operatorname{Spec} A$ is proper and $\mathscr{L}$ is an invertible sheaf on X, then $\mathscr{L}$ is very ample if and only if there exists a finite number of global sections $s_0, \dots, s_n$ of $\mathscr{L}$, with no common zeros such that the induced morphism*

$$X \to \mathbb{P}^n_A$$

is a closed embedding.

Theorem 2.54. *Suppose $\pi\colon X \to \operatorname{Spec} A$ is proper and $\mathscr{L}$ is an invertible sheaf on X. The following are equivalent*

- *(a) For some $N > 0$, $\mathscr{L}^{\otimes N}$ is very ample over A.*
- *(a') For all $n \gg 0$, $\mathscr{L}^{\otimes n}$ is very ample over A.*
- *(b) For all finite type quasicoherent sheaves $\mathscr{F}$, there is an $n_0 > 0$, such that for all $n > n_0$, $\mathscr{F}^{\otimes n}$ is globally generated.*
- *(c) As f runs over the global sections of $\mathscr{F}^{\otimes n}$ $(n > 0)$, the open subsets $X_f = \{p \in X \mid f(p) \neq 0\}$ form a base for the topology of X.*

3. Cohomology, arithmetic genus and Riemann-Roch theorem

In above sections, we see that a plenty of geometric information of a scheme X can be presented in Qch_X and Qch_X is an abelian category, which allows us to use method of homological algebra to get more algebraic invariants.

In this section, we will construct a sheaf cohomology, (̆C)ech cohomology, on Qch_X and algebraic invariants, arithmetic genus.

3.1. Čech cohomology.

Definition 3.1. Suppose X is a **quasicompact and separated** scheme, $\mathscr{F}$ is a quasicoherent sheaf on X and $\mathscr{U} = \{U_i\}_{i=1}^n$ is a finite collection of affine opne sets. Let I be a subset of $\{1, \dots, n\}$, $U_I = \cap_{i \in I} U_i$ and denote the cardinality of I by $|I|$, then we define the **Čech complex**

$$0 \to \prod_{|I|=1} \mathscr{F}(U_I) \to \cdots \to \prod_{|I|=i} \mathscr{F}(U_I) \to \prod_{|I|=i+1} \mathscr{F}(U_I) \to \dots \tag{6}$$

The maps are defined as follows. The map from $\mathscr{F}(U_I) \to \mathscr{F}(U_J)$ is 0 unless $J = I \cup \{i\}$ for some i. If j is the kth element of J, then the map is $(-1)^{k-1}$ times the restriction map $\operatorname{res}_{U_I, U_J}$. Then the sequence in (6) is indeed a complex according to the sheaf axiom.

Define $\mathrm{H}^i_{\mathscr{U}}(X, \mathscr{F})$ to be the ith cohomology group of the Čech complex (6).

Remark 3.2. According to sheaf axioms, $\mathrm{H}^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

Theorem 3.3. *The higher Čech cohomology $\mathrm{H}^i_{\mathscr{U}}(X, \mathscr{F})$ of an affine scheme vanishes for any affine cover $\mathscr{U}$, $i > 0$ and quasicoherent sheaf $\mathscr{F}$.*

Proof. See [Vak17] 18.2.4 □

Theorem 3.4. *Assume the assumption in Definition 3.1, $\mathrm{H}^i_{\mathscr{U}}(X, \mathscr{F})$ is independent of the choice of finite cover $\{U_i\}$. More precisely, for any two covers $\{U_i\} \subset \{V_i\}$, the maps*

$$\mathrm{H}^i_{\{V_i\}}(X, \mathscr{F}) \to \mathrm{H}^i_{\{U_i\}}(X, \mathscr{F})$$

induced by the natural map of Čech complexes (6) is an isomorphism.

