

variable
elimination

19 oct
2022

CSCI
373

c	g	
1	2	0.5
1	3	0.5
2	2	0
2	3	1
3	2	1
3	3	0

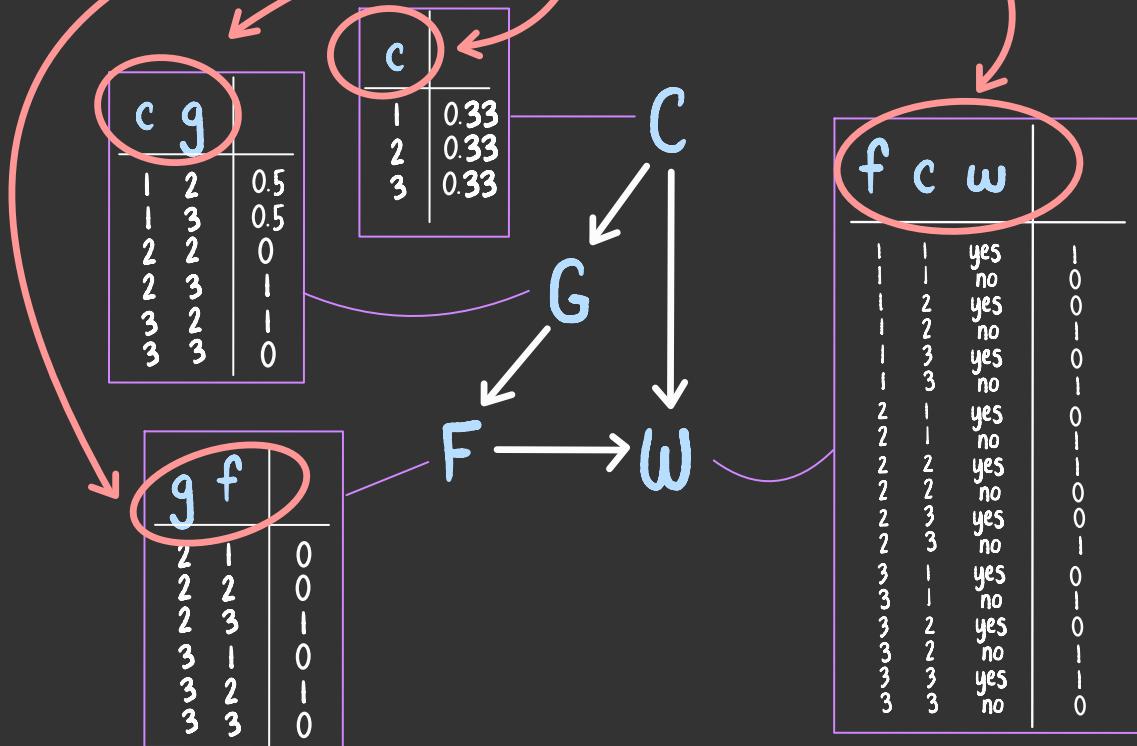
c	
1	0.33
2	0.33
3	0.33

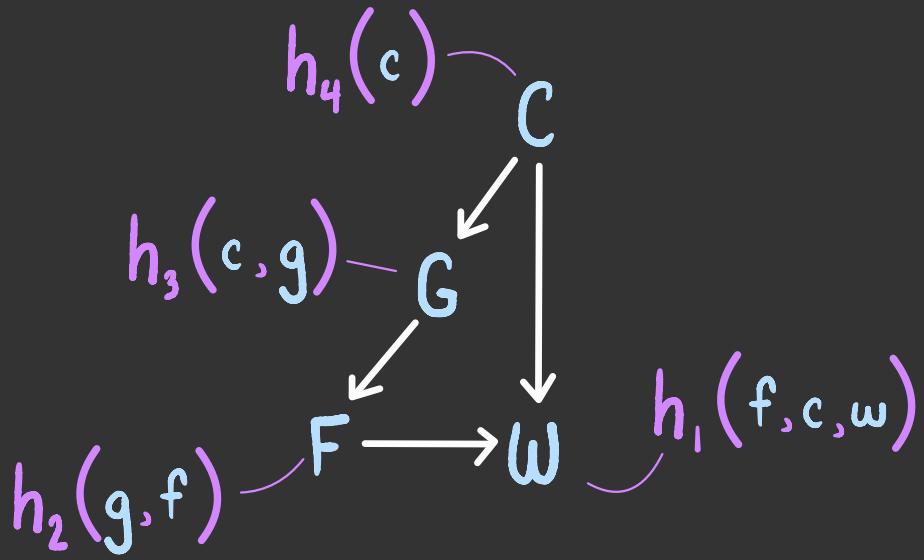
C
G
F → W

g	f	
2	1	0
2	2	0
2	3	1
3	1	0
3	2	1
3	3	0

f	c	w	
1	1	yes	1
1	1	no	0
1	2	yes	1
1	2	no	0
1	3	yes	0
1	3	no	1
2	1	yes	0
2	1	no	1
2	2	yes	1
2	2	no	0
2	3	yes	0
2	3	no	1
3	1	yes	0
3	1	no	1
3	2	yes	0
3	2	no	1
3	3	yes	1
3	3	no	0

these are multivariable functions





joint $P(w, f, g, c) = h_1(f, c, w) h_2(g, f) h_3(c, g) h_4(c)$

marginal $P(w) = \sum_c \sum_f \sum_g h_1(f, c, w) h_2(g, f) h_3(c, g) h_4(c)$

$$P(\omega) = \frac{\sum_{c} \sum_{f} \sum_{g} h_1(f, c, \omega) h_2(g, f) h_3(c, g) h_4(c)}{h_1(f, c, \omega)}$$

these are often referred
to as **factors**

$$P(\omega) = \sum_c \sum_f \sum_g h_1(f, c, \omega) h_2(g, f) h_3(c, g) h_4(c)$$

```

graph TD
    h4[h4(c)] --> c[c]
    h3[h3(c, g)] --> G[G]
    h2[h2(g, f)] --> F[F]
    F --> omega[omega]
    c --> h1[h1(f, c, omega)]

```

the algorithm we used
to compute this sum of
products is called
variable elimination

$$P(\omega) = \sum_{c} \sum_{f} \sum_{g} h_1(f, c, \omega) h_2(g, f) h_3(c, g) h_4(c)$$

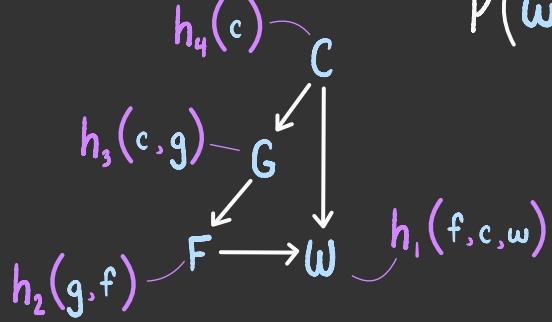
first, we choose an elimination order

elimination order: CFG

$$P(\omega) = \sum_c \sum_f \sum_g h_1(f, c, \omega) h_2(g, f) h_3(c, g) h_4(c)$$

then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order: CFG



$$P(\omega) = \sum_c \sum_f h_1(f, c, \omega) h_4(c) \frac{\sum_g h_2(g, f) h_3(c, g)}{h_5(f, c)}$$

then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order: CFG

$$P(\omega) = \sum_c \sum_f h_1(f, c, \omega) h_4(c) h_5(f, c)$$

```

graph TD
    h4[h4(c)] -- C --> C
    h3[h3(c, g)] -- G --> G
    h2[h2(g, f)] -- F --> F
    h1[h1(f, c, omega)] -- omega --> omega
    C --> G
  
```

then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order: CFG

$$P(\omega) = \sum_c h_4(c) \boxed{\sum_f h_1(f, c, \omega) h_5(f, c)}$$

$h_6(c, \omega)$

```

graph TD
    h4[h4(c)] -- C --> h3[h3(c, g)]
    h3 -- G --> h2[h2(g, f)]
    h2 -- F --> h1[h1(f, c, omega)]
    h1 -- omega --> h6[h6(c, omega)]

