variable elimination

19 oct 2022

CSCI 373

c	
1	0.33
2	0.33
3	0.33

c	g	
1	2	0.5
1	3	0.5
2	2	0
2	3	1
3	2	1
3	3	0

g	f	
2	1	0
2	2	0
2	3	1
3	1	0
3	2	1
3	3	0

f	c	w	
1	1	yes	1
1	1	no	0
1	2	yes	0
1	2	no	1
1	3	yes	0
1	3	no	1
2	1	yes	0
2	1	no	1
2	2	yes	1
2	2	no	0
2	3	yes	0
2	3	no	1
3	1	yes	0
3	1	no	1
3	2	yes	0
3	2	no	1
3	3	yes	1
3	3	no	0

these are multivariable functions

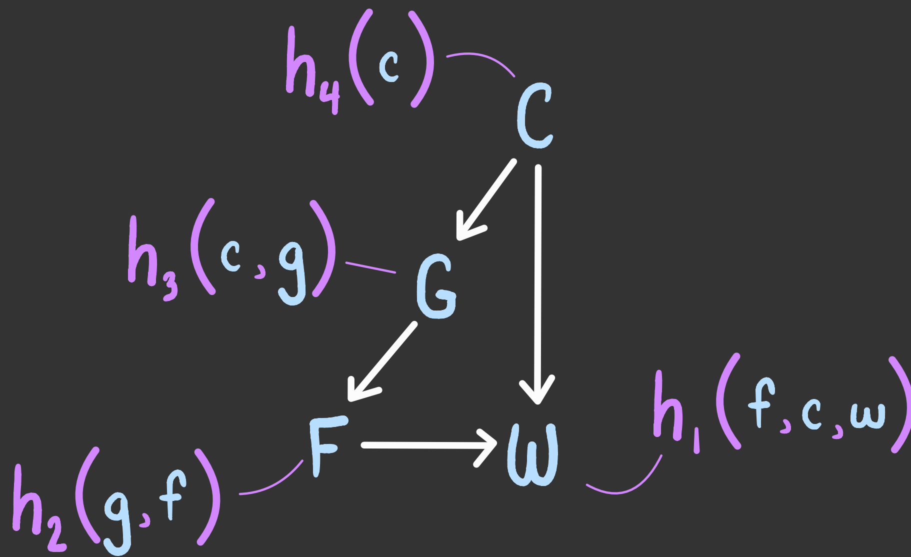
c	
1	0.33
2	0.33
3	0.33

c	g	
1	2	0.5
1	3	0.5
2	2	0
2	3	1
3	2	1
3	3	0

g	f	
2	1	0
2	2	0
2	3	1
3	1	0
3	2	1
3	3	0

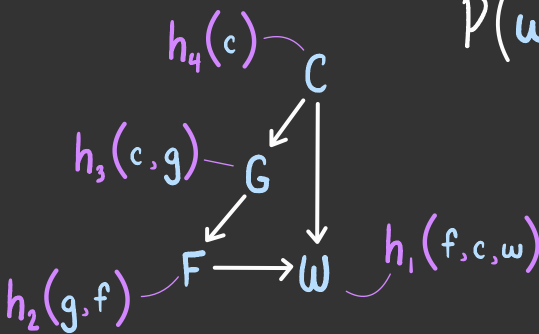
$$\begin{tikzcd} & C \arrow[dl] \arrow[dd] \\ G \arrow[d] & \\ F \arrow[r] & W \end{tikzcd}$$

f	c	w	
1	1	yes	1
1	1	no	0
1	2	yes	0
1	2	no	1
1	3	yes	0
1	3	no	1
2	1	yes	0
2	1	no	1
2	2	yes	1
2	2	no	0
2	3	yes	0
2	3	no	1
3	1	yes	0
3	1	no	1
3	2	yes	0
3	2	no	1
3	3	yes	1
3	3	no	0



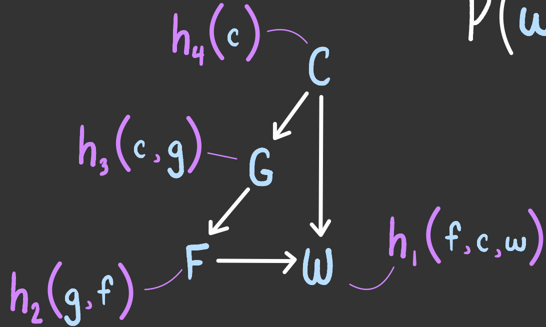
joint $P(w,f,g,c) = h_1(f,c,w)\,h_2(g,f)\,h_3(c,g)\,h_4(c)$

marginal $P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\,h_2(g,f)\,h_3(c,g)\,h_4(c)$



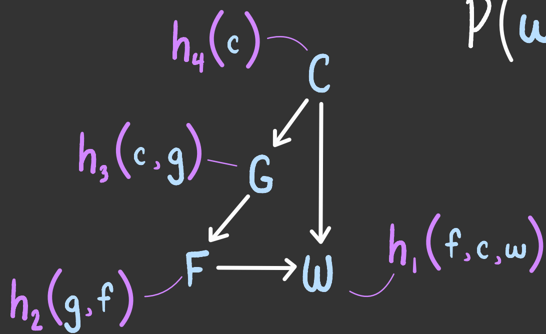
$$P(w) = \sum_c \sum_f \sum_g \underline{h_1(f,c,w)}\;\underline{h_2(g,f)}\;\underline{h_3(c,g)}\;\underline{h_4(c)}$$

these are often referred to as **factors**



$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c)$$

the algorithm we used to compute this sum of products is called variable elimination

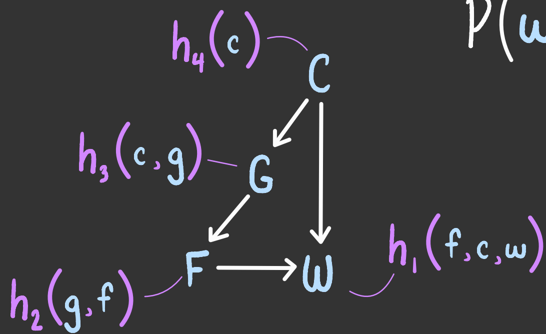


$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c)$$

first, we choose an elimination order

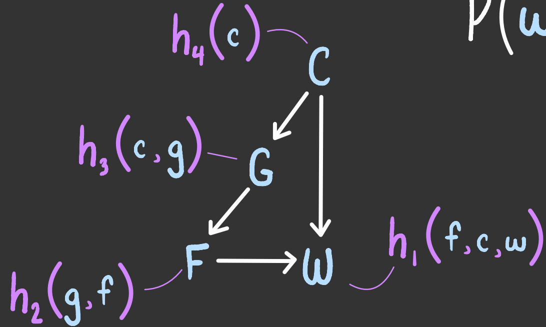
elimination order: CFG

$$P(w) = \sum_{c} \sum_{f} \sum_{g} h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c)$$



then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

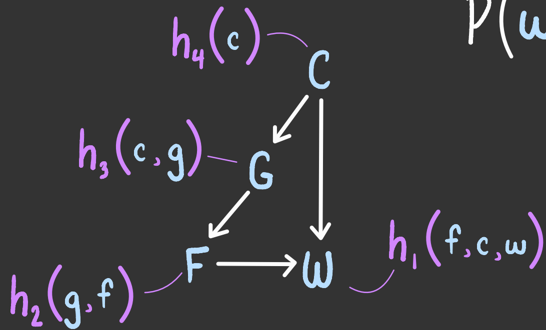
elimination order: CFG



$$P(w) = \sum_c \sum_f h_1(f,c,w)\, h_4(c) \underbrace{\boxed{\sum_g h_2(g,f)\, h_3(c,g)}}_{h_5(f,c)}$$

then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

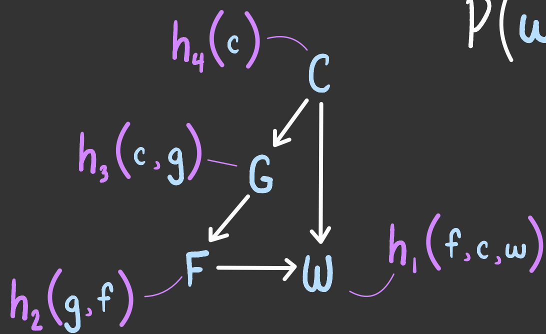
elimination order: CFG



$$P(w) = \sum_c \sum_f h_1(f,c,w)\, h_4(c)\, h_5(f,c)$$

then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

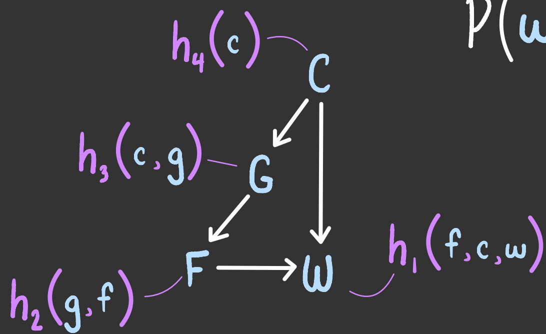
elimination order: CFG



$$P(w) = \sum_c h_4(c) \underbrace{\boxed{\sum_f h_1(f,c,w)\, h_5(f,c)}}_{h_6(c,w)}$$

then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

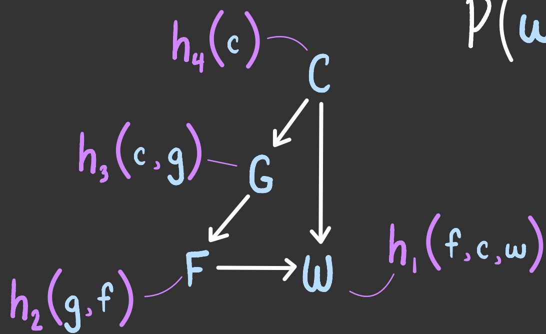
elimination order: CFG



$$P(w) = \sum_{c} h_4(c)\, h_6(c,w)$$

then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order: CFG

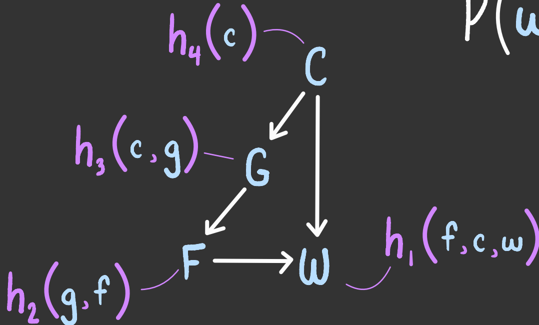


$$P(w) = \underbrace{\boxed{\sum_c h_4(c)\, h_6(c,w)}}_{h_7(w)}$$

then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

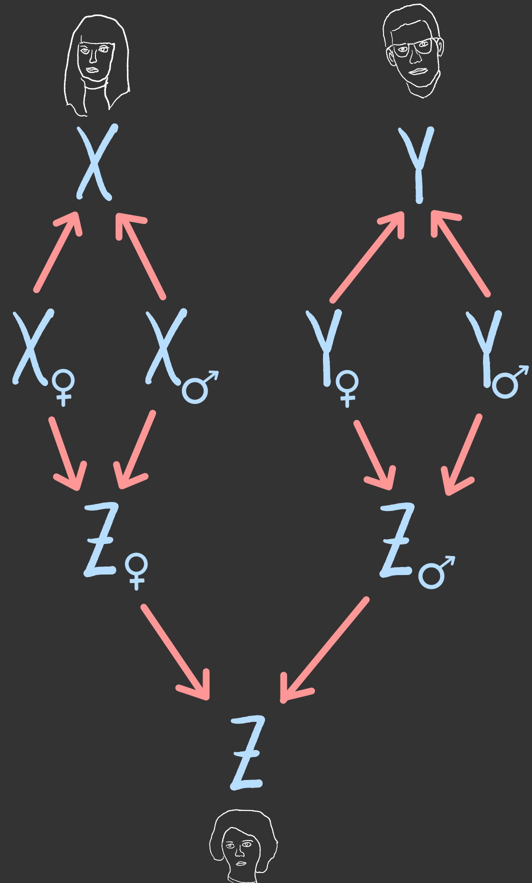
elimination order: CFG

$$P(w) = h_7(w)$$



then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order: CFG



let's try variable elimination on our **blood types** network

what is $P(X = A, Y = AB, Z = B)$?

