Lecture:
Representation and approximation of structured data

Part 3

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Ref: from Vetterli & Goyal, 'Teaching Signal Processing with Geometry'

Textbook: M. Vetterli, J. Kovacevic and V. Goyal, Foundations of Signal Processing, Cambridge University Press, 2014.
http://fourierandwavelets.org/

Discrete Fourier Transform

Time is ordered – from past to future.
Here we consider time discrete signals, a sequence of numbers indexed by integers.
As we saw before they form a vector space.

Operators that map a sequence to a sequence are called discrete-time systems.

Example 1: infinite sequences

$$x = [\dots \quad x_{-2} \quad x_{-1} \quad \boxed{x_0} \quad x_1 \quad x_2 \quad \dots]^\top,$$

Example 2: finite sequences of length $N \in \mathbb{N}$

$$x = [\boxed{x_0} \quad x_1 \quad x_2 \quad \dots \quad x_{N-1}]^\top$$

Example 3: circular extension of finite sequences (periodic extension)

$$x = [\dots \quad x_{N-1} \quad \underbrace{\boxed{x_0} \quad x_1 \quad \dots \quad x_{N-1}}_{\text{one period}} \quad x_0 \quad x_1 \quad \dots]^\top$$

Some examples (I):

Moving average operator:

$$y_n = \frac{1}{N} \sum_{k=-(N-1)/2}^{(N-1)/2} x_{n-k}, \qquad n \in \mathbb{Z}$$

where N is a small, odd positive integer. The local average reduces variations. This simple system is linear and shift-invariant since the same local averaging is performed at al indices *n*.

Autocorrelation *a* of a sequence *x*:

$$a_n = \sum_{k\in\mathbb{Z}} x_k x^*_{k-n} = \langle x_k, x_{k-n} \rangle_k$$

The autocorrelation satisfies

$$a_n = a^*_{-n},$$

$$a_0 = \sum_{k\in\mathbb{Z}} |x_k|^2 = \|x\|^2$$

And it measures the similarity of a sequence with respect to shifts of itself. For real *x* we have,

$$a_n = \sum_{k\in\mathbb{Z}} x_k x_{k-n} = a_{-n}$$

Some examples (II):

Cross-correlation c of two sequences x and y:

$$c_n = \sum_{k\in\mathbb{Z}} x_k y^*_{k-n} = \langle x_k, y_{k-n}\rangle_k$$

and is written as $c_{x,y,n}$ to specify the sequences involved. It satisfies

$$c_{x,y,n} = \left(\sum_{k\in\mathbb{Z}} y_{k-n} x^*_k\right)^* \stackrel{(a)}{=} \left(\sum_{m\in\mathbb{Z}} y_m x^*_{m+n}\right)^* = c^*_{y,x,-n},$$

where (a) follows from the change of variable $m = k - n$

For real-valued sequences x and y we have:

$$c_{x,y,n} = \sum_{k\in\mathbb{Z}} x_k y_{k-n} = c_{y,x,-n}.$$

Convolution of two sequences

DEFINITION 3.7 (CONVOLUTION) The *convolution* between sequences h and x is defined as

$$(Hx)_n \;=\; (h * x)_n \;=\; \sum_{k\in\mathbb{Z}} x_k h_{n-k} \;=\; \sum_{k\in\mathbb{Z}} x_{n-k} h_k, \tag{3.61}$$

where H is called the *convolution operator* associated with h.

When it is not clear from the context, we will use a subscript on the convolution operator, such as $*_n$, to denote the argument over which we perform the convolution (for example, $x_{n-m} *_n h_{\ell-n} = \sum_k x_{k-m} h_{\ell-n+k}$).

Convolution: some properties

Properties The convolution (3.61) satisfies the following properties:

(i) *Connection to the inner product:*

$$(h * x)_n \;=\; \sum_{k\in\mathbb{Z}} x_k h_{n-k} \;=\; \langle x_k,\, h^*_{n-k}\rangle_k.$$

(ii) *Commutativity:*

$$h * x \;=\; x * h.$$

(iii) *Associativity:*

$$g * (h * x) \;=\; g * h * x \;=\; (g * h) * x.$$

(iv) *Deterministic autocorrelation:*

$$a_n \;=\; \sum_{k\in\mathbb{Z}} x_k x^*_{k-n} \;=\; x_n *_n x^*_{-n}.$$

(v) *Shifting:* For any $k \in \mathbb{Z}$,

$$x_n *_n \delta_{n-k} \;=\; x_{n-k}.$$

Fourier Transformation

Motivation: change the representation of a signal/data/function
from time representation into frequency representation.

