

Random Variables

Motivation

Examples

Definitions

Visualization



What Matters

So far

coins, dice, cards, dominoes, marbles,

Often

subscribers clicks viewers yield weight sales

time congestion delay age temperature heart rate

GPA tuition assignment

income cost

Numbers!

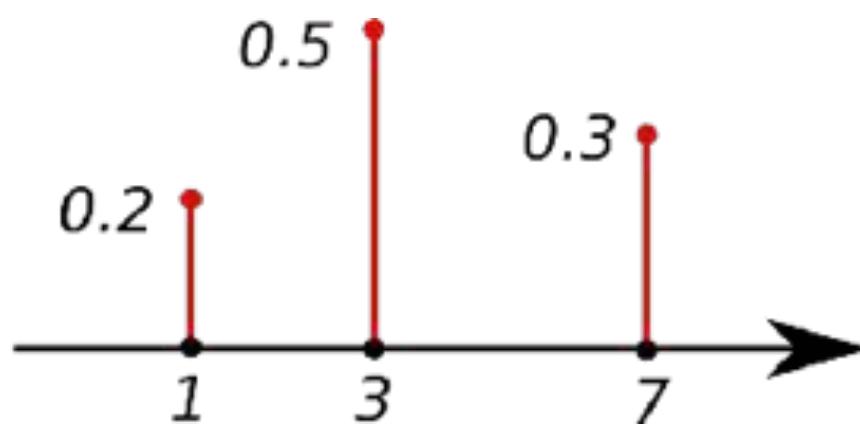
Random variable

Number-valued random outcome

Xtra with Numbers

Distr-
butio
n
 $p(x)$

View on a line



Express as function

$$p(x) = 1/x^2$$

Consider properties

Decreasing



concentrated

Random
Variable

X

Perform operations

$$X+1$$

$$X^2$$

Combine variables

$$X+Y$$

Consider properties

average value of X

Two Types

Size of sample space Ω

Ω is finite

$\{1, 2, 3\}$

$\{e, \pi\}$

or countably infinite

\mathbb{N}

\mathbb{Z}

Discrete

Ω is uncountably infinite

$[0,2]$

$(-1,3) \cup [4,5)$

\mathbb{R}

Continuous

Combination

$[0,2] \cup \{e, \pi\}$

Mixed

Begin with discrete

Been There

Several past examples had number outcomes

Outcome of a die roll

$\{1, \dots, 6\}$

Number of heads in 3 coin tosses

$\{0, \dots, 3\}$

Values of a domino tile

$\{0, \dots, 6\}$

Did not use numerical features

→ Use extensively

Familiar and new examples

General formulation

Heads

3 fair coins

$$\Omega = \{ \text{ttt}, \text{tth}, \text{tht}, \text{thh}, \text{htt}, \text{hth}, \text{hht}, \text{hhh} \}$$

