Aix-Marseille université

Lecture:
Representation and approximation of structured data

Kai Schneider

Master 1 Data Science, 2023/2024

Ref: from Vetterli & Goyal, ‘Teaching Signal Processing with Geometry’

Textbook: M. Vetterli, J. Kovacevic and V. Goyal, Foundations of Signal Processing, Cambridge University Press, 2014.
http://fourierandwavelets.org/

Organization:

Elevator talks (today, 2?)

Ametice: for the slides

Homework to do. See upcoming execise sheets

Email addresses? Did you get the zoom link?

Faculté des Sciences Aix-Marseille Université

|]-M1-DS Representation and approximation of ...

Accueil Tableau de bord Mes cours Cours actuel efoliAM Étudiants Enseignants

Mes cours › Sciences › Master › Mathématiques appliquées, statistique › Data Science (DS) › [21-22]-M1-DS RASD

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Outline

Modifier

Summary of the lecture and references Modifier

Overview

Goals of the lecture

- See that geometric notions unify (simplify!) signal processing
- Learn/review basics of Hilbert space view
- See Hilbert space view in action
- Learn about textbooks *Foundations of Signal Processing* and *Fourier and Wavelet Signal Processing*

Structure of the tutorial:

- Developing unified view of signal processing
- Hilbert space tools—Part I: Basics through projections
    - A few key results (best approximation and the projection theorem)
- Hilbert space tools—Part II: Bases through discrete-time systems
- Example lecture: Sampling made easy
- Textbooks and related materials

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Unifying principles

Signal processing has various dichotomies

- continuous time vs. discrete time
- infinite intervals vs. finite intervals
- periodic vs. aperiodic
- deterministic vs. stochastic

Each can placed in a common framework featuring geometry

Example payoffs:

- Unified understanding of best approximation (projection theorem)
- Unified understanding of Fourier domains
- Unified understanding of signal expansions (including sampling)

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Unifying framework: Hilbert spaces

Examples of Hilbert spaces:

- finite-dimensional vectors (basic linear algebra)
- sequences on $\{\ldots, -1, 0, 1, \ldots\}$ (discrete-time signals)
- sequences on $\{0, 1, \ldots, N-1\}$ (N-periodic discrete-time signals)
- functions on $(-\infty, \infty)$ (continuous-time signals)
- functions on $[0, T]$ (T-periodic continuous-time signals)
- scalar random variables

More abstraction. More mathematics. More difficult?

- With framework in place, can go farther, faster
- Leverage "real world" geometric intuition

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Mathematical rigor

Everything should be made as simple as possible, but no simpler.
– Common paraphrasing of Albert Einstein

Make everything as simple as possible without being wrong.
– Our variant for teaching

- Correct intuitions are separate from functional analysis details
- Teach the difference among
  - rigorously true, with elementary justification
  - rigorously true, justification not elementary (e.g., Poisson sum formula)
  - convenient and related to rigorous statements (e.g., uses of Dirac delta)

. . . if whether an airplane would fly or not depended on whether some function . . . was Lebesgue but not Riemann integrable, then I would not fly in it.
– Richard W. Hamming

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Vector spaces

A vector space generalizes easily beyond the $\mathbb{R}^2$ Euclidean plane

Axioms

- A vector space over a field of scalars $\mathbb{C}$ (or $\mathbb{R}$) is a set of vectors V together with operations
  - vector addition: $V \times V \to V$
  - scalar multiplication: $\mathbb{C} \times V \to V$

  that satisfy the following axioms:

  1. $x + y = y + x$
  2. $(x + y) + z = x + (y + z)$
  3. $\exists\, \mathbf{0} \in V$ s.t. $x + \mathbf{0} = x$ for all $x \in V$
  4. $\alpha(x + y) = \alpha x + \alpha y$
  5. $(\alpha + \beta)x = \alpha x + \beta x$
  6. $(\alpha\beta)x = \alpha(\beta x)$
  7. $0x = \mathbf{0}$ and $1x = x$

Vector spaces

Examples

- $\mathbb{C}^N$: complex (column) vectors of length N
- $\mathbb{C}^{\mathbb{Z}}$: sequences – discrete-time signals

  (write as infinite column vector)
- $\mathbb{C}^{\mathbb{R}}$: functions – continuous-time signals
- polynomials of degree at most K
- scalar random variables
- discrete-time stochastic processes

Vector spaces

Key notions

- Subspace
  - $S \subseteq V$ is a subspace when it is closed under vector addition and scalar multiplication:
    - For all $x, y \in S$, $x + y \in S$
    - For all $x \in S$ and $\alpha \in \mathbb{C}$, $\alpha x \in S$
- Span
  - S: set of vectors (could be infinite)
  - $\text{span}(S)$ = set of all finite linear combinations of vectors in S:

$$S = \left\{ \sum_{k=0}^{N-1} \alpha_k \varphi_k \mid \alpha_k \in \mathbb{C},\ \varphi_k \in S \text{ and } N \in \mathbb{N} \right\}$$

  - $\text{span}(S)$ is always a subspace

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Vector spaces

Key notions

- Linear independence
  - $S = \{\varphi_k\}_{k=0}^{N-1}$ is linearly independent when:

$$\sum_{k=0}^{N-1} \alpha_k \varphi_k = 0 \text{ only when } \alpha_k = 0 \text{ for all } k$$

  - If S is infinite, we need every finite subset to be linearly independent
- Dimension
  - $\dim(V) = N$ if V contains a linearly independent set with N vectors and every set with $N + 1$ or more vectors is linearly dependent
  - V is infinite dimensional if no such finite N exists

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Inner products

Inner products generalize angles (especially right angles) and orientation

Definition (Inner product)

- An inner product on vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying
  1. Distributivity: $\langle x + y, z \rangle \; = \; \langle x, z \rangle + \langle y, z \rangle$
  2. Linearity in the first argument: $\langle \alpha x, y \rangle \; = \; \alpha \langle x, y \rangle$
  3. Hermitian symmetry: $\langle x, y \rangle^* \; = \; \langle y, x \rangle$
  4. Positive definiteness: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$

- Note: $\langle x, \alpha y \rangle \; = \; \alpha^* \langle x, y \rangle$

Inner products

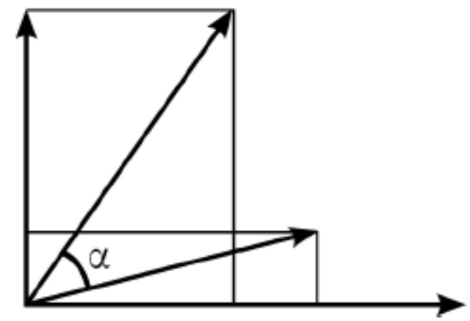
Examples

- On $\mathbb{C}^N$: $\langle x, y \rangle = \sum_{n=0}^{N-1} x_n y_n^* = y^* x$
- On $\mathbb{C}^{\mathbb{Z}}$: $\langle x, y \rangle = \sum_{n \in \mathbb{Z}} x_n y_n^* = y^* x$
- On $\mathbb{C}^{\mathbb{R}}$: $\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) \, dt$
- On $\mathbb{C}$-valued random variables: $\langle \mathrm{x}, \mathrm{y} \rangle = \mathrm{E}[\, \mathrm{x} \mathrm{y}^* \,]$

Geometry in inner product spaces

Drawn in $\mathbb{R}^2$ and true in general:

- $$\begin{aligned} \langle x, y \rangle &= x_1 y_1 + x_2 y_2 \\ &= \|x\| \, \|y\| \cos\alpha \\ &= \text{product of 2-norms times the cos of the angle between the vectors} \end{aligned}$$



- $\langle x, e_1 \rangle = x_1 = \|x\| \cos\alpha_x$

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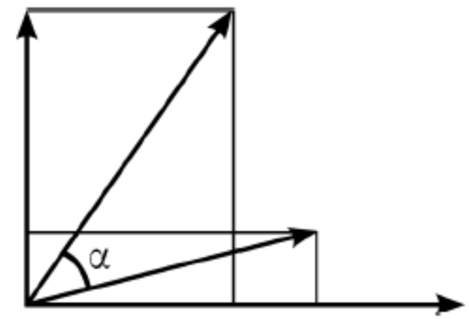

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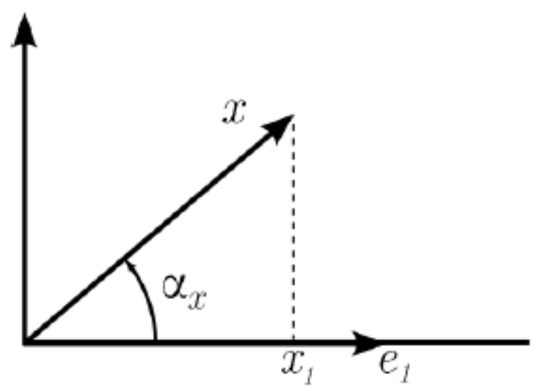

Orthogonality

Let $S = \{\varphi_i\}_{i \in \mathcal{I}}$ be a set of vectors

Definition (Orthogonality)

- x and y are orthogonal when $\langle x, y \rangle = 0$ written $x \perp y$
- S is orthogonal when for all $x, y \in S,\ x \neq y$ we have $x \perp y$
- S is orthonormal when it is orthogonal and for all $x \in S,\ \langle x, x \rangle = 1$
- x is orthogonal to S when $x \perp s$ for all $s \in S$, written $x \perp S$
- S_0 and S_1 are orthogonal when every $s_0 \in S_0$ is orthogonal to S_1, written $S_0 \perp S_1$