To this end we use sine and cosine functions, written efficiently using the complex exponential.

For infinite sequences x_n we have:

DEFINITION 3.11 (DISCRETE-TIME FOURIER TRANSFORM) The *discrete-time Fourier transform* of a sequence x is

$$X(e^{j\omega}) = \sum_{n\in\mathbb{Z}} x_n e^{-j\omega n}, \qquad \omega \in \mathbb{R}. \tag{3.80a}$$

It exists when (3.80a) converges for all $\omega \in \mathbb{R}$; we then call it the *spectrum* of x. The *inverse DTFT* of a 2π-periodic function $X(e^{j\omega})$ is

$$x_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega, \qquad n \in \mathbb{Z}. \tag{3.80b}$$

When the DTFT exists, we denote the DTFT pair as

$$x_n \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad X(e^{j\omega}).$$

Discrete Fourier transform (I)

For finite sequences x_n of length N we have,

DEFINITION 3.16 (DISCRETE FOURIER TRANSFORM) The *discrete Fourier transform* of a length-N sequence x is

$$X_k = (Fx)_k = \sum_{n=0}^{N-1} x_n W_N^{kn}, \qquad k \in \{0, 1, \ldots, N-1\}; \tag{3.163a}$$

we call it the *spectrum* of x. The *inverse DFT* of a length-N sequence X is

$$x_n = \frac{1}{N}(F^* X)_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k W_N^{-kn}, \qquad n \in \{0, 1, \ldots, N-1\}. \tag{3.163b}$$

We denote the DFT pair as

$$x_n \overset{\text{DFT}}{\longleftrightarrow} X_k.$$

Within the definition, we have introduced $F : \mathbb{C}^N \to \mathbb{C}^N$ to represent the linear DFT operator. The relationship between the inverse and the adjoint in (3.163b) is verified shortly.

$$W_N^{kn} = \exp(j(2\pi/N)kn) \quad \text{where} \quad j = \sqrt{-1}$$

Discrete Fourier transform (II)

Matrix view The DFT expression (3.163a) is the vector–vector product

$$X_k = \begin{bmatrix} 1 & W_N^k & \dots & W_N^{(N-1)k} \end{bmatrix} x,$$

where as usual $x \in \mathbb{C}^N$ is a column vector. By stacking the results for $k \in \{0, 1, \dots, N-1\}$ to produce the column vector $X \in \mathbb{C}^N$, we have

$$X = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} x.$$

$$W_N^{kn} = \exp(j(2\pi/N)kn) \quad \text{where} \quad j = \sqrt{-1}$$

Discrete Fourier transform (III)

Thus, the matrix $F \in \mathbb{C}^{N\times N}$ introduced in (3.163a) is

$$F = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}. \qquad (3.164a)$$

Using orthogonality we can easily verify that

$$F^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}. \qquad (3.164b)$$

Thus,

$$F^{-1} = \frac{1}{N}F^*, \qquad (3.164c)$$

This shows that the DFT is a unitary operator (up to a factor) and that F is a Vandermonde matrix

Discrete Fourier transform (IV)

DFT as analysis with an orthogonal basis The expression (3.163a) is an inner product $X_k = \langle x, \varphi_k \rangle$, where

$$\varphi_k = \begin{bmatrix} 1 & W_N^{-k} & \ldots & W_N^{-(N-1)k} \end{bmatrix}^\top = \begin{bmatrix} 1 & e^{j(2\pi/N)k} & \ldots & e^{j(2\pi/N)(N-1)k} \end{bmatrix}^\top. \tag{3.165}$$

Thus, the DFT of a length-N sequence is a set of N inner products obtained by applying the analysis operator associated with the basis $\{\varphi_k\}_{k=0}^{N-1}$ of $\mathbb{C}^N$.

