Aix–Marseille Université
Faculté des Sciences
Master M1 MAS
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TD-5 Representation and approximation of structured data

Exercise 1: Inner product computation by expansion sequences

We consider two sequences α and β in $\ell^2(\mathbb{N})$ and the corresponding functions

$$x(t) \;=\; \alpha_0 + \sqrt{2}\sum_{k=1}^{\infty} \alpha_k \, \cos(2\pi k t)$$

$$y(t) \;=\; \beta_0 + \sqrt{2}\sum_{k=1}^{\infty} \beta_k \, \cos(2\pi k t)$$

which are in the Hilbert space $L^2([-\frac{1}{2}, \frac{1}{2}])$.

Compute their inner product $\langle x, y\rangle$ and show that it can be computed via a simpler inner product between the expansion coefficients in $\ell^2(\mathbb{N})$, i.e., $\langle x, y\rangle = \langle \alpha, \beta\rangle$.

Exercise 2: Orthogonal projection with orthonormal basis

We consider three vectors in $\mathbb{C}^3$,

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \phi_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}, \quad \phi_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

Show that they are mutually orthogonal, are of length 1 and are linearly independent in $\mathbb{C}^3$. Thus they form an orthonormal basis for $\mathbb{C}^3$.

Now we define the two-dimensional subspace

$$S = \{[x_0 \quad x_1 \quad x_2]^T \in \mathbb{C}^3 \quad | \, x_1 = x_0 + x_2\}$$

with $S = \mathrm{span}\,(\{\phi_0, \phi_1\})$. The orthogonal projection onto S is given by

$$P_S x \;=\; \sum_{k=0}^{1} \langle x, \phi_k\rangle \, \phi_k \, .$$

Compute explicitely the orthogonal projection operator in matrix form, i.e., $\Phi\Phi^\star$ and verify that the matrix is idempotent and Hermitian.

Exercise 3: Gram–Schmidt orthogonalization

We consider three linearly independent vectors in $\mathbb{C}^3$,

$$x_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad x_1 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad x_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

Apply the Gram–Schmidt orthogonalization algorithm to compute an orthonormal basis $\{\phi_0, \phi_1, \phi_2\}$ in $\mathbb{C}^3$. Show that starting with a different vector yeilds a different orthonormal basis.

Exercise 4: Biorthogonal pair of bases in finite dimensions

Verify that

$$\phi_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \phi_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\widetilde{\phi}_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \widetilde{\phi}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \widetilde{\phi}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

yields a biorthogonal pair of bases for $\mathbb{C}^3$.

Show that any vector $x \in \mathbb{C}^3$ can be expanded into a biorthogonal basis as

$$x = \sum_{k=0}^{3} \langle x, \widetilde{\phi}_k \rangle \, \phi_k$$

Compute the expansion for $x = [-2 \; 3 \; 5]^T$ in this biorthogonal basis and give the matrix representation.