Design and Analysis of 2-Link Robotic Manipulator

Task 4

Introduction

This document presents the structured design and analysis of a 2-link robotic manipulator capable of lifting small objects weighing between 50g and 200g. It is organized into key sections that sequentially develop the theory and practical considerations for kinematics, torque, actuation, overload behavior, and efficiency.

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1 Kinematics (Forward and Inverse)

The 2-link robotic manipulator under consideration consists of two revolute joints, denoted by joint angles q_1 and q_2, and corresponding rigid links of lengths l_1 and l_2. This section presents both forward and inverse kinematic formulations, enabling the computation of the end-effector's position in the workspace and the joint angles required to reach a specific point, respectively.

Reference Frame and Joint Definitions

We define a global inertial frame $\{O\}$ with its origin located at the base of the manipulator. The x-axis points horizontally to the right, and the y-axis points vertically upwards.

- q_1: The angle between the first link and the global x-axis, measured counterclockwise.

- q_2: The relative angle between the second link and the first link, also measured counterclockwise.

The absolute orientation of the second link is thus given by $\theta_2 = q_1 + q_2$.

Forward Kinematics

To determine the Cartesian coordinates (x, y) of the end-effector, we use the following parametric equations derived from trigonometric projections of the links:

$$x = l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \tag{1}$$

$$y = l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \tag{2}$$

These equations assume ideal planar motion with no joint or link flexibility.

Inverse Kinematics

Given a desired position (x, y) within the reachable workspace, the inverse kinematics problem involves computing the corresponding joint angles (q_1, q_2).

We begin by solving for q_2 using the Law of Cosines:

$$\cos(q_2) = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2} \tag{3}$$

$$q_2 = \arccos\left(\cos(q_2)\right) \tag{4}$$

Then, using geometric construction and trigonometric identities:

$$q_1 = \arctan 2(y, x) - \arctan 2\left(l_2 \sin(q_2), l_1 + l_2 \cos(q_2)\right) \tag{5}$$

Note that two possible configurations (elbow-up and elbow-down) may exist for a single end-effector position. The `atan2` function ensures proper quadrant handling for accurate angle resolution.

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2 Torque Estimation for Joint Motion

To ensure accurate actuation and safe operation of the robotic arm, we derive the full dynamic equations of motion using the Euler-Lagrange formulation. The resulting expressions provide the torques τ_1 and τ_2 required at joints 1 and 2, respectively, accounting for link inertia, Coriolis effects, and gravity. Additionally, the payload mass m_p held at the end-effector is incorporated into the model.

System Parameters

- Link 1: mass m_1, length l_1, moment of inertia I_1 (about joint 1)
- Link 2: mass m_2, length l_2, moment of inertia I_2 (about joint 2)
- Payload at end-effector: mass m_p (in range 0.05–0.2 kg)

We assume the links are uniform rods:

$$I_1 = \frac{1}{3} m_1 l_1^2, \quad I_2 = \frac{1}{3} m_2 l_2^2$$

Kinetic Energy

The total kinetic energy T of the system includes:

- Rotational energy of both links
- Translational kinetic energy of their centers of mass
- Translational kinetic energy of the payload

$$T = \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} m_1 v_{c1}^2 + \frac{1}{2} I_2 (\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2} m_2 v_{c2}^2 + \frac{1}{2} m_p v_e^2$$

where:

- v_{c1}, v_{c2} are the velocities of the link centers of mass
- v_e is the velocity of the end-effector

Potential Energy

Let g be gravitational acceleration (typically 9.81 $\mathrm{m/s^2}$). The total potential energy is:

$$V = m_1 g y_{c1} + m_2 g y_{c2} + m_p g y_e$$

where:

- $y_{c1} = \frac{l_1}{2} \sin(q_1)$
- $y_{c2} = l_1 \sin(q_1) + \frac{l_2}{2} \sin(q_1 + q_2)$
- $y_e = l_1 \sin(q_1) + l_2 \sin(q_1 + q_2)$

Dynamic Model via Lagrangian

The Lagrangian is:

$$\mathcal{L} = T - V$$

We then compute the joint torques using the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \quad \text{for } i = 1, 2$$

Resulting Torque Equations

The full dynamic model can be written in matrix form as:

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \mathbf{M}(q)\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} + \mathbf{C}(q,\dot{q})\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} + \mathbf{G}(q)$$

Derivation of the Inertia Matrix $\mathbf{M}(q)$

To compute the joint torques due to link and payload inertia, we derive the inertia matrix $\mathbf{M}(q)$ using the Lagrangian formulation. This derivation includes the effects of both links and a payload at the end-effector.