$$|\Omega| = 8$$

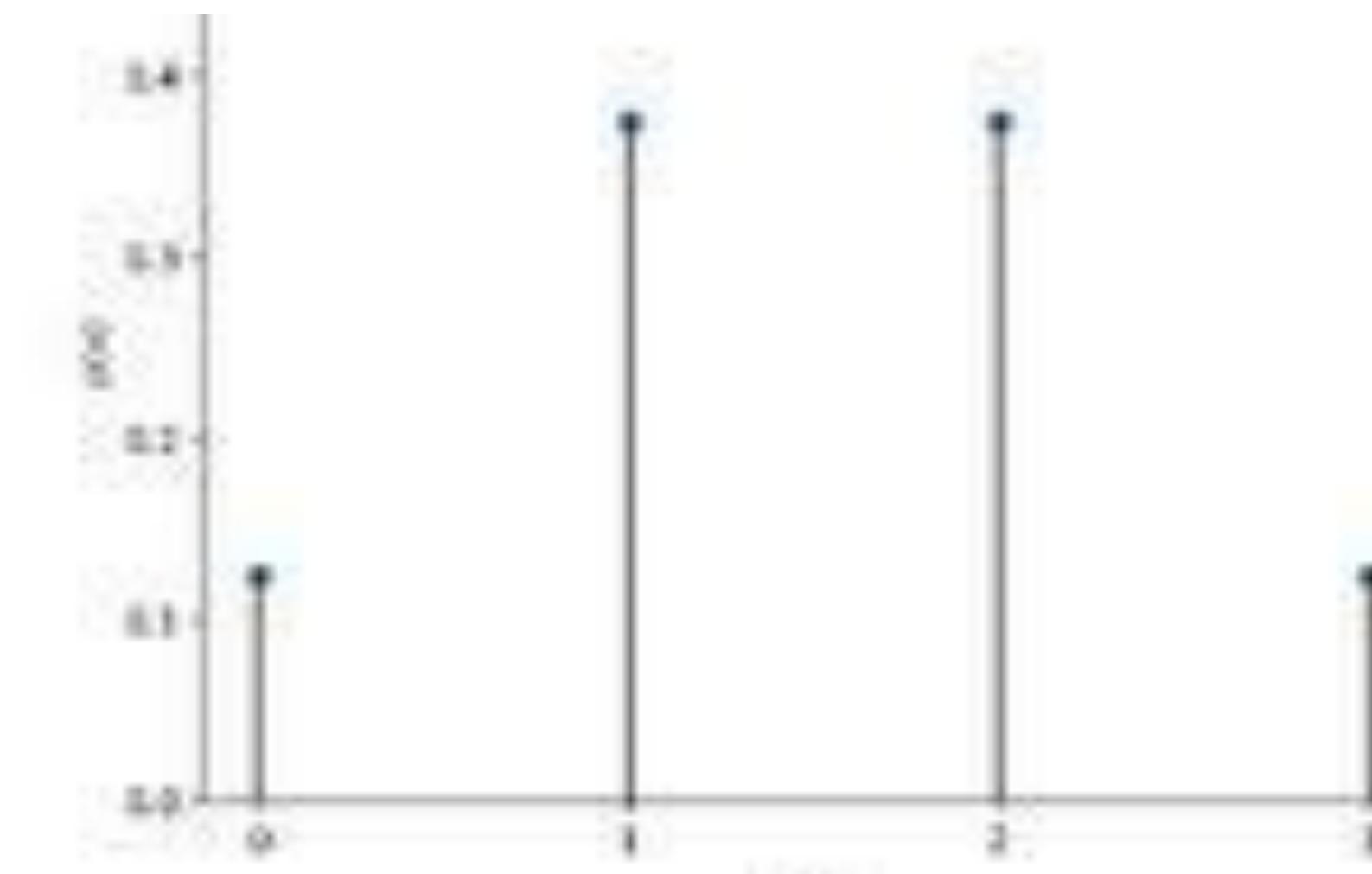
Equiprobable

$$p = 1/8$$

X

heads

x	Outcomes	p(x)
0	ttt	1/8
1	tth, tht, htt	3/8
2	thh, hth, hht	3/8
3	hhh	1/8



Specification

As before

Explicit

$$p(1)=.1 \quad p(2)=.2 \quad p(3)=.3 \quad p(4)=.4$$

Table

x	1	2	3	4
p(x)	.1	.2	.3	.4

With numbers

Function

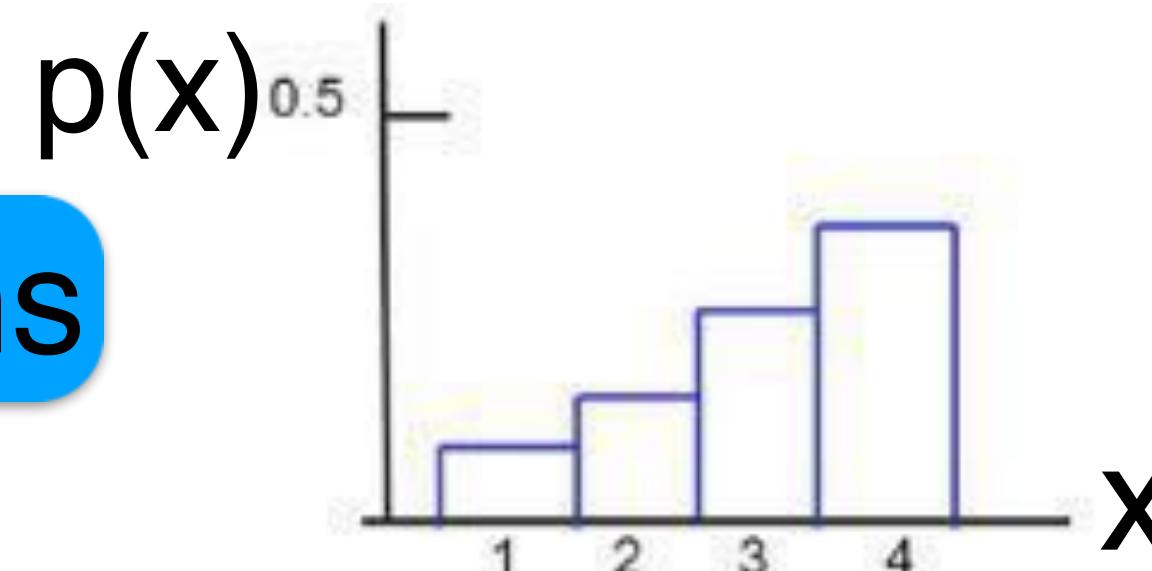
$$p(x) = x / 10$$

$$x \in \{1,2,3,4\}$$

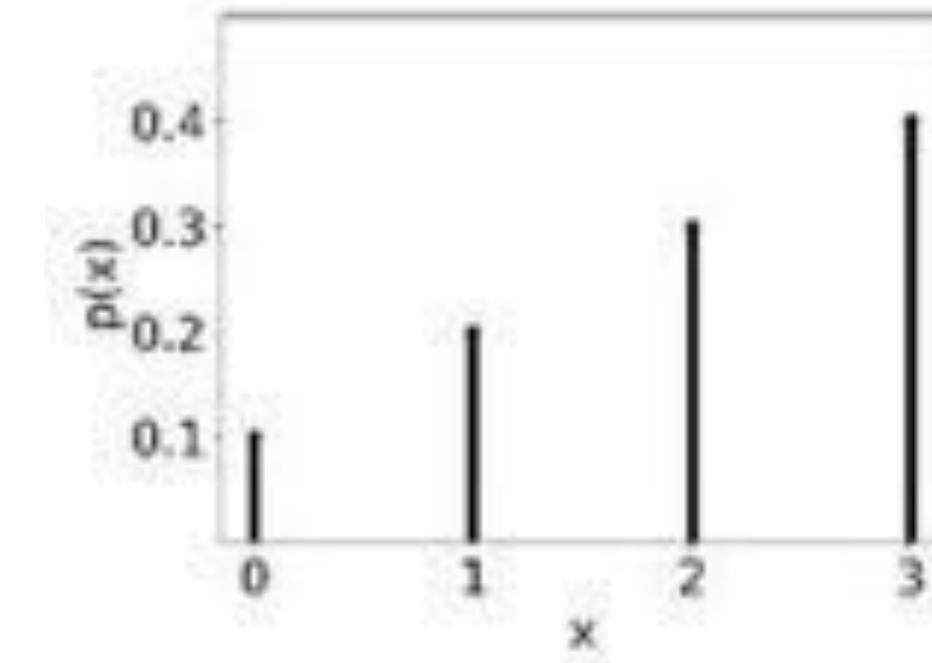


Histogram

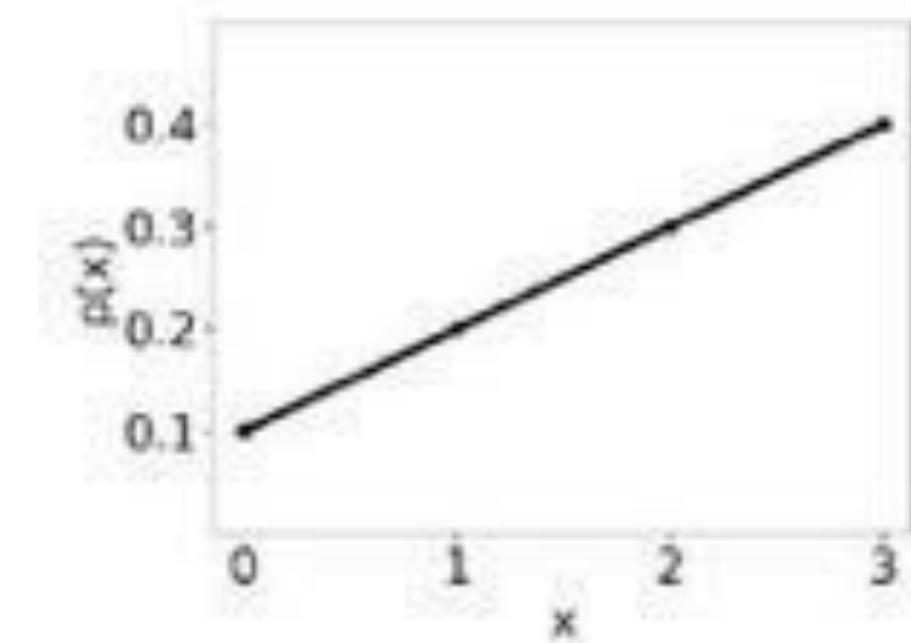
Graphs



Stem plot



Plot



Probability Mass Function

As before

pmf

$p : \Omega \rightarrow \mathbb{R}$

Specify Ω and p

Ω

Random variable $\rightarrow \subseteq \mathbb{R}$

Discrete \rightarrow finite or countably infinite

p

$p(x) \geq 0 \quad \forall x \in \Omega$

$$\sum_{x \in \Omega} p(x) = 1$$

If X is distributed according to p , we write $X \sim p$

Alternative Notation

Discrete

$$\Omega \subseteq \mathbb{R}$$

Often

$$\mathbb{Z}$$

$$\mathbb{N}$$

$$\mathbb{P}$$

$$\{1, \dots, n\}$$

$$p(x)$$

\rightarrow

$$p_x$$

$$p_i$$

$$p_i \geq 0$$

$$\sum_i p_i = 1$$

Types of Discrete Distributions

Finite

$$|\Omega| = n \in \mathbb{P}$$

Infinite

$$|\Omega| = \infty = \aleph_0$$

Finite Distributions

$|\Omega| = n$

Specify pmf

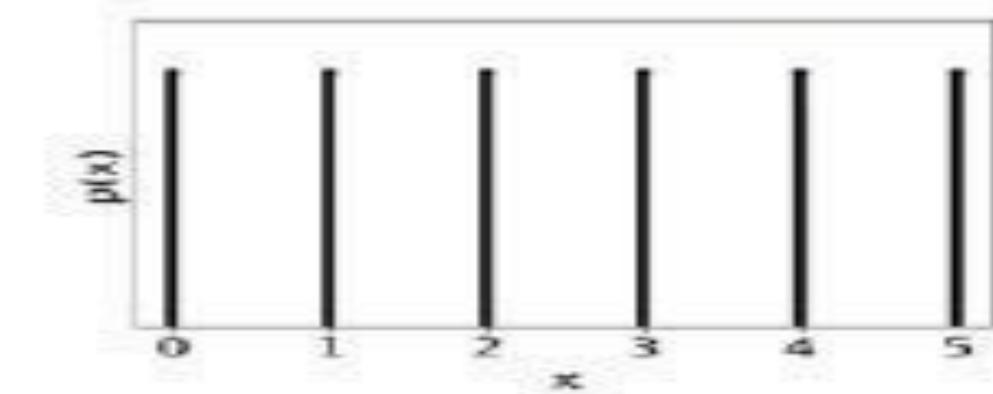
p_1, p_2, \dots, p_n

$$\forall 1 \leq i \leq n \quad p_i \geq 0$$

$$\sum_{i=1}^n p_i = 1$$

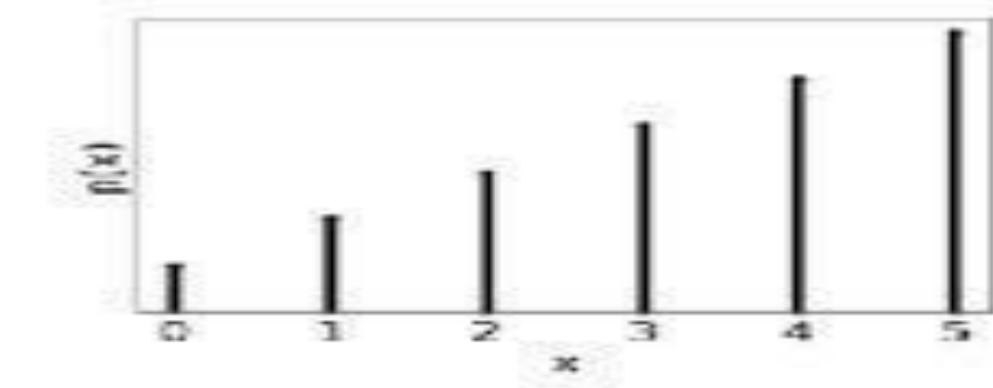
Uniform

$p_1 = p_2 = \dots = p_n = 1/n$



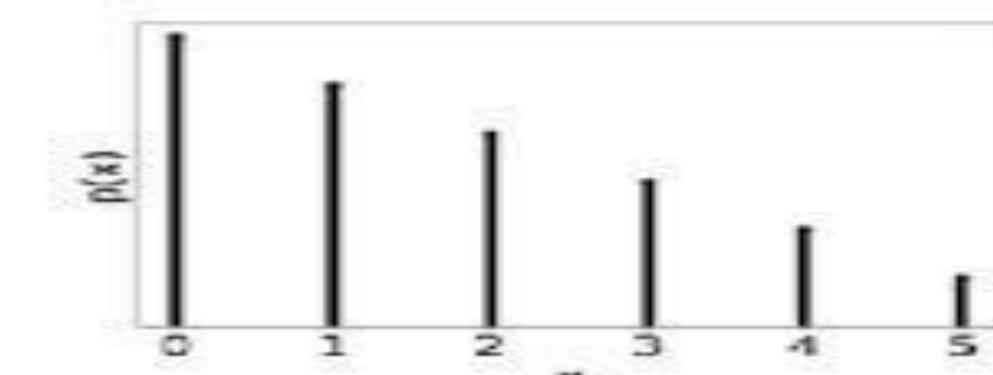
Increasing

$p_1 \leq p_2 \leq \dots \leq p_n$



Decreasing

$p_1 \geq p_2 \geq \dots \geq p_n$



Infinite Distributions

$|\Omega| = \infty$

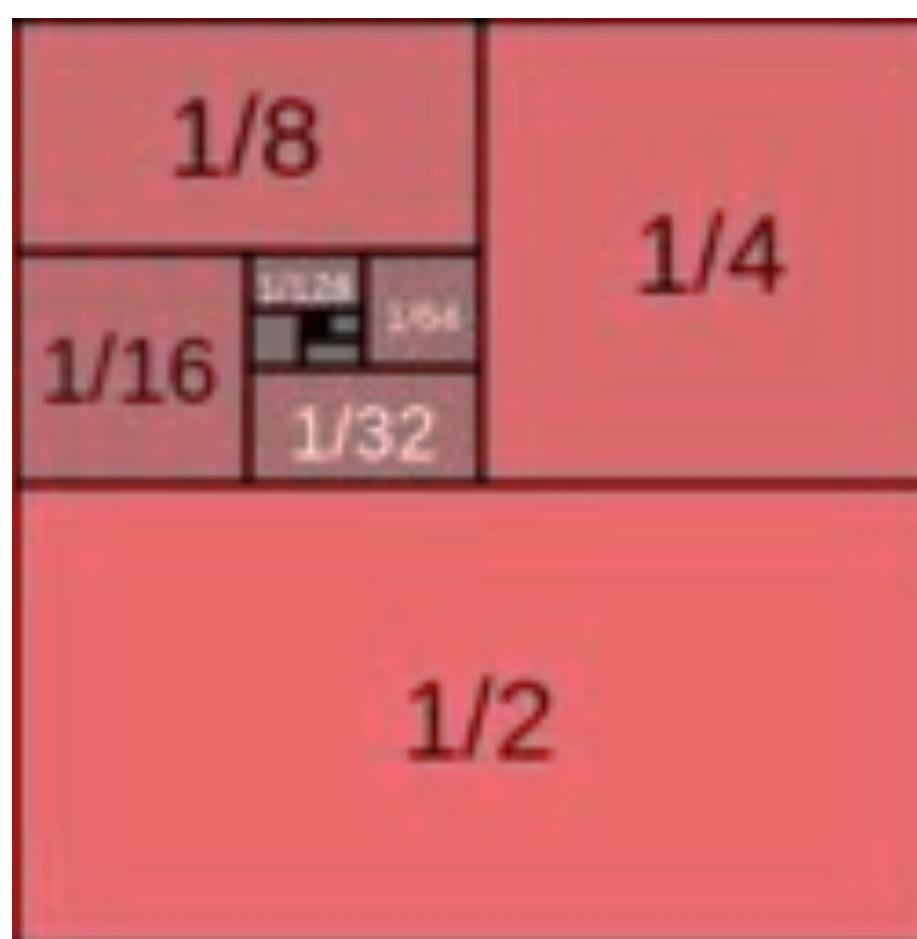
One-sided infinite

p_1, p_2, p_3, \dots

Cannot be uniform

$p = 0 \rightarrow \sum = 0$

$p > 0 \rightarrow \sum = \infty$



Cannot increase

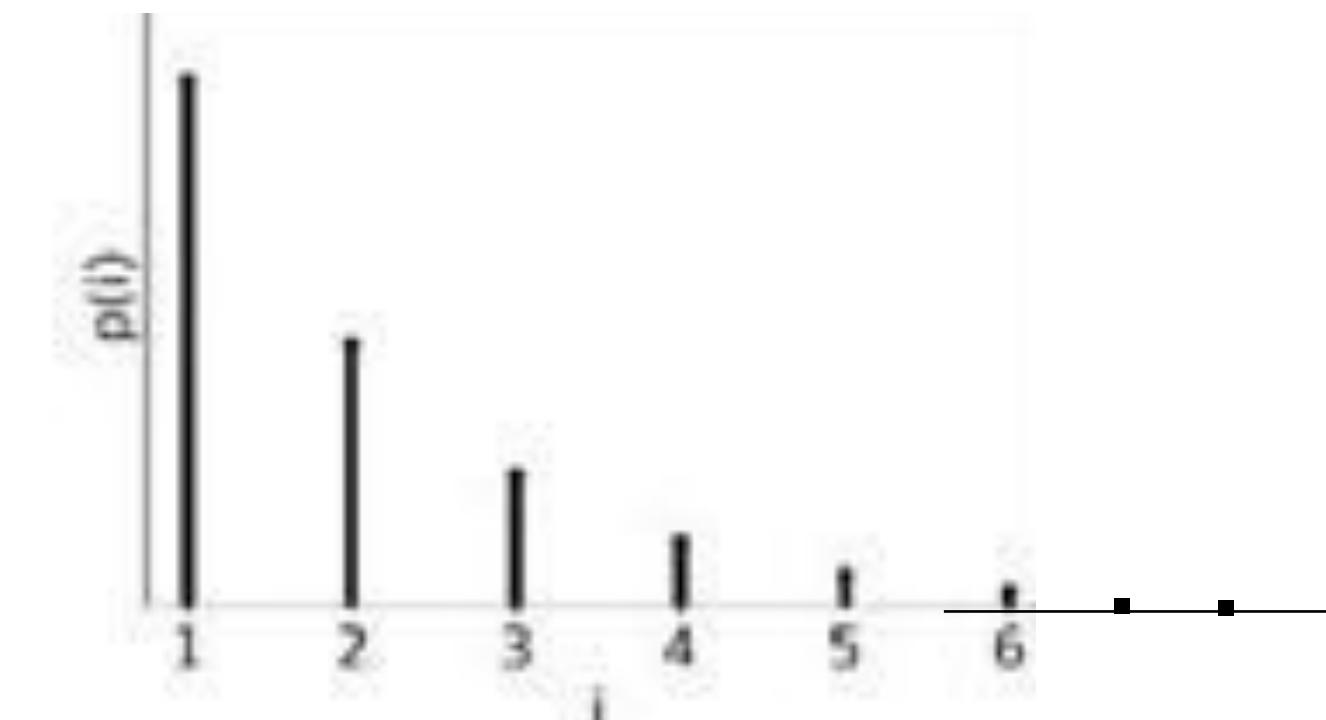
$p_i > 0 \rightarrow p_{i+1}, p_{i+2}, \dots > 0 \rightarrow \sum = \infty$

Can decrease

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

$$\sum_{i=1}^n \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$



Doubly infinite

$\dots, p_{-2}, p_{-1}, p_0, p_1, p_2, \dots$

$\dots, \frac{1}{8}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{8}, \dots$

Formal Definition

Random variable is a mapping $f : \Omega \rightarrow \mathbb{R}$

Simplify terminology, focus on math

Number-valued random experiment



Random Variables

Motivation

Numbers

Operations

Examples

Definitions

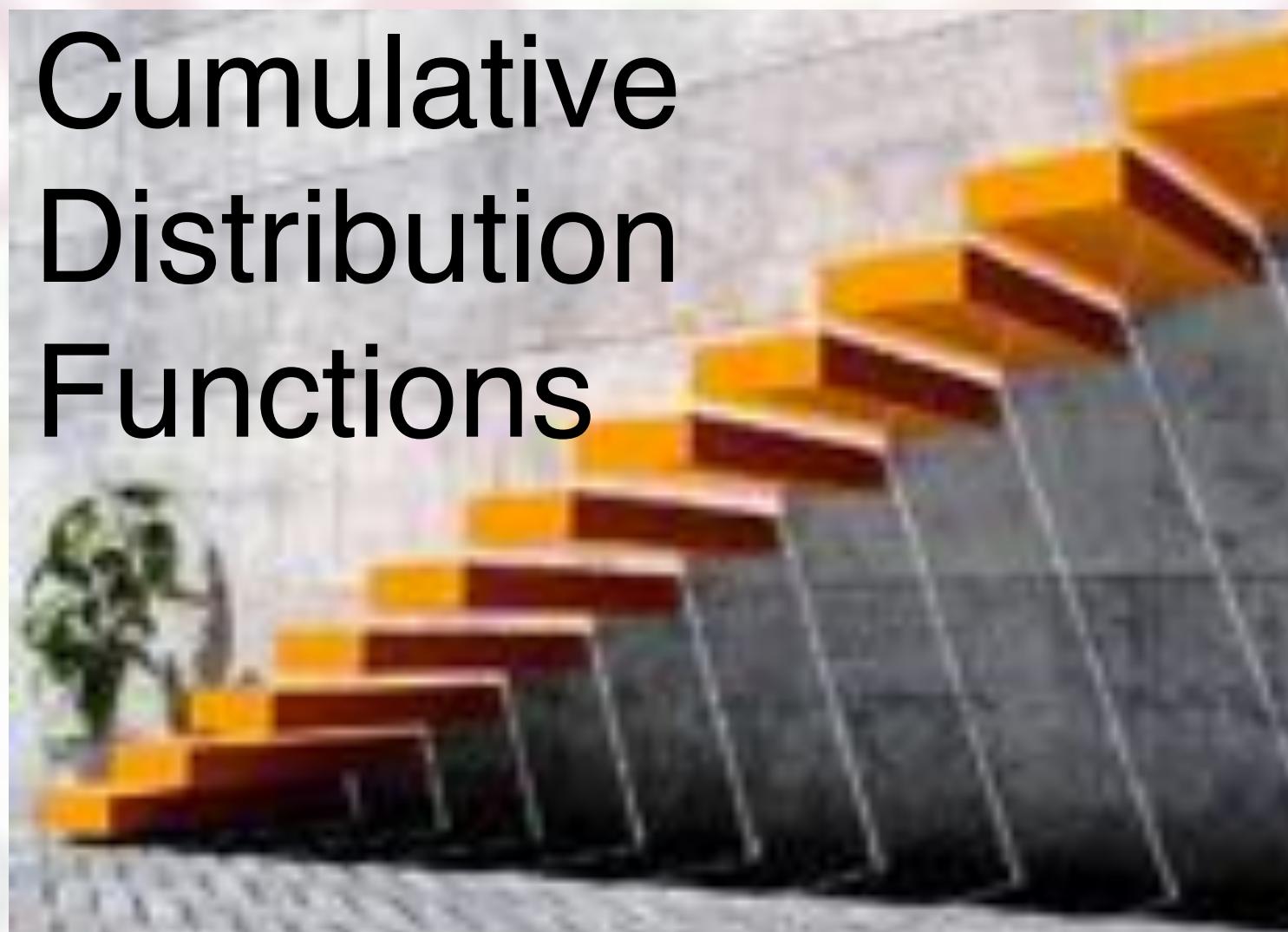
pmf

Visualization

Histogram

Plot

Stem



Cumulative
Distribution
Functions



Cumulative Distribution Functions



Areas of Interest

For random variable, often, interested in probability of intervals

Temperature between 20 and 80

Salary > 80K

GPA < 3.0

One function helps determine all interval probabilities

Cumulative Distribution Function

Probability mass function (pmf)

$$p: \Omega \rightarrow \mathbb{R}$$

Cumulative distribution function (cdf)

