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Neutrino Mass and Mixing with Discrete Symmetry *

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Abstract: This is a review article about neutrino mass and mixing and flavour model building strategies based on discrete family symmetry. After a pedagogical introduction and overview of the whole of neutrino physics, we focus on the PMNS mixing matrix and the latest global fits following the Daya Bay and RENO experiments which measure the reactor angle. We then describe the simple bimaximal, tri-bimaximal and golden ratio patterns of lepton mixing and the deviations required for a non-zero reactor angle, with solar or atmospheric mixing sum rules resulting from charged lepton corrections or residual trimaximal mixing. The different types of see-saw mechanism are then reviewed as well as the sequential dominance mechanism. We then give a mini-review of finite group theory, which may be used as a discrete family symmetry broken by flavons either completely, or with different subgroups preserved in the neutrino and charged lepton sectors. These two approaches are then reviewed in detail in separate chapters including mechanisms for flavon vacuum alignment and different model building strategies that have been proposed to generate the reactor angle. We then briefly review grand unified theories (GUTs) and how they may be combined with discrete family symmetry to describe all quark and lepton masses and mixing. Finally we discuss three model examples which combine an $SU(5)$ GUT with the discrete family symmetries A_4, S_4 and $\Delta(96)$.

Keywords: Beyond the Standard Model, Supersymmetric Models, Neutrino Physics.

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1. Introduction

1.1 Historical overview

In 1930 the Austrian physicist Wolfgang Pauli proposed the existence of particles called neutrinos, denoted as ν, as a "desperate remedy" to account for the missing energy in a type of radioactivity called beta decay. At the time physicists were puzzled because nuclear beta decay appeared to violate energy conservation. In beta decay, a neutron in an unstable nucleus transforms into a proton and emits an electron, where the radiated electron was found to have a continuous energy spectrum. This came as a great surprise to many physicists because other types of radioactivity involved gamma rays and alpha particles with discrete energies. Pauli deduced that some of the energy must have been taken away by a new particle emitted in the decay process, the neutrino, which carries energy and has spin 1/2, but which is massless, electrically neutral and very weakly interacting. Because neutrinos interact so weakly with matter, Pauli bet a case of champagne that nobody would ever detect one, and they became known as "ghost particles". Indeed it was not until a quarter of a century later, in 1956, that Pauli lost his bet and neutrinos were discovered when Clyde Cowan and Fred Reines detected antineutrinos emitted from a nuclear reactor at Savannah River in South Carolina, USA.

Since then, after decades of painstaking experimental and theoretical work, neutrinos have become enshrined as an essential part of the accepted quantum description of fundamental particles and forces, the Standard Model (SM) of particle physics. This is a highly successful theory in which elementary building blocks of matter are divided into three generations of two kinds of particle - quarks and leptons. It also includes three of the fundamental forces of Nature, the strong (g), electromagnetic (γ) and weak (W, Z) forces carried by spin 1 force carrying bosons (shown in parentheses) but does not include gravity. There are six flavours of quarks. The leptons consist of the charged electron e^-, muon μ^- and tau τ^-, together with three electrically neutral particles - the electron neutrino ν_e, muon neutrino ν_μ and tau neutrino ν_τ which are our main concern here.

The first clues that neutrinos have mass came from an experiment deep underground, carried out by an American scientist Raymond Davis Jr., detecting solar neutrinos [1]. It revealed only about one-third of the number predicted by theories of how the Sun works pioneered by John Bahcall [1]. The result puzzled both solar and neutrino physicists. Based on the original idea of neutrino oscillation, first introduced by Pontecorvo in 1957 [2] and independently by Maki, Nakagawa and Sakata in 1962 [3], some Russian researchers, Mikheyev and Smirnov, developing ideas proposed previously by Wolfenstein in the U.S., suggested that the solar neutrinos might be changing into something else. Only electron neutrinos are emitted by the Sun and they could be converting into muon and tau neutrinos which were not being detected by Davis's experiment. The precise mechanism for "solar neutrino oscillations" proposed by Mikheyev, Smirnov and Wolfenstein involved the resonant enhancement of neutrino oscillations due to matter effects in the Sun, and is known as the MSW effect [4].

Neutrino oscillations are analogous to coupled pendulums, where oscillations in one pendulum induce oscillations in another pendulum. The coupling strength is defined in terms of something called the "lepton mixing matrix" U which relates the basic Standard Model neutrino states, ν_e, ν_μ, ν_τ, associated with the electron, muon and tau, to the neutrino mass states ν_1, ν_2, and ν_3 with (real and positive) masses m_1, m_2, and m_3 [3],

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} . \tag{1.1}$$

According to quantum mechanics it is not necessary that the Standard Model states ν_e, ν_μ, ν_τ be identified in a one-one way with the mass eigenstates ν_1, ν_2, and ν_3, and the matrix elements of U give the quantum amplitude that a particular Standard Model state contains an admixture of a particular mass eigenstate. The probability that a particular neutrino mass state contains a particular SM state may be represented by colours as in Fig. 1. Note that neutrino oscillations are only sensitive to the differences between the squares of the neutrino masses $\Delta m^2_{ij} \equiv m^2_i - m^2_j$, and gives no information about the absolute value of the neutrino mass squared eigenvalues m^2_i. There are basically two patterns of neutrino mass squared orderings consistent with the atmospheric and solar data as shown in Fig. 1.

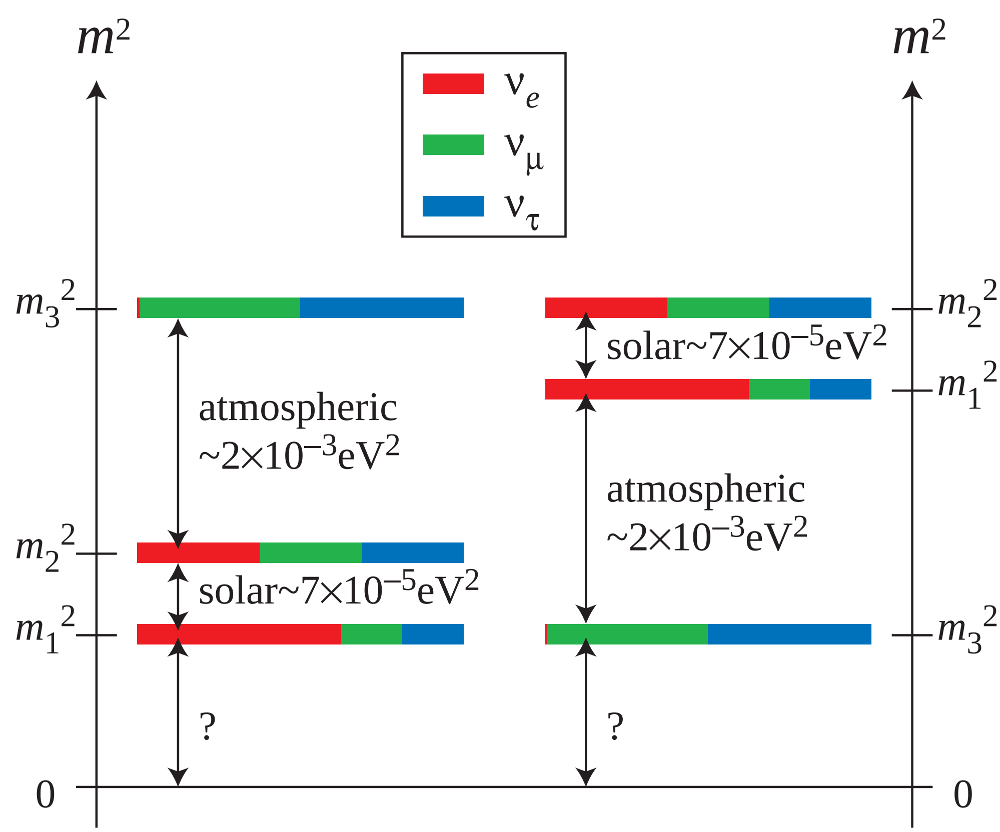

Figure 1: The probability that a particular neutrino mass state contains a particular SM state may be represented by colours as shown in the key. Note that neutrino oscillation experiments only determine the difference between the squared values of the masses. Also, while $m^2_2 > m^2_1$, it is presently unknown whether m^2_3 is heavier or lighter than the other two, corresponding to the left and right panels of the figure, referred to as normal or inverted mass squared ordering, respectively. Finally the value of the lightest neutrino mass (sometimes referred to as the neutrino mass scale) is presently unknown and is represented by a question mark in each case.

As with all quantum amplitudes, the matrix elements of U are expected to be complex numbers in general. The lepton mixing matrix U is also frequently referred to as the Maki-Nakagawa-Sakata (MNS) matrix $U_{\rm MNS}$ [3], and sometimes the name of Pontecorvo is added at the beginning to give $U_{\rm PMNS}$. The standard parameterisation of the PMNS matrix in terms of three angles and at least one complex phase, as recommended by the Particle Data Group (PDG) [5], will be discussed later.

Before getting into details, here is a quick executive summary of the implications of neutrino mass and mixing following from Fig. 1:

- Lepton flavour is not conserved, so the individual lepton numbers L_e, L_μ, L_τ are separately broken
- Neutrinos have tiny masses which are not very hierarchical
- Neutrinos mix strongly unlike quarks
- The SM parameter count is increased by at least 7 new parameters (3 neutrino masses, 3 mixing angles and at least one complex phase)
- It is the first (and so far only) new physics beyond the SM

The idea of neutrino oscillations was first confirmed in 1998 by the Japanese experiment Super-Kamiokande (SK) [6] which showed that there was a deficit of muon neutrinos reaching Earth when cosmic rays strike the upper atmosphere, the so-called "atmospheric neutrinos". Since most neutrinos pass through the Earth unhindered, Super-Kamiokande was able to detect muon neutrinos coming from above and below, and found that while the correct number of muon neutrinos came from above, only about a half of the expected number came from below. The results were interpreted as half the muon neutrinos from below oscillating into tau neutrinos over an oscillation length L of the diameter of the Earth, with the muon neutrinos from above having a negligible oscillation length, and so not having time to oscillate, yielding the expected number of muon neutrinos from above.

In 2002, the Sudbury Neutrino Observatory (SNO) in Canada spectacularly confirmed the flavour conversion in "solar neutrinos" [7]. The experiment measured both the flux of the electron neutrinos and the total flux of all three types of neutrinos. The SNO data revealed that physicists' theories of the Sun were correct after all, and the solar neutrinos ν_e were produced at the standard rate but were oscillating into ν_μ and ν_τ, with only about a third of the original ν_e flux arriving at the Earth.

Since then, neutrino oscillations consistent with solar neutrino observations have been seen using man made neutrinos from nuclear reactors at KamLAND in Japan [8] (which, for the first time, observed the periodic pattern characteristic for neutrino oscillations), and neutrino oscillations consistent with atmospheric neutrino observations have been seen using neutrino beams fired over hundreds of kilometres as in the K2K experiment in Japan [9], the Fermilab-MINOS experiment in the U.S. [10] or the CERN-OPERA experiment in Europe. Further long-baseline neutrino beam experiments are in the pipeline, and neutrino oscillation physics is entering the precision era, with superbeams and a neutrino factory on the horizon.

Following these results several research groups showed that the electron neutrino has a mixing matrix element of $|U_{e2}| \approx 1/\sqrt{3}$ which is the quantum amplitude for ν_e to contain an admixture of the mass eigenstate ν_2 corresponding to a massive neutrino of mass $m_2 \approx$ 0.008 electronvolts (eV) or greater (where $\sqrt{m_2^2 - m_1^2} \approx 0.008$ eV). By comparison the electron has a mass of about half a megaelectronvolt (MeV). Put another way, the mass state ν_2 contains roughly equal probabilities of ν_e, ν_μ and ν_τ sometimes called trimaximal mixing, corresponding to the three equal red, green and blue colours associated with m_2^2 in Fig. 1. The muon and tau neutrinos were observed to contain approximately equal amplitudes of the third neutrino ν_3 of mass m_3, $|U_{\mu 3}| \approx |U_{\tau 3}| \approx 1/\sqrt{2}$, where a normalised amplitude of $1/\sqrt{2}$ corresponds to a 1/2 fraction of ν_3 in each of ν_μ and ν_τ, leading to a maximal mixing and oscillation of $\nu_\mu \leftrightarrow \nu_\tau$. Put another way, the mass state ν_3 contains roughly equal probabilities of ν_μ and ν_τ called maximal mixing, corresponding to the two equal green and blue colours associated with m_3^2 in Fig. 1. Interestingly, the value of m_3 is not determined and it could be anywhere between zero and 0.3 eV, depending on the mass scale and ordering. Although at least one neutrino mass must be 0.05 eV or greater (where $\sqrt{|m_3^2 - m_2^2|} \approx 0.05$ eV), this could be either m_3 or m_2, as shown in Fig. 1.

According to the early results from the CHOOZ nuclear reactor experiment [11], the electron neutrino ν_e could only contain a very small amount of the third neutrino mass eigenstate ν_3, $|U_{e3}| < 0.2$. Evidence for non-zero U_{e3} was first provided by T2K, MINOS and Double Chooz [12]. Recently the Daya Bay [13], RENO [14], and Double Chooz [15] collaborations have measured $|U_{e3}| \approx 0.15$. Put another way, the mass state ν_3 has a probability of containing ν_e of about $(0.15)^2$, corresponding to the small amount of red colour associated with m_3^2 in Fig. 1. As we shall see, this element being non-zero excludes a number of simple mixing patterns and models which were previously proposed, and has led to a number of new developments.

1.2 Where we stand

The main experimental milestones from 1998-2012 may be summarised as follows:

- 1998 - SK confirms that atmospheric ν_μ are converted to another neutrino type, probably ν_τ consistent with near maximal mixing $|U_{\mu 3}| \approx |U_{\tau 3}| \approx 1/\sqrt{2}$

- 2002 - SK, SNO and the older neutrino experiments such as Homestake and the Gallium experiments results are combined in a global fit pointing towards the large (but non-maximal) mixing and conversion of solar neutrinos in the core of the Sun

- 2002 - SNO confirms that solar ν_e are converted to a linear combination of ν_μ and ν_τ with approximate trimaximal mixing $|U_{e2}| \approx |U_{\mu 2}| \approx |U_{\tau 2}| \approx 1/\sqrt{3}$

- 2004 - Reactor antineutrinos $\overline{\nu}_e$ are observed by KamLAND to oscillate with a probability consistent with the solar neutrino oscillations

- 2006 - Accelerator neutrinos ν_μ from Fermilab are observed over a long baseline (LBL) at MINOS with a disappearance probability consistent with the atmospheric oscillation results, providing a high precision confirmation of a similar observation from KEK to SK (K2K) in 2004

- 2010 - LBL accelerator neutrinos ν_μ from CERN appear at OPERA as ν_τ
- 2011 - T2K and MINOS observe and excess of accelerator neutrinos ν_μ appearing as ν_e, consistent with non-zero U_{e3}
- 2012 - Daya Bay, RENO and Double Chooz observe the disappearance of reactor antineutrinos $\overline{\nu}_e$ and measure $|U_{e3}| \approx 0.15$

1.3 The challenges ahead for experiment

Despite the above observations, neutrinos remain the least understood particles. Of the (at least) 7 new parameters which must be present due to neutrino mass and mixing, only 5 are currently measured, namely the three mixing angles and two mass squared differences. For example none of the CP violating phases are currently measured, although there are plans to measure one of these phases in next generation neutrino oscillation experiments. However, since the neutrino oscillations are only sensitive to mass squared differences, the lightest neutrino mass cannot be measured by oscillation experiments. Also the present experiments are not sensitive enough to uniquely determine the ordering of the neutrino square masses m_1^2, m_2^2, m_3^2, although it is known from the solution to the solar neutrino problem that $m_2^2 > m_1^2$. The neutrino mass scale (i.e. the mass of the lightest neutrino) is not known, although, as discussed later, there are cosmological reasons to believe that none of the neutrino masses can exceed about 0.3 eV. Hence the lightest neutrino mass should be somewhere between zero and 0.3 eV. However cosmology is not sensitive to whether neutrino mass is of the Dirac or Majorana kind.[1] In principle, if neutrinoless double beta decay were observed, it could simultaneously be used to measure both the lightest neutrino mass and show that it is Majorana (and future measurements could shed light on the additional phases associated with Majorana masses). However, despite our ignorance, we know that neutrino masses are much smaller than the other charged fermion masses, and this already represents something of a puzzle.

From the experimental perspective, the main known unknowns of neutrino mass and mixing may be summarised as:

- The neutrino mass squared ordering (normal or inverted)
- The neutrino mass scale (i.e. the mass of the lightest neutrino, presumably between zero and 0.3 eV)
- The nature of neutrino mass (Dirac or Majorana)
- The CP violating phase measurable in neutrino oscillations (the so-called Dirac phase δ, although it is also present if neutrino mass is Majorana)

[1] See Subsection 1.5 for the definition of Dirac and Majorana neutrino masses.

- The two possible further CP violating phases associated with Majorana neutrino masses (not present if neutrino mass is Dirac)

Neutrino physics has now entered the precision era, at least as far as the measured parameters are concerned. T2K is presently running [16] and will provide accurate measurements of the atmospheric neutrino mass squared difference and mixing angle, while NOνA [17], presently under construction, will provide complementary information about the mass ordering. Future neutrino oscillation experiments, under discussion [18], will give more accurate information about the mass squared splittings $\Delta m^2_{ij} \equiv m^2_i - m^2_j$, mixing angles, the mass squared ordering (commonly but incorrectly referred to as the "mass hierarchy"), and the neutrino mass scale (i.e. the mass of the lightest neutrino mass eigenstate, which will indeed decide if neutrino masses involve a significant mass hierarchy). The ultimate goal of oscillation experiments, however, is to measure the so far undetermined CP violating oscillation phase δ and there is considerable activity in this area [19] to determine the best way to do this.

1.4 The nature and scale of neutrino mass

Oscillation experiments are not by themselves capable of telling us anything about the nature or mass scale of neutrino mass. They can, however, shed light on the neutrino mass ordering as mentioned above. There are basically four ways to elucidate the mysteries surrounding neutrino masses.

1. Neutrinoless double beta decay experiments (for a recent review see e.g. [20]) effectively measure the 1-1 element of the Majorana neutrino mass matrix corresponding to
$$m_{\beta\beta} \equiv \Big| \sum_i U^2_{ei} m_i \Big|, \tag{1.2}$$
and can validate the Majorana nature of neutrinos. There was a claim of a signal in neutrinoless double beta decay corresponding to $m_{\beta\beta} = 0.11 - 0.56$ eV at 95% C.L. [21]. However this claim was criticised by two groups [22], and in turn this criticism has been refuted [23]. Experiments such as GERDA should report soon and decide this question [20].

2. Oscillation experiments can measure the sign of Δm^2_{32} and resolve normal from inverted mass squared orderings.

3. Independently of whether neutrinos are Dirac or Majorana, the Tritium beta decay experiment KATRIN [24] will tell us about the absolute scale of neutrino mass down to about 0.35 eV. Such experiments measure the "electron neutrino mass" defined by
$$m_{\nu_e} \equiv \sum_i |U_{ei}|^2 |m_i|. \tag{1.3}$$

4. More model dependently, cosmology can in principle probe the sum of neutrino masses, and hence the lightest neutrino mass m_{lightest}, down to very small values [25].

In future detection of energetic neutrinos from gamma ray bursts, neutrino telescopes could also provide important astrophysical information, and may provide another means of probing neutrino mass, and even quantum gravity [26].

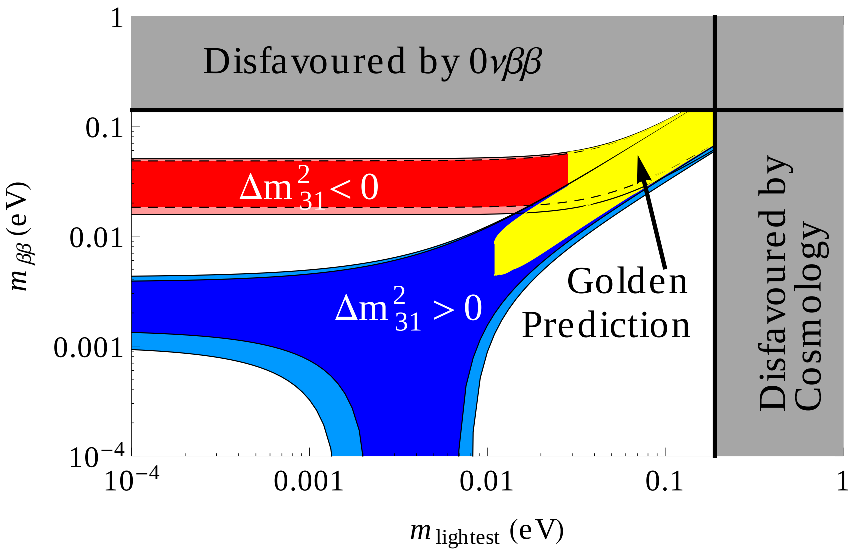

Figure 2: $m_{\beta\beta}$ vs. m_{lightest}: The red and light red regions represent the model independent values that the inverted neutrino mass ordering can take based on the central value and 1σ deviation of a recent global fit of neutrino parameters. The blue and light blue regions are the analogue of this for the normal neutrino mass ordering. The gold regions correspond to the golden ratio prediction for $m_{\beta\beta}$ in both the normal and inverted orderings, resulting from the A_5 inverse mass sum rule.

In Fig. 2 we show the allowed range of the effective mass parameter for neutrinoless double beta decay, $m_{\beta\beta}$, see Eq. (1.2), as a function of the lightest neutrino mass m_{lightest} for both the normal (blue) and inverted (red) neutrino mass squared orderings consistent with the one sigma range of parameters taken from a recent global fit as discussed in [27] from which this figure is taken.[2] Also shown (gold) is the restricted region in a model based on a discrete family symmetry A_5 which involves the golden ratio (GR) [27]. Such restricted regions follow from relations between the neutrino masses which can generally arise in models based on discrete family symmetry. For example, the neutrino masses may be related by a sum rule of the form,

$$\alpha m_1 + \beta m_2 = m_3, \tag{1.4}$$

where α, β are model dependent constants. If the model involves a see-saw mechanism, then the right-handed neutrino masses may be similarly related leading to inverse relationships between light physical neutrino masses of the form,

$$\frac{\gamma}{m_1} + \frac{\delta}{m_2} = \frac{1}{m_3}, \tag{1.5}$$

where γ, δ are model dependent constants.[3] In certain models analogous relations may

[2]Fig. 2 is generated using a code whose original version was developed in [28].

[3]Note that δ here is nothing to do with the CP violating phase denoted by the same Greek letter.

arise with the neutrino masses replaced by their square roots. All these mass sum rules have been recently studied in [29]. The A_5 GR model [27] mentioned above involves an inverse sum rule as in Eq. (1.5) with $\gamma = \delta = 1$. This may be compared to the A_4 model in [30] which involves an inverse mass sum rule with $\gamma = 1$ and $\delta = -2$, or the $\Delta(96)$ model in [31] with $\gamma = 1$ and $\delta = \pm 2i$, and so on. The appearance of the complex constant reminds us that in all the (inverse) mass sum rules there are CP violating phases associated with Majorana neutrino masses which are implicit.

1.5 The origin of neutrino mass

It is worth recalling why the observation of non-zero neutrino mass and mixing is evidence for new physics beyond the SM. The most intuitive way to understand why neutrino mass is forbidden in the Standard Model, is to understand that the Standard Model predicts that neutrinos always have a "left-handed" spin - rather like rifle bullets which spin counter clockwise to the direction of travel. In fact this property was first experimentally measured in 1958, two years after the neutrino was discovered, by Maurice Goldhaber, Lee Grodzins and Andrew Sunyar. More accurately, the "handedness" of a particle describes the direction of its spin vector along the direction of motion, and the neutrino being "left-handed" means that its spin vector always points in the opposite direction to its momentum vector. The fact that the neutrino is left-handed, written as ν_L, implies that it must be massless. If the neutrino has mass then, according to special relativity, it can never travel at the speed of light. In principle, a fast moving observer could therefore overtake the spinning massive neutrino and would see it moving in the opposite direction. To the observer, the massive neutrino would therefore appear right-handed. Since the Standard Model predicts that neutrinos must be strictly left-handed, it follows that neutrinos are massless in the Standard Model. It also follows that the discovery of neutrino mass implies new physics beyond the SM, with profound implications for particle physics and cosmology.

Neutrinos are massless in the Standard Model for three independent reasons:

- There are no right-handed neutrinos ν_R
- There are only Higgs doublets (and no Higgs triplets) of $SU(2)_L$
- There are only renormalisable term

In the SM, the three massless neutrinos ν_e, ν_μ, ν_τ are distinguished by separate lepton numbers L_e, L_μ, L_τ. Neutrinos and antineutrinos are distinguished by total conserved lepton number $L = L_e + L_\mu + L_\tau$. To generate neutrino mass we must relax one or more of the above three conditions. For example, by adding right-handed neutrinos the Higgs mechanism of the Standard Model can give neutrinos the same type of mass as the Dirac electron mass or other charged lepton and quark masses, which would generally break the separate lepton numbers L_e, L_μ, L_τ, but preserve the total lepton number L. However it is also possible for neutrinos to have a new type of mass of a type first proposed by Majorana, which would also break L. There exists a special case where total lepton number L is broken, but the combination $L_e - L_\mu - L_\tau$ is conserved; such a symmetry would give rise to a neutrino mass matrix with an inverted mass spectrum.

From the theoretical perspective, the main unanswered question is the origin of neutrino mass, and in particular the smallness of neutrino mass. The simplest possibility is that neutrinos have Dirac mass just like the electron mass in the SM, namely due to a term like $y_D\overline{L}H\nu_R$, where L is a lepton doublet containing ν_L, H is a Higgs doublet and ν_R is a right-handed neutrino. The observed smallness of neutrino masses implies that the Dirac Yukawa coupling y_D must be of order 10^{-12} to achieve a Dirac neutrino mass of about 0.1 eV. Advocates of Dirac masses point out that the electron mass already requires a Yukawa coupling y_e of about 10^{-6}, so we are used to such small Yukawa couplings. In this case, all that is required is to add right-handed neutrinos ν_R to the SM and we are done. Well, almost. It still needs to be explained why the ν_R have zero Majorana mass, after all they are gauge singlets and so nothing prevents them acquiring (large) Majorana mass terms $M_{RR}\nu_R\nu_R$ where M_{RR} could be as large as the Planck scale. Moreover, Majorana masses offer a unique (and testable) way to generate neutrino masses (since neutrinos do not carry electric charge) even without right-handed neutrinos. The simplest way to generate Majorana mass is via $y_M\Delta LL$ where Δ is a Higgs triplet and y_M is a Yukawa coupling associated with Majorana mass. Alternatively, at the effective level, Majorana neutrino mass can result from some additional dimension 5 operators which couple two lepton doublets L to two Higgs doublets H first proposed by Weinberg [32],

$$-\frac{1}{2}HL^T\kappa HL, \tag{1.6}$$

where κ has dimension $[\text{mass}]^{-1}$. This is a non-renormalisable operator, so it violates one of the tenets of the SM. In order to account for a neutrino mass of order 0.1 eV requires $\kappa \sim 10^{-14}$ GeV^{-1}. This suggests a new high energy mass scale M in physics, a small dimensionless coupling associated with κ, or both. There are basically five different proposals for the origin of neutrino mass:

- The see-saw mechanisms [33–35] (Weinberg operator e.g. from large Majorana mass $M = M_{RR}$ for right-handed neutrinos ν_R)
- R-parity violating supersymmetry [36] (Weinberg operator from TeV scale Majorana mass for neutralinos χ)
- TeV scale loop mechanisms [37, 38] (Majorana mass from extra Higgs doublets and singlets at the TeV scale)
- Extra dimensions [39] (Dirac mass with small y_D due to right-handed neutrinos ν_R in the bulk)
- String theory [40, 41] (new mechanisms for generating large Majorana mass for right-handed neutrinos ν_R from Planck or string scale physics)

These different mechanisms are reviewed in [42]. In this review we shall mainly focus on the see-saw mechanism which may be incorporated into a theory of flavour.

It has been one of the long standing goals of theories of particle physics beyond the Standard Model to predict quark and lepton masses and mixings. With the discovery of neutrino mass and mixing, this quest has received a massive impetus. Indeed, perhaps the greatest advance in particle physics over the past decade has been the discovery of neutrino mass and mixing involving large mixing. The largeness of the lepton mixing angles contrasts with the smallness of the quark mixing angles, and this observation, together with the smallness of neutrino masses, provides new and tantalising clues in the search for the origin of quark and lepton flavour. For example, it is amusing to note that the smallest lepton mixing may be related to the largest quark mixing, $|U_{e3}| \approx \theta_C/\sqrt{2}$ where θ_C is the Cabibbo angle. The quest to understand the origin of the three families of quarks and leptons and their pattern of masses and mixing parameters is called the flavour puzzle, and motivates the introduction of family symmetry. In particular, as we shall see, lepton mixing provides a motivation for discrete family symmetry, which will form the central part of this review. As we shall also see, such theories demand a high precision knowledge of the lepton mixing angles, beyond that currently achieved.

1.6 About this review

It should be mentioned at the outset that there are good and fairly up to date reviews already in the literature, for example: [43–45], although only the last one was written after Daya Bay and RENO. It should be remarked that the signal of another independent mass splitting from the LSND accelerator experiment [46] would either require a further light neutrino state with no weak interactions (a so-called “light sterile neutrino”) or some other non-standard physics. This effect has not been confirmed by a similar experiment KARMEN [47], and a subsequent experiment MiniBooNE [48] has not decisively resolved the issue. Since there is no solid evidence for light sterile neutrinos, in this review we shall not discuss this subject any further, but refer the interested reader to a recent dedicated discussion in [49]. Instead, in this review, we shall exclusively focus on the three active neutrino paradigm.

The starting point for the present review is the one that was written by one of us about ten years ago [50]. At that time it had just become apparent, after the first SNO results in 2002, that the solar mixing angle was large, which together with the large atmospheric mixing angle, meant that there were two large mixing angles in the lepton sector. The solar mixing being large effectively killed many neutrino mass models that had previously been proposed consistent with small solar mixing. This was actually the second great extinction of models, the first being after the discovery of a large atmospheric mixing angle by Super-Kamiokande in 1998. A decade after the last review, in 2012 we are now in an analogous position, namely that Daya Bay and RENO have just measured the reactor angle and shown it to be quite sizeable, killing neutrino mass models consistent with a very small reactor angle. Just as the review article in [50] was written shortly after the second great extinction in 2002, so the present review article is being written just after the third great extinction in 2012, so once again it is an opportune moment to identify new and surviving model species which may come to dominate the theory landscape over the coming years. In particular the fate of discrete family symmetry, which was to some extent motivated by the possibility of the reactor mixing being zero, or very small, will be fully discussed. We emphasise that we are now in the unique position in the history of neutrino physics of knowing not only that neutrino mass is real, and hence the Standard Model at least in its minimal formulation is incomplete, but also we finally have information on all three mixing angles.

As in the previous review, we focus on theoretical approaches to understanding neutrino masses and mixings in the framework of the *see-saw mechanism*, assuming three active neutrinos. The goal of such models is to account for two very large mixing angles, and one Cabibbo-sized mixing angle, and a pattern of neutrino masses consistent with observation. We give a strong emphasis to classes of models where large mixing angles can arise naturally and consistently with a neutrino mass hierarchy. We show that if one of the right-handed neutrinos contributes dominantly in the type I see-saw mechanism to the heaviest neutrino mass, and a second right-handed neutrino contributes dominantly to the second heaviest neutrino mass, then large atmospheric and solar mixing angles may be interpreted as simple ratios of Yukawa couplings. This is of course the sequential dominance (SD) mechanism [51–53]. Sequential dominance is not a model, it is a mechanism in search of a model. The conditions for sequential dominance, such as ratios of Yukawa couplings being of order unity for large mixing angles, and the required pattern of right-handed neutrino masses are put in by hand and require further theoretical input such as family symmetry. It is interesting to note that, without further constraints, SD generically has long predicted hierarchical neutrinos with a normal ordering and large reactor angle of order $|U_{e3}| \sim \mathcal{O}(m_2/m_3) \sim 0.2$. However, in order to achieve more precise predictions, SD needs to be combined with family symmetry.

The use of *discrete* family symmetry was mainly motivated by the hypothesis of exact tri-bimaximal (TB) mixing [54] defined by:

$$\begin{aligned} |U_{e3}| &= 0, \\ |U_{\mu 3}| &= |U_{\tau 3}| = 1/\sqrt{2}, \\ |U_{e2}| &= |U_{\mu 2}| = |U_{\tau 2}| = 1/\sqrt{3}. \end{aligned} \tag{1.7}$$

The TB mixing and discrete family symmetry approach gained much impetus over the past decade. Given that the measurement of the reactor mixing $|U_{e3}| \approx 0.15$ by Daya Bay and RENO kills exact TB mixing, the obvious question is what is the impact on the discrete family symmetry approach, which was largely inspired by TB mixing. This timely question will be addressed by the present review. There are a huge number of proposals in the literature, but not so many surviving the measurement of the reactor angle. The simple answer to the question is the discrete family symmetry approach is alive and kicking after Daya Bay and RENO. However, theorists have been forced to work harder to go beyond the simple mixing pattern of TB mixing which is now excluded. The simple discrete family symmetries proposed to account for TB mixing may still be viable as a leading order (LO) approximation, but higher order (HO) corrections may play a more important role than anticipated in many models. Alternatively, perhaps larger finite groups are relevant where the LO approximation already predicts a non-zero reactor angle. Yet another possibility is that the discrete family symmetry may be implemented indirectly, as in the sequential dominance approach, in new ways. All these interesting possibilities will be discussed.