System Assumptions

- Link 1: mass m_1, length l_1, CoM at $l_1/2$, inertia $I_1 = \frac{1}{3}m_1 l_1^2$
- Link 2: mass m_2, length l_2, CoM at $l_2/2$, inertia $I_2 = \frac{1}{3}m_2 l_2^2$
- Payload: point mass m_p at the end-effector

Velocity of Centers of Mass

The squared velocities of the centers of mass and payload contribute to the kinetic energy:

$$\begin{aligned}
\|\dot{\vec{r}}_{c1}\|^2 &= \left(\frac{l_1}{2}\dot{q}_1\right)^2 \\
\|\dot{\vec{r}}_{c2}\|^2 &= l_1^2\dot{q}_1^2 + \frac{l_2^2}{4}(\dot{q}_1+\dot{q}_2)^2 + l_1 l_2\cos(q_2)\dot{q}_1(\dot{q}_1+\dot{q}_2) \\
\|\dot{\vec{r}}_{p}\|^2 &= l_1^2\dot{q}_1^2 + l_2^2(\dot{q}_1+\dot{q}_2)^2 + 2l_1 l_2\cos(q_2)\dot{q}_1(\dot{q}_1+\dot{q}_2)
\end{aligned}$$

Kinetic Energy Expression

The total kinetic energy T is:

$$\begin{aligned}
T = &\frac{1}{2}I_1\dot{q}_1^2 + \frac{1}{2}m_1\left(\frac{l_1}{2}\dot{q}_1\right)^2 \\
&+ \frac{1}{2}I_2(\dot{q}_1+\dot{q}_2)^2 + \frac{1}{2}m_2\left[l_1^2\dot{q}_1^2 + \frac{l_2^2}{4}(\dot{q}_1+\dot{q}_2)^2 + l_1 l_2\cos(q_2)\dot{q}_1(\dot{q}_1+\dot{q}_2)\right] \\
&+ \frac{1}{2}m_p\left[l_1^2\dot{q}_1^2 + l_2^2(\dot{q}_1+\dot{q}_2)^2 + 2l_1 l_2\cos(q_2)\dot{q}_1(\dot{q}_1+\dot{q}_2)\right]
\end{aligned}$$

From Kinetic Energy to Inertia Matrix

The kinetic energy of a robotic system can always be written in a quadratic form with respect to joint velocities:

$$T = \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\mathbf{M}(q)\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}$$

where:

- $\dot{q}_i$ are the joint velocities,
- $\mathbf{M}(q)$ is the inertia matrix — a symmetric, positive definite matrix that depends on the geometry and mass distribution of the system.

How to Extract $\mathbf{M}(q)$: To find $\mathbf{M}(q)$, we compute the total kinetic energy T and group the terms based on joint velocity products:

- Coefficients of $\dot{q}_1^2$ go into M_{11},
- Coefficients of $\dot{q}_1\dot{q}_2$ and $\dot{q}_2\dot{q}_1$ go into $M_{12} = M_{21}$,
- Coefficients of $\dot{q}_2^2$ go into M_{22}.

Example: Suppose the total kinetic energy is:

$$T = A\dot{q}_1^2 + B\dot{q}_1\dot{q}_2 + C\dot{q}_2^2$$

This can be written in matrix form as:

$$T = \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix}2A & B \\ B & 2C\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$$

Therefore:

$$M_{11} = 2A, \quad M_{12} = M_{21} = B, \quad M_{22} = 2C$$

Application to the 2-Link Manipulator: When we derived the full kinetic energy T for the 2-link manipulator (links + payload), we had terms like:

- $\frac{1}{2}I_1\dot{q}_1^2$
- $\frac{1}{2}m_2\left(l_1^2\dot{q}_1^2 + l_1l_2\cos(q_2)\dot{q}_1(\dot{q}_1 + \dot{q}_2) + \cdots\right)$
- $\frac{1}{2}m_p\left(\cdots + 2l_1l_2\cos(q_2)\dot{q}_1(\dot{q}_1 + \dot{q}_2)\right)$

We then expand these expressions and reorganize them by velocity terms:

- Coefficients of $\dot{q}_1^2$ go into M_{11}
- Coefficients of $\dot{q}_1\dot{q}_2$ into M_{12}
- Coefficients of $\dot{q}_2^2$ into M_{22}

conclusion: This standard Lagrangian technique allows us to extract the complete inertia matrix $\mathbf{M}(q)$ directly from the kinetic energy expression — capturing the dynamic coupling and inertial behavior of the robotic manipulator. The inertia matrix M(q) is the matrix of coefficients of the **velocity squared and cross terms** in the total kinetic energy.

Once obtained, $\mathbf{M}(q)$ is used in the full dynamic model:

$$\mathbf{M}(q)\ddot{q} + \mathbf{C}(q,\dot{q})\dot{q} + \mathbf{G}(q) = \boldsymbol{\tau}$$

Inertia Matrix $\mathbf{M}(q)$:

$$\mathbf{M}(q) = \begin{bmatrix} a + 2b\cos(q_2) & d + b\cos(q_2) \\ d + b\cos(q_2) & d \end{bmatrix}$$

where:

$$a = I_1 + I_2 + m_1\left(\frac{l_1^2}{4}\right) + m_2\left(l_1^2 + \frac{l_2^2}{4} + l_1 l_2 \cos(q_2)\right) + m_p\left(l_1^2 + l_2^2 + 2l_1 l_2 \cos(q_2)\right)$$
$$b = m_2\left(\frac{l_1 l_2}{2}\right) + m_p l_1 l_2$$
$$d = I_2 + m_2\left(\frac{l_2^2}{4}\right) + m_p l_2^2$$

Coriolis and Centrifugal Matrix $\mathbf{C}(q, \dot{q})$:

Coriolis and Centrifugal Matrix $\mathbf{C}(q, \dot{q})$

The Coriolis and centrifugal forces arise due to the velocity-dependent inertial effects in the manipulator. They account for how the motion of one joint affects the torque required at another joint due to dynamic coupling.

Dynamic Model Overview: The full dynamic model of the 2-link planar manipulator is written as:

$$\mathbf{M}(q)\ddot{q} + \mathbf{C}(q, \dot{q})\dot{q} + \mathbf{G}(q) = \boldsymbol{\tau}$$

Coriolis and Centrifugal Terms: The matrix $\mathbf{C}(q, \dot{q})$ is defined such that:

$$\mathbf{C}(q, \dot{q})\dot{q} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

These terms come from the Christoffel symbols of the inertia matrix $\mathbf{M}(q)$. Each term C_{ij} is built using:

$$C_{ijk} = \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right)$$

Then:

$$C_{ij} = \sum_{k=1}^{2} C_{ijk}\dot{q}_k$$

Simplified Form for 2-Link Manipulator: Using symbolic simplification for a 2-link planar robot with payload, the Coriolis and centrifugal terms reduce to:

$$\mathbf{C}(q, \dot{q}) = \begin{bmatrix} -h\sin(q_2)\dot{q}_2 & -h\sin(q_2)(\dot{q}_1 + \dot{q}_2) \\ h\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}$$

where:

$$h = m_2\left(\frac{l_1 l_2}{2}\right) + m_p l_1 l_2$$

Interpretation:

- The terms with $\dot{q}_1 \dot{q}_2$ are **Coriolis** terms — they result from the coupling of motions between joints.
- The terms with $\dot{q}_2^2$ are **centrifugal** terms — they act to "pull outward" when a joint rotates.
- These forces do not exist in static motion and become significant during fast, coupled joint movements.

Usage in Simulation and Control: These velocity-dependent torques must be compensated in:

- High-speed motion planning
- Feedforward control design
- Accurate simulation of dynamic response

$$\mathbf{C}(q, \dot{q}) = \begin{bmatrix} -h \sin(q_2)\dot{q}_2 & -h \sin(q_2)(\dot{q}_1 + \dot{q}_2) \\ h \sin(q_2)\dot{q}_1 & 0 \end{bmatrix}, \quad h = b$$

Gravity Vector $\mathbf{G}(q)$

The gravity vector $\mathbf{G}(q)$ represents the torques at the joints required to counteract the weight of the links and payload due to gravity. These torques act along the axis of each joint and must be compensated by the motors during any static or dynamic task.

