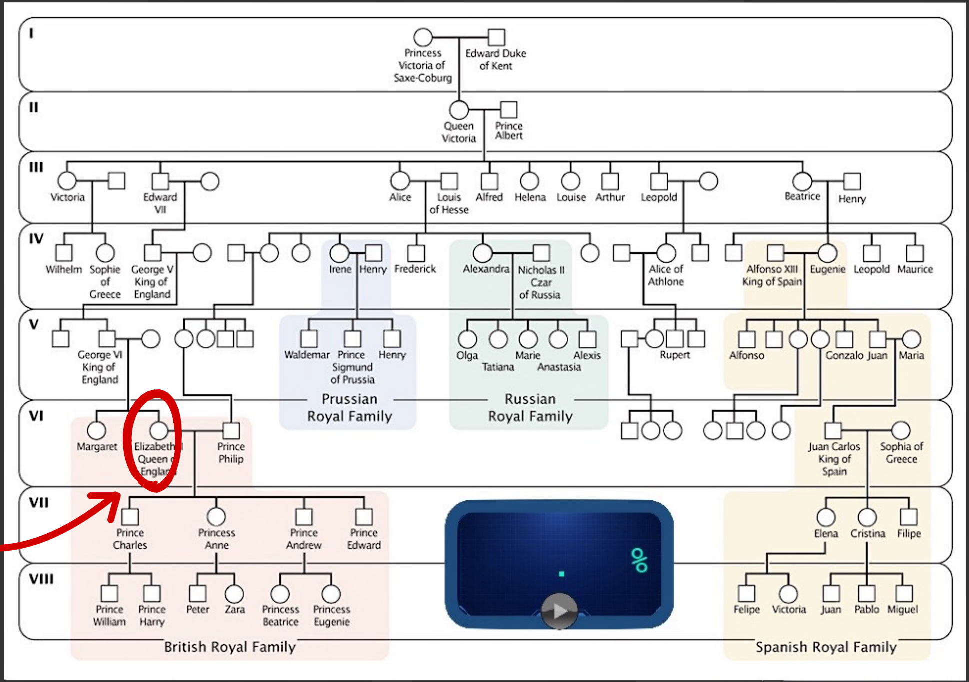
the junction tree algorithm

21 oct 2022

CSCI 373

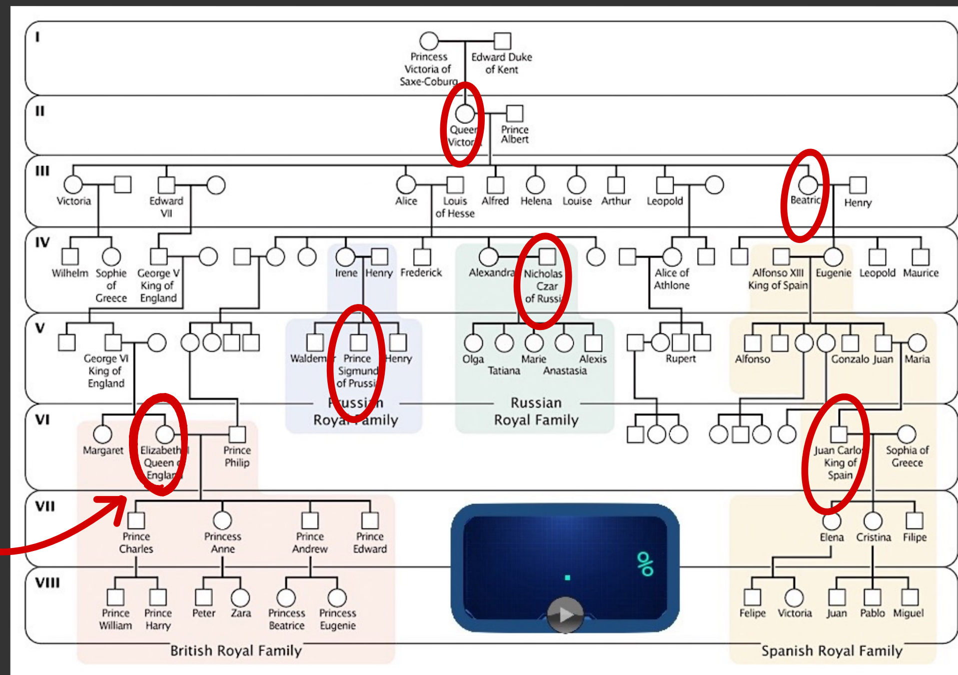
variable elimination computes a single marginal or conditional distribution

