BJSE Index **Summer 2023**

Technical Appendix

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1. **Index Design:** The algorithm should correctly calculate the value of the index at any give time t, given the price P and weight W of the top 30 stocks j, C_j, for $j = 1...30$. Assume at $t = 0$ the balance (value), $B_{t=0}$, of the index is, or normalized to, 100. The weights sum to 1 for any time t.

   On day 1, the balance of the index is 100, and the **balance allocation** (the money we should spend) for stock j at time $t = 0$ is:

$$\begin{aligned} A_{C_j,t=0} &= B_{t=0} * W_{C_{j,t=0},t=0} \\ &= 100 * W_{C_{j,t=0},t=0} \end{aligned} \tag{1}$$

   Then our **holdings** (number of shares we have), H, for C_J at time 0 is:

$$H_{C_{j,t=0},t=0} = \frac{A_{C_j,t=0}}{P_{C_{j,t=0},t=0}} \tag{2}$$

   So of course, the **balance** of index at $t = 0$ is simply the sum of holdings * price for each of stock j:

$$\begin{aligned} B_{t=0} &= \sum_{j=1}^{30} H_{C_{j,t=0},t=0} * P_{C_{j,t=0},t=0} \\ &= \sum_{j=1}^{30} \frac{A_{C_j,t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=0} \\ &= \sum_{j=1}^{30} \frac{100 * W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=0} \\ &= 100 * \sum_{j=1}^{30} W_{C_{j,t=0},t=0} \\ &= 100 \end{aligned} \tag{3}$$

   This seemingly redundant analysis is to demonstrate the flow of index balance calculation. We start from balance allocation, divide it by price to find holdings, and lastly multiply holdings by price again to arrive at our final balance. However, at $t = 1$, we would find that the price at the denominator in (3) does not cancel out:

$$\begin{aligned} B_{t=1} &= \sum_{j=1}^{30} H_{C_{j,t=0},t=0} * P_{C_{j,t=0},t=1} \\ &= \sum_{j=1}^{30} \frac{A_{C_j,t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=1} \\ &= \sum_{j=1}^{30} \frac{100 * W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=1} \end{aligned} \tag{4}$$

The equation in (4) demonstrates the naive case of index balance growth in the case of no rebalancing/reconstitution. Here, the only variable that have changed is P, and we have W and C_j to remain *constant* (the weights and index composition does not change with time). Which means in the case of **naive growth**:

$$\begin{aligned} B_t &= \sum_{j=1}^{30} \frac{100 * W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t} \\ &= f(P_{C_{j,t=0},t}) \end{aligned} \tag{5}$$

For the sake of clarity, we can write this formula for calculating B_t in matrix form:

$$\begin{aligned} B_t &= f(P_{C_{j,t=0},t}) \\ &= \begin{bmatrix} \frac{100*W_{C_{1,t=0},t=0}}{P_{C_{2,t=0},t=0}} & \frac{100*W_{C_{2,t=0},t=0}}{P_{C_{2,t=0},t=0}} & \dots \frac{100*W_{C_{30,t=0},t=0}}{P_{C_{30,t=0},t=0}} & \end{bmatrix} * \begin{bmatrix} P_{C_{1,t=0},t} \\ P_{C_{2,t=0},t} \\ \vdots \\ P_{C_{30,t=0},t} \end{bmatrix} \\ &= \begin{bmatrix} H_{C_{1,t=0},t=0} & H_{C_{2,t=0},t=0} & \dots H_{C_{30,t=0},t=0} & \end{bmatrix} * \begin{bmatrix} P_{C_{1,t=0},t} \\ P_{C_{2,t=0},t} \\ \vdots \\ P_{C_{30,t=0},t} \end{bmatrix} \\ &= \mathbf{H}^T \cdot \mathbf{P} \end{aligned} \tag{6}$$

