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Some fundamentals of Quantum Mechanics

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Introduction

Basic knowledge of the postulates of Quantum Mechanics, as summarised in the next section, will be assumed. However, an additional lecture could be arranged for students who do not have the necessary background in Quantum Mechanics. Elementary knowledge of Probability Theory, Vector Spaces and Group Theory will be useful.

1 Postulates of Quantum Mechanics

Quantum Mechanics is a physical theory which replaces Newtonian Mechanics and Classical Electromagnetism at atomic and sub–atomic levels. The basic principles of Quantum Mechanics can be summarized in the following postulates. These postulates provide a connection between the physical world and the mathematical formalism of Quantum Mechanics. These notes have been taken largely from **[1]** and **[4]**.

1.1 Postulate 1

To any isolated physical system there is associated a *Hilbert space*, called the *state space* of the system. The system is completely described by its *state vector*, which is a *unit* vector in the system's state space.

Recall the following properties of a Hilbert space, $\mathcal{H}$:
(i) It is a vector space over the complex numbers $\mathbf{C}$. We shall use Dirac's bra-ket notation and denote a vector in $\mathcal{H}$ by the *ket* $|\psi\rangle$.
(ii) It has a scalar product (or *inner product*) denoted by $\langle\psi|\phi\rangle$. A scalar product is a function that maps an ordered pair of vectors $|\phi\rangle, |\psi\rangle \in \mathcal{H}$ to $\mathbf{C}$. The *bra*-notation $\langle\psi|$ is used to denote the vector *dual* to the vector $|\psi\rangle$. In the case of an n-dimensional complex Hilbert space, a *ket* $|\psi\rangle$ can be considered as an n-dimensional column vector and a *bra* $\langle\phi|$ as an n-dimensional row vector. The scalar product $\langle\psi|\phi\rangle$ is then a complex number.

(iii) It is complete in the norm
$$||\psi|| = \langle\psi|\psi\rangle^{1/2}.$$
Since a state vector $|\psi\rangle$ is a unit vector, $\langle\psi|\psi\rangle = 1$. This is known as the **normalization condition** of state vectors.

Superposition Principle
If $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, then any state which is a superposition of these states, i.e., any state of the form
$$|\Psi\rangle = a|\psi\rangle + b|\phi\rangle, \tag{1}$$
where $a, b \in \mathbf{C}$, also belongs to $\mathcal{H}$. This is referred to as the Superposition Principle.

Properties of the scalar product

1. Positivity: $\langle\psi|\psi\rangle \geq 0$ with equality if and only if $|\psi\rangle = 0$.
2. Linearity:
$$\langle\phi|\left(a|\psi_1\rangle + b|\psi_2\rangle\right) = a\langle\phi|\psi_1\rangle + b\langle\phi|\psi_2\rangle$$
3. Skew-symmetry:
$$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$$

Global and Relative Phase of state vectors
Consider two states $|\Psi\rangle$ and $|\Phi\rangle$, where $|\Psi\rangle$ is given by (1) and $|\Phi\rangle = e^{i\theta}|\Psi\rangle$, θ being a real constant. These two states differ by the factor $e^{i\theta}$, of unit modulus, which is referred to as a *global phase factor*. These states describe the same physical state of a system. This is because measurements cannot distinguish between states which differ only by a global phase factor. Consequently, the state of a physical system is given by a *ray* in a Hilbert space, the latter being an equivalence class of vectors that differ by a complex scalar factor. If $|\phi\rangle \in \mathcal{H}$, then the ray is $\{c|\phi\rangle : c \in \mathbf{C}\}$. We choose a representative of this class to have unit norm : $\langle\psi|\psi\rangle = 1$. Note, however, that the relative phase between two states is of physical significance, i.e., the states $a|\Psi\rangle + b|\Phi\rangle$ and $a|\Psi\rangle + be^{i\theta}|\Phi\rangle$ do *not* represent the same physical state of the system.

Let us see the relevance of Postulate 1 in Quantum Information Theory. To do this, we first introduce the concept of a *qubit*.