The basis $\{\varphi_k\}_{k=0}^{N-1}$ is orthogonal (see (3.288c) in Appendix 3.A.1), and each element has norm $\sqrt{N}$. The associated dual basis is

$$\widetilde{\varphi}_k = \frac{1}{N}\varphi_k, \qquad k \in \{0, 1, \ldots, N-1\},$$

and the inverse DFT (3.163b) is obtained by applying the synthesis operator associated with this basis.

Discrete Fourier transform (V)

Relation of the DFT to the DTFT Given a length-N sequence x to analyze, we might first turn to what we already have for infinite sequences – the DTFT. However, since x is only N-dimensional, we should not need a function of a continuous variable $X(e^{j\omega})$ to characterize it. Choosing any N distinct samples within one 2π period of $X(e^{j\omega})$ allows recovery of x and thus contains all the information present. Choosing those sampling points to be

$$\omega_k = \frac{2\pi}{N}k, \qquad k \in \{0, 1, \ldots, N-1\} \tag{3.166a}$$

gives the DFT:

$$\begin{aligned} X(e^{j\omega})\Big|_{\omega=\omega_k} &\overset{(a)}{=} X(e^{j(2\pi/N)k}) \overset{(b)}{=} \sum_{n\in\mathbb{Z}} x_n e^{-j(2\pi/N)kn} \\ &\overset{(c)}{=} \sum_{n=0}^{N-1} x_n e^{-j(2\pi/N)kn} \overset{(d)}{=} X_k, \end{aligned} \tag{3.166b}$$

where (a) follows from the choice of sampling points; (b) from the definition of the DTFT, (3.80a); (c) from x being finite of length N; and (d) from the definition of the DFT (3.163a). Thus, sampling the DTFT uniformly results in the DFT.

Properties of the DFT (I)

Linearity The DFT operator F is a linear operator, or

$$\alpha x_n + \beta y_n \overset{\text{DFT}}{\longleftrightarrow} \alpha X_k + \beta Y_k. \tag{3.167}$$

Circular shift in time The DFT pair corresponding to a circular shift in time by n_0 is

$$x_{(n-n_0) \bmod N} \overset{\text{DFT}}{\longleftrightarrow} W_N^{kn_0} X_k. \tag{3.168}$$

Circular shift in frequency The DFT pair corresponding to a circular shift in frequency by k_0 is

$$W_N^{-k_0 n} x_n \overset{\text{DFT}}{\longleftrightarrow} X_{(k-k_0) \bmod N}. \tag{3.169}$$

As for the DTFT, a shift in frequency is often referred to as *modulation*.

Circular time reversal The DFT pair corresponding to circular time reversal $x_{-n \bmod N}$ is

$$x_{-n \bmod N} \overset{\text{DFT}}{\longleftrightarrow} X_{-k \bmod N}. \tag{3.170}$$

For a real x_n, the DFT of the time-reversed version $x_{-n \bmod N}$ is X_k^*.

N.B.: Modulo (mod) is the operation of finding the remainder when you divide two numbers.

Circular convolution (I)

DEFINITION 3.9 (CIRCULAR CONVOLUTION) The *circular convolution* between length-N sequences h and x is defined as

$$(Hx)_n = (h \circledast x)_n = \sum_{k=0}^{N-1} x_k h_{(n-k) \bmod N} = \sum_{k=0}^{N-1} x_{(n-k) \bmod N} h_k, \qquad (3.71)$$

where H is called the *circular convolution operator* associated with h.

The result of a circular convolution is a length-N sequence.

THEOREM 3.10 (EQUIVALENCE OF CIRCULAR AND LINEAR CONVOLUTIONS) Linear and circular convolutions between a length-M sequence x and a length-L sequence h are equivalent when the period of the circular convolution N satisfies

$$N \geq L + M - 1. \qquad (3.73)$$

Circular convolution (II)

Matrix view: similar as for (linear) convolution the circular convolution can be written as a matrix vector product

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} h_0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ h_1 & h_0 & h_{N-1} & \cdots & h_2 \\ h_2 & h_1 & h_0 & \cdots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 \end{bmatrix}}_{H} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = Hx$$

where H is a circulant matrix with h as its first column.
Example:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & 0 & h_2 & h_1 \\ h_1 & h_0 & 0 & 0 & 0 & h_2 \\ h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix}$$