Right angles (perpendicularity) extends beyond Euclidean geometry

Norm

Norms generalize length in ordinary Euclidean space

Definition (Norm)

- A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying
  1. Positive definiteness: $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$
  2. Positive scalability: $\|\alpha x\| = |\alpha| \, \|x\|$
  3. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ with equality if and only if $y = \alpha x$

- Any inner product induces a norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

- Not all norms are induced by an inner product

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Norms induced by inner products

Any inner product induces a norm: $\|x\| = \sqrt{\langle x, x\rangle}$

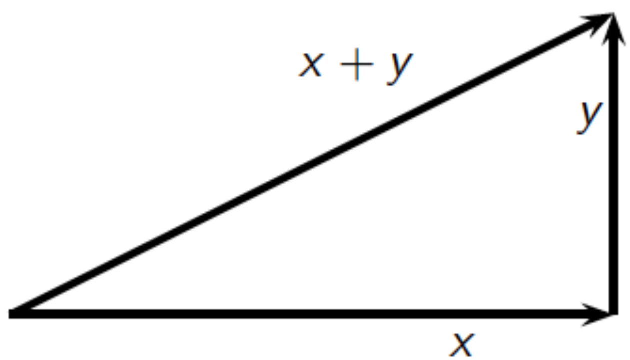
Examples

- On $\mathbb{C}^N$: $\|x\| = \sqrt{\langle x, x\rangle} = \left(\sum_{n=0}^{N-1} |x_n|^2\right)^{1/2}$
- On $\mathbb{C}^{\mathbb{Z}}$: $\|x\| = \sqrt{\langle x, x\rangle} = \left(\sum_{n\in\mathbb{Z}} |x_n|^2\right)^{1/2}$
- On $\mathbb{C}^{\mathbb{R}}$: $\|x\| = \sqrt{\langle x, x\rangle} = \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt\right)^{1/2}$
- On $\mathbb{C}$-valued random variables: $\|\mathrm{x}\| = \sqrt{\langle \mathrm{x}, \mathrm{x}\rangle} = \mathrm{E}\left[\, |\mathrm{x}|^2 \,\right]$

Norms induced by inner products

Properties

- Pythagorean theorem
  - $x \perp y \quad \Rightarrow \quad \|x+y\|^2 \;=\; \|x\|^2 + \|y\|^2$
  - $\{x_k\}_{k\in K}$ orthogonal $\quad \Rightarrow \quad \left\|\sum_{k\in K} x_k\right\|^2 = \sum_{k\in K} \|x_k\|^2$

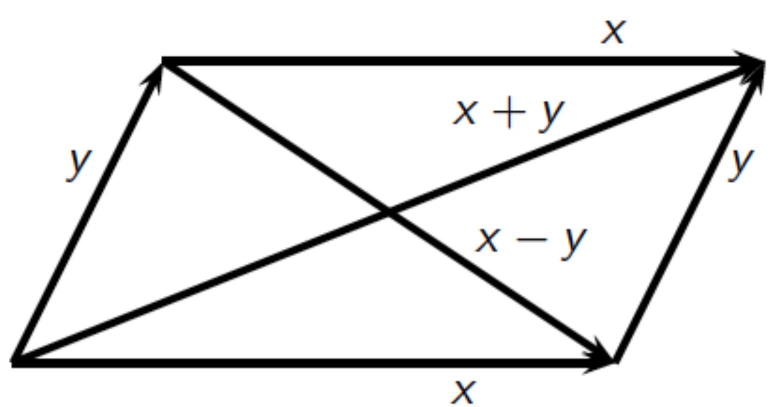


Norms induced by inner products

Properties

- Parallelogram Law

$$\|x+y\|^2 + \|x-y\|^2 \;=\; 2(\|x\|^2 + \|y\|^2)$$

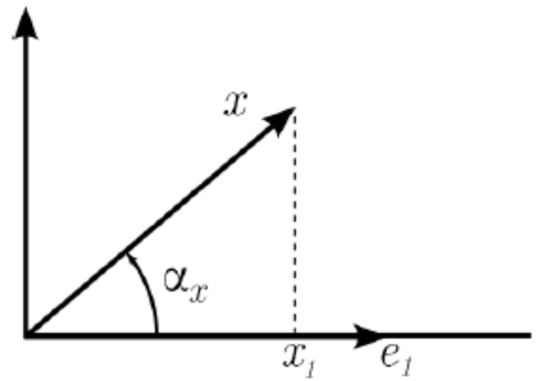


Norms induced by inner products

Properties

- Cauchy–Schwarz inequality

$$|\langle x, y\rangle| \;\leq\; \|x\|\,\|y\|$$



Norms not necessarily induced by inner products

Examples

- On $\mathbb{C}^N$: $\|x\|_p \;=\; \left(\sum_{n=0}^{N-1}|x_n|^p\right)^{1/p}, \quad p \in [1, \infty)$

- On $\mathbb{C}^{\mathbb{Z}}$: $\|x\|_p \;=\; \left(\sum_{n\in\mathbb{Z}}|x_n|^p\right)^{1/p}, \quad p \in [1, \infty)$

  $$\|x\|_\infty \;=\; \sup_{n\in\mathbb{Z}}|x_n|$$

- On $\mathbb{C}^{\mathbb{R}}$: $\|x\|_p \;=\; \left(\int_{-\infty}^{\infty}|x(t)|^p\, dt\right)^{1/p}, \quad p \in [1, \infty)$

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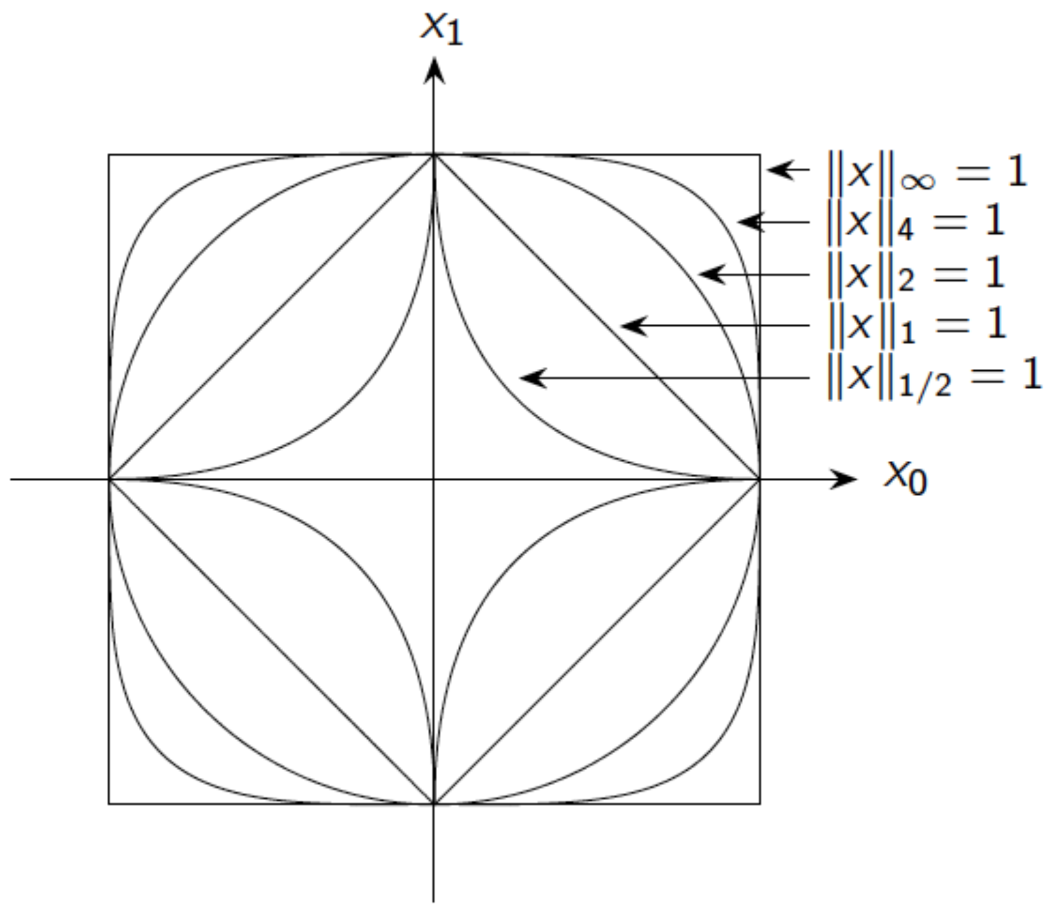
Only induced by inner products for $p = 2$

Geometry of ℓ^p: Unit balls



Valid norm (and convex unit ball) for $p \geq 1$; ordinary geometry for $p = 2$

Normed vector spaces

- A normed vector space is a set satisfying axioms of a vector space where the norm is finite
- $\ell^2(\mathbb{Z})$: square-summable sequences ("finite-energy discrete-time signals")

$$\|x\| = \left(\sum_{n\in\mathbb{Z}} |x_n|^2\right)^{1/2} < \infty$$

- $\mathcal{L}^2(\mathbb{R})$: square-integrable functions ("finite-energy continuous-time signals")

$$\|x\| = \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt\right)^{1/2} < \infty$$

x and y are the *same* when $\|x - y\| = 0$

No harm in considering only functions with finitely-many discontinuities

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Hilbert spaces: Convergence

Definition

A sequence of vectors $x_0, x_1, \ldots$ in a normed vector space V is said to converge to $v \in V$ when $\lim_{k\to\infty} \|v - x_k\| = 0$, or for any $\varepsilon > 0$, there exists K_ε such that $\|v - x_k\| < \varepsilon$ for all $k > K_\varepsilon$.

