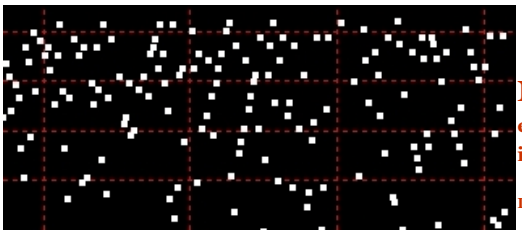

Fixing Fermat's Factorization Method

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The problem with Fermat's factorization method is that it requires, on average, many more iterations than does simple trial division. However, it turns out that there's a simple fix for this.

Fermat's factorization method says that by adding N to perfect squares (PSs) one will find eventually a sum N + PS that is itself a perfect square. When that happens one can know the factors of N by adding and subtracting the square roots of PS and N + PS.

First, one must ask "why use perfect squares at all?" if the factoring is ultimately done with the square roots of the perfect squares.

So let's consider for a moment how we apply this method to a composite with two prime factors (a *semiprime*):

Say N = 32951

If we begin with 1 + 32951 and continue through all the perfect squares (2*2 , 3*3, 4*4 and so on), we will find that 24649 (157*157) + 32951 finally equals another perfect square: 57600 (240*240).

So 24649 + 32951 = 57600

at which point we know that

240 + 157 = 397
240 - 157 = 83

and that

83 * 397 = 32951

All well and good, but factoring 32951 this way requires evaluating 157 perfect squares.

Can we speed this up? It turns out, Yes! Not just a bit but by a factor of 3 at least....

We know that if we're adding perfect squares to N, then if we are to begin the calculation without using perfect squares we would need to add integer square roots (counting numbers) to N - and, if we do that, we must begin with the square root of N.

The square root of 32951 = 181.524103082758681895679166559 19

Needless to say, it's not a perfect square *root* (and, by the way, if it was it would not be solvable using Fermat's factorization method).

Finding a Fix

Once we make the mental leap from perfect squares to perfect square roots, it's apparent that we're considering a counting problem. We need to approach this problem with two ideas in mind.

Idea #1
It's important we start counting and continue counting in the right place.

If we round up the square root to 182 and start counting up and down, we will get to 157 (-25) before 240 (+58) and never find the square roots we're looking for, since one without the other is useless.

So, as we count we need to have continuous 'course corrections' to be at the point midway between the unknown lower square root and the unknown upper

square root.

Idea #2
It's important we only count numbers once.

Consider this sequence:

Sqrt(32951 + 196) = 182.063175848385
Sqrt(32951 + 225) = 182.14280112044
Sqrt(32951 + 256) = 182.227879315982
Sqrt(32951 + 289) = 182.318402801253
Sqrt(32951 + 324) = 182.414363469547
Sqrt(32951 + 361) = 182.515752744797
Sqrt(32951 + 400) = 182.622561585364
Sqrt(32951 + 441) = 182.734780488007
Sqrt(32951 + 484) = 182.852399492049
Sqrt(32951 + 529) = 182.975408183723

Adding N to each of the successive perfect squares from 196 to 529 actually changes the square root of the sum by less than 1! And, of course, the bigger N is the greater the number of PSs needed to shift the square root by 1.

Clearly, we only need to evaluate Sqrt(32951 + 196) to know that 33147 is not a perfect square! The remaining 9 calcs are a completely wasted effort.

So we need a way of preemptively skipping the dud numbers, but how can we know ahead about this?

The following simple algorithm solves both these problems - actually, Step 3 does.

1. Determine the integer square root of N.

   sqrN = Int(Sqr(n))

2. Determine the square of Step 1.

   sqrN2 = sqrN * sqrN

   (Note that sqrN2 is initially the greatest perfect square less than N.)

3. Determine the integer square root of Step 2 *minus N*.

   PCS = Int(Sqr(Abs(sqrN2 - n)))

   (Let's say *PCS* means 'perfected counting square'.)

4. Determine if the sum of Step 2 and Step 3 is a factor of N.

   If mod(n, PCS + sqrN) = 0 then we have a factor, and we're done.

   (Similarly, PCS *minus* sqrN would produce the same result.)