Properties of the DFT (II)

Circular convolution in time The DFT pair corresponding to circular convolution in time is

$$(h \circledast x)_n \quad \overset{\text{DFT}}{\longleftrightarrow} \quad H_k X_k. \tag{3.171}$$

Circular convolution in frequency The DFT pair corresponding to circular convolution in frequency is

$$h_n x_n \quad \overset{\text{DFT}}{\longleftrightarrow} \quad \frac{1}{N}(H \circledast X)_k. \tag{3.172}$$

The circular convolution in frequency property (3.172) is dual to the convolution in time property (3.171).

Circular deterministic autocorrelation The DFT pair corresponding to the *circular deterministic autocorrelation* of a sequence x is

$$a_n \;=\; \sum_{k=0}^{N-1} x_k x^*_{(k-n) \bmod N} \quad \overset{\text{DFT}}{\longleftrightarrow} \quad A_k \;=\; |X_k|^2 \tag{3.173}$$

and satisfies

$$A_k \;=\; A_k^*, \tag{3.174a}$$

For a real x,

$$A_k \;=\; |X_k|^2 \;=\; A_{-k \bmod N}.$$

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Properties of the DFT (III)

Circular deterministic crosscorrelation The DFT pair corresponding to the *circular deterministic crosscorrelation* of sequences x and y is

$$c_n = \sum_{k=0}^{N-1} x_k y^*_{(k-n) \bmod N} \overset{\text{DFT}}{\longleftrightarrow} C_k = X_k Y^*_k \tag{3.175}$$

and satisfies

$$C_{x,y,k} = C^*_{y,x,k}. \tag{3.176a}$$

For real x and y,

$$C_{x,y,k} = X_k Y_{-k \bmod N} = C_{y,x,-k \bmod N}. \tag{3.176b}$$

Parseval equality The DFT operator F is a unitary operator (up to scaling) and thus preserves the Euclidean norm (up to scaling); see (2.56):

$$\|x\|^2 = \sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_k|^2 = \frac{1}{N}\|X\|^2 = \frac{1}{N}\|Fx\|^2. \tag{3.179}$$

This follows from $F/\sqrt{N}$ being a unitary matrix since $F^*F = NI$.

DFT properties	Time domain	DFT domain
Basic properties		
Linearity	$\alpha x_n + \beta y_n$	$\alpha X_k + \beta Y_k$
Circular shift in time	$x_{(n-n_0) \bmod N}$	$W_N^{kn_0} X_k$
Circular shift in frequency	$W_N^{-k_0 n} x_n$	$X_{(k-k_0) \bmod N}$
Circular time reversal	$x_{-n \bmod N}$	$X_{-k \bmod N}$
Circular convolution in time	$(h \circledast x)_n$	$H_k X_k$
Circular convolution in frequency	$h_n x_n$	$\frac{1}{N}(H \circledast X)_k$
Circular deterministic autocorrelation	$a_n = \sum_{k=0}^{N-1} x_k x^*_{(k-n) \bmod N}$	$A_k = \lvert X_k \rvert^2$
Circular deterministic crosscorrelation	$c_n = \sum_{k=0}^{N-1} x_k y^*_{(k-n) \bmod N}$	$C_k = X_k Y_k^*$
Parseval equality	$\lVert x \rVert^2 = \sum_{n=0}^{N-1} \lvert x_n \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X_k \rvert^2 = \frac{1}{N} \lVert X \rVert^2$	