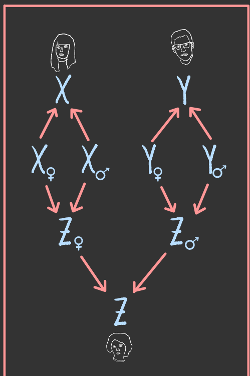
$$P(X=A,\ Y=AB,\ Z=B)$$

$$= \sum_{x_\female} \sum_{x_\male} \sum_{y_\female} \sum_{y_\male} \sum_{z_\female} \sum_{z_\male} P(X=A,\ Y=AB,\ Z=B,\ x_\female,\ x_\male,\ y_\female,\ y_\male,\ z_\female,\ z_\male)$$

$$P(X=A,\ Y=AB,\ Z=B)$$

total

$$= \sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}} P(X=A,\ Y=AB,\ Z=B,\ x_{\text{♀}},\ x_{\text{♂}},\ y_{\text{♀}},\ y_{\text{♂}},\ z_{\text{♀}},\ z_{\text{♂}})$$



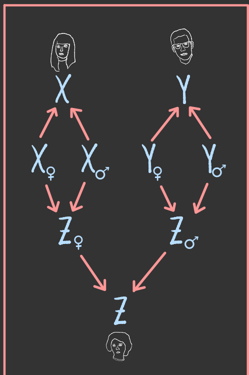
$$= \sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}} P(x_{\text{♀}})P(x_{\text{♂}})P(x \mid x_{\text{♀}}, x_{\text{♂}})P(y_{\text{♀}})P(y_{\text{♂}})P(y \mid y_{\text{♀}}, y_{\text{♂}})P(z_{\text{♀}} \mid x_{\text{♀}}, x_{\text{♂}})P(z_{\text{♂}} \mid y_{\text{♀}}, y_{\text{♂}})P(z \mid z_{\text{♀}}, z_{\text{♂}})$$

$$P(X=A,\ Y=AB,\ Z=B)$$

$$= \sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂}\sum_{z_♀}\sum_{z_♂} P(X=A,\ Y=AB,\ Z=B,\ x_♀,\ x_♂,\ y_♀,\ y_♂,\ z_♀,\ z_♂)$$

$$= \sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂}\sum_{z_♀}\sum_{z_♂} P(x_♀)P(x_♂)P(x \mid x_♀, x_♂)P(y_♀)P(y_♂)P(y \mid y_♀, y_♂)P(z_♀ \mid x_♀, x_♂)P(z_♂ \mid y_♀, y_♂)P(z \mid z_♀, z_♂)$$

$$= \sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂}\sum_{z_♀}\sum_{z_♂} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_4(y_♀)h_5(y_♂)h_6(y, y_♀, y_♂)h_7(z_♀, x_♀, x_♂)h_8(z_♂, y_♀, y_♂)h_9(z, z_♀, z_♂)$$



elimination order

$Z_♂\ Z_♀\ Y_♂\ Y_♀\ X_♂\ X_♀$

variable elimination

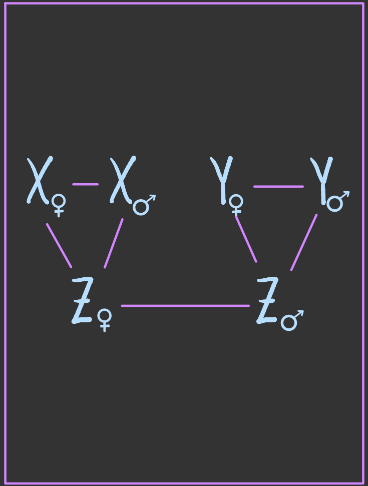
we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order

$Z_{\text{♂}}\ Z_{\text{♀}}\ Y_{\text{♂}}\ Y_{\text{♀}}\ X_{\text{♂}}\ X_{\text{♀}}$

$$\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}} h_1(x_{\text{♀}})\,h_2(x_{\text{♂}})\,h_3(x, x_{\text{♀}}, x_{\text{♂}})\,h_4(y_{\text{♀}})\,h_5(y_{\text{♂}})\,h_6(y, y_{\text{♀}}, y_{\text{♂}})\,h_7(z_{\text{♀}}, x_{\text{♀}}, x_{\text{♂}})\,h_8(z_{\text{♂}}, y_{\text{♀}}, y_{\text{♂}})\,h_9(z, z_{\text{♀}}, z_{\text{♂}})$$

two variables are adjacent if they appear in a common **factor**



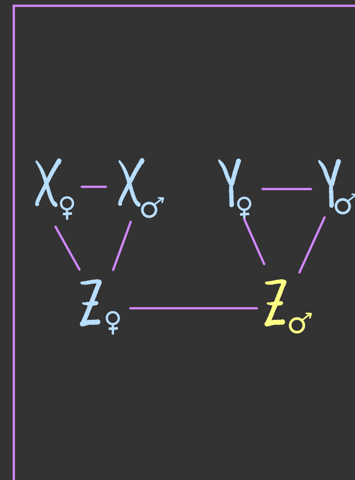
elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

$$\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}}\sum_{z_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x, x_{♀}, x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_7(z_{♀}, x_{♀}, x_{♂})h_8(z_{♂}, y_{♀}, y_{♀})h_9(z, z_{♀}, z_{♂})$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x, x_{♀}, x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_7(z_{♀}, x_{♀}, x_{♂})\sum_{z_{♂}} h_8(z_{♂}, y_{♀}, y_{♂})h_9(z, z_{♀}, z_{♂})$$

two variables are adjacent if they appear in a common **factor**



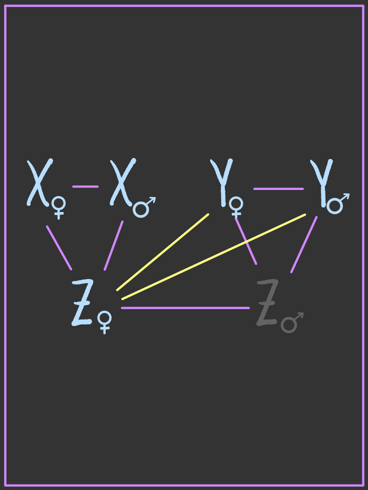
elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

$$\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}}\sum_{z_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x, x_{♀}, x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_7(z_{♀}, x_{♀}, x_{♂})h_8(z_{♂}, y_{♀}, y_{♂})h_9(z, z_{♀}, z_{♂})$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x, x_{♀}, x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_7(z_{♀}, x_{♀}, x_{♂})h_{10}(y_{♀}, y_{♂}, z, z_{♀})$$

two variables are adjacent if they appear in a common **factor**



elimination order

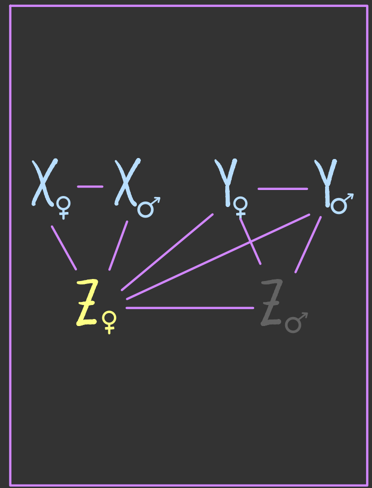
$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

$$\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}}\sum_{z_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x, x_{♀}, x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_7(z_{♀}, x_{♀}, x_{♂})h_8(z_{♂}, y_{♀}, y_{♂})h_9(z, z_{♀}, z_{♂})$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x, x_{♀}, x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_7(z_{♀}, x_{♀}, x_{♂})h_{10}(y_{♀}, y_{♂}, z, z_{♀})$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x, x_{♀}, x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})\sum_{z_{♀}} h_7(z_{♀}, x_{♀}, x_{♂})h_{10}(y_{♀}, y_{♂}, z, z_{♀})$$

two variables are adjacent if they appear in a common **factor**



elimination order

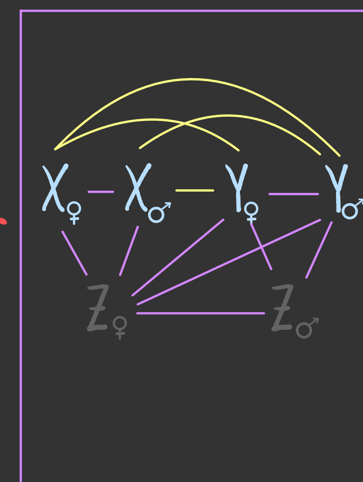
$Z_{\male}\ Z_{\female}\ Y_{\male}\ Y_{\female}\ X_{\male}\ X_{\female}$

$$\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}}\sum_{z_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$=\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_{10}(y_{\female},y_{\male},z,z_{\female})$$

$$=\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_{11}(x_{\female},x_{\male},y_{\female},y_{\male},z)$$

two variables are adjacent if they appear in a common factor



elimination order

$Z_{\male}\ Z_{\female}\ Y_{\male}\ Y_{\female}\ X_{\male}\ X_{\female}$

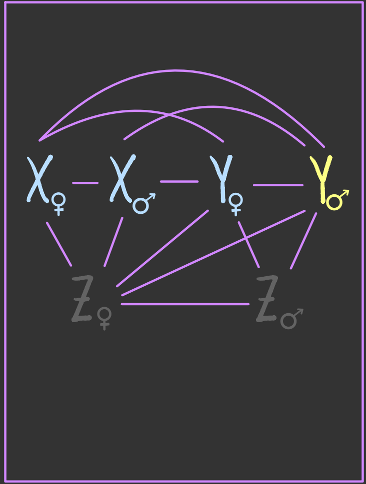
$$\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}}\sum_{z_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_{10}(y_{\female}, y_{\male}, z, z_{\female})$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_{11}(x_{\female}, x_{\male}, y_{\female}, y_{\male}, z)$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female}) \sum_{y_{\male}} h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_{11}(x_{\female}, x_{\male}, y_{\female}, y_{\male}, z)$$

two variables are adjacent if they appear in a common **factor**



elimination order

$Z_{\male}\ Z_{\female}\ Y_{\male}\ Y_{\female}\ X_{\male}\ X_{\female}$

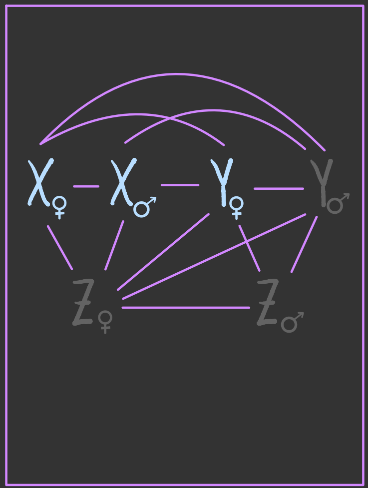
$$\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}}\sum_{z_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_{10}(y_{\female}, y_{\male}, z, z_{\female})$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_{11}(x_{\female}, x_{\male}, y_{\female}, y_{\male}, z)$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_{12}(x_{\female}, x_{\male}, y, y_{\female}, z)$$

two variables are adjacent if they appear in a common **factor**



elimination order

$Z_{\male}\ Z_{\female}\ Y_{\male}\ Y_{\female}\ X_{\male}\ X_{\female}$

$$\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}}\sum_{z_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