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

$$F(x) \stackrel{\text{def}}{=} P(X \in (-\infty, x])$$

$$\stackrel{\text{def}}{=} P(X \leq x)$$

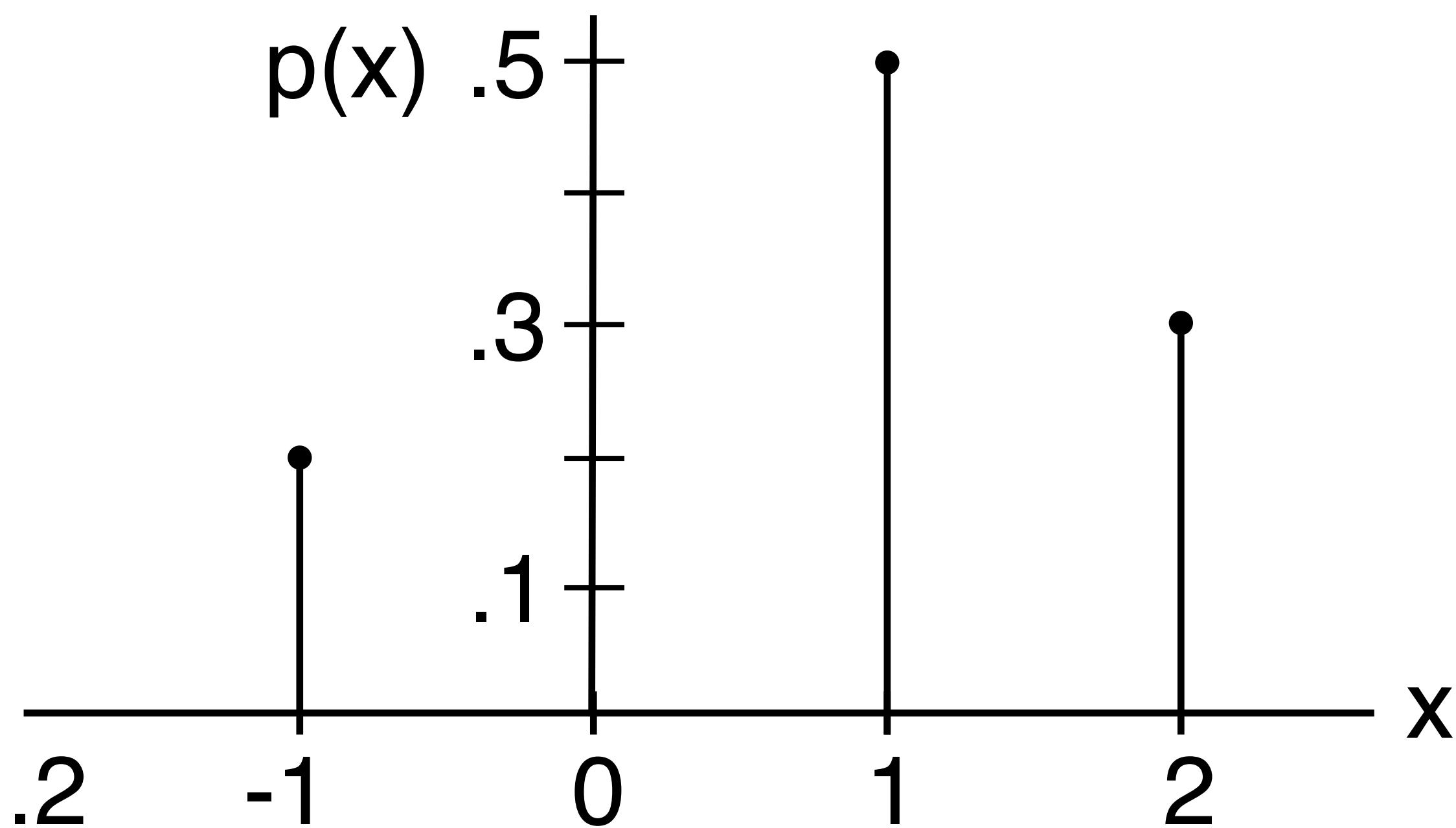
$$= \sum_{u \leq x} p(u)$$

X discrete, still F defined over \mathbb{R}

Example

PMF

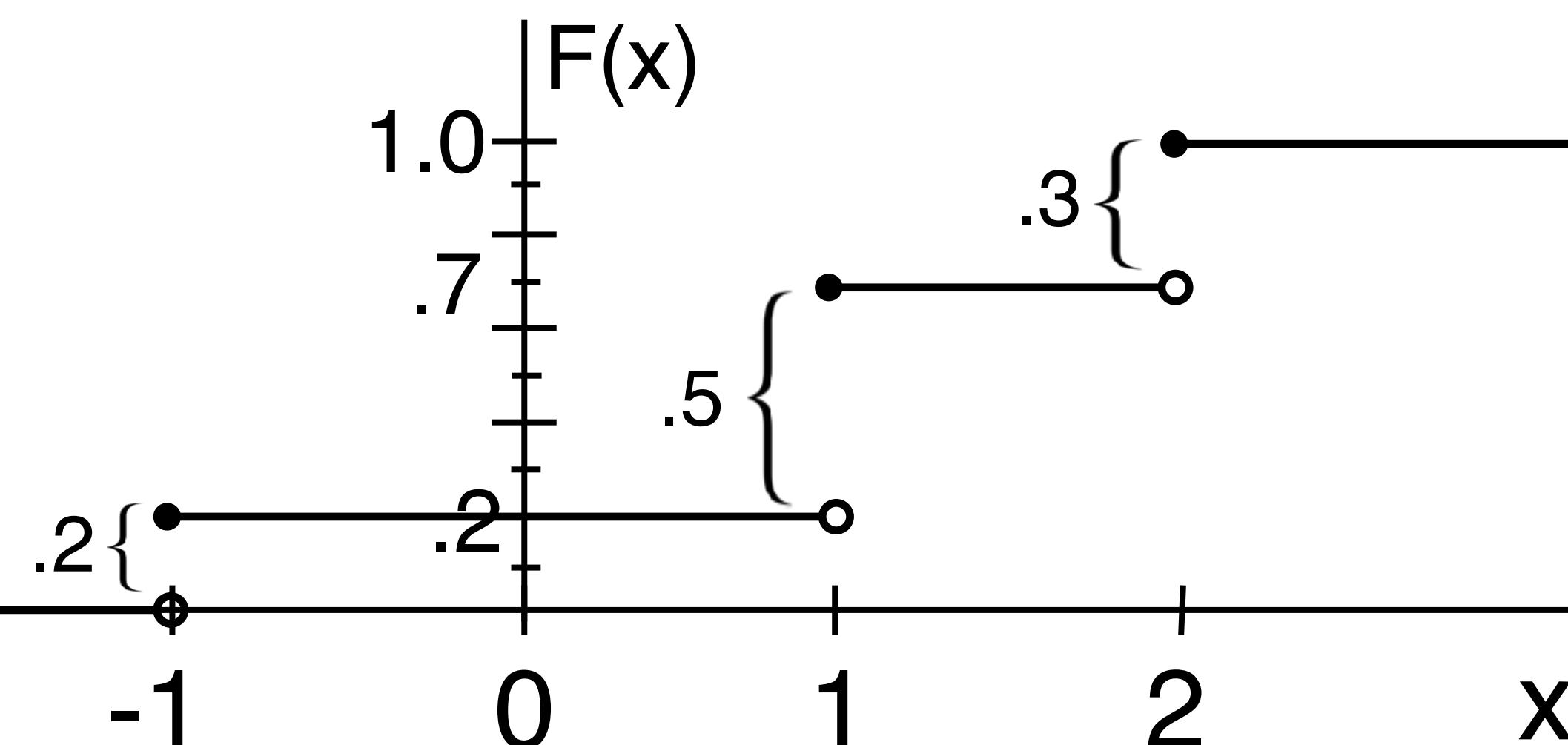
$$p(x) = \begin{cases} .2 & -1 \\ .5 & 1 \\ .3 & 2 \end{cases}$$



CDF

$$F(x) = P(X \leq x)$$

$$= \sum_{u \leq x} p(u)$$



Properties

Nondecreasing

$$x \leq y \rightarrow F(x) \leq F(y)$$

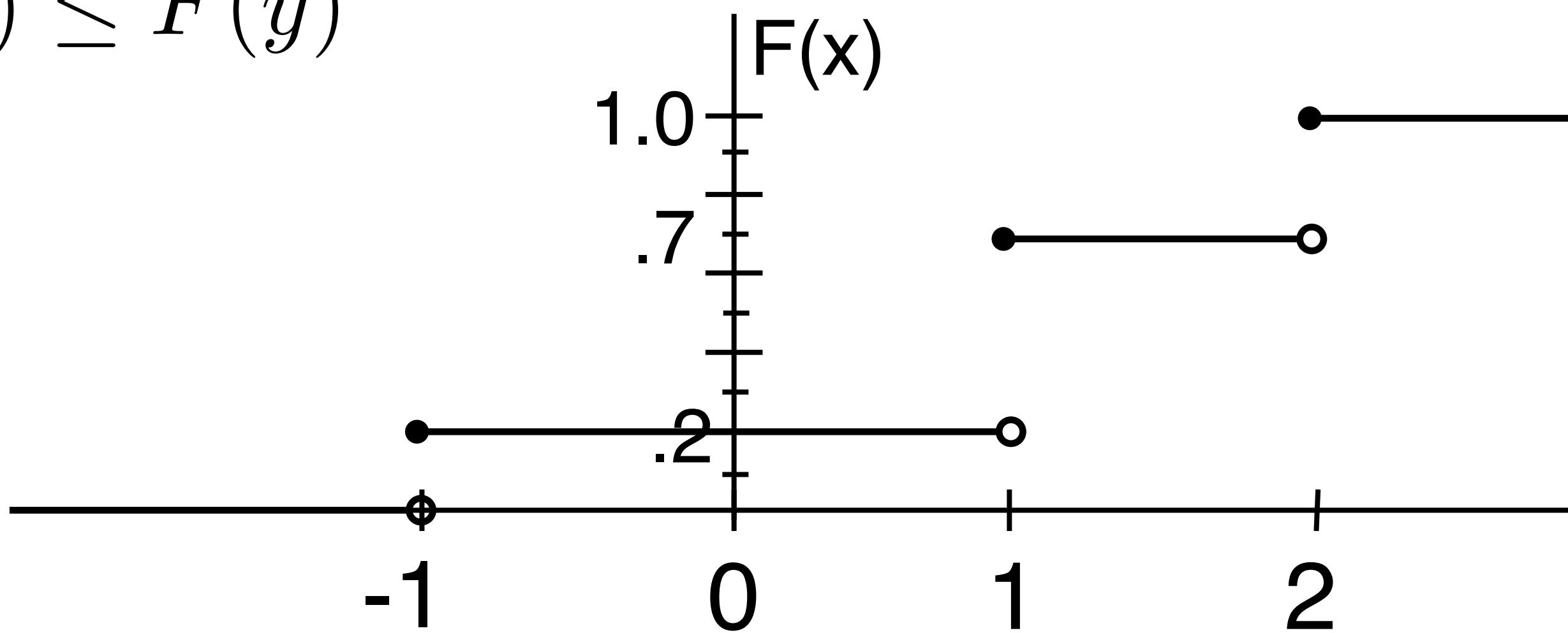
Limits

$$\lim_{x \rightarrow -\infty} F(x) = 0$$

$$\lim_{x \rightarrow \infty} F(x) = 1$$

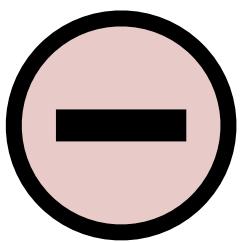
Right-continuous

$$\lim_{x \searrow a} F(x) = F(a)$$



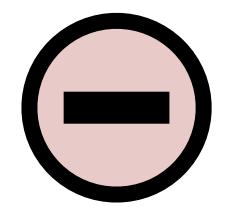
Interval Probabilities

$$P(X \leq a) = F(a) \quad - \text{ by definition}$$



$$P(X > a) = 1 - P(X \leq a) = 1 - F(a)$$

$$P(a < X \leq b) = P((X \leq b) - (X \leq a))$$



$$= P(X \leq b) - P(X \leq a)$$

$$= F(b) - F(a)$$

1963 Mr Average

2017 Mr Average

Expectations



What Matters

Important random-variable properties?

Range

Min & max values of X

Lowest & highest temperature / salary

$$x_{\min} = \min \{ x \in \Omega \mid p(x) > 0 \}$$

$$x_{\max} = \max \{ x \in \Omega \mid p(x) > 0 \}$$

Average

Average temperature / salary

Range average

$$\frac{x_{\min} + x_{\max}}{2} ?$$

Element average

$$\frac{1}{|\Omega|} \sum_{x \in \Omega} x ?$$

or over x s.t. $p(x) > 0$

Sample Mean

$\Omega = \{0, \dots, 100\}$

$p(0) = .8$

$p(90) = .1$

$p(100) = .1$

all other $p(x) = 0$

Range average

$(x_{\min} + x_{\max})/2$

$(0+100)/2 = 50$

Element average

positive probabilities

$(0+90+100)/3 = 63.3$

Ten samples

Typical

0, 0, 0, 0, 90, 0, 0, 0, 100, 0

Sample mean

$(8 \cdot 0 + 1 \cdot 90 + 1 \cdot 100)/10 = 190/10 = 19$

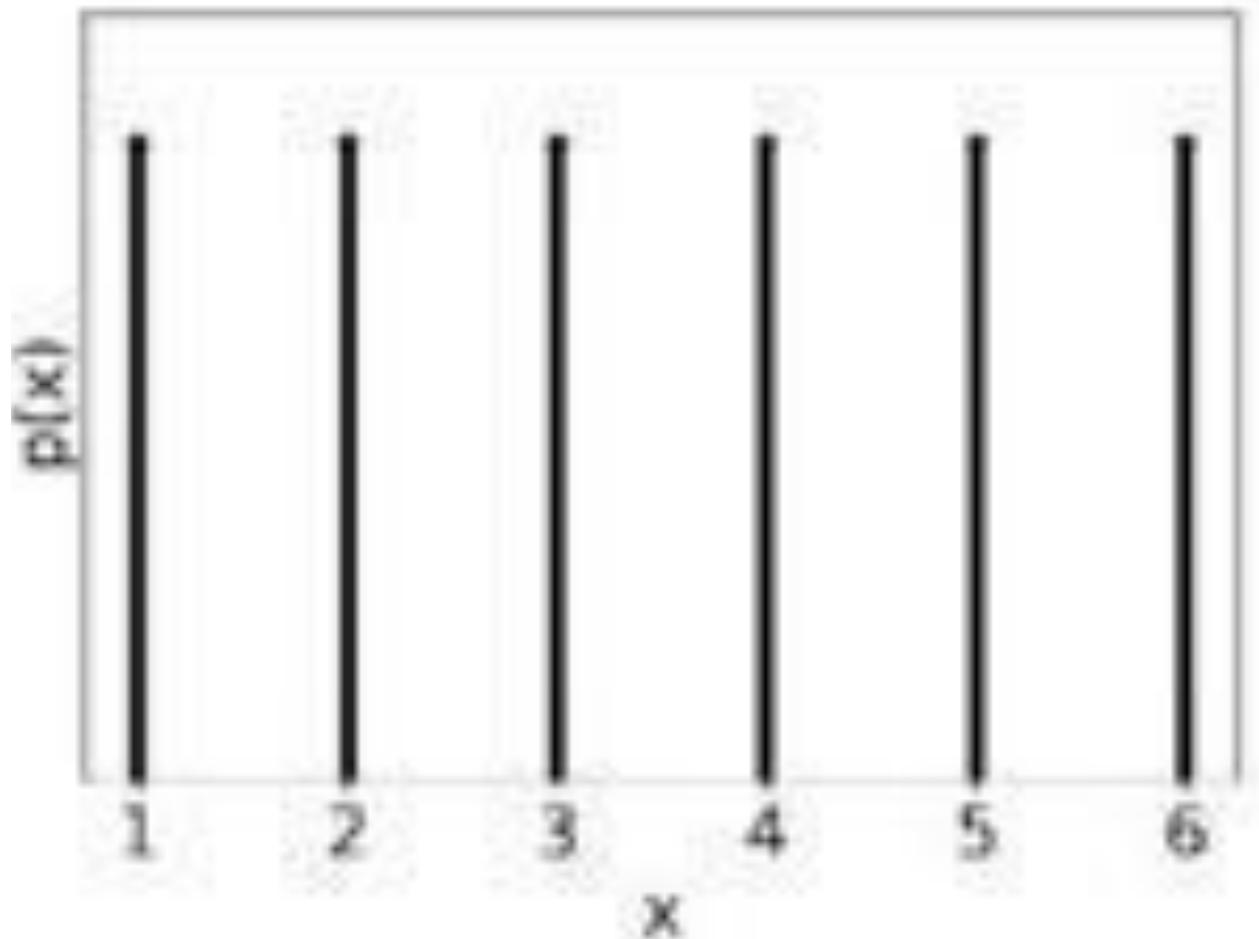
More representative of what we will observe

Fair Die

Roll a fair die $n \rightarrow \infty$ times

Average of the observed values = ?