- Choice of the norm in V is key

Example

For $k \in \mathbb{Z}^+$, let

$$x_k(t) = \begin{cases} 1, & \text{for } t \in [0, 1/k]; \\ 0, & \text{otherwise.} \end{cases}$$

$v(t) = 0$ for all t. Then for $p \in [1, \infty)$,

$$\|v - x_k\|_p = \left(\int_{-\infty}^{\infty} |v(t) - x_k(t)|^p \, dt\right)^{1/p} = \left(\frac{1}{k}\right)^{1/p} \xrightarrow{k\to\infty} 0,$$

For $p = \infty$: $\|v - x_k\|_\infty = 1$ for all k

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For $p = \infty$: $\|v - x_k\|_\infty \;=\; 1 \quad$ for all k

Hilbert spaces: Completeness

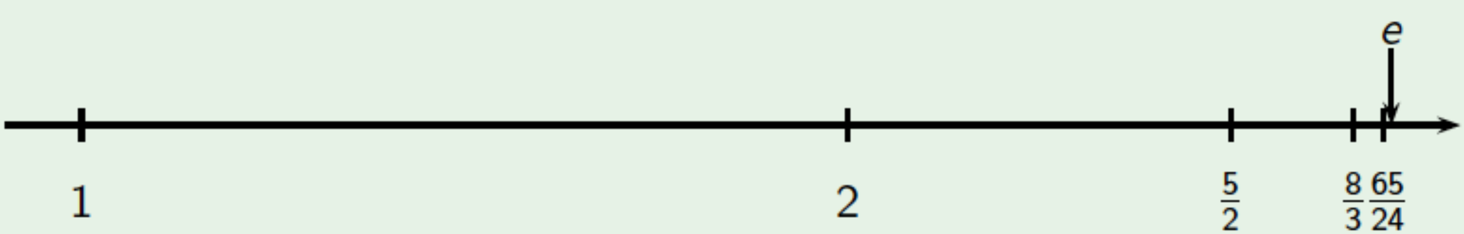
Definitions

- A sequence $\{x_n\}$ is a Cauchy sequence in a normed space when for any $\varepsilon > 0$, there exists k_ε such that $\|x_k - x_m\| < \varepsilon$ for all $k, m > k_\varepsilon$
- A normed vector space V is complete if every Cauchy sequence converges in V
- A complete normed vector space is called a Banach space
- A complete inner product space is called a Hilbert space

Hilbert spaces

Examples

- $\mathbb{Q}$ is not a complete space
  - $$\sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow \frac{\pi^2}{6} \quad \in \mathbb{R}, \notin \mathbb{Q}$$
  - $$\sum_{n=0}^{\infty} \frac{1}{n!} \rightarrow e \quad \in \mathbb{R}, \notin \mathbb{Q}$$



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Hilbert spaces

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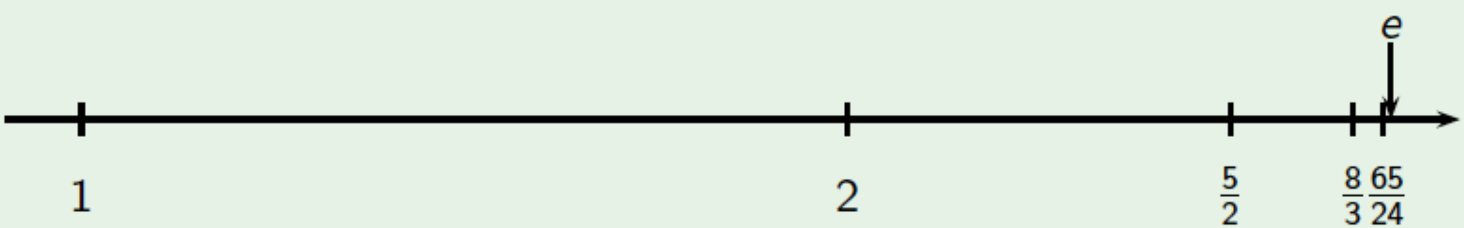

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Hilbert spaces

Examples

- All finite dimensional spaces are complete
- $\ell^p(\mathbb{Z})$ and $\mathcal{L}^p(\mathbb{R})$ are complete
  - $\ell^2(\mathbb{Z})$ and $\mathcal{L}^2(\mathbb{R})$ are Hilbert spaces
- $C^q([a,b])$, functions on $[a,b]$ with q continuous derivatives, are not complete except for $q=0$ under $\mathcal{L}^\infty$ norm

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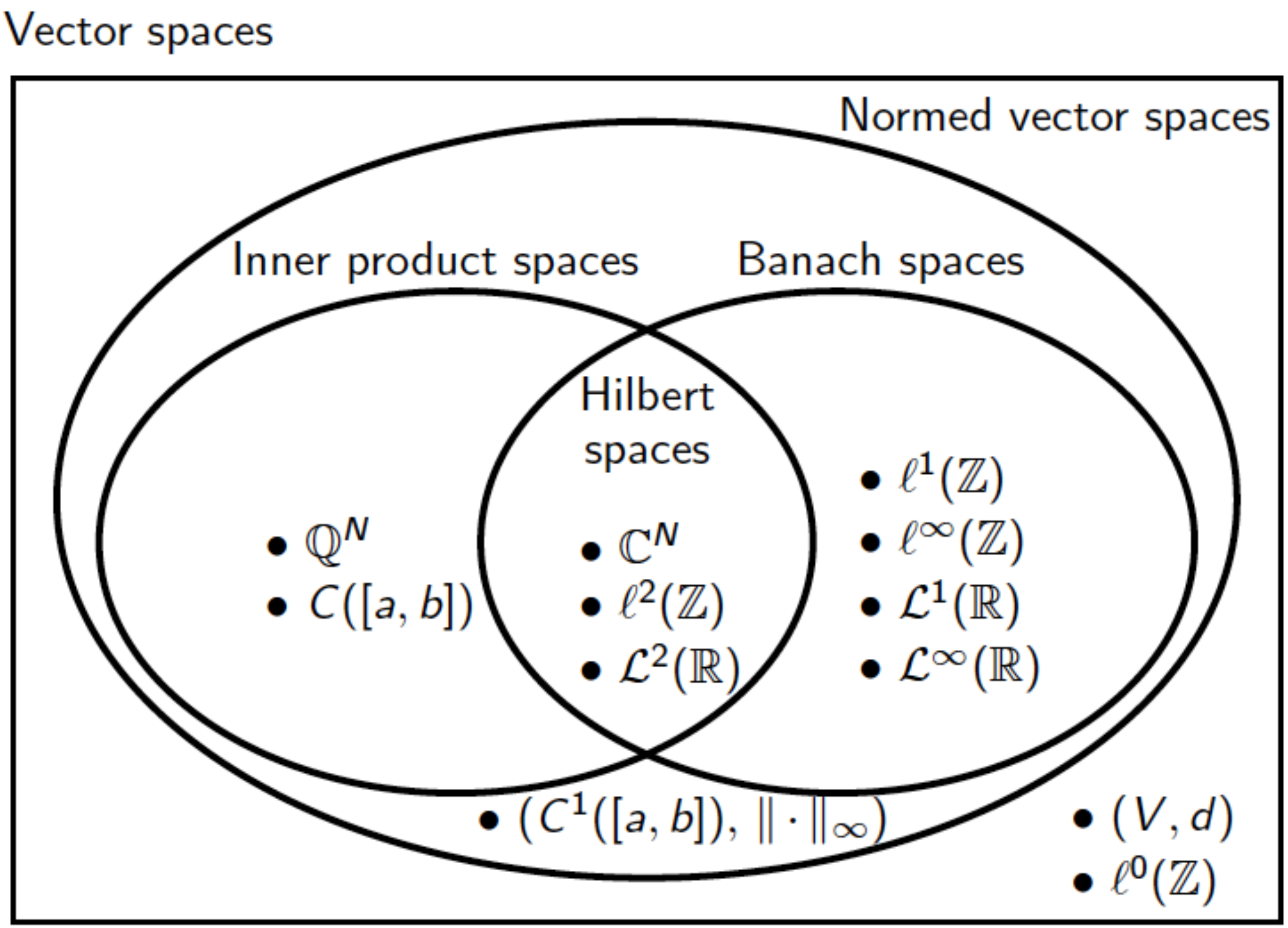
Summary on spaces



Linear operators

Linear operators generalize matrices

Definitions

- $A : H_0 \to H_1$ is a linear operator when for all $x, y \in H_0, \alpha \in \mathbb{C}$:
  1. Additivity: $A(x + y) = Ax + Ay$
  2. Scalability: $A(\alpha x) = \alpha(Ax)$
- Null space (subspace of H_0): $\mathcal{N}(A) = \{x \in H_0, Ax = 0\}$
- Range space (subspace of H_1): $\mathcal{R}(A) = \{Ax \in H_1, x \in H_0\}$
- Operator norm: $\|A\| = \sup_{\|x\|=1} \|Ax\|$
- A is bounded when: $\|A\| < \infty$
- Inverse: Bounded $B : H_1 \to H_0$ inverse of bounded A if and only if:
  $BAx = x$, for every $x \in H_0$
  $ABy = y$, for every $y \in H_1$

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- $A : H_0 \to H_1$ is a linear operator when for all $x, y \in H_0, \alpha \in \mathbb{C}$:
  1. Additivity: $A(x + y) = Ax + Ay$
  2. Scalability: $A(\alpha x) = \alpha(Ax)$
- Null space (subspace of H_0): $\mathcal{N}(A) = \{x \in H_0, Ax = 0\}$
- Range space (subspace of H_1): $\mathcal{R}(A) = \{Ax \in H_1, x \in H_0\}$
- Operator norm: $\|A\| = \sup_{\|x\|=1} \|Ax\|$
- A is bounded when: $\|A\| < \infty$
- Inverse: Bounded $B : H_1 \to H_0$ inverse of bounded A if and only if:
  $BAx = x$, for every $x \in H_0$
  $ABy = y$, for every $y \in H_1$