Where $\mathbf{H}^T$ is a 1×30 row vector and $\mathbf{P}$ is a 30×1 column vector. Therefore B_t is simply the dot product of $\mathbf{H}^T$ and $\mathbf{P}$, which means if we totally differentiate $B_t = f(P_{C_{j,t=0},t})$:

$$\begin{aligned} \mathrm{d}B_t &= \begin{bmatrix} \frac{\partial B_t}{\partial P_{C_{1,t=0},t}} & \frac{\partial B_t}{\partial P_{C_{2,t=0},t}} & \cdots \frac{\partial B_t}{\partial P_{C_{30,t=0},t}} \end{bmatrix} * \begin{bmatrix} \mathrm{d}P_{C_{1,t=0},t} \\ \mathrm{d}P_{C_{2,t=0},t} \\ \vdots \\ \mathrm{d}P_{C_{30,t=0},t} \end{bmatrix} \\ &= \begin{bmatrix} H_{C_{1,t=0},t=0} & H_{C_{2,t=0},t=0} & \ldots H_{C_{30,t=0},t=0} \end{bmatrix} * \begin{bmatrix} \mathrm{d}P_{C_{1,t=0},t} \\ \mathrm{d}P_{C_{2,t=0},t} \\ \vdots \\ \mathrm{d}P_{C_{30,t=0},t} \end{bmatrix} \end{aligned} \tag{7}$$

Which means the impact of any stock's price change is simply the magnitude of that change in price * by our holdings of the stock. In the next section, we will see how a more complicated index methodology impact our calculations and the comparative-static analysis.

On the third day, $t = 2$, we introduce **rebalancing**, where we allow an additional variable to change, namely the weights, $W_{C_{j,t=0},t}$. Notice that C_j, our top stocks, does not change with time, since rebalancing only implies simple change of stock weights but *not* the composition of index, so weights are associated with the same underlying stocks $C_{j,t=0}$ at each time t.

What's our new index balance at $t = 2$? The crux of this matter is realizing that, much like how we calculated the index value at $t = 0$, we need to some sort of root-level balance to work with. On the first day, that's the \$100 we supplied to start the index, and at $t = 2$, our rebalance day, the logic is the same — the current index balance $B_{t=2}$ is the root-level index value, and we cannot spend more than what we currently own.

Thus we first obtain the naive-growth balance at $t = 2$ to be our root-level balance from (5):

$$B_{t=2} = \sum_{j=1}^{30} \frac{100 * W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=2} \tag{8}$$

Then we can recursively solve for the updated allocation using our new weights:

$$A_{C_j,t=2} = B_{t=2} * W_{C_{j,t=0},t=2} \tag{9}$$

So our updated holdings under the new price are calculated as:

$$
\begin{aligned}
H_{C_{j,t=0},t=2} &= \frac{A_{C_j,t=2}}{P_{C_{j,t=0},t=2}} \\
&= \frac{B_{t=2} * W_{C_{j,t=0},t=2}}{P_{C_{j,t=0},t=2}} \\
&= \frac{(\sum_{j=1}^{30} \frac{100 * W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=2}) * W_{C_{j,t=0},t=2}}{P_{C_{j,t=0},t=2}}
\end{aligned} \tag{10}
$$

Which in the context of index balance, though the balance did not change at time $t = 3$, we *re-balanced* the index by changing the weights:

$$
\begin{aligned}
B_{t=2} &= \sum_{j=1}^{30} \frac{100 * W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=2} \\
&= \sum_{j=1}^{30} H_{C_{j,t=0},t=2} * P_{C_{j,t=0},t=2} \\
&= \sum_{j=1}^{30} \left(\frac{(\sum_{j=1}^{30} \frac{100 * W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=2}) * W_{C_{j,t=0},t=2}}{P_{C_{j,t=0},t=2}} \right) * P_{C_{j,t=0},t=2} \\
&= \sum_{j=1}^{30} \left(\sum_{j=1}^{30} \frac{100 * W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=2} \right) * W_{C_{j,t=0},t=2} \\
&= \sum_{j=1}^{30} \left(\sum_{j=1}^{30} H_{C_{j,t=0},t=0} * P_{C_{j,t=0},t=2} \right) * W_{C_{j,t=0},t=2} \\
&= \sum_{j=1}^{30} (H_{C_{1,t=0},t=0} * P_{C_{1,t=0},t=2} + ... + H_{C_{30,t=0},t=0} * P_{C_{30,t=0},t=2}) * W_{C_{j,t=0},t=2} \\
&= (H_{C_{1,t=0},t=0} * P_{C_{1,t=0},t=2} + ... + H_{C_{30,t=0},t=0} * P_{C_{30,t=0},t=2}) * W_{C_{1,t=0},t=2} + ... + \\
&\quad (H_{C_{1,t=0},t=0} * P_{C_{1,t=0},t=2} + ... + H_{C_{30,t=0},t=0} * P_{C_{30,t=0},t=2}) * W_{C_{30,t=0},t=2}
\end{aligned} \tag{11}
$$