probability that elizabeth is a carrier for **hemophilia**

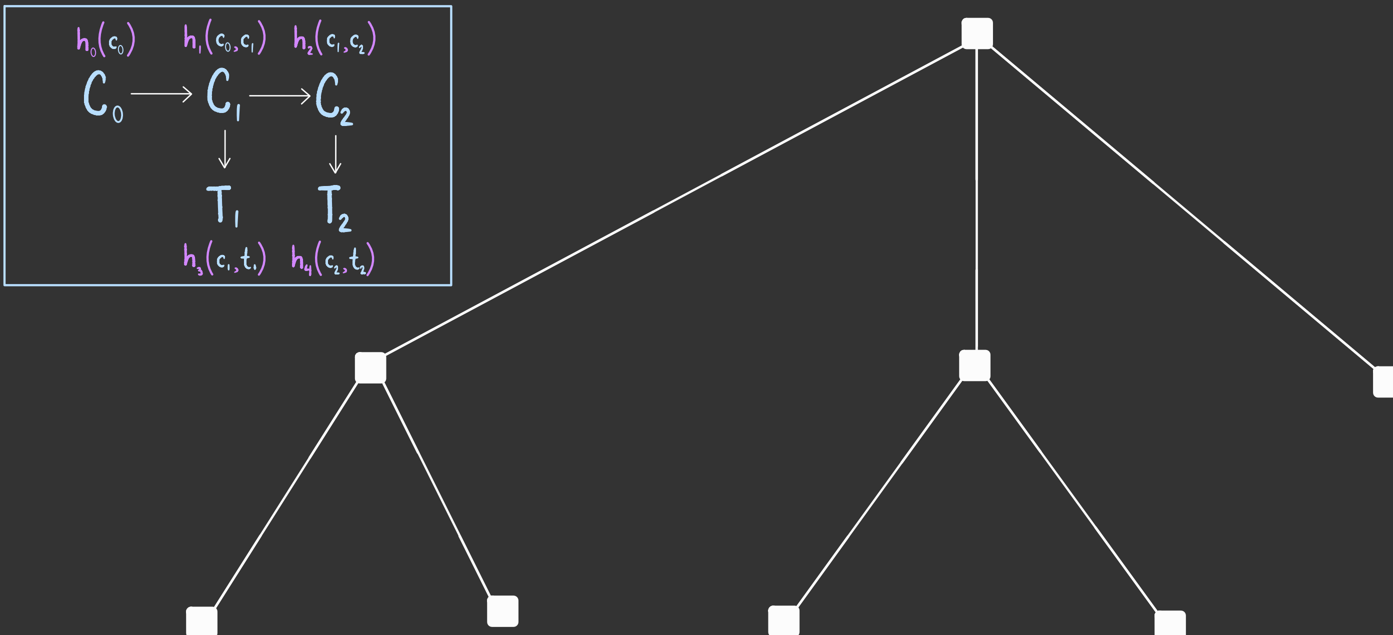


but what if wanted to compute these distributions for every variable of a network?

probability that PERSON is a carrier for hemophilia



step one: start with a tree whose leaves are labeled with the factors of the bayesian network