The qubit
The basic indivisible unit of classical information is the *bit*. It takes one of two possible values — 0 and 1. The corresponding unit of quantum information is called the "quantum bit" or **qubit**. A qubit is a vector in a 2-dimensional Hilbert space (referred to as the *single qubit space*). In analogy with the classical bit, we denote the elements of an orthonormal basis in the single qubit space as $|0\rangle$ and $|1\rangle$. The column- and row vector representations of these basis vectors are as follows:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

$$\langle 0| = (1 \;\; 0) \qquad \langle 1| = (0 \;\; 1)$$

Intuitively, the states $|0\rangle$ and $|1\rangle$ are analogous to the values 0 and 1 that a (classical) bit can take. However, there is an important difference between a qubit and a bit. Superpositions of the states $|0\rangle$ and $|1\rangle$ of the form

$$|\psi\rangle = a|0\rangle + b|1\rangle, \tag{3}$$

where $a, b \in \mathbf{C}$, can also exist. Any arbitrary state vector in the state space of a qubit can be expressed in the form (3). When in state $|\psi\rangle$, the qubit is neither definitely in state $|0\rangle$, nor definitely in state $|1\rangle$. The qubit is in the state $|0\rangle$ with probability $|a|^2$, and in the state $|1\rangle$ with probability $|b|^2$. The dual vector is

$$\langle\psi| = a^*\langle 0| + b^*\langle 1|. \tag{4}$$

The condition that $|\psi\rangle$ be a unit vector therefore requires that

$$|a|^2 + |b|^2 = 1. \tag{5}$$

We can exploit this relation to obtain a useful geometric representation of the state of a single qubit – the **Bloch sphere representation**. Eq. (5) allows us to express $|\psi\rangle$ as follows:

$$|\psi\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right],$$

where θ, γ and ϕ are real numbers. We can ignore the factor of $e^{i\gamma}$, since it is a global phase factor and has no observable effect, and write

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle.$$

The numbers θ and ϕ define a point on a sphere of unit radius called the *Bloch sphere*. It provides a useful means of visualizing the state of a single qubit. *This will be elaborated in an example sheet.*

Physical interpretation of a qubit

A state $|\psi\rangle$ given by (3) can be interpreted as the spin state of a spin-1/2 particle (e.g. an electron). Then $|0\rangle$ and $|1\rangle$ can be chosen to denote the spin-up ($|\uparrow\rangle$) and spin-down ($|\downarrow\rangle$) states of the particle along a particular axis, e.g. the z-axis. Then, through the Bloch sphere representation, a and b can be understood to describe the orientation of the spin in three-dimensional space (in terms of the polar angle θ and the azimuthal angle ϕ). Moreover, any operation on a qubit must preserve the unit norm of $|\psi\rangle$ and thus must be described by

2×2 unitary matrices. Note that any arbitrary unitary transformation acting on the state $|\psi\rangle$, except for the possible rotation of the global phase, is a rotation of the spin.

Observables

Another key concept of Quantum Mechanics is that of observables. An observable is a property of the physical system which can be measured (at least in principle).
Mathematically an observable is a linear, self-adjoint (or Hermitian) operator. A linear operator $\mathbf{A}$ acting on a Hilbert space $\mathcal{H}$ is a map:

$$\mathbf{A} : |\psi\rangle \to \mathbf{A}|\psi\rangle \;;\; \mathbf{A}(a|\psi\rangle + b|\phi\rangle) = a\mathbf{A}|\psi\rangle + b\mathbf{A}|\phi\rangle, \quad \text{for} \quad |\psi\rangle, |\phi\rangle \in \mathcal{H},\, a, b \in \mathbf{C}.$$

For an operator $\mathbf{A}$ acting on a Hilbert space $\mathcal{H}$ there exists a unique linear operator $\mathbf{A}^\dagger$ acting on $\mathcal{H}$ such that

$$\langle v|\mathbf{A}w\rangle = \langle \mathbf{A}^\dagger v|w\rangle.$$

The operator $\mathbf{A}^\dagger$ is the adjoint of $\mathbf{A}$. A linear operator $\mathbf{A}$ is represented by a matrix. Its adjoint $\mathbf{A}^\dagger$ is represented by the transpose of the complex conjugate of this matrix. An operator $\mathbf{A}$ is a self-adjoint (or Hermitian) operator if $\mathbf{A} = \mathbf{A}^\dagger$. From the definition of the adjoint it is easy to see that $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$. By convention, if $|\psi\rangle$ is a vector in the Hilbert space on which the operator $\mathbf{A}$ acts, then we define

$$|\psi\rangle^\dagger = \langle\psi|.$$

Hence,

$$(\mathbf{A}|\psi\rangle)^\dagger = \langle\psi|\mathbf{A}^\dagger.$$

There are four observables acting in the single qubit space, which are of particular significance in Quantum Information Theory. These are represented by the following 2×2 matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here σ_0 is the 2×2 identity matrix and σ_x, σ_y and σ_z are the Pauli matrices. The action of these operators on the basis vectors $|0\rangle$ and $|1\rangle$ of the single qubit space are:

$$\begin{aligned} \sigma_0|0\rangle &= |0\rangle \quad ; \quad \sigma_0|1\rangle = |1\rangle \\ \sigma_x|0\rangle &= |1\rangle \quad ; \quad \sigma_x|1\rangle = |0\rangle \\ \sigma_y|0\rangle &= i|1\rangle \quad ; \quad \sigma_y|1\rangle = -i|0\rangle \\ \sigma_z|0\rangle &= |0\rangle \quad ; \quad \sigma_y|1\rangle = -|1\rangle \end{aligned} \tag{6}$$

The Pauli matrices satisfy the following relations:

$$\sigma_x\sigma_y = i\sigma_z \,;\, \sigma_y\sigma_z = i\sigma_x \,;\, \sigma_z\sigma_x = i\sigma_y \,;$$

Heuristically, the action of the Pauli matrices on the state of a qubit can be interpreted as follows

σ_x : a bit flip ; σ_z : a phase flip ; $\sigma_y (= i\sigma_x\sigma_z)$: a combined (bit and phase) flip.

Eigenvectors and Eigenvalues
An eigenvector (or eigenstate) of a linear operator $\mathbf{A}$ acting on a Hilbert space $\mathcal{H}$ is a non-zero vector $|i\rangle$ such that

$$\mathbf{A}|i\rangle = a_i|i\rangle,$$

where a_i is a complex number known as the eigenvalue of $\mathbf{A}$ corresponding to the eigenstate $|i\rangle$. Using the properties of the scalar product one can show that (i) all eigenvalues of a linear self-adjoint operator are real, and (ii) eigenvectors corresponding to different eigenvalues of a linear self-adjoint operator are orthogonal. The orthonormality relation for the eigenvectors of an operator $\mathbf{A}$ is given by

$$\langle i|j\rangle = \delta_{ij},$$

$|i\rangle, |j\rangle$ being eigenvectors corresponding to eigenvalues a_i, a_j of $\mathbf{A}$, respectively. The set of eigenvalues $\{a_i\}$ defines the spectrum of $\mathbf{A}$. The set of its orthonormal eigenvectors $\{|i\rangle\}$ form a *complete set of orthonormal basis vectors* of $\mathcal{H}$. Any vector $|\psi\rangle \in \mathcal{H}$ can be expanded in this basis as follows:

$$|\psi\rangle = \sum_i c_i|i\rangle, \quad \text{where} \quad c_i \in \mathbf{C} \tag{7}$$

Eigenvalue decomposition of a Hermitian operator:
Let $\mathbf{A}$ be a linear Hermitian operator acting on a Hilbert space $\mathcal{H}$. Let $S_{\mathbf{A}} := \{|i\rangle\}$ denote a complete orthonormal set of eigenvectors of $\mathbf{A}$. Let the corresponding set of eigenvalues be denoted by $\{a_i\}$. Then $\mathbf{A}$ can be expressed as follows:

$$\begin{aligned} \mathbf{A} &= \sum_i a_i|i\rangle\langle i| && (8)\\ &= \sum_i a_i\mathbf{P}_i, && (9) \end{aligned}$$

where $\mathbf{P}_i$ is the orthogonal projection onto the space of eigenvectors with eigenvalue a_i. If a_i is non–degenerate, then $P_i = |i\rangle\langle i|$, the projection onto the corresponding eigenvector. The $\mathbf{P}_i$'s satisfy

$$\begin{aligned} \mathbf{P}_i\mathbf{P}_j &= \delta_{ij}\mathbf{P}_i \\ \mathbf{P}_i^\dagger &= \mathbf{P}_i. \end{aligned} \tag{10}$$

For any vector $|j\rangle \in S_{\mathbf{A}}$, the operator $\mathbf{A}$, defined through (8), yields the eigenvalue equation

$$\mathbf{A}|j\rangle = a_j|j\rangle.$$

Functions of a Hermitian operator:
For any Hermitian operator A acting on a finite-dimensional Hilbert space, there exists a unitary operator U such that

$$A = UA_dU^\dagger,$$

where $A_d = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, with λ_i, $i = 1, \ldots, d$ being the eigen-values of A. Then a function, $f(A)$, of the operator A is given by the following operator:

$$f(A) = Uf(A_d)U^\dagger,$$

where $f(A_d) = \text{diag}\{f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_m)\}$.
For example $f(A) = \sqrt{A}$ is defined in this manner.
Note: the above definition holds for any normal operator, i.e., for any operator A satisfying $A^\dagger A = AA^\dagger$.