```

then, we iteratively push
the summations as far to
the right as we can, then
compute and store all values
of the newly created factor

elimination order: CFG

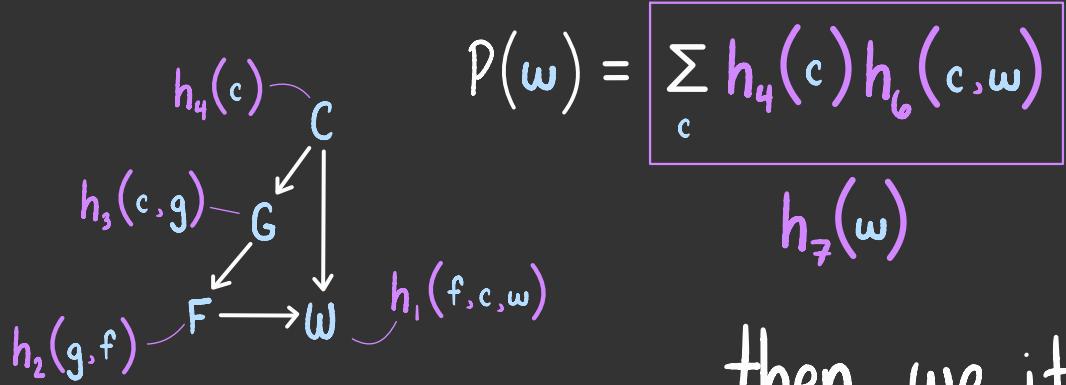
$$P(\omega) = \sum_c h_4(c) h_6(c, \omega)$$

```

graph TD
    P["P(ω)"] -- "h4(c)" --> c[c]
    P -- "h6(c, ω)" --> omega[ω]
    c --> G[G]
    G --> h3["h3(c, g)"]
    h3 --> F[F]
    F --> h2["h2(g, f)"]
    h2 --> omega
    h6 --> omega
  
```

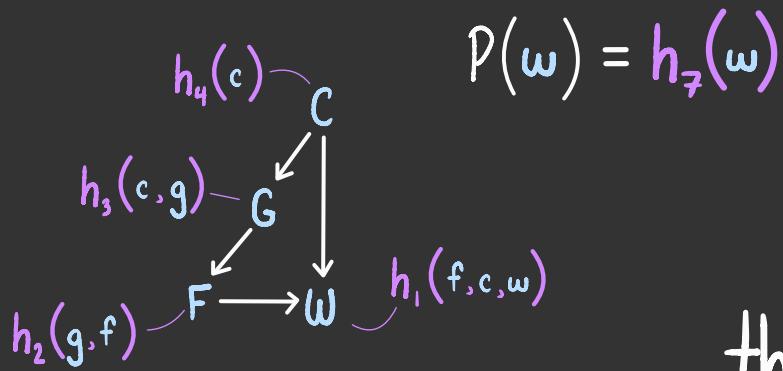
then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order: CFG



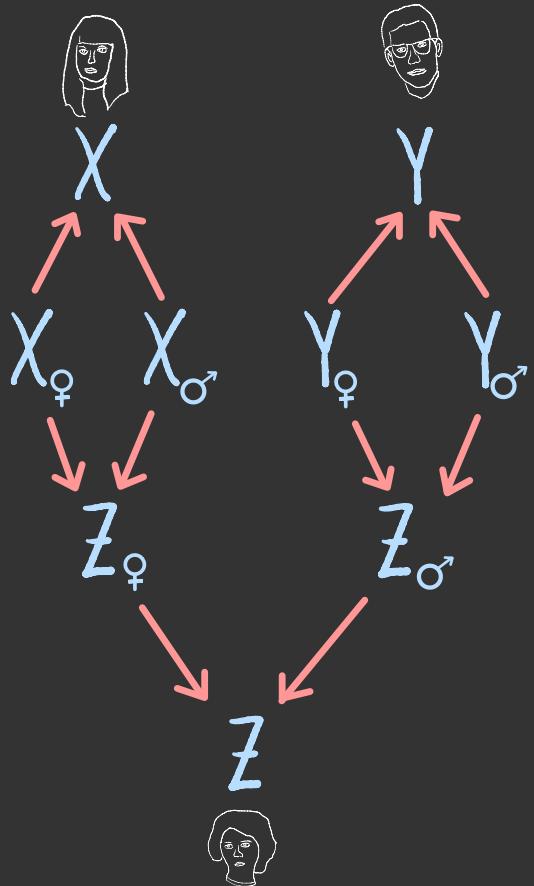
then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order: CFG



then, we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order: CFG



let's try variable elimination
on our **blood types** network

what is
 $P(X=A, Y=AB, Z=B)?$

$$P(X=A, Y=AB, Z=B)$$

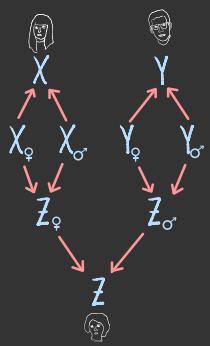
 total

$$= \sum_{x_q} \sum_{x_{\sigma'}} \sum_{y_q} \sum_{y_{\sigma'}} \sum_{z_q} \sum_{z_{\sigma'}} P(X=A, Y=AB, Z=B, x_q, x_{\sigma'}, y_q, y_{\sigma'}, z_q, z_{\sigma'})$$

$$P(X=A, Y=AB, Z=B)$$

total

$$= \sum_{x_q} \sum_{x_{o'}} \sum_{y_q} \sum_{y_{o'}} \sum_{z_q} \sum_{z_{o'}} P(X=A, Y=AB, Z=B, x_q, x_{o'}, y_q, y_{o'}, z_q, z_{o'})$$

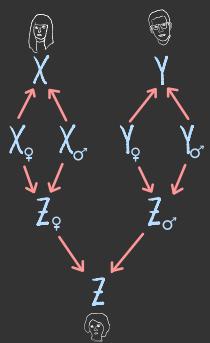


$$= \sum_{x_q} \sum_{x_{o'}} \sum_{y_q} \sum_{y_{o'}} \sum_{z_q} \sum_{z_{o'}} P(x_q) P(x_{o'}) P(x | x_q, x_{o'}) P(y_q) P(y_{o'}) P(y | y_q, y_{o'}) P(z_q) P(z_{o'}) P(z | z_q, z_{o'})$$

$$P(X=A, Y=AB, Z=B)$$

elimination order
 $Z_\varphi Z_\vartheta Y_\varphi Y_\vartheta X_\varphi X_\vartheta$

total



$$\begin{aligned}
 &= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} P(X=A, Y=AB, Z=B, x_\varphi, x_\sigma, y_\varphi, y_\sigma, z_\varphi, z_\sigma) \\
 &= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} P(x_\varphi) P(x_\sigma) P(x | x_\varphi, x_\sigma) P(y_\varphi) P(y_\sigma) P(y | y_\varphi, y_\sigma) P(z_\varphi | x_\varphi, x_\sigma) P(z_\sigma | y_\varphi, y_\sigma) P(z | z_\varphi, z_\sigma) \\
 &= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)
 \end{aligned}$$

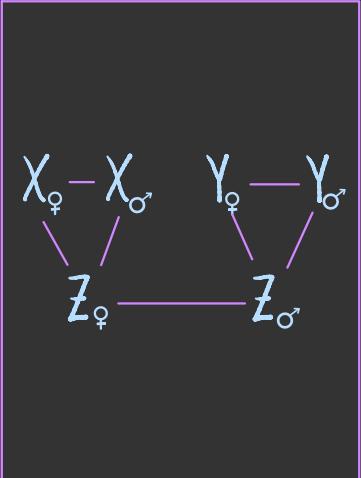
variable elimination

we iteratively push the summations as far to the right as we can, then compute and store all values of the newly created factor

elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

two variables
are adjacent
if they appear
in a common
factor
→

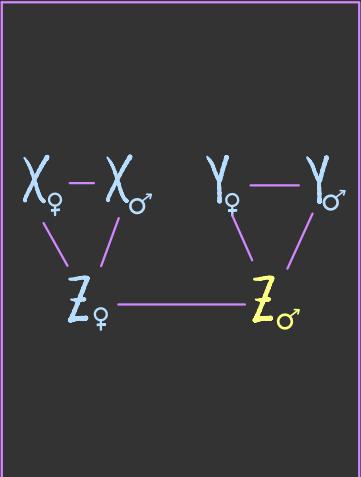


elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) \sum_{z_\sigma} h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