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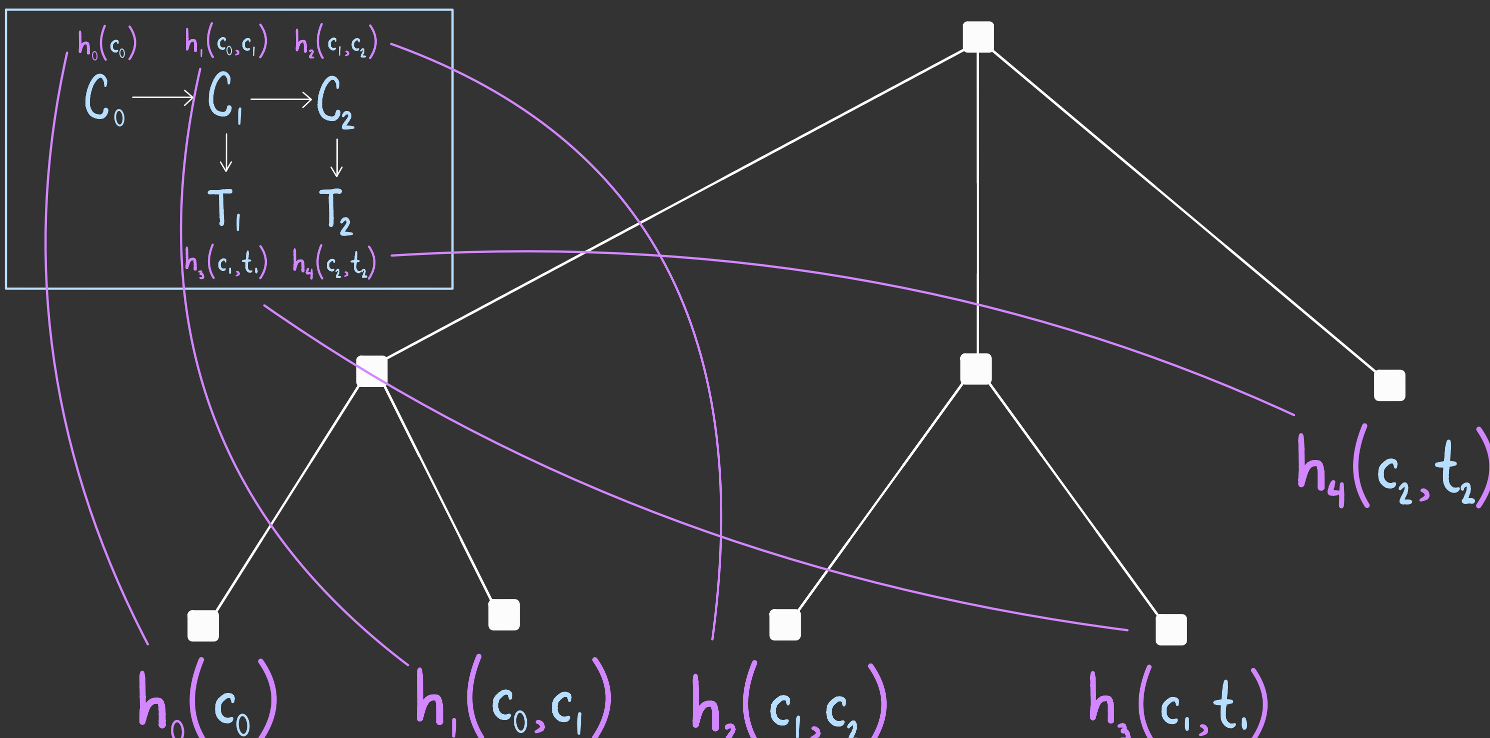