Thus on the day of index re-balancing, we find a new functional form for B_t:

$$
\begin{aligned}
B_t &= f(W_{C_{j,t=0},t}, P_{C_{j,t=0},t}) \\
&= \sum_{j=1}^{30} H_{C_{j,t=0},t} * P_{C_{j,t=0},t} \\
&= \sum_{j=1}^{30} \left(\sum_{j=1}^{30} H_{C_{j,t=0},t} * P_{C_{j,t=0},t} \right) * W_{C_{j,t=0},t}
\end{aligned} \tag{12}
$$

Which means that we can find the total differential to be:

$$
\begin{aligned}
\mathrm{d}B_t &= \begin{bmatrix} \frac{\partial B_t}{\partial P_{C_{1,t=0},t}} & \frac{\partial B_t}{\partial P_{C_{2,t=0},t}} & \cdots \frac{\partial B_t}{\partial P_{C_{30,t=0},t}} \end{bmatrix} * \begin{bmatrix} \mathrm{d}P_{C_{1,t=0},t} \\ \mathrm{d}P_{C_{2,t=0},t} \\ \vdots \\ \mathrm{d}P_{C_{30,t=0},t} \end{bmatrix} \\
&+ \begin{bmatrix} \frac{\partial B_t}{\partial W_{C_{1,t=0},t}} & \frac{\partial B_t}{\partial W_{C_{1,t=0},t}} & \cdots \frac{\partial B_t}{\partial W_{C_{1,t=0},t}} \end{bmatrix} * \begin{bmatrix} \mathrm{d}W_{C_{1,t=0},t} \\ \mathrm{d}W_{C_{1,t=0},t} \\ \vdots \\ \mathrm{d}W_{C_{1,t=0},t} \end{bmatrix}
\end{aligned} \tag{13}
$$

Where we can find $\frac{\partial B_t}{\partial P_{C_{j,t=0},t}}$ to be $\sum_j^{30} H_{C_{j,t=0},t} * W_{C_{j,t=0},t}$, the **sum of weighted holdings**, which we define as Γ_t.

$$
\begin{aligned}
\mathrm{d}B_t &= \begin{bmatrix} \Gamma_t & \ldots \Gamma_t \end{bmatrix} * \begin{bmatrix} \mathrm{d}P_{C_{1,t=0},t} \\ \vdots \\ \mathrm{d}P_{C_{30,t=0},t} \end{bmatrix} \\
&+ \begin{bmatrix} H_{C_{1,t=0},t=0} * P_{C_{1,t=0},t} & \ldots H_{C_{1,t=0},t=0} * P_{C_{30,t=0},t} \end{bmatrix} * \begin{bmatrix} \mathrm{d}W_{C_{1,t=0},t} \\ \vdots \\ \mathrm{d}W_{C_{30,t=0},t} \end{bmatrix}
\end{aligned} \tag{14}
$$

The total differential demonstrates that, when the price of token j drops on day t and our token weight is did not change (when, for example, on some date between our rebalancing dates), then $\mathrm{d}W_{C_{j,t=0},t} = 0$, then (14) simplifies to:

$$
\mathrm{d}B_t = \begin{bmatrix} \Gamma_t & \ldots \Gamma_t \end{bmatrix} * \begin{bmatrix} \mathrm{d}P_{C_{1,t=0},t} \\ \vdots \\ \mathrm{d}P_{C_{30,t=0},t} \end{bmatrix} \tag{15}
$$

Which means the price impact on the index balance of token j is just the holding of the token j at time t multiplied by its change in price. Note the crucial difference between equation (15) and (7), whereas the former uses the updated holdings, the latter uses the holdings of stock j at $t = 0$, which means the comparative-static yields very different result.