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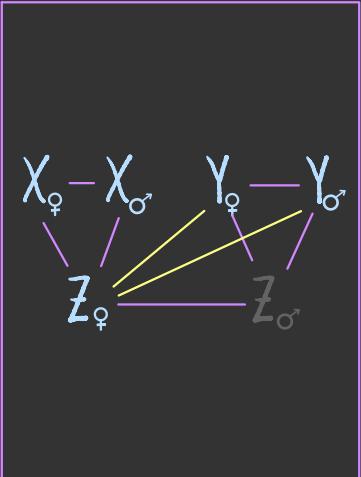


elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

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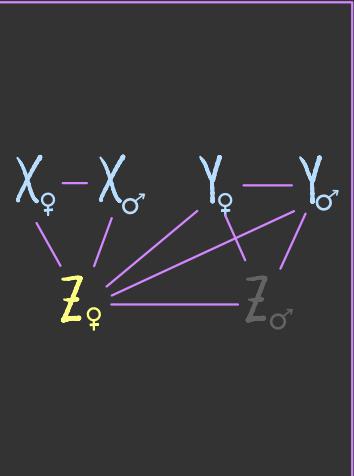



elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned} & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) \sum_{z_\varphi} h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \end{aligned}$$

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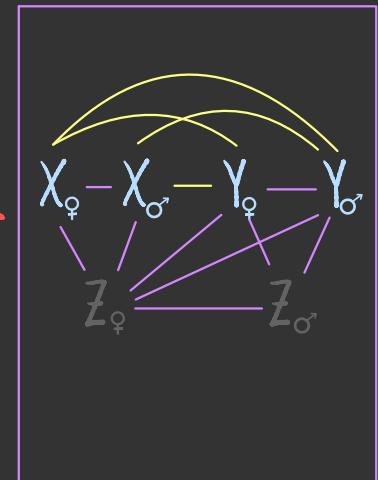


elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)
 \end{aligned}$$

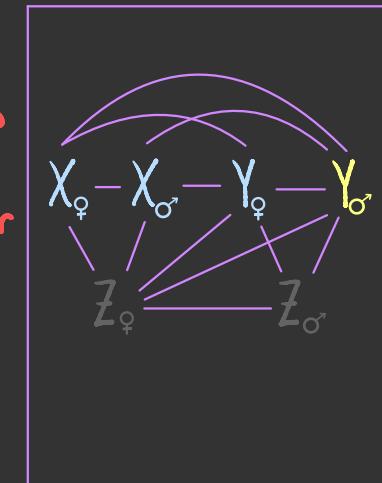
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elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) \sum_{y_\sigma} h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)
 \end{aligned}$$

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elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

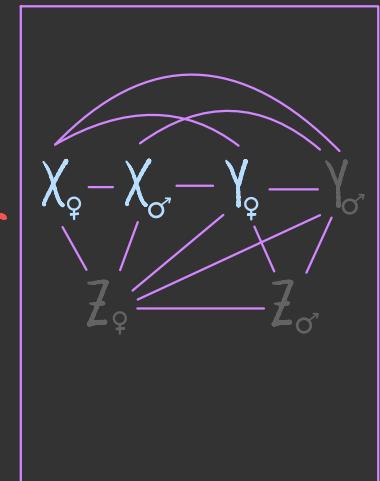
$$\sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y, y_\varphi, z)$$

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elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

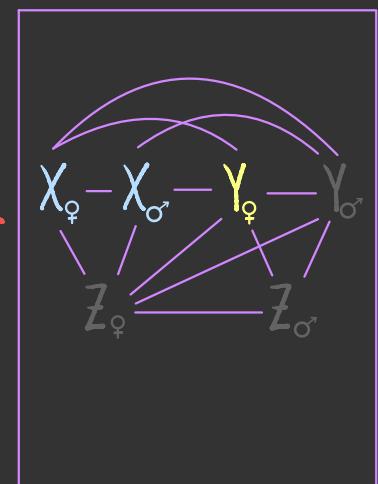
$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y, y_\varphi, z)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) \sum_{y_\varphi} h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y, y_\varphi, z)$$

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elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

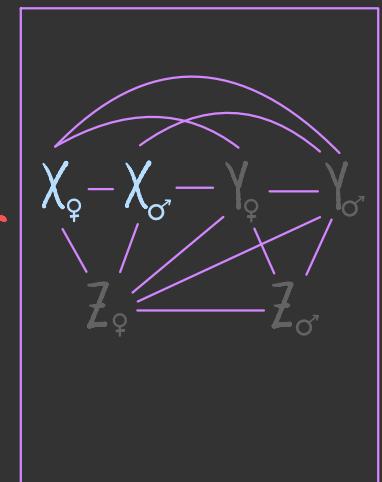
$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z)$$

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elimination order

$Z_{\sigma} Z_{\varphi} Y_{\sigma} Y_{\varphi} X_{\sigma} X_{\varphi}$

$$\sum_{x_{\varphi}} \sum_{x_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{x_{\varphi}} \sum_{x_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{z_{\varphi}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_{10}(y_{\varphi}, y_{\sigma}, z, z_{\varphi})$$

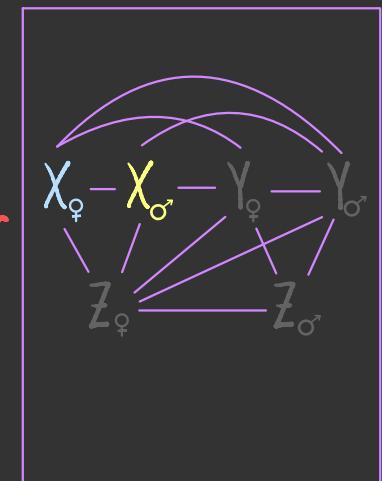
$$= \sum_{x_{\varphi}} \sum_{x_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_{11}(x_{\varphi}, x_{\sigma}, y_{\varphi}, y_{\sigma}, z)$$

$$= \sum_{x_{\varphi}} \sum_{x_{\sigma}} \sum_{y_{\varphi}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_{12}(x_{\varphi}, x_{\sigma}, y_{\varphi}, z)$$

$$= \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_{13}(x_{\varphi}, x_{\sigma}, y, z)$$

$$= \sum_{x_{\varphi}} h_1(x_{\varphi}) \sum_{x_{\sigma}} h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_{13}(x_{\varphi}, x_{\sigma}, y, z)$$

two variables
are adjacent
if they appear
in a common
factor



elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

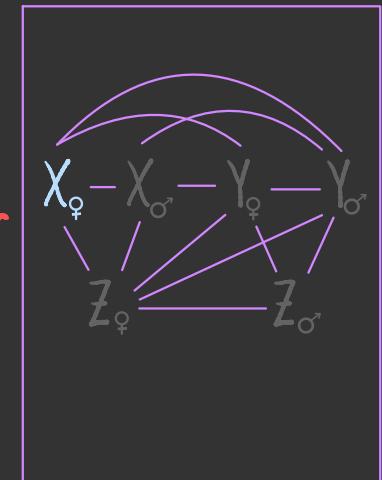
$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z)$$

$$= \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z)$$

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if they appear
in a common
factor



elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)$$

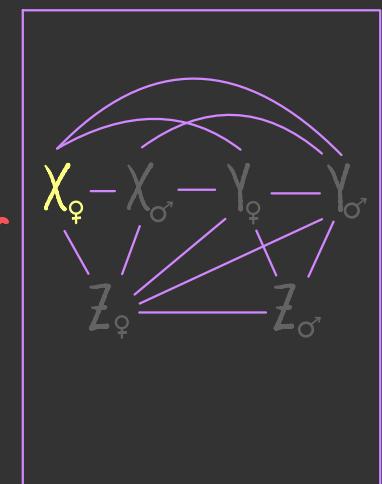
$$= \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z)$$

$$= \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z)$$

$$= \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z)$$

$$= \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z)$$

two variables
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factor



elimination order

$Z_{\sigma} Z_{\varphi} Y_{\sigma} Y_{\varphi} X_{\sigma} X_{\varphi}$

$$\sum_{x_{\varphi}} \sum_{x_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{x_{\varphi}} \sum_{x_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{z_{\varphi}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_{10}(y_{\varphi}, y_{\sigma}, z, z_{\varphi})$$

$$= \sum_{x_{\varphi}} \sum_{x_{\sigma}} \sum_{y_{\varphi}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_{11}(x_{\varphi}, x_{\sigma}, y_{\varphi}, y_{\sigma}, z)$$

$$= \sum_{x_{\varphi}} \sum_{x_{\sigma}} \sum_{y_{\varphi}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_{12}(x_{\varphi}, x_{\sigma}, y_{\varphi}, z)$$