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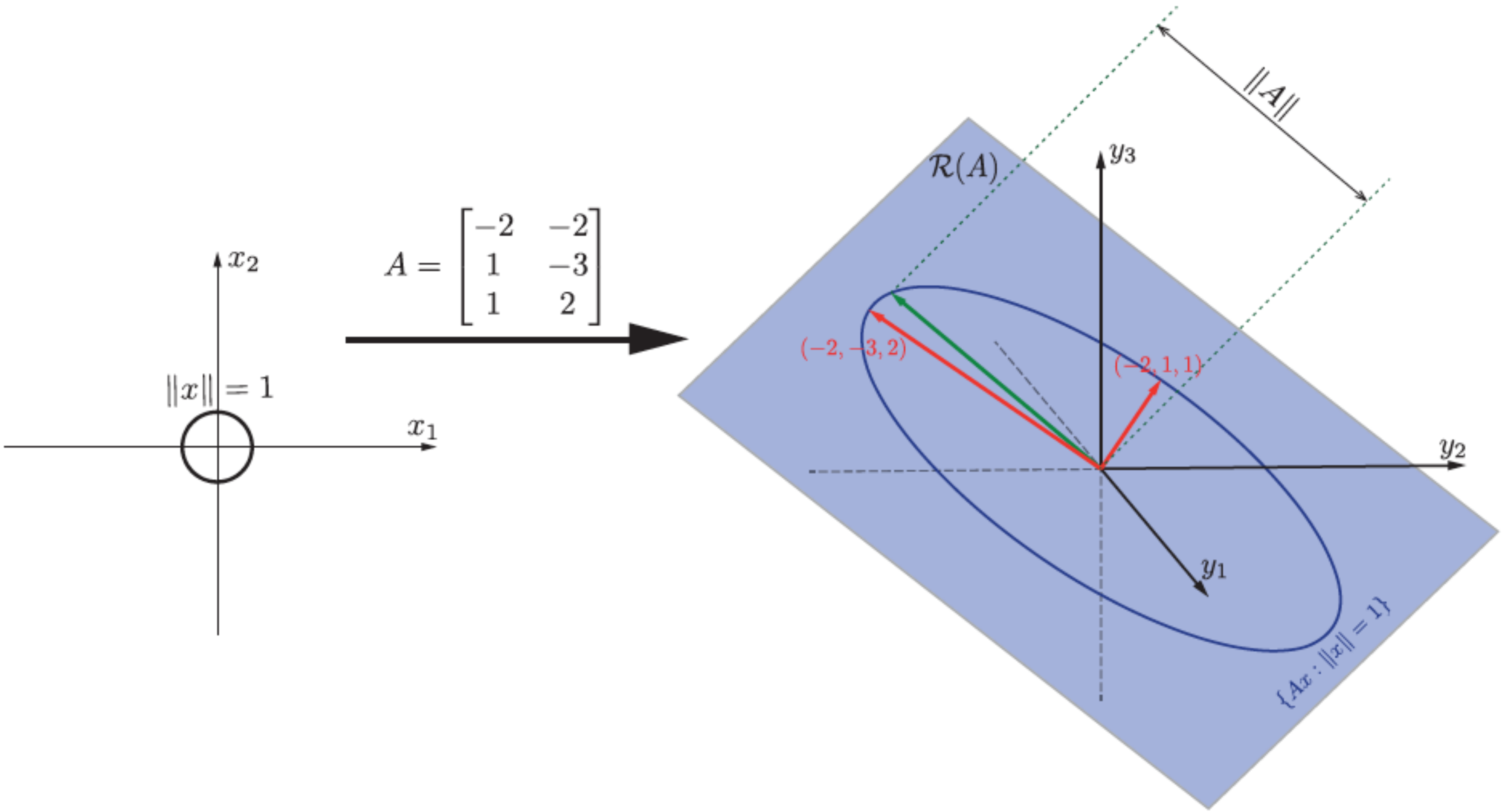
Linear operators: Illustration



- $\mathcal{R}(A)$ is the plane $5y_1 + 2y_2 + 8y_3 = 0$

Adjoint operators

Adjoint generalizes Hermitian transposition of matrices

Definition (Adjoint and self-adjoint operators)

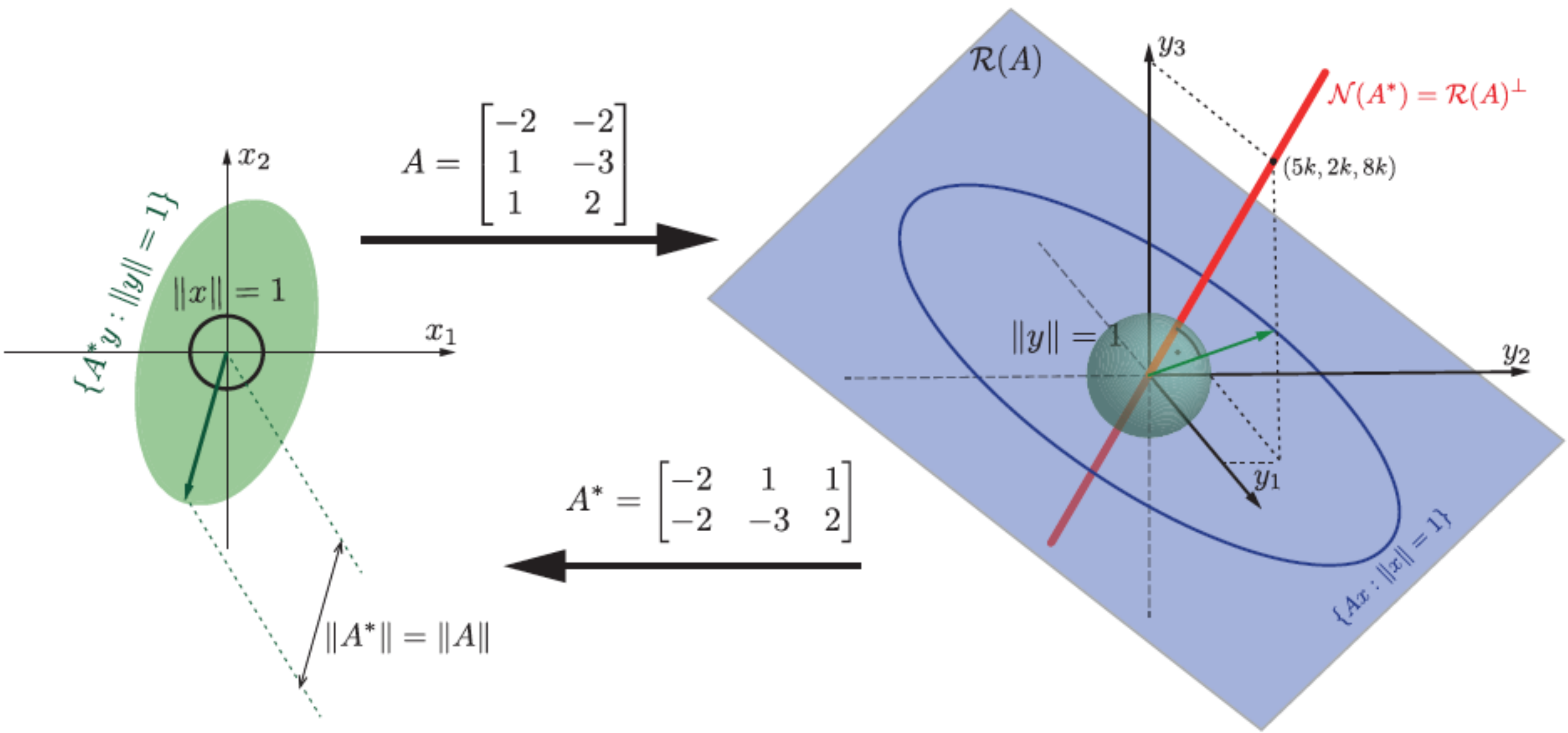
- $A^* : H_1 \to H_0$ is the adjoint of $A : H_0 \to H_1$ when

$$\langle Ax, y \rangle_{H_1} = \langle x, A^* y \rangle_{H_0} \text{ for every } x \in H_0, \ y \in H_1$$

- If $A = A^*, A$ is self-adjoint or Hermitian

- Note that $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$

Adjoint operator: Illustration



- $\mathcal{N}(A^*)$ is the line $\frac{1}{5}y_1 = \frac{1}{2}y_2 = \frac{1}{8}y_3$

Adjoint operators

Theorem (Adjoint properties)