Consider also the case where prices are held constant while weights changes, we found (14) simplifies to:

$$
\mathrm{d}B_t = \begin{bmatrix} H_{C_{1,t=0},t=0} * P_{C_{1,t=0},t} & \ldots H_{C_{1,t=0},t=0} * P_{C_{30,t=0},t} \end{bmatrix} * \begin{bmatrix} \mathrm{d}W_{C_{1,t=0},t} \\ \vdots \\ \mathrm{d}W_{C_{30,t=0},t} \end{bmatrix} \tag{16}
$$

Which would give us the impact of a potential weight change in the stocks that the index is holding. Note that however, by definition, the impact of the pure re-balancing is set to zero, as rebalancing is done with the current balance at time t.

On the fourth day, t=3, let's now consider the case with **reconstitution** along with rebalancing. In this case, stock $C_{j,t}$ could change over time (e.g. A stock could be, say $C_{j,t=2}$, but $C_{j,t=3}$ might become B stock).

We continue our analysis in a similar fashion — first calculate the root level balance $B_{t=3}$, then find the updated balance allocations for each token $A_{C_j,t=3}$, and finally find the updated stock holdings $H_{C_j,t=3}$.

Note that the root level balance is the value of the balance on $t = 3$, the amount of money we currently have, which is simply the product of holdings in the previous period (from (10)) multiplied by the current price:

$$\begin{aligned} B_{t=3} &= \sum_{j=1}^{30} H_{C_{j,t=2},t=2} * P_{C_{j,t=2},t=3} \\ &= \sum_{j=1}^{30} \frac{(\sum_{j=1}^{30} \frac{100*W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=2}) * W_{C_{j,t=0},t=2}}{P_{C_{j,t=0},t=2}} * P_{C_{j,t=2},t=3} \end{aligned} \tag{17}$$

Solving for the new balance allocation using (17):

$$\begin{aligned} A_{C_j,t=3} &= B_{t=3} * W_{C_{j,t=3},t=3} \\ &= (\sum_{j=1}^{30} H_{C_{j,t=2},t=2} * P_{C_{j,t=2},t=3}) * W_{C_{j,t=3},t=3} \\ &= (\sum_{j=1}^{30} \frac{(\sum_{j=1}^{30} \frac{100*W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=2}) * W_{C_{j,t=0},t=2}}{P_{C_{j,t=0},t=2}} * P_{C_{j,t=2},t=3}) * W_{C_{j,t=3},t=3} \end{aligned} \tag{18}$$

As well as the updated holdings with the new prices:

$$\begin{aligned} H_{C_{j,t=3},t=3} &= \frac{A_{C_j,t=3}}{P_{C_{j,t=3},t=3}} \\ &= \frac{(\sum_{j=1}^{30} H_{C_{j,t=2},t=2} * P_{C_{j,t=2},t=3}) * W_{C_{j,t=3},t=3}}{P_{C_{j,t=3},t=3}} \\ &= \frac{(\sum_{j=1}^{30} \frac{(\sum_{j=1}^{30} \frac{100*W_{C_{j,t=0},t=0}}{P_{C_{j,t=0},t=0}} * P_{C_{j,t=0},t=2}) * W_{C_{j,t=0},t=2}}{P_{C_{j,t=0},t=2}} * P_{C_{j,t=2},t=3}) * W_{C_{j,t=3},t=3}}{P_{C_{j,t=3},t=3}} \end{aligned} \tag{19}$$