Proof. See [Vak17] 18.2.2 □

Definition 3.5. Following the assumption and notations in previous theorem, we define the **Čech cohomology group** $\mathrm{H}^i(X, \mathscr{F}) := \mathrm{H}^i_{\mathscr{U}}(X, \mathscr{F})$

Suppose X is a separated and quasicompact A-scheme, then Čech cohomology groups satisfy the following properties

(1) Each H^i is a covariant functor $Qch_X \to \mathrm{Mod}_A$.
(2) The functor H^0 is Γ.
(3) If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of quasicoherent sheaves on X, then we have a long exact sequence of Čech cohomology groups

$$\begin{tikzcd} \mathrm{H}^{i-1}(X,\mathscr{H}) \arrow[r] & \mathrm{H}^i(X,\mathscr{F}) \arrow[r] & \mathrm{H}^i(X,\mathscr{G}) \arrow[r] & \mathrm{H}^i(X,\mathscr{H}) \arrow[dll, "\delta", rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\ & \mathrm{H}^{i+1}(X,\mathscr{F}) \arrow[r] & \mathrm{H}^{i+1}(X,\mathscr{G}) \arrow[r] & \mathrm{H}^{i+1}(X,\mathscr{H}) \arrow[r] & \mathrm{H}^{i+2}(X,\mathscr{F}) \end{tikzcd}$$

where we call such $\delta \colon \mathrm{H}^i(X, \mathscr{H}) \to \mathrm{H}^{i+1}(X, \mathscr{F})$ **connecting homomorphism**.
(4) If $\pi \colon X \to Y$ is a morphism of A-schemes, then there is a natural morphism
$$\mathrm{H}^i(Y, \pi_*\mathscr{F}) \to \mathrm{H}^i(X, \mathscr{F})$$
extending $\Gamma(Y, \pi_*\mathscr{F}) \to \Gamma(X, \mathscr{F})$.
(5) If $\pi \colon X \to Y$ is an affine morphism of A-schemes, the morphisms in (4) are isomorphisms.
(6) If X can be covered by n affine open sets, then $\mathrm{H}^i(X, \mathscr{F}) = 0$ for $i \geqslant n$ for all $\mathscr{F}$.

Theorem 3.6. *For any coherent sheaf $\mathscr{F}$ on a projective A-scheme where A is Noetherian, $\mathrm{H}^i(X, \mathscr{F})$ is a finitely generated A-module.*

Proof. See [Vak17] 18.1.4. □

Theorem 3.7. *Suppose X is a projective k-scheme, and $\mathscr{F}$ is a quasicoherent sheaf on X. Then $\mathrm{H}^i(X, \mathscr{F}) = 0$ for $i > \dim X$.*

3.2. **Arithmetic genus and Riemann-Roch theorem.** Let X be a separated quasicompact k-scheme and $\mathscr{F}$ is a quasicoherent sheaf on X, we denote $\mathrm{h}^i(X, \mathscr{F}) := \dim_k \mathrm{H}^i(X, \mathscr{F})$.

Suppose $\mathscr{F}$ is a coherent sheaf on a projective k-scheme X, then $\mathrm{h}^i(\mathscr{F})$ is finite according to Theorem 3.6. Define the **Euler characteristic** of $\mathscr{F}$ by

$$\chi(X, \mathscr{F}) := \sum_{i=0}^{\dim X} (-1)^i \mathrm{h}^i(X, \mathscr{F})$$

The Euler characteristic is an important algebraic invariant.

Proposition 3.1. *Suppose $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of coherent sheaves on a projective k-scheme X, then $\chi(X, \mathscr{G}) = \chi(X, \mathscr{F}) + \chi(X, \mathscr{H})$.*

Sketch proof. We just need to consider the associated long exact sequence of cohomology groups, then the result follows. □

Corollary 3.1. *Suppose there is an exact sequence of coherent sheaves*

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to \cdots \to \mathscr{F}_n \to 0$$

then

$$\sum_{i=1}^{n}(-1)^i \chi(X, \mathscr{F}_i) = 0$$

Definition 3.8. We define the **arithmetic genus** of a scheme X as $1-\chi(X, \mathscr{O}_X)$, denoted by $\mathrm{p}_a(X)$.