The layout of the remainder of the review article is as follows. In Section 2 we introduce and review the current status of neutrino masses and mixing angles. We also parameterise the PMNS mixing matrix in two different ways, whose equivalence is discussed in Appendix A. In Section 3 we discuss patterns of lepton mixing that have been proposed, starting with simple mixing patterns such as bimaximal (BM), tri-bimaximal (TB), bi-trimaximal (BT) and golden ratio (GR) mixing. These all may apply to the neutrino mixing, which is then corrected by charged lepton mixing corrections, to give acceptable PMNS mixing, leading to solar mixing angle sum rules. The closeness of the TB mixing pattern to the data suggests a parametrisation of the PMNS matrix in terms of deviations from TB mixing, which is also introduced. Using these deviations, several TB variants are introduced and discussed, including tri-bimaximal-reactor (TBR) mixing and trimaximal (TM) mixing in two forms, namely where the first and second columns of the PMNS matrix take the TB values, called TM1 and TM2, respectively. Section 4 is devoted to the see-saw mechanisms, which are central to this review, in both the simplest versions, called the type I, and including other types II and III, as well as alternative versions. We show how the type I see-saw mechanism may be applied to the hierarchical case in a very natural way using sequential dominance. Section 5 contains a mini-review of finite group theory and may be skipped by those readers who are already familiar with this subject. In Section 6 we give a pedagogical introduction to discrete family symmetry, and its direct or indirect implementation in model building. Section 7 is devoted to the direct model building approach in which different subgroups of the discrete family symmetry are preserved in the neutrino and charged lepton sectors, and discusses the associated vacuum alignments arising from the breaking of the discrete family symmetry using flavons. We also discuss the model building strategies following Daya Bay and RENO. Section 8 contains an analogous discussion for the indirect approach in which the discrete family symmetry is completely broken by flavons, but special vacuum alignments lead to particular mixing patterns, including new viable patterns with a non-zero reactor angle. In Section 9 we briefly review grand unified theories (GUTs) such as $SU(5)$ and how they may be combined with discrete family symmetry in order to account for all quark and lepton masses and mixing. In Section 10 we discuss three model examples which combine an $SU(5)$ GUT with the discrete family symmetries A_4, S_4 and $\Delta(96)$. Section 11 concludes the review. We also present appendices dealing with more technical issues which may provide useful model building tools. Appendix A proves the equivalence between different parametrisations of the neutrino mixing matrix and gives a useful dictionary. Appendix B gives the full three family neutrino oscillation formula in terms of deviations from TB mixing. Appendix C catalogues the generators and Clebsch-Gordan coefficients of A_4, S_4 and T_7.

2. Neutrino masses and mixing angles

The history of neutrino oscillations dates back to the work of Pontecorvo who in 1957 [2] proposed $\nu \to \bar{\nu}$ oscillations in analogy with $K \to \bar{K}$ oscillations, described as the mixing of two Majorana neutrinos. Pontecorvo was the first to realise that what we call the "electron neutrino" for example is really a linear combination of mass eigenstate neutrinos, and that this feature could lead to neutrino oscillations of the kind $\nu_e \to \nu_\mu$. Later on MSW proposed that such neutrino oscillations could be resonantly enhanced in the Sun [4]. The present section introduces the basic formalism of neutrino masses and mixing angles, and gives an up-to-date summary of the current experimental status of this fast moving field.

2.1 Three neutrino mixing ignoring phases

The minimal neutrino sector required to account for the atmospheric and solar neutrino oscillation data consists of three light physical neutrinos with left-handed flavour eigenstates, ν_e, ν_μ, and ν_τ, defined to be those states that share the same electroweak doublet as the corresponding left-handed charged lepton mass eigenstates. Within the framework of three-neutrino oscillations, the neutrino flavour eigenstates ν_e, ν_μ, and ν_τ are related to the neutrino mass eigenstates ν_1, ν_2, and ν_3 with mass m_1, m_2, and m_3, respectively, by a 3×3 unitary matrix called the lepton mixing matrix U_{PMNS} introduced in Eq. (1.1).

Assuming the light neutrinos are Majorana, U_{PMNS} can be parameterised in terms of three mixing angles θ_{ij} and three complex phases δ_{ij}. A unitary matrix has six phases but three of them are removed by the phase symmetry of the charged lepton masses. Since the neutrino masses are Majorana there is no additional phase symmetry associated with them, unlike the case of quark mixing where a further two phases may be removed.

To begin with, let us suppose that the phases are zero. Then the lepton mixing matrix may be parametrised by a product of three Euler rotations,

$$U_{\mathrm{PMNS}} = R_{23} R_{13} R_{12}, \tag{2.1}$$

where

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \qquad R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}, \qquad R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

with $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. Note that the allowed range of the angles is $0 \leq \theta_{ij} \leq \frac{\pi}{2}$.

Ignoring phases, the relation between the neutrino flavour eigenstates ν_e, ν_μ, and ν_τ and the neutrino mass eigenstates ν_1, ν_2, and ν_3 is therefore given as a product of three Euler rotations in Eq. (2.1) as depicted in Fig. 3.

2.2 Atmospheric neutrino mixing

In 1998, the Super-Kamiokande experiment published a paper [6] which represents a watershed in the history of neutrino physics. Super-Kamiokande measured the number of electron and muon neutrinos that arrive at the Earth's surface as a result of cosmic ray interactions in the upper atmosphere, which are referred to as "atmospheric neutrinos". While the number and angular distribution of electron neutrinos is as expected, Super-Kamiokande showed that the number of muon neutrinos is significantly smaller than expected and that the flux of muon neutrinos exhibits a strong dependence on the zenith angle. These observations gave compelling evidence that muon neutrinos undergo flavour oscillations and this in turn implies that at least one neutrino has a non-zero mass. The standard interpretation is that muon neutrinos are oscillating into tau neutrinos.

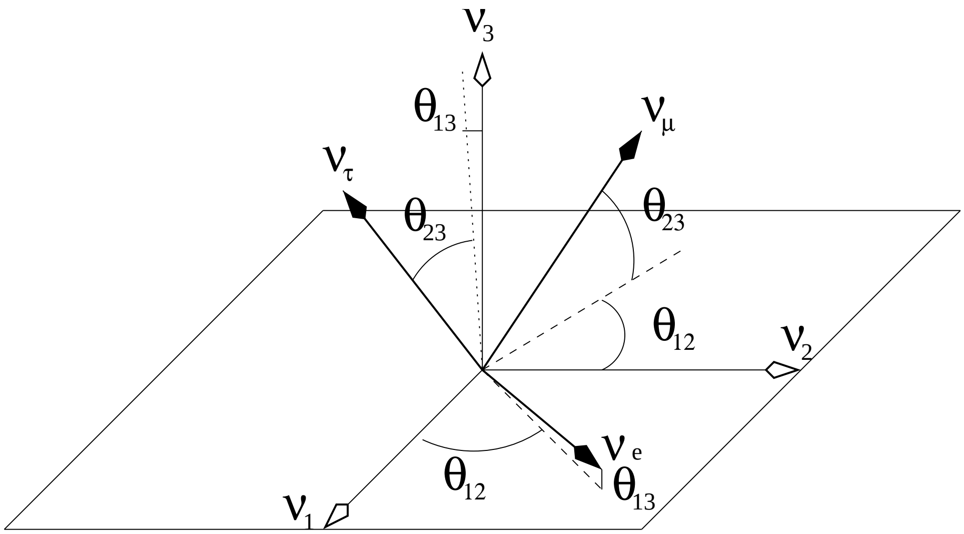

Figure 3: The relation between the neutrino flavour eigenstates ν_e, ν_μ, and ν_τ and the neutrino mass eigenstates ν_1, ν_2, and ν_3 in terms of the three mixing angles θ_{12}, θ_{13}, θ_{23}.

As a first approximation, one can set the reactor angle θ_{13} to zero, and assume that $|\Delta m^2_{32}| \gg |\Delta m^2_{21}|$. Current atmospheric neutrino oscillation data can then approximately be described by simple two-state mixing,

$$\begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \cos\theta_{23} & \sin\theta_{23} \\ -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \nu_2 \\ \nu_3 \end{pmatrix} , \tag{2.3}$$

and the two-state oscillation formula describing the probability that a ν_μ converts to a ν_τ,

$$P(\nu_\mu \to \nu_\tau) = \sin^2 2\theta_{23} \sin^2(1.27\Delta m^2_{32} L/E), \tag{2.4}$$

where

$$\Delta m^2_{ij} \equiv m^2_i - m^2_j \ , \tag{2.5}$$

and m_i are the physical neutrino mass eigenvalues associated with the mass eigenstates ν_i. Δm^2_{32} is in units of eV^2, the baseline L is in km and the beam energy E is in GeV. Note that the sign of Δm^2_{32}, and thus the mass ordering, cannot be determined from Eq. (2.4).

The data can be well described by approximately maximal atmospheric mixing $|U_{\mu 3}| \approx |U_{\tau 3}| \approx 1/\sqrt{2}$. This corresponds to

$$\sin\theta_{23} \approx 1/\sqrt{2}, \tag{2.6}$$

which means that the angle $\theta_{23} \approx \pi/4$ or 45° and we identify the heavy "atmospheric neutrino" of mass m_3 as being approximately

$$\nu_3 \approx \frac{\nu_\mu + \nu_\tau}{\sqrt{2}}. \tag{2.7}$$

2.3 Solar neutrino mixing

Super-Kamiokande was also sensitive to the electron neutrinos arriving from the Sun, the "solar neutrinos", and independently confirmed the reported deficit of such solar neutrinos long reported by other experiments. For example Davis's Homestake Chlorine experiment which began data taking in 1970 consisted of 615 tons of tetrachloroethylene, and uses radiochemical techniques to determine the ^{37}Ar production rate. The SAGE and Gallex experiments contained large amounts of ^{71}Ga which is converted to ^{71}Ge by low energy electron neutrinos arising from the dominant pp reaction in the Sun. The combined data from these and other experiments implied an energy dependent suppression of solar neutrinos which was interpreted as due to flavour oscillations. Taken together with the atmospheric data, this required that a second neutrino has a non-zero mass. The standard interpretation is that the electron neutrinos ν_e disappear due to an oscillation formula which involves a "solar neutrino" of mass m_2 given approximately by

$$\nu_2 \approx \frac{\nu_e + \nu_\mu - \nu_\tau}{\sqrt{3}}, \tag{2.8}$$

consistent with trimaximal solar mixing $|U_{e2}| \approx |U_{\mu 2}| \approx |U_{\tau 2}| \approx 1/\sqrt{3}$. This corresponds to

$$\sin\theta_{12} \approx 1/\sqrt{3}, \tag{2.9}$$

or $\theta_{12} \approx 35°$.

SNO measurements of charged current (CC) reaction on deuterium were sensitive exclusively to ν_e, while the neutral current (NC) reaction as well as the elastic scattering (ES) off electrons were also sensitive to ν_μ and ν_τ. The neutrino flux derived from the CC reactions was significantly smaller than the one obtained from NC and ES. This immediately disfavoured oscillations of ν_e to sterile neutrinos which would lead to a diminished flux of electron neutrinos, but equal CC, NC and ES fluxes. On the other hand the observations were consistent with oscillations of ν_e to active neutrinos ν_μ and ν_τ since this would lead to a larger NC and ES rate. The SNO analysis was nicely consistent with both the hypothesis that electron neutrinos from the Sun oscillate into other active flavours, and with the Standard Solar Model prediction. The results from SNO including the data taken with salt inserted into the detector to boost the efficiency of detecting the NC events [7], strongly favoured the large mixing angle (LMA) MSW solution. In other words, after SNO, there was no longer any solar neutrino problem: we had instead solar neutrino mass m_2!

KamLAND was a more powerful reactor experiment that measured $\bar{\nu}_e$ produced by surrounding nuclear reactors. KamLAND observed a signal of neutrino oscillations over the LMA MSW mass range, confirming the LMA MSW region "in the laboratory" [8]. KamLAND and SNO results when combined with other solar neutrino data especially that of Super-Kamiokande uniquely specify the LMA MSW [4] solar solution with three active light neutrino states, and approximately trimaximal solar mixing. This solution, which requires a careful treatment of the matter effects and the resulting asymmetry between the coherent forward scattering of the different neutrino flavour states, furthermore determines the sign of the mass squared splitting $\Delta m^2_{21} = m^2_2 - m^2_1$ to be positive. KamLAND has thus not only confirmed solar neutrino oscillations, but has also uniquely specified the LMA solar solution, heralding a new era in neutrino physics.

2.4 Reactor neutrino mixing

Until recently, the reactor angle θ_{13} was not measured, only limited by CHOOZ, a reactor experiment that failed to see any signal of neutrino oscillations over the Super-Kamiokande mass range. CHOOZ data from $\bar{\nu}_e \to \bar{\nu}_e$ disappearance not being observed provided a significant constraint on θ_{13} over the Super-Kamiokande preferred range of Δm^2_{32} [11]:

$$\sin^2 2\theta_{13} < 0.2. \tag{2.10}$$

Direct evidence for θ_{13} was first provided by T2K, MINOS and Double Chooz [12]. Recently the Daya Bay [13], RENO [14], and Double Chooz [15] collaborations have measured $\sin^2(2\theta_{13})$:

$$\begin{array}{cl} \text{Daya Bay}: & \sin^2(2\theta_{13}) = 0.089 \pm 0.010(\text{stat.}) \pm 0.005(\text{syst.})\ , \\ \text{RENO}: & \sin^2(2\theta_{13}) = 0.113 \pm 0.013(\text{stat.}) \pm 0.019(\text{syst.})\ , \\ \text{Double Chooz}: & \sin^2(2\theta_{13}) = 0.109 \pm 0.030(\text{stat.}) \pm 0.025(\text{syst.})\ . \end{array} \tag{2.11}$$

This corresponds to

$$|U_{e3}| = \sin\theta_{13} \approx 0.15, \tag{2.12}$$

or a reactor angle $\theta_{13} \approx 9^\circ$.

2.5 Three neutrino mixing including phases

If the reactor angle were zero then there would be no CP violation in neutrino oscillations. The measurement of the reactor angle means that we cannot ignore the presence of phases any more. Including the phases, assuming the light neutrinos are Majorana, $U_{\rm PMNS}$ can be parameterised in terms of three mixing angles θ_{ij}, a Dirac phase δ, together with two Majorana phases β_1, β_2, as follows [5],

$$U_{\rm PMNS} = R_{23} U_{13} R_{12} P_{12}, \tag{2.13}$$

where

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}, \qquad P_{12} = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.14}$$

and R_{23} and R_{12} were defined below Eq. (2.1), giving,

$$U_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix} P_{12}. \tag{2.15}$$

Alternatively the lepton mixing matrix may be expressed as a product of three complex Euler rotations [55],

$$U_{\rm PMNS} = U_{23} U_{13} U_{12}, \tag{2.16}$$

parameter	Forero et al	Fogli et al	Gonzalez-Garcia et al
$\Delta m^2_{21}[10^{-5}\text{eV}^2]$	7.62 ± 0.19	$7.54^{+0.26}_{-0.22}$	7.50 ± 0.185
$\Delta m^2_{31}[10^{-3}\text{eV}^2]$	$2.55^{+0.06}_{-0.09}$ $-2.43^{+0.06}_{-0.07}$	$2.43^{+0.06}_{-0.10}$ $-2.42^{+0.11}_{-0.07}$	$2.47^{+0.069}_{-0.067}$ $-2.43^{+0.042}_{-0.065}$
$\sin^2\theta_{12}$	$0.320^{+0.016}_{-0.017}$	$0.307^{+0.018}_{-0.016}$	0.30 ± 0.013
$\sin^2\theta_{23}$	$0.427^{+0.034}_{-0.027}$ & $0.613^{+0.022}_{-0.040}$ $0.600^{+0.026}_{-0.031}$	$0.386^{+0.024}_{-0.021}$ $0.392^{+0.039}_{-0.022}$	$0.41^{+0.037}_{-0.025}$ $0.41^{+0.037}_{-0.025}$ & $0.59^{+0.021}_{-0.022}$
$\sin^2\theta_{13}$	$0.0246^{+0.0029}_{-0.0028}$ $0.0250^{+0.0026}_{-0.0027}$	0.0241 ± 0.0025 $0.0244^{+0.0023}_{-0.0025}$	0.023 ± 0.0023
δ	$(0.80 \pm 1)\pi$ $-(0.03 \pm 1)\pi$	$(1.08^{+0.28}_{-0.31})\pi$ $(1.09^{+0.38}_{-0.26})\pi$	$(1.67^{+0.37}_{-0.77})\pi$

Table 1: Neutrino oscillation parameters summary. For Δm^2_{31}, $\sin^2\theta_{23}$, $\sin^2\theta_{13}$, and δ the upper (lower) row corresponds to normal (inverted) neutrino mass ordering. The best fit values and 1σ errors are shown. The subtleties associated with these numbers are discussed in the respective references Forero et al [56], Fogli et al [57] and Gonzalez-Garcia et al [58]. In particular [58] quotes two different global fits, depending on the assumptions made about reactor fluxes, where we only quote the first ("free flux"). Furthermore, the precise definition of the atmospheric neutrino mass splitting Δm^2_{31} differs slightly between the three global fits.

where

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{-i\delta_{23}} \\ 0 & -s_{23}e^{i\delta_{23}} & c_{23} \end{pmatrix}, \tag{2.17}$$

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix}, \tag{2.18}$$

$$U_{12} = \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_{12}} & 0 \\ -s_{12}e^{i\delta_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.19}$$

The equivalence of different parametrisations of the lepton mixing matrix, and the relation between them is discussed in [52] with the results summarised in Appendix A. If the neutrinos are Dirac, then the phases $\beta_1 = \beta_2 = 0$, but the phase δ remains.

2.6 Global fits

In Table 1 we give the global fits of the neutrino mixing parameters. For Δm^2_{31}, $\sin^2\theta_{23}$, $\sin^2\theta_{13}$, and δ the upper (lower) row corresponds to normal (inverted) neutrino mass ordering. The best fit values and 1σ errors are shown from Forero et al [56], Fogli et al [57] and

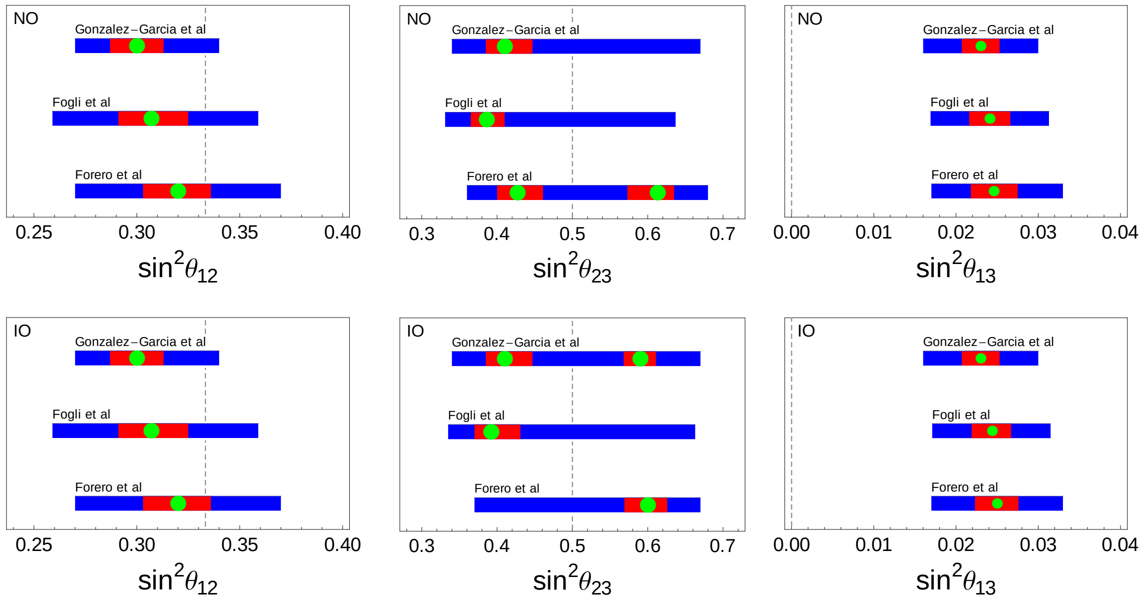

Figure 4: The mixing angles obtained from the three global fits [56–58]. The upper three panels correspond to the results for normal neutrino mass ordering (NO), while the lower three panels are for an inverted mass ordering (IO). Shown are the best fit values (green) as well as the 1σ (red) and 3σ (blue) intervals. Note that the solar angle is insensitive to the mass ordering.

Gonzalez-Garcia et al [58]. The results for the mixing angles are graphically contrasted in Fig. 4. We emphasise that this compilation is predominantly meant to illustrate some possibilities arising from present global fits. The reader is referred to the respective references for the subtleties associated with these numbers.

For the normal mass ordering, we shall take the average values and errors which approximately encompass the one sigma ranges of all three global fits (ignoring the solution of θ_{23} in the second octant found by Forero et al [56]):

$$\sin^2\theta_{12} = 0.31 \pm 0.02, \tag{2.20}$$
$$\sin^2\theta_{23} = 0.41 \pm 0.05, \tag{2.21}$$
$$\sin^2\theta_{13} = 0.024 \pm 0.003. \tag{2.22}$$

These values may be compared to the tri-bimaximal predictions $\sin^2\theta_{12} = 0.33$, $\sin^2\theta_{23} = 0.5$ and $\sin^2\theta_{13} = 0$, showing that TB mixing is excluded by the reactor angle. Alternatively we may write, remembering that these are one sigma ranges in the squares of the sines and not the sines themselves,

$$\sin\theta_{12} = 0.56 \pm 0.02, \tag{2.23}$$
$$\sin\theta_{23} = 0.64 \pm 0.04, \tag{2.24}$$
$$\sin\theta_{13} = 0.155 \pm 0.01. \tag{2.25}$$

In terms of the angles themselves we have, approximately, in round figures,

$$\theta_{12} = 34^\circ \pm 1^\circ, \tag{2.26}$$

$$\theta_{23} = 40^\circ \pm 3^\circ, \tag{2.27}$$

$$\theta_{13} = 9^\circ \pm 0.5^\circ. \tag{2.28}$$

A few comments are relevant about these angles. Firstly the errors are not linear, since, for one thing, the global fits are made in terms of the squares of the sines of the angles. Having said this, in the case of normal neutrino mass ordering, there is a preference for the atmospheric angle to be in the first octant (i.e. less than 45°) and hence not maximal mixing. Secondly, as already noted, the solar angle is still consistent with trimaximal mixing (i.e. 35.26°) but there is a preference for it to be slightly smaller.

3. Patterns of lepton mixing and sum rules

3.1 Simple forms of neutrino mixing

Below we give three examples of simple patterns of mixing in the neutrino sector which all have $s_{13} = 0$ and $s_{23} = c_{23} = 1/\sqrt{2}$. Inserting these values in Eq. (2.1) we obtain a PMNS matrix of the form,

$$U_0 = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{3.1}$$

where the zero subscript reminds us that this form has $\theta_{13} = 0$ (and $\theta_{23} = 45^\circ$).

For golden ratio (GR) mixing [59], the solar angle is given by $\tan\theta_{12} = 1/\phi$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio which implies $\theta_{12} = 31.7^\circ$. There is an alternative version where $\cos\theta_{12} = \phi/2$ and $\theta_{12} = 36^\circ$ [60], which we refer to as GR$'$ mixing.

For bimaximal (BM) mixing (see e.g. [61–63] and references therein), we insert $s_{12} = c_{12} = 1/\sqrt{2}$ ($\theta_{12} = 45^\circ$) into Eq. (3.1),

$$U_{\rm BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.2}$$

For tri-bimaximal (TB) mixing [54], alternatively we use $s_{12} = 1/\sqrt{3}$, $c_{12} = \sqrt{2/3}$ ($\theta_{12} = 35.26^\circ$) in Eq. (3.1),

$$U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.3}$$

These simple examples of neutrino mixing are now all excluded by the data. However they may still play a role in model building and we will revisit them when we consider charged lepton and other corrections below.

3.2 Deviations from tri-bimaximal mixing

From a theoretical or model building point of view, one significance of the global fits is that they exclude the tri-bimaximal lepton mixing pattern [54] in which the solar mixing angle is trimaximal, the atmospheric angle is maximal and the reactor angle vanishes. When comparing global fits to TB mixing it is convenient to express the $s$olar, $a$tmospheric and $r$eactor angles in terms of deviation parameters (s, a and r) from TB mixing [64]:

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}(1+s), \quad \sin\theta_{23} = \frac{1}{\sqrt{2}}(1+a), \quad \sin\theta_{13} = \frac{r}{\sqrt{2}}. \tag{3.4}$$

A related expansion is given in [65]. From these definitions we can write,

$$s = \sqrt{3}\sin\theta_{12} - 1, \quad a = \sqrt{2}\sin\theta_{23} - 1, \quad r = \sqrt{2}\sin\theta_{13}. \tag{3.5}$$

Using this last form, and Eqs. (2.23,2.24,2.25), we find the following values and ranges for the TB deviation parameters:

$$s = -0.03 \pm 0.03, \quad a = -0.10 \pm 0.05, \quad r = 0.22 \pm 0.01, \tag{3.6}$$

assuming a normal neutrino mass ordering. As well as showing that TB is excluded by the reactor angle being non-zero, Eq. (3.6) shows a preference (at the two sigma level) for the atmospheric angle to be below its maximal value and also a slight preference (at the one sigma level) for the solar angle to be below its trimaximal value. In general, this parametrisation shows that the solar angle must be quite close to trimaximal, while the atmospheric angle may be far from bimaximal, with the reactor angle necessarily very far from zero. In any expansion in terms of these parameters, it should be a good approximation to work to first order in s, and possibly a, although it is worth working to second order to obtain the most accurate results. In Appendix B the PMNS matrix is expanded to second order in r, s, a, and the neutrino oscillation formulae including matter effects are given to a similar level of approximation (results taken from [64]). Note that the global fit values in Eq. (3.6) are consistent with,

$$s = -S\lambda^2, \quad a = -A\lambda/2, \quad r = R\lambda, \tag{3.7}$$

where λ is the Wolfenstein parameter and S, A, R are numbers all of order unity. In fact present data is consistent with $S = A = R = 1$ at the one sigma level. In the next subsection we consider the simple case where $s = a = 0$ and $r = \lambda$ which is within the two sigma range.

3.3 Tri-bimaximal-Cabibbo mixing

The recent data is consistent with the remarkable relationship [66],

$$s_{13} = \frac{\sin\theta_C}{\sqrt{2}} = \frac{\lambda}{\sqrt{2}}, \tag{3.8}$$

where $\lambda = 0.2253 \pm 0.0007$ [5] is the Wolfenstein parameter. The above ansatz implies a reactor angle of

$$\theta_{13} \approx \frac{\theta_C}{\sqrt{2}} \approx 9.2^\circ, \tag{3.9}$$

where $\theta_C \approx 13^\circ$ is the Cabibbo angle. One may combine the relation in Eq. (3.8) with TB mixing to yield tri-bimaximal-Cabibbo (TBC) mixing [67]:

$$s_{13} = \frac{\lambda}{\sqrt{2}}, \quad s_{12} = \frac{1}{\sqrt{3}}, \quad s_{23} = \frac{1}{\sqrt{2}}. \tag{3.10}$$

This corresponds to $s = a = 0$ and $r = \lambda$ and leads to the following approximate form of the mixing matrix [67],

$$U_{\rm TBC} \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{4}\lambda^2) & \frac{1}{\sqrt{3}}(1-\frac{1}{4}\lambda^2) & \frac{1}{\sqrt{2}}\lambda e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\lambda e^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}\lambda e^{i\delta}) & \frac{1}{\sqrt{2}}(1-\frac{1}{4}\lambda^2) \\ \frac{1}{\sqrt{6}}(1-\lambda e^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}\lambda e^{i\delta}) & \frac{1}{\sqrt{2}}(1-\frac{1}{4}\lambda^2) \end{pmatrix} P_{12} + \mathcal{O}(\lambda^3), \tag{3.11}$$

corresponding to the mixing angles,

$$\theta_{13} \approx 9.2^\circ, \quad \theta_{12} = 35.26^\circ, \quad \theta_{23} = 45^\circ. \tag{3.12}$$

This is consistent with the data at the two sigma level since it ignores the preference for the atmospheric angle to be in the first octant.

3.4 Charged lepton contributions to neutrino masses and mixing angles

The mixing matrix in the lepton sector, the PMNS matrix $U_{\rm PMNS}$, is defined as the matrix which appears in the electroweak coupling to the W bosons expressed in terms of lepton mass eigenstates. With the mass matrices of charged leptons M_e and neutrinos m^ν_{LL} written as[4]

$$\mathcal{L} = -\overline{e_L} M_e e_R - \frac{1}{2}\overline{\nu_L} m^\nu_{LL} \nu^c_L + H.c.\ , \tag{3.13}$$

and performing the transformation from flavour to mass basis by

$$V_{e_L}\, M_e\, V^\dagger_{e_R} = \mathrm{diag}(m_e, m_\mu, m_\tau), \qquad V_{\nu_L}\, m^\nu_{LL}\, V^T_{\nu_L} = \mathrm{diag}(m_1, m_2, m_3), \tag{3.14}$$

the PMNS matrix is given by

$$U_{\rm PMNS} = V_{e_L} V^\dagger_{\nu_L}\ . \tag{3.15}$$

Here it is assumed implicitly that unphysical phases are removed by field redefinitions, and $U_{\rm PMNS}$ contains one Dirac phase and two Majorana phases. The latter are physical only in the case of Majorana neutrinos, for Dirac neutrinos the two Majorana phases can be absorbed as well.

As shown in [52,68–71] the lepton mixing matrix can be expanded in terms of neutrino and charged lepton mixing angles and phases to leading order in the charged lepton mixing angles which are assumed to be small,

$$\begin{aligned} s_{23} e^{-i\delta_{23}} &\approx s^\nu_{23} e^{-i\delta^\nu_{23}} - \theta^e_{23} c^\nu_{23} e^{-i\delta^e_{23}}\ , & (3.16)\\ \theta_{13} e^{-i\delta_{13}} &\approx \theta^\nu_{13} e^{-i\delta^\nu_{13}} - \theta^e_{13} c^\nu_{23} e^{-i\delta^e_{13}} - \theta^e_{12} s^\nu_{23} e^{i(-\delta^\nu_{23}-\delta^e_{12})}\ , & (3.17)\\ s_{12} e^{-i\delta_{12}} &\approx s^\nu_{12} e^{-i\delta^\nu_{12}} + \theta^e_{23} s^\nu_{12} e^{-i\delta^\nu_{12}} + \theta^e_{13} c^\nu_{12} s^\nu_{23} e^{i(\delta^\nu_{23}-\delta^e_{13})} - \theta^e_{12} c^\nu_{23} c^\nu_{12} e^{-i\delta^e_{12}}\ , & (3.18) \end{aligned}$$

[4] Although we have chosen to write a Majorana mass matrix, all relations in the following are independent of the Dirac or Majorana nature of neutrino masses.

where we have dropped the subscripts L for simplicity. Clearly θ_{13} receives important contributions not just from θ^{ν}_{13}, but also from the charged lepton angles θ^{e}_{12}, and θ^{e}_{13}. In models where θ^{ν}_{13} is extremely small, θ_{13} may originate almost entirely from the charged lepton sector. Charged lepton contributions could also be important in models where $\theta^{\nu}_{12} = \pi/4$, since charged lepton mixing angles may allow consistency with the LMA MSW solution.

Note that it is useful and possible to be able to diagonalise the mass matrices analytically, at least to first order in the small 13 mixing angle, but allowing the 23 and 12 angles to be large, while retaining all the phases. The procedure for doing this is discussed for a hierarchical and an inverted hierarchical neutrino mass matrix in [52]. The analytic results enable the separate mixing angles and phases associated with each of the unitary transformations V_{e_L} and $V^{\dagger}_{\nu_L}$ to be obtained in many useful cases of interest.