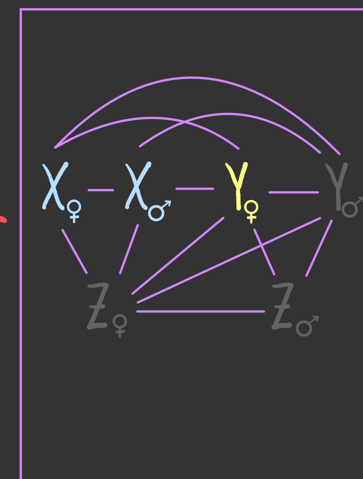
$$=\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_{10}(y_{\female},y_{\male},z,z_{\female})$$

$$=\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_{11}(x_{\female},x_{\male},y_{\female},y_{\male},z)$$

$$=\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_{12}(x_{\female},x_{\male},y,y_{\female},z)$$

$$=\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})\sum_{y_{\female}} h_4(y_{\female})h_{12}(x_{\female},x_{\male},y,y_{\female},z)$$

two variables are adjacent if they appear in a common factor



elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

$$\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}}\sum_{z_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_8(z_{♂},y_{♀},y_{♂})h_9(z,z_{♀},z_{♂})$$

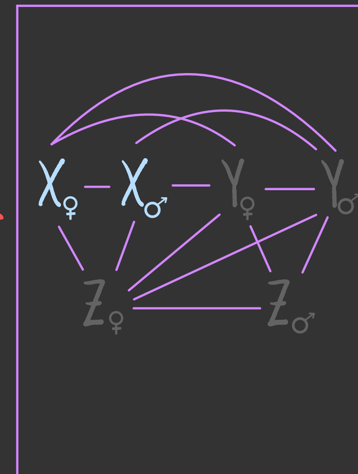
$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_{10}(y_{♀},y_{♂},z,z_{♀})$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_{11}(x_{♀},x_{♂},y_{♀},y_{♂},z)$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_{12}(x_{♀},x_{♂},y_{♀},z)$$

$$= \sum_{x_{♀}}\sum_{x_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_{13}(x_{♀},x_{♂},y,z)$$

two variables are adjacent if they appear in a common **factor**



elimination order

$Z_{\male}$ $Z_{\female}$ $Y_{\male}$ $Y_{\female}$ $X_{\male}$ $X_{\female}$

$$\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}}\sum_{z_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_{10}(y_{\female}, y_{\male}, z, z_{\female})$$

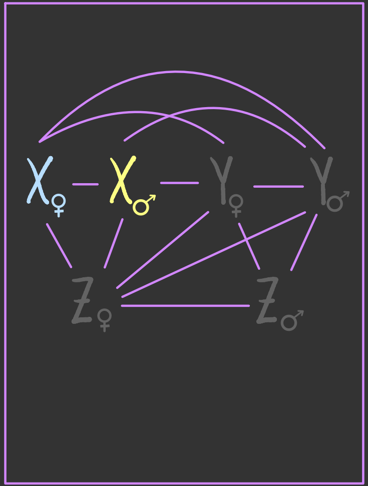
$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_{11}(x_{\female}, x_{\male}, y_{\female}, y_{\male}, z)$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_{12}(x_{\female}, x_{\male}, y_{\female}, z)$$

$$= \sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_{13}(x_{\female}, x_{\male}, y, z)$$

$$= \sum_{x_{\female}} h_1(x_{\female})\sum_{x_{\male}} h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_{13}(x_{\female}, x_{\male}, y, z)$$

two variables are adjacent if they appear in a common **factor**



elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

$$\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}}\sum_{z_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_8(z_{♂},y_{♀},y_{♂})h_9(z,z_{♀},z_{♂})$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_{10}(y_{♀},y_{♂},z,z_{♀})$$

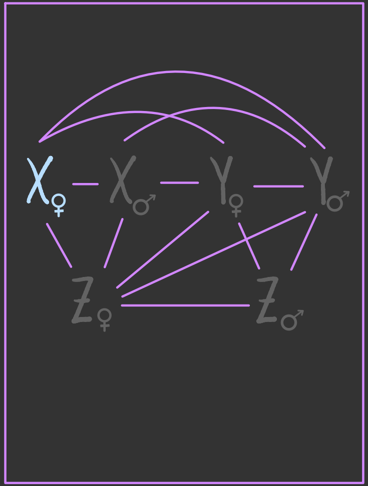
$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_{11}(x_{♀},x_{♂},y_{♀},y_{♂},z)$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_{12}(x_{♀},x_{♂},y_{♀},z)$$

$$= \sum_{x_{♀}}\sum_{x_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_{13}(x_{♀},x_{♂},y,z)$$

$$= \sum_{x_{♀}} h_1(x_{♀})h_{14}(x_{♀},x,y,z)$$

two variables are adjacent if they appear in a common **factor**



elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

$$\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}}\sum_{z_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_8(z_{♂},y_{♀},y_{♂})h_9(z,z_{♀},z_{♂})$$

$$=\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_{10}(y_{♀},y_{♂},z,z_{♀})$$

$$=\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_{11}(x_{♀},x_{♂},y_{♀},y_{♂},z)$$

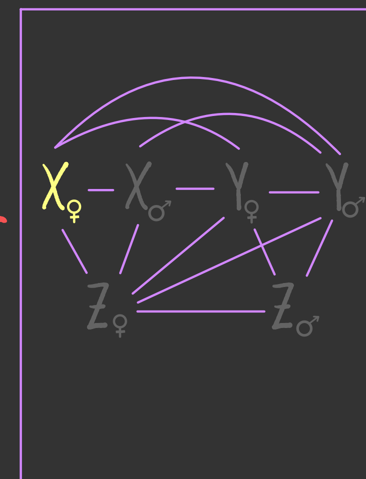
$$=\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_{12}(x_{♀},x_{♂},y_{♀},z)$$

$$=\sum_{x_{♀}}\sum_{x_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_{13}(x_{♀},x_{♂},y,z)$$

$$=\sum_{x_{♀}} h_1(x_{♀})h_{14}(x_{♀},x,y,z)$$

$$=\sum_{x_{♀}} h_1(x_{♀})h_{14}(x_{♀},x,y,z)$$

two variables are adjacent if they appear in a common factor



elimination order

$Z_{\text{♂}}\ Z_{\text{♀}}\ Y_{\text{♂}}\ Y_{\text{♀}}\ X_{\text{♂}}\ X_{\text{♀}}$

$$\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}} h_1(x_{\text{♀}})h_2(x_{\text{♂}})h_3(x,x_{\text{♀}},x_{\text{♂}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y,y_{\text{♀}},y_{\text{♂}})h_7(z_{\text{♀}},x_{\text{♀}},x_{\text{♂}})h_8(z_{\text{♂}},y_{\text{♀}},y_{\text{♂}})h_9(z,z_{\text{♀}},z_{\text{♂}})$$

$$=\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{z_{\text{♀}}} h_1(x_{\text{♀}})h_2(x_{\text{♂}})h_3(x,x_{\text{♀}},x_{\text{♂}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y,y_{\text{♀}},y_{\text{♂}})h_7(z_{\text{♀}},x_{\text{♀}},x_{\text{♂}})h_{10}(y_{\text{♀}},y_{\text{♂}},z,z_{\text{♀}})$$

$$=\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}} h_1(x_{\text{♀}})h_2(x_{\text{♂}})h_3(x,x_{\text{♀}},x_{\text{♂}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y,y_{\text{♀}},y_{\text{♂}})h_{11}(x_{\text{♀}},x_{\text{♂}},y_{\text{♀}},y_{\text{♂}},z)$$

$$=\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}}\sum_{y_{\text{♀}}} h_1(x_{\text{♀}})h_2(x_{\text{♂}})h_3(x,x_{\text{♀}},x_{\text{♂}})h_4(y_{\text{♀}})h_{12}(x_{\text{♀}},x_{\text{♂}},y_{\text{♀}},z)$$

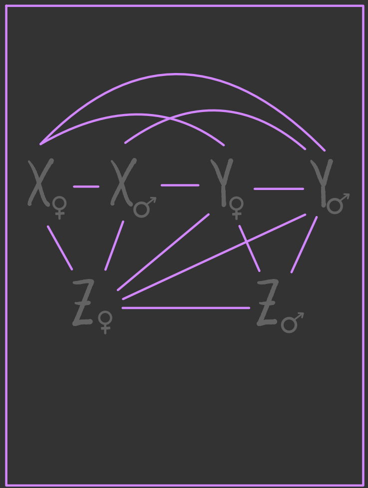
$$=\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}} h_1(x_{\text{♀}})h_2(x_{\text{♂}})h_3(x,x_{\text{♀}},x_{\text{♂}})h_{13}(x_{\text{♀}},x_{\text{♂}},y,z)$$

$$=\sum_{x_{\text{♀}}} h_1(x_{\text{♀}})h_{14}(x_{\text{♀}},x,y,z)$$

$$=h_{15}(x,y,z)$$

$$=P(x,y,z)$$

two variables are adjacent if they appear in a common **factor**



elimination order

$Z_{\female}\, Z_{\male}$ …

$$\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}}\sum_{z_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_{10}(y_{\female},y_{\male},z,z_{\female})$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_{11}(x_{\female},x_{\male},y_{\female},y_{\male},z)$$

$$= \sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_{12}(x_{\female},x_{\male},y_{\female},z)$$

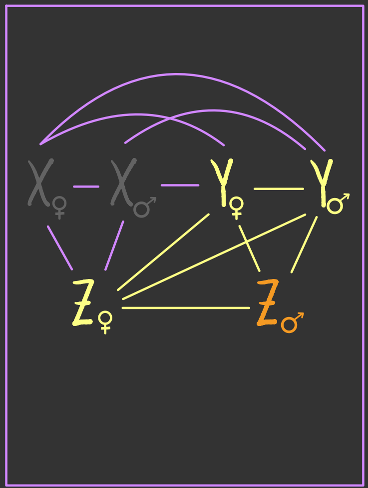
$$= \sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_{13}(x_{\female},x_{\male},y,z)$$

$$= \sum_{x_{\female}} h_1(x_{\female})h_{14}(x_{\female},x,y,z)$$

$$= h_{15}(x,y,z)$$

$$= P(x,y,z)$$

the cliques in this graph correspond to the factors we create



elimination order
$Z_\male \; Z_\female \; Y_\male \; Y_\female \; X_\male \; X_\female$

$$\sum_{x_\female}\sum_{x_\male}\sum_{y_\female}\sum_{y_\male}\sum_{z_\female}\sum_{z_\male} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_4(y_\female)h_5(y_\male)h_6(y,y_\female,y_\male)h_7(z_\female,x_\female,x_\male)h_8(z_\male,y_\female,y_\male)h_9(z,z_\female,z_\male)$$

$$=\sum_{x_\female}\sum_{x_\male}\sum_{y_\female}\sum_{y_\male}\sum_{z_\female} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_4(y_\female)h_5(y_\male)h_6(y,y_\female,y_\male)h_7(z_\female,x_\female,x_\male)h_{10}(y_\female,y_\male,z,z_\female)$$

$$=\sum_{x_\female}\sum_{x_\male}\sum_{y_\female}\sum_{y_\male} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_4(y_\female)h_5(y_\male)h_6(y,y_\female,y_\male)h_{11}(x_\female,x_\male,y_\female,y_\male,z)$$

$$=\sum_{x_\female}\sum_{x_\male}\sum_{y_\female} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_4(y_\female)h_{12}(x_\female,x_\male,y_\female,z)$$

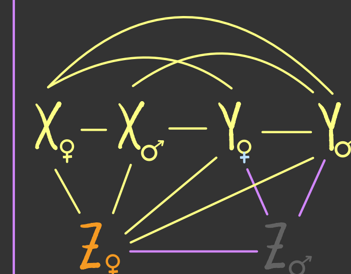
$$=\sum_{x_\female}\sum_{x_\male} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_{13}(x_\female,x_\male,y,z)$$

$$=\sum_{x_\female} h_1(x_\female)h_{14}(x_\female,x,y,z)$$

$$=h_{15}(x,y,z)$$

$$=P(x,y,z)$$

the cliques in this graph correspond to the factors we create



elimination order

$Z_♂\ Z_♀\ Y_♂\ Y_♀\ X_♂\ X_♀$

$$\sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂}\sum_{z_♀}\sum_{z_♂} h_1(x_♀)h_2(x_♂)h_3(x,x_♀,x_♂)h_4(y_♀)h_5(y_♂)h_6(y,y_♀,y_♂)h_7(z_♀,x_♀,x_♂)h_8(z_♂,y_♀,y_♂)h_9(z,z_♀,z_♂)$$

$$=\sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂}\sum_{z_♀} h_1(x_♀)h_2(x_♂)h_3(x,x_♀,x_♂)h_4(y_♀)h_5(y_♂)h_6(y,y_♀,y_♂)h_7(z_♀,x_♀,x_♂)h_{10}(y_♀,y_♂,z,z_♀)$$

$$=\sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂} h_1(x_♀)h_2(x_♂)h_3(x,x_♀,x_♂)h_4(y_♀)h_5(y_♂)h_6(y,y_♀,y_♂)\,h_{11}(x_♀,x_♂,y_♀,y_♂,z)$$

$$=\sum_{x_♀}\sum_{x_♂}\sum_{y_♀} h_1(x_♀)h_2(x_♂)h_3(x,x_♀,x_♂)h_4(y_♀)\,h_{12}(x_♀,x_♂,y_♀,z)$$