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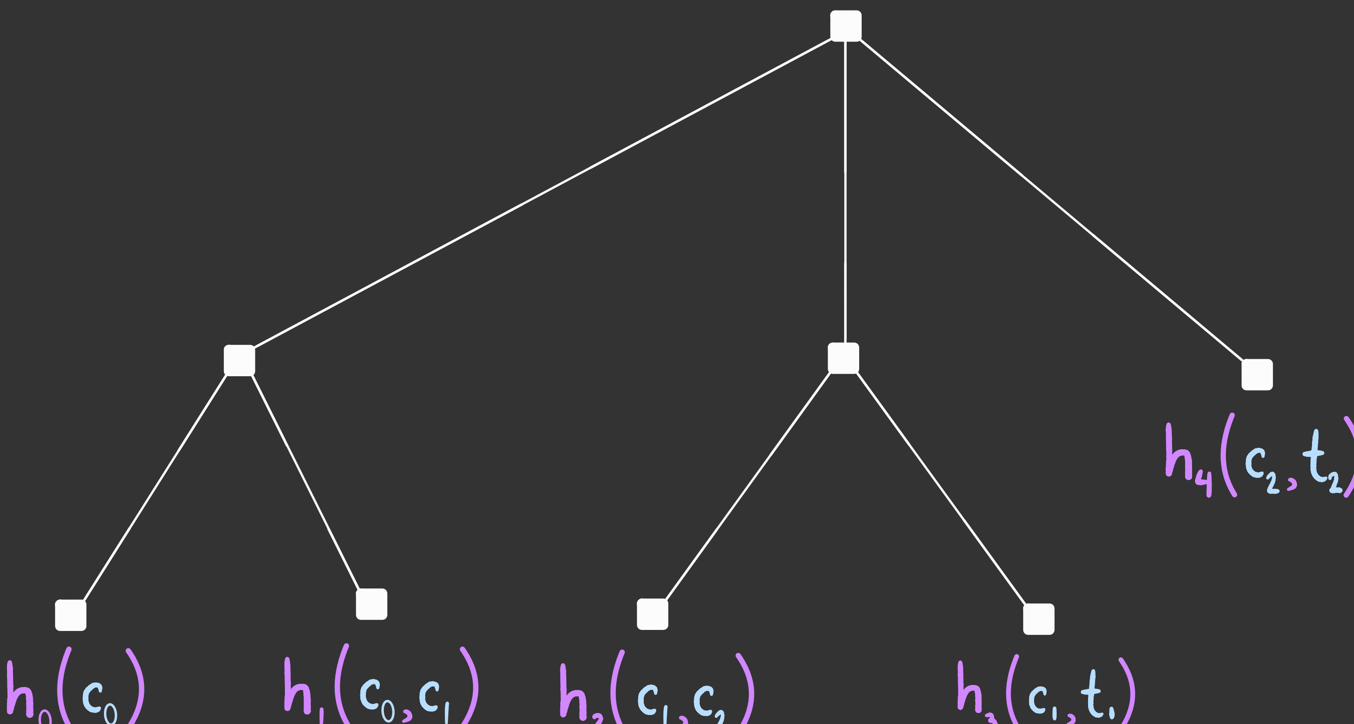
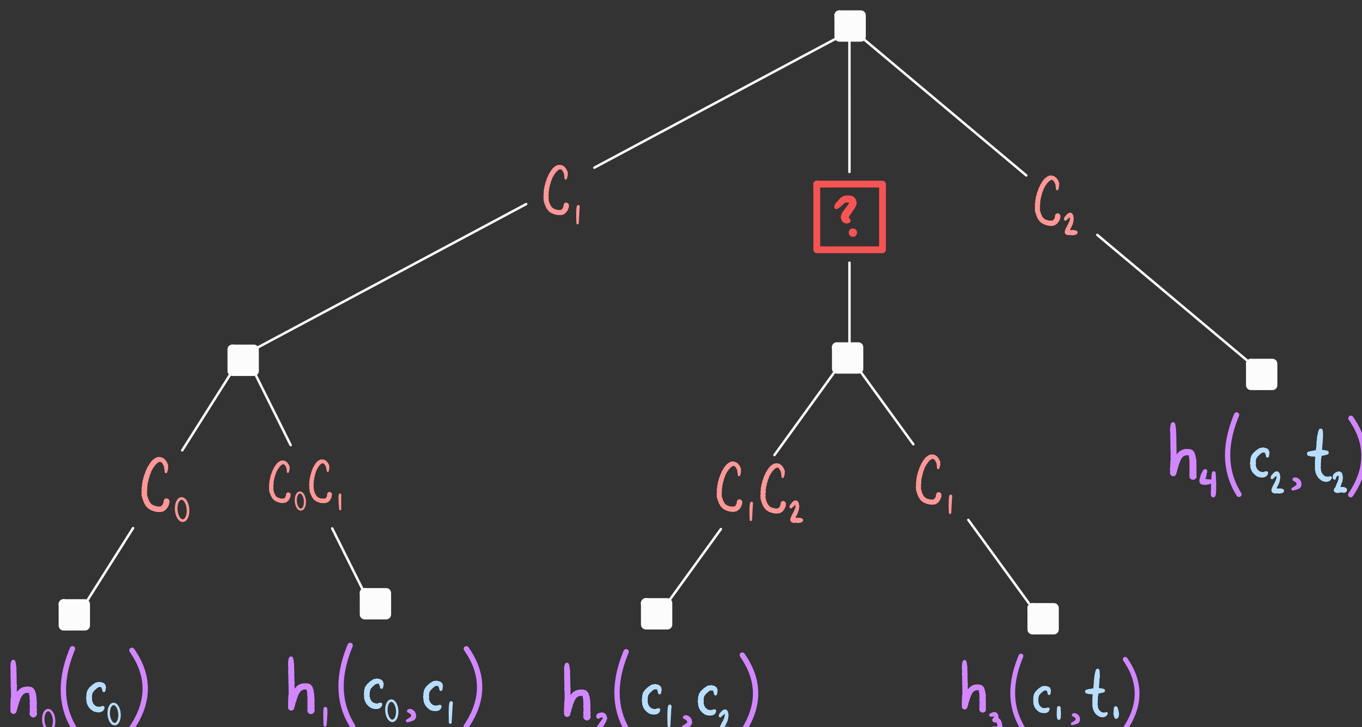