Potential Energy of the System

Let g denote gravitational acceleration (typically 9.81 m/s^2). We compute the total potential energy V from the vertical positions of the centers of mass (CoM) of each link and the payload.

Vertical positions:

$$\begin{aligned} y_{c1} &= \frac{l_1}{2} \sin(q_1) \\ y_{c2} &= l_1 \sin(q_1) + \frac{l_2}{2} \sin(q_1 + q_2) \\ y_p &= l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \end{aligned}$$

Total potential energy:

$$V = m_1 g y_{c1} + m_2 g y_{c2} + m_p g y_p$$

Gravity Torques from Potential Energy

The gravity vector is obtained by taking partial derivatives of V with respect to the joint angles:

$$G_i = \frac{\partial V}{\partial q_i} \quad \text{for } i = 1, 2$$

Thus, the gravity torque vector is:

$$\mathbf{G}(q) = \begin{bmatrix} \frac{\partial V}{\partial q_1} \\ \frac{\partial V}{\partial q_2} \end{bmatrix}$$

Final Expressions for Gravity Torques

After differentiating:

$$G_1 = g\left[\frac{m_1 l_1}{2}\cos(q_1) + m_2\left(l_1\cos(q_1) + \frac{l_2}{2}\cos(q_1+q_2)\right) + m_p\left(l_1\cos(q_1) + l_2\cos(q_1+q_2)\right)\right]$$

$$G_2 = g\left[\frac{m_2 l_2}{2}\cos(q_1+q_2) + m_p l_2\cos(q_1+q_2)\right]$$

So the gravity vector becomes:

$$\mathbf{G}(q) = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

Interpretation:

- G_1 includes all gravitational torques about joint 1, from all masses.
- G_2 includes only the components acting through joint 2.
- These torques are essential for holding static positions and must be counteracted by motor torque even when the robot is not moving.

Gravity Vector $\mathbf{G}(q)$:

$$\mathbf{G}(q) = \begin{bmatrix} g\left(\frac{m_1 l_1}{2} + m_2 l_1 + m_2\frac{l_2}{2}\cos(q_2) + m_p l_1 + m_p l_2\cos(q_2)\right)\sin(q_1+q_2) \\ g\left(\frac{m_2 l_2}{2} + m_p l_2\right)\sin(q_1+q_2) \end{bmatrix}$$

Joint Torque Calculation Over Motion

To compute the required torques at each joint during motion from an initial position q_{start} to a final position q_{end} within time t, we use the full dynamic model:

$$\boldsymbol{\tau}(t) = \mathbf{M}(q)\ddot{q}(t) + \mathbf{C}(q,\dot{q})\dot{q}(t) + \mathbf{G}(q)$$

Trajectory Assumptions

We assume:

- Motion starts and ends at rest: $\dot{q}(0) = \dot{q}(t) = 0$
- The trajectory follows a smooth acceleration profile, such as a trapezoidal or minimum-jerk trajectory
- For approximation, we can model acceleration as constant: $\ddot{q} \approx \frac{4\Delta q}{t^2}$ for rest-to-rest motion over time t

Angular Displacement

$$\Delta q_i = q_{i,\text{final}} - q_{i,\text{start}} \quad \text{for } i = 1, 2$$

Angular Acceleration Estimate

For a symmetric trapezoidal trajectory:

$$\ddot{q}_i = \frac{4\Delta q_i}{t^2}, \quad \dot{q}_i = \frac{2\Delta q_i}{t} \quad \text{(peak velocity)}$$

Torque Expression at Peak Load

$$\tau_i = \underbrace{M_{i1}\ddot{q}_1 + M_{i2}\ddot{q}_2}_{\text{Inertial torque}} + \underbrace{C_{i1}\dot{q}_1 + C_{i2}\dot{q}_2}_{\text{Coriolis and centrifugal torque}} + \underbrace{G_i}_{\text{Gravity torque}}$$

All terms are evaluated at the **intermediate configuration** (e.g., midpoint) of the trajectory, which is where torques typically peak.