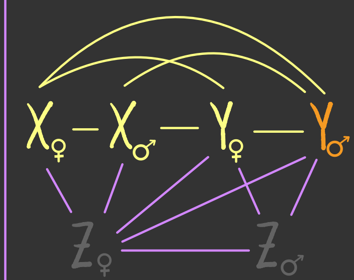
$$=\sum_{x_♀}\sum_{x_♂} h_1(x_♀)h_2(x_♂)h_3(x,x_♀,x_♂)\,h_{13}(x_♀,x_♂,y,z)$$

$$=\sum_{x_♀} h_1(x_♀)h_{14}(x_♀,x,y,z)$$

$$=h_{15}(x,y,z)$$

$$=P(x,y,z)$$

the cliques in this graph correspond to the factors we create



elimination order
$Z_\male\ Z_\female\ Y_\male\ Y_\female\ X_\male\ X_\female$

$$\sum_{x_\female}\sum_{x_\male}\sum_{y_\female}\sum_{y_\male}\sum_{z_\female}\sum_{z_\male} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_4(y_\female)h_5(y_\male)h_6(y,y_\female,y_\male)h_7(z_\female,x_\female,x_\male)h_8(z_\male,y_\female,y_\male)h_9(z,z_\female,z_\male)$$

$$= \sum_{x_\female}\sum_{x_\male}\sum_{y_\female}\sum_{y_\male}\sum_{z_\female} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_4(y_\female)h_5(y_\male)h_6(y,y_\female,y_\male)h_7(z_\female,x_\female,x_\male)h_{10}(y_\female,y_\male,z,z_\female)$$

$$= \sum_{x_\female}\sum_{x_\male}\sum_{y_\female}\sum_{y_\male} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_4(y_\female)h_5(y_\male)h_6(y,y_\female,y_\male)h_{11}(x_\female,x_\male,y_\female,y_\male,z)$$

$$= \sum_{x_\female}\sum_{x_\male}\sum_{y_\female} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_4(y_\female)h_{12}(x_\female,x_\male,y_\female,z)$$

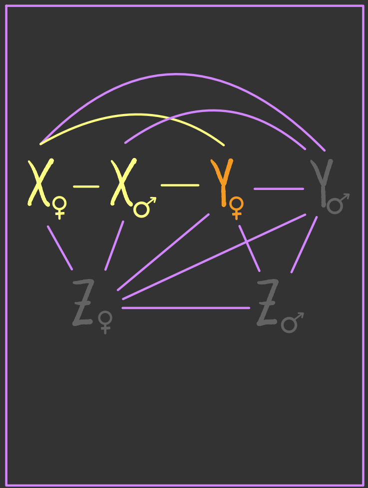
$$= \sum_{x_\female}\sum_{x_\male} h_1(x_\female)h_2(x_\male)h_3(x,x_\female,x_\male)h_{13}(x_\female,x_\male,y,z)$$

$$= \sum_{x_\female} h_1(x_\female)h_{14}(x_\female,x,y,z)$$

$$= h_{15}(x,y,z)$$

$$= P(x,y,z)$$

the cliques in this graph correspond to the factors we create



elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

$$\sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}}\sum_{z_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_8(z_{♂},y_{♀},y_{♂})h_9(z,z_{♀},z_{♂})$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{z_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_{10}(y_{♀},y_{♂},z,z_{♀})$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}}\sum_{y_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_{11}(x_{♀},x_{♂},y_{♀},y_{♂},z)$$

$$= \sum_{x_{♀}}\sum_{x_{♂}}\sum_{y_{♀}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_{12}(x_{♀},x_{♂},y_{♀},z)$$

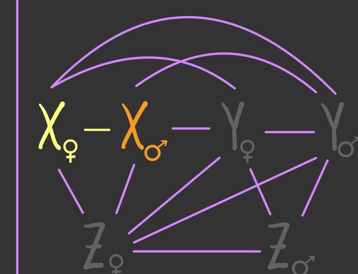
$$= \sum_{x_{♀}}\sum_{x_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_{13}(x_{♀},x_{♂},y,z)$$

$$= \sum_{x_{♀}} h_1(x_{♀})h_{14}(x_{♀},x,y,z)$$

$$= h_{15}(x,y,z)$$

$$= P(x,y,z)$$

the cliques in this graph correspond to the factors we create



elimination order

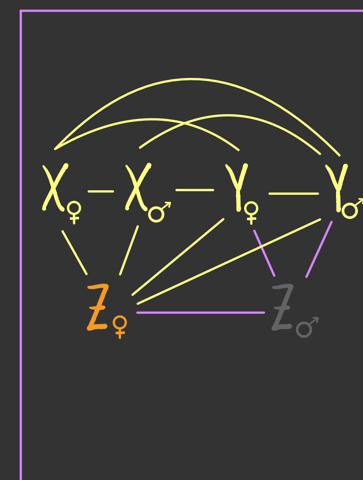
$Z_{\male}\ Z_{\female}\ Y_{\male}\ Y_{\female}\ X_{\male}\ X_{\female}$

$$\sum_{x_{\female}} \sum_{x_{\male}} \sum_{y_{\female}} \sum_{y_{\male}} \sum_{z_{\female}} \sum_{z_{\male}} h_1(x_{\female}) h_2(x_{\male}) h_3(x, x_{\female}, x_{\male}) h_4(y_{\female}) h_5(y_{\male}) h_6(y, y_{\female}, y_{\male}) h_7(z_{\female}, x_{\female}, x_{\male}) h_8(z_{\male}, y_{\female}, y_{\male}) h_9(z, z_{\female}, z_{\male})$$

$$= \sum_{x_{\female}} \sum_{x_{\male}} \sum_{y_{\female}} \sum_{y_{\male}} \sum_{z_{\female}} h_1(x_{\female}) h_2(x_{\male}) h_3(x, x_{\female}, x_{\male}) h_4(y_{\female}) h_5(y_{\male}) h_6(y, y_{\female}, y_{\male}) h_7(z_{\female}, x_{\female}, x_{\male}) h_{10}(y_{\female}, y_{\male}, z, z_{\female})$$

$$= \sum_{x_{\female}} \sum_{x_{\male}} \sum_{y_{\female}} \sum_{y_{\male}} h_1(x_{\female}) h_2(x_{\male}) h_3(x, x_{\female}, x_{\male}) h_4(y_{\female}) h_5(y_{\male}) h_6(y, y_{\female}, y_{\male}) h_{11}(x_{\female}, x_{\male}, y_{\female}, y_{\male}, z)$$

$$\boxed{h_{11}(x_{\female}, x_{\male}, y_{\female}, y_{\male}, z) = \sum_{z_{\female}} h_7(z_{\female}, x_{\female}, x_{\male}) h_{10}(y_{\female}, y_{\male}, z, z_{\female})}$$



this was the "hardest" factor to compute

elimination order

$Z_{\male}\ Z_{\female}\ Y_{\male}\ Y_{\female}\ X_{\male}\ X_{\female}$

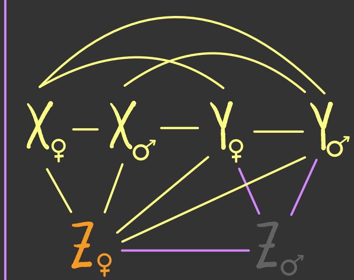
$$\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}}\sum_{z_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$=\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{z_{\female}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_{10}(y_{\female},y_{\male},z,z_{\female})$$

$$=\sum_{x_{\female}}\sum_{x_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})\underline{h_{11}(x_{\female},x_{\male},y_{\female},y_{\male},z)}$$

$$h_{11}(x_{\female},x_{\male},y_{\female},y_{\male},z) = \sum_{z_{\female}} h_7(z_{\female},x_{\female},x_{\male})h_{10}(y_{\female},y_{\male},z,z_{\female})$$

3 possible values



how many terms did we need to compute?

elimination order

$Z_{\male}\ Z_{\female}\ Y_{\male}\ Y_{\female}\ X_{\male}\ X_{\female}$

$$\sum_{x_{\female}} \sum_{x_{\male}} \sum_{y_{\female}} \sum_{y_{\male}} \sum_{z_{\female}} \sum_{z_{\male}} h_1(x_{\female}) h_2(x_{\male}) h_3(x, x_{\female}, x_{\male}) h_4(y_{\female}) h_5(y_{\male}) h_6(y, y_{\female}, y_{\male}) h_7(z_{\female}, x_{\female}, x_{\male}) h_8(z_{\male}, y_{\female}, y_{\male}) h_9(z, z_{\female}, z_{\male})$$

$$= \sum_{x_{\female}} \sum_{x_{\male}} \sum_{y_{\female}} \sum_{y_{\male}} \sum_{z_{\female}} h_1(x_{\female}) h_2(x_{\male}) h_3(x, x_{\female}, x_{\male}) h_4(y_{\female}) h_5(y_{\male}) h_6(y, y_{\female}, y_{\male}) h_7(z_{\female}, x_{\female}, x_{\male}) h_{10}(y_{\female}, y_{\male}, z, z_{\female})$$

$$= \sum_{x_{\female}} \sum_{x_{\male}} \sum_{y_{\female}} \sum_{y_{\male}} h_1(x_{\female}) h_2(x_{\male}) h_3(x, x_{\female}, x_{\male}) h_4(y_{\female}) h_5(y_{\male}) h_6(y, y_{\female}, y_{\male}) \underline{h_{11}(x_{\female}, x_{\male}, y_{\female}, y_{\male}, z)}$$

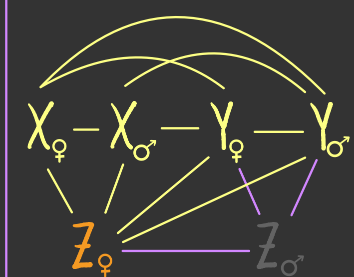
3^4 sums

a sum of 3 terms

$$\boxed{h_{11}(\overline{x_{\female}, x_{\male}, y_{\female}, y_{\male}}, z) = \sum_{z_{\female}} h_7(z_{\female}, x_{\female}, x_{\male})\, h_{10}(y_{\female}, y_{\male}, z, z_{\female})}$$

3 possible values

we needed to compute 3^5 terms



what if we use a

different

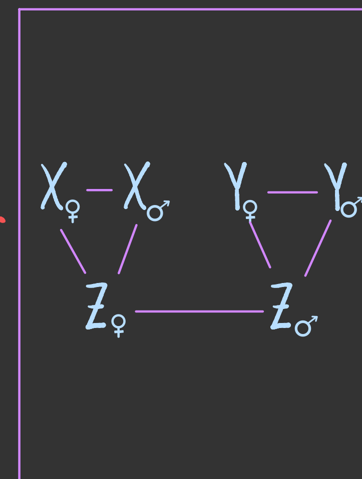
elimination order?

elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$

$$\sum_{z_{♀}} \sum_{z_{♂}} \sum_{y_{♀}} \sum_{y_{♂}} \sum_{x_{♀}} \sum_{x_{♂}} h_1(x_{♀})\, h_2(x_{♂})\, h_3(x, x_{♀}, x_{♂})\, h_4(y_{♀})\, h_5(y_{♂})\, h_6(y, y_{♀}, y_{♂})\, h_7(z_{♀}, x_{♀}, x_{♂})\, h_8(z_{♂}, y_{♀}, y_{♂})\, h_9(z, z_{♀}, z_{♂})$$

two variables are adjacent if they appear in a common **factor**



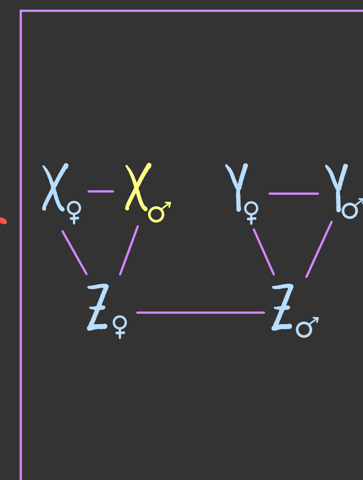
elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})\sum_{x_{\male}} h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_7(z_{\female},x_{\female},x_{\male})$$

two variables are adjacent if they appear in a common factor



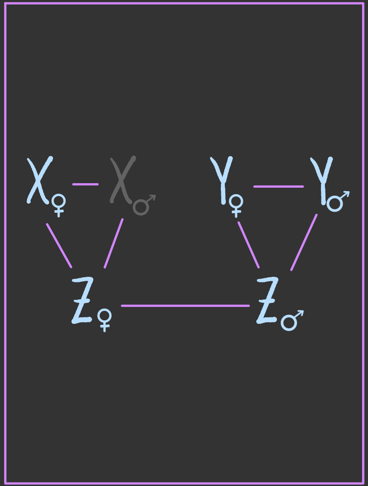
elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$

$$\sum_{z_{♀}} \sum_{z_{♂}} \sum_{y_{♀}} \sum_{y_{♂}} \sum_{x_{♀}} \sum_{x_{♂}} h_1(x_{♀})\, h_2(x_{♂})\, h_3(x, x_{♀}, x_{♂})\, h_4(y_{♀})\, h_5(y_{♂})\, h_6(y, y_{♀}, y_{♂})\, h_7(z_{♀}, x_{♀}, x_{♂})\, h_8(z_{♂}, y_{♀}, y_{♂})\, h_9(z, z_{♀}, z_{♂})$$

$$= \sum_{z_{♀}} \sum_{z_{♂}} \sum_{y_{♀}} \sum_{y_{♂}} \sum_{x_{♀}} h_1(x_{♀})\, h_4(y_{♀})\, h_5(y_{♂})\, h_6(y, y_{♀}, y_{♂})\, h_8(z_{♂}, y_{♀}, y_{♂})\, h_9(z, z_{♀}, z_{♂})\, h_{10}(x, x_{♀}, z_{♀})$$

two variables are adjacent if they appear in a common **factor**



elimination order

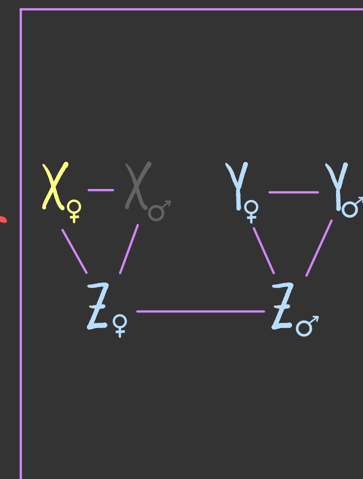
$X_{\text{♂}}\ X_{\text{♀}}\ Y_{\text{♂}}\ Y_{\text{♀}}\ Z_{\text{♂}}\ Z_{\text{♀}}$

$$\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}} h_1(x_{\text{♀}})\,h_2(x_{\text{♂}})\,h_3(x, x_{\text{♀}}, x_{\text{♂}})\,h_4(y_{\text{♀}})\,h_5(y_{\text{♂}})\,h_6(y, y_{\text{♀}}, y_{\text{♂}})\,h_7(z_{\text{♀}}, x_{\text{♀}}, x_{\text{♂}})\,h_8(z_{\text{♂}}, y_{\text{♀}}, y_{\text{♂}})\,h_9(z, z_{\text{♀}}, z_{\text{♂}})$$

$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{x_{\text{♀}}} h_1(x_{\text{♀}})\,h_4(y_{\text{♀}})\,h_5(y_{\text{♂}})\,h_6(y, y_{\text{♀}}, y_{\text{♂}})\,h_8(z_{\text{♂}}, y_{\text{♀}}, y_{\text{♂}})\,h_9(z, z_{\text{♀}}, z_{\text{♂}})\,h_{10}(x, x_{\text{♀}}, z_{\text{♀}})$$

$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}} h_4(y_{\text{♀}})\,h_5(y_{\text{♂}})\,h_6(y, y_{\text{♀}}, y_{\text{♂}})\,h_8(z_{\text{♂}}, y_{\text{♀}}, y_{\text{♂}})\,h_9(z, z_{\text{♀}}, z_{\text{♂}}) \sum_{x_{\text{♀}}} h_1(x_{\text{♀}})\,h_{10}(x, x_{\text{♀}}, z_{\text{♀}})$$

two variables are adjacent if they appear in a common **factor**



elimination order

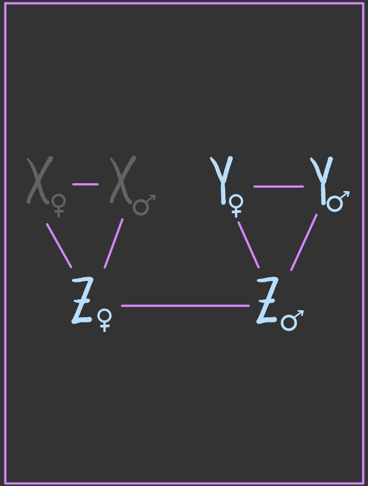
$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{10}(x,x_{\female},z_{\female})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})$$

two variables are adjacent if they appear in a common factor



elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

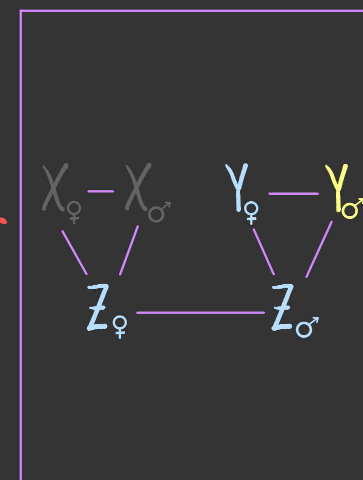
$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{10}(x,x_{\female},z_{\female})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}} h_4(y_{\female})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})\sum_{y_{\male}} h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})$$

two variables are adjacent if they appear in a common **factor**



elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

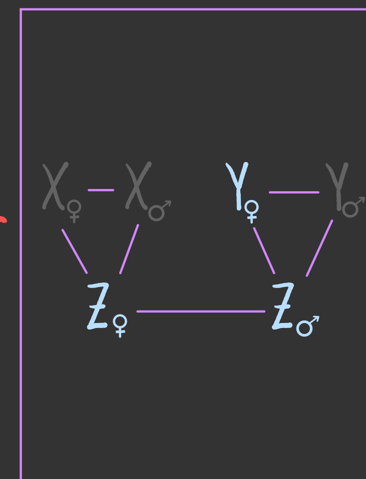
$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})h_{10}(x, x_{\female}, z_{\female})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})h_{11}(x, z_{\female})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}} h_4(y_{\female})h_9(z, z_{\female}, z_{\male})h_{11}(x, z_{\female})h_{12}(y, y_{\female}, z_{\male})$$

two variables are adjacent if they appear in a common **factor**



elimination order

$X_\male\ X_\female\ Y_\male\ Y_\female\ Z_\male\ Z_\female$

$$\sum_{z_\female}\sum_{z_\male}\sum_{y_\female}\sum_{y_\male}\sum_{x_\female}\sum_{x_\male} h_1(x_\female)h_2(x_\male)h_3(x, x_\female, x_\male)h_4(y_\female)h_5(y_\male)h_6(y, y_\female, y_\male)h_7(z_\female, x_\female, x_\male)h_8(z_\male, y_\female, y_\male)h_9(z, z_\female, z_\male)$$

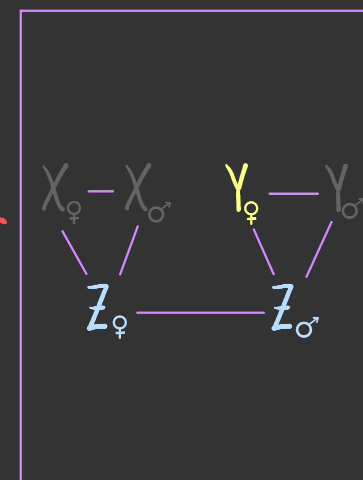
$$= \sum_{z_\female}\sum_{z_\male}\sum_{y_\female}\sum_{y_\male}\sum_{x_\female} h_1(x_\female)h_4(y_\female)h_5(y_\male)h_6(y, y_\female, y_\male)h_8(z_\male, y_\female, y_\male)h_9(z, z_\female, z_\male)h_{10}(x, x_\female, z_\female)$$

$$= \sum_{z_\female}\sum_{z_\male}\sum_{y_\female}\sum_{y_\male} h_4(y_\female)h_5(y_\male)h_6(y, y_\female, y_\male)h_8(z_\male, y_\female, y_\male)h_9(z, z_\female, z_\male)h_{11}(x, z_\female)$$

$$= \sum_{z_\female}\sum_{z_\male}\sum_{y_\female} h_4(y_\female)h_9(z, z_\female, z_\male)h_{11}(x, z_\female)h_{12}(y, y_\female, z_\male)$$

$$= \sum_{z_\female}\sum_{z_\male} h_9(z, z_\female, z_\male)h_{11}(x, z_\female)\sum_{y_\female} h_4(y_\female)h_{12}(y, y_\female, z_\male)$$

two variables are adjacent if they appear in a common **factor**



elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})$$

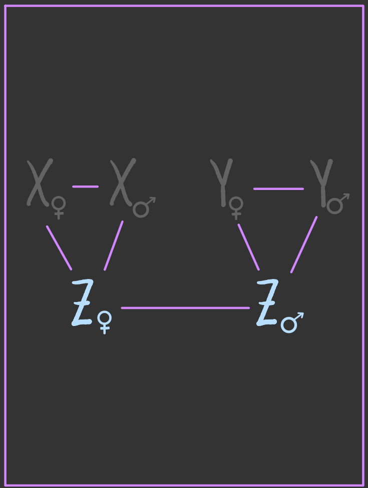
$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})h_{10}(x, x_{\female}, z_{\female})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})h_{11}(x, z_{\female})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}} h_4(y_{\female})h_9(z, z_{\female}, z_{\male})h_{11}(x, z_{\female})h_{12}(y, y_{\female}, z_{\male})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}} h_9(z, z_{\female}, z_{\male})h_{11}(x, z_{\female})h_{13}(y, z_{\male})$$

two variables are adjacent if they appear in a common **factor**



elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x, x_{\female}, x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_7(z_{\female}, x_{\female}, x_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})h_{10}(x, x_{\female}, z_{\female})$$

