The Long-Term Expected Rate of Return: Setting It Right

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A number of serious, widely held errors and misconceptions about the long-term expected rate of return need to be dispelled. First, this rate need not be approximated, because exact formulas for this estimate are easy to find and apply. Second, such an approximation could be quite misleading. This article offers simple methods and exact formulas to determine the expected value and variance of the n*-year horizon rate of return directly in terms of the one-year parameters. The probability distribution of that return is also brought out. Concrete examples illustrate these results.*

The key issue in investments is estimating expected return.

Fischer Black (1993, p. 36)

All analysts in pension funds, and all investors for that matter, wish they could know more about the future rate of return on an investment; "knowing more" means having a precise idea about the investment's expected value, its variance, and its probability distribution. Unfortunately, some serious errors and misconceptions about the n-year horizon expected return are widespread in the investment management discipline today. For example, in MacBeth (1995), the error is that an approximation of the n-year horizon expected return is the expected one-year return minus half of its variance.[1] In their textbook, Bodie, Kane, and Marcus (1993) go so far as to state that "[the approximation] is exact when returns are normally distributed." (1993, p. 799) In fact, there is no need to resort to any "approximation." This article provides simple formulas for the expected value and variance of the long-term rate of return.

The subject is important and contains a wealth of surprises. Suppose, for instance, that the expected inflation rate is 5 percent and you set out to protect yourself against inflation by buying a bond portfolio with a yearly expected return of 5 percent, so your expected *real* return is zero. Suppose also that the annual real returns are lognormally distributed with a variance of 2.25 percent. You will be more than surprised—you will be shocked—to discover that your 30-year expected real return is *minus* 1.07 percent and that the probability of having a *negative* 30-year return is about two-thirds.

Expected Value and Variance of the Long-Term Rate of Return

The development of formulas for the expected value and variance of the long-term rate of return relies on that workhorse of financial calculations—the lognormal assumption about the yearly dollar return—coupled with the hypothesis of independence of these rates.[2] Thus, $\log X_{t-1,\,t}$, the continuously compounded yearly return, is normally distributed, $N(\mu, \sigma^2)$.

The first task is to discover what the lognormal assumption implies in terms of expected value and variance of the yearly rate of return. The following notation will be used:

S_t	= a stock's value at end of year t
$R_{t-1,t}$	= $(S_t - S_{t-1})/S_{t-1}$, the yearly rate of return compounded once a year
$X_{t-1,t}$	= S_t/S_{t-1} or $1 + R_{t-1,t}$, the yearly dollar return, or 1 plus the yearly rate of return
$\log X_{t-1,t}$	= $\log(S_t/S_{t-1})$, the yearly continuously compounded rate of return
$R_{0,n}$	= $(S_n/S_0)^{1/n} - 1$, the n-year horizon rate of return; $R_{0,n}$ solves $S_n = S_0(1 + R_{0,n})^n$
$E \equiv E(X_{t-1,t})$	= $E(1 + R_{t-1,t})$, the expected value of the yearly dollar return, or 1 plus the expected value of the yearly rate of return

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$V \equiv \mathrm{var}(X_{t-1,t}) = \mathrm{var}(R_{t-1,t})$, the variance of the yearly dollar return, or the variance of the yearly rate of return

With this notation, we can write

$$\log\left(\frac{S_t}{S_{t-1}}\right) = \mu + \sigma Z, \tag{1}$$

where Z stands for the unit normal variable $N(0,1)$, so

$$\begin{aligned} X_{t-1,t} &= \frac{S_t}{S_{t-1}} \\ &= \exp(\mu + \sigma Z). \end{aligned} \tag{2}$$

The various moments of $X_{t-1,t}$ can easily be calculated: The first one is[3]

$$E(X_{t-1,t}) = \exp\left(\mu + \frac{\sigma^2}{2}\right). \tag{3}$$

Similarly, the variance of $X_{t-1,t}$ (or $R_{t-1,t}$) can be determined as

$$\mathrm{var}(R_{t-1,t}) = \exp\left(2\mu + \sigma^2\right) \times \left[\exp\left(\sigma^2\right) - 1\right]. \tag{4}$$