3.5 Solar mixing sum rules

In many models the neutrino mixing matrix has a simple form U_0 in Eq. (3.1), where $s^{\nu}_{23} = c^{\nu}_{23} = 1/\sqrt{2}$ and $s^{\nu}_{13} = 0$, while the charged lepton mixing matrix has a CKM-like structure, in the sense that V_{e_L} is dominated by a 12 mixing, i.e. its elements $(V_{e_L})_{13}$, $(V_{e_L})_{23}$, $(V_{e_L})_{31}$ and $(V_{e_L})_{32}$ are very small compared to $(V_{e_L})_{12}$ and $(V_{e_L})_{21}$, where in practice we take them to be zero. In this case we are led to a solar sum rule [69–71] derived from $U_{\rm PMNS} = V_{e_L} U_0$, which takes the form,

$$U_{\rm PMNS} = \begin{pmatrix} c^{e}_{12} & -s^{e}_{12}e^{-i\delta^{e}_{12}} & 0 \\ s^{e}_{12}e^{i\delta^{e}_{12}} & c^{e}_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c^{\nu}_{12} & s^{\nu}_{12} & 0 \\ -\frac{s^{\nu}_{12}}{\sqrt{2}} & \frac{c^{\nu}_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s^{\nu}_{12}}{\sqrt{2}} & -\frac{c^{\nu}_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & -\frac{s^{e}_{12}}{\sqrt{2}}e^{-i\delta^{e}_{12}} \\ \cdots & \cdots & \frac{c^{e}_{12}}{\sqrt{2}} \\ \frac{s^{\nu}_{12}}{\sqrt{2}} & -\frac{c^{\nu}_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.19}$$

The important point to notice is that the 3-1, 3-2 and 3-3 elements of $U_{\rm PMNS}$ in Eq. (3.19) are uncorrected by charged lepton corrections and are the same as those of U_0, and also the 1-3 element of $U_{\rm PMNS}$ has a simple form. By comparing Eq. (3.19) to the PDG parametrisation of $U_{\rm PMNS}$ in Eq. (2.15) we find the relations,

$$s_{13} = \frac{s^{e}_{12}}{\sqrt{2}}\,, \tag{3.20}$$

$$s_{23}c_{13} = \frac{c^{e}_{12}}{\sqrt{2}}\,, \tag{3.21}$$

$$c_{23}c_{13} = \frac{1}{\sqrt{2}}\,, \tag{3.22}$$

$$|s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta}| = \frac{s^{\nu}_{12}}{\sqrt{2}}\,, \tag{3.23}$$

$$|s_{23}c_{12} + s_{13}c_{23}s_{12}e^{i\delta}| = \frac{c^{\nu}_{12}}{\sqrt{2}}\,. \tag{3.24}$$

Using Eq. (3.22) we see that, to leading order in θ_{13}, the atmospheric angle is unchanged from its maximal value by the assumed form of the charged lepton corrections. To this approximation, it is then straightforward to expand these results to obtain the more useful approximate form of the sum rule [69–71],

$$\theta_{12} \approx \theta_{12}^{\nu} + \theta_{13} \cos\delta. \tag{3.25}$$

Given the accurate determination of the reactor angle in Eq. (2.28) ($\theta_{13} \approx 9^\circ \pm 0.5^\circ$) and the solar angle Eq. (2.26) ($\theta_{12} \approx 34^\circ \pm 1^\circ$) the sum rule in Eq. (3.25) yields a favoured range of $\cos\delta$ for each of the cases $\theta_{12}^{\nu} = 35.26^\circ, 45^\circ, 31.7^\circ, 36^\circ$ for the cases of TB, BM, GR, GR$'$, namely $\cos\delta \approx -0.2, -1, 0.2, -0.2$, or $\cos\delta \approx -\lambda, -1, \lambda, -\lambda$, respectively. For example, for TB neutrino mixing, the sum rule in Eq. (3.25) may be written compactly as,

$$s \approx r\, \cos\delta. \tag{3.26}$$

In order to obtain the values in Eq. (3.7), namely $s \approx -\lambda^2$ from $r \approx \lambda$, we need to have $\cos\delta \approx -\lambda$.

This approach relies on a Cabibbo-sized charged lepton mixing angle as is clear from Eq. (3.20) which, together with Eq. (3.8), shows that we need $s_{12}^e \approx \lambda$ in order to account for the observed reactor angle, starting from one of the simple classic patterns of neutrino mixing. This is not straightforward to achieve in realistic models [67, 72], which would typically prefer smaller charged lepton mixing angles such as $s_{12}^e \approx \lambda/3$. This suggests that the neutrino mixing angle θ_{13}^{ν} is not zero, but has some non-zero value closer to the observed reactor angle. In the next subsection we consider this possibility.

3.6 Atmospheric mixing sum rules

In this subsection we consider simple alternative patterns related to TB mixing which allow a non-zero reactor angle. When looking for variants of TB mixing, it is useful to start from the general expansion around TB mixing in Eq. (3.4) [64], which to leading order, gives a PMNS mixing matrix, as in Eq. (B.1),

$$U_{\rm PMNS} \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix} P_{12}. \tag{3.27}$$

Clearly TB mixing in Eq. (3.3) corresponds to $s = a = r = 0$. If we set $s = a = 0$ but retain a non-zero value of r then this defines tri-bimaximal-reactor mixing (TBR) [73],

$$U_{\rm TBR} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}}(1-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \end{pmatrix} P_{12}. \tag{3.28}$$

This is very constrained, in particular it requires maximal atmospheric mixing $a = 0$. We can maintain trimaximal (TM) mixing defined by $s = 0$ but relax maximal atmospheric mixing, allowing both a non-zero a and r,

$$U_{\rm TM} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1-a+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-a-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+a-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1+a+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix} P_{12}. \tag{3.29}$$

There are two interesting special cases of TM mixing in which the first or second column of the mixing matrix reduce to those of the first or second column of the TB mixing matrix, referred to as TM1 and TM 2 mixing, namely,

$$U_{\rm TM1} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} r e^{-i\delta} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1-\frac{3}{2} r e^{i\delta}) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}(1+\frac{3}{2} r e^{i\delta}) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix} P_{12}, \tag{3.30}$$

with

$$a = r\cos\delta, \tag{3.31}$$

and

$$U_{\rm TM2} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} r e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\frac{3}{2} r e^{i\delta}) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1-\frac{3}{2} r e^{i\delta}) & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix} P_{12}, \tag{3.32}$$

with

$$a = -(r/2)\cos\delta. \tag{3.33}$$

These TM1 and TM2 relations, both with $s = 0$, are examples of atmospheric sum rules to first order in λ. In order to obtain the values in Eq. (3.7), namely $a \approx -\lambda/2$ with $r \approx \lambda$, we see that TM1 predicts $\cos\delta \approx -1/2$ and TM2 predicts $\cos\delta \approx 1$.

The above atmospheric sum rules are valid to leading order in λ. The exact TM1 relations (for both a and s) are obtained by equating PMNS elements to the first column of the TB mixing matrix:

$$c_{12}c_{13} = \sqrt{\frac{2}{3}}, \tag{3.34}$$

$$|c_{23}s_{12} + s_{13}s_{23}c_{12}e^{i\delta}| = \frac{1}{\sqrt{6}}, \tag{3.35}$$

$$|s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta}| = \frac{1}{\sqrt{6}}, \tag{3.36}$$

where these lead to Eq. (3.31) when expanded to leading order.

The exact sum rule relations for TM2 are obtained by equating PMNS elements to the second column of the TB mixing matrix:

$$s_{12}c_{13} = \frac{1}{\sqrt{3}}, \tag{3.37}$$

$$|c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta}| = \frac{1}{\sqrt{3}}, \tag{3.38}$$

$$|s_{23}c_{12} + s_{13}c_{23}s_{12}e^{i\delta}| = \frac{1}{\sqrt{3}}, \tag{3.39}$$

where these lead to Eq. (3.33) when expanded to leading order.

From Eqs. (3.34,3.37) we see that to leading order in s_{13} the solar angle is unchanged from its TB value for both TM1 and TM2, corresponding to $s = 0$ as discussed earlier, but to second order in s_{13}, the solar angle deviates and this deviation is different for TM1 and TM2.

4. The see-saw mechanisms

The starting point of the see-saw mechanisms is the dimension 5 operator in Eq. (1.6) which we repeat below,

$$-\frac{1}{2}HL^T\kappa HL. \tag{4.1}$$

One might wonder if it is possible to simply write down an operator by hand similar to Eq. (1.6), without worrying about its origin. In fact, historically, such an operator was introduced suppressed by the Planck scale (rather than the right-handed neutrino mass scales) by Weinberg in order to account for small neutrino masses [74]. The problem is that such a Planck scale suppressed operator would lead to neutrino masses of order 10^{-5}eV which are too small to account for the two heavier neutrino masses (though it could account for the lightest neutrino mass). To account for the heaviest neutrino mass requires a dimension 5 operator suppressed by a mass scale of order 3×10^{14} GeV if the dimensionless coupling of the operator is of order unity, and the Higgs vacuum expectation value (VEV) is equal to that of the Standard Model.

There are several different kinds of see-saw mechanism in the literature. In this review we shall focus on the simplest type I see-saw mechanism, which we shall introduce below. However for completeness we shall also discuss the type II and III see-saw mechanisms and the double see-saw mechanism, as well as the inverse and linear see-saw mechanisms.

4.1 Type I see-saw

Before discussing the see-saw mechanism it is worth first reviewing the different types of neutrino mass that are possible. So far we have been assuming that neutrino masses are Majorana masses of the form

$$m^{\nu}_{LL}\overline{\nu_L}\nu^c_L \ , \tag{4.2}$$

where ν_L is a left-handed neutrino field and ν^c_L is the CP conjugate of a left-handed neutrino field, in other words a right-handed antineutrino field. Such Majorana masses are possible since both the neutrino and the antineutrino are electrically neutral and so Majorana masses are not forbidden by electric charge conservation. For this reason a Majorana mass for the electron would be strictly forbidden. However such Majorana neutrino masses violate lepton number conservation. Assuming only Higgs *doublets*, they are forbidden in the Standard Model at the renormalisable level by gauge invariance. The idea of the simplest version of the see-saw mechanism is to assume that such terms are zero to begin with, but are generated effectively, after right-handed neutrinos are introduced [33].

If we introduce right-handed neutrino fields then there are two sorts of additional neutrino mass terms that are possible. There are additional Majorana masses of the form

$$M_{RR}\overline{\nu^c_R}\nu_R \ , \tag{4.3}$$

where ν_R is a right-handed neutrino field and ν^c_R is the CP conjugate of a right-handed neutrino field, in other words a left-handed antineutrino field. In addition there are Dirac masses of the form

$$m_{LR}\overline{\nu_L}\nu_R \ . \tag{4.4}$$

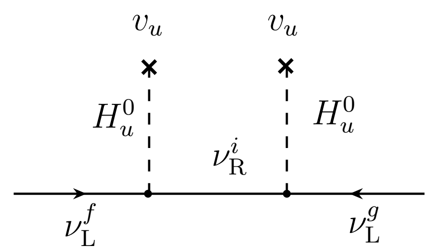

Figure 5: Diagram illustrating the type I see-saw mechanism.

Such Dirac mass terms conserve lepton number, and are not forbidden by electric charge conservation even for the charged leptons and quarks.

Once this is done then the types of neutrino mass discussed in Eqs. (4.3,4.4) (but not Eq. (4.2) since we assume no Higgs triplets) are permitted, and we have the mass matrix

$$\left(\overline{\nu_L} \; \overline{\nu^c_R} \right) \begin{pmatrix} 0 & m_{LR} \\ m_{LR}^T & M_{RR} \end{pmatrix} \begin{pmatrix} \nu^c_L \\ \nu_R \end{pmatrix} . \tag{4.5}$$

Since the right-handed neutrinos are electroweak singlets the Majorana masses of the right-handed neutrinos M_{RR} may be orders of magnitude larger than the electroweak scale. In the approximation that $M_{RR} \gg m_{LR}$ the matrix in Eq. (4.5) may be diagonalised to yield effective Majorana masses of the type in Eq. (4.2),

$$m^\nu_{LL} = -m_{LR} M_{RR}^{-1} m_{LR}^T \; . \tag{4.6}$$

The effective left-handed Majorana masses m^ν_{LL} are naturally suppressed by the heavy scale M_{RR}. In a one family example if we take $m_{LR} \sim M_W$ (where M_W is the mass of the W boson) and $M_{RR} \sim M_{\rm GUT}$ then we find $m^\nu_{LL} \sim 10^{-3}$ eV which looks good for solar neutrinos. Atmospheric neutrino masses would require a right-handed neutrino with a mass below the GUT scale.

The type I see-saw mechanism is illustrated diagrammatically in Fig. 5. It can be formally derived from the following Lagrangian

$$\mathcal{L} = -\overline{\nu_L} m_{LR} \nu_R - \frac{1}{2} \nu_R^T M_{RR} \nu_R + H.c. \; , \tag{4.7}$$

where ν_L represents left-handed neutrino fields (arising from electroweak doublets), ν_R represents right-handed neutrino fields (arising from electroweak singlets), in a matrix notation where the m_{LR} matrix elements are typically of order the charged lepton masses, while the M_{RR} matrix elements may be much larger than the electroweak scale, and maybe up to the Planck scale. The number of right-handed neutrinos is not fixed, but the number of left-handed neutrinos is equal to three. Below the mass scale of the right-handed neutrinos we can integrate them out using the equations of motion

$$\frac{d\mathcal{L}}{d\nu_R} = 0 \; , \tag{4.8}$$

which gives

$$\nu_R^T = -\overline{\nu_L} m_{LR} M_{RR}^{-1} \; , \qquad \nu_R = -M_{RR}^{-1} m_{LR}^T \overline{\nu_L}^T \; . \tag{4.9}$$

Substituting back into the original Lagrangian we find

$$\mathcal{L} = -\frac{1}{2}\overline{\nu_L} m^{\nu}_{LL} \nu^c_L + H.c. \ , \tag{4.10}$$

with m^{ν}_{LL} as in Eq. (4.6).

4.2 Minimal see-saw extension of the Standard Model

We now briefly discuss what the Standard Model looks like, assuming a minimal see-saw extension. In the Standard Model Dirac mass terms for charged leptons and quarks are generated from Yukawa type couplings to Higgs doublets whose vacuum expectation value gives the Dirac mass term. Neutrino masses are zero in the Standard Model because right-handed neutrinos are not present, and also because the Majorana mass terms in Eq. (4.2) require Higgs triplets in order to be generated at the renormalisable level. The simplest way to generate neutrino masses from a renormalisable theory is to introduce right-handed neutrinos, as in the type I see-saw mechanism, which we assume here. The Lagrangian for the lepton sector of the Standard Model containing three right-handed neutrinos with heavy Majorana masses is[5]

$$\mathcal{L}_{\rm mass} = -\left[\epsilon_{\alpha\beta}\tilde{Y}^{ij}_e H^{\alpha}_d L^{\beta}_i e^c_j - \epsilon_{\alpha\beta}\tilde{Y}^{ij}_{\nu} H^{\alpha}_u L^{\beta}_i \nu^c_j + \frac{1}{2}\nu^c_i \tilde{M}^{ij}_{RR}\nu^c_j\right] + H.c. \ , \tag{4.11}$$

where $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$, and the remaining notation is standard except that the 3 right-handed neutrinos ν^i_R have been replaced by their CP conjugates ν^c_i, and $\tilde{M}^{ij}_{RR}$ is a complex symmetric Majorana matrix. When the two Higgs doublets get their VEVs $< H^2_u >= v_u$, $< H^1_d >= v_d$, where the ratio of VEVs is defined to be $\tan\beta \equiv v_u/v_d$, we find the terms

$$\mathcal{L}_{\rm mass} = -v_d \tilde{Y}^{ij}_e e_i e^c_j - v_u \tilde{Y}^{ij}_{\nu} \nu_i \nu^c_j - \frac{1}{2}\tilde{M}^{ij}_{RR}\nu^c_i \nu^c_j + H.c. \ . \tag{4.12}$$

Replacing CP conjugate fields we can write in a matrix notation

$$\mathcal{L}_{\rm mass} = -\overline{e_L} v_d \tilde{Y}^*_e e_R - \overline{\nu_L} v_u \tilde{Y}^*_{\nu}\nu_R - \frac{1}{2}\nu^T_R \tilde{M}^*_{RR}\nu_R + H.c. \ . \tag{4.13}$$

It is convenient to work in the diagonal charged lepton basis

$$\mathrm{diag}(m_e, m_{\mu}, m_{\tau}) = V_{e_L} v_d \tilde{Y}^*_e V^{\dagger}_{e_R} \ , \tag{4.14}$$

and the diagonal right-handed neutrino basis

$$\mathrm{diag}(M_1, M_2, M_3) = V_{\nu_R}\tilde{M}^*_{RR} V^T_{\nu_R} \ , \tag{4.15}$$

where $V_{e_L}, V_{e_R}, V_{\nu_R}$ are unitary transformations. In this basis the neutrino Yukawa couplings are given by

$$Y_{\nu} = V_{e_L}\tilde{Y}^*_{\nu} V^T_{\nu_R} \ , \tag{4.16}$$

[5] We introduce two Higgs doublets to pave the way for the supersymmetric Standard Model. For the same reason we express the Standard Model Lagrangian in terms of left-handed fields, replacing right-handed fields ψ_R by their CP conjugates ψ^c, where the subscript R has been dropped. In the case of the minimal standard see-saw model with only one Higgs doublet, the other Higgs doublet in Eq. (4.11) is obtained by charge conjugation, i.e. $H_d \equiv H^c_u$.

and the Lagrangian in this basis is

$$\begin{aligned}\mathcal{L}_{\rm mass} = & - \ (\overline{e_L}\ \overline{\mu_L}\ \overline{\tau_L}) {\rm diag}(m_e, m_\mu, m_\tau)(e_R\, \mu_R\, \tau_R)^T \\ & - \ (\overline{\nu_{eL}}\ \overline{\nu_{\mu L}}\ \overline{\nu_{\tau L}}) v_u Y_\nu (\nu_{R1}\, \nu_{R2}\, \nu_{R3})^T \\ & - \ \frac{1}{2}(\nu_{R1}\, \nu_{R2}\, \nu_{R3}) {\rm diag}(M_1, M_2, M_3)(\nu_{R1}\, \nu_{R2}\, \nu_{R3})^T + H.c.\ . \end{aligned} \tag{4.17}$$

After integrating out the right-handed neutrinos (the see-saw mechanism) we find

$$\begin{aligned}\mathcal{L}_{\rm mass} = & - \ (\overline{e_L}\ \overline{\mu_L}\ \overline{\tau_L}) {\rm diag}(m_e, m_\mu, m_\tau)(e_R\, \mu_R\, \tau_R)^T \\ & - \ \frac{1}{2}(\overline{\nu_{eL}}\ \overline{\nu_{\mu L}}\ \overline{\nu_{\tau L}}) m^\nu_{LL} ({\nu_e}^c_L\, {\nu_\mu}^c_L\, {\nu_\tau}^c_L)^T + H.c.\ , \end{aligned} \tag{4.18}$$

where the light effective left-handed Majorana neutrino mass matrix in the above basis is given by the following see-saw formula which is equivalent to Eq. (4.6),

$$m^\nu_{LL} = -v_u^2\, Y_\nu\, {\rm diag}(M_1^{-1}, M_2^{-1}, M_3^{-1})\, Y_\nu^T\ . \tag{4.19}$$

In this basis the type I see-saw mechanism reproduces the dimension 5 operator in Eq. (4.1) with

$$\kappa = Y_\nu\, {\rm diag}(M_1^{-1}, M_2^{-1}, M_3^{-1})\, Y_\nu^T\ . \tag{4.20}$$

4.3 Sequential right-handed neutrino dominance

In this subsection we show how the type I see-saw mechanism may lead to a neutrino mass hierarchy with large solar and atmospheric mixing angles, and a reactor angle of order m_2/m_3 via a simple and natural mechanism known as sequential dominance (SD).[6] First consider the case of single right-handed neutrino dominance where only one right-handed neutrino ν_3^c of heavy Majorana mass M_3 is present in the see-saw mechanism, namely the one responsible for the atmospheric neutrino mass m_3 [51, 52]. We work in the basis of the previous subsection where the right-handed neutrinos and charged lepton mass matrices are diagonal. If the single right-handed neutrino couples to the three lepton doublets L_i in the diagonal charged lepton mass basis as,

$$H_u(dL_e + eL_\mu + fL_\tau)\nu_3^c, \tag{4.21}$$

where d, e, f are Yukawa couplings (assumed real for simplicity[7]), and H_u is the Higgs doublet, where it is assumed that $d \ll e, f$. so that the see-saw mechanism yields the atmospheric neutrino mass,

$$m_3 \approx (e^2 + f^2)\frac{v_u^2}{M_3}, \tag{4.22}$$

where $v_u = \langle H_u \rangle$. Then the reactor and atmospheric angles are approximately given by simple ratios of Yukawa couplings [51, 52],

$$\theta_{13} \approx \frac{d}{\sqrt{e^2+f^2}}, \qquad \tan\theta_{23} \approx \frac{e}{f}. \tag{4.23}$$

[6]SD should not be confused with an alternative mechanism proposed by Smirnov which is based on the premise that there are no large mixing angles in the Yukawa sector, and does not discuss any mechanism for achieving this involving right-handed neutrinos [75].

[7]The full results including phases are discussed in [51, 52] and summarised for the case of $d = 0$ in [53].

According to SD [51, 52] the solar neutrino mass and mixing are accounted for by introducing a second right-handed neutrino ν_2^c with mass M_2 which couples to the three lepton doublets L_i in the diagonal charged lepton mass basis as,

$$H_u(aL_e + bL_\mu + cL_\tau)\nu_2^c, \tag{4.24}$$

where a, b, c are Yukawa couplings (assumed real for simplicity). Then the second right-handed neutrino is mainly responsible for the solar neutrino mass, providing

$$(a, b, c)^2/M_2 \ll (e, f)^2/M_3, \tag{4.25}$$

which is the basic SD condition. Assuming this, then the see-saw mechanism leads to the solar neutrino mass,

$$m_2 \approx \left(a^2 + (c_{23}b - s_{23}c)^2\right)\frac{v_u^2}{M_2}, \tag{4.26}$$

and the solar neutrino mixing is approximately given by a simple ratios of Yukawa couplings [51, 52],

$$\tan\theta_{12} \approx \frac{a}{(c_{23}b - s_{23}c)}. \tag{4.27}$$

There is an additional contribution to the reactor angle of the form [51, 52],

$$\Delta\theta_{13} \approx \frac{a(eb + fc)}{(e^2 + f^2)^{3/2}}\frac{M_3}{M_2} \sim \mathcal{O}(m_2/m_3). \tag{4.28}$$

There may also be a third right-handed neutrino but with completely subdominant contributions to the see-saw mechanism, and hence it may ignored to leading order.[8]

Let us summarise what SD achieves. With the assumption in Eq. (4.25), SD predicts a neutrino mass hierarchy, together with solar and atmospheric mixing angles which are independent of neutrino mass. Since they only involve ratios of Yukawa couplings they may easily be large. On the other hand the reactor angle has two contributions, one proportional to a ratio of Yukawa couplings which may be small if $d \ll e$, while the other one gives a contribution $\mathcal{O}(m_2/m_3)$ which is by itself of the correct magnitude to account for the reactor angle (even if $d = 0$). The origin of these conditions and assumptions may be due to family symmetry as we will discuss.

4.4 Other see-saw mechanisms

One might also wonder if the see-saw mechanism with right-handed neutrinos is the only possibility? In fact it is possible to generate the dimension 5 operator in Eq. (1.6) by the exchange of heavy Higgs triplets of $SU(2)_L$, referred to as the type II see-saw mechanism [34] or the exchange of heavy $SU(2)_L$ triplet fermions, referred to as the type III see-saw mechanism [35].

In the type II see-saw the general neutrino mass matrix is given by

$$\begin{pmatrix} \overline{\nu_L} & \overline{\nu_R^c} \end{pmatrix} \begin{pmatrix} m_{LL}^{II} & m_{LR} \\ m_{LR}^T & M_{RR} \end{pmatrix} \begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix}. \tag{4.29}$$

[8] The contributions of the third sub-subdominant right-handed neutrino to the mixing angles has been considered in [76].

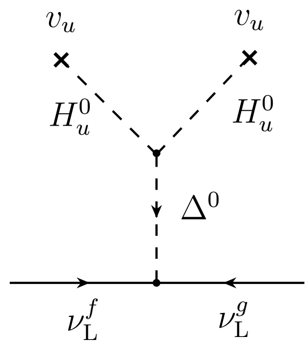

Figure 6: Diagram leading to a type II contribution m_{LL}^{II} to the neutrino mass matrix via an induced VEV of the neutral component of a triplet Higgs Δ.

Under the assumption that the mass eigenvalues M_i of M_{RR} are very large compared to the components of m_{LL}^{II} and m_{LR}, the mass matrix can approximately be diagonalised yielding effective Majorana masses

$$m_{LL}^{\nu} \approx m_{LL}^{II} + m_{LL}^{I} \ , \tag{4.30}$$

with

$$m_{LL}^{I} = -m_{LR}\, M_{RR}^{-1}\, m_{LR}^T \ , \tag{4.31}$$

for the light neutrinos. The direct mass term m_{LL}^{II} can also provide a naturally small contribution to the light neutrino masses if it stems e.g. from a see-saw suppressed induced VEV, see Fig. 6. The general case, where both possibilities are allowed, is also referred to as the type II see-saw mechanism.

Alternatively the see-saw can be implemented in a two-stage process by introducing additional neutrino singlets beyond the three right-handed neutrinos that we have considered so far. It is useful to distinguish between "right-handed neutrinos" ν_R which carry $B-L$ and perhaps form $SU(2)_R$ doublets with right-handed charged leptons, and "neutrino singlets" S which have no Yukawa couplings to the left-handed neutrinos, but which may couple to ν_R. If the singlets have Majorana masses M_{SS}, but the right-handed neutrinos have a zero Majorana mass $M_{RR} = 0$, the see-saw mechanism may proceed via mass couplings of singlets to right-handed neutrinos M_{RS}. In the basis (ν_L^c, ν_R, S) the mass matrix is

$$\begin{pmatrix} 0 & m_{LR} & 0 \\ m_{LR}^T & 0 & M_{RS} \\ 0 & M_{RS}^T & M_{SS} \end{pmatrix} . \tag{4.32}$$

There are two different cases often considered:

(i) Assuming $M_{SS} \gg M_{RS}$ the light physical left-handed Majorana neutrino masses are then,

$$m_{LL}^{\nu} = m_{LR} M_{RR}^{-1} m_{LR}^T \ , \tag{4.33}$$

where

$$M_{RR} = M_{RS} M_{SS}^{-1} M_{RS}^T . \tag{4.34}$$

This is called the double see-saw mechanism [40]. It is often used in GUT or string inspired neutrino mass models to explain why M_{RR} is below the GUT or string scale.

(ii) Assuming $M_{SS} \ll M_{RS}$, the matrix has one light and two heavy quasi-degenerate states for each generation. The mass matrix of the light physical left-handed Majorana neutrino masses is,

$$m^{\nu}_{LL} = m_{LR} M^{T\,-1}_{RS} M_{SS} M^{-1}_{RS} m^{T}_{LR}, \tag{4.35}$$

which has a double suppression.

In the limit that $M_{SS} \to 0$ all neutrinos become massless and lepton number symmetry is restored. Close to this limit one may have acceptable light neutrino masses for $M_{RS} \sim$ 1 TeV, allowing a testable low energy see-saw mechanism referred to as the inverse see-saw mechanism. If one allows the 1-3 elements of Eq. (4.32) to be filled in [77] then one obtains another version of the low energy see-saw mechanism called the linear see-saw mechanism.

5. Finite group theory in a nutshell

In this section, we give a first introduction to finite group theory, using the permutation group of three objects S_3 as an example, and later generalising the discussion to include all finite groups with triplet representations. Readers who are familiar with finite group theory may wish to skip this section.

5.1 Group multiplication table

Non-Abelian discrete symmetries appear to play an important role in understanding the physics of flavour. In order for this pedagogical review to be self-contained, we give a brief introduction into the main mathematical concepts of finite group theory. Many more details can be found, for instance, in the recent textbook by Ramond [78] which provides an excellent survey of the topic particularly aimed at physicists.

Finite groups G consist of a finite number of elements g together with a binary operation that maps two elements onto one element of G. In the following we use the term multiplication for such an operation. By definition, a group must include an identity element e as well as the inverse g^{-1} of a given element g. Furthermore, the multiplication must be associative, meaning that the product of three elements satisfies $(g_1 g_2) g_3 = g_1 (g_2 g_3)$. Groups are called Abelian if $g_1 g_2 = g_2 g_1$ for all group elements, while the elements of non-Abelian groups do not satisfy this trivial commutation relation in general. We shall only be interested in non-Abelian groups from now on.

The most basic way of defining a group is given in terms of the multiplication table, where the result of each product of two elements is listed. In the case of the smallest non-Abelian finite group, the permutation group S_3, we have:

S_3	e	a_1	a_2	b_1	b_2	b_3
e	e	a_1	a_2	b_1	b_2	b_3
a_1	a_1	a_2	e	b_2	b_3	b_1
a_2	a_2	e	a_1	b_3	b_1	b_2
b_1	b_1	b_3	b_2	e	a_2	a_1
b_2	b_2	b_1	b_3	a_1	e	a_2
b_3	b_3	b_2	b_1	a_2	a_1	e

The six elements are classified into the identity element e, elements b_i whose square is e and finally elements a_i for which the square does not yield e but, as can be seen easily, the cube does. It is generally true for any finite group that there exists some exponent n for each element g such that $g^n = e$. The smallest exponent for which this holds is called the order of the element g. This is not to be confused with the order of a group G which simply means the number of elements contained in G.

5.2 Group presentation

Clearly, the definition of a finite group in terms of its multiplication table becomes cumbersome very quickly with increasing order of G. It is therefore necessary to find a more compact way of defining G. Noticing that all six elements of S_3 can be obtained by multiplying only a subset of all elements, we arrive at the notion of generators of a group. Denoting $a_1 = a$ and $b_1 = b$, we obtain $a_2 = a^2$ as well as $b_2 = ab$ and $b_3 = ba$. In other words, a and b generate the group S_3. Being the group of permutations on three objects which is isomorphic to the group of symmetry transformations of an equilateral triangle, a corresponds to a 120° rotation and b to a reflection. This observation leads to the definition of S_3 using the so-called presentation

$$< a\,,\, b \mid a^3 \;=\; b^2 \;=\; e\,,\, bab^{-1} \;=\; a^{-1} > \,, \tag{5.1}$$

where the generators have to respect the rules listed on the right. Depending the these presentation rules, a group can be defined uniquely in a compact way. Unfortunately, such an abstract definition of a group is not very useful for physical applications as it does not show the possible irreducible representations of the group. We therefore quickly continue our journey through the fields of finite group theory towards the important notion of character tables.

5.3 Character table

In order to understand the meaning of a character table, is it mandatory to introduce the idea of conjugacy classes and irreducible representations. Conjugacy classes are subsets of elements of G which are obtained from collecting all those elements related to a given element g_i by conjugation gg_ig^{-1}, for all $g \in G$. The union of all possible conjugacy classes is nothing but the set of all elements of G. In the case of S_3 we find three different classes,

$$\begin{aligned}
1C^1(1) &= \{g\,1\,g^{-1}\,|\,g \in S_3\} &=& \{1\}\ , \\
2C^3(a) &= \{g\,a\,g^{-1}\,|\,g \in S_3\} &=& \{a,\, a^2\}\ , \\
3C^2(b) &= \{g\,b\,g^{-1}\,|\,g \in S_3\} &=& \{b,\, ab,\, ba\}\ .
\end{aligned} \tag{5.2}$$

Here we have used the notation $N_i C^{n_i}(g_i)$, where g_i is an element of the class, N_i gives the number of different elements contained in that class, and n_i denotes the order of these elements, which is identical for all $g g_i g^{-1}$ with $g \in G$.

The other ingredient for constructing a character table is the set of possible irreducible representations of the group G. In general non-Abelian groups can be realised in terms of $r \times r$ matrices, where the positive integers r depend on the group. Then, the abstract generators of a group are promoted to matrices which satisfy the presentation rules. Such matrix representations are called reducible if there exists a basis in which the $r \times r$ matrices of all generators of G can be brought into the same block diagonal form. If this is not possible, the representation is called irreducible. Clearly, the trivial singlet representation $\mathbf{1}$, where all generators of G are identically 1, satisfies any presentation rule and is thus an irreducible representation of all groups. This trivial example shows that irreducible representations do not necessarily have to be faithful, i.e. multiplying the matrices corresponding to the group generators can give a smaller number of different matrices than the order of G. In the case of S_3, the irreducible representations compatible with the presentation rules of Eq. (5.1) take the form

$$\begin{aligned}
\mathbf{1}:&\ a = 1\ , \qquad b = 1\ , \\
\mathbf{1}':&\ a = 1\ , \qquad b = -1\ , \\
\mathbf{2}:&\ a = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}\ , \qquad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ .
\end{aligned} \tag{5.3}$$

The fact that S_3 has three irreducible representations and also three conjugacy classes is not a coincidence. It is generally true that the number of irreducible representations of a finite group is equal to the number of its conjugacy classes. Moreover, summing up the squares of the dimensions of all irreducible representations always yields the order of the group G. For example, in S_3 we get $1^2 + 1^2 + 2^2 = 6$. These two facts can be used to work out all irreducible representations of a given group G.