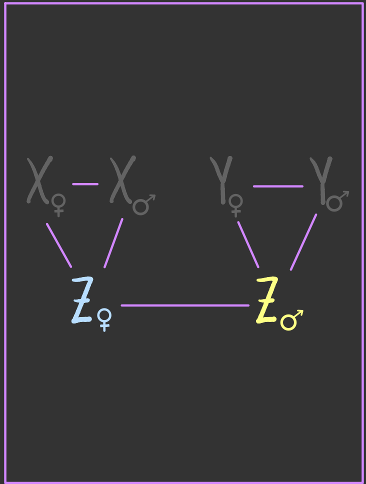
$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_4(y_{\female})h_5(y_{\male})h_6(y, y_{\female}, y_{\male})h_8(z_{\male}, y_{\female}, y_{\male})h_9(z, z_{\female}, z_{\male})h_{11}(x, z_{\female})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}} h_4(y_{\female})h_9(z, z_{\female}, z_{\male})h_{11}(x, z_{\female})h_{12}(y, y_{\female}, z_{\male})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}} h_9(z, z_{\female}, z_{\male})h_{11}(x, z_{\female})h_{13}(y, z_{\male})$$

$$= \sum_{z_{\female}} h_{11}(x, z_{\female}) \sum_{z_{\male}} h_9(z, z_{\female}, z_{\male})h_{13}(y, z_{\male})$$

two variables are adjacent if they appear in a common **factor**



elimination order

$X_{\text{♂}}\ X_{\text{♀}}\ Y_{\text{♂}}\ Y_{\text{♀}}\ Z_{\text{♂}}\ Z_{\text{♀}}$

$$\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}} h_1(x_{\text{♀}})h_2(x_{\text{♂}})h_3(x, x_{\text{♀}}, x_{\text{♂}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y, y_{\text{♀}}, y_{\text{♂}})h_7(z_{\text{♀}}, x_{\text{♀}}, x_{\text{♂}})h_8(z_{\text{♂}}, y_{\text{♀}}, y_{\text{♂}})h_9(z, z_{\text{♀}}, z_{\text{♂}})$$

$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{x_{\text{♀}}} h_1(x_{\text{♀}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y, y_{\text{♀}}, y_{\text{♂}})h_8(z_{\text{♂}}, y_{\text{♀}}, y_{\text{♂}})h_9(z, z_{\text{♀}}, z_{\text{♂}})h_{10}(x, x_{\text{♀}}, z_{\text{♀}})$$

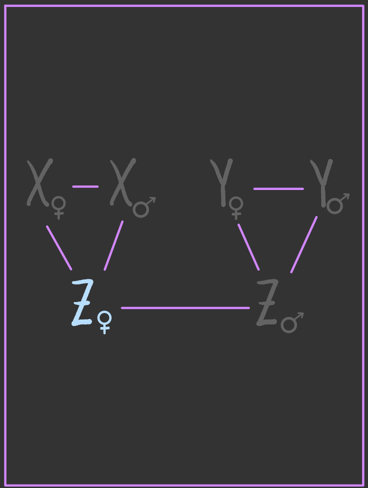
$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}} h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y, y_{\text{♀}}, y_{\text{♂}})h_8(z_{\text{♂}}, y_{\text{♀}}, y_{\text{♂}})h_9(z, z_{\text{♀}}, z_{\text{♂}})h_{11}(x, z_{\text{♀}})$$

$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}} h_4(y_{\text{♀}})h_9(z, z_{\text{♀}}, z_{\text{♂}})h_{11}(x, z_{\text{♀}})h_{12}(y, y_{\text{♀}}, z_{\text{♂}})$$

$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}} h_9(z, z_{\text{♀}}, z_{\text{♂}})h_{11}(x, z_{\text{♀}})h_{13}(y, z_{\text{♂}})$$

$$= \sum_{z_{\text{♀}}} h_{11}(x, z_{\text{♀}})h_{14}(y, z, z_{\text{♀}})$$

two variables are adjacent if they appear in a common **factor**



elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{10}(x,x_{\female},z_{\female})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})$$

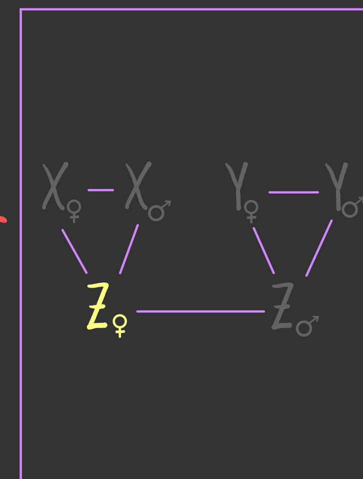
$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}} h_4(y_{\female})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})h_{12}(y,y_{\female},z_{\male})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}} h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})h_{13}(y,z_{\male})$$

$$=\sum_{z_{\female}} h_{11}(x,z_{\female})h_{14}(y,z,z_{\female})$$

$$=\sum_{z_{\female}} h_{11}(x,z_{\female})h_{14}(y,z,z_{\female})$$

two variables are adjacent if they appear in a common **factor**



elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$

$$\sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{x_{♀}}\sum_{x_{♂}} h_1(x_{♀})\,h_2(x_{♂})\,h_3(x, x_{♀}, x_{♂})\,h_4(y_{♀})\,h_5(y_{♂})\,h_6(y, y_{♀}, y_{♂})\,h_7(z_{♀}, x_{♀}, x_{♂})\,h_8(z_{♂}, y_{♀}, y_{♂})\,h_9(z, z_{♀}, z_{♂})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{x_{♀}} h_1(x_{♀})\,h_4(y_{♀})\,h_5(y_{♂})\,h_6(y, y_{♀}, y_{♂})\,h_8(z_{♂}, y_{♀}, y_{♂})\,h_9(z, z_{♀}, z_{♂})\,h_{10}(x, x_{♀}, z_{♀})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}} h_4(y_{♀})\,h_5(y_{♂})\,h_6(y, y_{♀}, y_{♂})\,h_8(z_{♂}, y_{♀}, y_{♂})\,h_9(z, z_{♀}, z_{♂})\,h_{11}(x, z_{♀})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}} h_4(y_{♀})\,h_9(z, z_{♀}, z_{♂})\,h_{11}(x, z_{♀})\,h_{12}(y, y_{♀}, z_{♂})$$

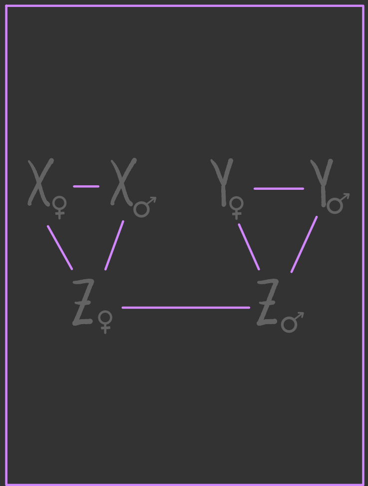
$$= \sum_{z_{♀}}\sum_{z_{♂}} h_9(z, z_{♀}, z_{♂})\,h_{11}(x, z_{♀})\,h_{13}(y, z_{♂})$$

$$= \sum_{z_{♀}} h_{11}(x, z_{♀})\,h_{14}(y, z, z_{♀})$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

two variables are adjacent if they appear in a common **factor**



elimination order

$X_{\text{♂}}\ X_{\text{♀}}\ Y_{\text{♂}}\ Y_{\text{♀}}\ Z_{\text{♂}}\ Z_{\text{♀}}$

$$\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}} h_1(x_{\text{♀}})h_2(x_{\text{♂}})h_3(x,x_{\text{♀}},x_{\text{♂}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y,y_{\text{♀}},y_{\text{♂}})h_7(z_{\text{♀}},x_{\text{♀}},x_{\text{♂}})h_8(z_{\text{♂}},y_{\text{♀}},y_{\text{♂}})h_9(z,z_{\text{♀}},z_{\text{♂}})$$

$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{x_{\text{♀}}} h_1(x_{\text{♀}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y,y_{\text{♀}},y_{\text{♂}})h_8(z_{\text{♂}},y_{\text{♀}},y_{\text{♂}})h_9(z,z_{\text{♀}},z_{\text{♂}})h_{10}(x,x_{\text{♀}},z_{\text{♀}})$$

$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}} h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y,y_{\text{♀}},y_{\text{♂}})h_8(z_{\text{♂}},y_{\text{♀}},y_{\text{♂}})h_9(z,z_{\text{♀}},z_{\text{♂}})h_{11}(x,z_{\text{♀}})$$

$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}} h_4(y_{\text{♀}})h_9(z,z_{\text{♀}},z_{\text{♂}})h_{11}(x,z_{\text{♀}})h_{12}(y,y_{\text{♀}},z_{\text{♂}})$$

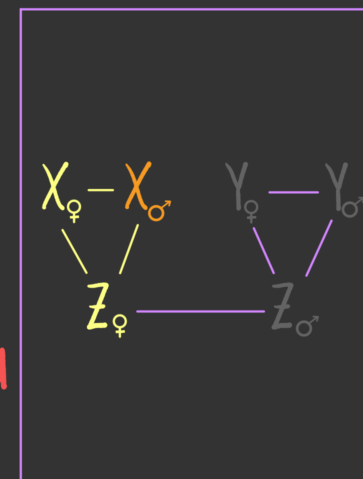
$$= \sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}} h_9(z,z_{\text{♀}},z_{\text{♂}})h_{11}(x,z_{\text{♀}})h_{13}(y,z_{\text{♂}})$$

$$= \sum_{z_{\text{♀}}} h_{11}(x,z_{\text{♀}})h_{14}(y,z,z_{\text{♀}})$$

$$= h_{15}(x,y,z)$$

$$= P(x,y,z)$$

the cliques in this graph correspond to the factors we created



elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{10}(x,x_{\female},z_{\female})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}} h_4(y_{\female})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})h_{12}(y,y_{\female},z_{\male})$$

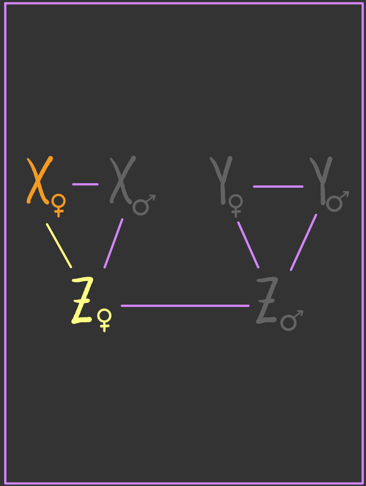
$$= \sum_{z_{\female}}\sum_{z_{\male}} h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})h_{13}(y,z_{\male})$$

$$= \sum_{z_{\female}} h_{11}(x,z_{\female})h_{14}(y,z,z_{\female})$$

$$= h_{15}(x,y,z)$$

$$= P(x,y,z)$$

the cliques in this graph correspond to the factors we created



elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$

$$\sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{x_{♀}}\sum_{x_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x, x_{♀}, x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_7(z_{♀}, x_{♀}, x_{♂})h_8(z_{♂}, y_{♀}, y_{♂})h_9(z, z_{♀}, z_{♂})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{x_{♀}} h_1(x_{♀})h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_8(z_{♂}, y_{♀}, y_{♂})h_9(z, z_{♀}, z_{♂})h_{10}(x, x_{♀}, z_{♀})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}} h_4(y_{♀})h_5(y_{♂})h_6(y, y_{♀}, y_{♂})h_8(z_{♂}, y_{♀}, y_{♂})h_9(z, z_{♀}, z_{♂})h_{11}(x, z_{♀})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}} h_4(y_{♀})h_9(z, z_{♀}, z_{♂})h_{11}(x, z_{♀})h_{12}(y, y_{♀}, z_{♂})$$

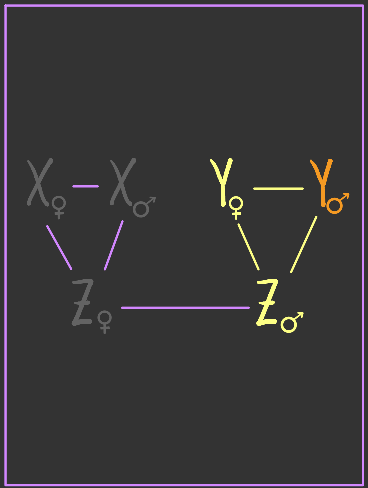
$$= \sum_{z_{♀}}\sum_{z_{♂}} h_9(z, z_{♀}, z_{♂})h_{11}(x, z_{♀})h_{13}(y, z_{♂})$$

$$= \sum_{z_{♀}} h_{11}(x, z_{♀})h_{14}(y, z, z_{♀})$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

the cliques in this graph correspond to the factors we created



elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$

$$\sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{x_{♀}}\sum_{x_{♂}} h_1(x_{♀})h_2(x_{♂})h_3(x,x_{♀},x_{♂})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_7(z_{♀},x_{♀},x_{♂})h_8(z_{♂},y_{♀},y_{♂})h_9(z,z_{♀},z_{♂})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}}\sum_{x_{♀}} h_1(x_{♀})h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_8(z_{♂},y_{♀},y_{♂})h_9(z,z_{♀},z_{♂})h_{10}(x,x_{♀},z_{♀})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}}\sum_{y_{♂}} h_4(y_{♀})h_5(y_{♂})h_6(y,y_{♀},y_{♂})h_8(z_{♂},y_{♀},y_{♂})h_9(z,z_{♀},z_{♂})h_{11}(x,z_{♀})$$

$$= \sum_{z_{♀}}\sum_{z_{♂}}\sum_{y_{♀}} h_4(y_{♀})h_9(z,z_{♀},z_{♂})h_{11}(x,z_{♀})h_{12}(y,y_{♀},z_{♂})$$