Total Torque Expressions

Now, compute the total torques at each joint:

$$\begin{aligned}
\tau_1 &= M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + (-h\sin q_2)\dot{q}_2 \cdot \dot{q}_1 + (-h\sin q_2)(\dot{q}_1 + \dot{q}_2) \cdot \dot{q}_2 + G_1 \\
\tau_2 &= M_{12}\ddot{q}_1 + M_{22}\ddot{q}_2 + h\sin q_2 \cdot \dot{q}_1 \cdot \dot{q}_1 + G_2
\end{aligned}$$

This is the Final expression for torque to be generated by motors in respective positions.

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3 Motor Selection Based on Torque Requirements

The motors at each joint must be capable of delivering the required torque throughout the manipulator's motion. This includes inertial torque, Coriolis/centrifugal effects, and gravitational compensation — especially under the worst-case payload condition of $m_p =$ 200 g.

Torque Requirement Summary

From Section 2, the total required torque at each joint is given by:

$$\tau_i = \underbrace{\sum_{j=1}^{2} M_{ij}(q)\ddot{q}_j}_{\text{Inertial Torque}} + \underbrace{\sum_{j=1}^{2} C_{ij}(q,\dot{q})\dot{q}_j}_{\text{Coriolis/Centrifugal Torque}} + \underbrace{G_i(q)}_{\text{Gravitational Torque}}$$

We take the torque values computed using the worst-case configuration (maximum payload and full horizontal reach) as the basis for motor selection.

Selection Criteria

The selected motors must satisfy the following requirements:

- **Torque Capability:** Continuous torque rating τ_{rated} must be greater than the maximum required torque:
  $$\tau_{\text{rated}} \geq \tau_{\max} \cdot \text{SF}$$
  where SF is a safety factor (typically 1.5 to 2.0).
- **Speed Capability:** The motor must also support the required angular speed:
  $$\omega_i = \max(\dot{q}_i)$$
- **Form Factor and Weight:** Motor size and weight must be acceptable, especially for joint 2 where added mass affects dynamic load on joint 1.
- **Control Precision:** High-resolution encoders or integrated feedback may be required for accurate joint positioning and smooth motion.

Recommended Torque Margin Calculation

Let:

$$\begin{aligned} \tau_{1,\max} &= \text{Maximum evaluated torque at joint 1 with } m_p = 0.2 \text{ kg} \\ \tau_{2,\max} &= \text{Maximum evaluated torque at joint 2 with } m_p = 0.2 \text{ kg} \end{aligned}$$

Then select motors such that:

$$\tau_{1,\text{rated}} \geq \tau_{1,\max} \cdot \text{SF}, \quad \tau_{2,\text{rated}} \geq \tau_{2,\max} \cdot \text{SF}$$

Drive System Note

If gearboxes are used, the required output torque is reduced by the gear ratio G, and motor torque is:

$$\tau_{\text{motor}} = \frac{\tau_{\text{joint}}}{G}$$

But this also reduces speed and may introduce backlash.

Motor Type Considerations

- **Joint 1:** Requires higher torque; typically a high-torque servo motor or NEMA stepper with gear reduction.
- **Joint 2:** Lower torque; can use compact servo or geared micro motor.

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4 Effect of Unexpected Overload

In practical operation, the manipulator may unintentionally attempt to lift objects that exceed the expected payload range (50–200 g). This section analyzes how such an overload affects the joint dynamics, actuator limits, and system safety.

Overload Scenario

Let the expected payload be $m_p \in [0.05, 0.2]$ kg. Consider a case where the arm encounters a significantly heavier object, for example:

$$m_p^{\text{actual}} = 0.4 \text{ kg} \quad (2\times \text{ overload})$$

This mass was not included in the motor selection or torque profiling. As a result, the actual torque demands at both joints increase beyond nominal.

Torque Amplification

From the dynamic torque equation:

$$\boldsymbol{\tau} = \mathbf{M}(q)\ddot{q} + \mathbf{C}(q, \dot{q})\dot{q} + \mathbf{G}(q)$$

The gravitational torque increases linearly with payload:

$$\Delta\tau_G \propto m_p \cdot l_2 \cdot \cos(q_1 + q_2)$$

Similarly, inertial terms scale with m_p during acceleration.