Turn now to the n-year horizon rate of return:

$$\begin{aligned} R_{0,n} &= \left(\frac{S_n}{S_0}\right)^{1/n} - 1 \\ &= \left(\frac{S_1}{S_0} \times \frac{S_2}{S_1} \times \ldots \times \frac{S_n}{S_{n-1}}\right)^{1/n} - 1. \end{aligned} \tag{5}$$

Consequently,

$$\begin{aligned} 1 + R_{0,n} &= \prod_{t=1}^{n} X_{t-1,t}^{1/n} \\ &= \prod_{t=1}^{n} \exp\left(\frac{1}{n}\log X_{t-1,t}\right) \\ &= \exp\left(\frac{1}{n}\sum_{t=1}^{n}\log X_{t-1,t}\right). \end{aligned} \tag{6}$$

Because each $\log X_{t-1,t}$ is normal $N(\mu, \sigma^2)$, $\frac{1}{n}\sum_{t=1} \log X_{t-1,t}$ is normal, $N[\mu, (\sigma^2/n)]$. Therefore,

$$1 + R_{0,n} = \exp\left[\mu + \left(\frac{\sigma}{\sqrt{n}}\right)Z\right], Z \sim N(0,1). \tag{7}$$

The expected value of the n-horizon rate of return can now be easily determined to be

$$E(R_{0,n}) = \exp\left(\mu + \frac{\sigma^2}{2n}\right) - 1, \tag{8}$$

and its standard deviation is

$$\begin{aligned} \sigma(R_{0,n}) &= \exp\left(\mu + \frac{\sigma^2}{2n}\right)\left[\exp\left(\frac{\sigma^2}{n}\right) - 1\right]^{1/2} \\ &= [1 + E(R_{0,n})]\left[\exp\left(\frac{\sigma^2}{n}\right) - 1\right]^{1/2}. \end{aligned} \tag{9}$$

Note that when n increases indefinitely, this standard deviation tends to zero, as it should.[4]

Equations 8 and 9 are expressed in terms of μ and σ^2, the mean and variance of the continuously compounded yearly rate of return, but formulas directly expressed in terms of the yearly dollar return's expected value (that is, $E[X_{t-1,t}] = 1 + E[R_{t-1,t}] \equiv E$) and variance ($\mathrm{var}[R_{t-1,t}] \equiv V$) would be more convenient. For that purpose, the system of Equations 3 and 4 can be solved for μ and σ^2 in terms of E and V. The result is

$$\mu = \log E - \frac{1}{2}\log\left(1 + \frac{V}{E^2}\right) \tag{10}$$

and

$$\sigma^2 = \log\left(1 + \frac{V}{E^2}\right). \tag{11}$$

Plugging these values into Equations 8 and 9 produces

$$E(R_{0,n}) = E \times \left(1 + \frac{V}{E^2}\right)^{(1/2)[(1/n)-1]} - 1 \tag{12}$$

and

$$\sigma(R_{0,n}) = [1 + E(R_{0,n})]\left[\left(1 + \frac{V}{E^2}\right)^{1/n} - 1\right]^{1/2}. \tag{13}$$

The formulas shown as Equations 12 and 13 are ready to use. Note that the long-term expected return is always a decreasing function of the horizon, n; in the limit, the return tends toward $E(R_{0,\infty}) = E \times (1 + V/E^2)^{-1/2} - 1$; this result will soon be of use.

Probability Distribution of the *n*-Year Rate

In the basic formula expressed in Equation 7, 1 *plus* the horizon rate of return is a lognormal distribution with parameters μ and σ^2/n; that is, the logarithm of the n-year horizon *dollar* return, $\log(1 + R_{0,n})$, is normally distributed. The implication is that $[\log(1 + R_{0,n}) - \mu]/(\sigma/\sqrt{n})$ is a unit normal variable; therefore, the probability that $R_{0,n}$ is between any two values a and b is

$$\Phi\left[\frac{\log(1+b) - \mu}{\sigma/\sqrt{n}}\right] - \Phi\left[\frac{\log(1+a) - \mu}{\sigma/\sqrt{n}}\right],$$

where Φ is the cumulative probability distribution of the unit normal variable.