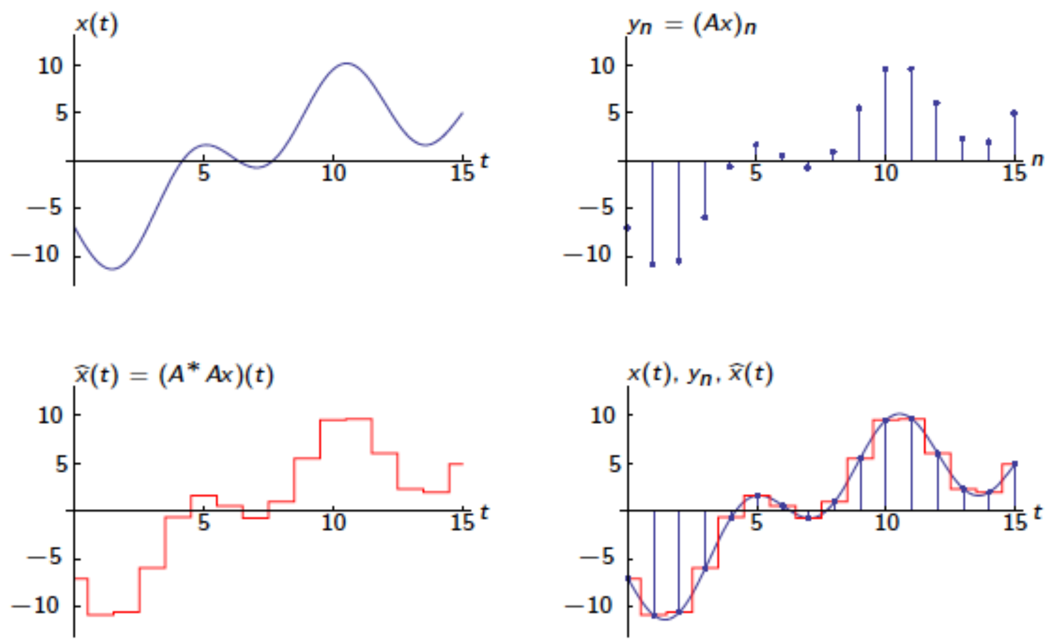
Let $A : H_0 \to H_1$ be a bounded linear operator

1. A^* *exists and is unique*
2. $(A^*)^* = A$
3. AA^* *and* A^*A *are self-adjoint*
4. $\|A^*\| = \|A\|$
5. *If* A *is invertible,* $(A^{-1})^* = (A^*)^{-1}$
6. *If* $B : H_0 \to H_1$ *is bounded,* $(A + B)^* = A^* + B^*$
7. *If* $B : H_1 \to H_2$ *is bounded,* $(BA)^* = A^* B^*$

Adjoint operators: Local averaging

$$A : \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{Z}) \qquad (Ax)_k = \int_{k-1/2}^{k+1/2} x(t)dt$$

$$\begin{aligned}
\langle Ax,\, y\rangle_{\ell^2} &= \sum_{n\in\mathbb{Z}} (Ax)_n\, y_n^* = \sum_{n\in\mathbb{Z}} \left(\int_{n-1/2}^{n+1/2} x(t)\, dt \right) y_n^* = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t) y_n^*\, dt \\
&= \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t)\widehat{x}^*(t)\, dt = \int_{-\infty}^{\infty} x(t)\widehat{x}^*(t)\, dt = \langle x,\, \widehat{x}\rangle_{\mathcal{L}^2} = \langle x,\, A^* y\rangle_{\mathcal{L}^2}
\end{aligned}$$



Unitary operators

Definition (Unitary operators)

- A bounded linear operator $A : H_0 \to H_1$ is unitary when:
  1. A is invertible
  2. A preserves inner products: $\langle Ax, Ay \rangle_{H_1} = \langle x, y \rangle_{H_0}$ for every $x, y \in H_0$

- If A is unitary, then $\|Ax\|^2 = \|x\|^2$
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Projection operators

Definition (Projection, orthogonal projection, oblique projection)

- P is idempotent when $P^2 = P$
- A projection operator is a bounded linear operator that is idempotent
- An orthogonal projection operator is a self-adjoint projection operator
- An oblique projection operator is not self adjoint

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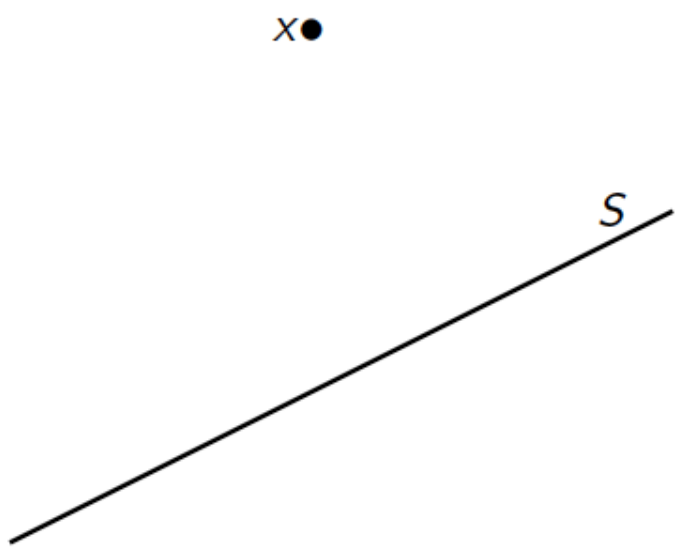
Theorem

- *If $A : H_0 \to H_1$, $B : H_1 \to H_0$ bounded and A is a left inverse of B, then BA is a projection operator. If $B = A^*$ then, $BA = A^* A$ is an orthogonal projection*

Best approximation: Euclidean geometry

- x is a point in Euclidean space
- S is a line in Euclidean space



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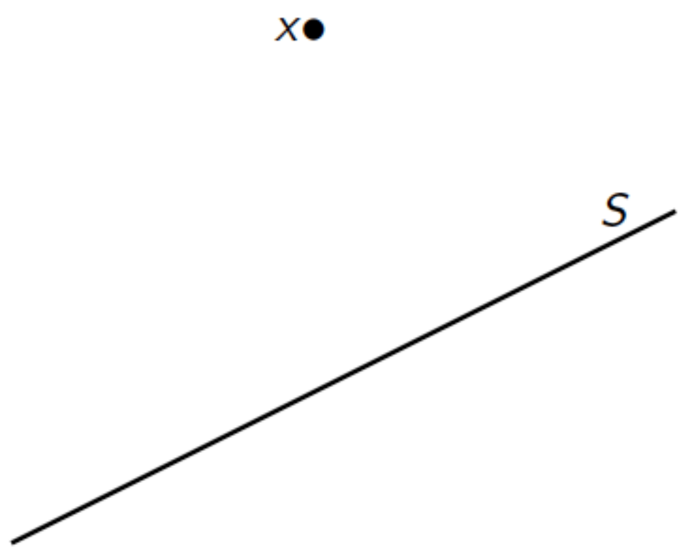

- Nearest point problem: Find $\widehat{x} \in S$ that is closest to x

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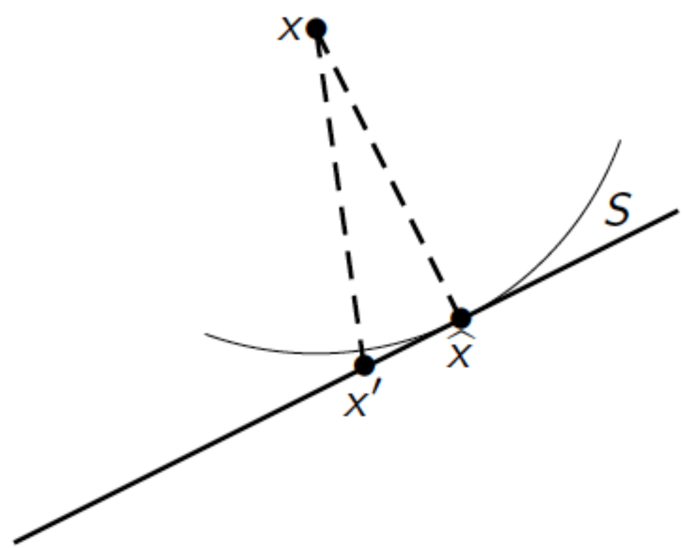

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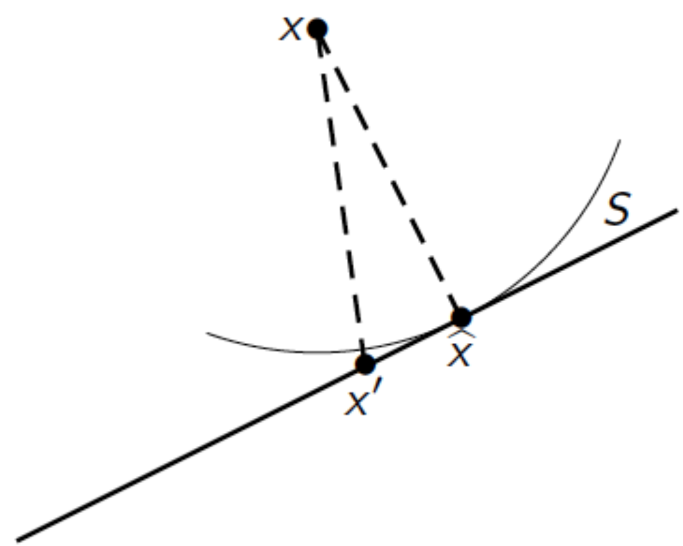

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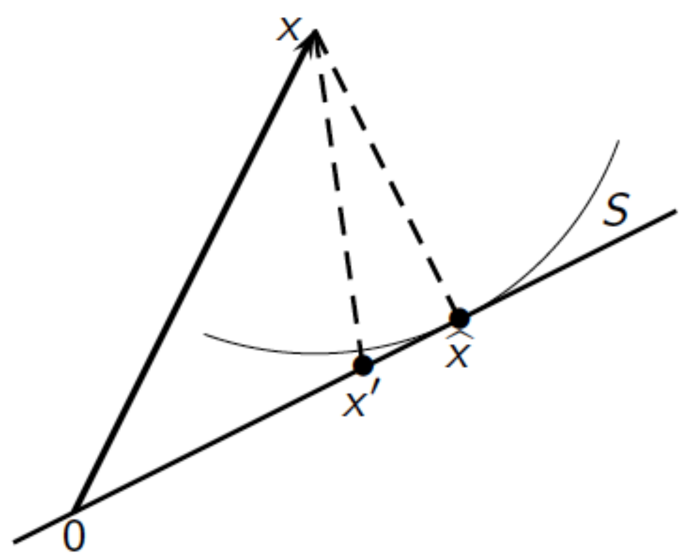