5. If not, add 1 to Step 1 (that is, sqrN+1) and repeat.

So how does this method fare with 32951? It reduces the number of required iterations to 59 from 157, as follows:

Cnt	sqrN	PCS	sqrN-PCS	sqrN+PCS
1	182	13	169	195
2	183	23	160	206
3	184	30	154	214
4	185	35	150	220
5	186	40	146	226
6	187	44	143	231
7	188	48	140	236
8	189	52	137	241
9	190	56	134	246
10	191	59	132	250
11	192	62	130	254
12	193	65	128	258
13	194	68	126	262
14	195	71	124	266
15	196	73	123	269

16	197	76	121	273
17	198	79	119	277
18	199	81	118	280
19	200	83	117	283
20	201	86	115	287
21	202	88	114	290
22	203	90	113	293
23	204	93	111	297
24	205	95	110	300
25	206	97	109	303
26	207	99	108	306
27	208	101	107	309
28	209	103	106	312
29	210	105	105	315
30	211	107	104	318
31	212	109	103	321
32	213	111	102	324
33	214	113	101	327
34	215	115	100	330
35	216	117	99	333
36	217	118	99	335
37	218	120	98	338
38	219	122	97	341
39	220	124	96	344
40	221	126	95	347
41	222	127	95	349
42	223	129	94	352
43	224	131	93	355
44	225	132	93	357
45	226	134	92	360
46	227	136	91	363
47	228	137	91	365
48	229	139	90	368
49	230	141	89	371
50	231	142	89	373
51	232	144	88	376
52	233	146	87	379
53	234	147	87	381
54	235	149	86	384
55	236	150	86	386
56	237	152	85	389
57	238	153	85	391
58	239	155	84	394
59	240	157	83	397

The most important thing to notice is the PCS variable:

- *It is by definition always half the difference of the potential factors (sqrN-PCS and sqrN+PCS).*
- *It is increasing in uneven intervals, skipping a lot of numbers to begin with but just a few near the end.*
- *The PCS variable doesn't just speed up Fermat's method by a factor of 3 or better, it actually makes the method faster than trial division for 'symmetrical semiprimes' of larger magnitudes.**

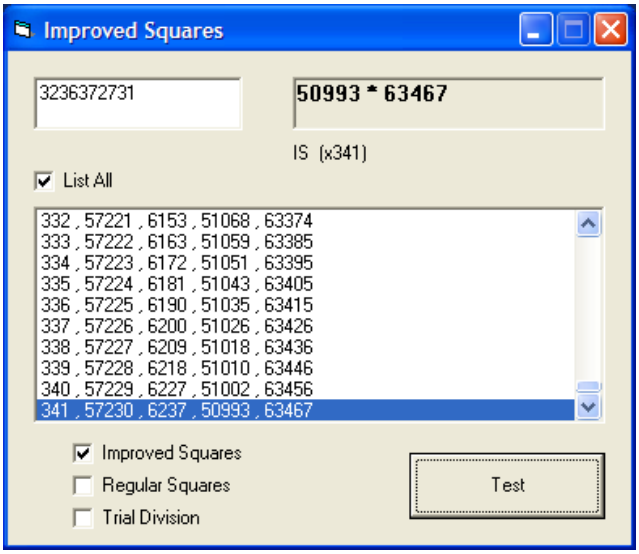
*OK, I acknowledge that trial div still has the edge for this baby example. Obviously, try something bigger for a fair test, say 9 or 10 digits. Remember that we're interested in the difficult ones - those with only 2 prime factors that are similar in size to the square root. A systematic survey by magnitude will be coming soon.

Now you're ready to try this out for yourself. I've written some simple illustrative code comparing the "Improved Squares" method with the standard method and with trial division.

Instructions

To try this code, you can use any Microsoft Office VBA macro editor, such as for Excel.

Add 1 text box, 1 list box, 1 command button, 2 labels, and 4 checkboxes to make a form like this:



Then copy and paste the following code and try to run it...

```
Option Explicit
Private blEsc As Boolean

Private Sub Command1_Click()
On Error GoTo errout
    Dim n As Variant, sN As Variant, sqrN As Variant, sqrN2 As Variant
    Dim PCS As Variant, f1 As Variant, f2 As Variant
    Dim cnt As Variant, cnt2 As Variant, ocnt As Variant
    Dim blfirst As Boolean

    If Len(Text1) = 0 Then
        MsgBox "Enter a number to factor.", vbCritical
        Text1.SetFocus
        Exit Sub
    End If
    If Check2.Value = 0 And Check3.Value = 0 And Check4.Value = 0 Then
        MsgBox "No factoring method is selected.", vbCritical
        Check2.SetFocus
        Exit Sub
    End If
```

The results don't lie...!