Each value $\sim n/6$ times



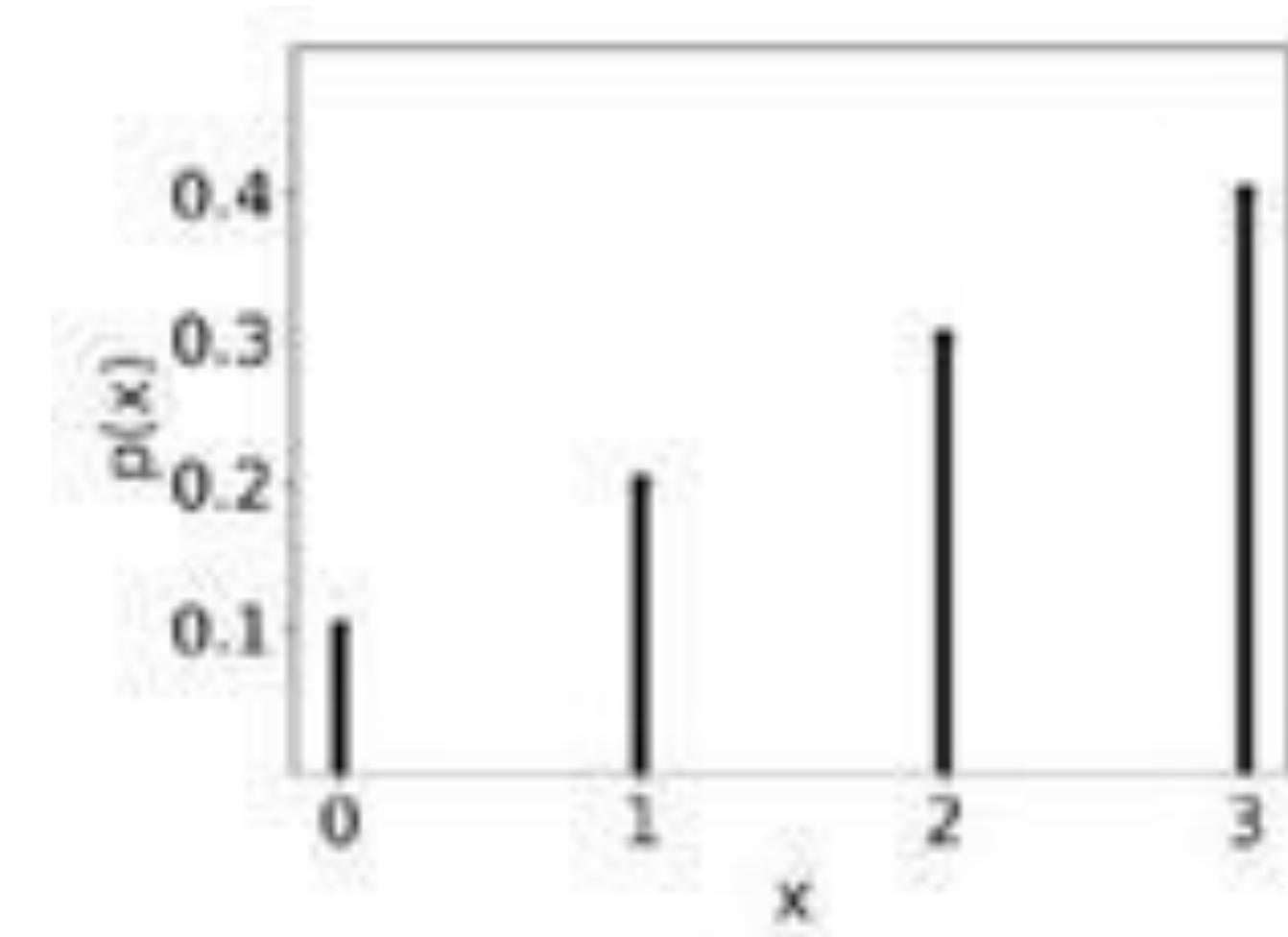
$$\frac{\frac{n}{6} \cdot 1 + \frac{n}{6} \cdot 2 + \dots + \frac{n}{6} \cdot 6}{n} = \frac{1 + \dots + 6}{6} = \frac{1}{6} \frac{(1+6) \cdot 6}{2} = 3.5$$

1,...,6 → Average = 3.5



4-Sided Die

Side	Prob	Appear
1	.1	.1n
2	.2	.2n
3	.3	.3n
4	.4	.4n
	<hr/> 1	<hr/> n



Average

$$= \frac{.1n \cdot 1 + .2n \cdot 2 + .3n \cdot 3 + .4n \cdot 4}{n}$$

$$= 0.1 \cdot 1 + 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4 = 3$$

Arithmetic average

$$(1+2+3+4)/4 = 2.5$$

Probabilities skew to the right

Expectation

In $n \rightarrow \infty$ samples

x will appear

$p(x) \cdot n$ times

$$\text{Average} = \frac{\sum_{x} [P(x) \cdot n] \cdot x}{n} = \sum_{x} P(x) \cdot x \stackrel{\text{def}}{=} E(X)$$

Expectation

Mean

$E(X)$ also denoted

EX

μ_x

μ

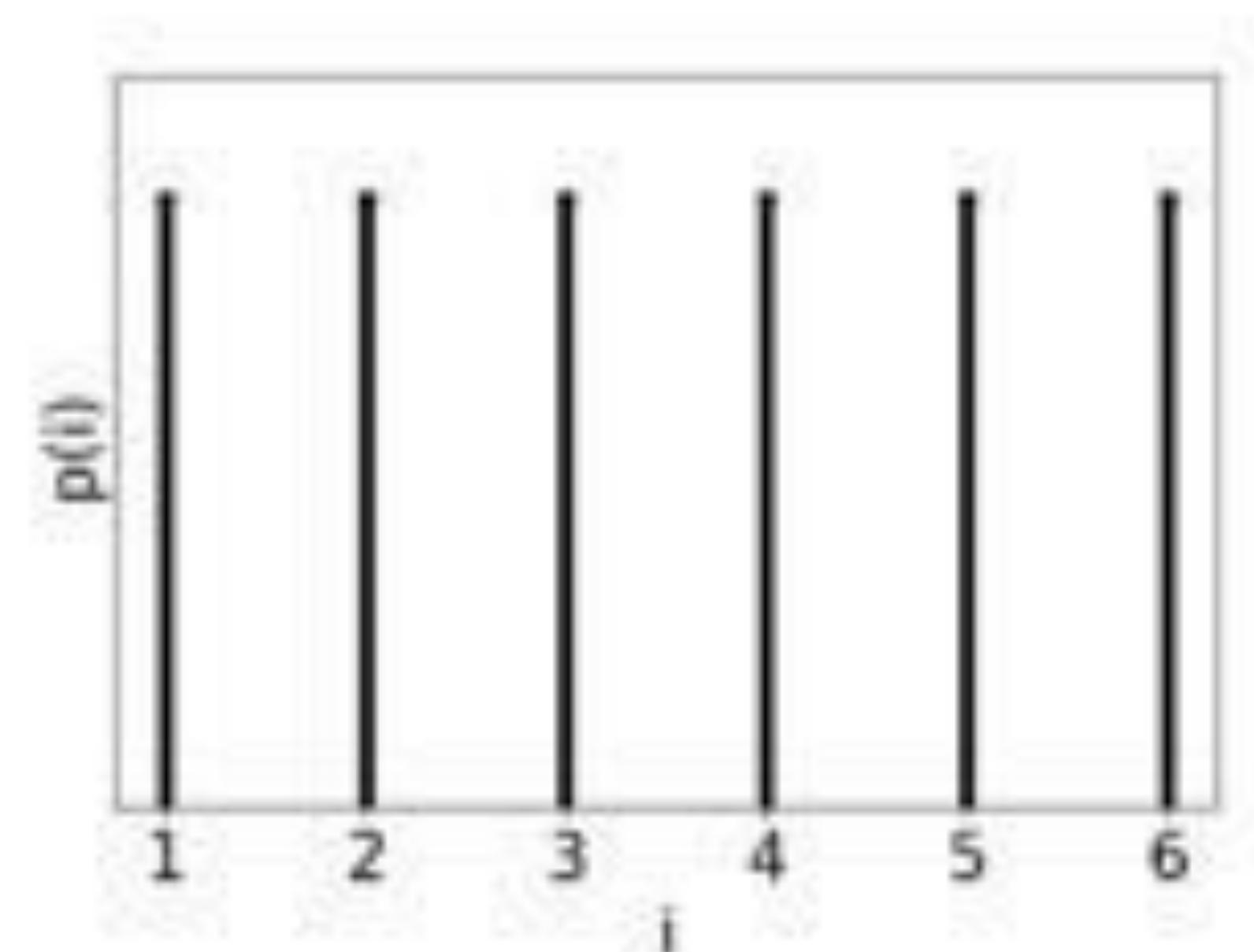
Not random

constant

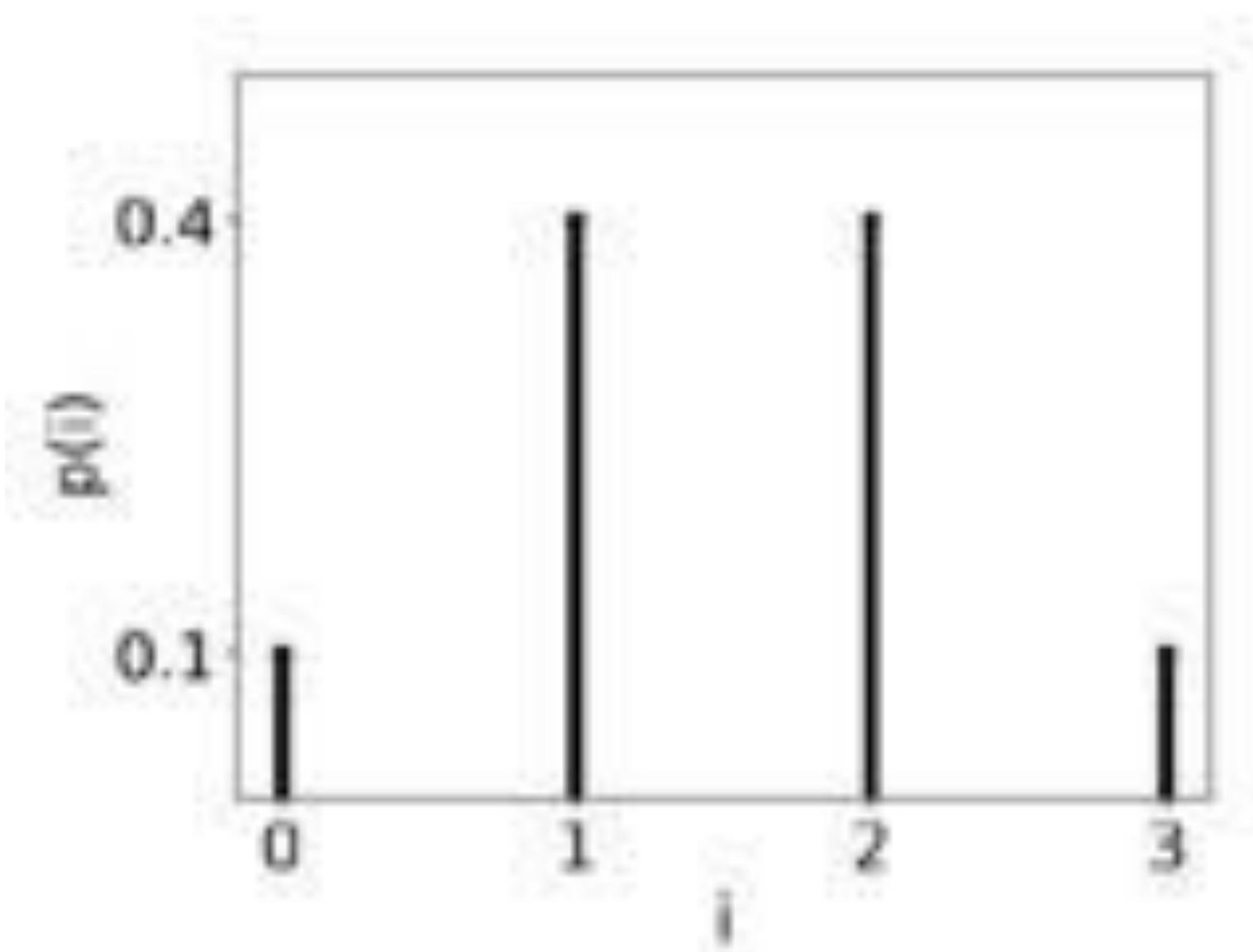
property of the distribution

Fair Die

$$\begin{aligned}E(X) &= \sum_{i=1}^6 P(i) \cdot i \\&= \sum_{i=1}^6 \frac{1}{6} \cdot i \\&= \frac{1 + 2 + \dots + 6}{6} \\&= \frac{1}{6} \frac{(1 + 6) \cdot 6}{2} \\&= \frac{7}{2} = 3.5 \quad \checkmark\end{aligned}$$



4 Sided- Die



$$E(X) = \sum_{i=1}^4 p_i \cdot i$$

$$= 0.1 \cdot 1 + 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4$$

$$= 3$$



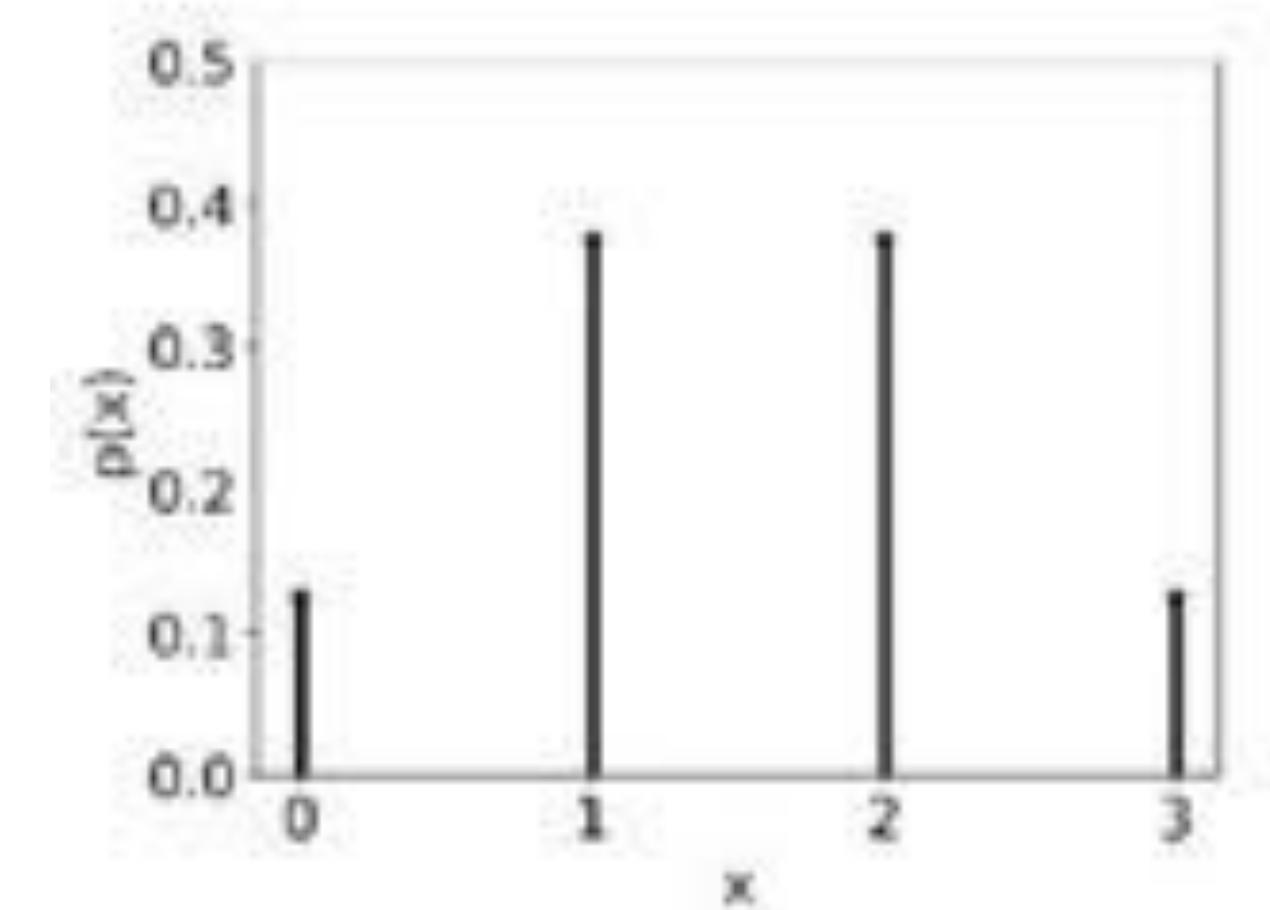
3 Coins

Toss a coin 3 times

X - # heads

$E(X) = ?$

x	outcomes	p(x)
0	ttt	$\frac{1}{8}$
1	tth, tht, htt	$\frac{3}{8}$
2	thh, hth, hht	$\frac{3}{8}$
3	hhh	$\frac{1}{8}$