In the case of irreducible representations $\mathbf{r}$ with $r > 1$, the explicit matrix form of the generators depends on the basis. In order to obtain a basis independent quantity, one defines the character $\chi_{g_i}^{[\mathbf{r}]}$ of the matrix representation of a group element g_i to be its trace. Since the elements of a conjugacy class are all related by $g g_i g^{-1}$ with $g \in G$, it is meaningful to speak of the character $\chi_i^{[\mathbf{r}]}$ of the elements of a conjugacy class i. Therefore one can define the (quadratic) character table where the rows list the irreducible representations and the columns show the conjugacy classes. Using Eq. (5.3), we easily find the following character table of S_3.

S_3	$1C^1(1)$	$2C^3(a)$	$3C^2(b)$
$\chi_i^{[\mathbf{1}]}$	1	1	1
$\chi_i^{[\mathbf{1}']}$	1	1	-1
$\chi_i^{[\mathbf{2}]}$	2	-1	0

Defining a group in terms of its character table is much more suitable for physical applications than the previous two definitions. First, it immediately lists all possible irreducible representations which might be used in constructing particle physics models. Secondly, it is also straightforward to extract the Kronecker products of a finite group G from its character table.

5.4 Kronecker products and Clebsch-Gordan coefficients

Multiplying arbitrary irreducible representations $\mathbf{r}$ and $\mathbf{s}$

$$\mathbf{r} \otimes \mathbf{s} \;=\; \sum_{\mathbf{t}} d(\mathbf{r},\mathbf{s},\mathbf{t})\;\mathbf{t}\;, \tag{5.4}$$

one can calculate the multiplicity $d(\mathbf{r},\mathbf{s},\mathbf{t})$ with which the irreducible representation $\mathbf{t}$ occurs in the product by

$$d(\mathbf{r},\mathbf{s},\mathbf{t}) \;=\; \frac{1}{N}\sum_i N_i \cdot \chi_i^{[\mathbf{r}]}\chi_i^{[\mathbf{s}]}\chi_i^{[\mathbf{t}]^*}\;, \tag{5.5}$$

where the sum is over all classes. N denotes the order of the group G and the asterisk indicates complex conjugation. With this, we obtain the following non-trivial Kronecker products from the S_3 character table,

$$\begin{aligned}
\mathbf{1}' \otimes \mathbf{1}' &= \mathbf{1}\;,\\
\mathbf{1}' \otimes \mathbf{2} \;&= \mathbf{2}\;,\\
\mathbf{2}\, \otimes \mathbf{2} \;&= \mathbf{1} \;+\; \mathbf{1}' \;+\; \mathbf{2}\;.
\end{aligned}$$

The Kronecker products are necessarily independent of the bases of the irreducible representations $\mathbf{r}$ with $r > 1$. When formulating and spelling out the details of a model, particular bases have to be chosen by hand. With the bases fixed, it is possible to work out the basis dependent Clebsch-Gordan coefficients of a group. Denoting the components of the two multiplet of a product by α_i and β_j, the resulting representation with components γ_k are obtained from,

$$\gamma_k = \sum_{i,j} c_{ij}^k\;\alpha_i\,\beta_j\;, \tag{5.6}$$

where c_k^{ij} are the Clebsch-Gordan coefficients. These are determined by the required transformation properties of the components γ_k under the group generators. In the case of S_3, using the basis of Eq. (5.3), one gets,

$$\begin{aligned}
\mathbf{1}' \otimes \mathbf{1}' &\rightarrow \mathbf{1} && \alpha\beta \ , \\
\mathbf{1}' \otimes \mathbf{2} &\rightarrow \mathbf{2} && \alpha \begin{pmatrix} \beta_1 \\ -\beta_2 \end{pmatrix} \ , \\
\mathbf{2} \otimes \mathbf{2} &\rightarrow \mathbf{1} && \alpha_1\beta_2 + \alpha_2\beta_1 \ , \\
\mathbf{2} \otimes \mathbf{2} &\rightarrow \mathbf{1}' && \alpha_1\beta_2 - \alpha_2\beta_1 \ , \\
\mathbf{2} \otimes \mathbf{2} &\rightarrow \mathbf{2} && \begin{pmatrix} \alpha_2\beta_2 \\ \alpha_1\beta_1 \end{pmatrix} \ ,
\end{aligned}$$

where α_i refers to the first factor of the Kronecker product and β_j to the second. We conclude our discussion of the most important concepts in finite group theory by pointing out that – due to the choice of convenient bases – a representation which is real (that is for which there exists a basis where all generators are explicitly real) may have complex generators. This is for instance the case for the doublet of S_3 in the basis of Eq. (5.3).

5.5 Finite groups with triplet representations

For applications in flavour physics, we are interested in finite groups with triplet representations. They can be found among the subgroups of $SU(3)$ and fall into four classes [44, 79]:[9]

- Groups of the type $(Z_n \times Z_m) \rtimes S_3$
- Groups of the type $(Z_n \times Z_m) \rtimes Z_3$
- The simple groups A_5 and $PSL_2(7)$ [81] plus a few more "exceptional" groups [82]
- The double covers of the tetrahedral (A_4), octahedral (S_4) and icosahedral (A_5) groups

The latter are subgroups of $SU(2)$, whose triplet representations are identical to the triplets of the respective rotation groups (which in turn are already included in the other classes). Many of the physically useful symmetries are special cases within these general classes. For instance, S_4, the natural symmetry of tri-bimaximal mixing in direct models, see Subsection 6.3, is isomorphic to $\Delta(6n^2) = (Z_n \times Z_n) \rtimes S_3$ with $n = 2$. The presentation rules of $\Delta(6n^2)$ can be given in terms of four[10] generators, a, b, c, d [83],

$$\begin{gathered}
a^3 = b^2 = (ab)^2 = c^n = d^n = 1 \ , \qquad cd = dc \ , \\
aca^{-1} = c^{-1}d^{-1} \ , \quad ada^{-1} = c \ , \quad bcb^{-1} = d^{-1} \ , \quad bdb^{-1} = c^{-1} \ .
\end{gathered} \tag{5.7}$$

[9] Subgroups of $U(3)$ can be derived from $SU(3)$ [44], however, a complete classification is still lacking [80].

[10] In principle, the presentation can be easily formulated with only three generators by expressing either c or d in terms of the other three generators.

The dimensions of all irreducible representations can only take values 1, 2, 3 or 6.[11] A faithful triplet representation is found, e.g., in the following set of matrices [83],

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \quad b = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad c = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^{-1} \end{pmatrix} , \tag{5.8}$$

where $\eta = e^{\frac{2\pi i}{n}}$. With $n = 2$ this triplet representation is explicitly real, and therefore does not correspond to the basis in which the S_4 order three generator T is diagonal and complex, cf. Section 6. To make connection to the S_4 triplet generators S, U and T as listen in Appendix C, we have to perform the basis transformation [85],

$$S = w\, d\, w^{-1} , \qquad U = w\, (aba^{-1})\, w^{-1} , \qquad T = w\, a\, w^{-1} , \tag{5.9}$$

where

$$w = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} , \quad \text{with} \quad \omega = e^{\frac{2\pi i}{3}} . \tag{5.10}$$

This shows how the tri-bimaximal Klein symmetry $Z_2 \times Z_2$ of the neutrino mass matrix in the diagonal charged lepton basis, generated by S and U of Eq. (6.8), is inherited from $\Delta(24) = (Z_2 \times Z_2) \rtimes S_3$: one Z_2 factor (namely S) originates from the first factor, $Z_2 \times Z_2$, and the other (namely U) is derived from the second, S_3. We remark in passing that the smallest group within the series $\Delta(6n^2)$ containing sextets is $\Delta(96)$.

Another series of groups can be obtained from the presentation of Eq. (5.7) by simply dropping the generator b, and consequently all conditions involving b [86]. This results in the groups $\Delta(3n^2) = (Z_n \times Z_n) \rtimes Z_3$ which only allow for irreducible representations of dimension 1 and 3. The case with $n = 2$ generates the tetrahedral group A_4, and the faithful triplet representation is the same as in the case of S_4 only without the b or U generator, cf. Eqs. (5.8,5.9). With $n = 3$ we obtain the group $\Delta(27)$ which has also been applied successfully as a family symmetry in indirect models [87–90].

A third series is obtained from the second class of groups, $(Z_n \times Z_m) \rtimes Z_3$, by setting $m = 1$. The presentation of this series of groups $T_n = Z_n \rtimes Z_3$ reads [91]

$$a^3 = c^n = 1 , \qquad aca^{-1} = c^k , \tag{5.11}$$

where the integer k must satisfy $1 + k + k^2 = 0 \mod n$. One can easily check that, with $\eta = e^{\frac{2\pi i}{n}}$, a faithful triplet representation is given by

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \quad c = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{k^2} \end{pmatrix} . \tag{5.12}$$

Popular examples of such groups include T_7 [92] and T_{13} [93], both of which do not include a Z_2 subgroup so that the Klein symmetry of the neutrino mass matrix cannot be obtained

[11] As shown in [84], this is also true for the more general series of groups $(Z_n \times Z_m) \rtimes S_3$.

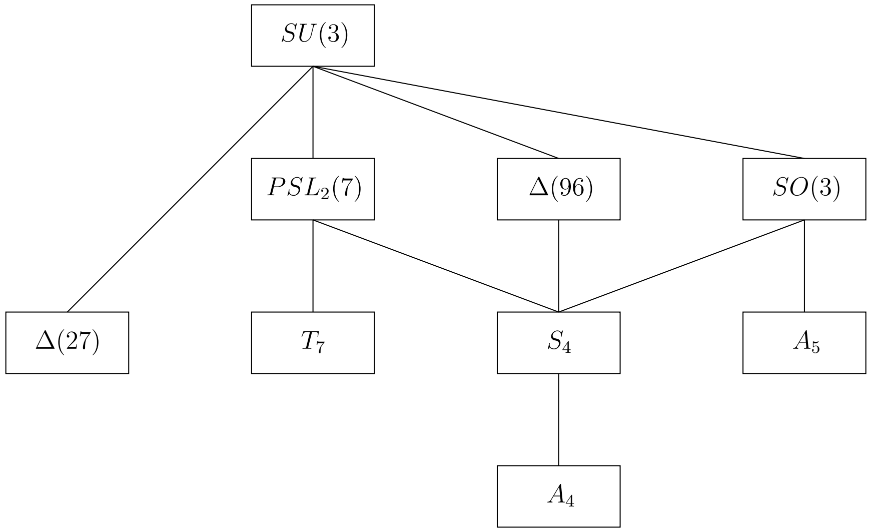

Figure 7: Examples of subgroups of $SU(3)$ with triplet representations discussed in this review. A line connecting two groups indicates that the smaller is a subgroup of the bigger one.

from these groups in a direct or semi-direct way, see Subsection 6.3. Yet, from the model building point of view it can still be useful to change to a basis in which the order three generator becomes diagonal [94], analogously to the case of S_4. In Appendix C we list the generators and Clebsch-Gordan coefficients of the groups S_4, A_4 and T_7 in the T diagonal basis. Their relation to $SU(3)$ and some of its subgroups is schematically illustrated in Fig. 7.

6. Discrete family symmetries and model building approaches

6.1 Family symmetries and flavons

The masses and mixings of the three families of quarks and leptons result from the form of the respective Yukawa matrices formulated in the flavour basis. Is there an organising principle which dictates the family structure of these Yukawa couplings? While this review takes the view that the observed mass and mixing patterns can be traced back to a family symmetry, we remark that some authors answer this question negatively, referring to a landscape of parameter choices out of which Nature has picked one that is compatible with the experimental measurements. In particular, the observation of a large reactor angle has been interpreted as a sign for an anarchical neutrino mass matrix [95]. Following the symmetry approach, it is clear that the family symmetry must be broken in order to generate the observed non-trivial structures. This is achieved by means of Higgs-type fields. These so-called flavon fields ϕ are neutral under the SM gauge group and break the family symmetry spontaneously by acquiring a VEV. This VEV in turn introduces an expansion parameter

$$\epsilon = \frac{\langle\phi\rangle}{\Lambda} \,, \tag{6.1}$$

(Λ denotes a high energy mass scale) which can be used to derive hierarchical Yukawa matrices, possibly with texture zeroes.

Family symmetries are sometimes also called horizontal symmetries, as opposed to GUT symmetries which unify different members within a family. It is possible to impose Abelian or non-Abelian family symmetries. The former choice goes back to the old idea of Froggatt and Nielsen [96] to explain the hierarchies of the quark masses and mixings in terms of an underlying $U(1)_{\rm FN}$ symmetry. In such a framework, the three generations of (left- and right-handed) quark fields $q_{L,R}$ (where we do not distinguish up- from down-type quarks) carry different charges under $U(1)_{\rm FN}$ such that the usual Yukawa terms have positive integer charges. Depending on the involved generations, this can be compensated by multiplying n_{ij} factors of the flavon field ϕ which conventionally has a $U(1)_{\rm FN}$ charge assignment of -1. The resulting terms which give rise to the usual Yukawa interactions then take the form

$$c_{ij} \left(\frac{\phi}{\Lambda}\right)^{n_{ij}} \overline{q_L}_i \, q_{Rj} \; H \; , \tag{6.2}$$

where H is the Higgs doublet, i and j are family indices and c_{ij} denote undetermined order one coefficients. Inserting the flavon VEV then generates the Yukawa couplings $Y_{ij} = c_{ij}\epsilon^{n_{ij}}$ which become hierarchical if the $U(1)_{\rm FN}$ charges are chosen appropriately. We emphasise that this approach is mainly useful for explaining hierarchical structures as the order one coefficients c_{ij} remain unspecified. Nonetheless, there have been recent proposals to adopt extensions of the Froggatt-Nielsen idea, involving additional generation dependent Z_n symmetries, in order to make qualitative predictions for the lepton sector as well, in particular aiming to explain the so-called bi-large neutrino mixing pattern [97–99].

In order to accurately describe non-hierarchical family structures such as the observed peculiar lepton mixing pattern, it is necessary to impose a non-Abelian family symmetry. The three generations of quarks and leptons can then be unified into suitable multiplets (i.e. irreducible representations) of the family symmetry G. An intimate connection of all three families is provided if G contains triplet representations, $\psi = (\psi_1, \psi_2, \psi_3)^T \sim \mathbf{3}$. Requiring irreducible triplet representations, the possible choices for G are limited to $U(3)$ and subgroups thereof. To illustrate the idea, we sketch the essential steps using the example of $SU(3)$ [100], applied to the Weinberg operator $HL^T\kappa LH$, cf. Eq. (1.6). The three generations of lepton doublets are unified into a triplet of $SU(3)$ while the Higgs doublet H is assumed to be a singlet of G. In order to construct an $SU(3)$ invariant operator, a flavon field ϕ transforming in the $\overline{\mathbf{3}}$ of $SU(3)$ can be introduced, leading to the term

$$HL^T \phi\phi^T LH \; . \tag{6.3}$$

The VEV of the flavon field ϕ will now correspond to a vector with a particular alignment, i.e. $\langle\phi\rangle \propto (a,b,c)^T$, where a,b,c take numerical values dictated by the scalar potential. Inserting this vacuum configuration into the factor $\phi\phi^T$ of Eq. (6.3) will generate a contribution to the neutrino mass matrix which is proportional to

$$\begin{pmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{pmatrix} . \tag{6.4}$$

Assuming simple flavon alignments, it is possible to relate all entries of the neutrino mass matrix in a particular way so that special mixing patterns can be explained. In practice, at least two flavons with different alignments have to be imposed in order to avoid degeneracies in the neutrino masses. Although it is possible to obtain simple and predictive flavon alignments in models based on continuous family symmetries such as $SU(3)$ and $SO(3)$, the problem of vacuum alignment can be solved in a significantly simpler and more natural way by imposing a *discrete* family symmetry instead. In the following we therefore focus our attention on non-Abelian discrete family symmetries with triplets.

6.2 The Klein symmetry of the neutrino mass matrix

The PMNS mixing is dictated by the structure of charged and neutral lepton mass matrices in a weak eigenstate basis. More precisely it is obtained as the mismatch of the transformations on the two left-handed lepton states needed to bring the charged lepton and the neutrino mass matrices into diagonal form. In order to easily reach a physical interpretation, it is convenient to work in a basis in which the charged leptons are diagonal, or approximately diagonal. The latter is useful in GUT model building where the non-diagonal hierarchical down-type quark mass matrix, required for the observed CKM mixing, is directly related to the charged lepton mass matrix which, as a consequence, is also not completely diagonal. The total PMNS mixing will then be predominantly determined by the neutrino mass matrix, and small charged lepton corrections, see Subsection 3.4, will have to be taken into account separately, leading to characteristic mixing sum rules as explained in Subsection 3.5.

With this in mind, one can hope to obtain clues on the nature of the underlying family symmetry by studying the symmetry properties of the neutrino mass matrix in the basis of (approximately) diagonal charged leptons. Assuming neutrinos to be Majorana rather than Dirac particles, their mass matrix is always symmetric under a Klein symmetry $Z_2 \times Z_2$. This follows from the obvious observation that the diagonalised neutrino mass matrix $m_{LL}^{\nu,\mathrm{diag}}$ is left invariant under the transformation

$$\widetilde{K}_{p,q}^T \, m_{LL}^{\nu,\mathrm{diag}} \, \widetilde{K}_{p,q} \;=\; m_{LL}^{\nu,\mathrm{diag}} \; , \qquad \text{with} \quad \widetilde{K}_{p,q} \;=\; \begin{pmatrix} (-1)^p & 0 & 0 \\ 0 & (-1)^q & 0 \\ 0 & 0 & (-1)^{p+q} \end{pmatrix} , \tag{6.5}$$

where p and q take the integer values 0 and 1. The explicit form of the Klein symmetry of the non-diagonalised neutrino mass matrix m_{LL}^{ν}, expressed in terms of 3×3 matrices, can then be determined as

$$K_{p,q} \;=\; U_{\mathrm{PMNS}}^{*} \widetilde{K}_{p,q} U_{\mathrm{PMNS}}^{T} \; , \tag{6.6}$$

where U_{PMNS} is (approximately) the unitary PMNS mixing matrix. The matrices $K_{p,q}$ form a group of four elements whose squares yield the identity element $K_{0,0}$. The fact that the neutrino mass matrix is symmetric under a transformation by $K_{p,q}$ can be easily verified using Eq. (3.14),

$$\begin{aligned} K_{p,q}^T \, m_{LL}^\nu \, K_{p,q} &= U_{\rm PMNS} \widetilde{K}_{p,q}^T (U_{\rm PMNS}^\dagger \, m_{LL}^\nu \, U_{\rm PMNS}^*) \widetilde{K}_{p,q} U_{\rm PMNS}^T \\ &= U_{\rm PMNS} (\widetilde{K}_{p,q}^T \, m_{LL}^{\nu,{\rm diag}} \, \widetilde{K}_{p,q}) U_{\rm PMNS}^T \\ &= U_{\rm PMNS} \, m_{LL}^{\nu,{\rm diag}} \, U_{\rm PMNS}^T \\ &= \; m_{LL}^\nu \; . \end{aligned} \tag{6.7}$$

We point out that the $Z_2 \times Z_2$ symmetry of Eq. (6.6) exists for any choice of PMNS mixing. In the remainder of this review we will denote the two generators of this Klein symmetry by S and U. Particularly simple forms of these generators are obtained when $U_{\rm PMNS}$ features a simple mixing pattern. For instance, in the case of tri-bimaximal mixing, $U_{\rm PMNS} = U_{\rm TB}$, we find

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \; , \qquad U = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{6.8}$$

Such a symmetry of the neutrino mass matrix is only meaningful if the charged leptons are (approximately) diagonal. Therefore, it is useful to consider also the (approximate) symmetry of the charged lepton mass matrix M_e. As charged leptons are Dirac particles, one has to consider the square $M_e M_e^\dagger$ which – if diagonal – is symmetric under a general phase transformation T,

$$T^\dagger \left(M_e M_e^\dagger \right) T \; = \; M_e M_e^\dagger \; , \qquad \text{with} \quad T \; = \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{m}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{m}} \end{pmatrix} \; , \tag{6.9}$$

and m being an integer. The smallest value of m which also enforces a diagonal $M_e M_e^\dagger$ is $m = 3$. Choosing $m = 2$ would leave room for non-vanishing off-diagonal entries in the 2-3 block of the squared charged lepton mass matrix, which however could be removed by imposing a second T-type generator with permuted diagonal entries.

We conclude the discussion of the symmetries of the mass matrices by emphasising that the T symmetry of the charged leptons can hold exactly in models which are only concerned about the lepton sector. In GUT models, which additionally include the quarks, such a T symmetry is usually only valid approximately.

6.3 The direct model building approach

Family symmetry models can be classified according to the origin of the symmetry of the neutrino mass matrix. The neutrino Klein symmetry can arise as a residual symmetry of the underlying family symmetry G, in other words, the four elements $K_{p,q}$ of Eq. (6.6) are also elements of the imposed family symmetry. Models of this type are called direct models [101].

In such models, the neutrino mass term involves flavon fields ϕ^ν whose vacuum alignments break the family symmetry G down to the remnant Klein symmetry of Eq. (6.6).

Schematically this can be expressed as

$$\mathcal{L}^\nu \;\sim\; \frac{\phi^\nu}{\Lambda^2} L H_u L H_u \;, \qquad \text{with} \quad S\langle\phi^\nu\rangle = U\langle\phi^\nu\rangle = \langle\phi^\nu\rangle \;, \tag{6.10}$$

where the flavon enters only linearly, and the lepton doublet L with hypercharge $-1/2$ transforms as a triplet $\mathbf{3}$ under the family symmetry, while the up-type Higgs doublet H_u with hypercharge $+1/2$ is a singlet $\mathbf{1}$ of G.[12] Depending on the family symmetry, there are several neutrino-type flavons ϕ^ν which contribute to the neutrino mass matrix and furnish different representations of G, typically also including a triplet representation. Therefore, S and U stand for the respective Klein generators in the representation of ϕ^ν. In particular, if the group theory allows LH_uLH_u to be contracted to a singlet $\mathbf{1}$ of G, then it is possible to introduce a flavon in the $\mathbf{1}$ of G with $S = U = 1$. Such a trivial flavon would clearly not break the Klein symmetry, in fact, it does not even break G. The same Klein symmetry can be realised straightforwardly in direct models based on the type I see-saw mechanism. There, the right-handed neutrinos ν^c transform as a $\overline{\mathbf{3}}$ of G so that the Dirac neutrino term does not involve a flavon field, and therefore does not break G at all, whereas the Majorana neutrino mass term involves the S and U preserving flavons linearly,

$$\mathcal{L}^\nu \sim L\nu^c H_u + \phi^\nu \nu^c \nu^c \;. \tag{6.11}$$

Application of the type I see-saw formula yields an effective light neutrino mass matrix which is again symmetric under S and U. Analogously, the charged lepton sector often involves a flavon ϕ^ℓ which breaks G (approximately) to the symmetry generated by T,

$$\mathcal{L}^\ell \;\sim\; \frac{\phi^\ell}{\Lambda} L \ell^c H_d \;, \qquad \text{with} \quad T\langle\phi^\ell\rangle \approx \langle\phi^\ell\rangle \;. \tag{6.12}$$

The direct approach is schematically illustrated in Fig. 8. However, we remark that according to our classification, the charged lepton sector may be diagonal simply by construction.

Assuming the T symmetry to be exact, the class of direct models clearly requires G to contain both the Klein symmetry of the neutrino sector as well as the symmetry of the charged leptons. The minimal symmetry group for which this is satisfied can be determined by simply calculating all possible products of the matrices S, U and T. In the tri-bimaximal case, i.e. with the generators of Eq. (6.8) for the neutrino Klein symmetry, and a charged lepton symmetry T of Eq. (6.9) with $m = 3$ one obtains a total of 24 different matrices which form a finite group isomorphic to S_4, the group of permutation on four objects. It is interesting to note that a different choice of m does not necessarily yield a finite group. In fact, with S and U of Eq. (6.8) one can easily show, using the computer algebra programme GAP [102], that no finite group is obtained for reasonably small $m \neq 3$. We have checked the validity of this statement for $m \leq 30$, which is even true for the case with $m = 2$. In that sense, S_4 is the natural symmetry group of tri-bimaximal mixing in direct models. Of course, any group that contains S_4 as a subgroup could be applied equally well.

In addition to the pure class of direct models, there are semi-direct models in which one Z_2 factor of the Klein symmetry can be identified as a subgroup of G, while the other

[12] We use the hypercharge convention such that $Q = T_3 + Y$.

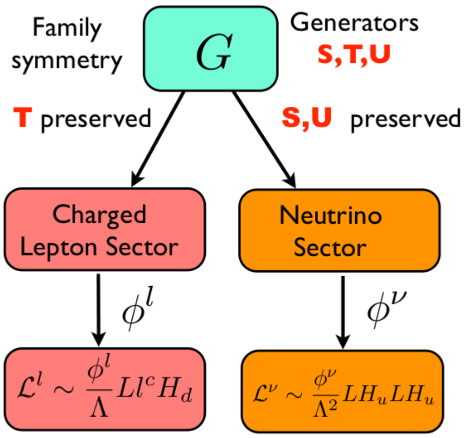

Figure 8: A sketch of the direct model building approach. The charged lepton sector is (approximately) diagonal either due to a remnant (approximate) T symmetry or simply by construction.

Z_2 factor arises accidentally. The flavons of semi-direct models appear linearly in the neutrino mass term, similar to Eq. (6.10), and break G down to one of its Z_2 subgroups. An example of such a model is provided by the famous Altarelli-Feruglio A_4 model of tri-bimaximal mixing [30, 103]. A_4 is the group of even permutations on four object, and as such a subgroup of S_4. It can be obtained from S_4 by simply dropping the U generator. Not being a part of the underlying family symmetry, it is therefore evident that the U symmetry of Eq. (6.8) must arise accidentally.

6.4 The indirect model building approach

In the class of indirect models, no Z_2 factor of the Klein symmetry of Eq. (6.6) forms a subgroup of G. Models of this class are typically based on the type I see-saw mechanism together with the assumption of sequential dominance, see Subsection 4.3. Here, the main role of the family symmetry consists in relating the Yukawa couplings d, e, f of Eq. (4.21) as well as a, b, c of Eq. (4.24) by introducing triplet flavon fields which acquire special vacuum configurations. The directions of the flavon alignments are determined by the G symmetric operators of the flavon potential [101].

Working in a basis where both the charged leptons as well as the right-handed neutrinos are diagonal, the leptonic flavour structure is encoded in the Dirac neutrino Yukawa operator. The triplet flavons ϕ_i^ν of indirect models enter linearly in this term,

$$\mathcal{L}^\nu \;\sim\; \sum_i \frac{\phi_i^\nu}{\Lambda} L\nu_i^c H_u + M_i \nu_i^c \nu_i^c \;, \tag{6.13}$$

where Λ is a cut-off scale and the sum is over the number of right-handed neutrinos. The lepton doublet L with hypercharge $-1/2$ transforms as a triplet of G, while the right-handed neutrinos ν_i^c and the up-type Higgs doublet with hypercharge $+1/2$ are all singlets of G. Adopting the notation of Subsection 4.3, extended to include a third right-handed neutrino ν_1^c, we obtain the Dirac neutrino Yukawa matrix by inserting the flavon VEVs into Eq. (6.13). Suppressing the dimensionless couplings of the Dirac neutrino terms for notational clarity, we get

$$Y^\nu \;=\; \begin{pmatrix} a' & a & d \\ b' & b & e \\ c' & c & f \end{pmatrix} \sim \frac{1}{\Lambda} \begin{pmatrix} \langle\phi_1^\nu\rangle_1 & \langle\phi_2^\nu\rangle_1 & \langle\phi_3^\nu\rangle_1 \\ \langle\phi_1^\nu\rangle_2 & \langle\phi_2^\nu\rangle_2 & \langle\phi_3^\nu\rangle_2 \\ \langle\phi_1^\nu\rangle_3 & \langle\phi_2^\nu\rangle_3 & \langle\phi_3^\nu\rangle_3 \end{pmatrix} . \tag{6.14}$$

The columns of the Dirac neutrino Yukawa matrix are therefore proportional to the vacuum alignments of the flavons fields ϕ_i^ν. The effective Majorana operators of the light neutrinos can be derived from this using the see-saw formula of Eq. (4.6), yielding

$$\mathcal{L}_{\text{eff}}^\nu \;\sim\; L^T \sum_{i=1}^{3} \left(\frac{\langle\phi_i^\nu\rangle}{\Lambda} \cdot \frac{1}{M_i} \cdot \frac{\langle\phi_i^\nu\rangle^T}{\Lambda} \right) L H_u H_u \; . \tag{6.15}$$

Note that the flavons enter the effective neutrino mass terms quadratically. In models with sequential dominance, the three contributions to the effective light neutrino mass matrix are hierarchical, and it is often possible to ignore one term (e.g. $i = 1$) such that the sum contains only one dominant and one subdominant contribution. In the class of indirect models, the PMNS mixing pattern thus becomes a question of the alignment vectors $\langle\phi_i^\nu\rangle$. For instance, a neutrino mass matrix that gives rise to tri-bimaximal mixing can be obtained using the flavon alignments

$$\frac{\langle\phi_1^\nu\rangle}{\Lambda} \propto \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} , \qquad \frac{\langle\phi_2^\nu\rangle}{\Lambda} \propto \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \qquad \frac{\langle\phi_3^\nu\rangle}{\Lambda} \propto \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} . \tag{6.16}$$

Note that the resulting columns of the Dirac neutrino Yukawa matrix are proportional to the columns of the unitary (in the present case tri-bimaximal) mixing matrix. Such a property of the Dirac neutrino Yukawa matrix is generally called form dominance [104].[13] Furthermore these alignments are left invariant under the action of the S and U generators of Eq. (6.8), up to an irrelevant sign which drops out due to the quadratic appearance of each flavon in Eq. (6.15). Since the family symmetry G does not contain the neutrino Klein symmetry, its primary role is then to explain the origin of these or similarly simple flavon alignments. We schematically illustrated the indirect approach in Fig. 9.

For completeness we also mention that there are a few special models not based on the type I see-saw mechanism, in which the Klein symmetry of the neutrino sector is not a subgroup of the family symmetry, see e.g. [89,94]. In such models, the flavon enters linearly in the neutrino mass term, and the Klein symmetry arises accidentally from a combination of the Clebsch-Gordan coefficients and suitable flavon alignments of ϕ^ν.

6.5 Comments on the classification into direct and indirect models

Before continuing to discuss the details of the family symmetry breaking, we compare the direct approach with the indirect one, see Figs. 8 and 9. This classification is purely based

[13] Exact form dominance entails vanishing leptogenesis [105].

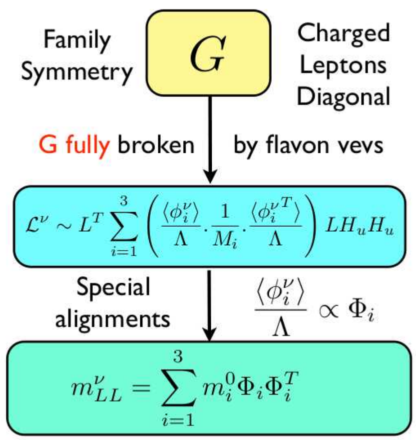

Figure 9: A sketch of the indirect model building approach. The charged lepton sector is (approximately) diagonal by construction.

on the origin of the $Z_2 \times Z_2$ Klein symmetry of the neutrino sector, formulated in a basis of (approximately) diagonal charged leptons. In direct models, this symmetry, generated by the order two elements S and U, arises as a subgroup of G, whereas this is not the case for indirect models. In both approaches, the family symmetry G has to be broken spontaneously by flavon fields acquiring a VEV. The flavon vacuum configuration of direct models is dictated by the requirement that S and U be preserved. In indirect models based on the type I see-saw, the vacuum alignment of the flavons enters in the columns of the Dirac neutrino Yukawa matrix, thereby generating contributions to the effective light neutrino mass matrix of the form proportional to $\langle \phi^\nu \rangle \langle \phi^\nu \rangle^T$.

We emphasise that the charged leptons have to be (approximately) diagonal by construction for the purpose of this classification. In the framework of direct models, this can be enforced by demanding a subgroup of G, generated by T, to be (approximately) preserved in the charged lepton sector. However, in grand unified models such a T symmetry cannot be exact as it would then apply to the quarks as well. This in turn would entail a phenomenologically unacceptable quark sector without CKM mixing.

7. Direct model building

In models based on a family symmetry G, the Dirac Yukawa and the Majorana couplings are typically generated dynamically from G invariant operators involving one or more flavon fields. In general, these flavons can transform in any of the irreducible representations of G. For non-Abelian discrete symmetries, the choice is limited to a finite set of representations. With flavons transforming as multiplets of the family symmetry G, the breaking of G and with it the family structure of the Dirac Yukawa and the Majorana couplings crucially depends on the *alignment* of the flavon VEVs. In this section we discuss general strategies for identifying useful flavons alignments in direct models where the family symmetry G is broken to a particular subgroup in the neutrino sector. Furthermore, we give explicit examples which illustrate ways of deriving vacuum alignments from flavon potentials. We remark that throughout this section we assume a diagonal or approximately diagonal charged lepton mass matrix which may or may not arise as a result of an (approximately) unbroken subgroup of G.