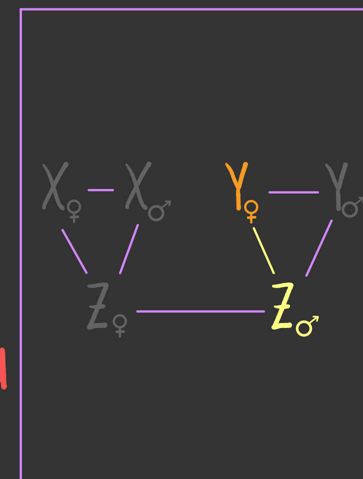
$$= \sum_{z_{♀}}\sum_{z_{♂}} h_9(z,z_{♀},z_{♂})h_{11}(x,z_{♀})h_{13}(y,z_{♂})$$

$$= \sum_{z_{♀}} h_{11}(x,z_{♀})h_{14}(y,z,z_{♀})$$

$$= h_{15}(x,y,z)$$

$$= P(x,y,z)$$

the cliques in this graph correspond to the factors we created



elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{10}(x,x_{\female},z_{\female})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}} h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}} h_4(y_{\female})h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})h_{12}(y,y_{\female},z_{\male})$$

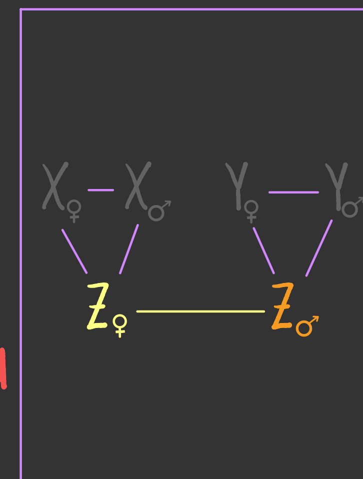
$$=\sum_{z_{\female}}\sum_{z_{\male}} h_9(z,z_{\female},z_{\male})h_{11}(x,z_{\female})h_{13}(y,z_{\male})$$

$$=\sum_{z_{\female}} h_{11}(x,z_{\female})h_{14}(y,z,z_{\female})$$

$$= h_{15}(x,y,z)$$

$$= P(x,y,z)$$

the cliques in this graph correspond to the factors we created



elimination order

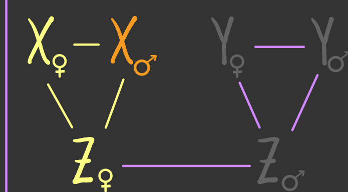
$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_7(z_{\female},x_{\female},x_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})$$

$$=\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})h_4(y_{\female})h_5(y_{\male})h_6(y,y_{\female},y_{\male})h_8(z_{\male},y_{\female},y_{\male})h_9(z,z_{\female},z_{\male})h_{10}(x,x_{\female},z_{\female})$$

$$\boxed{h_{10}(x,x_{\female},z_{\female})=\sum_{x_{\male}} h_2(x_{\male})h_3(x,x_{\female},x_{\male})h_7(z_{\female},x_{\female},x_{\male})}$$

this was the "hardest" factor to compute



elimination order
$X_{\text{♂}}\ X_{\text{♀}}\ Y_{\text{♂}}\ Y_{\text{♀}}\ Z_{\text{♂}}\ Z_{\text{♀}}$

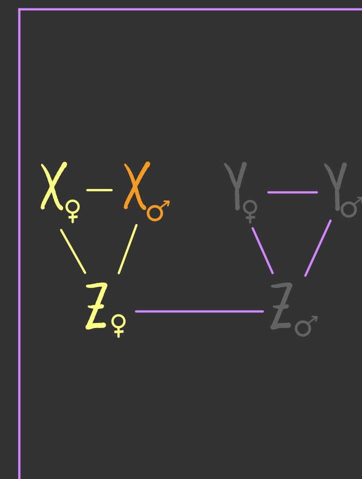
$$\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{x_{\text{♀}}}\sum_{x_{\text{♂}}} h_1(x_{\text{♀}})h_2(x_{\text{♂}})h_3(x,x_{\text{♀}},x_{\text{♂}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y,y_{\text{♀}},y_{\text{♂}})h_7(z_{\text{♀}},x_{\text{♀}},x_{\text{♂}})h_8(z_{\text{♂}},y_{\text{♀}},y_{\text{♂}})h_9(z,z_{\text{♀}},z_{\text{♂}})$$

$$=\sum_{z_{\text{♀}}}\sum_{z_{\text{♂}}}\sum_{y_{\text{♀}}}\sum_{y_{\text{♂}}}\sum_{x_{\text{♀}}} h_1(x_{\text{♀}})h_4(y_{\text{♀}})h_5(y_{\text{♂}})h_6(y,y_{\text{♀}},y_{\text{♂}})h_8(z_{\text{♂}},y_{\text{♀}},y_{\text{♂}})h_9(z,z_{\text{♀}},z_{\text{♂}})\underline{h_{10}(x,x_{\text{♀}},z_{\text{♀}})}$$

$$h_{10}(x,x_{\text{♀}},z_{\text{♀}})=\sum_{x_{\text{♂}}} h_2(x_{\text{♂}})h_3(x,x_{\text{♀}},x_{\text{♂}})h_7(z_{\text{♀}},x_{\text{♀}},x_{\text{♂}})$$

3 possible values

how many terms did we need to compute?



elimination order

$X_{\male}\ X_{\female}\ Y_{\male}\ Y_{\female}\ Z_{\male}\ Z_{\female}$

$$\sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}}\sum_{x_{\male}} h_1(x_{\female})\,h_2(x_{\male})\,h_3(x, x_{\female}, x_{\male})\,h_4(y_{\female})\,h_5(y_{\male})\,h_6(y, y_{\female}, y_{\male})\,h_7(z_{\female}, x_{\female}, x_{\male})\,h_8(z_{\male}, y_{\female}, y_{\male})\,h_9(z, z_{\female}, z_{\male})$$

$$= \sum_{z_{\female}}\sum_{z_{\male}}\sum_{y_{\female}}\sum_{y_{\male}}\sum_{x_{\female}} h_1(x_{\female})\,h_4(y_{\female})\,h_5(y_{\male})\,h_6(y, y_{\female}, y_{\male})\,h_8(z_{\male}, y_{\female}, y_{\male})\,h_9(z, z_{\female}, z_{\male})\,\underline{h_{10}(x, x_{\female}, z_{\female})}$$

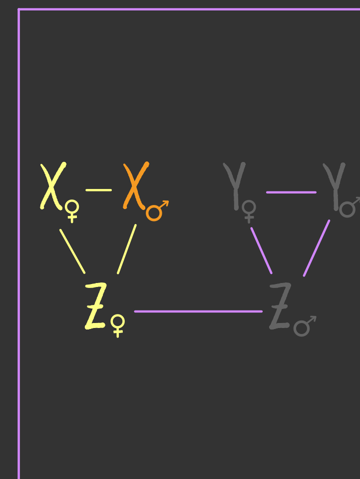
3^2 sums

a sum of 3 terms

$$\boxed{h_{10}(x, \underline{x_{\female}, z_{\female}}) = \sum_{x_{\male}} h_2(x_{\male})\,h_3(x, x_{\female}, x_{\male})\,h_7(z_{\female}, x_{\female}, x_{\male})}$$

3 possible values

we needed to compute 3^3 terms



elimination order

$Z_{\text{♂}}\ Z_{\text{♀}}\ Y_{\text{♂}}\ Y_{\text{♀}}\ X_{\text{♂}}\ X_{\text{♀}}$

to compute the most difficult factor, we needed to compute

3^5 terms

3^4 sums a sum of 3 terms

$$h_{11}(\overline{x_{\text{♀}}, x_{\text{♂}}, y_{\text{♀}}, y_{\text{♂}}}, z) = \sum_{z_{\text{♀}}} h_7(z_{\text{♀}}, x_{\text{♀}}, x_{\text{♂}})\, h_{10}(y_{\text{♀}}, y_{\text{♂}}, z, z_{\text{♀}})$$

elimination order

$X_{\text{♂}}\ X_{\text{♀}}\ Y_{\text{♂}}\ Y_{\text{♀}}\ Z_{\text{♂}}\ Z_{\text{♀}}$

to compute the most difficult factor, we needed to compute

3^3 terms

3^2 sums a sum of 3 terms

$$h_{10}(x, \overline{x_{\text{♀}}, z_{\text{♀}}}) = \sum_{x_{\text{♂}}} h_2(x_{\text{♂}})\, h_3(x, x_{\text{♀}}, x_{\text{♂}})\, h_7(z_{\text{♀}}, x_{\text{♀}}, x_{\text{♂}})$$

elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$

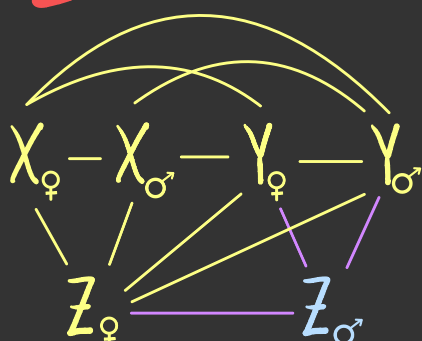
to compute the most difficult factor, we needed to compute

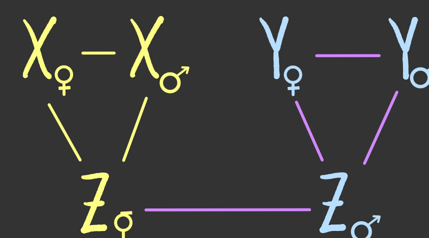
3^5 terms

to compute the most difficult factor, we needed to compute

3^3 terms

size of largest clique





elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$

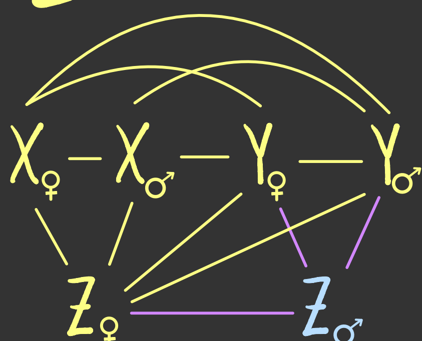
to compute the most difficult factor, we needed to compute