We again evaluate B_t, this time with $C_{j,t}$ changing:

$$
\begin{aligned}
B_{t=2} &= \sum_{j=1}^{30} H_{C_{j,t=3},t=3} * P_{C_{j,t=3},t=3} \\
&= \sum_{j=1}^{30} \Big(\frac{(\sum_{j=1}^{30} H_{C_{j,t=2},t=2} * P_{C_{j,t=2},t=3}) * W_{C_{j,t=3},t=3}}{P_{C_{j,t=3},t=3}} \Big) * P_{C_{j,t=3},t=3} \\
&= \sum_{j=1}^{30} \Big(\sum_{j=1}^{30} H_{C_{j,t=2},t=2} * P_{C_{j,t=2},t=3} \Big) * W_{C_{j,t=3},t=3} \\
&= \sum_{j=1}^{30} (H_{C_{1,t=2},t=2} * P_{C_{1,t=2},t=3} + ... + H_{C_{30,t=2},t=2} * P_{C_{30,t=2},t=3}) * W_{C_{j,t=3},t=3} \\
&= (H_{C_{1,t=2},t=2} * P_{C_{1,t=2},t=3} + ... + H_{C_{30,t=2},t=2} * P_{C_{30,t=2},t=3}) * W_{C_{1,t=3},t=3} + ...+ \\
&\quad (H_{C_{1,t=2},t=2} * P_{C_{1,t=2},t=3} + ... + H_{C_{30,t=2},t=2} * P_{C_{30,t=2},t=3}) * W_{C_{30,t=3},t=3}
\end{aligned} \tag{20}
$$

With the *last* rebalancing/reconstitution day $t = l$ (the last time the holdings are determined), we find the functional form to be:

$$
\begin{aligned}
B_t &= f(W_{C_{j,t},t}, P_{C_{j,t},t}) \\
&= \sum_{j=1}^{30} H_{C_{j,t},t} * P_{C_{j,t},t} \\
&= \sum_{j=1}^{30} \Big(\sum_{j=1}^{30} H_{C_{j,t=l},t} * P_{C_{j,t=l},t} \Big) * W_{C_{j,t},t}
\end{aligned} \tag{21}
$$

Which is to say, the balance in the current period is the sum of the holdings of the last basket of top stocks and * the current price of those stocks * the current weights assigned to them.

Proceeding with our comparative statics, $\Gamma_t = \frac{\partial B_t}{\partial P_{C_{j,t=l},t}}$ is $\sum_j^{30} H_{C_{j,t=l},t} * W_{C_{j,t},t}$, the sum of weighted holdings, and similarly we find the total differentials to be:

$$
\begin{aligned}
\mathrm{d}B_t &= \begin{bmatrix} \Gamma_t & \dots \Gamma_t \end{bmatrix} * \begin{bmatrix} \mathrm{d}P_{C_{1,t=l},t} \\ \vdots \\ \mathrm{d}P_{C_{30,t=l},t} \end{bmatrix} \\
&+ \begin{bmatrix} H_{C_{1,t=l},t} * P_{C_{1,t=l},t} & \dots H_{C_{1,t=l},t} * P_{C_{30,t=l},t} \end{bmatrix} * \begin{bmatrix} \mathrm{d}W_{C_{1,t},t} \\ \vdots \\ \mathrm{d}W_{C_{30,t},t} \end{bmatrix}
\end{aligned} \tag{22}
$$

The index buy or sell $C_{j,t=3}$ depending on whether $H_{C_{j,t=3},t=3} >< H_{C_{j,t=3},t=2}$, and here, since we assumed, say $C_{j,t=3} \neq C_{j,t=2}$, that means $H_{C_{j,t=3},t=2} = 0$, thus we buy $C_{j,t=3}$.

Note that in (17), $C_{j,t=2} = C_{j,t=0}$, the last time the index is *rebalanced/reconstituted*. Thus generalizing, on any arbitrary date $t = n$:

$$B_{t=n} = \sum_{j=1}^{30} H_{C_{j,t=l},t=l} * P_{C_{j,t=n},t=n} \tag{23}$$

And on any given *current* rebalancing date, $t = c \subset n$, we *additionally* update our balance allocation and holdings for any given $C_{j,t=c}$:

$$\begin{aligned} A_{C_j,t=c} &= B_{t=n} * W_{C_{j,t=c},t=c} \\ H_{C_{j,t=c},t=c} &= \frac{A_{C_j,t=c}}{P_{C_{j,t=c},t=c}} \end{aligned} \tag{24}$$