7.1 Flavon alignments in direct models

In direct models, flavons enter linearly in the terms of the neutrino Lagrangian as shown in Eq. (6.10). These operators are invariant under the full family symmetry G. When G is broken spontaneously by the flavon fields acquiring a VEV, the smaller $Z_2 \times Z_2$ Klein symmetry, generated by the order two elements $S, U \in G$, still remains intact. The criteria on the vacuum alignment of the involved flavon VEVs can therefore be formulated by the condition

$$S\langle\phi^\nu\rangle = U\langle\phi^\nu\rangle = \langle\phi^\nu\rangle \ . \tag{7.1}$$

With flavons generally furnishing different representations $\mathbf{r}$ of the family group G, the explicit matrix form of S and U clearly differs for different $\mathbf{r}$. For a given representation, it is then straightforward to calculate the form of the alignments which satisfy Eq. (7.1). This procedure can be repeated for all other representations $\mathbf{r}$, but not all of them will necessarily yield a solution to Eq. (7.1), meaning that flavons transforming in such representations cannot be adopted in the considered case.

We discuss this strategy of identifying the structure of the flavon alignments in an explicit example for the purpose of illustration. Consider the case of an S_4 family symmetry. The generators S and U of the tri-bimaximal Klein symmetry are listed in Appendix C for all five irreducible representations, cf. also Eq. (6.8). The S generator of the $\mathbf{1}$, $\mathbf{1'}$ and $\mathbf{2}$ are all trivial, i.e. the identity element. Therefore any vacuum configuration of flavons transforming in these representations will leave invariant the Z_2 symmetry associated with S. The second Z_2 symmetry of the Klein symmetry, generated by U is always broken by the VEV of a flavon transforming in the $\mathbf{1'}$ since $U = -1$ in this case, while it is left intact by a flavon in the $\mathbf{1}$ of S_4. For the two-dimensional representation, one quickly finds that the flavon alignment has to be proportional to $(1, 1)^T$ in order not to break the U symmetry. Turning to the $\mathbf{3}$ of S_4, invariance under U entails a flavon alignment of the form $(0, 1, -1)^T$. Applying the S transformation on such an alignment yields $(0, -1, 1)^T$, hence, this alignment does not satisfy Eq. (7.1) as it is not an eigenvector of S with eigenvalue $+1$. Finally, we discuss flavons transforming in the $\mathbf{3'}$ representation of S_4. The most general alignment which is left invariant under the U transformation reads $(a, b, b)^T$. Demanding $S(a, b, b)^T = (a, b, b)^T$ fixes $a = b$, showing that an alignment proportional to $(1, 1, 1)^T$ leaves invariant both S and U. Collecting the results of this discussion, we have shown that flavons transforming as a $\mathbf{1'}$ and $\mathbf{3}$ of S_4 cannot be used to break the family symmetry S_4 down to the tri-bimaximal Klein symmetry. On the other hand, flavon fields in the $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3'}$ representations can be adopted, where the latter two have to be aligned as

$$\langle\phi^\nu_{\mathbf{2}}\rangle = \varphi^\nu_{\mathbf{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \qquad \langle\phi^\nu_{\mathbf{3'}}\rangle = \varphi^\nu_{\mathbf{3'}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ , \tag{7.2}$$

in order to leave invariant the Z_2 symmetries associated with S and U. Here φ denotes the overall VEV of a flavon ϕ. Inserting all three flavons into Eq. (6.10), assuming the lepton doublets L to transform in the **3** representation, we end up with a neutrino mass matrix which comprises three terms,

$$m^\nu_{LL} \;\approx\; \left[\varphi^\nu_{\mathbf{3}'}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix} + \varphi^\nu_{\mathbf{1}}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + \varphi^\nu_{\mathbf{2}}\begin{pmatrix}0&1&1\\1&1&0\\1&0&1\end{pmatrix}\right]\frac{v_u^2}{\Lambda^2}\;. \tag{7.3}$$

Using the matrices S and U of Eq. (6.8), one can easily check explicitly that $S^T m^\nu_{LL} S = U^T m^\nu_{LL} U = m^\nu_{LL}$ as required. Clearly, the alignments of Eq. (7.2) depend on the chosen basis. In particular the basis of the doublet representation could have been chosen differently without affecting the basis of the triplet representation (which we fixed by demanding a diagonal T generator). This, however, would also change the Clebsch-Gordan coefficients such that the form of the neutrino mass matrix in Eq. (7.3) remains unchanged. We emphasise that the same procedure of identifying the flavon alignments of direct models can be applied to arbitrary choices of the Klein symmetry.

7.2 Vacuum alignment mechanism in direct models

Having determined the alignments required in a given direct models, the next step is to derive them from minimising a flavon potential. In the context of direct models, the most popular and perhaps natural approach to tackle the problem of the flavon alignment is provided by the so-called F-term alignment mechanism [30,103]. The idea is to couple the flavons to so-called driving fields in a supersymmetric setup. Like flavons, driving fields are neutral under the SM gauge group and transform in general in a non-trivial way under the family symmetry G. Introducing a $U(1)_R$ symmetry under which the chiral supermultiplets containing the SM fermions carry charge $+1$, allows to distinguish flavons from driving fields by assigning a charge of $+2$ to the latter while keeping the former neutral. With this $U(1)_R$ charge assignment, the driving fields can only appear linearly in the superpotential and cannot couple to the SM fermions. The set of superpotential operators involving the driving fields X_i is usually referred to as the driving or simply flavon potential W_{flavon}. Assuming that supersymmetry remains unbroken at the scale where the flavons develop their VEVs, we can obtain the flavon alignments from the terms of W_{flavon} by setting the F-terms of the driving fields to zero, i.e.

$$-F^*_{X_i} = \frac{W_{\text{flavon}}}{X_i} = 0\;, \tag{7.4}$$

by which the scalar potential is minimised.

To illustrate the F-term alignment mechanism we give two simple examples based on the family symmetry group S_4. First, consider a driving field $X_{\mathbf{1}}$ and a flavon field $\phi_{\mathbf{2}}$ transforming in the **1** and **2** representations of S_4, respectively. Expanding the resulting term of the driving superpotential in terms of the component fields $\phi_{\mathbf{2},i}$ we obtain

$$X_{\mathbf{1}}\phi_{\mathbf{2}}\phi_{\mathbf{2}} \;=\; X_{\mathbf{1}}(\phi_{\mathbf{2},1}\phi_{\mathbf{2},2}+\phi_{\mathbf{2},2}\phi_{\mathbf{2},1}) = 2X_{\mathbf{1}}\phi_{\mathbf{2},1}\phi_{\mathbf{2},2}\;. \tag{7.5}$$

The F-term condition of Eq. (7.4) then gives rise to the following two solutions,

$$\langle \phi_{\mathbf{2}} \rangle \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \qquad \langle \phi_{\mathbf{2}} \rangle \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{7.6}$$

Notice that these two alignments are related by the S_4 symmetry transformation U, while a transformation induced by T does not change the alignment but only the phase of the VEV. It is a general feature of any G symmetric theory that one particular solution for the flavon alignments will automatically imply a whole set of solutions which are related by symmetry transformations. However, the reverse need not be true, i.e. there may be cases in which two or more solutions exist which are not related through symmetry transformations.

As a second example let us consider the alignments of Eq. (7.2). One possible way to derive these using the F-term alignment mechanism consists in introducing two driving fields, one transforming in the $\mathbf{3}$ of S_4, the other in the $\mathbf{3}'$ [106]. The corresponding terms of the flavon superpotential then read

$$g_0 X_{\mathbf{3}} \phi^\nu_{\mathbf{3}'} \phi^\nu_{\mathbf{2}} \ + \ X_{\mathbf{3}'} \left(g_1 \phi^\nu_{\mathbf{3}'} \phi^\nu_{\mathbf{3}'} + g_2 \phi^\nu_{\mathbf{3}'} \phi^\nu_{\mathbf{2}} + g_3 \phi^\nu_{\mathbf{3}'} \phi^\nu_{\mathbf{1}} \right) \ , \tag{7.7}$$

where g_i are dimensionless coupling constants. Denoting the VEVs of $\phi^\nu_{\mathbf{3}'}$, $\phi^\nu_{\mathbf{2}}$ and $\phi^\nu_{\mathbf{1}}$ by c_i, b_i and a, respectively, the F-term conditions take the form

$$g_0 \left[b_1 \begin{pmatrix} c_2 \\ c_3 \\ c_1 \end{pmatrix} - b_2 \begin{pmatrix} c_3 \\ c_1 \\ c_2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ , \tag{7.8}$$

$$2 g_1 \begin{pmatrix} c_1^2 - c_2 c_3 \\ c_3^2 - c_1 c_2 \\ c_2^2 - c_3 c_1 \end{pmatrix} + g_2 \left[b_1 \begin{pmatrix} c_2 \\ c_3 \\ c_1 \end{pmatrix} + b_2 \begin{pmatrix} c_3 \\ c_1 \\ c_2 \end{pmatrix} \right] + g_3 a \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ . \tag{7.9}$$

Restricting to solutions in which all of the three flavons develop a VEV, Eq. (7.8) requires non-zero values for all b_i and all c_i. Using this, it is straightforward to find the most general solution to the set of F-term equations. Up to symmetry transformations, we obtain

$$\langle \phi^\nu_{\mathbf{3}'} \rangle = \varphi^\nu_{\mathbf{3}'} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ , \qquad \langle \phi^\nu_{\mathbf{2}} \rangle = \varphi^\nu_{\mathbf{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \qquad \varphi^\nu_{\mathbf{2}} = -\frac{g_3}{2 g_2} \varphi^\nu_{\mathbf{1}} \ . \tag{7.10}$$

We remark that the trivial vacuum, that is the vacuum configuration where none of the flavons develops a VEV, typically provides a solution to the F-term equations as well. This can be eliminated by including soft supersymmetry breaking effects. Then the scalar potential relevant for the flavon alignments takes the general form

$$V_{\text{flavon}} \ = \ \sum_i \left| \frac{W_{\text{flavon}}}{X_i} \right|^2 + \left| \frac{W_{\text{flavon}}}{\phi_i} \right|^2 + m^2_{X_i} |X_i|^2 + m^2_{\phi_i} |\phi_i|^2 + \ \cdots \ , \tag{7.11}$$

where $m^2_{X_i}$ and $m^2_{\phi_i}$ denote the soft breaking masses of the driving fields X_i and the flavons ϕ_i. The dots stand for additional soft breaking terms. Assuming positive $m^2_{X_i}$, the driving fields do not develop a VEV. As a consequence, the operators which involve a driving field, i.e. those represented by the second term of Eq. (7.11), vanish. The first term, on the other hand, only depends on the flavon fields. This together with negative $m^2_{\phi_i}$ removes the trivial vacuum configuration and the flavons acquire a VEV [30, 103].

Alternatively, it is in principle also possible to add an explicit mass scale in the flavon potential which will then drive the flavon VEVs to non-zero values. For instance, the cube of the S_4 doublet flavon $\phi_{\mathbf{2}}$ of Eq. (7.6) can be contracted to an S_4 singlet with a non-vanishing VEV. Introducing a driving field $X'_{\mathbf{1}}$ one could therefore write down the driving terms

$$X'_{\mathbf{1}} \left[\frac{(\phi_{\mathbf{2}})^3}{M} - M^2 \right] , \tag{7.12}$$

where M denotes an explicit mass scale. Since M is a pure (dimensionful) number, the driving field $X'_{\mathbf{1}}$ and with it $(\phi_{\mathbf{2}})^3$ must be completely neutral under any imposed symmetry. In particular, they must not carry charges under extra so-called shaping symmetries which are typically introduced in concrete models to separate the flavons of different sectors. In the given example, a Z_3 shaping symmetry under which the flavon $\phi_{\mathbf{2}}$ (as well as the driving field $X_{\mathbf{1}}$) carries charge $+1$ allows for the coexistence of the alignment term of Eq. (7.5) together with the term of Eq. (7.12) which explictly drives the flavon VEV to non-zero values. Assuming a CP conserving high energy theory where all parameters of the model can be chosen to be real, driving terms of the form of Eq. (7.12) could generate spontaneous CP violation where the values of the CP violating phases are constrained to a finite number of choices [107].

7.3 Direct models after Daya Bay and RENO

The method of identifying the flavon alignments of direct models using Eq. (7.1) can be applied to any mixing pattern. Yet, until recently, the main focus was limited to only a few simple cases, namely tri-bimaximal, bimaximal and golden ratio mixing, see Subsection 3.1, all of which predict $\theta_{13} = 0°$. The observation of a sizable reactor neutrino mixing angle of about 9° by the Daya Bay and RENO collaborations in early 2012, preceded by first hints for a non-zero θ_{13} from the T2K, MINOS and Double Chooz experiments in 2011, has now ruled out these simple mixing patterns. This fact seems to call into question the direct model building approach. However, it is worth recalling that a vanishing reactor angle has long been compatible with experimental data, and hence there was no need to consider more complicated mixing patterns. The situation has now changed, and new strategies for constructing direct models have to be conceived.

There are two main paths one can pursue. The first is based on *small groups* like e.g. S_4 and A_5, and leads to the simple TB, BM or GR mixing patterns with vanishing reactor angle at leading order. To render such models compatible with a sizable θ_{13} it is critical to discuss higher order corrections which break the simple structure of the mixing matrix. This situation is depicted in the central branch of Fig. 10 for the family symmetry S_4. Depending on which group elements are selected for the symmetries of the charged lepton sector (T) and the neutrino sector (S, U), it is possible to obtain either TB or BM mixing from S_4 at leading order. Analogously, the leading order GR mixing derived from A_5

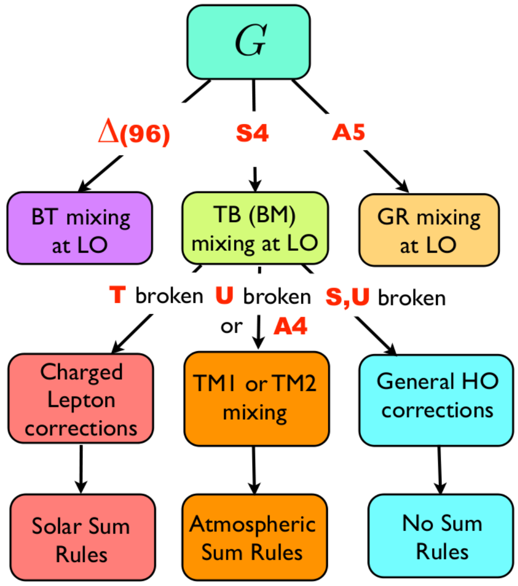

Figure 10: Possible strategies for constructing direct models after Daya Bay and RENO. Adopting small family symmetries G which predict simple leading order (LO) mixing patters with $\theta_{13} = 0$ (e.g. S_4, A_5), requires higher order (HO) corrections. Larger family symmetries can give rise to richer LO mixing patterns with non-zero θ_{13} (e.g. $\Delta(96)$). The A_4 family symmetry refers to the semi-direct case as discussed in the text. In this diagram, we have used the acronyms BT=bi-trimaximal, TB=tri-bimaximal, BM=bimaximal, GR=golden ratio, TM=trimaximal.

can be perturbed by higher order effects (not shown explicitly in Fig. 10). In general, higher order corrections are guaranteed to perturb the leading order structure by only small contributions. The breaking of the leading order structure can happen either in the charged lepton or the neutrino sector. The former entails charged lepton corrections of the simple leading order mixing patterns, which give rise to solar mixing sum rules as discussed in Subsection 3.5. If the breaking occurs in the neutrino sector, it is possible to break either one or both Z_2 factors of the leading order Klein symmetry. As the U symmetry typically enforces $\theta_{13} = 0$ in these models, it is necessary to break U in either case. Demanding S to remain a good symmetry at higher order, gives rise to atmospheric mixing sum rules, see Subsection 3.6, while breaking also S leads to arbitrary and unpredictive higher order corrections. In Subsection 10.2 we will present a concrete $S_4 \times SU(5)$ model of tri-bimaximal mixing at leading order, augmented by higher order corrections which break U but not S. This model yields the trimaximal neutrino mixing pattern TM2, see Eq. (3.32), which can accommodate a sizable reactor angle.

The second strategy of constructing direct models compatible with a sizable reactor angle makes use of *larger groups* such as $\Delta(96)$, see left branch of Fig. 10. Such groups are capable of predicting richer leading order mixing patterns (e.g. bi-trimaximal mixing [31]) as they contain non-standard Klein symmetries, generated by more complicated forms of the elements S and/or U [108, 109]. As before, higher order effects can correct these leading order mixing patterns. Charged lepton corrections induced by a breaking of the T generator give rise to new solar sum rules. Indeed, in the $\Delta(96)\times SU(5)$ model discussed in Subsection 10.3, the charged lepton corrections are essential in driving the resulting reactor angle to a physically viable value. In the neutrino sector, it is generally possible to break either S or U, however, in typical models, the symmetry associated with S stabilises the solar angle at a phenomenologically viable value. In practice, it should therefore be the U generator which gets broken at higher order, leading to new atmospheric sum rules.

Before turning to the breaking of the family symmetry in indirect models, we comment on the case of semi-direct models. As mentioned at the end of Subsection 6.3, the Altarelli-Feruglio A_4 model [30, 103] provides an example of a semi-direct model. While the tri-bimaximal S symmetry of the neutrino mass matrix forms part of the family symmetry, the tri-bimaximal U symmetry arises accidentally due to the absence of flavons in the $\mathbf{1}'$ and $\mathbf{1}''$ representations of A_4. Introducing such neutrino type flavons in the non-trivial one-dimensional representations, a situation which is tantamount to breaking the U generator in S_4 (see Fig. 10), generates contributions to the neutrino mass matrix which are not of tri-bimaximal form [106, 110]. However, as can be seen from Appendix C, they respect the original S symmetry, thus enforcing the trimaximal mixing pattern TM2, see Eq. (3.32). The accidental tri-bimaximal U symmetry, on the other hand, gets broken by the VEVs of the flavons in the $\mathbf{1}'$ and $\mathbf{1}''$ representations, which allows to accommodate arbitrary values of θ_{13}. The relative smallness of the reactor angle, compared to the solar and atmospheric angles, remains unaccounted for and must therefore be understood as a result of a mild tuning of parameters.

A similar semi-direct approach was taken by Hernandez and Smirnov [111] in an effort to accommodate a sizable reactor angle. Focusing on the relevant von Dyck groups A_4, S_4 and A_5, they demand the T symmetry of the charged leptons and (only) one Z_2 factor of the neutrino Klein symmetry[14] to arise as unbroken subgroups of the underlying family symmetry. This strategy allows to identify viable mixing patterns in which a given column of the PMNS matrix is completely determined by the properties of the imposed symmetry group. For the successful cases, these columns are identical (in some cases up to permutations of the rows) to either the first or the second column of the bimaximal, the tri-bimaximal and the golden ratio mixing patterns, cf. Subsection 3.1, [113]. With the other two columns of the mixing matrix unspecified, the reactor angle can be regarded as a free parameter which, together with the CP phase δ, gives rise to predictions for the other two mixing angles, expressed in the form of (exact) sum rules.

8. Indirect model building

8.1 Flavon alignments in indirect models

The vast majority of indirect models is formulated in the framework of the type I see-saw mechanism where the right-handed neutrino and the charged lepton mass matrices

[14] Assuming a diagonal charged lepton sector, correlations of the neutrino mixing parameters arising from requiring only one Z_2 factor were previously derived in [112].

are both diagonal, see Subsection 6.4. The lepton mixing arises from the structure of the Dirac neutrino Yukawa matrix, which in turn originates from the alignment of the flavon fields ϕ_i^ν. With the lepton doublet L furnishing a triplet representation $\mathbf{3}$ of the family symmetry G, the neutrino flavons typically transform as a $\overline{\mathbf{3}}$ of G.[15] The family indices are then contracted to the G singlet in the familiar $SU(3)$ way, showing that the columns of the Dirac neutrino Yukawa matrix are proportional to the alignments of the flavon fields ϕ_i^ν, as presented in Eq. (6.14). Application of the see-saw formula gives rise to an effective light neutrino mass matrix of the form

$$m_{LL}^\nu \;=\; \sum_{i=1}^{3} m_i^0 \Phi_i \Phi_i^T \;, \tag{8.1}$$

where $\Phi_i \propto \frac{\langle \phi_i^\nu \rangle}{\Lambda}$ denotes a dimensionless vector normalised to one. From the model building perspective, the direction of these vectors in flavour space depends on the alignment of the flavon VEVs. In general one can distinguish two cases. In this subsection we focus on the case where the flavon alignments are orthogonal to each other. The situation where this is not the case will be treated in Subsection 8.3.

Under the assumption that Φ_i and Φ_j are orthogonal for $i \neq j$, the light neutrino mass matrix m_{LL}^ν of Eq. (8.1) is diagonalised by a unitary PMNS mixing matrix with columns Φ_i^*. The resulting eigenvalues are simply m_i^0. This scenario, in which the columns of the Dirac neutrino Yukawa matrix are proportional to the columns of the PMNS mixing matrix, is called form dominance (FD) [104]. An example of this is provided by the alignments of Eq. (6.16),

$$\Phi_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} , \qquad \Phi_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \qquad \Phi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} , \tag{8.2}$$

generating the famous tri-bimaximal mixing pattern. It is important to notice that FD is a general concept which applies to arbitrary orthogonal vectors Φ_i. In principle, one could therefore choose the Φ_i directions so as to yield the experimentally observed mixing matrix. However, when it comes to building a model of flavour, a crucial ingredient is the justification of the assumed flavon alignments, see Subsection 8.2. Therefore, in practice, only "simple" alignments are adopted in explicit models. With the parameters m_i^0 of Eq. (8.1) being completely independent of the vectors Φ_i (which arise from some flavon alignment mechanism), it is clear that the mixing matrix does not depend on the masses. In indirect models, FD thus implies form diagonalisability [104, 114].[16]

[15] If the triplet $\mathbf{3}$ is real, L and ϕ_i^ν transform in the same representation. In indirect models, the basis of the triplet representation of G must then be chosen explicitly real. Note that for this reason the Clebsch-Gordan coefficients of S_4 and A_4 given in Appendix C are not applicable in indirect models. For a basis suitable for indirect models, see Eq. (5.8) and the discussion thereafter.

[16] However, this is not necessarily the case in direct models, see [106], where the columns of the Dirac neutrino Yukawa matrix – in the basis of diagonal right-handed neutrinos – are not related to the flavon alignments in a simple way.

A special case of FD is obtained if the three contributions to the neutrino mass matrix of Eq. (8.1) feature a hierarchy $m_1^0 \ll m_2^0 \ll m_3^0$. In such a scenario, which is called sequential dominance [51, 52, 76], see Subsection 4.3, the first term, and with it the vector Φ_1, can be ignored to good approximation. In fact, one can even remove the flavon ϕ_1^ν from the theory altogether. This would set m_1^0 automatically to zero, without affecting the pattern of the 3×3 mixing. The latter can be understood by realising that the first column of the mixing matrix is uniquely determined by requiring orthogonality to the other two columns Φ_2 and Φ_3. As above, SD is a general concept applicable to arbitrary two (or three) orthogonal flavon alignments. Choosing Φ_2 and Φ_3 as given in Eq. (8.2) leads to constrained sequential dominance (CSD) [70], and predicts tri-bimaximal neutrino mixing.

8.2 Vacuum alignment mechanism in indirect models

We have discussed in Subsection 7.2 how flavons of direct models can be aligned using the F-term alignment mechanism. In indirect models, the same mechanism is available, however, if a triplet representation of the family symmetry is real, it is mandatory to work in a basis where this is explicitly realised, i.e. where all group generators are real. Applications of the F-term alignment mechanism in indirect models can be found e.g. in [107, 115, 116]. In addition to the usual F-term alignment mechanism, indirect models offer an elegant alternative possibility for achieving particular flavon vacuum configurations. This so-called D-term alignment mechanism, as the name suggests, was first implemented in supersymmetric models [87, 117], however it is also possible to apply it in a non-supersymmetric context.

The starting point is a flavon scalar potential field which may or may not arise in a supersymmetric model from D-terms,

$$V \;=\; -\,m^2 \sum_i {\phi^i}^\dagger \phi^i \;+\; \lambda \left(\sum_i {\phi^i}^\dagger \phi^i \right)^2 \;+\; \Delta V \;, \tag{8.3}$$

where the index i labels the components of a particular flavon triplet ϕ and

$$\Delta V \;=\; \kappa \sum_i {\phi^i}^\dagger \phi^i {\phi^i}^\dagger \phi^i \;. \tag{8.4}$$

Ignoring the term ΔV in Eq. (8.3), the potential features an $SU(3)$ symmetry and, as a consequence, no direction of the flavon alignment would be preferred. Inclusion of the term ΔV breaks the $SU(3)$ symmetry of the potential and leads to minima which single out particular vacuum alignments. With the scale of the flavon VEV depending on m^2, λ and κ, it is sufficient to consider the extrema of the quartic term in Eq. (8.4) for a unit vector Φ. If $\kappa > 0$, it is necessary to *minimise* the sum $\sum_i |\Phi^i|^4$, leading to the solution

$$\kappa > 0 \qquad \longrightarrow \qquad \Phi_+ = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\vartheta_1} \\ e^{i\vartheta_2} \\ e^{i\vartheta_3} \end{pmatrix} \;, \tag{8.5}$$

where ϑ_i are arbitrary phases.[17] Such an alignment is of the form of Φ_2 in Eq. (8.2). In fact, in indirect models, where the alignment of Eq. (8.5) appears as a column of the Dirac neutrino Yukawa matrix, the phases ϑ_i can be removed by a field redefinition of the charged leptons. In the case where $\kappa < 0$, the sum $\sum_i |\Phi^i|^4$ has to be *maximised.* This gives rise to the alignment

$$\kappa < 0 \qquad \longrightarrow \qquad \Phi_- = \begin{pmatrix} e^{i\vartheta_1} \\ 0 \\ 0 \end{pmatrix} , \tag{8.6}$$

and permutations thereof. Such alignments are typically useful for constructing a diagonal charged lepton sector. They are furthermore necessary to obtain the alignments Φ_3 in Eq. (8.2) via $SU(3)$ invariant orthogonality conditions. Introducing a new flavon field ϕ which couples to the flavons ϕ_+ (with alignment $\Phi_+ = \Phi_2$) and ϕ_- (with alignment Φ_-) as

$$\kappa' \left| \sum_i {\phi_-^i}^\dagger \phi^i \right|^2 + \kappa'' \left| \sum_i {\phi_+^i}^\dagger \phi^i \right|^2 , \tag{8.7}$$

we generate the alignment $\langle \phi \rangle \propto \Phi_3$ if κ' and κ'' are taken to be positive. An alignment proportional to Φ_1 of Eq. (8.2) can be obtained subsequently from orthogonality conditions involving flavons with alignments along the directions Φ_2 and Φ_3.

The preceding discussion illustrates the importance of the $SU(3)$ breaking term in Eq. (8.3). It is therefore natural to identify finite groups G which have the operator in Eq. (8.4) as an invariant. Obviously, the family symmetry G must admit at least one triplet representation, with generators which are symmetry transformations of Eq. (8.4). As was shown in [101], possible candidate symmetries include the groups $\Delta(3n^2)$ [82, 86], $\Delta(6n^2)$ [82, 83] and T_n [91], cf. also Eqs. (5.8,5.12).

All these symmetries allow for at least two quartic invariants of type $\mathbf{3\bar{3}3\bar{3}}$, namely the $SU(3)$ invariant and the operator of Eq. (8.4). However, four of them have additional independent quartic invariants. These are $\Delta(24) = S_4$ with one extra invariant, as well as $\Delta(12) = A_4$, $\Delta(27)$ and $\Delta(54)$ with two additional invariants each [101]. These new invariants may spoil the structure of the vacuum derived from ΔV of Eq. (8.4) unless they are sufficiently suppressed. From this perspective, the groups $\Delta(3n^2)$ and $\Delta(6n^2)$ with $n > 3$, as well as the groups T_n are preferred candidates for the underlying discrete family symmetry of indirect models.

We conclude the discussion of the alignments in indirect models with a possible alternative to the invariant of Eq. (8.4), which has not received any attention yet. A cubic term of the form $\phi^1\phi^2\phi^3$ is left invariant under the groups $\Delta(3n^2)$ and T_n. As such a term is generally not real, the new term in the flavon potential reads

$$\Delta V = \kappa(\phi^1\phi^2\phi^3 + H.c.) , \tag{8.8}$$

[17] In principle these phases could be rotated away by an $SU(3)$ transformation, however, this would generally change the basis of the assumed discrete symmetry. Yet, in specific models, these phases can typically be absorbed into a redefinition of the physical fields that accompany these flavons.

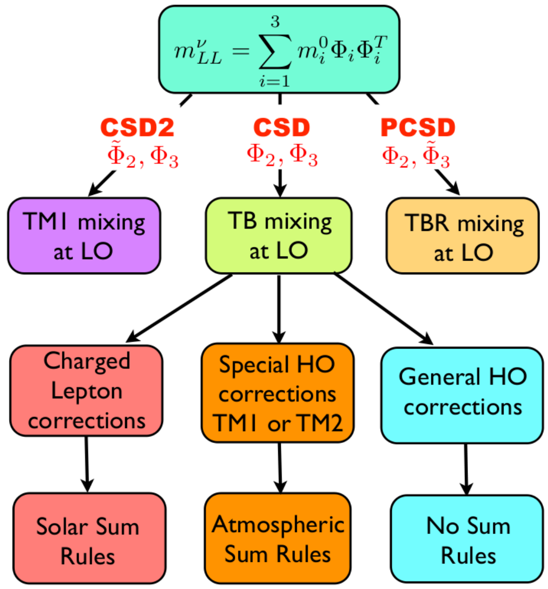

Figure 11: Possible strategies for constructing indirect models after Daya Bay and RENO. Starting from constrained sequential dominance (CSD) it is possible to add higher order (HO) corrections. Alternatively, it is possible to modify the flavon alignments of CSD at leading order, thus giving rise to constrained sequential dominance 2 (CSD2) and partially constrained sequential dominance (PCSD). In this diagram, we have used the acronyms TM=trimaximal, TB=tri-bimaximal, TBR=tri-bimaximal-reactor, LO=leading order.

replacing the operator in Eq. (8.4). One can easily show that, with $\kappa < 0$, such a term in the flavon potential would generate an alignment of type Φ_+, see Eq. (8.5), with $\vartheta_3 = -(\vartheta_1 + \vartheta_2)$.

8.3 Indirect models after Daya Bay and RENO

So far, we have only discussed the flavon alignments of indirect models leading to tri-bimaximal mixing. The measurement of a reactor angle θ_{13} of around $9°$ by the Daya Bay and RENO collaborations has now ruled out models of accurate tri-bimaximal mixing. This fact demands a modification or extension of the common strategies for constructing indirect models.

As for direct models, there are two principle paths one can pursue. The first builds on existing indirect models of tri-bimaximal mixing which arise in the framework of constrained sequential dominance, see the central branch of Fig. 11. Such a leading order structure must be broken by higher order corrections, which can stem from either the charged lepton or the neutrino sector. The former case requires a breaking of the accidental T symmetry and leads to solar mixing sum rules as discussed in Subsection 3.5. Alternatively, the higher order corrections can break the accidental tri-bimaximal U symmetry in the neutrino sector, entailing atmospheric mixing sum rules, see Subsection 3.6. Breaking the tri-bimaximal Klein symmetry of the neutrino sector completely gives rise to arbitrary and therefore unpredictive corrections to the mixing angles. An example for higher order corrections which break the U symmetry can be constructed using the flavon alignments of constrained sequential dominance, proportional to Φ_2 and Φ_3, see Eq. (8.2), and add a small perturbation along the direction Φ_1 to Φ_3, see also [118]. The form of the two flavon alignments can then be written as

$$\frac{\langle\phi_2\rangle}{\Lambda} \propto \Phi_2 , \qquad \frac{\langle\phi_3'\rangle}{\Lambda} \propto \Phi_3' = \Phi_3 + \epsilon\Phi_1 , \tag{8.9}$$

with $\epsilon \ll 1$. Note that these two vectors are still orthogonal to each other, hence the conditions of form dominance are satisfied. With these alignments, it is straightforward to show that the resulting light neutrino mass matrix

$$m_{LL}^{\nu} = m_2^0 \Phi_2 \Phi_2^T + m_3^0 \Phi_3' \Phi_3'^T , \tag{8.10}$$

breaks the original U symmetry while continuing to respect the S symmetry. The latter can be easily verified by noticing that $(1,1,1)^T$ is still an eigenvector of m_{LL}^{ν} in Eq. (8.10), meaning that the second column of the tri-bimaximal PMNS mixing matrix, i.e. Φ_2, remains unchanged. Hence the trimaximal mixing structure TM2, see Eq. (3.32), is achieved, which stabilises the solar angle. On the other hand, the breaking of the U symmetry of the neutrino mass matrix by higher order effects allows to accommodate non-zero θ_{13}.