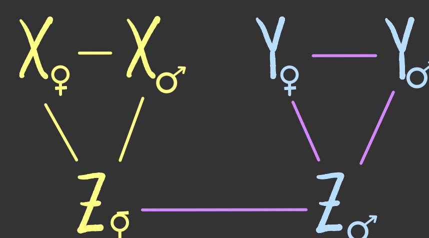
3^5 terms

to compute the most difficult factor, we needed to compute

3^3 terms

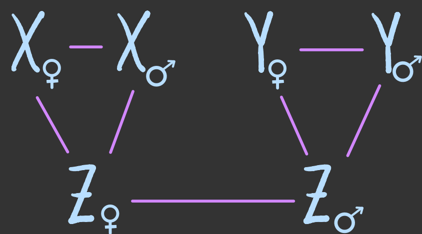
size of variable domains





elimination order

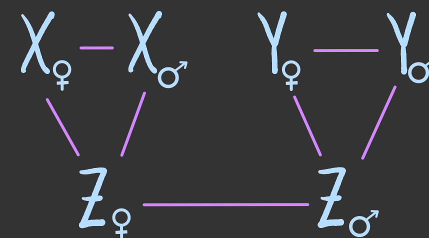
$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$



max : 3

elimination order

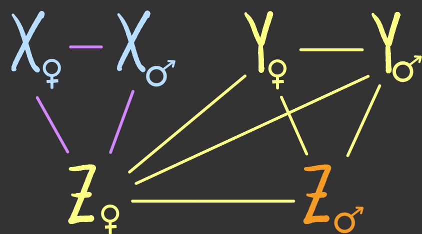
$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$



max : 3

elimination order

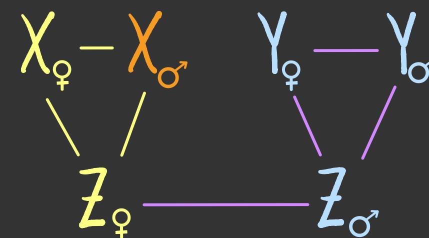
$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$



max : ~~3~~ 4

elimination order

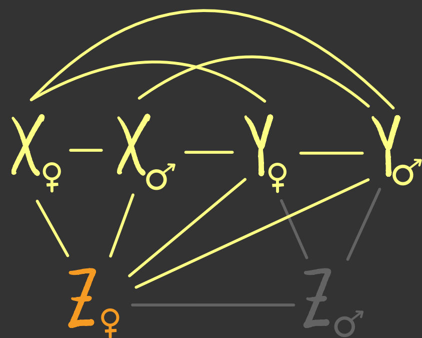
$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$



max : 3

elimination order

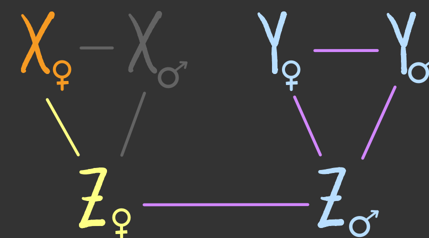
$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$



max : ~~3~~ ~~4~~ 5

elimination order

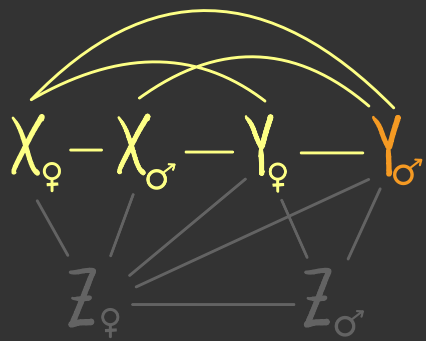
$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$



max : 3

elimination order

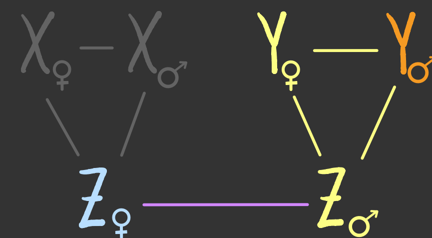
$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$



max : ~~3~~ ~~4~~ 5

elimination order

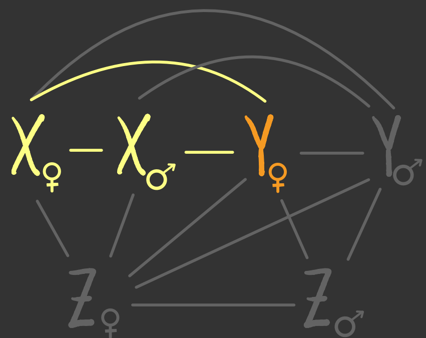
$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$



max : 3

elimination order

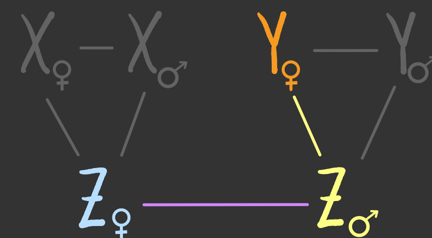
$Z_♂ \; Z_♀ \; Y_♂ \; Y_♀ \; X_♂ \; X_♀$



max : ~~3~~ ~~4~~ 5

elimination order

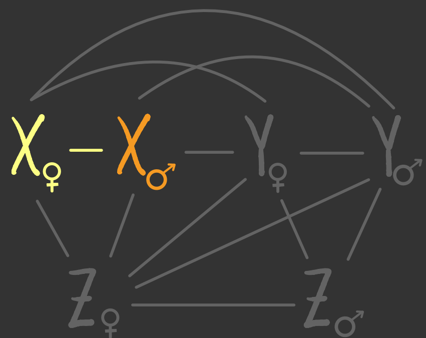
$X_♂ \; X_♀ \; Y_♂ \; Y_♀ \; Z_♂ \; Z_♀$



max : 3

elimination order

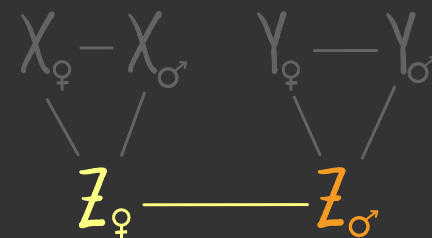
$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$



max : ~~3~~ ~~4~~ 5

elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$



max : 3

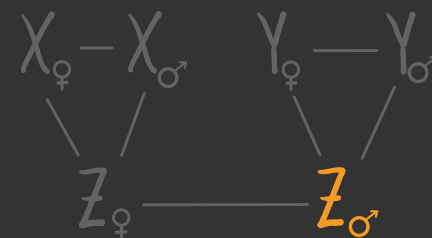
elimination order

$Z_{♂}$ $Z_{♀}$ $Y_{♂}$ $Y_{♀}$ $X_{♂}$ $X_{♀}$

max : ~~3~~ ~~4~~ 5

elimination order

$X_{♂}$ $X_{♀}$ $Y_{♂}$ $Y_{♀}$ $Z_{♂}$ $Z_{♀}$



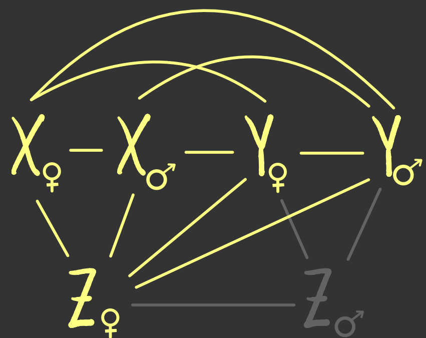
max : 3

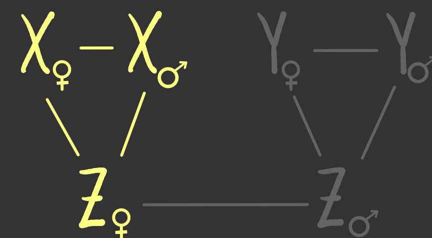
elimination order

$Z_{♂}\ Z_{♀}\ Y_{♂}\ Y_{♀}\ X_{♂}\ X_{♀}$

elimination order

$X_{♂}\ X_{♀}\ Y_{♂}\ Y_{♀}\ Z_{♂}\ Z_{♀}$





width : 5

width : 3

this called the **width** of the elimination order

let w be the width of the elimination order
let d be the size of the largest variable domain
the most difficult factor requires us to sum $O(d^w)$ terms

$$\sum_{x_♀} \sum_{x_♂} \sum_{y_♀} \sum_{y_♂} \sum_{z_♀} h_1(x_♀) h_2(x_♂) h_3(x, x_♀, x_♂) h_4(y_♀) h_5(y_♂) h_6(y, y_♀, y_♂) h_7(z_♀, x_♀, x_♂) h_{10}(y_♀, y_♂, z, z_♀)$$

$$= \sum_{x_♀} \sum_{x_♂} \sum_{y_♀} \sum_{y_♂} h_1(x_♀) h_2(x_♂) h_3(x, x_♀, x_♂) h_4(y_♀) h_5(y_♂) h_6(y, y_♀, y_♂) h_{11}(x_♀, x_♂, y_♀, y_♂, z)$$

$$= \sum_{x_♀} \sum_{x_♂} \sum_{y_♀} h_1(x_♀) h_2(x_♂) h_3(x, x_♀, x_♂) h_4(y_♀) h_{12}(x_♀, x_♂, y_♀, z)$$

$$= \sum_{x_♀} \sum_{x_♂} h_1(x_♀) h_2(x_♂) h_3(x, x_♀, x_♂) h_{13}(x_♀, x_♂, y, z)$$

$$= \sum_{x_♀} h_1(x_♀) h_{14}(x_♀, x, y, z)$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

let w be the width of the elimination order

let d be the size of the largest variable domain

the most difficult factor requires us to sum $O(d^w)$ terms

$$\sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂}\sum_{z_♀} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_4(y_♀)h_5(y_♂)h_6(y, y_♀, y_♂)h_7(z_♀, x_♀, x_♂)h_{10}(y_♀, y_♂, z, z_♀)$$

$$= \sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_4(y_♀)h_5(y_♂)h_6(y, y_♀, y_♂)h_{11}(x_♀, x_♂, y_♀, y_♂, z)$$

$$= \sum_{x_♀}\sum_{x_♂}\sum_{y_♀} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_4(y_♀)h_{12}(x_♀, x_♂, y_♀, z)$$

$$= \sum_{x_♀}\sum_{x_♂} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_{13}(x_♀, x_♂, y, z)$$

$$= \sum_{x_♀} h_1(x_♀)h_{14}(x_♀, x, y, z)$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

we compute a new factor for each variable we

eliminate

let w be the width of the elimination order

let d be the size of the largest variable domain

the most difficult factor requires us to sum $O(d^w)$ terms

$$\sum_{x_♀} \sum_{x_♂} \sum_{y_♀} \sum_{y_♂} \sum_{z_♀} h_1(x_♀) h_2(x_♂) h_3(x, x_♀, x_♂) h_4(y_♀) h_5(y_♂) h_6(y, y_♀, y_♂) h_7(z_♀, x_♀, x_♂) h_{10}(y_♀, y_♂, z, z_♀)$$

$$= \sum_{x_♀} \sum_{x_♂} \sum_{y_♀} \sum_{y_♂} h_1(x_♀) h_2(x_♂) h_3(x, x_♀, x_♂) h_4(y_♀) h_5(y_♂) h_6(y, y_♀, y_♂) h_{11}(x_♀, x_♂, y_♀, y_♂, z)$$

$$= \sum_{x_♀} \sum_{x_♂} \sum_{y_♀} h_1(x_♀) h_2(x_♂) h_3(x, x_♀, x_♂) h_4(y_♀) h_{12}(x_♀, x_♂, y_♀, z)$$

$$= \sum_{x_♀} \sum_{x_♂} h_1(x_♀) h_2(x_♂) h_3(x, x_♀, x_♂) h_{13}(x_♀, x_♂, y, z)$$

$$= \sum_{x_♀} h_1(x_♀) h_{14}(x_♀, x, y, z)$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

let n be the number of variables in the bayesian network

we compute up to n new factors

let w be the width of the elimination order
let d be the size of the largest variable domain
the most difficult factor requires us to sum $O(d^w)$ terms

$$\sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂}\sum_{z_♀} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_4(y_♀)h_5(y_♂)h_6(y, y_♀, y_♂)h_7(z_♀, x_♀, x_♂)h_{10}(y_♀, y_♂, z, z_♀)$$

$$= \sum_{x_♀}\sum_{x_♂}\sum_{y_♀}\sum_{y_♂} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_4(y_♀)h_5(y_♂)h_6(y, y_♀, y_♂)h_{11}(x_♀, x_♂, y_♀, y_♂, z)$$

$$= \sum_{x_♀}\sum_{x_♂}\sum_{y_♀} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_4(y_♀)h_{12}(x_♀, x_♂, y_♀, z)$$

$$= \sum_{x_♀}\sum_{x_♂} h_1(x_♀)h_2(x_♂)h_3(x, x_♀, x_♂)h_{13}(x_♀, x_♂, y, z)$$

$$= \sum_{x_♀} h_1(x_♀)h_{14}(x_♀, x, y, z)$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

let n be the number of variables in the bayesian network
we compute up to n new factors

so the worst-case runtime is $O(nd^w)$