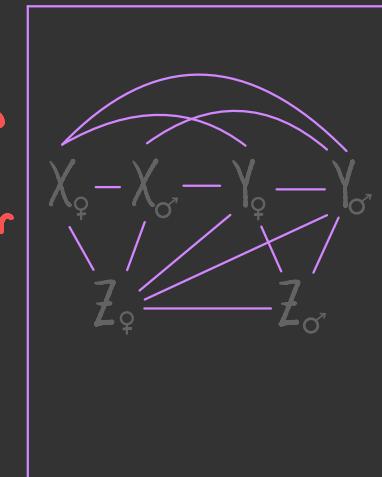
$$= \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_{13}(x_{\varphi}, x_{\sigma}, y, z)$$

$$= \sum_{x_{\varphi}} h_1(x_{\varphi}) h_{14}(x_{\varphi}, x, y, z)$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

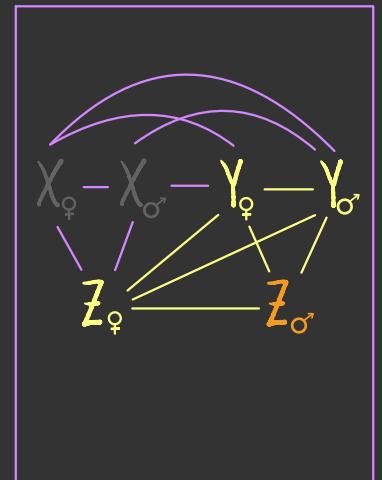
two variables
are adjacent
if they appear
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factor



elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z) \\
 = & \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z) \\
 = & h_{15}(x, y, z) \\
 = & P(x, y, z)
 \end{aligned}$$

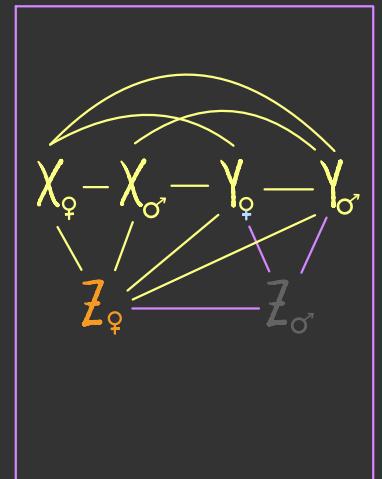
the cliques in this graph correspond to the factors we create



elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z) \\
 = & \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z) \\
 = & h_{15}(x, y, z) \\
 = & P(x, y, z)
 \end{aligned}$$

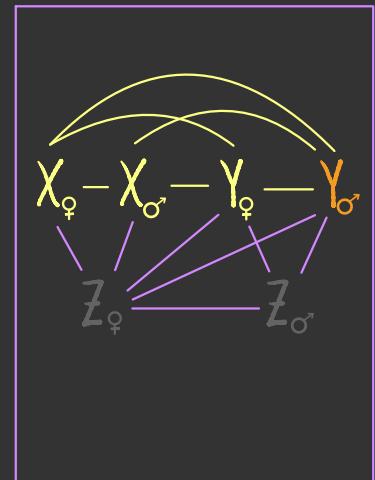
the cliques in this graph correspond to the factors we create



elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z) \\
 = & \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z) \\
 = & h_{15}(x, y, z) \\
 = & P(x, y, z)
 \end{aligned}$$

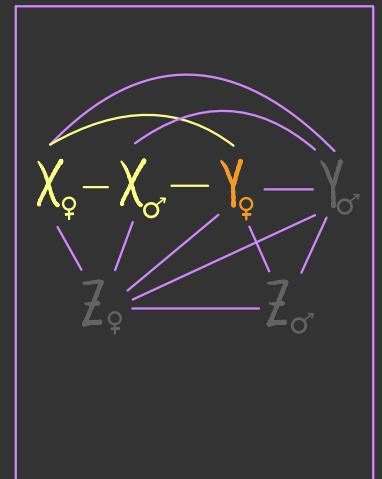
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elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z) \\
 = & \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z) \\
 = & h_{15}(x, y, z) \\
 = & P(x, y, z)
 \end{aligned}$$

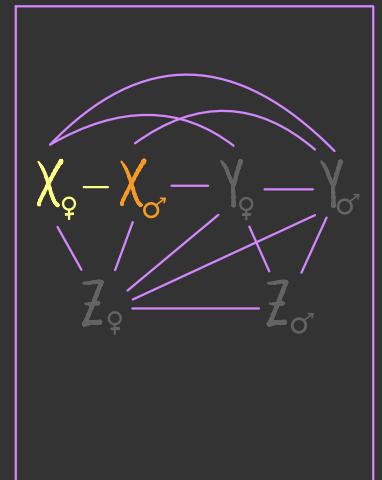
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elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z) \\
 = & \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z) \\
 = & h_{15}(x, y, z) \\
 = & P(x, y, z)
 \end{aligned}$$

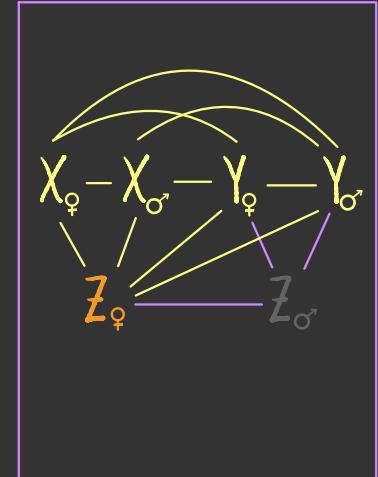
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elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) \underbrace{h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)}_{}
 \end{aligned}$$

$$h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) = \sum_{z_\varphi} h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$



this was the "hardest" factor to compute

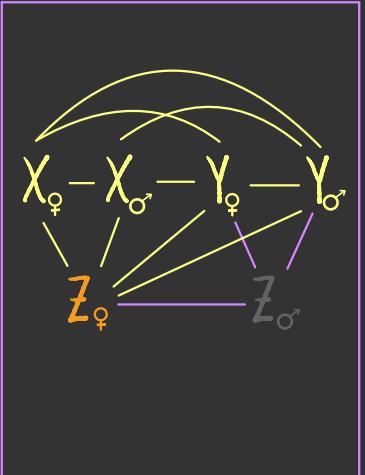
elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) \underbrace{h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)}_{}
 \end{aligned}$$

$$h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) = \sum_{z_\varphi} h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

3 possible values

how many terms did we need to compute?



elimination order
 $Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

$$\begin{aligned}
 & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} \sum_{z_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\
 = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) \underbrace{h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z)}_{\text{a sum of 3 terms}}
 \end{aligned}$$

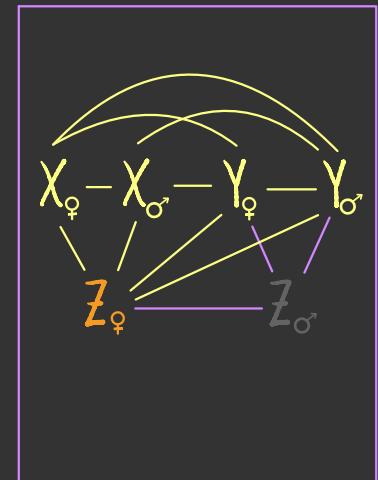
3^4 sums

a sum of 3 terms

$$h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) = \sum_{z_\varphi} h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

3 possible values

we needed to compute
 3^5 terms

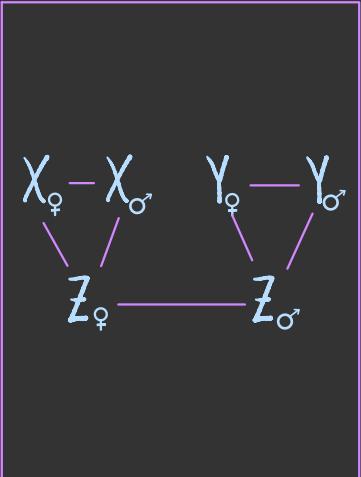


what if we use a
different
elimination order?