Trace and Partial Trace of an operator The trace of an operator A acting on a finite-dimesnional Hilbert Space $\mathcal{H}$, with $\dim\mathcal{H} = d$, is given by

$$\text{Tr}A = \sum_{i=1}^{d} \langle \alpha_i|A|\alpha_i\rangle, \tag{11}$$

where $\{|\alpha_i\rangle\}_{i=1}^d$ denotes an orthonormal basis of $\mathcal{H}$. Note that the trace is independent of the choice of the basis. Hence, for any unitary operator U, $\text{Tr}(UAU^\dagger) = \text{Tr}A$. The following properties are satisfied by the trace of an operator: *(i)* Linearity: For two operators A and B, and complex constants c_1 and c_2, $\text{Tr}(c_1A + c_2B) = c_1\text{Tr}A + c_2\text{Tr}B$; *(ii)* $\text{Tr}(A^\dagger) = \text{Tr}A$; *(iii)* Cyclicity: $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$.

Partial Trace: Let X_{AB} denote an operator acting on a bipartite Hilbert Space $\mathcal{H}_A \otimes \mathcal{H}_B$. Then taking its partial trace on the Hilbert space $\mathcal{H}_B$ (denoted as Tr_B) yields an operator X_A acting on $\mathcal{H}_A$. It is defined as follows:

$$X_A := \text{Tr}_B X_{AB} = \sum_i \langle e_i^B|X_{AB}|e_i^B\rangle, \tag{12}$$

where $\{|e_i^B\rangle\}_{i=1}^{d_B}$ denotes an orthonormal basis of $\mathcal{H}_B$. (Here $d_B = \dim\mathcal{H}_B$).
Note that the trace over the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ of a composite system ABC can be decomposed into partial traces on the individual subsystems. If X_{ABC} is an operator acting on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, then

$$\begin{aligned} \text{Tr}_{AB}X_{ABC} &= \text{Tr}_A(\text{Tr}_B(X_{ABC})) \\ \text{Tr}X_{ABC} &= \text{Tr}_C(\text{Tr}_A(\text{Tr}_B(X_{ABC}))) = \text{Tr}(\text{Tr}_A(\text{Tr}_B(X_{ABC}))) \end{aligned} \tag{13}$$

This can be easily seen by using the fact that if $\{|\alpha_k^A\rangle\}_k$ and $\{|e_i^B\rangle\}_i$ denote orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, then $\{|\alpha_k^A\rangle \otimes |e_i^B\rangle\}_{i,k}$ is an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_B$.

Completeness Relation:
If $|u\rangle$ is a vector in a Hilbert space $\mathcal{H}$ and $|v\rangle$ is a vector in a Hilbert space $\mathcal{H}'$, then

$$Q := |v\rangle\langle u|, \tag{14}$$

is a linear operator from $\mathcal{H}$ to $\mathcal{H}'$ whose action on any vector $|w\rangle \in \mathcal{H}$ is as follows:

$$Q|w\rangle := (|v\rangle\langle u|)(|w\rangle) = |v\rangle\langle u|w\rangle \equiv \langle u|w\rangle|v\rangle \tag{15}$$

The RHS of (18) is said to be the *outer product representation* of the operator Q. We can take linear combinations of outer product operators as follows: $\sum_i a_i|u_i\rangle\langle v_i|$ (where $a_i \in \mathbf{C}$) is the linear operator, which when acting on $|w\rangle$ yields the output $\sum_i a_i\langle v_i|w\rangle|u_i\rangle$.
The outer product representation can be used to yield an important result known as the **completeness relation** for orthonormal basis vectors. If $\{|i\rangle\}$ denotes a complete set of orthonormal vectors in a Hilbert space $\mathcal{H}$, then

$$\sum_i |i\rangle\langle i| = \mathbf{I}, \tag{16}$$

where $\mathbf{I}$ is the identity operator. This can be seen as follows. Any vector $|u\rangle \in \mathcal{H}$ can be written as $|u\rangle = \sum_i u_i|i\rangle$ where $u_i = \langle i|u\rangle \in \mathbf{C}$. Hence

$$(\sum_i |i\rangle\langle i|)|u\rangle = \sum_i |i\rangle\langle i|u\rangle = \sum_i u_i|i\rangle = |u\rangle. \tag{17}$$

This holds for any vector $|u\rangle \in \mathcal{H}$, and hence (16) follows.