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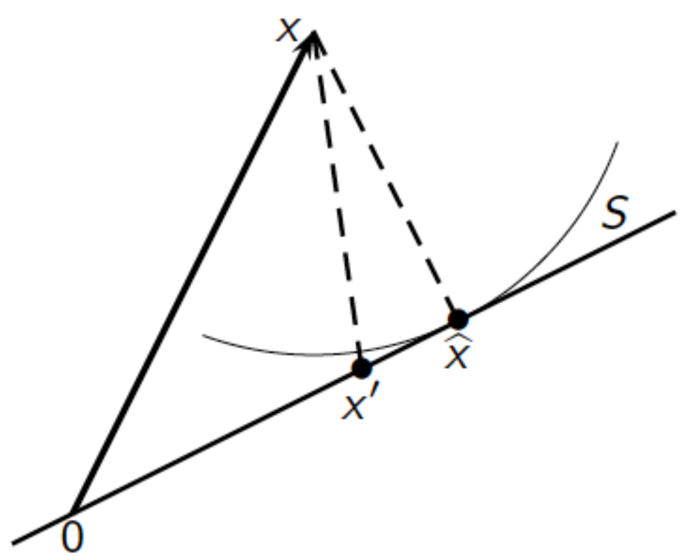
Best approximation: Hilbert space geometry

- S closed subspace of a Hilbert space
- Best approximation problem:

Find $\widehat{x} \in S$ that is closest to x

$$\widehat{x} = \operatorname*{argmin}_{s \in S} \|x - s\|$$



Best approximation by orthogonal projection

Theorem (Projection theorem)

Let S be a closed subspace of Hilbert space H and let $x \in H$.

- *Existence: There exists $\widehat{x} \in S$ such that $\|x - \widehat{x}\| \leq \|x - s\|$ for all $s \in S$*
- *Orthogonality: $x - \widehat{x} \perp S$ is necessary and sufficient to determine $\widehat{x}$*
- *Uniqueness: $\widehat{x}$ is unique*
- *Linearity: $\widehat{x} = Px$ where P is a linear operator*
- *Idempotency: $P(Px) = Px$ for all $x \in H$*
- *Self-adjointness: $P = P^*$*

All "nearest vector in a subspace" problems in Hilbert spaces are the same

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Example 1: Least-square polynomial approximation

- Consider: $x(t) = \cos(\frac{3}{2}\pi t) \in \mathcal{L}^2([0,1])$
- Find the degree-1 polynomial closest to x (in $\mathcal{L}^2$ norm)
- Solution: Use orthogonality

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$$0 \;=\; \langle x(t) - \widehat{x}(t),\, 1\rangle_t \;=\; \int_0^1 \left(\cos(\tfrac{3}{2}\pi t) - (a_0 + a_1 t)\right)\, 1\, dt \;=\; -\frac{2}{3\pi} - a_0 - \frac{1}{2}a_1$$

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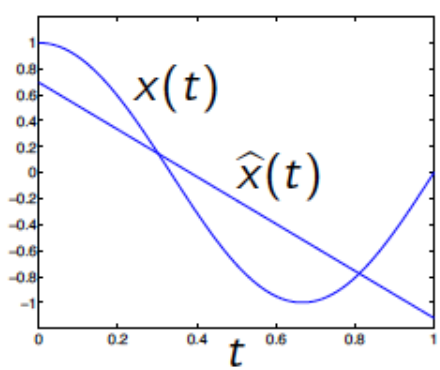
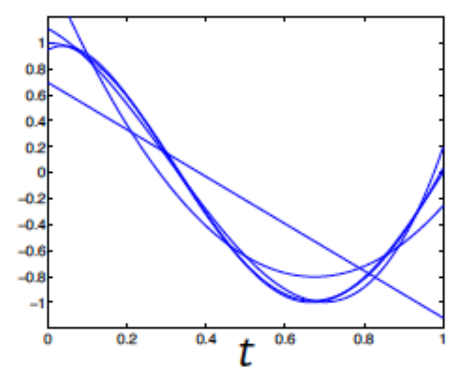


Approx. with degree 1 polynomial



Approx. with higher degree polynomials

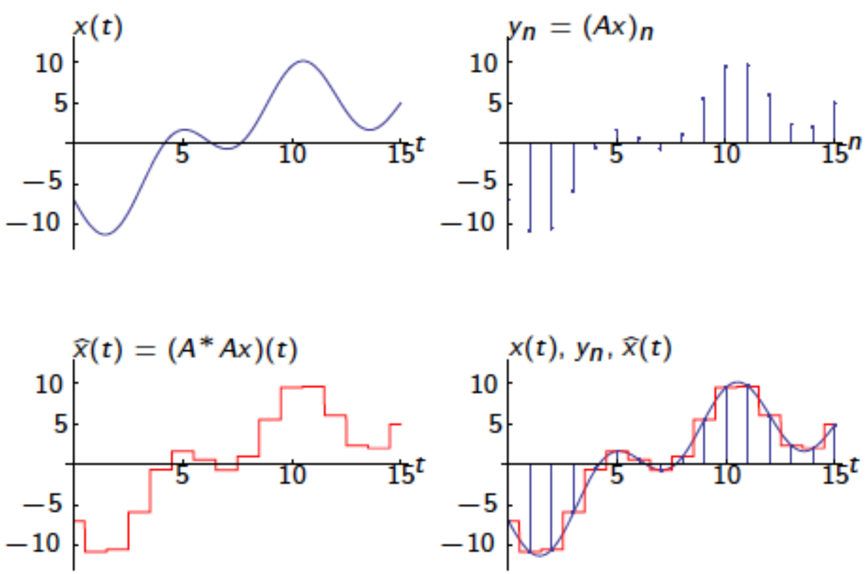
Example 2: Best piecewise-constant approximation

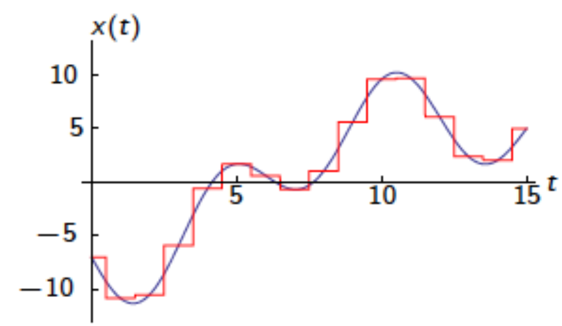
- Local averaging

$$A : \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{Z}) \qquad (Ax)_k = \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} x(t)dt$$

  has adjoint $A^* : \ell^2(\mathbb{Z}) \to \mathcal{L}^2(\mathbb{R})$ that produces staircase function

- AA^* is identity, so A^*A is orthogonal projection





Bases

Definition (Basis)

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset V$ is a basis when
  1. Φ is linearly independent and
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- Expansion formula: for any $x \in V$, $\quad x = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k$

$$\begin{aligned} \{\alpha_k\}_{k\in\mathcal{K}} &: \quad \text{is unique} \\ \alpha_k &: \quad \text{expansion coefficients} \end{aligned}$$

Example

- The standard basis for $\mathbb{R}^N$

$e_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \quad k = 0, \ldots, N-1$

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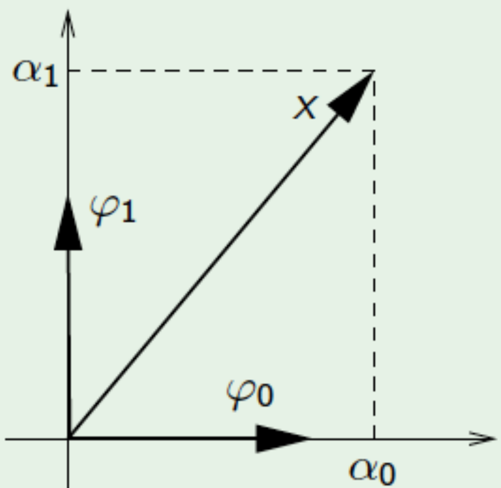
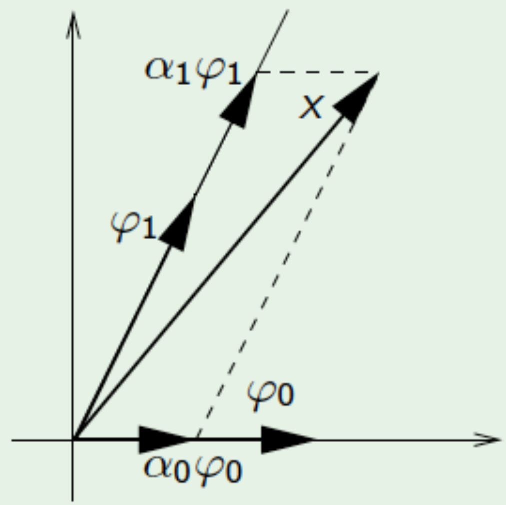
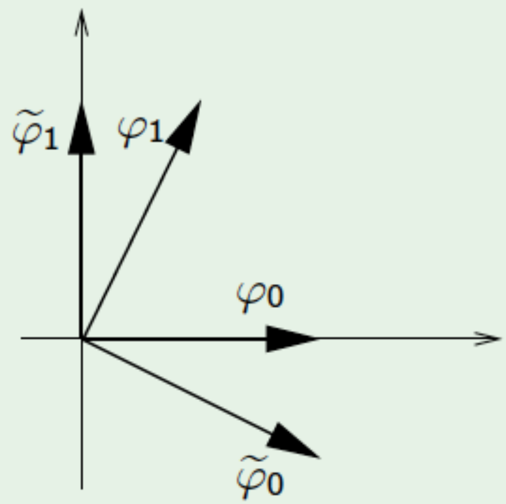
Examples