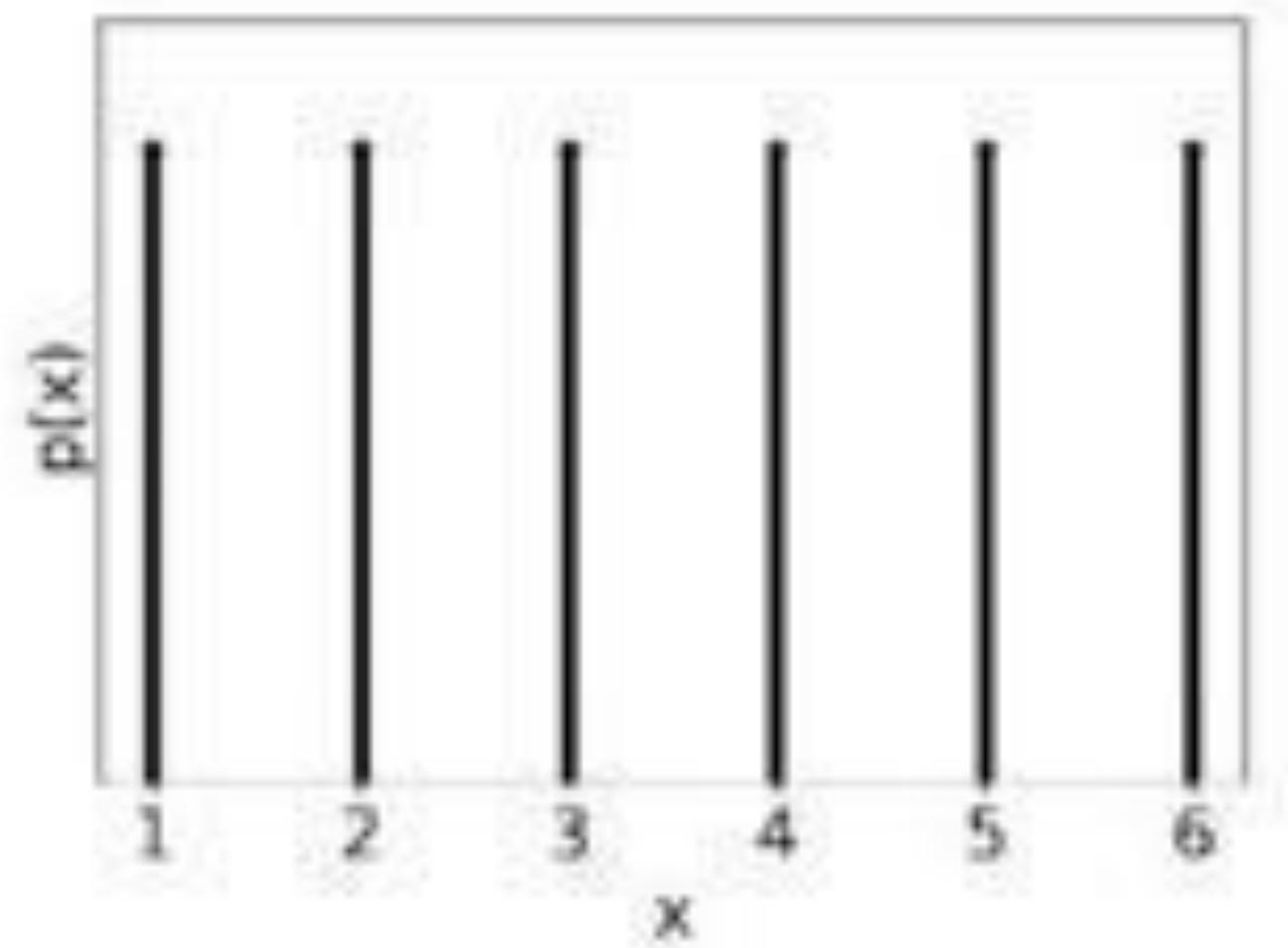
$$\sum P(x) \cdot x = \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3 = 1.5$$

heads ranges from 0 to 3, on average 1.5

Uniform Variables

X uniform over Ω

$$p(x) = \frac{1}{|\Omega|}$$



$$E(X) = \sum_{x \in \Omega} p(x) \cdot x = \sum_{x \in \Omega} \frac{1}{|\Omega|} \cdot x = \frac{1}{|\Omega|} \sum_{x \in \Omega} x$$

$E(X)$ is the arithmetic average of elements in Ω



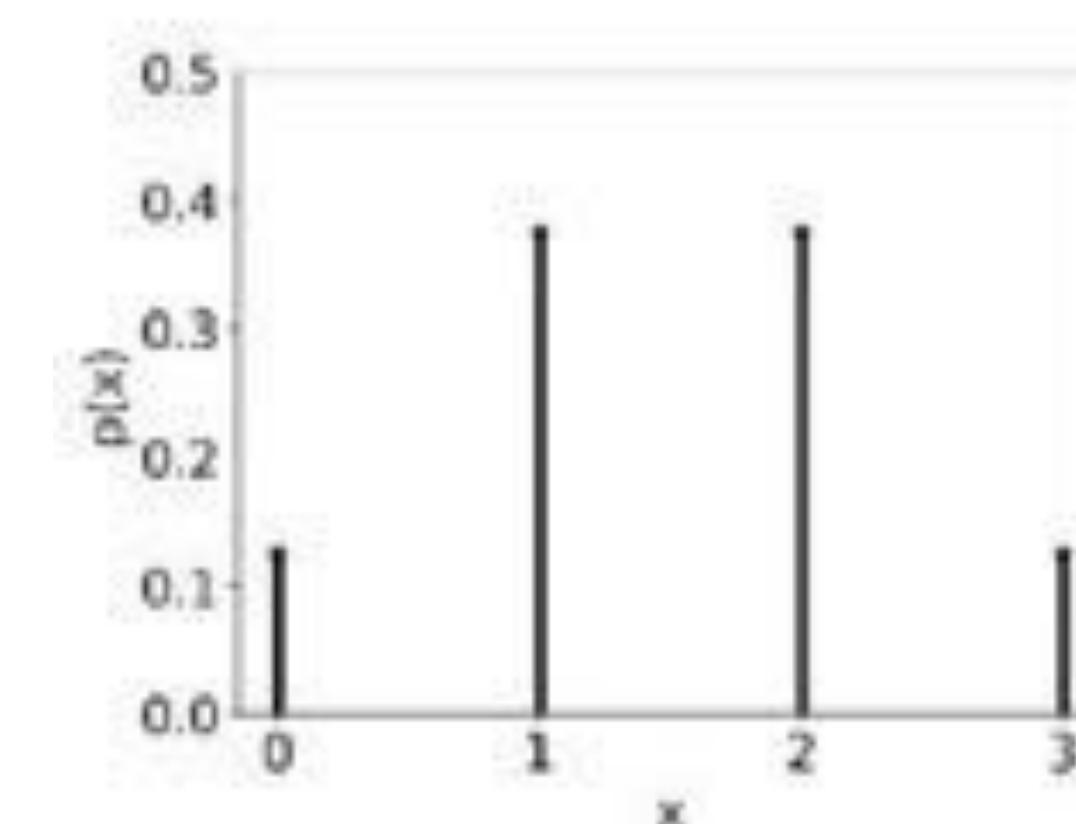
$$E(X) = \frac{1+2+...+6}{6} = 3.5$$

Symmetry

A distribution p is symmetric around a if for all $x > 0$, $p(a+x) = p(a-x)$

If p is symmetric around a , then $E(X) = a$

x	outcomes	P(x)
0	ttt	$\frac{1}{8}$
1	tth, tht, htt	$\frac{3}{8}$
2	thh, hth, hht	$\frac{3}{8}$
3	hhh	$\frac{1}{8}$



Symmetric around 1.5

$E(X) = 1.5$

Properties

$E(X)$

Despite notation

Not random

Number

Property of distribution

$$E(X) = 1.5$$

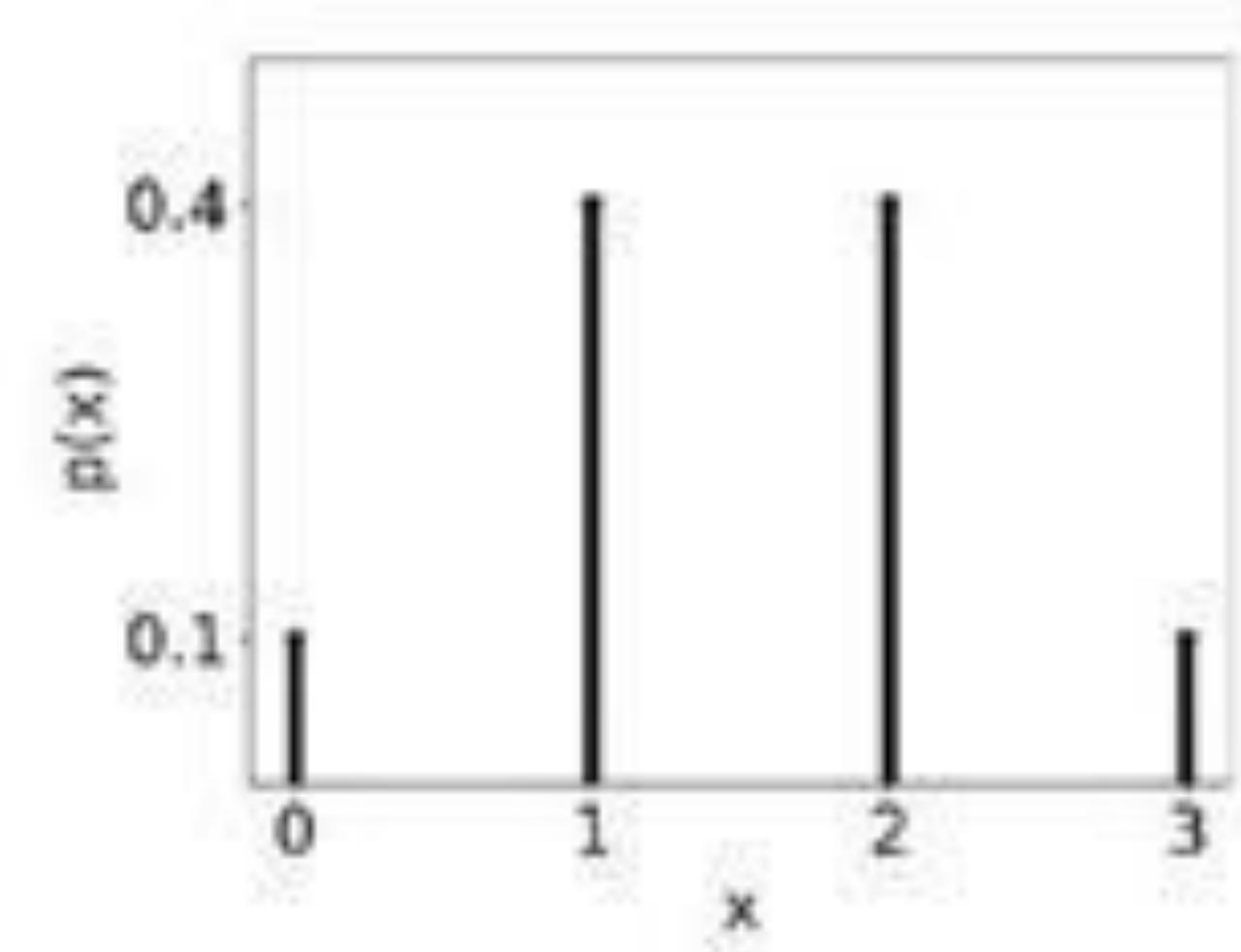
$$x_{\min} \leq E(X) \leq x_{\max}$$

$$= \text{ iff } X = c$$

$$0 \leq E(X) \leq 3$$

X is a constant, namely $X=c \rightarrow E(X)=c$

$$E(E(X)) = E(X)$$



Is Expectation Expected?

$\mu = EX$ - expectation of X

Do we expect to see it?

Is p_μ high?

Not necessarily

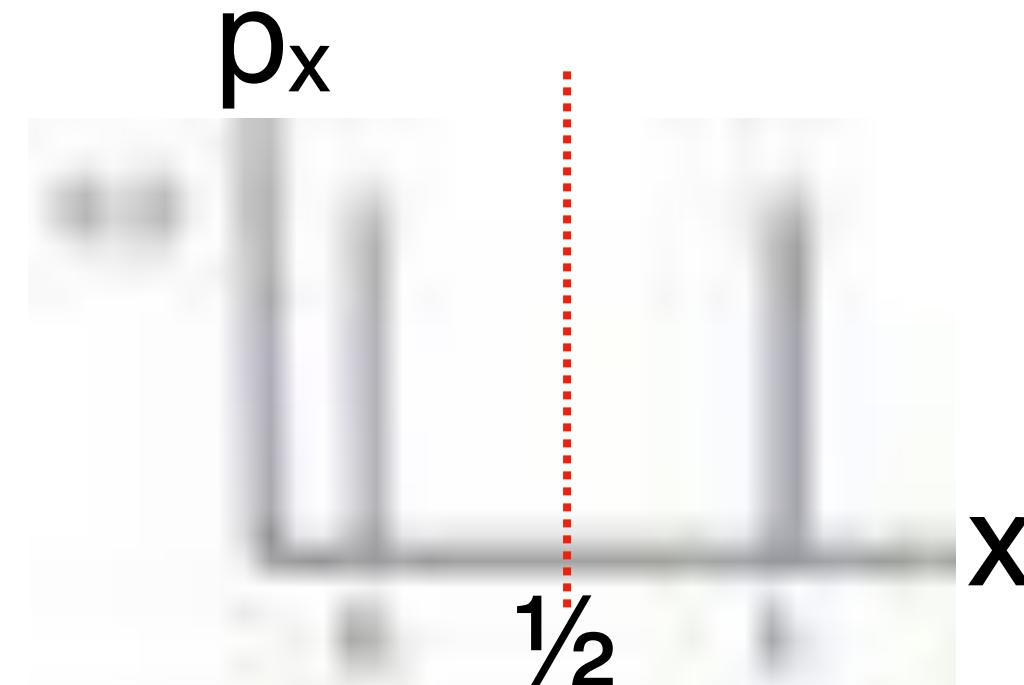
We may never see it!

$X \in \{0,1\}$

$p_0 = p_1 = 0.5$

$$EX = 0 \cdot p_0 + 1 \cdot p_1 = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Symmetric around $\frac{1}{2}$



$\frac{1}{2}$ will never happen!