elimination order
 $X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

$$\sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

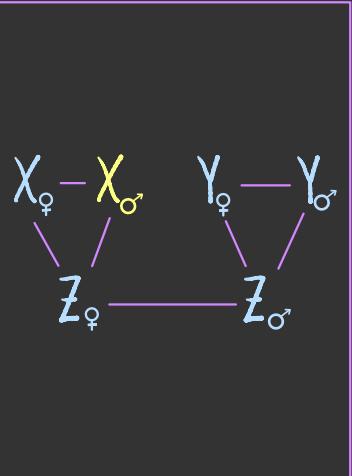
two variables
are adjacent
if they appear
in a common
factor
→



elimination order
 $X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\begin{aligned}
 & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) \\
 = & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) \sum_{x_{\sigma}} h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_8(z_{\varphi}, x_{\varphi}, x_{\sigma})
 \end{aligned}$$

two variables
 are adjacent
 if they appear
 in a common
 factor



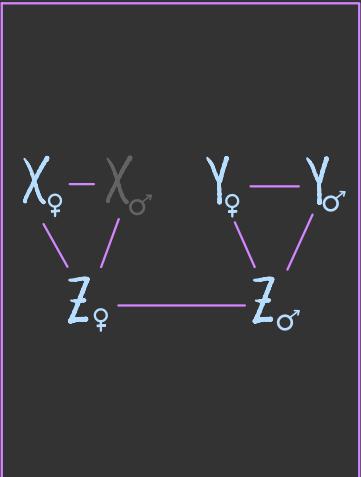
elimination order

$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

two variables
are adjacent
if they appear
in a common
factor



elimination order

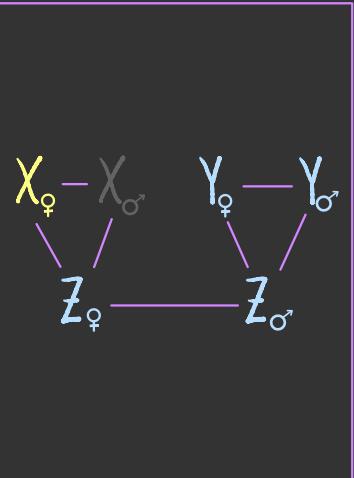
$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) \sum_{x_{\varphi}} h_1(x_{\varphi}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

two variables
are adjacent
if they appear
in a common
factor



elimination order

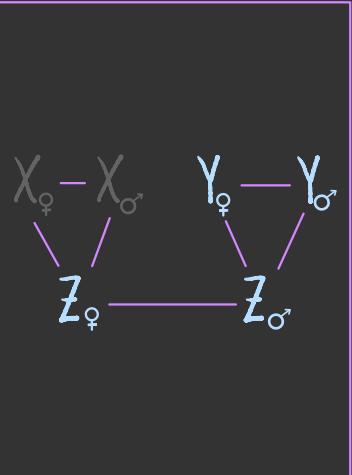
$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi})$$

two variables
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factor



elimination order

$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

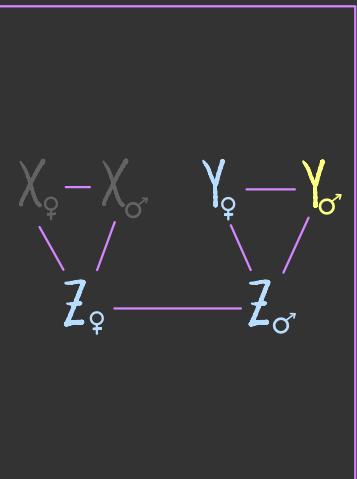
$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) \sum_{y_{\sigma}} h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma})$$

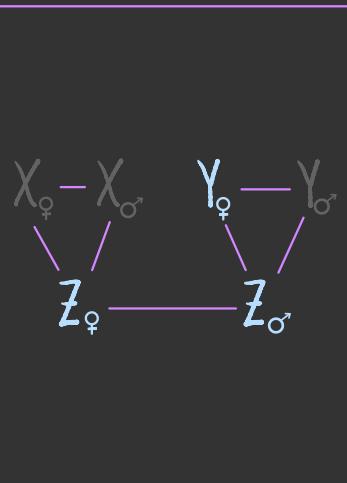
two variables
are adjacent
if they appear
in a common
factor



elimination order
 $X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\begin{aligned}
 & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) \\
 = & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi}) \\
 = & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) \\
 = & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma})
 \end{aligned}$$

two variables
are adjacent
if they appear
in a common
factor



elimination order

$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

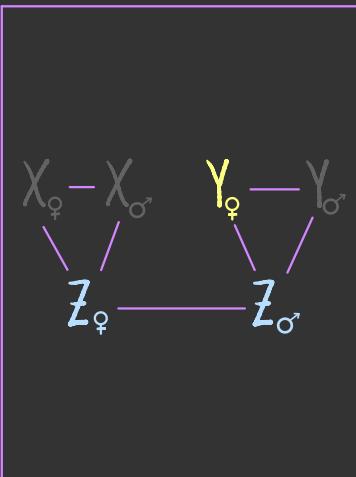
$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) \sum_{y_{\varphi}} h_4(y_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma})$$

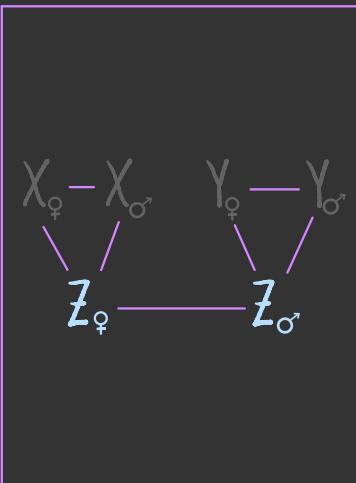
two variables
are adjacent
if they appear
in a common
factor



elimination order
 $X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\begin{aligned}
 & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) \\
 = & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi}) \\
 = & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) \\
 = & \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma}) \\
 = & \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{13}(y, z_{\sigma})
 \end{aligned}$$

two variables
are adjacent
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factor



elimination order

$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

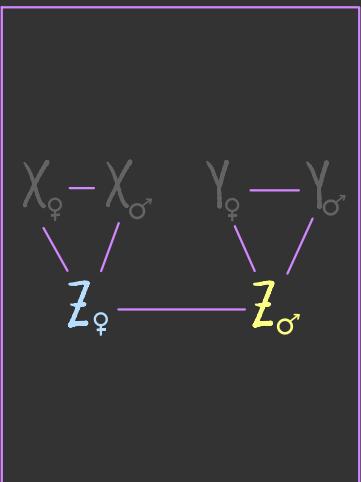
$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{13}(y, z_{\sigma})$$

$$= \sum_{z_{\varphi}} h_{11}(x, z_{\varphi}) \sum_{z_{\sigma}} h_9(z, z_{\varphi}, z_{\sigma}) h_{13}(y, z_{\sigma})$$

two variables
are adjacent
if they appear
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factor



elimination order

$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

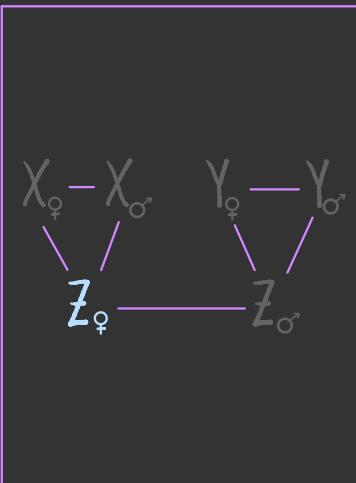
$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{13}(y, z_{\sigma})$$

$$= \sum_{z_{\varphi}} h_{11}(x, z_{\varphi}) h_{14}(y, z, z_{\varphi})$$

two variables
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factor



elimination order

$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi})$$

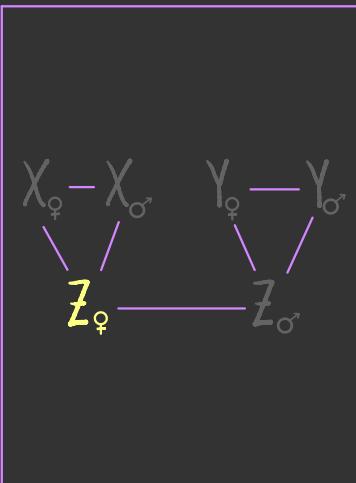
$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{13}(y, z_{\sigma})$$

$$= \sum_{z_{\varphi}} h_{11}(x, z_{\varphi}) h_{14}(y, z, z_{\varphi})$$

$$= \sum_{z_{\varphi}} h_{11}(x, z_{\varphi}) h_{14}(y, z, z_{\varphi})$$

two variables
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elimination order