Orthonormal basis



Biorthogonal basis



. . . with dual basis

Operators associated with bases

Definition (Basis synthesis operator)

- Synthesis operator
  - $\Phi : \ell^2(\mathcal{K}) \to H \qquad \Phi\alpha = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k$
  - Adjoint: Let $\alpha \in \ell^2(\mathbb{Z})$ and $y \in H$

$$\langle \Phi\alpha,\, y\rangle \;=\; \left\langle \sum_{k\in\mathcal{K}} \alpha_k\varphi_k,\, y \right\rangle \;=\; \sum_{k\in\mathcal{K}} \alpha_k \langle y,\, \varphi_k\rangle^*$$

Definition (Basis analysis operator)

- Analysis operator
  - $\Phi^* : H \to \ell^2(\mathcal{K}) \qquad (\Phi^* x)_k = \langle x, \varphi_k\rangle, \quad k \in \mathcal{K}$

- Note that the analysis operator is the adjoint of the synthesis operator

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Orthonormal bases

Definition (Orthonormal basis)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset H$ is an orthonormal basis for H when
  1. Φ is a basis for H and
  2. Φ is an orthonormal set

     $\langle \varphi_i, \varphi_k \rangle \;=\; \delta_{i-k}$ for all $i, k \in \mathcal{K}$

- If Φ is an orthogonal set, then it is linearly independent
- If $\overline{\text{span}}(\Phi) = H$ and Φ is an orthogonal set, then Φ is an orthogonal basis for H

  If we also have $\|\varphi_k\| = 1$, then Φ is an orthonormal basis

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Orthonormal basis expansions

Definition (Orthonormal basis expansions)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ orthonormal basis for H, then for any $x \in H$:

$$\alpha_k \;=\; \langle x, \varphi_k \rangle \text{ for } k \in \mathcal{K}, \quad \text{or} \quad \alpha \;=\; \Phi^* x, \quad \text{and } \alpha \text{ is unique}$$

- Synthesis: $x \;=\; \sum_{k \in \mathcal{K}} \langle x, \varphi_k \rangle \varphi_k \;=\; \Phi \alpha \;=\; \Phi \Phi^* x$

Example

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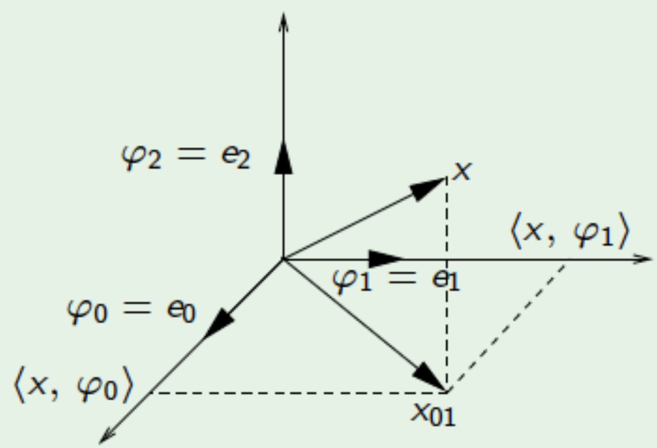

Orthonormal basis: Parseval equality

Theorem (Parseval's equalities)

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}$ *orthonormal basis for H*

$$\|x\|^2 = \sum_{k \in \mathcal{K}} |\langle x, \varphi_k \rangle|^2 = \|\Phi^* x\|^2 = \|\alpha\|^2$$

- *In general:*

$$\langle x, y \rangle = \langle \Phi^* x, \Phi^* y \rangle = \langle \alpha, \beta \rangle$$

$$\text{where} \quad \alpha_k = \langle x, \varphi_k \rangle, \quad \beta_k = \langle y, \varphi_k \rangle$$

Orthonormal basis: Parseval equality

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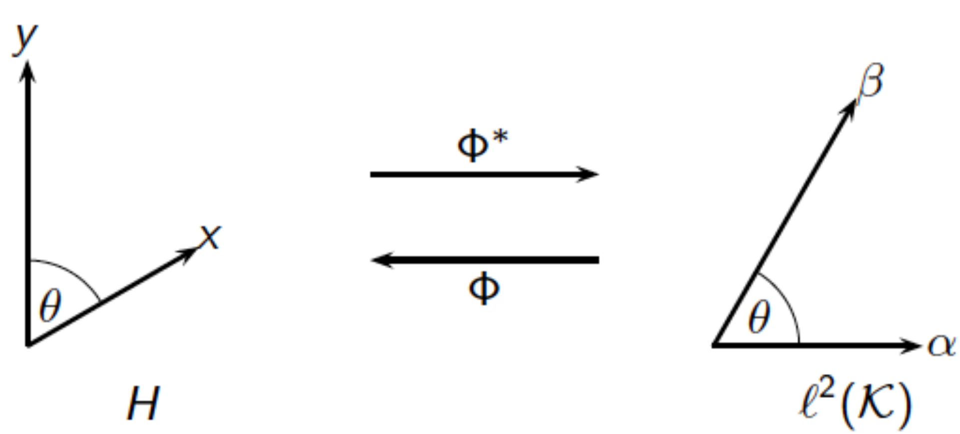
$$\|x\|^2 \;=\; \sum_{k\in\mathcal{K}} |\langle x, \varphi_k\rangle|^2 \;=\; \|\Phi^* x\|^2 \;=\; \|\alpha\|^2$$

- *In general:*

$$\langle x, y\rangle \;=\; \langle \Phi^* x, \Phi^* y\rangle \;=\; \langle \alpha, \beta\rangle$$

$$\textit{where} \quad \alpha_k \;=\; \langle x, \varphi_k\rangle, \quad \beta_k \;=\; \langle y, \varphi_k\rangle$$



Orthonormal basis: Parseval equality

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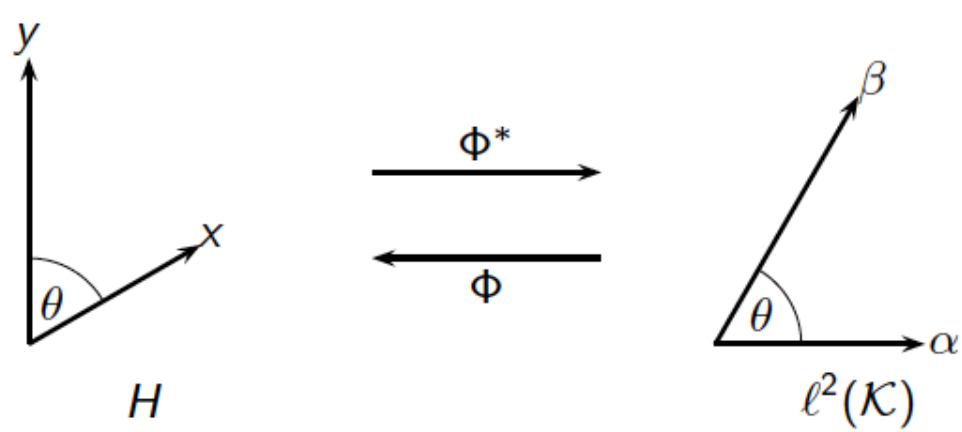

isometry

Orthogonal projection and decomposition

Theorem

- $\Phi = \{\varphi_k\}_{k \in \mathcal{I}} \subset H, \quad \mathcal{I} \subset \mathcal{K}$

  $$P_{\mathcal{I}} x = \sum_{k \in \mathcal{I}} \langle x, \varphi_k \rangle \varphi_k = \Phi_{\mathcal{I}} \Phi_{\mathcal{I}}^* x$$

  is the orthogonal projection of x onto $S_{\mathcal{I}} = \overline{\mathrm{span}}(\{\varphi_k\}_{k \in \mathcal{I}})$

- Φ induces an orthogonal decomposition

  $$H = \bigoplus_{k \in \mathcal{K}} S_{\{k\}} \quad \text{where } S_{\{k\}} = \mathrm{span}(\varphi_k)$$

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Biorthogonal pairs of bases

Definition

Biorthogonal pairs of bases

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset H$ and $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}} \subset H$ is a biorthogonal pair of bases when
  1. Φ and $\widetilde{\Phi}$ are both bases for H
  2. Φ and $\widetilde{\Phi}$ are biorthogonal: $\langle \varphi_i, \widetilde{\varphi}_k\rangle = \delta_{i-k}$ for all $i, k \in \mathcal{K}$
- Roles of Φ and $\widetilde{\Phi}$ are interchangeable

Example

$$\varphi_0 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \varphi_1 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \varphi_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \Phi = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}$$

$$\widetilde{\varphi}_0 = \begin{bmatrix}0\\1\\-1\end{bmatrix}, \quad \widetilde{\varphi}_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad \widetilde{\varphi}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \Phi^{-1} = \begin{bmatrix}0 & 1 & -1\\-1 & 1 & 0\\1 & -1 & 1\end{bmatrix}$$