Conversely, an interval can easily be determined for $R_{0,n}$ spanning a given probability—for instance, 95 percent:

$$\text{prob}\left[\mu - 1.96\sigma/\sqrt{n} < \log(1+R_{0,n}) < \mu + 1.96\sigma/\sqrt{n}\right] = 0.95. \tag{14}$$

Equivalently:

$$\text{prob}\left[\exp\left(\mu - 1.96\sigma/\sqrt{n}\right) - 1 < R_{0,n} < \exp\left(\mu + 1.96\sigma/\sqrt{n}\right) - 1\right] = 0.95. \tag{15}$$

Thus, a 95 percent probability interval for the n-horizon rate of return is given by $[\exp(\mu - 1.96\sigma/\sqrt{n}) - 1,\ \exp(\mu + 1.96\sigma/\sqrt{n}) - 1]$; in turn, and if needed, this interval can be expressed directly in terms of E and V by using Equations 10 and 11. Before looking at examples of applications, consider how misleading the so-called approximation mentioned in the introduction turns out to be.

A Misleading Error

The approximation $E(R_{0,n}) = E - V/2$ constitutes a wrong and badly misleading result for a number of reasons.

First and foremost, one would expect any approximation to yield the right value at least for one set of values of the relevant variables, but the approximation is independent of the horizon and, therefore, *never* gives the right result.

Second, the error resulting from using the approximation can be substantial. Consider, for instance, the example $E(R_{0,1}) = 10$ percent (that is, $E[X_{0,1}] \equiv E = E[R_{0,1}] + 1 = 1.1$), $\sigma(R_{0,1}) = 20$ percent (that is, $\sigma^2[R_{0,1}] \equiv V = 0.04$), and a horizon of two years. The correct expected two-year horizon annual return, from Equation 12, is $E(R_{0,2}) = 9.1$ percent, but the approximation gives $E(R_{0,1}) - V/2 = 8$ percent—off the mark by 110 basis points (bps) a year. For a longer horizon ($n = 10$), the right result would be $E(R_{0,10}) = 8.4$ percent; the error still amounts to 40 bps.

Third, the approximation conceals the all-important property that a horizon expected rate of return is *always a decreasing function* of the horizon and tends toward a lower limit when n goes to infinity. This general property is independent of the probability distribution underlying $X_{t-1,t}$. The limit is the infinite-horizon expected rate of return, $E(R_{0,\infty})$, equal to $\exp[E(\log X_{t-1,t})] - 1$, which turns out to be the geometric mean of the probability distribution of $X_{t-1,t}$, minus 1. The reasons behind this important result are provided in the final section.

Fourth, in the important case in which $X_{t-1,t}$ is assumed to be lognormal, the approximation is always lower than the lowest value that the correct value can ever take, which is the infinite-horizon expected return. Indeed, $E - 1 - V/2$ is always smaller than $E(R_{0,\infty})$, equal here to $E \times (1 + V/E^2)^{-1/2} - 1$. For instance, in the case of $E(R_{0,1}) = 10$ percent and $\sigma(R_{0,1}) = 20$ percent, the approximation is 8 percent whereas $E(R_{0,\infty})$ is 8.23 percent. So, the approximation does provide one assurance: If we use it, the error we will make is *at least* 23 bps! Consequently, not only is the approximation misleading, it is completely useless. We would *always* be better off replacing it with $E(R_{0,\infty})$!

Correcting the Error

An example will demonstrate application of the right formulas, after which, the formulas will be used to correct the results given by MacBeth.