Many samples \rightarrow average = $\frac{1}{2}$

EX - average of large sample

Not necessarily likely

May not be observed at all

Infinite Expectation

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

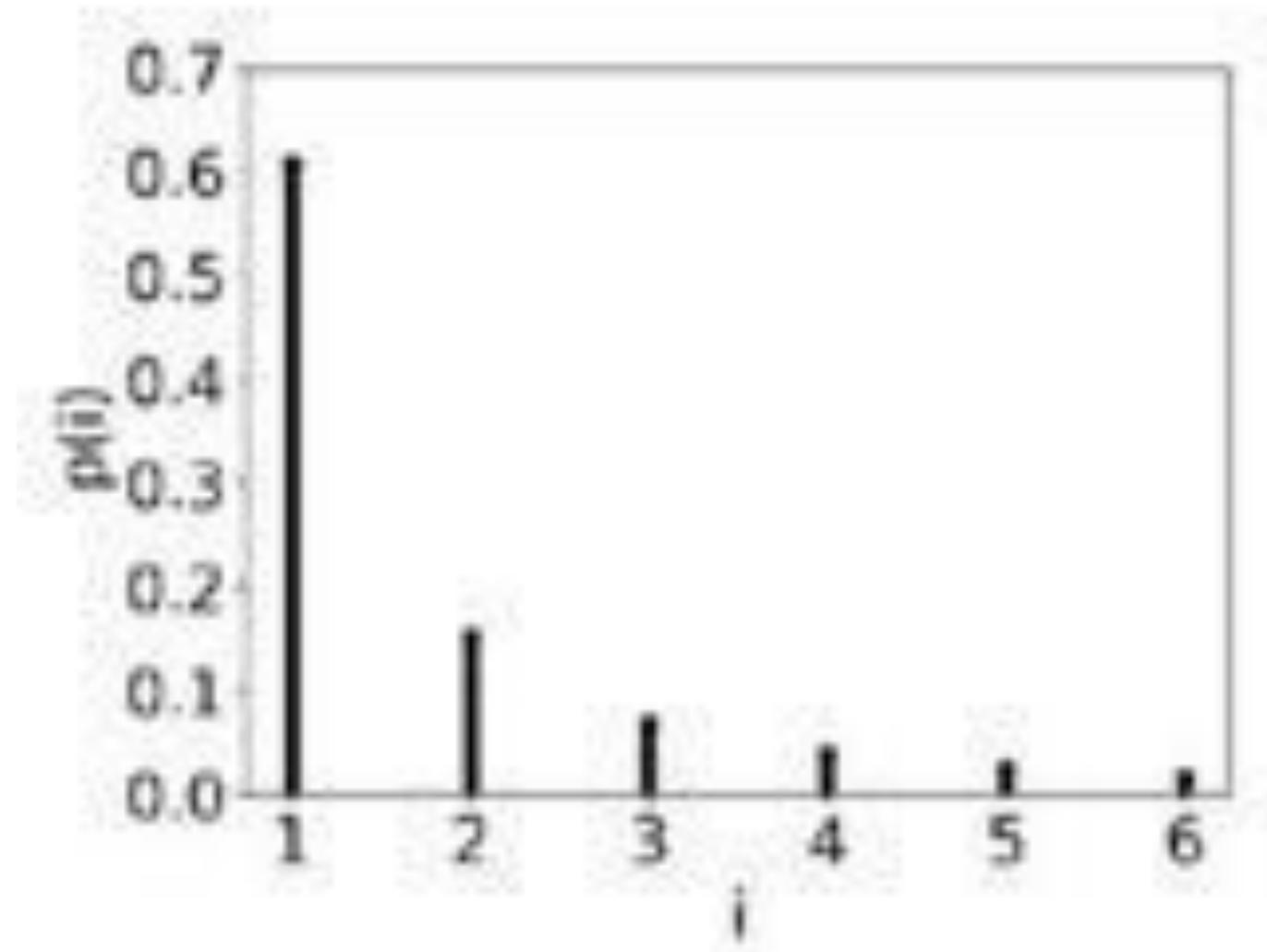
Basel problem

Euler → famous

$$\frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} = 1$$

$$p_i = \frac{6}{\pi^2} \cdot \frac{1}{i^2}$$

probability distribution over \mathbb{P}



$$E(X) = \sum_{i=1}^{\infty} i \cdot p_i = \frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

Many samples

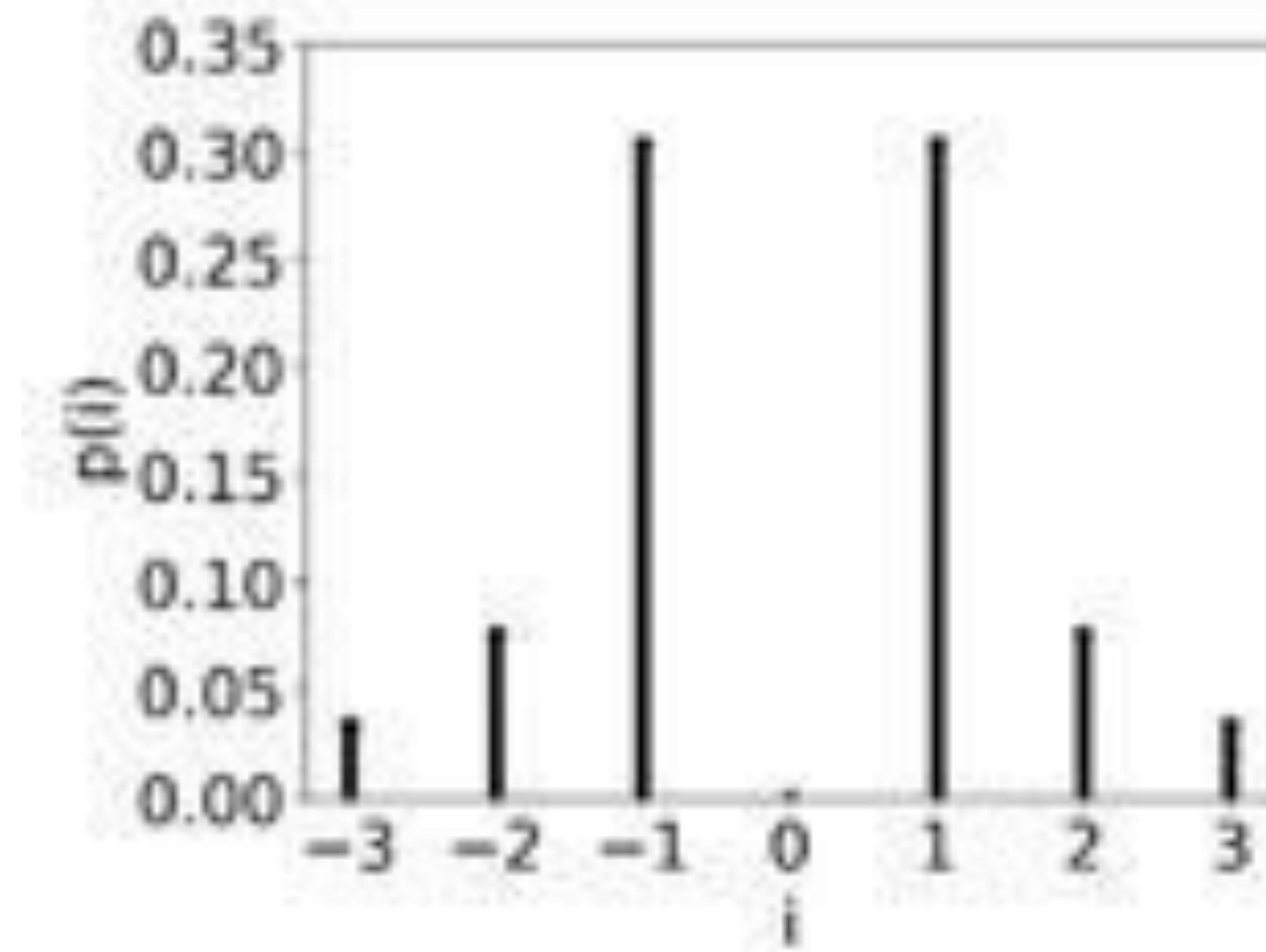
Average will go to ∞

Undefined Expectation

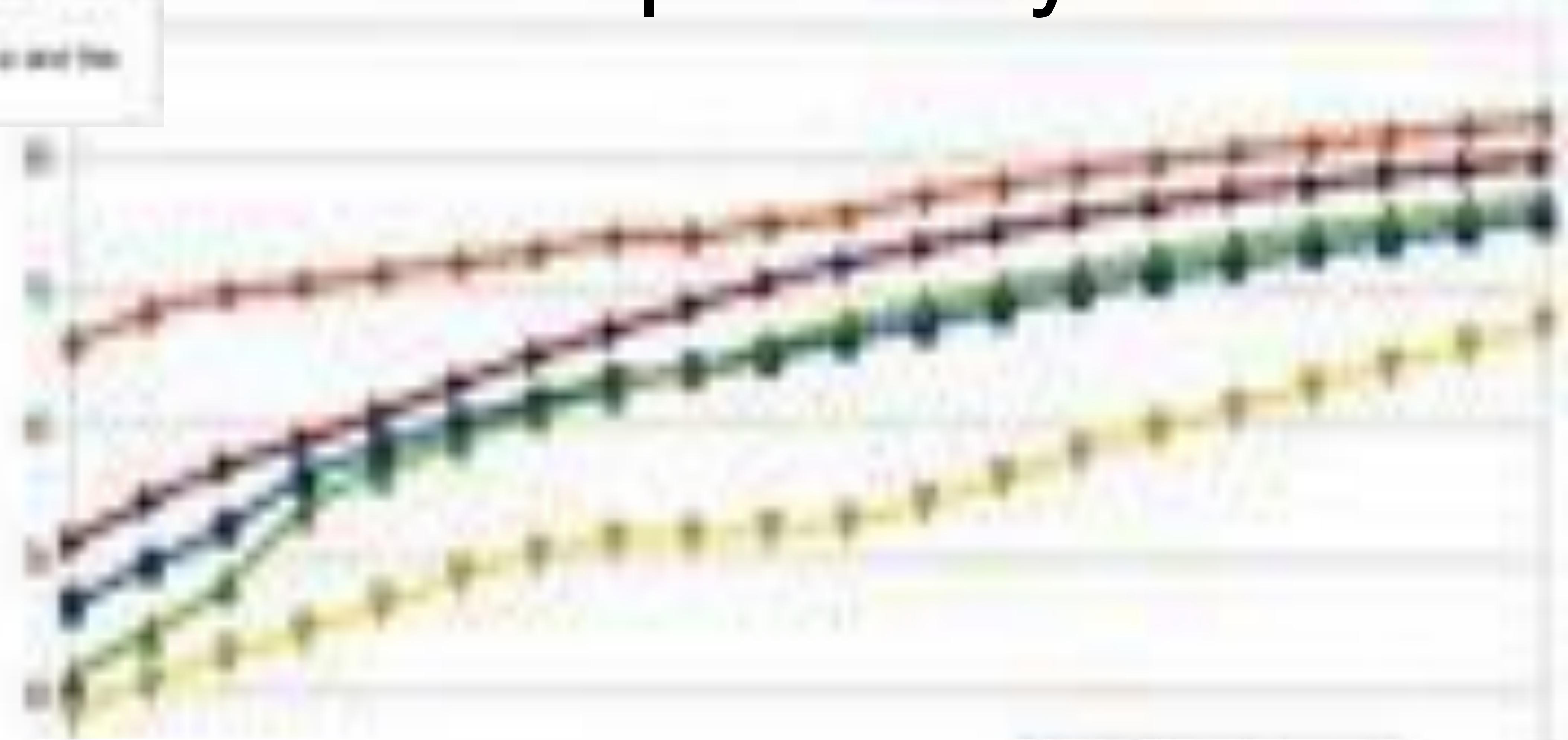
$$p_i = \frac{3}{\pi^2} \cdot \frac{1}{i^2} \quad \text{for } i \neq 0$$

$E(X) = \infty - \infty$

Undefined



Life Expectancy



1963 Mr Average

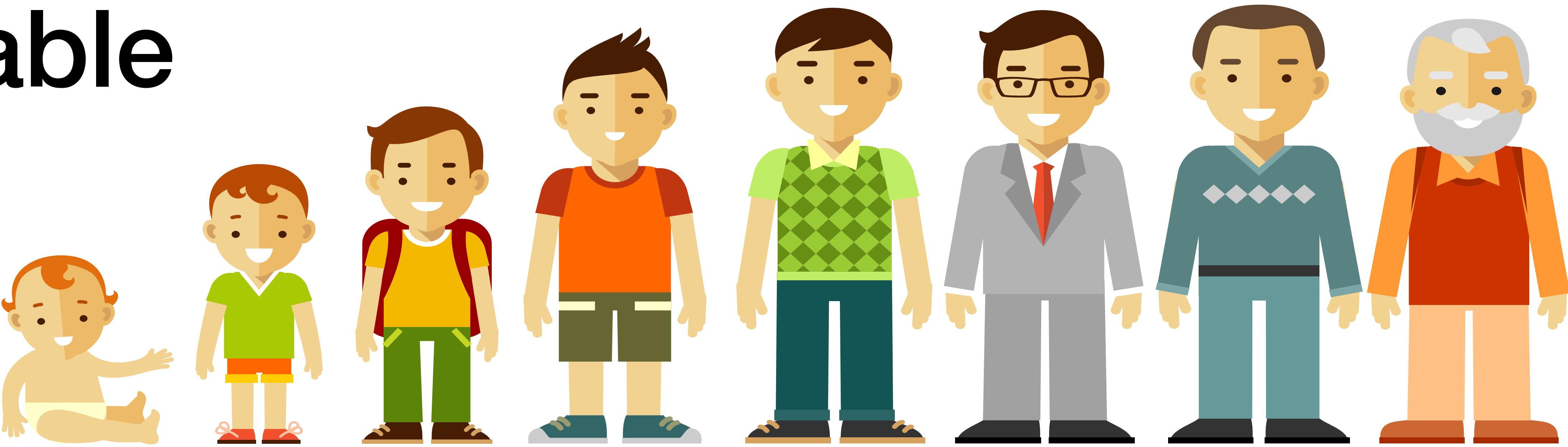
2017 Mr Average

Expectation



Expectations of Functions
of Random Variables

Variable



Modifications (aka functions)