The second strategy of constructing indirect models with sizable θ_{13} is based on new alignments at leading order. In the following we present two examples (see the right and the left branch of Fig. 11): partially constrained sequential dominance (PCSD) [73] and constrained sequential dominance 2 (CSD2) [116]. Both scenarios make use of two flavon triplets whose alignments are *not* orthogonal to each other, in contrast to the previously discussed cases.

PCSD

Partially constrained sequential dominance was first proposed in [73] as a simple modification of CSD, where one flavon is aligned along the original Φ_2 direction, while the alignment of the other flavon is assumed to deviate slightly from the Φ_3 direction by filling the zero of the first component with $\epsilon \ll 1$, i.e.

$$\Phi_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} , \qquad \widetilde{\Phi}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}\epsilon\\1\\-1\end{pmatrix} . \tag{8.11}$$

Note that Φ_2 and $\widetilde{\Phi}_3$ are not orthogonal to each other, hence PCSD violates form dominance at linear order in ϵ. Inserting these two alignments into Eq. (8.1) yields the effective neutrino mass matrix

$$m_{LL}^{\nu} = \frac{m_2^0}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} + \frac{m_3^0}{2}\begin{pmatrix}\epsilon^2&\epsilon&-\epsilon\\\epsilon&1&-1\\-\epsilon&-1&1\end{pmatrix} . \tag{8.12}$$

This matrix – with non-zero ϵ – is no longer diagonalised by a the tri-bimaximal mixing matrix. Assuming $|\epsilon| \approx 0.2$ as well as a normal neutrino mass hierarchy, i.e. $|m_2^0| \approx |\epsilon m_3^0|$, analytic expressions for the mixing parameters valid to second order in ϵ were derived in [119]. These results show that to first order in ϵ, the tri-bimaximal solar and atmospheric mixing angle predictions are maintained while the reactor angle takes a value of order ϵ. Therefore, PCSD gives rise to tri-bimaximal-reactor mixing, see Eq. (3.28), at leading order. A special case of TBR mixing is obtained if the parameter ϵ can be identified with the Wolfenstein parameter $\lambda = 0.2253 \pm 0.0007$. As discussed in [67], such a situation results in a reactor angle which satisfies $\sin\theta_{13} = \frac{\lambda}{\sqrt{2}}$, leading to $\theta_{13} \approx 9.2^\circ$, a value remarkably close to the one measured by Daya Bay and RENO, see Subsection 3.3.

The alignment of the flavon $\widetilde{\phi}_3$ in Eq. (8.11) can be achieved through both the F-term and the D-term alignment mechanism. The starting point are the simple alignments proportional to

$$\Phi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \Phi_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{8.13}$$

as obtained for instance in Eq. (8.2) and Eq. (8.6), respectively. The alignment in the direction of $\widetilde{\Phi}_3$ then arises from successive orthogonality conditions as follows. Imposing orthogonality of the VEV of an auxiliary flavon ϕ_a with $\langle\phi_3\rangle$ and $\langle\phi_x\rangle$ yields

$$\langle\phi_a\rangle \perp \langle\phi_3\rangle \quad \text{and} \quad \langle\phi_a\rangle \perp \langle\phi_x\rangle \qquad \to \qquad \frac{\langle\phi_a\rangle}{\Lambda} \propto \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{8.14}$$

Requiring the alignment of the flavon $\widetilde{\phi}_3$ to be orthogonal to the alignment of this auxiliary flavon, we find the general structure

$$\langle\widetilde{\phi}_3\rangle \perp \langle\phi_a\rangle \qquad \to \qquad \frac{\langle\widetilde{\phi}_3\rangle}{\Lambda} \propto \begin{pmatrix} n_1 \\ n_2 \\ -n_2 \end{pmatrix}, \tag{8.15}$$

where n_1 and n_2 can take arbitrary values. For $n_1 \ll n_2$, this is nothing but the alignment in the direction of $\widetilde{\Phi}_3$. The hierarchy between n_1 and n_2 may either be a consequence of mild tuning or, under certain assumptions, result from a combination of a renormalisable and a non-renormalisable term, where, after contracting the family indices, the former is proportional to n_1 while the latter is proportional to n_2. The necessary mass suppression of the non-renormalisable term then naturally suppresses $\epsilon = \frac{n_1}{n_2} \ll 1$ [119]. In order to establish a connection of ϵ and λ, one can envisage scenarios in which the flavon $\widetilde{\phi}_3$ appears in both the neutrino as well as in the quark sector. In the latter, $\widetilde{\phi}_3$ has to be responsible for generating the Cabibbo mixing [67].

CSD2

Constrained sequential dominance 2, proposed in [116], assumes two flavon fields in the neutrino sector. One is aligned along the direction of Φ_3 of Eq. (8.2), while the alignment of the other flavon is a relatively simple variation of Φ_2 of Eq. (8.2), explicitly

$$\widetilde{\Phi}_2 \;=\; \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix} \quad \text{or} \quad \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}\;, \qquad \Phi_3 \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}\;. \tag{8.16}$$

Analogous to the case of PCSD, the alignments of the two flavons, pointing in the directions of $\widetilde{\Phi}_2$ and Φ_3, are not orthogonal to each other, implying that CSD2 violates form dominance as well. In the following, we only present the discussion of the first $\widetilde{\Phi}_2$ alignment of Eq. (8.16); the alternative case of the second alignment can be treated analogously, leading to almost identical results. Inserting the alignments of $\widetilde{\phi}_2$ and ϕ_3 into Eq. (8.1) generates the neutrino mass matrix

$$m^\nu_{LL} \;=\; \frac{m_2^0}{5}\begin{pmatrix}1&2&0\\2&4&0\\0&0&0\end{pmatrix} \;+\; \frac{m_3^0}{2}\begin{pmatrix}0&0&0\\0&1&-1\\0&-1&1\end{pmatrix}\;, \tag{8.17}$$

which is not of tri-bimaximal structure. Yet, one can immediately verify that $(-2,1,1)^T$, i.e. the first column of the tri-bimaximal mixing matrix, is still an eigenvector of m^ν_{LL}. This shows that CSD2 necessarily leads to the trimaximal mixing pattern TM1, see Eq. (3.30). With the assumption of a normal neutrino mass hierarchy, i.e. with $|m_2^0| \approx |\epsilon m_3^0|$, analytic expressions for the mixing parameters valid to second order in ϵ were derived in [116]. These results explicitly confirm that the solar angle maintains its tri-bimaximal value at linear order in ϵ, while the deviations of the reactor and the atmospheric mixing angles from their tri-bimaximal values are proportional to ϵ, leading to the linear mixing sum rule of Eq. (3.31).

The alignment of $\widetilde{\phi}_2$ in Eq. (8.16) can be derived from a set of orthogonality conditions similar to the situation in PCSD. In CSD2 we start from the simple alignments proportional to

$$\Phi_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}, \qquad \Phi_y = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \Phi_z = \begin{pmatrix}0\\0\\1\end{pmatrix}\;, \tag{8.18}$$

see Eqs. (8.2,8.6). Demanding orthogonality of the flavon VEV $\langle\widetilde{\phi}_2\rangle$ with $\langle\phi_1\rangle$ and $\langle\phi_z\rangle$ generates an alignment in the direction of the first $\widetilde{\Phi}_2$ vector in Eq. (8.16),

$$\langle\widetilde{\phi}_2\rangle \perp \langle\phi_1\rangle \quad \text{and} \quad \langle\widetilde{\phi}_2\rangle \perp \langle\phi_z\rangle \qquad \to \qquad \frac{\langle\widetilde{\phi}_2\rangle}{\Lambda} \propto \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix}\;. \tag{8.19}$$

The second $\widetilde{\Phi}_2$ vector in Eq. (8.16) can be obtained similarly by using Φ_y instead of Φ_z.

9. Grand unified theories of flavour

9.1 Grand unified theories

One of the exciting things about the discovery of neutrino masses and mixing angles is that this provides additional information about the flavour problem - the problem of understanding the origin of three families of quarks and leptons and their masses and mixing

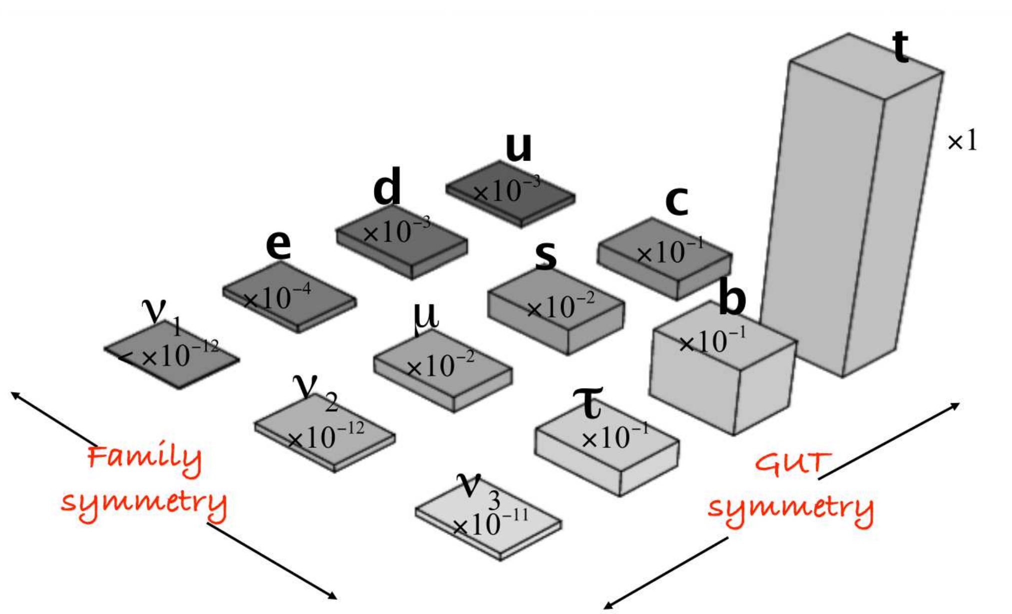

Figure 12: The fermion masses are here represented by a lego plot. We have multiplied the masses of the bottom, charm and tau by 10, the strange and muon by 10^2, the up and down by 10^3, the electron by 10^4 to make the lego blocks visible. Assuming a normal neutrino mass hierarchy, we have multiplied the third neutrino mass by 10^{11} and the second neutrino mass by 10^{12} to make the lego blocks visible. This underlines how incredibly light the neutrinos are. The symmetry groups $G_{\rm GUT}$ and $G_{\rm FAM}$ act in the indicated directions.

angles. In the framework of the see-saw mechanism, new physics beyond the Standard Model is required to violate lepton number and generate right-handed neutrino masses which may be as large as the GUT scale. This is also exciting since it implies that the origin of neutrino masses is also related to some GUT symmetry group $G_{\rm GUT}$, which unifies the fermions within each family as shown in Fig. 12. Some possible candidate unified gauge groups are shown in Fig. 13.

Let us take $G_{\rm GUT} = SU(5)$ as an example. Each family of quarks (with colour r, b, g) and leptons fits nicely into $SU(5)$ representations of left-handed (L) fermions, $F = \overline{\mathbf{5}}$ and $T = \mathbf{10}$

$$F = \begin{pmatrix} d^c_r \\ d^c_b \\ d^c_g \\ e^- \\ -\nu_e \end{pmatrix}_L , \qquad T = \begin{pmatrix} 0 & u^c_g & -u^c_b & u_r & d_r \\ . & 0 & u^c_r & u_b & d_b \\ . & . & 0 & u_g & d_g \\ . & . & . & 0 & e^c \\ . & . & . & . & 0 \end{pmatrix}_L , \tag{9.1}$$

where c denotes CP conjugated fermions. The $SU(5)$ representations $F = \overline{\mathbf{5}}$ and $T = \mathbf{10}$

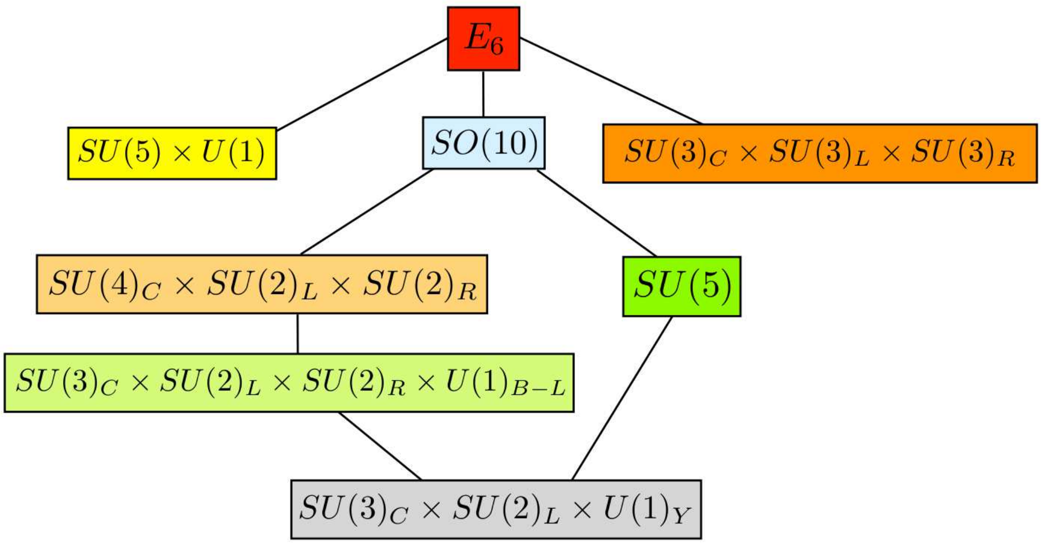

Figure 13: Some possible candidate unified gauge groups.

decompose into multiplets of the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ as $F = (d^c, L)$, corresponding to,

$$\overline{\mathbf{5}} = (\overline{\mathbf{3}}, \mathbf{1}, 1/3) \oplus (\mathbf{1}, \overline{\mathbf{2}}, -1/2), \tag{9.2}$$

and $T = (u^c, Q, e^c)$, corresponding to,

$$\mathbf{10} = (\overline{\mathbf{3}}, \mathbf{1}, -2/3) \oplus (\mathbf{3}, \mathbf{2}, 1/6) \oplus (\mathbf{1}, \mathbf{1}, 1). \tag{9.3}$$

Thus a complete quark and lepton SM family (Q, u^c, d^c, L, e^c) is accommodated in the $F = \overline{\mathbf{5}}$ and $T = \mathbf{10}$ representations, with right-handed neutrinos, whose CP conjugates are denoted as ν^c, being singlets of $SU(5)$, $\nu^c = \mathbf{1}$. The Higgs doublets H_u and H_d which break electroweak symmetry in a two Higgs doublet model are contained in the $SU(5)$ multiplets $H_{\mathbf{5}}$ and $H_{\overline{\mathbf{5}}}$.

The Yukawa couplings for one family of quarks and leptons are given by,

$$y_u H_{\mathbf{5}i} T_{jk} T_{lm} \epsilon^{ijklm} + y_\nu H_{\mathbf{5}i} F^i \nu^c + y_d H^i_{\overline{\mathbf{5}}} T_{ij} F^j, \tag{9.4}$$

where ϵ^{ijklm} is the totally antisymmetric tensor of $SU(5)$ with $i, j, j, k, l = 1, \ldots, 5$, which decompose into the SM Yukawa couplings

$$y_u H_u Q u^c + y_\nu H_u L \nu^c + y_d (H_d Q d^c + H_d e^c L). \tag{9.5}$$

Notice that the Yukawa couplings for down quarks and charged leptons are equal at the GUT scale. Generalising this relation to all three families we find the $SU(5)$ prediction for Yukawa matrices,

$$Y_d = Y_e^T, \tag{9.6}$$

which is successful for the third family, but fails badly for the first and second families. Georgi and Jarlskog [120] suggested to include a higher Higgs representation $H_{\overline{\mathbf{45}}}$ which is responsible for the 2-2 entry of the down and charged lepton Yukawa matrices. Dropping $SU(5)$ indices for clarity,

$$(Y_d)_{22} H_{\overline{\mathbf{45}}} T_2 F_2, \tag{9.7}$$

decomposes into the second family SM Yukawa couplings

$$(Y_d)_{22}(H_d Q_2 d_2^c - 3 H_d e_2^c L_2), \tag{9.8}$$

where the factor of -3 is an $SU(5)$ Clebsch-Gordan coefficient.[18] Assuming a hierarchical Yukawa matrix with a zero Yukawa element (texture) in the 1-1 position, results in the GUT scale Yukawa relations,

$$y_b = y_\tau, \quad y_s = \frac{y_\mu}{3}, \quad y_d = 3y_e, \tag{9.9}$$

which, after renormalisation group running effects are taken into account, are consistent with the low energy masses. The precise viability of these relations has been widely discussed in the light of recent progress in lattice theory which enable more precise values of quark masses to be determined, especially the strange quark mass (see, e.g., [121]). In supersymmetric (SUSY) theories with low values of the ratio of Higgs vacuum expectation values, the relation for the third generation $y_b = y_\tau$ at the GUT scale remains viable, but a viable GUT scale ratio of y_μ/y_s is more accurately achieved within SUSY $SU(5)$ GUTs using a Clebsch factor of 9/2, as proposed in [122], which is 50% higher than the Georgi-Jarlskog prediction of 3.

9.2 Combining GUTs and family symmetry

As already remarked in Subsection 3.3, it is a remarkable fact that the smallest leptonic mixing angle, the reactor angle, is of a similar magnitude to the largest quark mixing angle, the Cabibbo angle, indeed they may even be equal to each other up to a factor of $\sqrt{2}$. Such relationships may be a hint of a connection between leptonic mixing and quark mixing, where such a connection might be achieved using GUTs [123,124]. For example, the Georgi-Jarlskog relations discussed above already lead to the left-handed charged lepton mixing angle having a simple relation with the right-handed down-type quark mixing angle $\theta_{12}^{e_L} \approx \theta_{12}^{d_R}/3$ where the approximation assumes hierarchical Yukawa matrices, with the 1-1 elements being approximately zero. If the upper 2×2 Yukawa matrices are symmetric (as motivated by the successful Gatto-Sartori-Tonin (GST) relation [125] which relates the 12 mixing $\theta_{12}^{d_{L,R}}$ to the down and strange mass by $\theta_{12}^{d_{L,R}} \approx \sqrt{m_d/m_s}$) then we may drop the

[18] In this setup, H_d is the light linear combination of the electroweak doublets contained in $H_{\overline{\mathbf{5}}}$ and $H_{\overline{\mathbf{45}}}$.

L, R subscripts and this relation simply becomes $\theta^e_{12} = \theta^d_{12}/3$. In large classes of models, including those discussed later, the quark mixing originates predominantly from the down-type quark sector, in which case this relation becomes $\theta^e_{12} = \theta_C/3$. If one starts from TB mixing in the neutrino sector, resulting from some discrete family symmetry, then, using the results in Subsection 3.4 such a charged lepton correction results in a reactor angle in the lepton sector of $\theta_{13} \approx \theta_C/(3\sqrt{2})$ as discussed for example in [70]. This is a factor of 3 too small to account for the observed reactor angle, but it illustrates how the reactor angle could possibly be related to the Cabibbo angle using GUTs. Indeed it has been suggested that perhaps the charged lepton mixing angle is exactly equal to the Cabibbo angle in some GUT model, leading to $\theta_{13} \approx \theta_C/\sqrt{2}$ [67, 72, 126]. However it is non-trivial to reconcile such large charged lepton mixing with the successful relationships between charged lepton and down-type quark masses, and it seems more likely that charged lepton mixing is not entirely responsible for the reactor angle.

The above discussion provides an additional motivation for combining GUTs with discrete family symmetry in order to account for the reactor angle. Putting these two ideas together we are suggestively led to a framework of new physics beyond the Standard Model based on commuting GUT and family symmetry groups,

$$G_{\mathrm{GUT}} \times G_{\mathrm{FAM}}. \tag{9.10}$$

There are many possible candidate GUT and family symmetry groups some of which are listed in Table 2. Unfortunately the model dependence does not end there, since the details of the symmetry breaking vacuum plays a crucial role in specifying the model and determining the masses and mixing angles, resulting in many models. These models may be classified according to the particular GUT and family symmetry they assume as shown in Table 2.

Another complication is that the masses and mixing angles determined in some high energy theory must be run down to low energies using the renormalisation group equations. Large radiative corrections are seen when the see-saw parameters are tuned, since the spectrum is sensitive to small changes in the parameters, and this effect is sometimes used to magnify small mixing angles into large ones.

In natural models with a normal mass hierarchy based on SD the parameters are not tuned, since the hierarchy and large atmospheric and solar angles arise naturally as discussed in the previous section. Therefore in SD models the radiative corrections to neutrino masses and mixing angles are only expected to be a few per cent, and this has been verified numerically.

10. Model examples

In this section we give three examples of SUSY GUTs of flavour based on the (semi-)direct approach, which can account for all quark and lepton masses and mixing, including the observed reactor angle which only gets a small correction from charged lepton mixing. The first example is based on the minimal family symmetry A_4 combined with the minimal GUT $SU(5)$. This is actually a semi-direct model, since only half the Klein symmetry is

G_{GUT} / G_{FAM}	$SU(2)_L \times U(1)_Y$	$SU(5)$	$SO(10)$
S_3	[127]		[140]
A_4	[38, 106, 110, 111, 116, 119, 128, 129]	[136, 137]	
T'		[138]	
S_4	[62, 106, 111, 129, 130]	[63, 139]	[141]
A_5	[27, 111]		
T_7	[94, 131]		
$\Delta(27)$	[90]		[142]
$\Delta(96)$	[108, 132]	[31]	
D_N	[133]		
Q_N	[134]		
other	[135]		

Table 2: Some candidate GUT and discrete family symmetry groups, and the papers that use these symmetries to successfully describe the solar, atmospheric and reactor neutrino data.

contained in A_4, resulting in TM2 mixing, but, like all semi-direct models, it cannot explain the relative smallness of the reactor angle. The second example does explain the smallness of the reactor angle compared to the atmospheric or solar angles by embedding the A_4 into an S_4 family symmetry. This allows a direct model where the Klein symmetry resulting from S_4 is half broken by a rather special higher order correction, resulting again in TM2 mixing as in the A_4 case. The third model based on $\Delta(96)$ is an example of a direct model with a larger family symmetry where the Klein symmetry corresponds to bi-trimaximal mixing in which the reactor angle is already non-zero at the leading order, and where the small charged lepton correction with an assumed zero phase brings it into agreement with the Daya Bay and RENO measurements of $\theta_{13} \approx 9^\circ$.

It is worth mentioning that all three models require a $U(1)_R$ symmetry in order to achieve the desired flavon vacuum alignment, see Subsection 7.2. The $U(1)_R$ charges of the different superfields are assigned in a standard way, i.e. the quark and lepton superfields carry charge $+1$, while the Higgs and the flavon fields are neutral. After supersymmetry breaking the models therefore feature a residual discrete symmetry, called R-parity, forbidding the dangerous lepton and baryon number violating terms LLe^c, LQd^c and $u^c d^c d^c$ which, if simultaneously present, would mediate rapid proton decay in conflict with experimental bounds on the proton lifetime. R-parity does, however, not forbid the effective operators $QQQL$ and $u^c u^c d^c e^c$. In principle, these could endanger the stability of the proton in the presented models. However, whether or not this leads to a conflict with existing constraints, is a subtle quantitative question which depends on the details of the underlying renormalisable theory (set of messenger fields and their masses). For an extensive review on proton stability in grand unified theories we refer the reader to [143].

	matter fields					Higgs fields			flavon fields									
	ν^c	F	T_1	T_2	T_3	$H_{\mathbf{5}}$	$H_{\overline{\mathbf{5}}}$	$H_{\overline{\mathbf{45}}}$	$\phi^\nu_{\mathbf{3}}$	$\phi^\nu_{\mathbf{1}}$	$\phi^\nu_{\mathbf{1}'}$	$\phi^\nu_{\mathbf{1}''}$	$\phi^q_{\mathbf{3}}$	$\phi^q_{\mathbf{1}}$	$\phi^q_{\mathbf{1}'}$	$\phi^q_{\mathbf{1}''}$	$\widetilde{\phi}^q_{\mathbf{1}'}$	$\widetilde{\phi}^q_{\mathbf{1}}$
$SU(5)$	$\mathbf{1}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{10}$	$\mathbf{10}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}''$	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{1}'$	$\mathbf{1}$
$U(1)$	1	−1	3	3	0	0	−1	−2	−2	−2	−2	−2	2	−1	−1	−1	−5	2
Z_2	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0
Z_3	1	2	2	0	0	0	1	1	1	1	1	1	0	1	2	2	2	0
Z_5	1	4	0	0	0	0	1	1	3	3	3	3	0	0	0	0	0	0

Table 3: The charge assignments of the matter, Higgs and flavon superfields in the $A_4 \times SU(5)$ model of [137]. The shaping symmetry $U(1) \times Z_2 \times Z_3 \times Z_5$ constrains the set of operators allowed in the superpotential.

10.1 $A_4 \times SU(5)$: a semi-direct model of trimaximal mixing

Our first example of a $G_{\rm FAM} \times G_{\rm GUT}$ model with large θ_{13} is based on the Altarelli-Feruglio A_4 model of leptons [30, 103]. Working in a supersymmetric $SU(5)$ setting, the three matter families of $F = \overline{\mathbf{5}}$ and $T = \mathbf{10}$, see Subsection 9.1, transform under A_4 as $\mathbf{3}$ and $\mathbf{1}'', \mathbf{1}', \mathbf{1}$, respectively. The see-saw mechanism is implemented in the model by introducing right-handed neutrinos ν^c living in the $\mathbf{3}$ of A_4. The Higgs fields $H_{\mathbf{5}}$, $H_{\overline{\mathbf{5}}}$ and $H_{\overline{\mathbf{45}}}$ furnish the one-dimensional A_4 representations $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$. The latter gives rise to the intriguing Georgi-Jarlskog (GJ) relations [120]. The family symmetry breaking flavon fields are $SU(5)$ singlets and can be divided into fields which appear in the neutrino sector $\phi^\nu_{\mathbf{r}}$ and fields which appear in the quark sector $\phi^q_{\mathbf{r}}$. The A_4 family symmetry is enriched by the shaping symmetry $U(1) \times Z_2 \times Z_3 \times Z_5$ in order to control the coupling of the flavons to the different matter sectors. The complete charge assignments of the matter, Higgs and flavon superfields is presented in Table 3.

A discussion of all the different aspects of the model, including the vacuum alignment, can be found in [137]. Here we mainly focus our attention on the neutrino sector. The leading order renormalisable operators of the neutrino superpotential which are invariant under all imposed symmetries of the model, see Table 3, are

$$W_\nu \;\sim\; F\nu^c H_{\mathbf{5}} + (\phi^\nu_{\mathbf{3}} + \phi^\nu_{\mathbf{1}} + \phi^\nu_{\mathbf{1}'} + \phi^\nu_{\mathbf{1}''})\,\nu^c\nu^c \;, \tag{10.1}$$

where dimensionless order one coupling coefficients are suppressed. Note that the Dirac Yukawa term does not involve any flavon field, hence, the Dirac neutrino Yukawa matrix does not break the A_4 symmetry. Inserting the Higgs VEV v_u (the VEV of the electroweak doublet contained in $H_{\mathbf{5}}$) then generates an A_4 invariant Dirac neutrino mass matrix m_{LR}. As a consequence, the mixing in the neutrino sector originates solely from the right-handed neutrino mass matrix whose form depends on the chosen basis of A_4 as well as the vacuum configuration of the flavon fields. Working in the A_4 basis of Appendix C, it has been shown in [137] that the two triplet flavon alignments

$$\langle \phi^{\nu}_{\mathbf{3}} \rangle \;=\; \varphi^{\nu}_{\mathbf{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;, \qquad \langle \phi^{q}_{\mathbf{3}} \rangle \;=\; \varphi^{q}_{\mathbf{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;, \tag{10.2}$$

can be obtained from the F-term alignment mechanism along the lines of [30]. This necessitates the introduction of a $U(1)_R$ symmetry, a set of driving fields, see Subsection 7.2, as well as the auxiliary flavon field $\widetilde{\phi}^{q}_{\mathbf{1}}$ which has the same charges as the triplet flavon $\phi^{q}_{\mathbf{3}}$, except for A_4. With these alignments as well as VEVs for the one-dimensional flavon fields, using the Clebsch-Gordan coefficients in Appendix C, the Dirac neutrino mass matrix and the right-handed neutrino mass matrix take the form[19]

$$m_{LR} \;\approx\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u \;, \tag{10.3}$$

$$M_{RR} \;\approx\; \varphi^{\nu}_{\mathbf{3}} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \varphi^{\nu}_{\mathbf{1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \varphi^{\nu}_{\mathbf{1'}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \varphi^{\nu}_{\mathbf{1''}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{10.4}$$

The effective light neutrino mass matrix emerges from these by applying the type I see-saw formula of Eq. (4.6), i.e.

$$m^{\nu}_{LL} = -m_{LR} M^{-1}_{RR} m^{T}_{LR} \;. \tag{10.5}$$

Due to the trivial structure of the Dirac neutrino mass matrix m_{LR}, the neutrino mixing matrix is identical to the unitary matrix which diagonalises M_{RR} (and automatically also M^{-1}_{RR}), except for a permutation of the second and the third row [106]. We remark that the light neutrino masses are not related by a mass sum rule since the right-handed neutrino mass term involves four independent flavon fields. The inverse mass sum rule for A_4 quoted in Eq. (1.5) with $\gamma = 1$ and $\delta = -2$ can only be recovered by removing the flavons $\phi^{\nu}_{\mathbf{1'}}$ and $\phi^{\nu}_{\mathbf{1''}}$, which in turn corresponds to the well-known case of tri-bimaximal neutrino mixing.

Rather than diagonalising M_{RR} explicitly, let us discuss its symmetries. The first two terms in Eq. (10.4) are symmetric under the tri-bimaximal Klein generators S and U of Eq. (6.8). The third and the fourth terms break the tri-bimaximal structure, however, in a special way. It is straightforward to prove explicitly that $S^T M_{RR} S = M_{RR}$ is still respected. A simple way of seeing this is by noticing that all neutrino flavon VEVs remain unchanged under the A_4 transformation S, see Appendix C. On the other hand, the U matrix of Eq. (6.8) does not form part of A_4. In order for M_{RR} to be also symmetric under

[19] As the mass term of the right-handed neutrinos in the superpotential of Eq. (10.1) involves the CP conjugate fields ν^c, the mass matrix in the conventions of Subsection 4.1 is obtained using the complex conjugate flavons VEVs $\varphi^{\nu *}_{\mathbf{r}}$, cf. also Eq. (4.13). However, for notational clarity, we drop the * here and in the following.

U, hence entailing tri-bimaximal neutrino mixing, one would have to require $\varphi^\nu_{\mathbf{1}'} = \varphi^\nu_{\mathbf{1}''}$. In [30, 103] this condition among the a priori unrelated VEVs is realised by not including flavons in the $\mathbf{1}'$ and $\mathbf{1}''$ representations of A_4 in the first place. Alternatively, the two non-trivial one-dimensional representations can be unified into a doublet of S_4, see Appendix C. A suitable VEV alignment of such a doublet can relate the two components such as to generate a right-handed neutrino mass matrix of tri-bimaximal form, see e.g. [106]. In general, however, $\varphi^\nu_{\mathbf{1}'} \neq \varphi^\nu_{\mathbf{1}''}$ and there is no accidental U symmetry in an A_4 model with neutrino flavons in all possible representations of the family symmetry.

With m_{LR} being invariant under the full A_4 family symmetry and M_{RR} being symmetric under S, the type I see-saw mechanism generates a light effective neutrino mass matrix which also respects the S symmetry. This symmetry can be translated to a particular mixing pattern by considering an eigenvector $\vec{v}$ of S with eigenvalue $+1$. One can easily check that the only solution to $S\vec{v} = \vec{v}$ is of the form $\vec{v} \propto (1,1,1)^T$. Using this and the invariance of the light neutrino mass matrix m^ν_{LL} under S as well as $S^T = S$, we obtain

$$m^\nu_{LL}\vec{v} \;=\; Sm^\nu_{LL}S\vec{v} \;=\; Sm^\nu_{LL}\vec{v} \;, \tag{10.6}$$

which shows that $m^\nu_{LL}\vec{v}$ is an eigenvector of S with eigenvalue $+1$, and thus

$$m^\nu_{LL}\vec{v} \;\propto\; \vec{v} \;. \tag{10.7}$$

As $\vec{v} \propto (1,1,1)^T$ is an eigenvector of the neutrino mass matrix, the normalised vector $\frac{\vec{v}}{|\vec{v}|}$ corresponds to a column of the neutrino mixing matrix. Except for being orthogonal to $\vec{v}$, the other two columns are not specified by the S symmetry. In order to be meaningful for physics, the vector $\frac{\vec{v}}{|\vec{v}|}$ has to be identified with the second column of the neutrino mixing matrix, so that the trimaximal pattern TM2 of Eq. (3.32) ensues. As mentioned earlier, this special mixing pattern allows for a large reactor angle while keeping the solar angle at its tri-bimaximal value at leading order.

