

## TD-2 Representation and approximation of structured data

### Exercise 1: Inner product and norm

We consider the complex-valued vector space  $\mathbb{C}^N$  with  $N = 2$  and consider the four vectors  $u = [3, -1]^T$ ,  $v = [2, -5]^T$ ,  $x = [2 + 3i, -1 - 2i]^T$  and  $y = [-2 + 5i, 3 + 6i]^T$ .

- Compute the norms  $(\|\cdot\|_2)$  of the four vectors and construct the corresponding unit vectors.
- Compute the following inner products:  $\langle u, v \rangle$  and  $\langle x, y \rangle$ .
- Compute the following distances:  $d(u, v)$  and  $d(x, y)$ .

Note that  $i = \sqrt{-1}$ .

### Exercise 2: Projection, bases and frames

A) We consider the unit vector  $\phi = [1/\sqrt{2}, 1/\sqrt{2}]^T$  and the two vectors  $x = [2, 1]^T$  and  $[-2, 1]^T$  in  $\mathbb{R}^2$ .

- Compute the orthogonal projection  $\hat{x} = \langle x, \phi \rangle \phi$  of  $x$  and  $y$  onto  $\phi$ .
- Show that  $x - \hat{x}$  is orthogonal to  $\phi$ .
- Make a graphical illustration.

B) Show that the vectors  $e_1 = [1, 0, 0]^T$ ,  $e_2 = [0, 1, 0]^T$  and  $e_3 = [0, 0, 1]^T \in \mathbb{R}^3$  are linearly independent, i.e.  $\sum_{k=1}^3 \alpha_k e_k = 0$ , if and only if  $\alpha_k = 0$  for  $k = 1, 2, 3$ . Show also that any vector  $x \in \mathbb{R}^3$  can be represented using these three vectors.

C) Show that the vectors  $\phi_1 = [1, 0]^T$ ,  $\phi_2 = [0, 1]^T$  and  $\phi_3 = [-1, -1]^T$  are linearly dependent and that any vector  $x \in \mathbb{R}^2$  can be represented using the three vectors. Construct the corresponding frame.

D) We consider the set of vectors  $\Phi = \{\phi_k\}_{k \in \mathbb{N}} \subset \mathbb{C}^{[-1/2, 1/2]}$ , where  $\phi_0(t) = 1$  and  $\phi_k(t) = \sqrt{2} \cos(2\pi kt)$  for  $k = 1, 2, \dots$

- Make a graphical illustration of  $\phi_0, \phi_1$  and  $\phi_2$ .
- Show that the set of vectors  $\Phi$  is orthogonal, i.e.  $\langle \phi_k, \phi_\ell \rangle = \delta_{k,\ell}$ .
- Show that the set of vectors is orthogonal to the set of odd functions  $S_{\text{odd}} = \{s | s(t) = -s(-t) \text{ for } t \in [-1/2, 1/2]\}$

### Exercise 3: Norms

Verify if  $\|x\|_\alpha$  satisfies for  $\alpha = 0, 1, 2$  and  $\infty$  the properties of a norm.

Note that the  $\ell^0$  'norm' is defined by  $\|x\|_0 = \lim_{p \rightarrow 0} \sum_k |x|^p$  and counts the number of non zero entries of a vector, here we define  $0^0 = 0$ . For  $1 < p < \infty$  the corresponding  $\ell^p$  norms are defined as  $\|x\|_p = \sum_k |x_k|^p$  and for  $p = \infty$  we have  $\|x\|_\infty = \sup_k |x_k|$ .