Remark 3.9. Recall that we can compute topological genus by Euler characteristic, the arithmetic case is an analgous case and surprisely, if we take X as a $\mathbb{C}$-curve, then the set of $\mathbb{C}$-valued points $X(\mathbb{C})$ is a compact Riemannian surface, and the topological genus of $X(\mathbb{C})$ is the same as arthmetic genus of X.

3.2.1. *Riemann-Roch theorem for line bundles on regular projective curves.* Suppose $D := \sum_{p \in C} a_p[p]$ is a divisor on a regular projective curve C over a field k (where $a_p \in \mathbb{Z}$ and p is a closed point). Define the **degree of** D by

$$\deg D := \sum a_p \deg p$$

where $\deg p$ is the degree of the field extension of the residue field over k.

Theorem 3.10 (Riemann-Roch)**.** *Assume the above assumption, then*

$$\chi(X, \mathscr{O}_C(D)) = \deg D + \chi(C, \mathscr{O}_C)$$

Proof. When $\deg D = 0$, this is obviously ture. Argue by induction, we just need to show

$$\chi(C, \mathscr{O}_C(D)) = \deg p + \chi(X, \mathscr{O}_C(D-p))$$

Claim the following sequence is exact (recall Example 2.46) :

$$0 \longrightarrow \mathscr{O}_C(-p) \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_C|_p \longrightarrow 0$$

We check the exactness stalk by stalk and we may assume take an affine open neighborhood of each point. When $q \neq p \in C$, then $\mathscr{O}_C|_{p,q} = 0$ according to the definition and by take an affine open neighborhood U of q that does not contain p, hence $\mathscr{O}_C(-p)|_U \cong \mathscr{O}_U$ then $\mathscr{O}_C(-p)_q \cong \mathscr{O}_{C,q}$ clearly. If we consider the stalk at p, let $\mathfrak{m}_p$ be the maximal ideal of $\mathscr{O}_C$, then according to Definiton 2.32, $\mathscr{O}_C(-p)_p = \mathfrak{m}_p$. Thus the result follows. **Actually, a direct way to show its exactness is to view $\mathscr{O}_C(-p)$ as the sheaf of ideals for closed subscheme p.**

Now we tensor the exact sequence by $\mathscr{O}_C(D)$, since $\mathscr{O}_C(D)$ is locally free, we get another short exact sequence

$$0 \longrightarrow \mathscr{O}_C(-p) \otimes_{\mathscr{O}_C} \mathscr{O}_C(D) \longrightarrow \mathscr{O}_C \otimes_{\mathscr{O}_C} \mathscr{O}_C(D) \longrightarrow \mathscr{O}_C|_p \otimes_{\mathscr{O}_C} \mathscr{O}_C(D) \longrightarrow 0$$

Then according to Proposition 3.1, we have

$$\chi(X, \mathscr{O}_C(D)) = \chi(X, \mathscr{O}_C(D-p)) + \chi(X, \mathscr{O}_C|_p \otimes_{\mathscr{O}_C} \mathscr{O}_C(D))$$

Note that $\mathscr{O}_C|_p \otimes_{\mathscr{O}_C} \mathscr{O}_C(D)$ vanishes at higher cohomology groups, because $\mathscr{O}_C|_p$ only support at p and

$$\mathrm{H}^0(X, \mathscr{O}_C|_p \otimes_{\mathscr{O}_C} \mathscr{O}_C(D)) = \mathrm{H}^0(p, \mathscr{O}_C|_p \otimes_{\mathscr{O}_C} \mathscr{O}_C(D)) = \deg_k k(p) \otimes_k k = [k(p) : k]$$

Then it is done. ∎

Remark 3.11. As every invertible sheaf $\mathscr{L}$ is of the form $\mathscr{O}_C(D)$ fro some D (recall Corollary 2.2, the last paragraph of Example 2.36).

Remark 3.12. There is another equivalent statement of Riemann-Roch theorem

$$\mathrm{h}^0(C, \mathscr{L}) - \mathrm{h}^1(C, \mathscr{L}) = \deg \mathscr{L} - \mathrm{p}_a(C) + 1$$

and the degree of $\mathscr{L}$ is $\chi(C, \mathscr{L}) - \chi(C, \mathscr{O}_C)$.

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