Example. Consider the case in the preceding section in which $E(R_{0,1}) = 10$ percent and $\sigma(R_{0,1}) = 20$ percent. $E(R_{0,10})$ in that case was 8.4 percent, and Equation 13 can be used to find the standard deviation of $R_{0,10}$ to be 6.2 percent.

Suppose we want to know the probability that the 10-year horizon return will be negative. We first have to translate E (equal to 1.10) and V (equal to 0.04) into μ and σ. Using Equations 10 and 11, we get $\mu = 7.90$ percent, $\sigma^2 = 3.25$ percent, and $\sigma = 18.03$ percent. The probability of $R_{0,10}$ being lower than $b = 0$ is equal to

$$\Phi\left(\frac{\log(1+b) - \mu}{\sigma/\sqrt{n}}\right)\Big|_{b=0} = \Phi\left(\frac{-7.90\%}{18.03\%/\sqrt{10}}\right) = \Phi(-1.38) \approx 8\%.$$

Similarly, the 95 percent confidence interval for $R_{0,10}$ is

$$\exp(\mu - 1.96\sigma/\sqrt{n}) - 1, \exp(\mu + 1.96\sigma/\sqrt{n}) - 1 = -3.2\% \text{ and } +21.0\%.$$

On the other hand, $R_{0,10}$ has a 68 percent probability of being between 2.2 percent and 14.6 percent.

The MacBeth Case. MacBeth addressed an important issue: What is the error made by practitioners when they (wrongly) consider a portfolio's long-term expected return to be the weighted average of the expected returns of its components? His answer to this interesting question was unfortunately flawed by using the ill-fated approximation. The correct answer, given MacBeth's data, is as follows:

The portfolio has equal shares of stocks and bonds, and the properties of the returns are

$$E\left(R_{t-1,t}^{stocks}\right) = 0.123; \sigma\left(R_{t-1,t}^{stocks}\right) = 0.205$$

$$E\left(R_{t-1,t}^{bonds}\right) = 0.059; \sigma\left(R_{t-1,t}^{bonds}\right) = 0.084$$

$$\text{cov}\left(R_{t-1,t}^{stocks}, R_{t-1,t}^{bonds}\right) = 0.003788$$

$$E\left(R_{t-1,t}^{portfolio}\right) = 0.091; \sigma\left(R_{t-1,t}^{portfolio}\right) = 0.119.$$

Applying Equation 12 with $E = E\left(X_{t-1,t}^{portfolio}\right) = E\left(R_{t-1,t}^{portfolio}\right) + 1 = 1.091$ gives the 20-year horizon expected return for the portfolio as $E\left(R_{0,20}^{portfolio}\right) = 8.49$ percent. On the other hand, the expected return for the stock portfolio is $E\left(R_{0,20}^{stocks}\right) = 10.57$ percent; for the bond portfolio, it is $E\left(R_{0,20}^{bonds}\right) = 5.59$ percent, for an average of 8.08 percent, which is off the mark by 41 bps. Using the ill-fated approximation, MacBeth obtained $E\left(R_{0,20}^{portfolio}\right) = 8.4$ percent, $E\left(R_{0,20}^{stocks}\right) = 10.2$ percent, and $E\left(R_{0,20}^{bonds}\right) = 5.6$ percent, for a mean of 7.9 percent.

MacBeth then concluded that using the weighted average of the stock and bond rates produced an error of 8.4 percent – 7.9 percent = 50 bps. Using the approximation, however, made him overstate the error by 25 percent.

More importantly, the approximation yields especially bad results for volatile investments, such as stocks: The correct expected long-term return of the stock portfolio is, as noted above, 10.57 percent, not 10.20 percent.

In short, because an exact and simple formula exists for computing long-term expected rates of return (Equation 12), practitioners have no reason to use the faulty approximation.

Shape of the Distribution

The hypothesis of a lognormal distribution for the yearly dollar return, $X_{t-1,t}$, implies that the continuously compounded yearly rate of return, $\log X_{t-1,t}$, is normal. This hypothesis stems from one of the most important results (if not the most important result) of statistics, the Central Limit Theorem (CLT).

