Latent variable modelling, assignement 2

Jean-Marc Freyermuth

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Exercise 1: Linex loss function

The Linex loss function is defined as:

$$L(\theta, \hat{\theta}) = e^{a(\hat{\theta}-\theta)} - a(\hat{\theta} - \theta) - 1,\ a \neq 0$$

- Comment the behaviour of the loss function accordingly to the value of a.
- Consider the Normal mean problem, with known variance and Normal prior on the mean. Find the Bayes estimator of the mean under the Linex loss function.

Exercice 2: Prior specification

Suppose $x = (x_1, \ldots, x_n)$ are iid random variables from an exponential distribution with mean θ.

- Derive the Jeffreys prior for θ; find the associated posterior. In which family it belongs to ?
- Derive the Jeffreys prior for $\alpha = 1/\theta$; find the associated posterior. In which family it belongs to ?

Exercice 3: more on Prior specification

Consider observations arising from Poisson distribution with parameter θ:

$$\begin{aligned} x|\theta &\sim Poi(\theta) \\ p(x|\theta) &= \frac{\theta^x}{x!} e^{-\theta}. \end{aligned}$$

- Give the conjugate prior family.
- Determine the Jeffreys prior.
- Find the maximum entropy prior using as reference prior $\pi_J(\theta)$, under the constraints that the prior mean and variance of θ are both 1. (do not solve for the values of the constraint).
- Under the Jeffreys prior find the Maximum a posteriori.

Exercice 4: Binomial model and mixture prior specification

We consider a Binomial model with a prior distribution

$$
\begin{aligned}
& y|\theta \sim Bin(n,\theta) \\
& p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}.
\end{aligned}
$$

You consider the standard Beta prior not flexible enough for your purpose and propose the following mixture prior.

$$
\begin{aligned}
\pi(\theta) &= \sum_{j=1}^{J} c_j \pi_j(\theta),\ \sum_{j=1}^{J} c_j = 1. \\
\pi_j(\theta) &= \frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)}\theta^{\alpha_j-1}(1-\theta)^{\beta_j-1}1_{\{\theta\in(0,1)\}}.
\end{aligned}
$$

Show that you can write the posterior as

$$
\pi(\theta|y) = \sum_{j=1}^{J} \tilde{c}_j \pi_j(\theta|y),
$$

where $\tilde{c}_j$ are weights to determine.

- Give the Bayesian estimator under the quadratic risk.

Bayesian Linear regression

Consider the Gaussian linear regression model. We aim to compute the Maximum a Posteriori (MAP) estimate for pointwise parameter estimation.

Assume independent prior distributions on the vector of parameters. Specifically, consider the following priors:

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Uniform prior
Gaussian prior
Laplace prior
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Prove that the Bayesian MAP estimators under these priors correspond to the following respective optimization problems:

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The least squares solution (for the uniform prior),
The ridge regression solution (for the Gaussian prior),
The Lasso regression solution (for the Laplace prior).
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Hint:

- $p(\boldsymbol{\beta}) \propto 1$
- $p(\boldsymbol{\beta}) \propto \prod_{p=1}^{P} \mathcal{N}(\beta_p; 0, \alpha^2)$
- $p(\boldsymbol{\beta}) \propto \prod_{p=1}^{P} \mathcal{L}(\beta_p; \mu, \alpha),\ \mathcal{L}(x; \mu, \alpha) = \frac{1}{2\alpha}\exp\left\{-\frac{|x-\mu|}{\alpha}\right\}$