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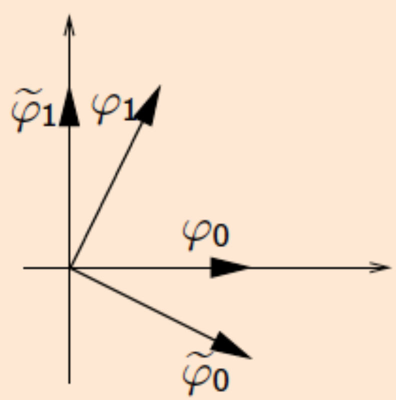
Biorthogonal basis expansion

Theorem

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$, $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *biorthogonal pair of bases for H*
- *Any $x \in H$ has expansion coefficients*

$$\alpha_k = \langle x, \widetilde{\varphi}_k\rangle \text{ for } k \in \mathcal{K}, \quad \text{or} \quad \alpha = \widetilde{\Phi}^* x$$

- *Synthesis:* $x = \sum_{k\in\mathcal{K}} \langle x, \widetilde{\varphi}_k\rangle \varphi_k = \Phi\alpha = \Phi\widetilde{\Phi}^* x$
- *Also* $x = \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \widetilde{\varphi}_k$



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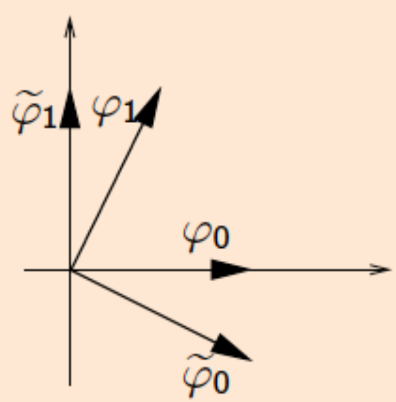

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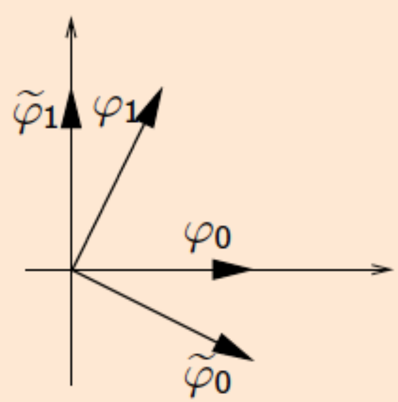
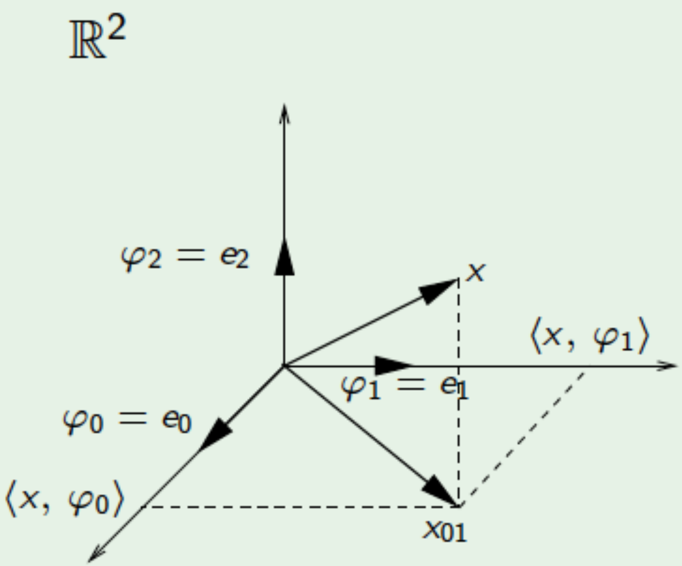


Frames: Overcomplete representations

$$x = \sum_{k \in \mathcal{J}} \langle x, \varphi_k \rangle \varphi_k = \Phi\Phi^* x \quad \text{without } \{\varphi_k\}_{k \in \mathcal{J}} \text{ being linearly independent}$$

Example



- Given $\Phi = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$
- $\{\varphi_0, \varphi_1, \varphi_2\}$ is a frame for $\mathbb{R}^2$

Change of basis: Orthonormal basis

- How are the expansion coefficients in two orthonormal bases related?
- Assume $x = \Phi\alpha = \Psi\beta$
- Then $\beta = \Psi^* x = \Psi^*\Phi\alpha$
- Change of basis from Φ to Ψ that maps α to β is the operator

$$C_{\Phi,\Psi} : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \quad \text{s.t.} \quad C_{\Phi,\Psi} = \Psi^*\Phi$$

- As a matrix

$$C_{\Phi,\Psi} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \psi_{-1} \rangle & \langle \varphi_0, \psi_{-1} \rangle & \langle \varphi_1, \psi_{-1} \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_0 \rangle & \boxed{\langle \varphi_0, \psi_0 \rangle} & \langle \varphi_1, \psi_0 \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \psi_1 \rangle & \langle \varphi_0, \psi_1 \rangle & \langle \varphi_1, \psi_1 \rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

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Matrix representation of operator: Orthonormal basis

- Let $y = Ax$ with $A : H \to H$
- How are expansion coefficients of x and y related?
  - $\{\varphi_k\}_{k \in \mathcal{K}}$ orthonormal basis of H
  - $x = \Phi\alpha, \quad y = \Phi\beta$
- Matrix representation allows computation of A directly on coefficient sequences

$$\Gamma : \ell^2(\mathcal{K}) \to \ell^2(\mathcal{K}) \quad \text{s.t.} \quad \beta = \Gamma\alpha$$

- As a matrix:

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Matrix representation of operator: Orthonormal basis

$$
\begin{tikzcd}
H & & H \\
x \arrow[rr, "A"] \arrow[d, bend right, "\Phi^*"'] & & y \arrow[d, bend right, "\Phi^*"'] \\
\alpha \arrow[rr, "\Gamma"'] \arrow[u, bend right, "\Phi"'] & & \beta \arrow[u, bend right, "\Phi"'] \\
\ell^2(\mathcal{K}) & & \ell^2(\mathcal{K})
\end{tikzcd}
$$

- When orthonormal bases are used, matrix representation of A^* is Γ^*

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Example: Averaging Operator I

Example

- Let $A : H_0 \to H_1$, $$y(t) = Ax(t) = \frac{1}{2}\int_{2\ell}^{2(\ell+1)} x(\tau)\, d\tau$$ for $2\ell \le t < 2(\ell+1)$, $\ell \in \mathbb{Z}$

  H_0: piecewise-constant, finite-energy functions with breakpoints at integers

  H_1: piecewise-constant, finite-energy functions with breakpoints at even integers

- Given $\chi_I(t) = \begin{cases} 1, & \text{for } t \in I; \\ 0, & \text{otherwise} \end{cases}$

  Let
  $$\Phi = \{\varphi_k(t)\}_{k\in\mathbb{Z}} = \{\chi_{[k,k+1)}(t)\}_{k\in\mathbb{Z}}$$
  $$\Psi = \{\psi_i(t)\}_{i\in\mathbb{Z}} = \{\tfrac{1}{\sqrt{2}}\chi_{[2i,2(i+1))}(t)\}_{i\in\mathbb{Z}}$$

  be orthonormal bases for H_0, H_1 respectively

Example: Averaging Operator II

Example (Cont.)

- $A\varphi_0(t) \;=\; \frac{1}{2}\,\chi_{[0,2)}(t) \quad\Rightarrow\quad \langle A\varphi_0,\, \psi_0\rangle \;=\; \int_0^2 \frac{1}{2}\frac{1}{\sqrt{2}}\,d\tau \;=\; \frac{1}{\sqrt{2}}$
- Then $\Gamma \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{1} & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

Simplicity of matrix representation depends on the basis!

Discrete-time systems

- A linear system $A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ has a matrix representation H with respect to the standard basis
- For a linear shift-invariant (LSI) system, the matrix H is Toeplitz:

$$y = \begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & h_0 & h_{-1} & h_{-2} & h_{-3} & h_{-4} & \cdots \\ \cdots & h_1 & h_0 & h_{-1} & h_{-2} & h_{-3} & \cdots \\ \cdots & h_2 & h_1 & \boxed{h_0} & h_{-1} & h_{-2} & \cdots \\ \cdots & h_3 & h_2 & h_1 & h_0 & h_{-1} & \cdots \\ \cdots & h_4 & h_3 & h_2 & h_1 & h_0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{H} \begin{bmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = Hx$$

Discrete-time systems

- Matrix representation of A^* is H^* [Note: using orthonormal basis]

$$
H^* = \begin{bmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & & \iddots \\
\ddots & h_0^* & h_1^* & h_2^* & h_3^* & h_4^* & \\
\ddots & h_{-1}^* & h_0^* & h_1^* & h_2^* & h_3^* & \ddots \\
\ddots & h_{-2}^* & h_{-1}^* & \boxed{h_0^*} & h_1^* & h_2^* & \ddots \\
\ddots & h_{-3}^* & h_{-2}^* & h_{-1}^* & h_0^* & h_1^* & \ddots \\
 & h_{-4}^* & h_{-3}^* & h_{-2}^* & h_{-1}^* & h_0^* & \ddots \\
\iddots & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
$$

- Adjoint of filtering by h_n is filtering by h_{-n}^*

Discrete-time systems: Periodic sequences

- For an N-periodic setting, the matrix H is circulant:

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} h_0 & h_{N-1} & h_{N-2} & \dots & h_1 \\ h_1 & h_0 & h_{N-1} & \dots & h_2 \\ h_2 & h_1 & h_0 & \dots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \dots & h_0 \end{bmatrix}}_{H} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = Hx$$

- Equally well in both cases (and continuous time as well):
  - Eigenvectors (-sequences, -signals) lead to diagonal representation of H
  - Fourier transform follows logically from the class of operators
  - Convolution theorem follows logically from the definition of the Fourier transform