Consider the daily continuously compounded rate of return on an asset, $\log(S_j/S_{j-1})$. Suppose we have no indication as to the nature of the probability distribution of the $\log(S_j/S_{j-1})$ variables. Assume, however, that these variables are independent of one another and that they have finite variance. Suppose also that we can, in the usual way, estimate both $E[\log(S_j/S_{j-1})] \equiv m$ and $\text{var}[\log(S_j/S_{j-1})] \equiv s^2$ from our sample. Assume a year contains v trading days (for instance, $v = 250$). Then,

$$\log\left(\frac{S_t}{S_{t-1}}\right) = \sum_{j=1}^{v} \log\left(\frac{S_j}{S_{j-1}}\right). \tag{16}$$

That is, the continuously compounded yearly return is simply the sum of the continuously compounded daily rates of return.

The CLT ensures that $\log(S_t/S_{t-1})$ converges toward a normal variable when v becomes large (250 is very large) with mean $vm \equiv \mu$ and variance $vs^2 \equiv \sigma^2$.

The important point is that the theorem applies whatever the probability distribution of $\log(S_j/S_{j-1})$. For the CLT to apply, only independence and finite variance of each $\log(S_j/S_{j-1})$ are needed. Therefore,

$$\begin{aligned} \log\left(\frac{S_t}{S_{t-1}}\right) &= \log(X_{t,t-1}) \\ &\sim N(vm, vs^2) \\ &\equiv N(\mu, \sigma^2), \end{aligned} \tag{17}$$

hence, the vital hypothesis of the normality of the continuously compounded yearly return $\log X_{t-1,t}$.

The CLT at Work for You Again

Suppose you are not quite sure about the exact nature of the probability distribution of the yearly continuously compounded rate of return, $\log X_{t-1,t}$. In particular, you are not sure that it is normal, which implies that you are not sure about the lognormality of $X_{t-1,t}$. Nevertheless, you perform the calculations for the long-term expected return, $E(R_{0,n})$, under the assumption that $X_{t-1,t}$ is lognormal.

Now suppose you want to know the kind of error you might have made by relying on a distribution that was not necessarily the true one. To find out, you conduct the following experiment: Assume that the true probability distribution of $\log X_{t-1,t}$ is *uniform* rather than normal. As shown in the appendix, this assumption implies that the underlying probability density of $X_{t-1,t}$ is hyperbolic—which is quite a departure from the lognormal distribution—and that the expected value of $1 + R_{0,n}$ (denoted $Y_{0,n}$) is therefore

$$\begin{aligned} &E(Y_{0,n}) | \log X_{t-1,t} \sim \text{uniform}[a,b] \\ &= \left(\frac{n}{2\sqrt{3}\sigma}\left\{\exp\left[\left(\mu + \sqrt{3}\sigma\right)/n\right] - \exp\left[\left(\mu - \sqrt{3}\sigma\right)/n\right]\right\}\right)^n \\ &= \left\{\frac{n}{b-a}\left[\exp(b/n) - \exp(a/n)\right]\right\}^n, \end{aligned} \tag{18}$$

where $a = \mu - \sqrt{3}\sigma$ and $b = \mu + \sqrt{3}\sigma$.

This formula looks exceedingly different from the result corresponding to the normal distribution for $\log X_{t-1,t}$:

$$\begin{aligned} E(Y_{0,n}) | \log X_{t-1,t} &\sim N(\mu, \sigma^2) \\ &= \exp\left[\mu + \left(\frac{\sigma^2}{2n}\right)\right] \\ &= \exp\left[\left(\frac{b+a}{2}\right) + \frac{(b-a)^2}{24n}\right], \end{aligned} \tag{19}$$

which applies in the case of the normal distribution.

Now apply Equations 18 and 19 to the following example: Let $E(\log X_{t-1,t}) = \mu = 10$ percent and $\sigma(\log X_{t-1,t}) = \sigma = 20$ percent. Surprisingly, when the horizon is as short as *two years, the first four digits* of the long-term expected dollar return, $Y_{0,n}$, are the same in both cases (see **Table 1**). From five years onward, six digits are the same.