Further, if $\{|i\rangle\}_{i=1}^d$ is an orthonormal basis of a d-dimensional Hilbert space $\mathcal{H}$, then any self-adjoint operator A acting on $\mathcal{H}$ can be written as

$$A = \sum_{i,j=1}^d a_{ij}|i\rangle\langle j|, \tag{18}$$

with $a^*_{ji} = a_{ij}$ (since $A = A^\dagger$ and $A^\dagger = \sum_{i,j} a^*_{ij}|j\rangle\langle i| = \sum_{i,j} a^*_{ji}|i\rangle\langle j|$).
Then, using the linearity of the trace we obtain

$$\mathrm{Tr}A = \sum_{i,j} a_{ij}\mathrm{Tr}|i\rangle\langle j| = \sum_{i,j} a_{ij}\langle j|i\rangle = \sum_i c_{ii}. \tag{19}$$

Commutator and Anticommutator
The commutator of two operators $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}. \tag{20}$$

If $[\mathbf{A}, \mathbf{B}] = 0$, i.e., $\mathbf{AB} = \mathbf{BA}$, then we say that $\mathbf{A}$ and $\mathbf{B}$ commute. Similarly, the anticommutator of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\{\mathbf{A}, \mathbf{B}\} = \mathbf{AB} + \mathbf{BA}. \tag{21}$$

The operators $\mathbf{A}$ and $\mathbf{B}$ anticommute if $\{\mathbf{A}, \mathbf{B}\} = 0$. Many important properties of pairs of operators can be deduced from their commutator and anticommutator. If $\mathbf{A}$ and $\mathbf{B}$ are two self-adjoint operators then $[\mathbf{A}, \mathbf{B}] = 0$ if and only if there exists an orthonormal basis such that both $\mathbf{A}$ and $\mathbf{B}$ are diagonal with respect to that basis and hence have spectral representations of the form

$$\mathbf{A}_i = \sum_i a_i|i\rangle\langle i| \quad ; \quad \mathbf{B}_i = \sum_i b_i|i\rangle\langle i|.$$

The operators are said to be *simultaneously diagonalizable* and $\{|i\rangle\}$ denotes the set of their simultaneous eigenstates.

1.2 Postulate 2

The evolution of an isolated (closed) quantum system is described by a unitary transformation. If a system is in a state $|\psi(t_1)\rangle$ at time t_1 and a state $|\psi(t_2)\rangle$ at a later time t_2, then

$$|\psi(t_2)\rangle = \mathbf{U}(t_1, t_2)|\psi(t_1)\rangle,$$

where $\mathbf{U}(t_1, t_2)$ is a unitary operator which depends only on t_1 and t_2. More precisely, the time evolution of a state vector is governed by the *Schrödinger equation*:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathbf{H}|\psi(t)\rangle,$$

where $\mathbf{H}$ is a self-adjoint operator, called the *Hamiltonian*, which generates the unitary transformation ($\hbar$ is a constant called the Planck's constant). In particular, for a time-independent Hamiltonian, we have

$$\mathbf{U}(t_1, t_2) = e^{-\frac{i}{\hbar}\mathbf{H}(t_2 - t_1)}.$$

Note that the unitary evolution of a closed system is *deterministic*. Given $|\psi(0)\rangle$, the theory predicts the state $|\psi(t)\rangle$ at all later times t.

An isolated (closed) quantum system has a *unitary* evolution. However, when an experiment is done to find out the properties of the system, there is an interaction between the system and the experimentalists and their equipment (i.e., the external physical world). So the system is no longer closed and its evolution is not necessarily unitary. The following postulate provides a means of describing the effects of a measurement on a quantum-mechanical system.

1.3 Postulate 3

In Quantum Mechanics one measures an observable. The numerical outcome of a measurement of an observable $\mathbf{A}$ is an eigenvalue a_n (say) of $\mathbf{A}$. Immediately after the measurement the system is in an eigenstate of $\mathbf{A}$ with the measured eigenvalue a_n. If the system is in a state $|\psi\rangle$ just before the measurement, then the probability that the outcome is a_n is given by

$$p(a_n) = \langle\psi|\mathbf{P}_n|\psi\rangle,$$

where $\mathbf{P}_n$ is the orthogonal projection onto the space of eigenvectors of $\mathbf{A}$ with eigenvalue a_n. If the measured value is a_n, then, as a result of the measurement, the normalized state of the system becomes

$$\frac{\mathbf{P}_n|\psi\rangle}{\langle\psi|\mathbf{P}_n|\psi\rangle^{1/2}}.$$

This prescription tells us that if the measurement is repeated immediately, the measured value is again a_n, this time with probability one.