Implications

- **Motor saturation:** If $\tau_{\text{actual}} > \tau_{\text{rated}}$, the motor cannot track the desired motion $\rightarrow$ trajectory error or stalling.
- **Thermal stress:** Prolonged high torque increases motor temperature, risking damage.
- **Mechanical strain:** Joints, mounts, and gearbox could exceed design loads.
- **Controller instability:** If not properly bounded, large position or velocity errors could destabilize a PID or feedforward controller.

Safety and Mitigation Strategies

- **Torque sensors or current monitoring** to detect excessive loads in real time.
- **Emergency stop thresholds** for motor current or joint torque.
- **Compliance control (impedance/admittance)** to absorb shocks and prevent rigid responses to overloads.
- **Overload detection logic** that compares estimated vs. commanded torque and switches to a fail-safe state.

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5 Energy Efficiency Optimization

Minimizing energy consumption in robotic manipulators is critical for performance, battery life, thermal stability, and actuator longevity. This section develops a formal basis for optimizing energy usage and presents the most effective strategy for doing so in a 2-link arm: **minimum-torque trajectory optimization**.

1. Total Mechanical Energy Cost

The total actuator energy is estimated by the integral of mechanical power over time:

$$E = \int_0^T \boldsymbol{\tau}^T(t) \cdot \dot{\boldsymbol{q}}(t)\ dt$$

Substituting full dynamics:

$$\boldsymbol{\tau}(t) = \mathbf{M}(q)\ddot{q} + \mathbf{C}(q,\dot{q})\dot{q} + \mathbf{G}(q)$$

Then,

$$E = \int_0^T \left[\dot{q}^T\mathbf{M}(q)\ddot{q} + \dot{q}^T\mathbf{C}(q,\dot{q})\dot{q} + \dot{q}^T\mathbf{G}(q)\right] dt$$

These are interpreted as:

- **Inertial Power**: required to accelerate links and payload
- **Coriolis Power**: power loss due to joint coupling and inertial interactions
- **Gravitational Power**: work done against gravity

2. Optimization Objective

The energy-optimal trajectory problem is posed as:

$$\min_{q(t)} \int_0^T \|\boldsymbol{\tau}(t)\|^2 dt \quad \text{subject to:} \begin{cases} q(0) = q_{\text{start}}, \quad q(T) = q_{\text{end}} \\ \dot{q}(0) = \dot{q}(T) = 0 \end{cases}$$

This results in a nonlinear trajectory optimization problem over $q(t)$, often solved using:

- Euler-Lagrange or Pontryagin's minimum principle
- Direct collocation (e.g., using optimization libraries)
- Time-scaling techniques using polynomial interpolation

3. Best Method: Minimum Torque-Squared Trajectory

Define the cost function:

$$J = \int_0^T \boldsymbol{\tau}^T(t)\boldsymbol{\tau}(t)\, dt$$

This is minimized when:

- Trajectory is smooth (low $\ddot{q}$ and $\dot{q}$)
- Joint velocities and accelerations are scaled according to dynamic coupling
- Payload and mass distribution are considered in $\mathbf{M}(q)$ and $\mathbf{G}(q)$

Implementation Outline

1. Choose a time-scaling function $s(t) \in [0, 1]$ with fixed duration T
2. Represent $q(t) = q_{\text{start}} + \Delta q \cdot f(s(t))$ using $f(s)$ (e.g., 5th-order polynomial)
3. Plug $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$ into torque expression
4. Minimize J numerically over coefficients of $f(s)$ using optimization tools

4. Real-Time Adaptation (Optional)

If onboard sensing estimates payload m_p, the planner can:

- Adapt T (duration) to reduce peak torque
- Re-optimize trajectory to minimize energy for detected m_p
- Switch to energy-conserving low-torque control mode

The most effective method to reduce energy usage in a 2-link manipulator is to use **minimum-torque-squared trajectory optimization**, adjusted in real time based on sensed system parameters. This accounts for full system dynamics and adapts to varying mass and geometry for maximum efficiency.