$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma})$$

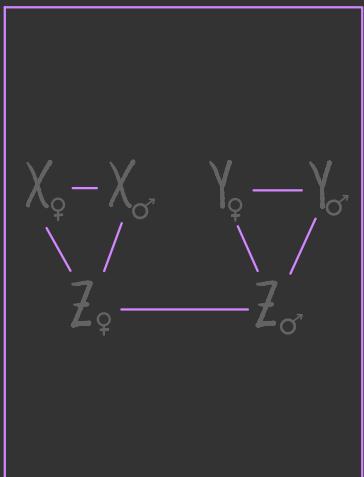
$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{13}(y, z_{\sigma})$$

$$= \sum_{z_{\varphi}} h_{11}(x, z_{\varphi}) h_{14}(y, z, z_{\varphi})$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

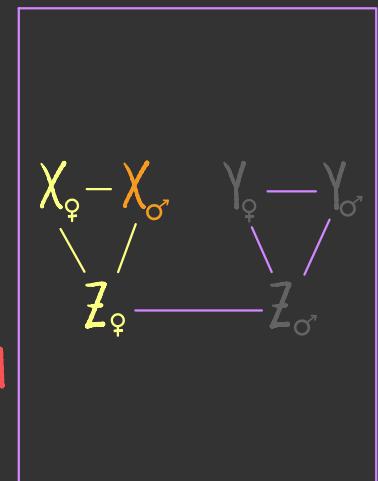
two variables
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elimination order
 $X_{\sigma} X_{\sigma'} Y_{\sigma} Y_{\sigma'} Z_{\sigma} Z_{\sigma'}$

$$\begin{aligned}
 & \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} \sum_{x_{\sigma}} \sum_{x_{\sigma'}} h_1(x_{\sigma}) h_2(x_{\sigma'}) h_3(x, x_{\sigma}, x_{\sigma'}) h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_7(z_{\sigma}, x_{\sigma}, x_{\sigma'}) h_8(z_{\sigma'}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'}) \\
 = & \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} \sum_{x_{\sigma}} h_1(x_{\sigma}) h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_8(z_{\sigma}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{10}(x, x_{\sigma}, z_{\sigma'}) \\
 = & \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_8(z_{\sigma}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma}) \\
 = & \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} h_4(y_{\sigma}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma}) h_{12}(y, y_{\sigma}, z_{\sigma'}) \\
 = & \sum_{z_{\sigma}} \sum_{z_{\sigma'}} h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma}) h_{13}(y, z_{\sigma'}) \\
 = & \sum_{z_{\sigma}} h_{11}(x, z_{\sigma}) h_{14}(y, z, z_{\sigma}) \\
 = & h_{15}(x, y, z) \\
 = & P(x, y, z)
 \end{aligned}$$

the cliques in this graph correspond to the factors we created



elimination order

$X_{\sigma} X_{\varphi} Y_{\sigma} Y_{\varphi} Z_{\sigma} Z_{\varphi}$

$$\sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} \sum_{x_{\sigma}} h_1(x_{\varphi}) h_2(x_{\sigma}) h_3(x, x_{\varphi}, x_{\sigma}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_7(z_{\varphi}, x_{\varphi}, x_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} \sum_{x_{\varphi}} h_1(x_{\varphi}) h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{10}(x, x_{\varphi}, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} \sum_{y_{\sigma}} h_4(y_{\varphi}) h_5(y_{\sigma}) h_6(y, y_{\varphi}, y_{\sigma}) h_8(z_{\sigma}, y_{\varphi}, y_{\sigma}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi})$$

$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} \sum_{y_{\varphi}} h_4(y_{\varphi}) h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{12}(y, y_{\varphi}, z_{\sigma})$$

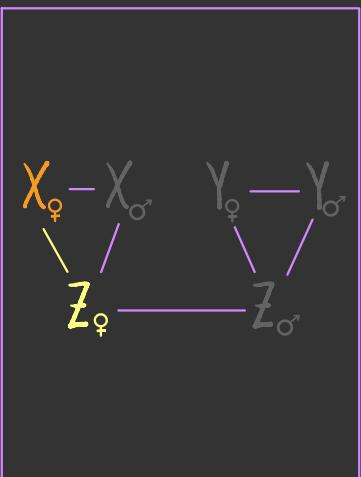
$$= \sum_{z_{\varphi}} \sum_{z_{\sigma}} h_9(z, z_{\varphi}, z_{\sigma}) h_{11}(x, z_{\varphi}) h_{13}(y, z_{\sigma})$$

$$= \sum_{z_{\varphi}} h_{11}(x, z_{\varphi}) h_{14}(y, z, z_{\varphi})$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

the cliques in this graph correspond to the factors we created



elimination order

$X_{\sigma} X_{\sigma'} Y_{\sigma} Y_{\sigma'} Z_{\sigma} Z_{\sigma'}$

$$\sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} \sum_{x_{\sigma}} \sum_{x_{\sigma'}} h_1(x_{\sigma}) h_2(x_{\sigma'}) h_3(x, x_{\sigma}, x_{\sigma'}) h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_7(z_{\sigma}, x_{\sigma}, x_{\sigma'}) h_8(z_{\sigma'}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'})$$

$$= \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} \sum_{x_{\sigma}} h_1(x_{\sigma}) h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_8(z_{\sigma}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{10}(x, x_{\sigma}, z_{\sigma})$$

$$= \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_8(z_{\sigma}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma})$$

$$= \sum_{z_{\sigma}} \sum_{y_{\sigma}} h_4(y_{\sigma}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma}) h_{12}(y, y_{\sigma}, z_{\sigma'})$$

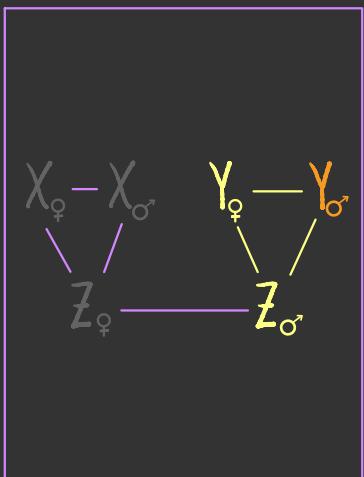
$$= \sum_{z_{\sigma}} \sum_{z_{\sigma'}} h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma}) h_{13}(y, z_{\sigma'})$$

$$= \sum_{z_{\sigma}} h_{11}(x, z_{\sigma}) h_{14}(y, z, z_{\sigma})$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

the cliques in this graph correspond to the factors we created



elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

$$\sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) h_{10}(x, x_\varphi, z_\varphi)$$

$$= \sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) h_{11}(x, z_\varphi)$$

$$= \sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} h_4(y_\varphi) h_9(z, z_\varphi, z_\sigma) h_{11}(x, z_\varphi) h_{12}(y, y_\varphi, z_\sigma)$$

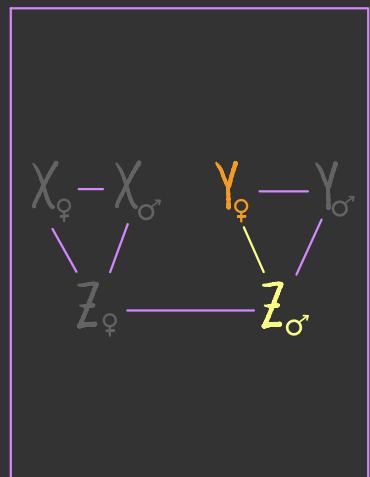
$$= \sum_{z_\varphi} \sum_{z_\sigma} h_9(z, z_\varphi, z_\sigma) h_{11}(x, z_\varphi) h_{13}(y, z_\sigma)$$

$$= \sum_{z_\varphi} h_{11}(x, z_\varphi) h_{14}(y, z, z_\varphi)$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

the cliques in this graph correspond to the factors we created



elimination order

$X_{\sigma} X_{\sigma'} Y_{\sigma} Y_{\sigma'} Z_{\sigma} Z_{\sigma'}$

$$\sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} \sum_{x_{\sigma}} \sum_{x_{\sigma'}} h_1(x_{\sigma}) h_2(x_{\sigma'}) h_3(x, x_{\sigma}, x_{\sigma'}) h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_7(z_{\sigma}, x_{\sigma}, x_{\sigma'}) h_8(z_{\sigma'}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'})$$

$$= \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} \sum_{x_{\sigma}} h_1(x_{\sigma}) h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_8(z_{\sigma}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{10}(x, x_{\sigma}, z_{\sigma})$$

$$= \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} \sum_{y_{\sigma'}} h_4(y_{\sigma}) h_5(y_{\sigma'}) h_6(y, y_{\sigma}, y_{\sigma'}) h_8(z_{\sigma}, y_{\sigma}, y_{\sigma'}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma})$$

$$= \sum_{z_{\sigma}} \sum_{z_{\sigma'}} \sum_{y_{\sigma}} h_4(y_{\sigma}) h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma}) h_{12}(y, y_{\sigma}, z_{\sigma'})$$