It is now important to notice that the TM2 mixing sum rule of Eq. (3.33) applies only to the neutrino sector. In other words, the TB deviation parameters as well as the CP phase should carry a neutrino index, i.e. s^ν, a^ν, r^ν and δ^ν. In order to find the deviation parameters of the physical PMNS matrix, it is necessary to add the effect of the charged lepton corrections. As is demonstrated in [137], the mass matrices of the charged fermions in the $A_4 \times SU(5)$ model are given by

$$M_u \sim \begin{pmatrix} \overline{\epsilon}^6 & \overline{\epsilon}^6 & \overline{\epsilon}^3 \\ \overline{\epsilon}^6 & \overline{\epsilon}^3 & \overline{\epsilon}^3 \\ \overline{\epsilon}^3 & \overline{\epsilon}^3 & 1 \end{pmatrix} v_u \;, \quad M_d \sim \begin{pmatrix} \epsilon^6 & \epsilon^4 & \epsilon^4 \\ \epsilon^4 & \epsilon^3 & \epsilon^3 \\ \epsilon^7 & \epsilon^6 & \epsilon \end{pmatrix} v_d \;, \quad M_e \sim \begin{pmatrix} -3\epsilon^6 & \epsilon^4 & \epsilon^7 \\ \epsilon^4 & -3\epsilon^3 & -3\epsilon^6 \\ \epsilon^4 & -3\epsilon^3 & \epsilon \end{pmatrix} v_d \;, \tag{10.8}$$

where the scales of the flavon VEVs were assumed to be

$$\varphi^q_{\mathbf{r}} \;\sim\; \widetilde{\varphi}^q_{\mathbf{1}} \;\sim\; \epsilon M \;, \qquad \widetilde{\varphi}^q_{\mathbf{1}'} \;\sim\; \epsilon^2 M \;. \tag{10.9}$$

M denotes a messenger mass which we allow to vary for the up-type and the down-type quarks, thus justifying a different expansion parameter in M_u ($\overline{\epsilon}$) and $M_{d,e}$ (ϵ). The factors of -3 in M_e correspond to an $SU(5)$ Clebsch-Gordan coefficient which originates from Georgi-Jarlskog terms of the form $FTH_{\overline{\mathbf{45}}}$ [120], multiplied by appropriate products of flavon fields. Finally, v_d denotes the VEV of the light combination of the electroweak doublets contained in $H_{\overline{\mathbf{5}}}$ and $H_{\overline{\mathbf{45}}}$.

With the structure of M_e given in Eq. (10.8), the only significant left-handed charged lepton mixing V_{e_L} is the 12 mixing $\theta^e_{12} \approx \epsilon/3$. The parameter ϵ can be approximated by the Wolfenstein parameter λ as the effect of the left-handed up-type quark mixing on the CKM matrix is negligible. Combining the TM2 mixing of the neutrino sector and charged lepton corrections with $\theta^e_{12} \approx \lambda/3$, see Subsection 3.4, leads to the sum rule bounds [137]

$$|a| \lesssim \frac{1}{2}\left(r + \frac{\lambda}{3}\right) |\cos\delta| \ , \qquad |s| \lesssim \frac{\lambda}{3} \ , \tag{10.10}$$

where s, a, r are the physical TB deviation parameters of the PMNS matrix and δ denotes the physical CP phase.

We conclude the discussion of the $A_4 \times SU(5)$ model of trimaximal neutrino mixing by pointing out that this framework does not provide any explanation for the suppression of the reactor angle compared to the solar or atmospheric angles. Therefore, this model relies on mild tuning of parameters. In the next subsection we show how to obtain such a suppression in the context of an S_4 model of tri-bimaximal mixing in which the U symmetry gets broken by higher order corrections.

10.2 $S_4 \times SU(5)$: a direct model of tri-bimaximal mixing with corrections

In this subsection we present the main ingredients of the supersymmetric $S_4 \times SU(5)$ model of [139]. It is based on an earlier direct model [144] which has been ruled out by the measurement of $\theta_{13} \approx 9^\circ$. In order to accommodate this experimental result, the model of [144] has simply been augmented with an extra S_4 singlet flavon field η. The three families of $SU(5)$ matter multiplets $F = \overline{\mathbf{5}}$ and $T = \mathbf{10}$ transform under S_4 as $\mathbf{3}$ and $\mathbf{2} + \mathbf{1}$, respectively. We furthermore introduce three right-handed neutrinos ν^c which are unified in the $\mathbf{3}$ of S_4 and allow for the type I see-saw mechanism. The Higgs sector is S_4 blind and comprises the standard $SU(5)$ Higgses in the $\mathbf{5}$ and $\overline{\mathbf{5}}$, plus an additional Georgi-Jarlskog Higgs in the $\overline{\mathbf{45}}$. The family symmetry is broken by a set of flavon fields transforming in various representations of S_4. In order to control the coupling of the flavon fields to different matter sectors, we impose a global $U(1)$ shaping symmetry. The complete charge assignments of matter, Higgs and flavon fields are listed in Table 4.

With the model formulated at the effective level, it is straightforward to derive the leading operators of the matter superpotential which are invariant under all imposed symmetries. Assuming a generic messenger mass M of order the GUT scale, and suppressing all dimensionless order one coupling coefficients, we find

$$W \sim T_3 T_3 H_{\mathbf{5}} + \frac{1}{M} TT\phi^u_{\mathbf{2}} H_{\mathbf{5}} + \frac{1}{M^2} TT\phi^u_{\mathbf{2}}\widetilde{\phi}^u_{\mathbf{2}} H_{\mathbf{5}} \tag{10.11}$$

$$+\frac{1}{M} FT_3\phi^d_{\mathbf{3}} H_{\overline{\mathbf{5}}} + \frac{1}{M^2}(F\widetilde{\phi}^d_{\mathbf{3}})_{\mathbf{1}}(T\phi^d_{\mathbf{2}})_{\mathbf{1}} H_{\overline{\mathbf{45}}} + \frac{1}{M^3}(F\phi^d_{\mathbf{2}}\phi^d_{\mathbf{2}})_{\mathbf{3}}(T\widetilde{\phi}^d_{\mathbf{3}})_{\mathbf{3}} H_{\overline{\mathbf{5}}} \tag{10.12}$$

$$+F\nu^c H_{\mathbf{5}} + \nu^c\nu^c\phi^\nu_{\mathbf{1}} + \nu^c\nu^c\phi^\nu_{\mathbf{2}} + \nu^c\nu^c\phi^\nu_{\mathbf{3'}} + \frac{1}{M}\nu^c\nu^c\phi^d_{\mathbf{2}}\eta \ . \tag{10.13}$$

	matter fields				Higgs fields			flavon fields									
	T_3	T	F	ν^c	$H_{\mathbf{5}}$	$H_{\overline{\mathbf{5}}}$	$H_{\overline{\mathbf{45}}}$	$\phi^u_{\mathbf{2}}$	$\widetilde{\phi}^u_{\mathbf{2}}$	$\phi^d_{\mathbf{3}}$	$\widetilde{\phi}^d_{\mathbf{3}}$	$\phi^d_{\mathbf{2}}$	$\phi^\nu_{\mathbf{3'}}$	$\phi^\nu_{\mathbf{2}}$	$\phi^\nu_{\mathbf{1}}$	η
$SU(5)$	$\mathbf{10}$	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
S_4	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{2}$	$\mathbf{3'}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$
$U(1)$	0	5	4	-4	0	0	1	-10	0	-4	-11	1	8	8	8	7

Table 4: The charge assignments of the matter, Higgs and flavon superfields in the $S_4 \times SU(5)$ model of [139]. The $U(1)$ shaping symmetry constrains the set of operators allowed in the superpotential.

The terms in Eq. (10.13), may be compared to the neutrino sector of the A_4 model in Eq. (10.1). In the S_4 model it is the last term highlighted in red colour which provides the source of the higher order correction to the right-handed neutrino mass matrix which is essential in generating a large reactor angle. In principle, all independent invariant products of the S_4 representations have to be considered for each of these terms; in practice, there is often only one possible choice. In our example, the second and the third term of Eq. (10.12) would give rise to several independent terms. However, the contractions specified by the subscripts **1** and **3** single out a unique choice. Within a given UV completion, the existence and non-existence of certain messenger fields can justify such a construction.

The Yukawa matrices are generated when the flavon fields acquire their VEVs. The explicit form of these matrices depends on the S_4 basis which we choose as given in Appendix C. Adopting the F-term alignment mechanism which requires to introduce a $U(1)_R$ symmetry as well as new driving fields, see Subsection 7.2, is has been shown in [139, 144] that the following alignments can be obtained,

$$\langle \phi^u_{\mathbf{2}} \rangle = \varphi^u_{\mathbf{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \langle \widetilde{\phi}^u_{\mathbf{2}} \rangle = \widetilde{\varphi}^u_{\mathbf{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{10.14}$$

$$\langle \phi^d_{\mathbf{3}} \rangle = \varphi^d_{\mathbf{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \langle \widetilde{\phi}^d_{\mathbf{3}} \rangle = \widetilde{\varphi}^d_{\mathbf{3}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} , \quad \langle \phi^d_{\mathbf{2}} \rangle = \varphi^d_{\mathbf{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{10.15}$$

$$\langle \phi^\nu_{\mathbf{3'}} \rangle = \varphi^\nu_{\mathbf{3'}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \quad \langle \phi^\nu_{\mathbf{2}} \rangle = \varphi^\nu_{\mathbf{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad \langle \phi^\nu_{\mathbf{1}} \rangle = \varphi^\nu_{\mathbf{1}} \, . \tag{10.16}$$

Inserting these vacuum alignments and the Higgs VEVs v_u and v_d yields a diagonal up-type quark mass matrix $M_u \approx \mathrm{diag}\,(\varphi^u_{\mathbf{2}}\widetilde{\varphi}^u_{\mathbf{2}}/M^2, \varphi^u_{\mathbf{2}}/M\,, 1)\, v_u$ as well as down-type quark and charged lepton mass matrices

$$M_d \approx \begin{pmatrix} 0 & (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & -(\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 \\ -(\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & \varphi^d_{\mathbf{2}}\widetilde{\varphi}^d_{\mathbf{3}}/M^2 & -\varphi^d_{\mathbf{2}}\widetilde{\varphi}^d_{\mathbf{3}}/M^2 + (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 \\ 0 & 0 & \varphi^d_{\mathbf{3}}/M \end{pmatrix} v_d \ , \quad (10.17)$$

$$M_e \approx \begin{pmatrix} 0 & -(\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & 0 \\ (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & -3\,\varphi^d_{\mathbf{2}}\widetilde{\varphi}^d_{\mathbf{3}}/M^2 & 0 \\ -(\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & 3\,\varphi^d_{\mathbf{2}}\widetilde{\varphi}^d_{\mathbf{3}}/M^2 + (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & \varphi^d_{\mathbf{3}}/M \end{pmatrix} v_d \ . \quad (10.18)$$

The factors of -3 in M_e originate from the second term of Eq. (10.12) involving the Georgi-Jarlskog Higgs field $H_{\overline{\mathbf{45}}}$ [120]. Note that the 1-2 and 2-1 entries, which originate from the same superpotential term, have identical absolute values; together with the zero texture in the 1-1 entry, this allows for a simple realisation of the GST relation in the $S_4 \times SU(5)$ model. In the neutrino sector we find the Dirac neutrino mass matrix and the right-handed neutrino mass matrix

$$m_{LR} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u \,, \quad M_{RR} \approx \begin{pmatrix} \varphi^\nu_{\mathbf{1}} + 2\varphi^\nu_{\mathbf{3'}} & \varphi^\nu_{\mathbf{2}} - \varphi^\nu_{\mathbf{3'}} + \frac{\varphi^d_{\mathbf{2}}\langle\eta\rangle}{M} & \varphi^\nu_{\mathbf{2}} - \varphi^\nu_{\mathbf{3'}} \\ \varphi^\nu_{\mathbf{2}} - \varphi^\nu_{\mathbf{3'}} + \frac{\varphi^d_{\mathbf{2}}\langle\eta\rangle}{M} & \varphi^\nu_{\mathbf{2}} + 2\varphi^\nu_{\mathbf{3'}} & \varphi^\nu_{\mathbf{1}} - \varphi^\nu_{\mathbf{3'}} \\ \varphi^\nu_{\mathbf{2}} - \varphi^\nu_{\mathbf{3'}} & \varphi^\nu_{\mathbf{1}} - \varphi^\nu_{\mathbf{3'}} & \varphi^\nu_{\mathbf{2}} + 2\varphi^\nu_{\mathbf{3'}} + \frac{\varphi^d_{\mathbf{2}}\langle\eta\rangle}{M} \end{pmatrix} . \quad (10.19)$$

It is clear from Eqs. (10.17-10.19) that the fermion masses and mixings are solely determined by the scales of the flavon VEVs. In order to achieve viable GUT scale hierarchies of the quark masses and mixing angles [121], we have to assume

$$\varphi^u_{\mathbf{2}} \sim \widetilde{\varphi}^u_{\mathbf{2}} \sim \lambda^4 M \ , \qquad \varphi^d_{\mathbf{3}} \sim \lambda^2 M \ , \quad \widetilde{\varphi}^d_{\mathbf{3}} \sim \lambda^3 M \ , \quad \varphi^d_{\mathbf{2}} \sim \lambda M \ , \quad (10.20)$$

where λ denotes the Wolfenstein parameter. With these magnitudes, the charged fermion mass matrices are fixed completely,

$$M_u \sim \begin{pmatrix} \lambda^8 & 0 & 0 \\ 0 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} v_u \ , \quad M_d \sim \begin{pmatrix} 0 & \lambda^5 & \lambda^5 \\ \lambda^5 & \lambda^4 & \lambda^4 \\ 0 & 0 & \lambda^2 \end{pmatrix} v_d \ , \quad M_e \sim \begin{pmatrix} 0 & \lambda^5 & 0 \\ \lambda^5 & 3\lambda^4 & 0 \\ \lambda^5 & 3\lambda^4 & \lambda^2 \end{pmatrix} v_d \ . \quad (10.21)$$

Due to the GJ factor of -3 and the texture zero in the 1-1 entry, we obtain viable charged lepton masses. With the vanishing off-diagonals in the third column of M_e, there is only a non-trivial 12 mixing in the left-handed charged lepton mixing V_{e_L}, see Subsection 3.4. This mixing, $\theta^e_{12} \approx \lambda/3$, will contribute to the total PMNS mixing as a charged lepton correction.

Turning to the neutrino sector, we first observe that the Dirac neutrino Yukawa term does not involve any flavon field. As the family symmetry S_4 remains unbroken by m_{LR}, the mixing pattern of the effective light neutrino mass matrix m^ν_{LL} (obtained from the type I see-saw mechanism) is exclusively determined by the structure of M_{RR}. Dropping the higher order terms which are written in red, we note that the leading order structure of M_{RR}, and with it m^ν_{LL}, is of tri-bimaximal form.[20] This can be easily seen by verifying that

[20] Similar to the $A_4 \times SU(5)$ model of Subsection 10.1, the masses of the light neutrinos are not related by any mass sum rule as the right-handed neutrino mass matrix M_{RR} is generated from the VEVs of three independent flavon fields.

the flavon alignments of Eq. (10.16) are left invariant under the S and U transformations of Appendix C. This leading order tri-bimaximal structure yields light neutrino masses $m^\nu \sim 0.1\,\text{eV}$ if we set $\varphi^\nu_{\mathbf{1},\mathbf{2},\mathbf{3}'} \sim \lambda^4 M$. As we want to break the TB Klein symmetry by means of the flavon η at higher order, we set $\langle\eta\rangle \sim \lambda^4 M$. Then the TB breaking effect is suppressed by one power of λ compared to the leading order. The effective flavon $\phi^d_{\mathbf{2}}\eta$ transforms as an S_4 doublet with an alignment proportional to $(1,0)^T$. As can be seen from the S_4 generators of the doublet representation, see Appendix C, this alignment breaks the U symmetry but respects S. This directly proves that M_{RR} as well as m^ν_{LL} are both invariant under S, which in turn entails the TM2 neutrino mixing pattern, where the second column of the mixing matrix is proportional to $(1,1,1)^T$, cf. Eq. (3.32). The physical PMNS matrix is obtained from multiplying the TM2 neutrino mixing with the left-handed charged lepton mixing, see Eq. (3.15). As a result, we find the same sum rule bound as in the $A_4 \times SU(5)$ model of Subsection 10.1, given explicitly in Eq. (10.10) [139].

In summary, the measurement of large θ_{13} has ruled out the original $S_4 \times SU(5)$ model [144] which predicted accurate tri-bimaximal neutrino mixing plus small charged lepton corrections. A modest extension of the particle content can induce a breaking of the U symmetry of the TB Klein symmetry at relative order λ. The required new flavon field allows for large θ_{13} and does not destroy the successful predictions of the original model, i.e. it does not have any significant effects on the quark or flavon sectors of the model.

10.3 $\boldsymbol{\Delta(96) \times SU(5)}$: a direct model of bi-trimaximal mixing

The direct model discussed in this subsection is based on the observation that larger family symmetry groups can contain physically interesting $Z_2 \times Z_2$ subgroups which, in the basis of diagonal charged leptons, differ from the well-known TB Klein symmetry [108,109]. The first model of leptons adopting the family symmetry $\Delta(96)$ was constructed in [132]. Here we present the first (supersymmetric) grand unified model based on $\Delta(96) \times SU(5)$ [31].

The group $\Delta(96)$ is a member of the series of groups $\Delta(6n^2)$ [83] with $n = 4$. Like its subgroup $S_4 = \Delta(6\cdot 2^2)$, it can be obtained from three generators S, T, U. The group has ten irreducible representations: $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$, $\overline{\mathbf{3}}$, $\widetilde{\mathbf{3}}$, $\mathbf{3}'$, $\overline{\mathbf{3}}'$, $\widetilde{\mathbf{3}}'$, $\mathbf{6}$. The generators of the one- and two-dimensional representations are identical to the corresponding S_4 representations, see Appendix C. The triplet $\mathbf{3}$ is generated by

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad U = \frac{1}{3}\begin{pmatrix} -1+\sqrt{3} & -1-\sqrt{3} & -1 \\ -1-\sqrt{3} & -1 & -1+\sqrt{3} \\ -1 & -1+\sqrt{3} & -1-\sqrt{3} \end{pmatrix}, \quad T = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{10.22}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Notice that S is identical to the tri-bimaximal S symmetry of Eq. (6.8). Invariance of the neutrino sector of a $\Delta(96)$ model under S therefore implies TM2 neutrino mixing. The complex conjugate representation $\overline{\mathbf{3}}$ is generated by the matrices of Eq. (10.22) with $T \to T^*$. The third triplet $\widetilde{\mathbf{3}}$ is obtained from

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U = \frac{1}{3}\begin{pmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{pmatrix}, \qquad T = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \tag{10.23}$$

	matter fields				Higgs fields			flavon fields							
	T_3	T	F	ν^c	$H_{\mathbf{5}}$	$H_{\overline{\mathbf{5}}}$	$H_{\overline{\mathbf{45}}}$	$\phi^u_{\mathbf{2}}$	$\widetilde{\phi}^u_{\mathbf{2}}$	$\phi^d_{\overline{\mathbf{3}}}$	$\widetilde{\phi}^d_{\overline{\mathbf{3}}}$	$\phi^d_{\mathbf{2}}$	$\phi^\nu_{\overline{\mathbf{3}}'}$	$\phi^\nu_{\widetilde{\mathbf{3}}'}$	$\phi^\nu_{\widetilde{\mathbf{3}}}$
$SU(5)$	$\mathbf{10}$	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\Delta(96)$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{2}$	$\overline{\mathbf{3}}'$	$\widetilde{\mathbf{3}}'$	$\widetilde{\mathbf{3}}$
$U(1)$	0	x	y	$-y$	0	0	z	$-2x$	0	$-y$	$-x-y-2z$	z	$2y$	$2y$	w
Z_3	0	0	2	1	0	1	1	0	0	0	0	0	1	1	1

Table 5: The charge assignments of the matter, Higgs and flavon superfields in the $\Delta(96) \times SU(5)$ model of [31]. The $U(1)$ shaping symmetry is defined by four independent integers x, y, z, and w.

The generators of the three representations $\mathbf{3}$, $\overline{\mathbf{3}}$, $\widetilde{\mathbf{3}}$ all have determinant $+1$. This is not so for the other three representations $\mathbf{3}'$, $\overline{\mathbf{3}}'$, $\widetilde{\mathbf{3}}'$ which can be obtained from the unprimed triplets by simply changing the overall sign of the corresponding U generator. Concerning the sextet representation and for more details on the group theory of $\Delta(96)$ such as Kronecker products and Clebsch-Gordan coefficients, we refer to the extensive appendix of [31].

The construction of the $\Delta(96) \times SU(5)$ model follows closely the logic of the $S_4 \times SU(5)$ model in [144]. In particular, the flavons of the up-type and the down-type quark sector are almost identical, and the GJ mechanism is also implemented along with the GST relation. The complete charge assignments of the matter, Higgs and flavon fields of the $\Delta(96)$ model [31] are listed in Table 5. In addition to a $U(1)$ shaping symmetry – defined in terms of suitably chosen integers x, y, z, w – a Z_3 factor has been introduced to forbid dangerous terms in the superpotential which, otherwise, would be allowed.

With the particle content and the symmetries of Table 5, the leading order operators of the matter superpotential take the form

$$
\begin{aligned}
W \;\sim\;& T_3 T_3 H_{\mathbf{5}} + \frac{1}{M} T T \phi^u_{\mathbf{2}} H_{\mathbf{5}} + \frac{1}{M^2} T T \phi^u_{\mathbf{2}} \widetilde{\phi}^u_{\mathbf{2}} H_{\mathbf{5}} && (10.24)\\
&+ \frac{1}{M} F T_3 \phi^d_{\overline{\mathbf{3}}} H_{\overline{\mathbf{5}}} + \frac{1}{M^2} (F \widetilde{\phi}^d_{\overline{\mathbf{3}}})_{\mathbf{1}} (T \phi^d_{\mathbf{2}})_{\mathbf{1}} H_{\overline{\mathbf{45}}} + \frac{1}{M^3} (F \phi^d_{\mathbf{2}} \phi^d_{\mathbf{2}})_{\mathbf{3}} (T \widetilde{\phi}^d_{\overline{\mathbf{3}}})_{\overline{\mathbf{3}}} H_{\overline{\mathbf{5}}} && (10.25)\\
&+ F \nu^c H_{\mathbf{5}} + \nu^c \nu^c \phi^\nu_{\overline{\mathbf{3}}'} + \nu^c \nu^c \phi^\nu_{\widetilde{\mathbf{3}}'} \,. && (10.26)
\end{aligned}
$$

Dimensionless order one coupling coefficients are suppressed, and M is a generic messenger mass scale. The subscripts on the parentheses denote the specific contractions being taken from the $\Delta(96)$ tensor product contained inside the parentheses.

As has been elaborated in [31], the F-term alignment mechanism can be used to derive the vacuum alignment of the flavon fields as[21]

$$\langle\phi^u_{\mathbf{2}}\rangle = \varphi^u_{\mathbf{2}}\begin{pmatrix}0\\1\end{pmatrix}, \quad \langle\widetilde{\phi}^u_{\mathbf{2}}\rangle = \widetilde{\varphi}^u_{\mathbf{2}}\begin{pmatrix}0\\1\end{pmatrix}, \tag{10.27}$$

$$\langle\phi^d_{\mathbf{3}}\rangle = \varphi^d_{\mathbf{3}}\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \langle\widetilde{\phi}^d_{\mathbf{3}}\rangle = \widetilde{\varphi}^d_{\mathbf{3}}\begin{pmatrix}0\\1\\-1\end{pmatrix}, \quad \langle\phi^d_{\mathbf{2}}\rangle = \varphi^d_{\mathbf{2}}\begin{pmatrix}1\\0\end{pmatrix}, \tag{10.28}$$

$$\langle\phi^\nu_{\mathbf{3}'}\rangle = \varphi^\nu_{\mathbf{3}'}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \langle\phi^\nu_{\widetilde{\mathbf{3}}'}\rangle = \varphi^\nu_{\widetilde{\mathbf{3}}'}\begin{pmatrix}\omega^2\\-\frac{1}{2}\\\omega\end{pmatrix}. \tag{10.29}$$

Inserting the flavon and the Higgs VEVs yields the charged fermion mass matrices $M_u \approx \mathrm{diag}\,(\varphi^u_{\mathbf{2}}\widetilde{\varphi}^u_{\mathbf{2}}/M^2, \varphi^u_{\mathbf{2}}/M\,, 1)\, v_u$ and

$$M_d \approx \begin{pmatrix} 0 & (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & -(\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 \\ (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & \varphi^d_{\mathbf{2}}\widetilde{\varphi}^d_{\mathbf{3}}/M^2 - (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & -\varphi^d_{\mathbf{2}}\widetilde{\varphi}^d_{\mathbf{3}}/M^2 \\ 0 & 0 & \varphi^d_{\mathbf{3}}/M \end{pmatrix} v_d\,, \tag{10.30}$$

$$M_e \approx \begin{pmatrix} 0 & (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & 0 \\ (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & -3\,\varphi^d_{\mathbf{2}}\widetilde{\varphi}^d_{\mathbf{3}}/M^2 - (\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & 0 \\ -(\varphi^d_{\mathbf{2}})^2\widetilde{\varphi}^d_{\mathbf{3}}/M^3 & 3\,\varphi^d_{\mathbf{2}}\widetilde{\varphi}^d_{\mathbf{3}}/M^2 & \varphi^d_{\mathbf{3}}/M \end{pmatrix} v_d\,. \tag{10.31}$$

In the neutrino sector we obtain the Dirac neutrino mass matrix and the right-handed neutrino mass matrix

$$m_{LR} \approx \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} v_u\,, \qquad M_{RR} \approx \begin{pmatrix} -2\varphi^\nu_{\mathbf{3}'} + \omega\varphi^\nu_{\widetilde{\mathbf{3}}'} & \varphi^\nu_{\mathbf{3}'} + \omega^2\varphi^\nu_{\widetilde{\mathbf{3}}'} & \varphi^\nu_{\mathbf{3}'} - \frac{1}{2}\varphi^\nu_{\widetilde{\mathbf{3}}'} \\ \varphi^\nu_{\mathbf{3}'} + \omega^2\varphi^\nu_{\widetilde{\mathbf{3}}'} & -2\varphi^\nu_{\mathbf{3}'} - \frac{1}{2}\varphi^\nu_{\widetilde{\mathbf{3}}'} & \varphi^\nu_{\mathbf{3}'} + \omega\varphi^\nu_{\widetilde{\mathbf{3}}'} \\ \varphi^\nu_{\mathbf{3}'} - \frac{1}{2}\varphi^\nu_{\widetilde{\mathbf{3}}'} & \varphi^\nu_{\mathbf{3}'} + \omega\varphi^\nu_{\widetilde{\mathbf{3}}'} & -2\varphi^\nu_{\mathbf{3}'} + \omega^2\varphi^\nu_{\widetilde{\mathbf{3}}'} \end{pmatrix}. \tag{10.32}$$

The mass matrices M_u, M_d and M_e of the $\Delta(96)$ model are almost identical to those of the S_4 model presented in Subsection 10.2. Indeed, with a similar hierarchy of flavon VEVs,

$$\varphi^u_{\mathbf{2}} \sim \widetilde{\varphi}^u_{\mathbf{2}} \sim \lambda^4 M\,, \qquad \varphi^d_{\mathbf{3}} \sim \lambda^2 M\,, \quad \widetilde{\varphi}^d_{\mathbf{3}} \sim \lambda^3 M\,, \quad \varphi^d_{\mathbf{2}} \sim \lambda M\,, \tag{10.33}$$

one obtains the charged fermion mass matrices of Eq. (10.21). As in Subsection 10.2, the left-handed charged lepton mixing is described by a unitary matrix with angles $\theta^e_{13} \approx \theta^e_{23} \approx 0$ and $\theta^e_{12} \approx \lambda/3$ as well as a general phase δ^e_{12}, see e.g. Eq. (A.7).

In the neutrino sector, the right-handed neutrino mass matrix M_{RR} involves the VEVs of only two flavon fields. The three eigenvalues of M_{RR} are therefore related, leading to the mass sum rule for the light neutrinos reported in Eq. (1.5) with $\gamma = 1$ and $\delta = \pm 2i$. M_{RR}

[21] The alignments presented in Eq. (10.29) do not break the neutrino Klein symmetry generated by S and U. While $\langle\phi^\nu_{\mathbf{3}'}\rangle$ corresponds to the most general such alignment of a flavon in the $\overline{\mathbf{3}}'$, this is not the case for $\langle\phi^\nu_{\widetilde{\mathbf{3}}'}\rangle$. It is straightforward to verify that the most general alignment of a flavon in the $\widetilde{\mathbf{3}}'$ of $\Delta(96)$ which satisfies $S\langle\phi^\nu_{\widetilde{\mathbf{3}}'}\rangle = U\langle\phi^\nu_{\widetilde{\mathbf{3}}'}\rangle = \langle\phi^\nu_{\widetilde{\mathbf{3}}'}\rangle$, with U being the negative of the matrix shown in Eq. (10.23), takes the form $\langle\phi^\nu_{\widetilde{\mathbf{3}}'}\rangle \propto \left(v_1, \frac{v_1+v_3}{2}, v_3\right)^T$.

is furthermore symmetric under the Klein symmetry generated by S and U of Eq. (10.22). This can be shown either explicitly by calculating $S^T M_{RR} S = U^T M_{RR} U = M_{RR}$, or by realising that the alignments of the two neutrino flavons $\phi^{\nu}_{\mathbf{3}'}$ and $\phi^{\nu}_{\bar{\mathbf{3}}'}$ of Eq. (10.29) are left invariant under both S and U (in the appropriate representations). With m_{LR} being proportional to the identity matrix, the effective light neutrino mass matrix m^{ν}_{LL}, obtained from the type I see-saw formula, is symmetric under S and U as well. This particular Klein symmetry which originates from the family symmetry $\Delta(96)$ corresponds to the so-called bi-trimaximal mixing pattern [31, 108] in the neutrino sector,

$$U^{\nu}_{\rm BT} = \begin{pmatrix} a_+ & \frac{1}{\sqrt{3}} & a_- \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} , \quad \text{with} \;\; a_{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right) . \tag{10.34}$$

The bi-trimaximal mixing structure, which is a special form of TM2 mixing, does not involve any CP phases and translates to the following values of the neutrino mixing angles[22]

$$\theta^{\nu}_{13} = \sin^{-1}(a_-) \approx 12.2^\circ \; , \qquad \theta^{\nu}_{12} = \theta^{\nu}_{23} = \tan^{-1}\left(\sqrt{3}-1\right) \approx 36.2^\circ \; . \tag{10.35}$$

It is a remarkable fact that θ^{ν}_{13} deviates from the experimentally measured value of the reactor angle $\theta_{13} \approx 9^\circ$ by about 3°, a deviation which is typical for charged lepton corrections to a vanishing 13 neutrino mixing angle in $SU(5)$ GUTs with implemented Georgi-Jarlskog mechanism. Indeed, taking into account the left-handed charged lepton 12 mixing with $\theta^e_{12} \approx \lambda/3$ and phase δ^e_{12}, see Subsection 3.4, leads to the relation

$$\sin\theta_{13} \approx a_- - \tfrac{1}{\sqrt{3}}\theta^e_{12}\cos\delta^e_{12} \; . \tag{10.36}$$

As the charged lepton correction to θ_{13} has to be negative in order to hit the measured value, one must assume $\delta^e_{12} \approx 0$. This entails a vanishing Dirac CP phase

$$\delta \approx 0 \; . \tag{10.37}$$

The final PMNS mixing matrix is then real (up to Majorana phases) and has the form,

$$U_{\rm PMNS} \approx \begin{pmatrix} a_+ c^e_{12} + \frac{1}{\sqrt{3}} s^e_{12} & \frac{1}{\sqrt{3}} c^e_{12} - \frac{1}{\sqrt{3}} s^e_{12} & a_- c^e_{12} - \frac{1}{\sqrt{3}} s^e_{12} \\ a_+ s^e_{12} - \frac{1}{\sqrt{3}} c^e_{12} & \frac{1}{\sqrt{3}} s^e_{12} + \frac{1}{\sqrt{3}} c^e_{12} & a_- s^e_{12} + \frac{1}{\sqrt{3}} c^e_{12} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} , \tag{10.38}$$

leading to the following phenomenologically viable lepton mixing angles,

$$\theta_{13} \approx 9.6^\circ \; , \qquad \theta_{12} \approx 32.7^\circ \; , \qquad \theta_{23} \approx 36.9^\circ \; . \tag{10.39}$$

The charged lepton corrections are thus crucial for direct models based on a $\Delta(96)$ family symmetry. Furthermore, a specific choice of the phase δ^e_{12} is required, which must eventually be explained in a more complete model, for example along the lines of the models proposed in [107], see also end of Subsection 7.2.

[22] Notice that this leading order result is a realisation of bi-large mixing as defined in [97].