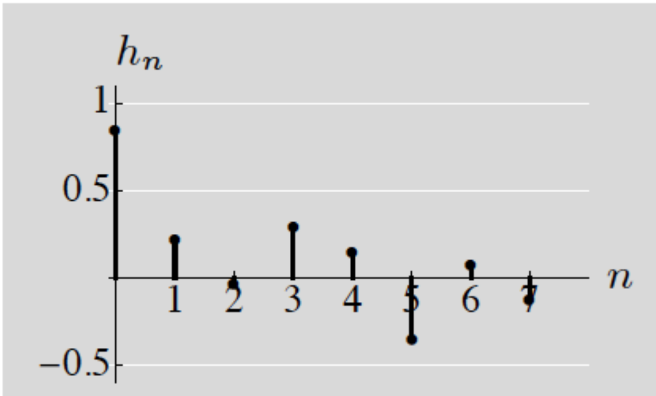
DFT properties	Time domain	DFT domain
Related sequences		
Conjugate	x_n^*	$X^*_{-k \bmod N}$
Conjugate, time-reversed	$x^*_{-n \bmod N}$	X_k^*
Real part	$\Re(x_n)$	$(X_k + X^*_{-k \bmod N})/2$
Imaginary part	$\Im(x_n)$	$(X_k - X^*_{-k \bmod N})/(2j)$
Conjugate-symmetric part	$(x_n + x^*_{-n \bmod N})/2$	$\Re(X_k)$
Conjugate-antisymmetric part	$(x_n - x^*_{-n \bmod N})/(2j)$	$\Im(X_k)$
Symmetries for real x		
X conjugate symmetric		$X_k = X^*_{-k \bmod N}$
Real part of X even		$\Re(X_k) = \Re(X_{-k \bmod N})$
Imaginary part of X odd		$\Im(X_k) = -\Im(X_{-k \bmod N})$
Magnitude of X even		$\|X_k\| = \|X_{-k \bmod N}\|$
Phase of X odd		$\arg X_k = -\arg X_{-k \bmod N}$

DFT properties	Time domain	DFT domain
Common transform pairs		
Kronecker delta sequence	δ_n	1
Shifted Kronecker delta sequence	$\delta_{(n-n_0) \bmod N}$	$W_N^{kn_0}$
Constant sequence	1	$N\delta_k$
Geometric sequence	α^n	$(1-\alpha W_N^{kN})/(1-\alpha W_N^k)$
Periodic sinc sequence (ideal lowpass filter)	$\sqrt{\frac{k_0}{N}}\frac{\mathrm{sinc}(\pi n k_0/N)}{\mathrm{sinc}(\pi n/N)}$	$\begin{cases} \sqrt{\frac{N}{k_0}}, & \left\lvert k-\frac{N}{2}\right\rvert \geq \frac{k_0-1}{2}; \\ 0, & \text{otherwise.} \end{cases}$
Box sequence	$\begin{cases} \frac{1}{\sqrt{n_0}}, & \left\lvert n-\frac{N}{2}\right\rvert \geq \frac{n_0-1}{2}; \\ 0, & \text{otherwise.} \end{cases}$	$\sqrt{n_0}\,\frac{\mathrm{sinc}(\pi n_0 k/N)}{\mathrm{sinc}(\pi k/N)}$

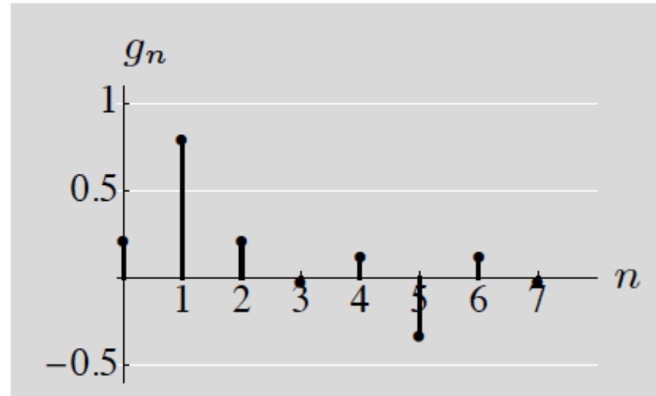
For all sequences in in the above transforms we have

$n = 0, 1, \ldots, N-1$, and $k = 0, 1, \ldots, N-1$

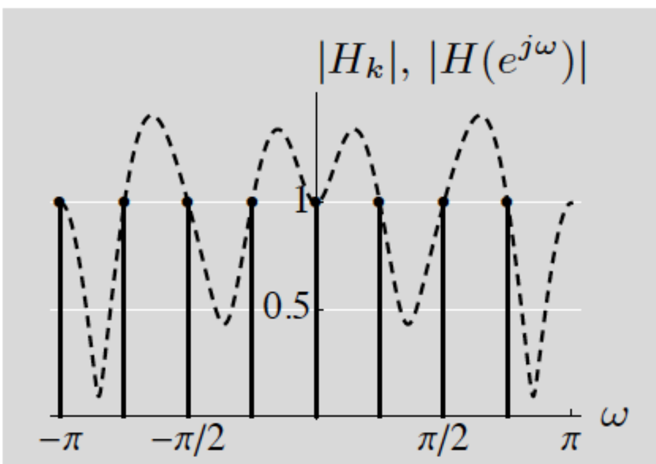
Example DFT and DTFT (I)