Table 1. Comparison of $E(R_{0,n})$ when $\log X_{t-1,t}$ Is Either Normal or Uniform
($E[\log X_{t-1,t}] = 10$ percent; $\sigma[\log X_{t-1,t}] = 20$ percent)

Horizon, n (years)	$E(R_{0,n})$ when $\log X_{t-1,t} \sim$ Normal	$E(R_{0,n})$ when $\log X_{t-1,t} \sim$ Uniform
2	0.116278	0.116267
4	0.110711	0.110709
6	0.108861	0.108861
8	0.107937	0.107937
10	0.107383	0.107383
20	0.106277	0.106277
30	0.105908	0.105908
100	0.105392	0.105392

The explanation of this remarkable result lies, of course, in the strength of the CLT. Indeed, the geometric average of the dollar yearly returns can be written as

$$\begin{aligned} Y_{0,n} &= \prod_{t=1}^{n} X_{t-1,t}^{1/n} \\ &= \exp\left[\left(\frac{1}{n}\right)\sum_{t=1}^{n} \log X_{t-1,t}\right], \end{aligned} \tag{20}$$

and the CLT can be applied to the power of the exponential. Thus, $Y_{0,n}$ is approximately

$$Y_{0,n} \approx \exp\left[E(\log X_{t-1,t}) + \frac{\sigma(\log X_{t-1,t})}{\sqrt{n}} Z\right], Z \sim N(0,1), \tag{21}$$

and the expected value of $Y_{0,n}$ is[5]

$$E(Y_{0,n}) \approx \exp\left[E(\log X_{t-1,t}) + \frac{\sigma^2(\log X_{\log\, t-1,t})}{2n}\right]. \tag{22}$$

In most cases (this example is such a case), the average in the power of the exponential converges extremely quickly.[6] Hence, the result here: The distribution of $Y_{0,n}$ becomes lognormal early in the investment period even if the continuously compounded return is far from normal (a uniform distribution in this example).

Determining the limit of the long-term dollar return when the horizon tends to infinity, which was promised earlier, is now easy—thanks again to the CLT. Indeed, Equation 22 makes immediately clear that $E(Y_{0,n})$ converges toward $\exp[E(\log X_{t-1,t})]$, which is none other than the geometric mean of the distribution of $X_{t-1,t}$.[7] In the example here, the distribution is hyperbolic if $\log X_{t-1,t}$ is uniform and lognormal if $\log X_{t-1,t}$ is normal. This property can be extended to *all* moments of $Y_{0,n}$: Each kth moment of $Y_{0,n}$ tends toward the kth geometric moment of the yearly dollar returns distribution.[8]

Conclusion

What matter most when forecasting expected long-term returns are the two first moments of the yearly returns, *not* their probability distributions. Whenever you assume yearly returns to be independent and to have finite variance, do not worry about their probability densities, and let the Central Limit Theorem do the work for you.

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Appendix

The expected long-term return when the continuously compounded yearly return is uniform is determined as follows:

If $\log X_{t-1,t}$ is uniform on $[a, b]$, the following relationships exist:

$$\begin{aligned} E(\log X_{t-1,t}) &\equiv \mu \\ &= \frac{a+b}{2} \end{aligned} \tag{A1}$$

$$\begin{aligned} \sigma(\log X_{t-1,t}) &\equiv \sigma \\ &= \frac{b-a}{2\sqrt{3}}. \end{aligned} \tag{A2}$$