11. Conclusion

Neutrino physics has progressed at a breathtaking rate over the last decade and a half, as discussed in Section 1, with a major discovery almost every other year, as can be seen by glancing at the milestones in Subsection 1.2. The big experimental result of 2012 is the measurement by Daya Bay and RENO of the reactor angle, which has had a major impact on neutrino physics, making the early measurement of CP violation possible. It has also ruled out a large number of models which predicted the reactor angle to be zero, including simple patterns of lepton mixing known as BM, TB or GR which are presented in Subsection 3.1. As discussed in Section 6, these simple patterns can result from discrete symmetries, reviewed in Section 5, such as A_4, S_4 or A_5. On the other hand, anarchy expected the reactor angle to be rather large and this is what was observed. Does this mean that we should give up on the symmetry approach in favour of anarchy?

In this review the idea of giving up on the symmetry approach in favour of anarchy is given short shrift for good reason, namely some simple mixing patterns remain a good approximate characterisation of lepton mixing. For example, we have seen that it is possible to describe the current pattern of lepton mixing by simply perturbing around TB mixing. This perturbation can be parameterised in terms of the reactor (r), solar (s) and atmospheric (a) deviations from TB mixing introduced in Subsection 3.2, where such perturbations are no larger than the Wolfenstein parameter λ, which apparently coincides with the reactor parameter r. Indeed TBC mixing provides a good approximation to the observed lepton mixing.

From the symmetry perspective, the measurement of the reactor angle has caused theorists to work harder to explain the observed deviations from TB mixing, which in the near future could also include the atmospheric deviation parameter a and the solar deviation parameter s. Indeed there are already hints from the global fits to oscillation data that both these parameters could be non-zero, as discussed in Subsection 2.6. The models based on small discrete family symmetries such as A_4, S_4 or A_5 may be viewed as predicting BM, TB or GR lepton mixing only at the LO in the absence of HO operator corrections or charged lepton corrections, not to mention renormalisation group running or canonical normalisation corrections. Indeed there are many sources of corrections that can modify the naive simple mixing patterns apparently predicted by discrete family symmetry.

If we are very lucky then it is possible that only a special subset of these corrections are important, as discussed in Section 3. For example, if only Cabibbo-like charged lepton corrections are important then this leads to solar mixing sum rules. Although, in the framework of GUT models, it might seem natural to equate charged lepton corrections to the Cabibbo angle, this assumption is at odds with the simplest type of relations between charged lepton masses and down-type quark masses, where such relations would prefer the charged lepton mixing angle to be about a third of the Cabibbo angle. Alternatively, in the absence of charged lepton corrections, if only certain kinds of HO corrections are important then TB mixing could be reduced down to TM1 or TM2 mixing, where only half of the original Klein symmetry of the neutrino mass matrix is broken, and this leads to atmospheric sum rules. In either case, solar and atmospheric sum rules provide interesting relations involving the deviation parameters r, s, a together with $\cos\delta$ which can be tested in future neutrino oscillation experiments.

We have distinguished between two general approaches of using discrete family symmetry to build realistic models, referred to as direct or indirect. In the direct approach the Klein symmetry of the neutrino mass matrix, discussed in Subsection 6.2, is identified as a subgroup of the discrete family symmetry (with a different subgroup preserved in the charged lepton sector), while in the indirect approach the Klein symmetry emerges accidentally after the discrete family symmetry is completely broken. Both direct and indirect approaches rely on spontaneous breaking of the discrete family symmetry via new scalar fields which develop VEVs, referred to here as flavons, as discussed generally in Subsection 6.1.

A common feature of both the simplest direct or indirect models is the use of the type I see-saw mechanism, reviewed in Section 4, with form dominance. In fact the different types of see-saw mechanism are generally reviewed along with the sequential dominance mechanism. As discussed in Subsection 8.1, form dominance corresponds to the columns of the Dirac mass matrix being proportional to the columns of the PMNS matrix in the basis where the right-handed neutrino mass matrix is diagonal. The virtue is that this usually leads to a form diagonalisable physical neutrino mass matrix, with mixing angles being independent of neutrino masses. In the case of the simplest indirect models, form dominance is usually identified with constrained sequential dominance featuring a normal mass hierarchy with the lightest neutrino mass being approximately zero. The downside of form dominance is that, since the columns of the Dirac mass matrix are orthogonal, leptogenesis is identically zero so corrections to form dominance may be required.

The direct approach is discussed in detail in Section 7, including flavon vacuum alignment and model building strategies following Daya Bay and RENO. In the direct approach it is possible to achieve the simple patterns of lepton mixing, namely BM, TB or GR as an approximation to lepton mixing using small discrete family symmetries such as A_4, S_4 or A_5, and then consider the effect of corrections as discussed above. If we are lucky then such corrections could either preserve the Klein symmetry in the neutrino sector and break the symmetry in the charged lepton sector, leading to solar sum rules, or preserve half the original Klein symmetry in the neutrino sector, leading to TM mixing and atmospheric sum rules. Indeed it is possible to start with only half the original Klein symmetry in the neutrino sector arising as a subgroup of the family symmetry, as in A_4 for example, which we refer to as the semi-direct approach. More generally, however, the whole original Klein symmetry will be broken by HO corrections, and also charged lepton, renormalisation group and canonical normalisation corrections will also be present. Alternatively it is possible to use larger discrete family symmetries in which the Klein symmetry already gives a non-zero reactor angle at the LO. For example $\Delta(96)$ can give bi-trimaximal mixing, where the reactor angle is non-zero and the solar and atmospheric angles start out equal, with all angles receiving modest charged lepton corrections.

An analogous discussion for the indirect approach is provided in Section 8. An important difference is that TB mixing at the LO can be achieved not only from groups in which the Klein symmetry $Z_2 \times Z_2$ can be embedded but also in groups of odd order such as $\Delta(27)$ or T_7, or more generally infinite classes of groups. The types of correction to TB mixing discussed above are also possible for the indirect case, perhaps leading to solar and atmospheric sum rules if we are lucky. However, the indirect approach offers new alternatives to achieving a reactor angle already at the LO, without resorting to large discrete family symmetry groups, by using new vacuum alignments to construct the neutrino mass matrix. Such new alignments are more easily achieved with smaller groups since the discrete family symmetry is completely broken and we are freed from the requirement of identifying the Klein symmetry as a subgroup. Examples of new vacuum alignments, which can be achieved even in A_4, include PCSD and CSD2, where PCSD can give the successful TBC mixing if the misalignment parameter is identified with the Wolfenstein parameter.

We have already explained that we prefer the symmetry approach to, say, anarchy, since simple symmetric mixing patterns still provide a good approximation to reality. However there is a deeper reason why we prefer to use symmetry, namely to address the flavour problem. In our view, to abandon the symmetry approach would completely miss the opportunity provided by neutrino physics of elucidating the flavour problem, the problem of all quark and lepton masses and mixing parameters, including CP violating phases. The history of physics, if it tells us anything at all, teaches us that symmetry and unification have always provided a guiding light in the path to understanding deep problems in physics. Therefore in this review we have considered models based not only on discrete family symmetry, but also using GUTs, in order to address the flavour problem. Motivated by such considerations, in Section 9 we have briefly reviewed grand unified theories and how they may be combined with discrete family symmetry to describe all quark and lepton masses and mixing, tabulating some recent examples of this kind.

Finally in Section 10 we have discussed three model examples which combine an $SU(5)$ GUT with the discrete family symmetries A_4, S_4 and $\Delta(96)$. These models are presented in sufficient detail to illustrate the complexity of the current state of the art of GUTs of flavour that is required to account for all quark and lepton masses and mixing. Critics will use these examples as evidence that the effort of constructing such models is not worth the trouble and question what has been achieved by all this complexity. They will also point out that the number of input parameters in these models exceeds the number of mass and mixing parameters that we are trying to explain. However this last observation misses the point. What is relevant is the number of predictions (or postdictions) not the number of parameters. Typically each of these models contains half a dozen relationships such as the GST and GJ relations which agree with experiment, and neutrino mass and mixing sum rules giving predictions for future neutrino experiments. Thus one feels that something has been understood by constructing these models, including the mass hierarchy and origin of the three families as well as the milder neutrino hierarchy with bi-large lepton mixing.

The flavour problem is not going away, in fact since 1998 it has got significantly worse with at least seven new parameters arising from the neutrino sector on top of the three charged lepton masses and the ten flavour parameters from the quark sector. However the neutrino parameters also provide some clues such as small neutrino masses, bi-large mixing and a Cabibbo-like reactor angle. There is a ghost of a chance that these clues may be just enough to allow us to unlock the whole flavour puzzle. We are not there yet, but the hope is that the details of the admittedly rather complicated models given in this review may inform and inspire new young researchers to do better.

It is still not too late for theorists to redeem their past failures to successfully predict anything in the neutrino sector by making a genuine prediction which can be tested by experiment. Indeed there is still room to make predictions for the solar and atmospheric deviation parameters s, a as well as $\cos\delta$, or to relate them via sum rules to the reactor parameter r since all these parameters can and will be measured in future high precision neutrino experiments. Also one can make predictions for the pattern of neutrino masses including their ordering and their scale. A crucial question is whether neutrinos are Dirac or Majorana particles. In the former case neutrino mass could have the same origin as that of the quarks and charged leptons, while in the latter case something qualitatively different may be involved such as the see-saw mechanism for example. In the absence of experimental information about these questions, we must admit that we do not yet understand the origin of neutrino mass, which remains one of the biggest unsolved mysteries of the Standard Model.

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A. Equivalence of different parametrisations

In this appendix we exhibit the equivalence of different parametrisations of the lepton mixing matrix. A 3×3 unitary matrix may be parametrised by 3 angles and 6 phases. We shall find it convenient to parametrise a unitary matrix $V^\dagger$ by:[23]

$$V^\dagger = P_2 R_{23} R_{13} P_1 R_{12} P_3, \tag{A.1}$$

where R_{ij} are a sequence of real rotations corresponding to the Euler angles θ_{ij}, and P_i are diagonal phase matrices. The Euler matrices are given by

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}, \quad R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.2}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. The phase matrices are given by

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}, \quad P_3 = \begin{pmatrix} e^{i\omega_1} & 0 & 0 \\ 0 & e^{i\omega_2} & 0 \\ 0 & 0 & e^{i\omega_3} \end{pmatrix}. \tag{A.3}$$

[23] It is convenient to define the parametrisation of $V^\dagger$ rather than V because the lepton mixing matrix involves $V^\dagger_{\nu_L}$ and the neutrino mixing angles will play a central rôle.

By commuting the phase matrices to the left, it is not difficult to show that the parametrisation in Eq. (A.1) is equivalent to

$$V^\dagger = PU_{23}U_{13}U_{12}, \tag{A.4}$$

where $P = P_1P_2P_3$ and

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{-i\delta_{23}} \\ 0 & -s_{23}e^{i\delta_{23}} & c_{23} \end{pmatrix}, \tag{A.5}$$

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix}, \tag{A.6}$$

$$U_{12} = \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_{12}} & 0 \\ -s_{12}e^{i\delta_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.7}$$

where

$$\delta_{23} = \chi + \omega_2 - \omega_3, \tag{A.8}$$

$$\delta_{13} = \omega_1 - \omega_3, \tag{A.9}$$

$$\delta_{12} = \omega_1 - \omega_2. \tag{A.10}$$

The matrix U is an example of a unitary matrix, and as such it may be parametrised by either of the equivalent forms in Eqs. (A.1) or (A.4). If we use the form in Eq. (A.4) then the phase matrix P on the left may always be removed by an additional charged lepton phase rotation $\Delta V_{e_L} = P^\dagger$, which is always possible since right-handed charged lepton phase rotations can always make the charged lepton masses real. Therefore U can always be parametrised by

$$U = U_{23}U_{13}U_{12}, \tag{A.11}$$

which involves just three irremovable physical phases δ_{ij}. In this parametrisation the Dirac phase δ which enters the CP odd part of neutrino oscillation probabilities is given by

$$\delta = \delta_{13} - \delta_{23} - \delta_{12}. \tag{A.12}$$

Another common parametrisation of the lepton mixing matrix is

$$U = R_{23}U_{13}R_{12}P_0, \tag{A.13}$$

where

$$P_0 = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.14}$$

and in Eq. (A.13) U_{13} is of the form in Eq. (A.6) but with δ_{13} replaced by the Dirac phase δ. The parametrisation in Eq. (A.13) can be transformed into the parametrisation in Eq. (A.11) by commuting the phase matrix P_0 in Eq. (A.13) to the left, and then removing the phases on the left-hand side by charged lepton phase rotations. The two parametrisations are then related by the phase relations

$$\delta_{23} = \beta_2, \tag{A.15}$$
$$\delta_{13} = \delta + \beta_1, \tag{A.16}$$
$$\delta_{12} = \beta_1 - \beta_2. \tag{A.17}$$

The use of the parametrisation in Eq. (A.13) is widespread in the literature, however it is often more convenient to use the parametrisation in Eq. (A.11) which is trivially related to Eq. (A.13) by the above phase relations.

B. Deviations from TB mixing to second order

B.1 PMNS matrix expansion

The PMNS matrix when expanded to first order in the three real parameters r, s, a defined in Eq. (3.4) reads [64]

$$U_{\mathrm{PMNS}} \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix} P_{12}. \tag{B.1}$$

The second order corrections to the PMNS matrix elements are [64],

$$\begin{aligned}
\Delta U_{e1} &\approx \sqrt{\frac{2}{3}}(-\frac{1}{4}r^2 - \frac{3}{8}s^2), \\
\Delta U_{e2} &\approx \frac{1}{\sqrt{3}}(-\frac{1}{4}r^2), \\
\Delta U_{e3} &\approx 0, \\
\Delta U_{\mu 1} &\approx -\frac{1}{\sqrt{6}}(\frac{1}{2}rse^{i\delta} - rae^{i\delta} + sa + a^2), \\
\Delta U_{\mu 2} &\approx \frac{1}{\sqrt{3}}(-\frac{1}{2}rse^{i\delta} - \frac{1}{2}rae^{i\delta} + \frac{1}{2}sa - \frac{3}{8}s^2 - a^2), \\
\Delta U_{\mu 3} &\approx \frac{1}{\sqrt{2}}(-\frac{1}{4}r^2), \\
\Delta U_{\tau 1} &\approx \frac{1}{\sqrt{6}}(\frac{1}{2}rse^{i\delta} + rae^{i\delta} + sa), \\
\Delta U_{\tau 2} &\approx -\frac{1}{\sqrt{3}}(\frac{1}{2}rse^{i\delta} - \frac{1}{2}rae^{i\delta} - \frac{1}{2}sa - \frac{3}{8}s^2), \\
\Delta U_{\tau 3} &\approx \frac{1}{\sqrt{2}}(-\frac{1}{4}r^2 - a^2).
\end{aligned} \tag{B.2}$$

The Jarlskog CP invariant to second order is then given by [64]

$$J \approx (\frac{r}{6} + \frac{rs}{12})\sin\delta. \tag{B.3}$$

B.2 Neutrino oscillations in matter

In this appendix we present the complete formulae for neutrino oscillations in the presence of matter of constant density to second order in the quantities r, s, a and Δ_{21}, where it is assumed that $\Delta_{21} \ll 1$. Here $\Delta_{ij} = 1.27\Delta m^2_{ij} L/E$ with L the oscillation length in km, E the beam energy in GeV, and $\Delta m^2_{ij} = m^2_i - m^2_j$ in eV^2. We write $\Delta = \Delta_{31}$, $\alpha = \frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and $A = \frac{VL}{2\Delta}$ where V is the potential expressed in units of eV as

$$V \approx 7.56 \times 10^{-14} \rho\, N_e, \tag{B.4}$$

where ρ is the matter density of the Earth in units of $\mathrm{g/cm}^3$ and $N_e \approx 0.5$ is the number of electrons per nucleon in the Earth. The constant density approximation is good when the neutrino beam only passes through the Earth's crust where $\rho \approx 3\ \mathrm{g/cm}^3$ or the Earth's mantle where $\rho \approx 4.5\ \mathrm{g/cm}^3$.

The complete set of neutrino oscillation probabilities for electron neutrino or muon neutrino beams in the presence of matter of constant density to second order in the parameters r, s, a and α are [64]:

$$P_{ee} = 1 - \frac{8}{9}\alpha^2 \frac{\sin^2 A\Delta}{A^2} - 2r^2 \frac{\sin^2(A-1)\Delta}{(A-1)^2}, \tag{B.5}$$

$$\begin{aligned} P_{e\mu} = {} & \frac{4}{9}\alpha^2 \frac{\sin^2 A\Delta}{A^2} + r^2 \frac{\sin^2(A-1)\Delta}{(A-1)^2} \\ & + \frac{4}{3} r\alpha \cos(\Delta - \delta) \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)}, \end{aligned} \tag{B.6}$$

$$\begin{aligned} P_{e\tau} = {} & \frac{4}{9}\alpha^2 \frac{\sin^2 A\Delta}{A^2} + r^2 \frac{\sin^2(A-1)\Delta}{(A-1)^2} \\ & - \frac{4}{3} r\alpha \cos(\Delta - \delta) \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)}, \end{aligned} \tag{B.7}$$

$$\begin{aligned} P_{\mu e} = {} & \frac{4}{9}\alpha^2 \frac{\sin^2 A\Delta}{A^2} + r^2 \frac{\sin^2(A-1)\Delta}{(A-1)^2} \\ & + \frac{4}{3} r\alpha \cos(\Delta + \delta) \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)}, \end{aligned} \tag{B.8}$$

$$
\begin{aligned}
P_{\mu\mu} &= 1-(1-4a^2)\sin^2\Delta+\frac{2}{3}(1-s)\alpha\Delta\sin 2\Delta\\
&-\frac{4}{9}\alpha^2\frac{\sin^2 A\Delta}{A^2}-\frac{4}{9}\alpha^2\Delta^2\cos 2\Delta\\
&+\frac{4}{9}\alpha^2\frac{1}{A}\left(\sin\Delta\frac{\sin A\Delta}{A}\cos(A-1)\Delta-\frac{\Delta}{2}\sin 2\Delta\right)\\
&-r^2\frac{\sin^2(A-1)\Delta}{(A-1)^2}\\
&-\frac{1}{A-1}r^2\left(\sin\Delta\cos A\Delta\frac{\sin(A-1)\Delta}{(A-1)}-\frac{A}{2}\Delta\sin 2\Delta\right)\\
&-\frac{4}{3}r\alpha\cos\delta\cos\Delta\frac{\sin A\Delta}{A}\frac{\sin(A-1)\Delta}{(A-1)},
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
P_{\mu\tau} &= (1-4a^2)\sin^2\Delta-\frac{2}{3}(1-s)\alpha\Delta\sin 2\Delta+\frac{4}{9}\alpha^2\Delta^2\cos 2\Delta\\
&-\frac{4}{9}\alpha^2\frac{1}{A}\left(\sin\Delta\frac{\sin A\Delta}{A}\cos(A-1)\Delta-\frac{\Delta}{2}\sin 2\Delta\right)\\
&+\frac{1}{A-1}r^2\left(\sin\Delta\cos A\Delta\frac{\sin(A-1)\Delta}{(A-1)}-\frac{A}{2}\Delta\sin 2\Delta\right)\\
&+\frac{4}{3}r\alpha\sin\delta\sin\Delta\frac{\sin A\Delta}{A}\frac{\sin(A-1)\Delta}{(A-1)}.
\end{aligned}
\tag{B.10}
$$

C. Generators and Clebsch-Gordan coefficients of S_4, A_4 and T_7

In this section we list the generators of the groups S_4, A_4 and T_7 in the basis where the order three generator is diagonal. As this basis is particularly convenient for model building purposes, we state the corresponding (basis dependent) Clebsch-Gordan coefficients for all non-trivial Kronecker products. We first consider the two intimately linked groups S_4 and A_4, before discussing the group T_7.

C.1 The groups S_4 and A_4

The permutation group S_4 can be defined in terms of three generators S, T and U satisfying the presentation rules [144]

$$
S^2=T^3=U^2=(ST)^3=(SU)^2=(TU)^2=(STU)^4=1\ . \tag{C.1}
$$

Dropping the generator U and with it all relations involving U, we obtain the presentation of the alternating group A_4.

The triplet matrix representations of the three S_4 generators in the T-diagonal basis can be obtained from Eq. (5.9). Noticing that the b generator (corresponding to U) in Eq. (5.7) occurs only quadratically, we immediately find another triplet representation by changing the overall sign of U. The obtained irreducible representations are called **3** and **3′**, respectively. Likewise we find the two singlet representations **1** and **1′**. Summing up the square of the dimensions of these representations, $1^2+1^2+3^2+3^2=20$, shows that there exists only one more irreducible representation, namely the doublet **2**. Its matrix representation is presented, together with the other irreducible representations in the following table.

S_4	A_4	S	T	U
$\mathbf{1},\mathbf{1}'$	$\mathbf{1}$	1	1	± 1
$\mathbf{2}$	$\begin{pmatrix} \mathbf{1}'' \\ \mathbf{1}' \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{3},\mathbf{3}'$	$\mathbf{3}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\mp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

The same table also shows the representations of the S_4 subgroup A_4, generated by S and T only. Dropping the U generator, it is clear that both triplets of S_4 coincide with the single A_4 triplet. Likewise, the two S_4 singlets correspond to the trivial singlet of A_4. The S_4 doublet, on the other hand, becomes reducible once the U generator is removed. Hence, it decomposes into two separate non-trivial irreducible representations of A_4, $\mathbf{1}''$ and $\mathbf{1}'$.

The non-trivial S_4 product rules in the T-diagonal basis are listed below, where we use the number of primes within the expression

$$\boldsymbol{\alpha}^{(\prime)} \otimes \boldsymbol{\beta}^{(\prime)} \;\to\; \boldsymbol{\gamma}^{(\prime)} \;, \tag{C.2}$$

to classify the results. We denote this number by n, e.g. in $\mathbf{3}\otimes\mathbf{3}' \to \mathbf{3}'$ we get $n = 2$.

$$\mathbf{1}^{(\prime)} \otimes \mathbf{1}^{(\prime)} \;\to\; \mathbf{1}^{(\prime)} \quad \left\{ n = \text{even} \quad \begin{array}{l} \mathbf{1} \otimes \mathbf{1} \;\to\; \mathbf{1} \\ \mathbf{1}' \otimes \mathbf{1}' \;\to\; \mathbf{1} \\ \mathbf{1} \otimes \mathbf{1}' \;\to\; \mathbf{1}' \end{array} \right\} \alpha\beta \;,$$

$$\mathbf{1}^{(\prime)} \otimes \mathbf{2} \;\to\; \mathbf{2} \quad \left\{ \begin{array}{ll} n = \text{even} & \mathbf{1} \otimes \mathbf{2} \;\to\; \mathbf{2} \\ n = \text{odd} & \mathbf{1}' \otimes \mathbf{2} \;\to\; \mathbf{2} \end{array} \right\} \alpha \begin{pmatrix} \beta_1 \\ (-1)^n \beta_2 \end{pmatrix},$$

$$\mathbf{1}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \;\to\; \mathbf{3}^{(\prime)} \quad \left\{ n = \text{even} \quad \begin{array}{l} \mathbf{1} \otimes \mathbf{3} \;\to\; \mathbf{3} \\ \mathbf{1}' \otimes \mathbf{3}' \;\to\; \mathbf{3} \\ \mathbf{1} \otimes \mathbf{3}' \;\to\; \mathbf{3}' \\ \mathbf{1}' \otimes \mathbf{3} \;\to\; \mathbf{3}' \end{array} \right\} \alpha \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix},$$

$$\mathbf{2} \otimes \mathbf{2} \;\to\; \mathbf{1}^{(\prime)} \quad \left\{ \begin{array}{ll} n = \text{even} & \mathbf{2} \otimes \mathbf{2} \;\to\; \mathbf{1} \\ n = \text{odd} & \mathbf{2} \otimes \mathbf{2} \;\to\; \mathbf{1}' \end{array} \right\} \alpha_1\beta_2 + (-1)^n \alpha_2\beta_1 \;,$$

$$\mathbf{2} \otimes \mathbf{2} \;\to\; \mathbf{2} \quad \left\{ n = \text{even} \quad \mathbf{2} \otimes \mathbf{2} \;\to\; \mathbf{2} \right\} \begin{pmatrix} \alpha_2\beta_2 \\ \alpha_1\beta_1 \end{pmatrix},$$

$$\mathbf{2} \otimes \mathbf{3}^{(\prime)} \to \mathbf{3}^{(\prime)} \quad \left\{ \begin{array}{ll} n = \text{even} & \begin{array}{l} \mathbf{2} \otimes \mathbf{3} \to \mathbf{3} \\ \mathbf{2} \otimes \mathbf{3}' \to \mathbf{3}' \end{array} \\ n = \text{odd} & \begin{array}{l} \mathbf{2} \otimes \mathbf{3} \to \mathbf{3}' \\ \mathbf{2} \otimes \mathbf{3}' \to \mathbf{3} \end{array} \end{array} \right\} \alpha_1 \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_1 \end{pmatrix} + (-1)^n \alpha_2 \begin{pmatrix} \beta_3 \\ \beta_1 \\ \beta_2 \end{pmatrix},$$

$$\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \to \mathbf{1}^{(\prime)} \quad \left\{ n = \text{even} \quad \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \to \mathbf{1} \\ \mathbf{3}' \otimes \mathbf{3}' \to \mathbf{1} \\ \mathbf{3} \otimes \mathbf{3}' \to \mathbf{1}' \end{array} \right\} \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \ ,$$

$$\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \to \mathbf{2} \quad \left\{ \begin{array}{ll} n = \text{even} & \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \to \mathbf{2} \\ \mathbf{3}' \otimes \mathbf{3}' \to \mathbf{2} \end{array} \\ n = \text{odd} & \mathbf{3} \otimes \mathbf{3}' \to \mathbf{2} \end{array} \right\} \begin{pmatrix} \alpha_2\beta_2 + \alpha_3\beta_1 + \alpha_1\beta_3 \\ (-1)^n(\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1) \end{pmatrix},$$

$$\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \to \mathbf{3}^{(\prime)} \quad \left\{ n = \text{odd} \quad \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \to \mathbf{3}' \\ \mathbf{3} \otimes \mathbf{3}' \to \mathbf{3} \\ \mathbf{3}' \otimes \mathbf{3}' \to \mathbf{3}' \end{array} \right\} \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix},$$

$$\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \to \mathbf{3}^{(\prime)} \quad \left\{ n = \text{even} \quad \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \to \mathbf{3} \\ \mathbf{3}' \otimes \mathbf{3}' \to \mathbf{3} \\ \mathbf{3} \otimes \mathbf{3}' \to \mathbf{3}' \end{array} \right\} \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}.$$

The A_4 Clebsch-Gordan coefficients can be obtained from these expressions by simply dropping all primes and identifying the components of the S_4 doublet $\mathbf{2}$ as the $\mathbf{1}''$ and $\mathbf{1}'$ representations of A_4. We thus find the non-trivial A_4 products, explicitly,

$$\mathbf{1}' \otimes \mathbf{1}'' \to \mathbf{1} \qquad \alpha\beta \ ,$$

$$\mathbf{1}' \otimes \mathbf{3} \to \mathbf{3} \qquad \alpha \begin{pmatrix} \beta_3 \\ \beta_1 \\ \beta_2 \end{pmatrix},$$

$$\mathbf{1}'' \otimes \mathbf{3} \to \mathbf{3} \qquad \alpha \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_1 \end{pmatrix},$$

$$\mathbf{3} \otimes \mathbf{3} \to \mathbf{1} \qquad \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \ ,$$

$$\mathbf{3} \otimes \mathbf{3} \to \mathbf{1}' \qquad \alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 \ ,$$

$$\mathbf{3} \otimes \mathbf{3} \to \mathbf{1}'' \qquad \alpha_2\beta_2 + \alpha_3\beta_1 + \alpha_1\beta_3 \ ,$$

$$\mathbf{3} \otimes \mathbf{3} \to \mathbf{3} + \mathbf{3} \quad \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix} + \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}.$$

C.2 The group T_7

The Frobenius group $T_7 = Z_7 \rtimes Z_3$ is obtained from two generators a and c obeying the presentation, see e.g. [81],

$$< a, c \,|\, a^3 = c^7 = 1\,,\; aca^{-1} = c^2 > \quad . \tag{C.3}$$

Notice that with $k = 2$ in Eq. (5.11), the condition $1 + 2 + 2^2 = 0 \bmod 7$ holds. A triplet representation with non-diagonal order three generator is given in Eq. (5.12), where $\eta = e^{\frac{2\pi i}{7}}$. To diagonalise a, we apply the basis transformation of Eq. (5.10) (followed by the phase transformation T^2 in order to bring the c generator into a more appealing form), resulting in the matrix representation of the triplet **3** as shown in the below table. Clearly, T_7 also contains another triplet representation given by the complex conjugate $\overline{\mathbf{3}}$ of the **3**. Furthermore there are three singlet representations.

	a	c
$\mathbf{1}$	1	1
$\mathbf{1}'$	ω	1
$\mathbf{1}''$	ω^2	1
$\mathbf{3}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{\eta}{3}\begin{pmatrix} 1+\eta+\eta^3 & \omega^2+\omega\eta+\eta^3 & \omega+\omega^2\eta+\eta^3 \\ \omega+\omega^2\eta+\eta^3 & 1+\eta+\eta^3 & \omega^2+\omega\eta+\eta^3 \\ \omega^2+\omega\eta+\eta^3 & \omega+\omega^2\eta+\eta^3 & 1+\eta+\eta^3 \end{pmatrix}$
$\overline{\mathbf{3}}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\frac{\eta^6}{3}\begin{pmatrix} 1+\eta^6+\eta^4 & \omega+\omega^2\eta^6+\eta^4 & \omega^2+\omega\eta^6+\eta^4 \\ \omega^2+\omega\eta^6+\eta^4 & 1+\eta^6+\eta^4 & \omega+\omega^2\eta^6+\eta^4 \\ \omega+\omega^2\eta^6+\eta^4 & \omega^2+\omega\eta^6+\eta^4 & 1+\eta^6+\eta^4 \end{pmatrix}$

Although the order-seven generators of the triplet representations look rather involved, the Clebsch-Gordan coefficients take a relatively simple form. Omitting the trivial products, i.e. those involving the singlet **1** as well as products of only one-dimensional irreducible representations, the product rules are reported below. Again, we use the convention that the components of the first representation of any given product $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ are denoted by α_i while we use β_i for the components of the second representation.

$$\mathbf{1}' \otimes \mathbf{3} \;\to\; \mathbf{3} \quad \alpha \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_1 \end{pmatrix} , \qquad \mathbf{1}'' \otimes \mathbf{3} \;\to\; \mathbf{3} \quad \alpha \begin{pmatrix} \beta_3 \\ \beta_1 \\ \beta_2 \end{pmatrix} ,$$

$$\mathbf{1}' \otimes \overline{\mathbf{3}} \;\to\; \overline{\mathbf{3}} \quad \alpha \begin{pmatrix} \beta_3 \\ \beta_1 \\ \beta_2 \end{pmatrix} , \qquad \mathbf{1}'' \otimes \overline{\mathbf{3}} \;\to\; \overline{\mathbf{3}} \quad \alpha \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_1 \end{pmatrix} ,$$

$$
\begin{aligned}
\mathbf{3}\otimes\overline{\mathbf{3}} \;&\to\; \mathbf{1} && \alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3 \ , \\
\mathbf{3}\otimes\overline{\mathbf{3}} \;&\to\; \mathbf{1}' && \alpha_1\beta_2+\alpha_2\beta_3+\alpha_3\beta_1 \ , \\
\mathbf{3}\otimes\overline{\mathbf{3}} \;&\to\; \mathbf{1}'' && \alpha_1\beta_3+\alpha_2\beta_1+\alpha_3\beta_2 \ , \\
\mathbf{3}\otimes\overline{\mathbf{3}} \;&\to\; \mathbf{3} && \begin{pmatrix}\alpha_1\beta_1+\omega^2\alpha_2\beta_2+\omega\alpha_3\beta_3\\ \alpha_1\beta_3+\omega^2\alpha_2\beta_1+\omega\alpha_3\beta_2\\ \alpha_1\beta_2+\omega^2\alpha_2\beta_3+\omega\alpha_3\beta_1\end{pmatrix} \ , \\
\mathbf{3}\otimes\overline{\mathbf{3}} \;&\to\; \overline{\mathbf{3}} && \begin{pmatrix}\alpha_1\beta_1+\omega\alpha_2\beta_2+\omega^2\alpha_3\beta_3\\ \alpha_3\beta_1+\omega\alpha_1\beta_2+\omega^2\alpha_2\beta_3\\ \alpha_2\beta_1+\omega\alpha_3\beta_2+\omega^2\alpha_1\beta_3\end{pmatrix} \ , \\
\mathbf{3}\otimes\mathbf{3} \;&\to\; \mathbf{3} && \begin{pmatrix}\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\ \omega(\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3)\\ \omega^2(\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1)\end{pmatrix} \ , \\
\mathbf{3}\otimes\mathbf{3} \;&\to\; \overline{\mathbf{3}}+\overline{\mathbf{3}} && \begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\ 2\alpha_2\beta_2-\alpha_3\beta_1-\alpha_1\beta_3\\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix}+\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_3\beta_1-\alpha_1\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix} \ .
\end{aligned}
$$

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