step two: label the edges with their separators, the set of variables that appear in factors on both sides of the edge



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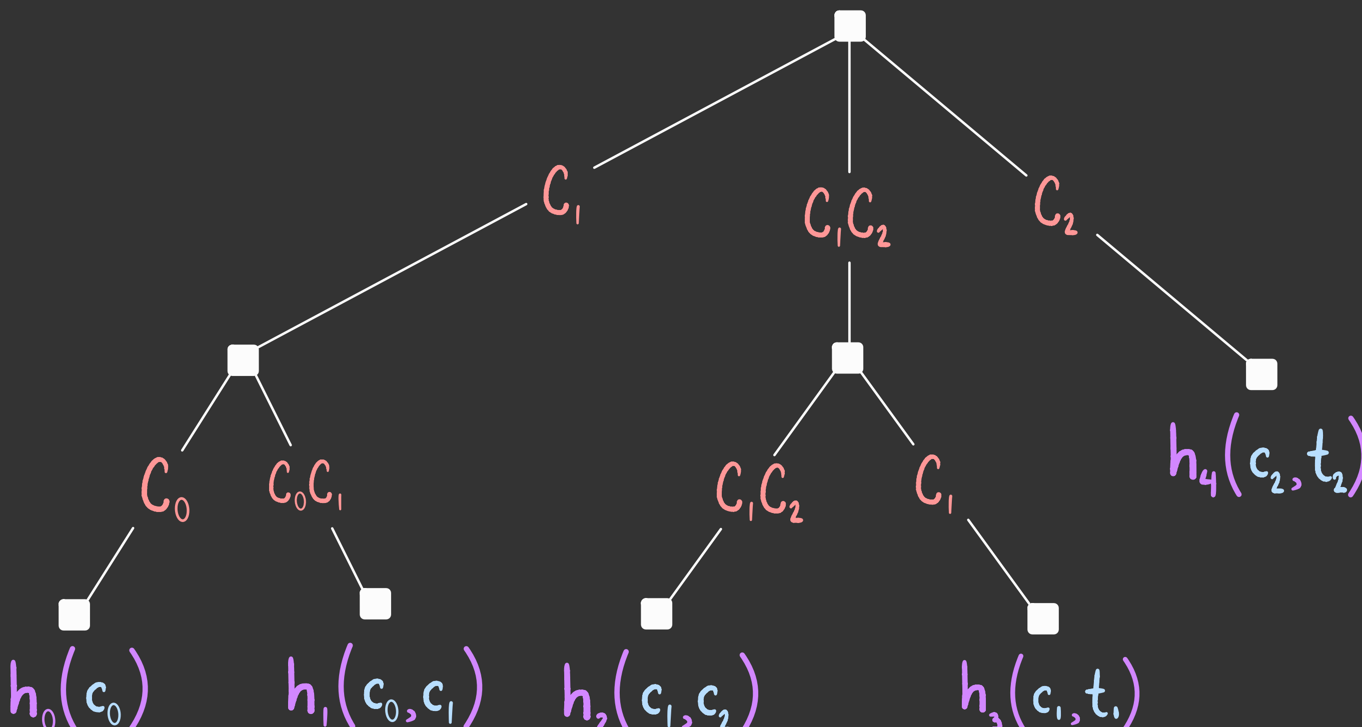
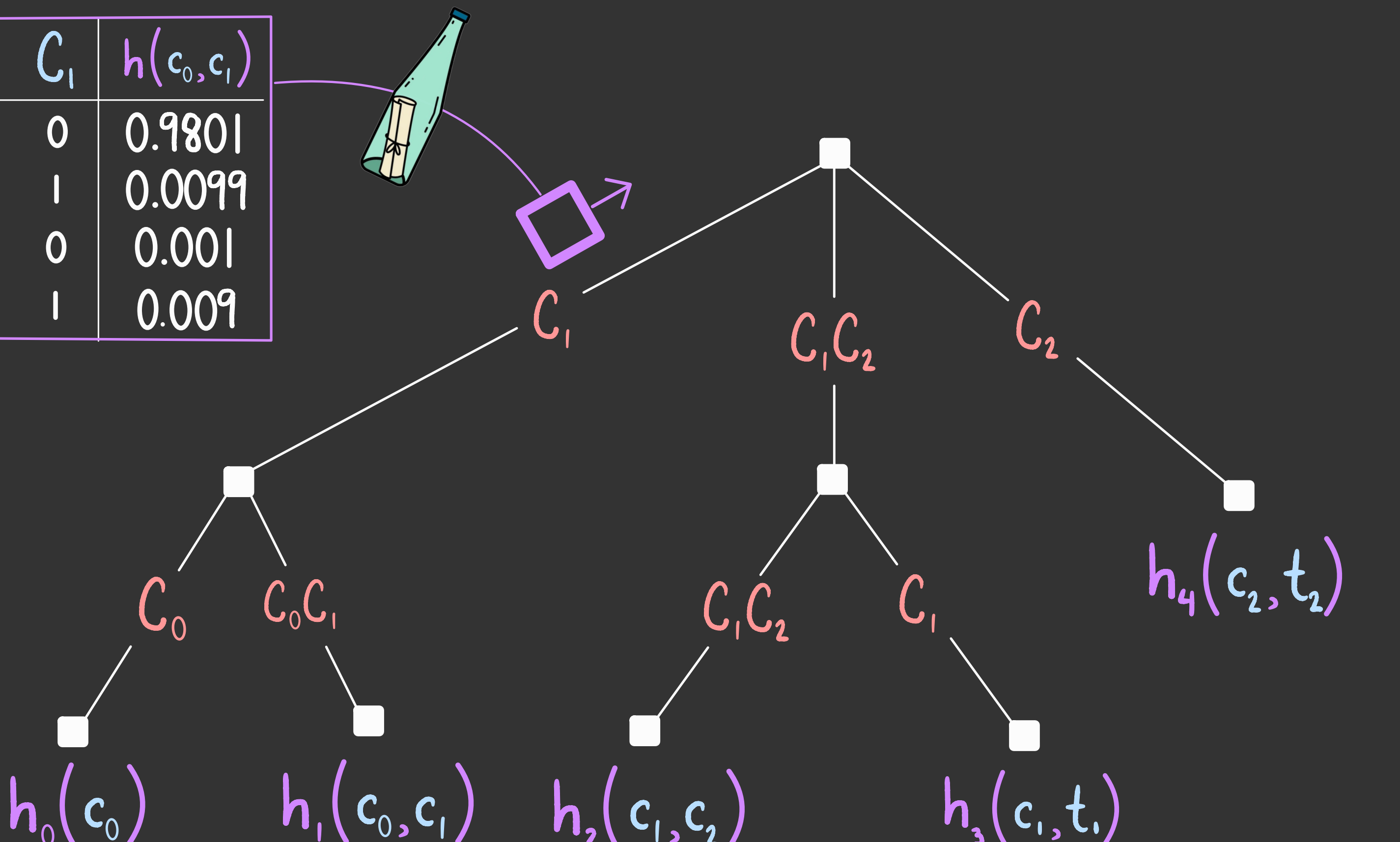