Conversely, a and b are related to μ and σ by

$$a = \mu - \sqrt{3}\sigma \tag{A3}$$

and

$$b = \mu + \sqrt{3}\sigma. \tag{A4}$$

For simplicity, denote $U \equiv \log X_{t-1,t}$. If $h(u)$ is the density function of U and $f(x)$ is the density function of $X_{t-1,t}$, the well-known relationship holds:

$$f(x) = h(u)\left|\frac{du}{dx}\right| = \left(\frac{1}{b-a}\right)\left(\frac{1}{x}\right);$$

hence, the density function of $X_{t-1,t}$ is a hyperbola defined between $\exp(a)$ and $\exp(b)$. Because of the independence of the $X_{t-1,t}$'s, the expected value of the n-year dollar return $Y_{0,n}$ can be written as

$$E(Y_{0,n}) = E\left(\prod_{t=1}^{n} X_{t-1,t}^{1/n}\right) = \prod_{t=1}^{n} E\left(X_{t-1,t}^{1/n}\right) = \left[E\left(X_{t-1,t}^{1/n}\right)\right]^n.$$

Now evaluate each $E\left(X_{t-1,t}^{1/n}\right)$ in this case:

$$\begin{aligned} E(X_{t-1,t}^{1/n}) &= E\left\{\exp\left[\left(\frac{1}{n}\right)\log X_{t-1,t}\right]\right\} \\ &= E\left[\exp\left(\frac{1}{n}\right)U\right] \\ &= \int_a^b \exp\left[\left(\frac{1}{n}\right)u\right]\left(\frac{1}{b-a}\right)du \\ &= \frac{n}{b-a}\left[\exp\left(\frac{b}{n}\right) - \exp\left(\frac{a}{n}\right)\right]. \end{aligned} \tag{A5}$$

Finally,

$$E(Y_{0,n}) = \left\{\frac{n}{b-a}\left[\exp\left(\frac{b}{n}\right) - \exp\left(\frac{a}{n}\right)\right]\right\}^n, \tag{A6}$$

which is the first part of Equation 18 in the text; the second part of Equation 18 is obtained by plugging Equations A3 and A4 into Equation A6.

Notes

1. It is unfortunate that MacBeth had recourse to this approximation, because the thrust of his article was absolutely in the right direction: It cautioned the reader against the far too frequent error of confusing a portfolio's long-term expected return with the weighted average of the expected long-term rates of each of the portfolio's components. As will be shown later, the approximation had the effect in his example of overstating somewhat the size of the error made by using the weighted average of the components' rates of return—a further reason *not* to use it.
2. These hypotheses have profound support, which will be explained later in the article.
3. A very efficient way to perform this type of calculation can be applied throughout this article: Observe that the first moment of S_t/S_{t-1}—namely, $E[\exp(\mu + \sigma Z)]$, with $Z \sim N(0,1)$—is simply $\exp(\mu) \times E[\exp(\sigma Z)]$, a linear transformation of the moment-generating function of the unit normal distribution, where σ plays the role of the parameter. Therefore, the first moment is equal to $\exp(\mu) \times \exp(\sigma^2/2) = \exp(\mu+\sigma^2/2)$. This procedure can be generalized: Let the kth moment of a lognormal variable be $E\{\exp[k(\mu+\sigma Z)]\} = \exp(k\mu) \times E[\exp(k\sigma Z)]$. This moment is simply a linear transformation of the moment-generating function of Z, where $k\sigma$ now plays the role of the parameter; hence, it is equal to $\exp[k\mu + (1/2)k^2\sigma^2]$. For instance, the second moment is equal to $\exp(2\mu + 2\sigma^2)$, and the variance of $X_{t-1,t} = S_t/S_{t-1}$ becomes $\exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) = \exp(2\mu + \sigma^2) \times [\exp(\sigma^2) - 1]$, as stated in Equation 4.
4. For an approach entirely consistent with these formulas, see Ibbotson (1996).
5. To see this result, again use the fact that $E(Y_{0,n})$ is a linear transformation of the moment-generating function of Z, where $[\sigma(\log X_{t-1,t})]/n$ plays the role of the parameter.
6. For a good illustration of the power of the CLT, see the spectacular diagrams in Pitman (1993, pp. 199–202).
7. See Hines (1983) for a discussion of this property.
8. For a definition and properties of geometric moments of order k, see de La Grandville (1997).

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