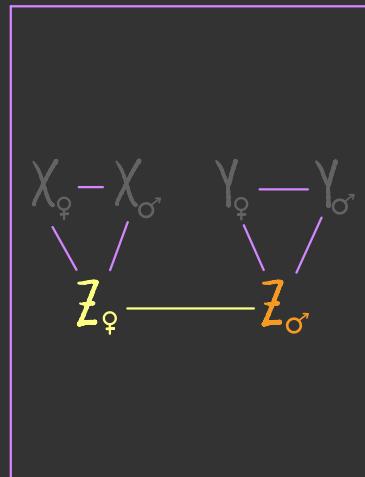
$$= \sum_{z_{\sigma}} \sum_{z_{\sigma'}} h_9(z, z_{\sigma}, z_{\sigma'}) h_{11}(x, z_{\sigma}) h_{13}(y, z_{\sigma'})$$

$$= \sum_{z_{\sigma}} h_{11}(x, z_{\sigma}) h_{14}(y, z, z_{\sigma})$$

$$= h_{15}(x, y, z)$$

$$= P(x, y, z)$$

the cliques in this graph correspond to the factors we created



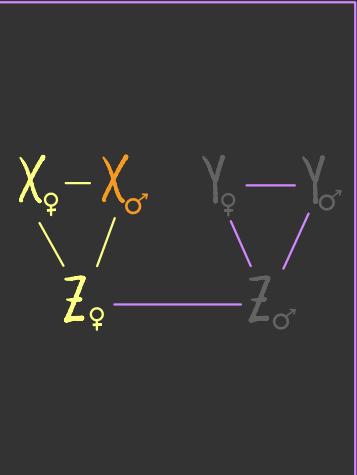
elimination order
 $X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

$$\sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{x_\varphi} h_1(x_\varphi) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) h_{10}(x, x_\varphi, z_\varphi)$$

$$h_{10}(x, x_\varphi, z_\varphi) = \sum_{x_\sigma} h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma)$$

this was the "hardest" factor to compute



elimination order
 $X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

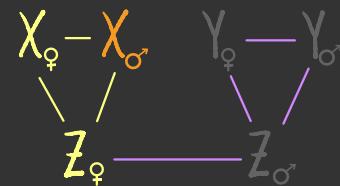
$$\sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{x_\varphi} h_1(x_\varphi) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) h_{10}(x, x_\varphi, z_\varphi)$$

$$h_{10}(x, x_\varphi, z_\varphi) = \sum_{x_\sigma} h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma)$$

3 possible values

how many terms did we need to compute?



elimination order
 $X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

$$\sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma)$$

$$= \sum_{z_\varphi} \sum_{z_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{x_\varphi} h_1(x_\varphi) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_8(z_\sigma, y_\varphi, y_\sigma) h_9(z, z_\varphi, z_\sigma) h_{10}(x, x_\varphi, z_\varphi)$$

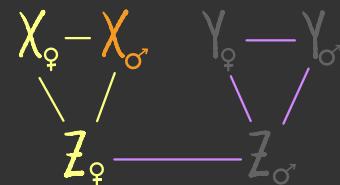
3^2 sums

a sum of 3 terms

$$h_{10}(x, x_\varphi, z_\varphi) = \sum_{x_\sigma} h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma)$$

3 possible values

we needed to compute
 3^3 terms



elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

to compute the most difficult factor, we needed to compute

3^5 terms

3^4 sums

a sum of 3 terms

$$h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) = \sum_{z_\varphi} h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi)$$

3^2 sums

a sum of 3 terms

$$h_{10}(x, x_\varphi, z_\varphi) = \sum_{x_\sigma} h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma)$$

elimination order

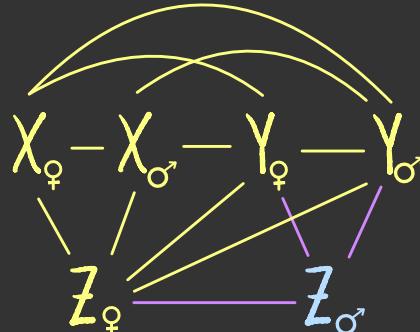
$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

to compute the most difficult factor, we needed to compute

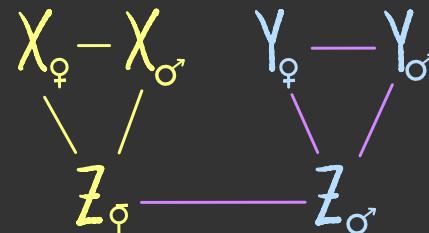
3^5 terms



to compute the most difficult factor, we needed to compute

3^3 terms

size of largest clique



elimination order

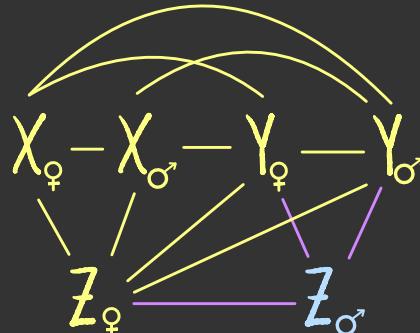
$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

to compute the most difficult factor, we needed to compute

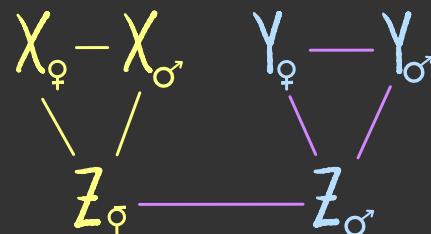
3^5 terms



to compute the most difficult factor, we needed to compute

3^3 terms

size of variable domains

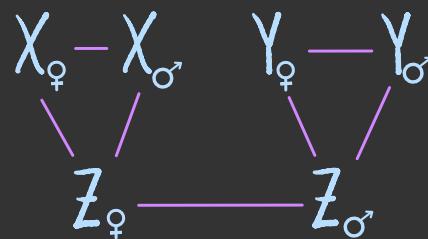
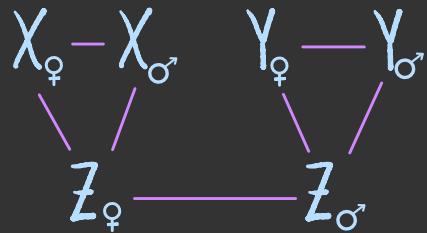


elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$



max: 3

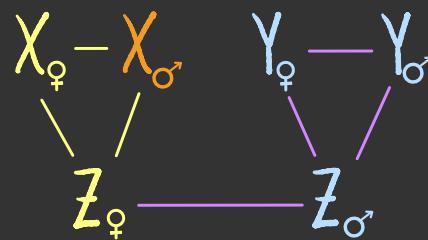
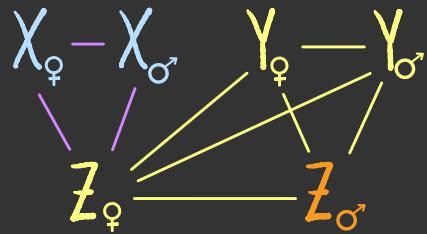
max: 3

elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$

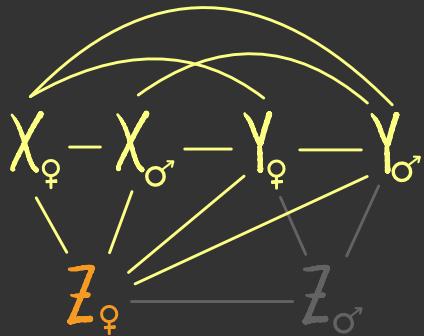


max: 4

max: 3

elimination order

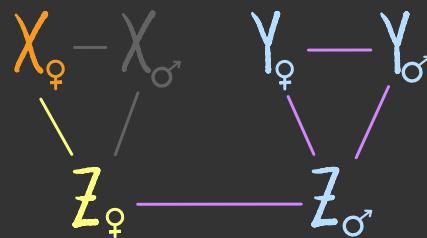
Z_σ Z_φ Y_σ Y_φ X_σ X_φ



max: ~~5~~ 5

elimination order

X_σ X_φ Y_σ Y_φ Z_σ Z_φ



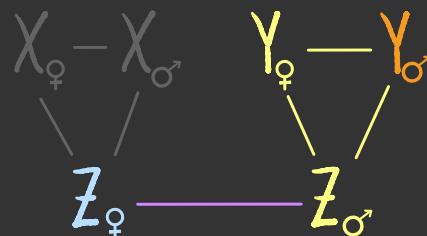
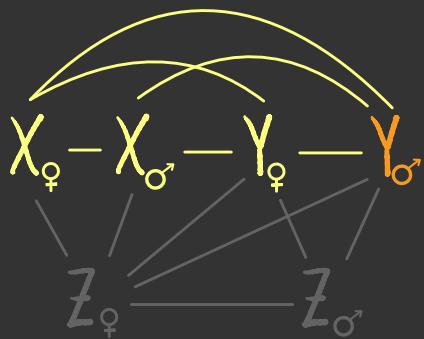
max: 3

elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$



max: ~~5~~ 5

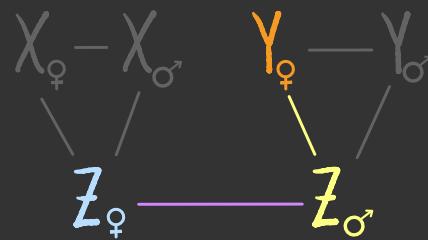
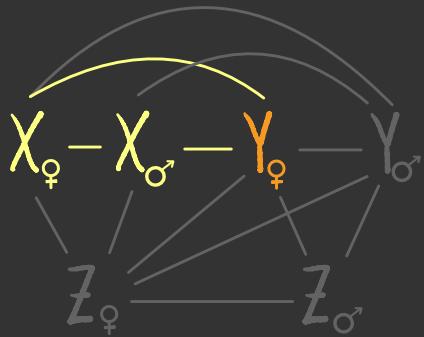
max: 3

elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$



max: ~~5~~ 5

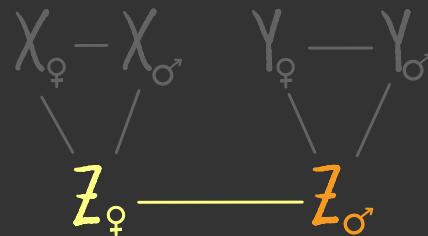
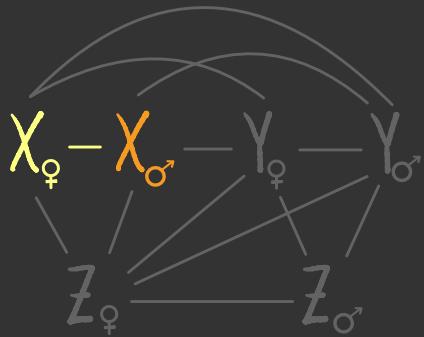
max: 3

elimination order

Z_σ Z_φ Y_σ Y_φ X_σ X_φ

elimination order

X_σ X_φ Y_σ Y_φ Z_σ Z_φ

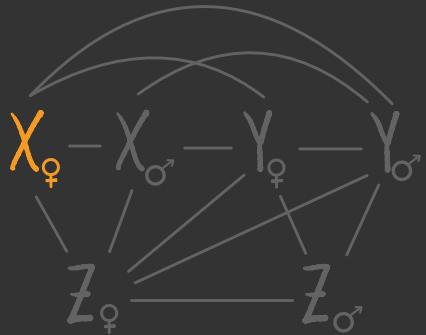


max: ~~8~~ ~~5~~ 5

max: 3

elimination order

Z_σ Z_φ Y_σ Y_φ X_σ X_φ



max: ~~8~~ ~~5~~ 5

elimination order

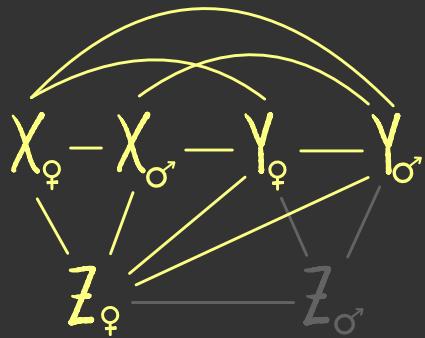
X_σ X_φ Y_σ Y_φ Z_σ Z_φ



max: 3

elimination order

$Z_\sigma Z_\varphi Y_\sigma Y_\varphi X_\sigma X_\varphi$

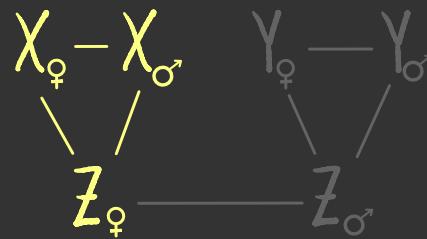


width: 5

this called the width of the elimination order

elimination order

$X_\sigma X_\varphi Y_\sigma Y_\varphi Z_\sigma Z_\varphi$



width : 3

let w be the width of the elimination order

let d be the size of the largest variable domain

the most difficult factor requires us to sum $O(d^w)$ terms

$$\begin{aligned} & \sum_{x_q} \sum_{x_o} \sum_{y_q} \sum_{y_o} \sum_{z_q} h_1(x_q) h_2(x_o) h_3(x, x_q, x_o) h_4(y_q) h_5(y_o) h_6(y, y_q, y_o) h_7(z_q, x_q, x_o) h_{10}(y_q, y_o, z, z_q) \\ = & \sum_{x_q} \sum_{x_o} \sum_{y_q} \sum_{y_o} h_1(x_q) h_2(x_o) h_3(x, x_q, x_o) h_4(y_q) h_5(y_o) h_6(y, y_q, y_o) h_{11}(x_q, x_o, y_q, y_o, z) \\ = & \sum_{x_q} \sum_{x_o} \sum_{y_q} h_1(x_q) h_2(x_o) h_3(x, x_q, x_o) h_4(y_q) h_{12}(x_q, x_o, y_q, z) \\ = & \sum_{x_q} \sum_{x_o} h_1(x_q) h_2(x_o) h_3(x, x_q, x_o) h_{13}(x_q, x_o, y, z) \\ = & \sum_{x_q} h_1(x_q) h_{14}(x_q, x, y, z) \\ = & h_{15}(x, y, z) \\ = & P(x, y, z) \end{aligned}$$

let w be the width of the elimination order

let d be the size of the largest variable domain

the most difficult factor requires us to sum $O(d^w)$ terms

$$\begin{aligned} & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z) \\ = & \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z) \\ = & h_{15}(x, y, z) \\ = & P(x, y, z) \end{aligned}$$

we compute a new factor
for each variable we
eliminate

let w be the width of the elimination order

let d be the size of the largest variable domain

the most difficult factor requires us to sum $O(d^w)$ terms

$$\begin{aligned} & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z) \\ = & \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z) \\ = & h_{15}(x, y, z) \\ = & P(x, y, z) \end{aligned}$$

let n be the number of variables
in the bayesian network
we compute up to n new factors

let w be the width of the elimination order

let d be the size of the largest variable domain

the most difficult factor requires us to sum $O(d^w)$ terms

$$\begin{aligned} & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} \sum_{z_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_7(z_\varphi, x_\varphi, x_\sigma) h_{10}(y_\varphi, y_\sigma, z, z_\varphi) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} \sum_{y_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_5(y_\sigma) h_6(y, y_\varphi, y_\sigma) h_{11}(x_\varphi, x_\sigma, y_\varphi, y_\sigma, z) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} \sum_{y_\varphi} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_4(y_\varphi) h_{12}(x_\varphi, x_\sigma, y_\varphi, z) \\ = & \sum_{x_\varphi} \sum_{x_\sigma} h_1(x_\varphi) h_2(x_\sigma) h_3(x, x_\varphi, x_\sigma) h_{13}(x_\varphi, x_\sigma, y, z) \\ = & \sum_{x_\varphi} h_1(x_\varphi) h_{14}(x_\varphi, x, y, z) \\ = & h_{15}(x, y, z) \\ = & P(x, y, z) \end{aligned}$$

so the worst-case runtime is $O(nd^w)$

let n be the number of variables
in the bayesian network
we compute up to n new factors