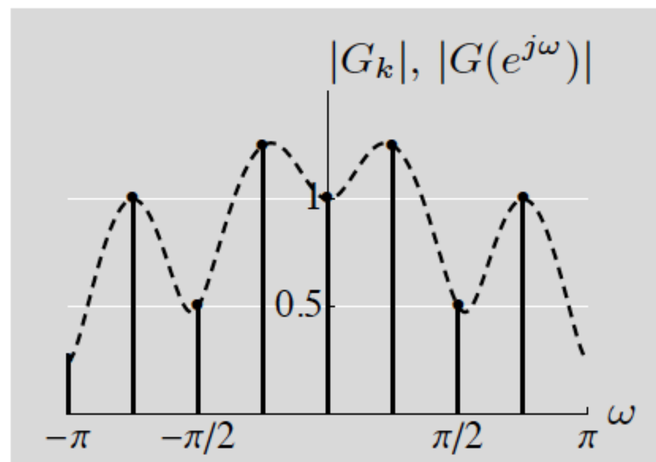
(a) Filter h.



(d) Filter g.

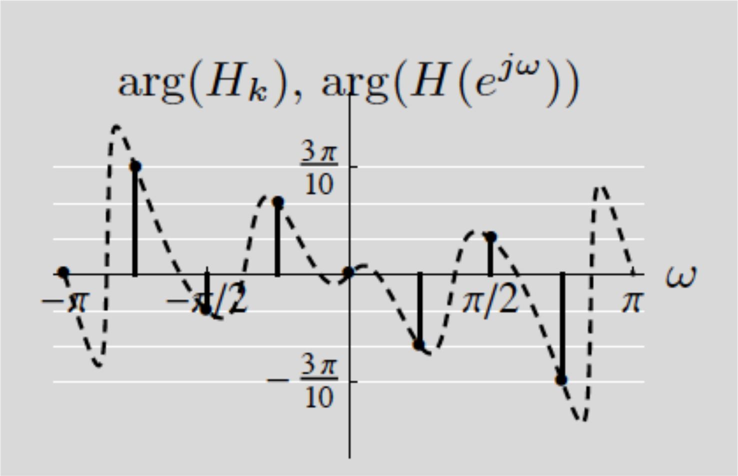


(b) Magnitudes of DFT and DTFT.

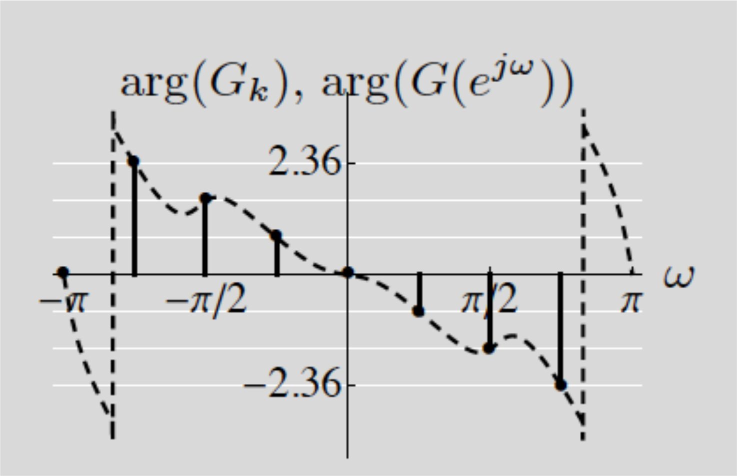


(e) Magnitudes of DFT and DTFT.

Example DFT and DTFT (II)



(c) Phases of DFT and DTFT.



(f) Phases of DFT and DTFT.

What are wavelets?
A short primer