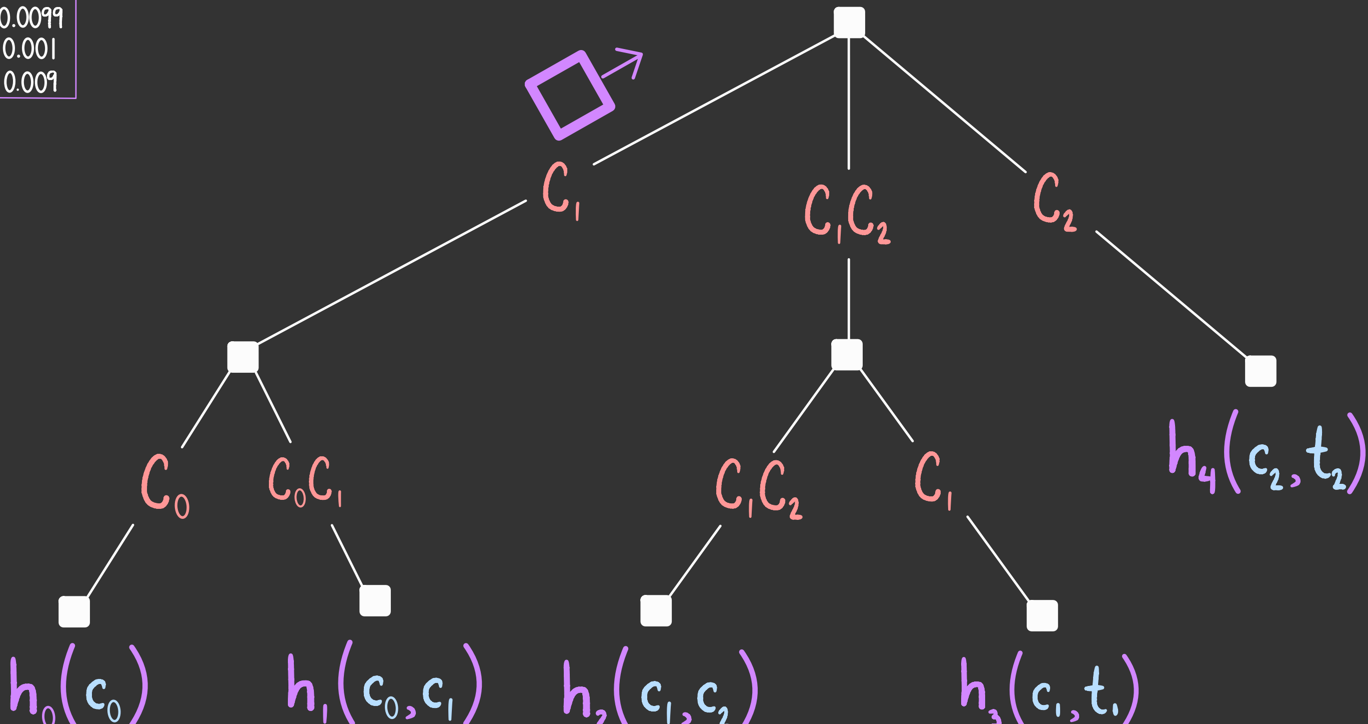
step three: the nodes start sending factors to their neighbors