Functions of a Random Variable

Random variables X take values in \mathbb{R}

Often interested in related variable

$$Y = g(X)$$

$g: \mathbb{R} \rightarrow \mathbb{R}$ is a fixed function

X

Random salary in \$

\$10 raise

$$Y = X + 10$$

10% raise

$$Y = 1.1X$$

→ CEO

$$Y = X^2$$

Deterministic Functions

$$Y = g(X)$$

g is a **deterministic** function over \mathbb{R} (or Ω)

$$Y = X + 3$$

All randomness in Y derives from X

Deterministically modified by g

$$X = 5$$

$$Y = 8$$

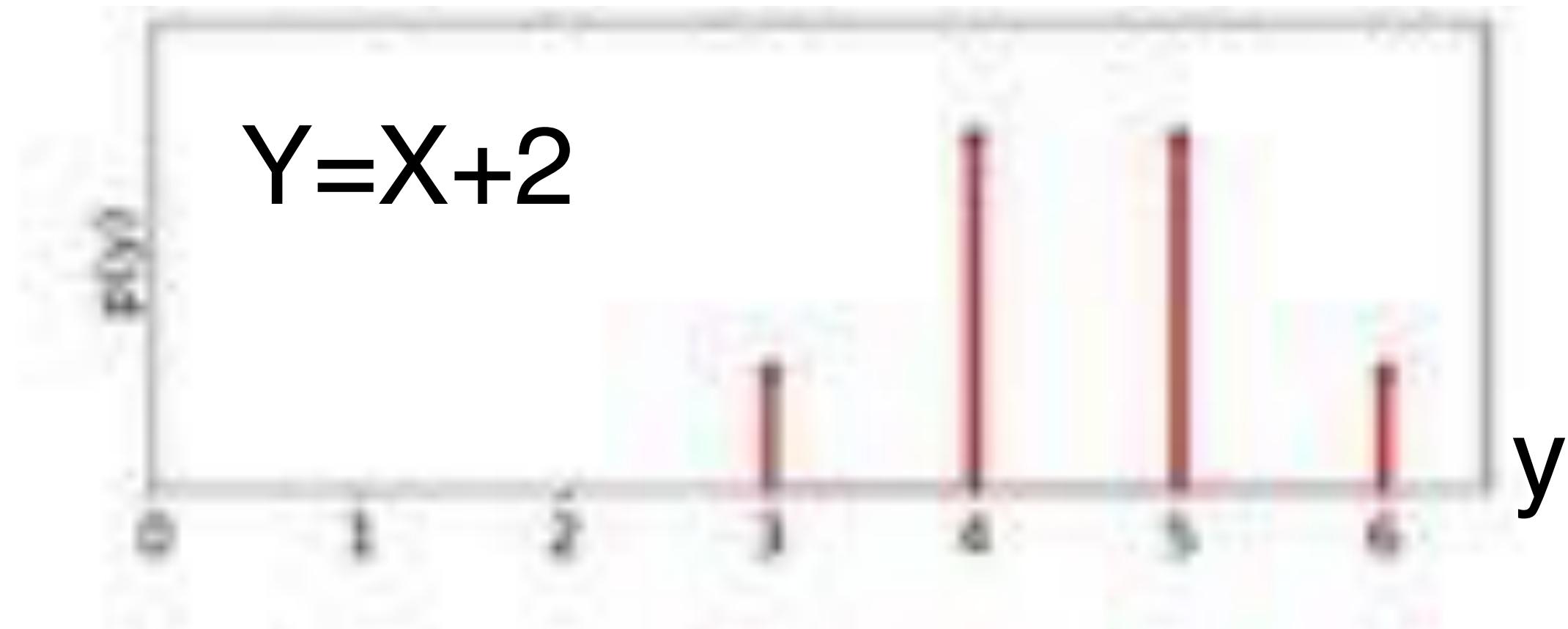
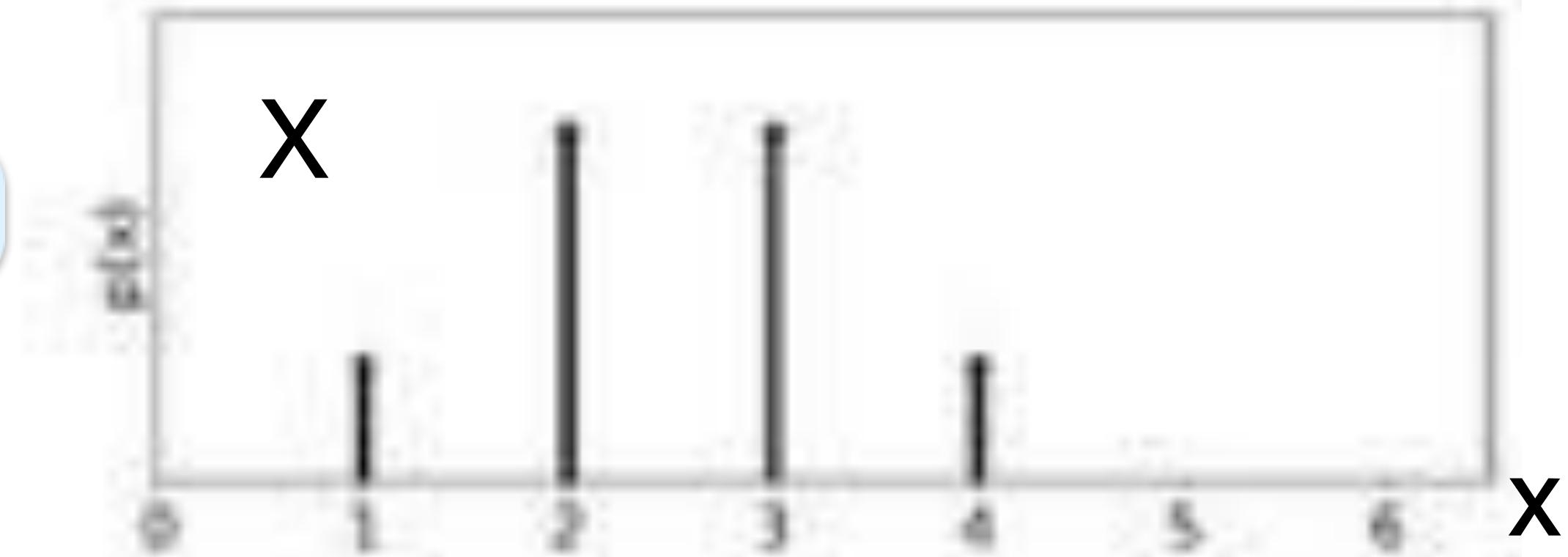
Translation

Add constant b to X

$$Y = X + b$$

$$P(Y=y) = P(X+b=y) = P(X=y-b)$$

Translate X by b



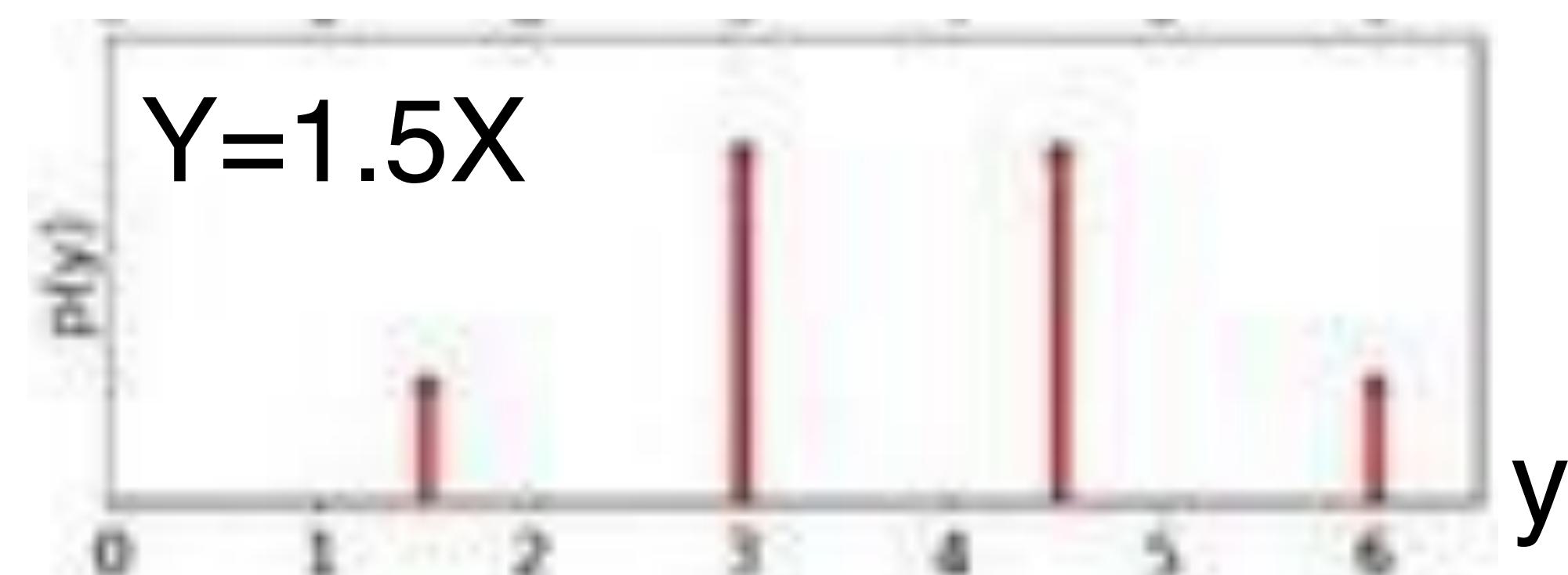
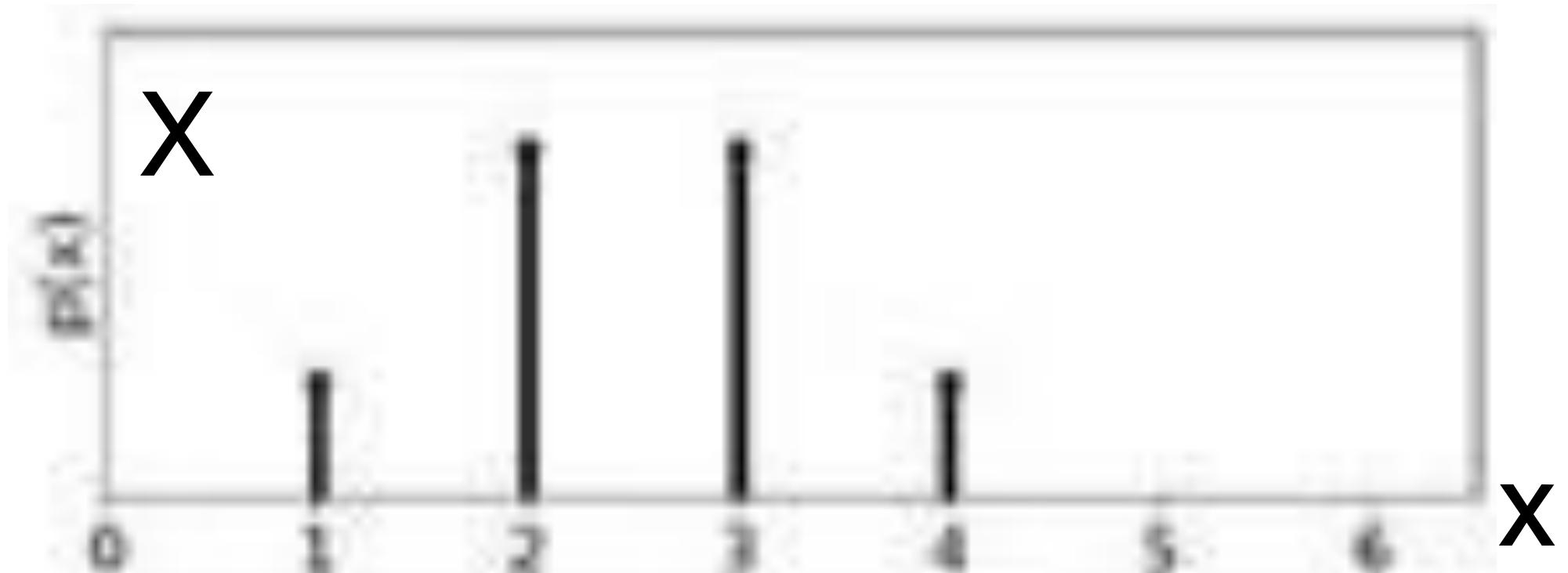
Scaling

Multiply X by a constant b

Scale X by a factor b

$$Y = b \cdot X$$

$$P(Y=y) = P(bX=y) = P(X=y/b)$$



Two Square Examples

Square is 1-1

X	x	0	1	2
$p(X = x)$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$Y = X^2$$

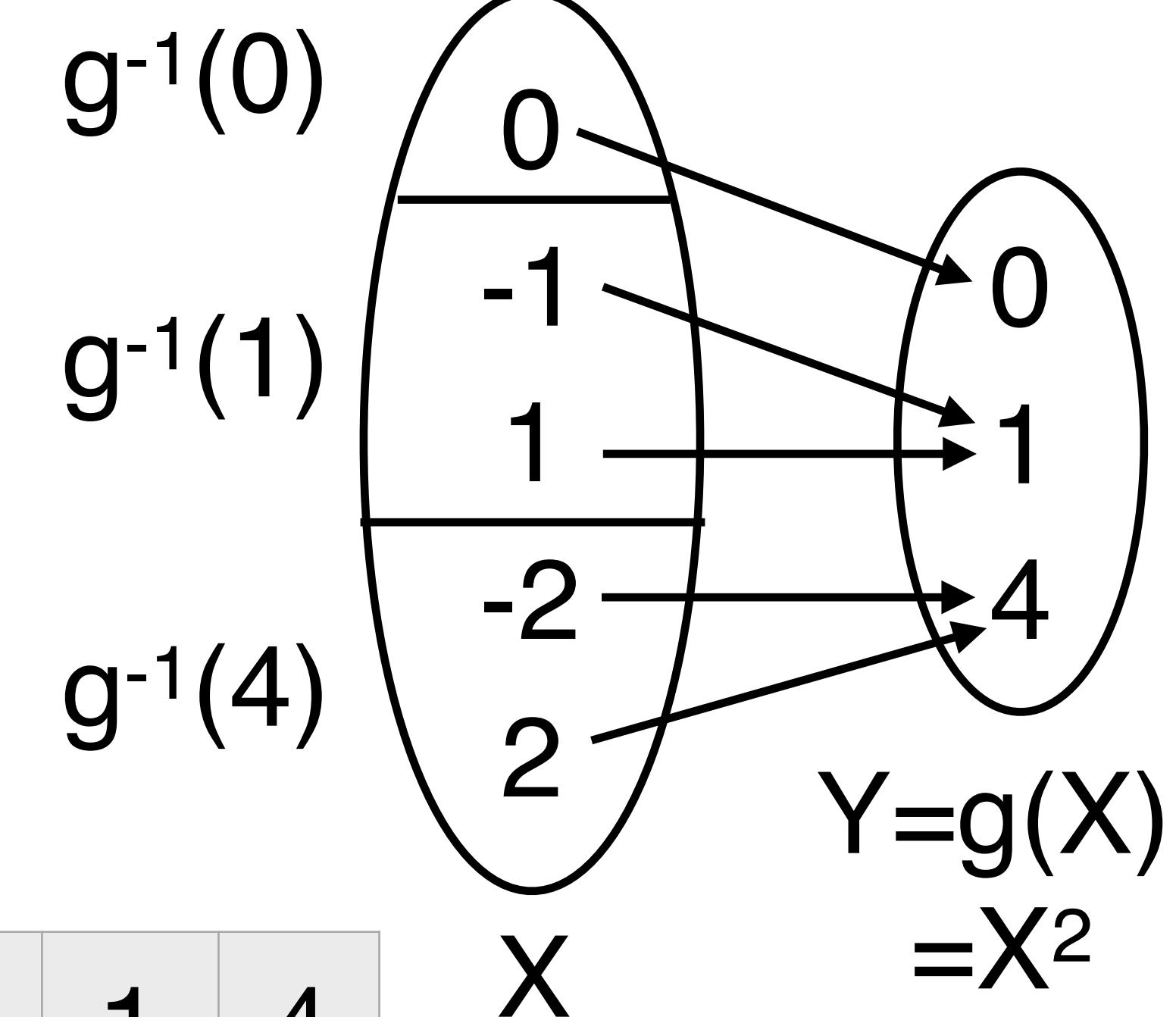
y	0	1	4
$p(Y = y)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Square is many to 1

X	x	-2	-1	0	1	2
$p(X = x)$		$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$$Y = X^2$$

y	0	1	4
$p(Y = y)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$



$$P(Y=y) = P(g(X)=y) = P(X \in g^{-1}(y)) = \sum_{x \in g^{-1}(y)} P(X=x)$$

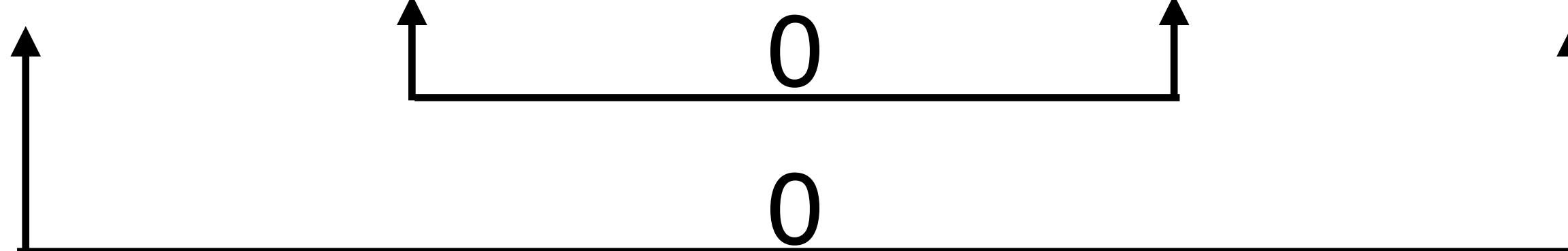
Expectation of Functions of Random Variables