Note that measurement in Quantum Mechanics is *probabilistic*, i.e., Quantum Mechanics assigns probabilities to different possible outcomes of a measurement. Moreover, measurement *disturbs* the state of a system, taking it to an eigenstate of the measured observable. In particular, if $\mathbf{A}$ and $\mathbf{B}$ are two self-adjoint operators which do not commute, then a measurement of $\mathbf{A}$ will necessarily influence the outcome of a subsequent measurement of $\mathbf{B}$. The fact that acquiring information about a quantum system inevitably disturbs the state of the system leads to important differences between Classical– and Quantum Information Theory.

1.4 Postulate 4

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems.

Tensor Products

The *tensor product* is a way of putting vector spaces together to form larger vector spaces. If $\mathcal{H}_1$ is an m-dimensional Hilbert space and $\mathcal{H}_2$ is an n-dimensional Hilbert space, then their tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is an mn-dimensional Hilbert space. The elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ are linear combinations of tensor products $|u\rangle \otimes |v\rangle$ of elements $|u\rangle$ of $\mathcal{H}_1$ and $|v\rangle$ of $\mathcal{H}_2$. In particular if $\{|i\rangle\}$ and $\{|j\rangle\}$ are orthonormal bases for the Hilbert Spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ then $\{|i\rangle \otimes |j\rangle\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. We often use the abbreviated notation $|uv\rangle$ for the tensor product $|u\rangle \otimes |v\rangle$. For example, if $\mathcal{H}$ is a two–dimensional Hilbert space with basis vectors $|0\rangle$ and $|1\rangle$ then $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a unit vector in $\mathcal{H} \otimes \mathcal{H}$.

Suppose $|u\rangle$ and $|v\rangle$ are vectors in $\mathcal{H}_1$ and $\mathcal{H}_2$, and A_1 and A_2 are linear operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then we can define a linear operator $A_1 \otimes A_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by the equation

$$(A_1 \otimes A_2)(|u\rangle \otimes |v\rangle) = A_1|u\rangle \otimes A_2|v\rangle.$$

More generally,

$$(A_1 \otimes A_2)(\sum_i a_i|u_i\rangle \otimes |v_i\rangle) = \sum_i a_i A_1|u_i\rangle \otimes A_2|v_i\rangle.$$

The inner products on the Hilbert Spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ can be used to define a natural inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\Big(\sum_i a_i|u_i\rangle \otimes |v_i\rangle, \sum_j b_j|u'_j\rangle \otimes |v'_j\rangle\Big) = \sum_{ij} a_i^* b_j \langle u_i|u'_j\rangle \langle v_i|v'_j\rangle.$$

A convenient representation of the tensor product of two operators A and B (and equivalently of two vectors $|u\rangle$ and $|v\rangle$) is the *Kronecker product representation*. If A is an $m \times n$ matrix, and B is a $p \times q$ matrix, then $A \otimes B$ is an $mp \times nq$ matrix. If $a_{ij}, i = 1, \ldots, m, j = 1, \ldots n$ denote the elements of the matrix A, then in this representation $A \otimes B$ is given by the matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & \cdots & a_{mn}B \end{pmatrix}, \tag{22}$$

Here $a_{11}B$ denotes a $p \times q$ submatrix whose entries are proportional to the matrix B, with an overall proportionality constant a_{11}.
Similarly, the tensor products of the vectors $|0\rangle$ and $|1\rangle$ defined through (2) is given by

$$|0\rangle \otimes |1\rangle = (0, 1, 0, 0)^T,$$

where T denotes the transpose.

For further properties of tensor products see e.g. **[1]**.

References

Here is a list of textbooks and lecture notes that might be useful for preliminary reading.

[1] *Quantum Computation and Quantum Information* by M.A. Nielsen and I.L. Chuang (Cambridge University Press)

[2] *Quantum Computing* by Jozef Gruska (McGraw Hill)

[3] *Lecture notes on Quantum Computation*, by John Preskill (available online at **http://www.theory.caltech.edu/people/preskill/ph229/**).