Expectation Reminder

x	-2	-1	0	1	2
p(x)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$$E(X) = \sum_x p(x) \cdot x$$

$$= -2 \cdot \frac{1}{5} + -1 \cdot \frac{1}{5} + 0 \cdot \frac{1}{5} + 1 \cdot \frac{1}{5} + 2 \cdot \frac{1}{5} = 0$$



“By Symmetry”

Expectation of a Square

X	x	-2	-1	0	1	2
p(x)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	

$Y = X^2$

	y	0	1	4
p(y)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	

$$P(Y = 0) = P(X^2 = 0) = P(X = 0) = \frac{1}{5}$$

$$P(Y = 1) = P(X^2 = 1) = P(X \in \{-1, 1\}) = \frac{2}{5}$$

$$P(Y = 4) = P(X^2 = 4) = P(X \in \{-2, 2\}) = \frac{2}{5}$$

$$E(Y) = \frac{1}{5} \cdot 0 + \frac{2}{5} \cdot 1 + \frac{2}{5} \cdot 4 = \frac{10}{5} = 2$$

Alternative Formulation

$$E(Y) = \sum_y y \cdot P(Y=y)$$

$$= \sum_y y \cdot P(X \in g^{-1}(y))$$

$$= \sum_y y \sum_{x \in g^{-1}(y)} p(x)$$

$$= \sum_y \sum_{x \in g^{-1}(y)} y \cdot p(x)$$

$$= \sum_y \sum_{x \in g^{-1}(y)} g(x) \cdot p(x)$$

$$= \sum_x g(x) \cdot p(x)$$

Example

Visualize

Square Again

X	x	-2	-1	0	1	2
X	p(x)	1/5	1/5	1/5	1/5	1/5

Y = X ²	y	0	1	4
Y = X ²	p(y)	1/5	2/5	2/5

$$E(Y) = \sum_{y=0,1,4} y \cdot p(Y=y) = \frac{1}{5} \cdot 0 + \frac{2}{5} \cdot 1 + \frac{2}{5} \cdot 4 = 10/5 = 2$$

$$\begin{aligned} E(Y) &= \sum_x x^2 \cdot p(x) \\ &= (-2)^2 \cdot \frac{1}{5} + (-1)^2 \cdot \frac{1}{5} + 0^2 \cdot \frac{1}{5} + 1^2 \cdot \frac{1}{5} + 2^2 \cdot \frac{1}{5} \\ &= \frac{4}{5} + \frac{1}{5} + \frac{1}{5} + \frac{4}{5} = 2 \end{aligned}$$

Visualization

$$E(Y) = \sum_y y \cdot P(Y=y)$$

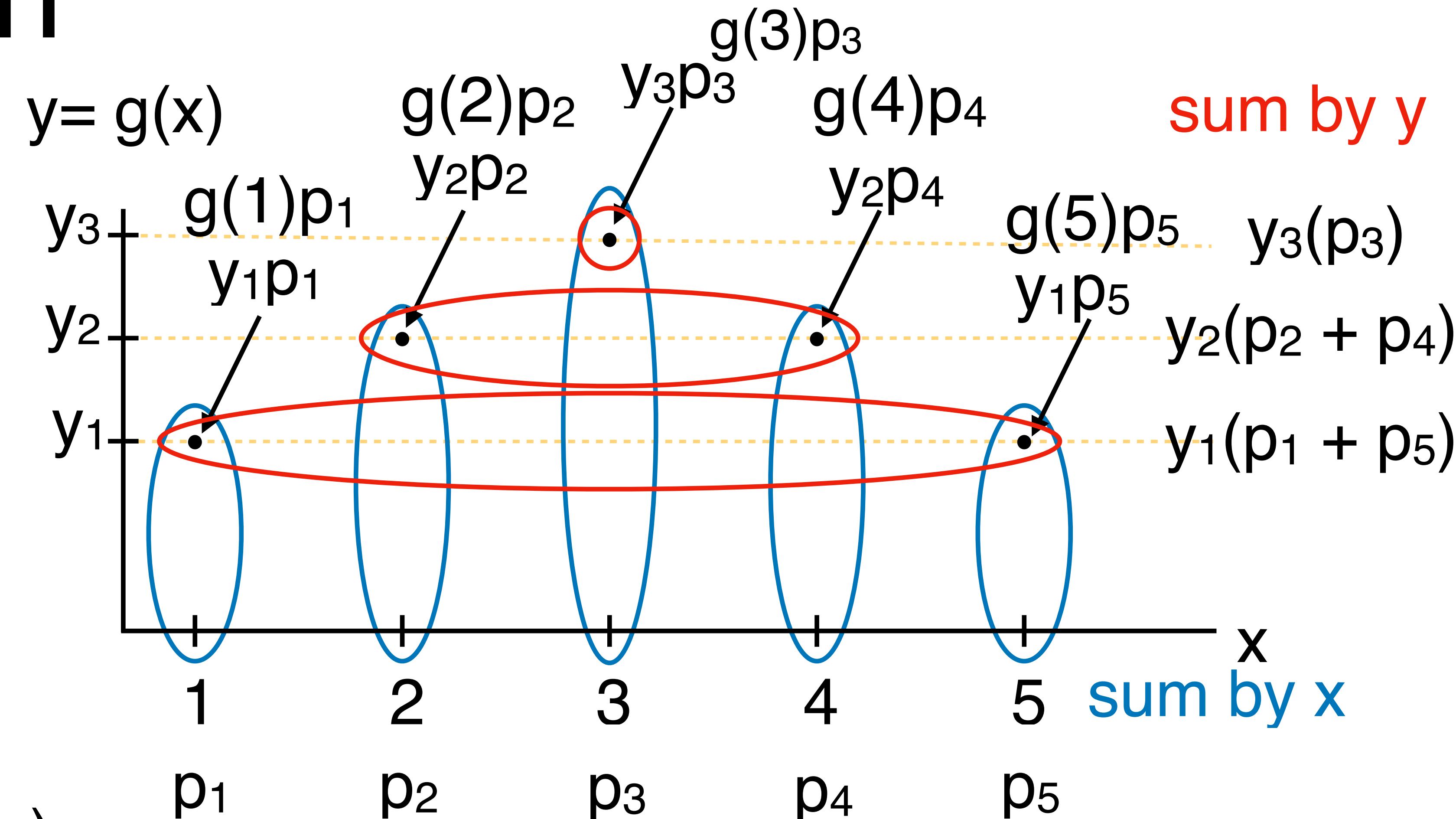
$$= \sum_y y \cdot P(X \in g^{-1}(y))$$

$$= \sum_y y \sum_{x \in g^{-1}(y)} p(x)$$

$$= \sum_y \sum_{x \in g^{-1}(y)} y \cdot p(x)$$

$$= \sum_y \sum_{x \in g^{-1}(y)} g(x) \cdot p(x)$$

$$= \sum_x g(x) \cdot p(x)$$



y: Fewer multiplication

x: Properties (next)

General Formulas

Constant Addition

$$E(X + b) = \sum p(x) \cdot (x + b)$$

$$= \sum p(x) \cdot x + \sum p(x) \cdot b$$

$$= E(X) + b \cdot \sum p(x)$$

$$= E(X) + b$$

x	0	1
p(x)	1 - p	p

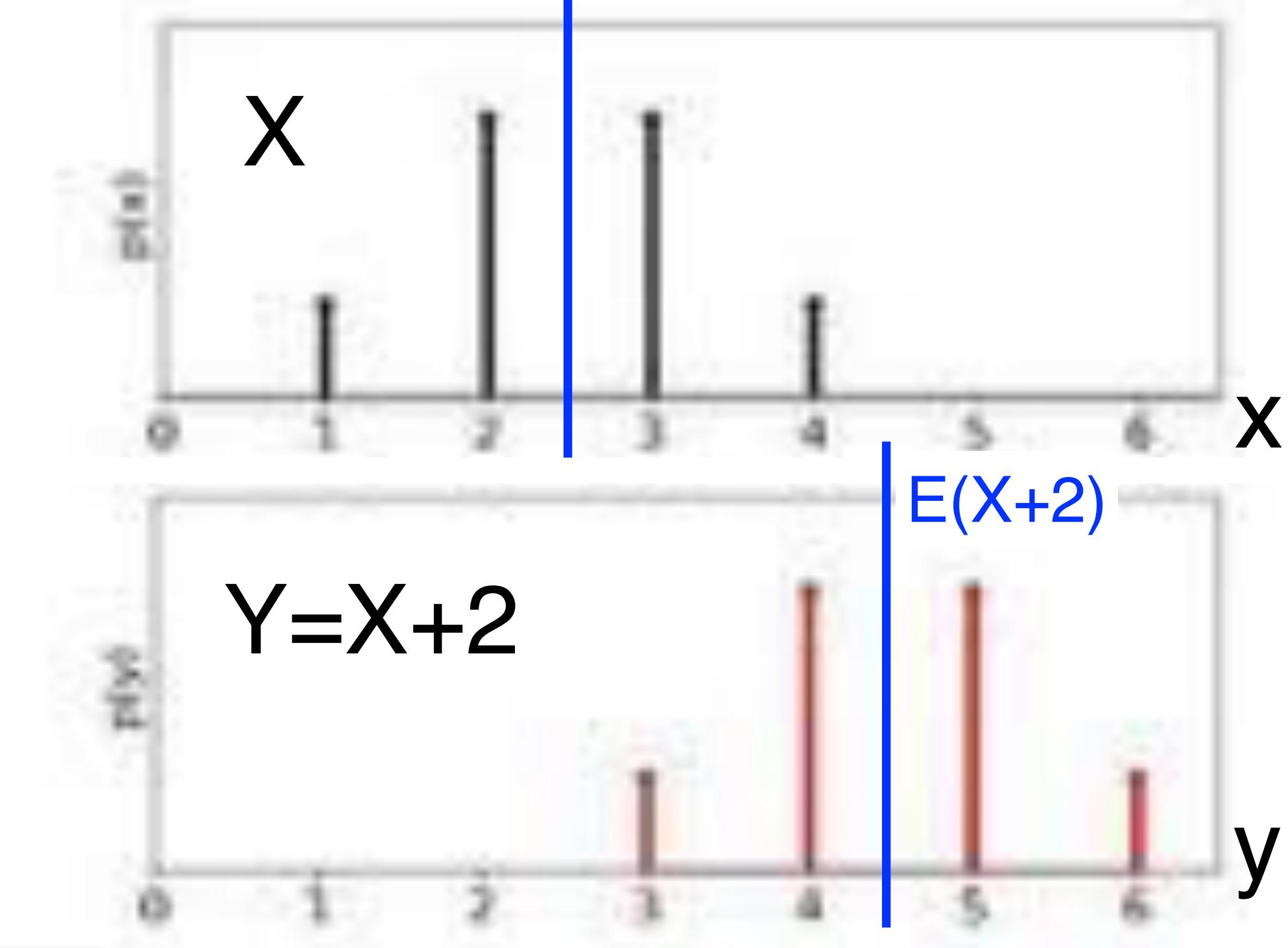
Bernoulli p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

$$E(X + 2) = (1 - p) \cdot (0 + 2) + p \cdot (1 + 2)$$

$$= 2 - 2p + 3p$$

$$= p + 2 = E(X) + 2$$



Constant Multiplication

$$E(aX) = \sum p(x) \cdot (ax)$$

$$= a \sum p(x) \cdot x$$

$$= aE(X)$$

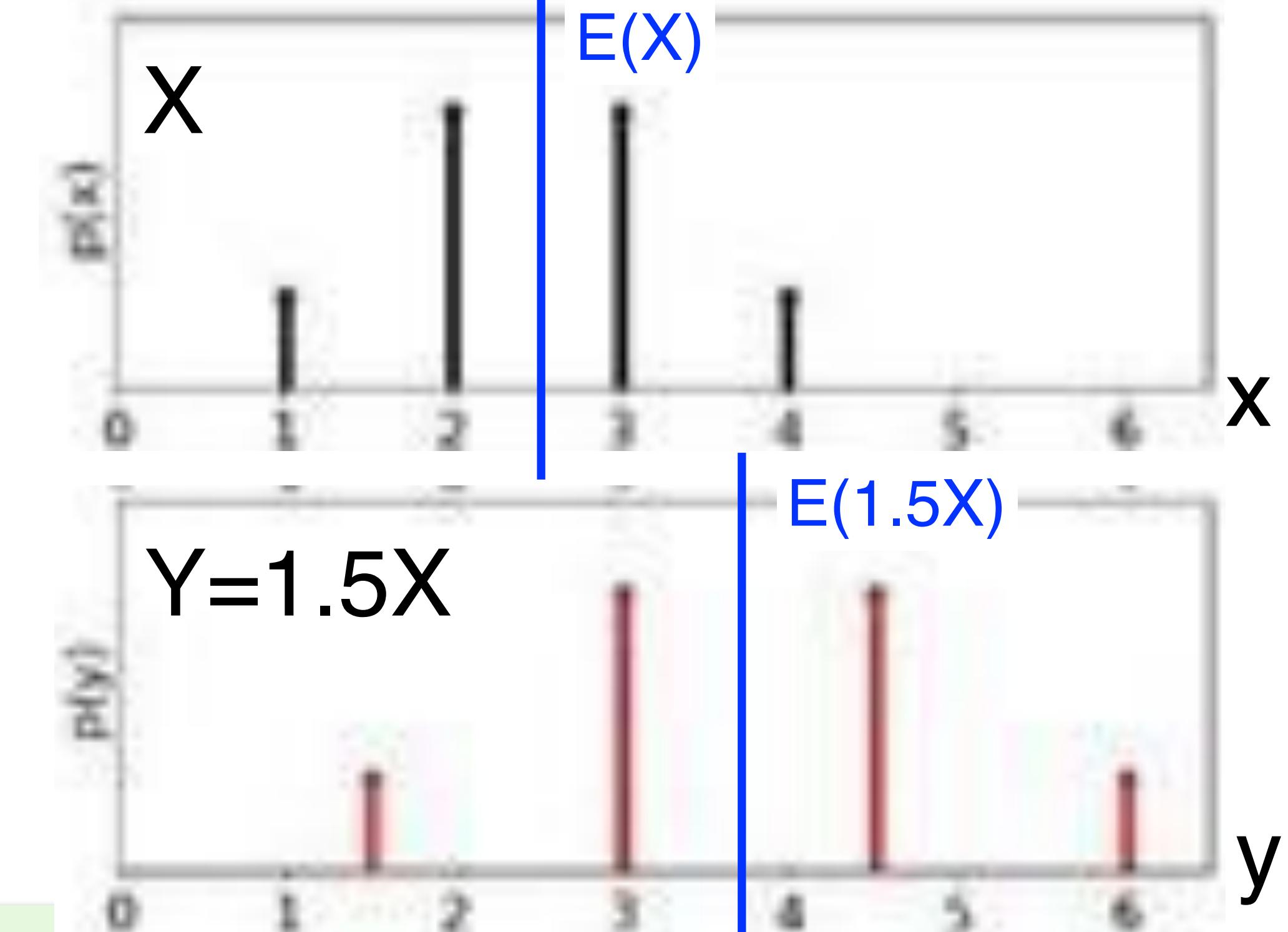
Bernoulli p

x	0	1
p(x)	1 - p	p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

$$E(3X) = (1 - p) \cdot (3 \cdot 0) + p \cdot (3 \cdot 1)$$

$$= 3p = 3E(X)$$



Linearity of Expectation

$$E(aX + b) = E(aX) + b$$

$$= a E(X) + b$$

Bernoulli p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

x	0	1
p(x)	1 - p	p

$$E(2X + 3) = (1 - p)(2 \cdot 0 + 3) + p(2 \cdot 1 + 3)$$

$$= 3 - 3p + 5p$$

$$= 2p + 3 = 2E(X) + 3$$

Expectation of Functions of Variables

Next: Variance