C_0	C_1	$h(c_0, c_1)$
0	0	0.9801
0	1	0.0099
1	0	0.001
1	1	0.009



step three: but only the separator variables survive the journey
the rest are marginalized out using 

C_0	C_1	$h(c_0, c_1)$
0	0	0.9801
0	1	0.0099
1	0	0.001
1	1	0.009

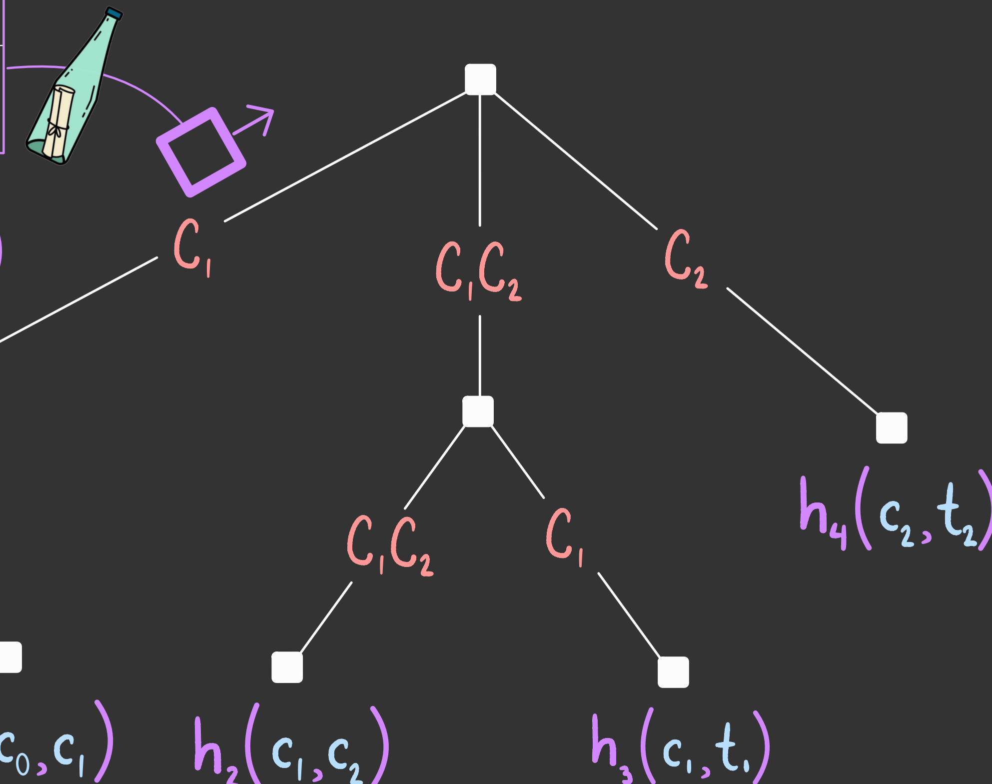


step three: but only the separator variables survive the journey
the rest are marginalized out using total

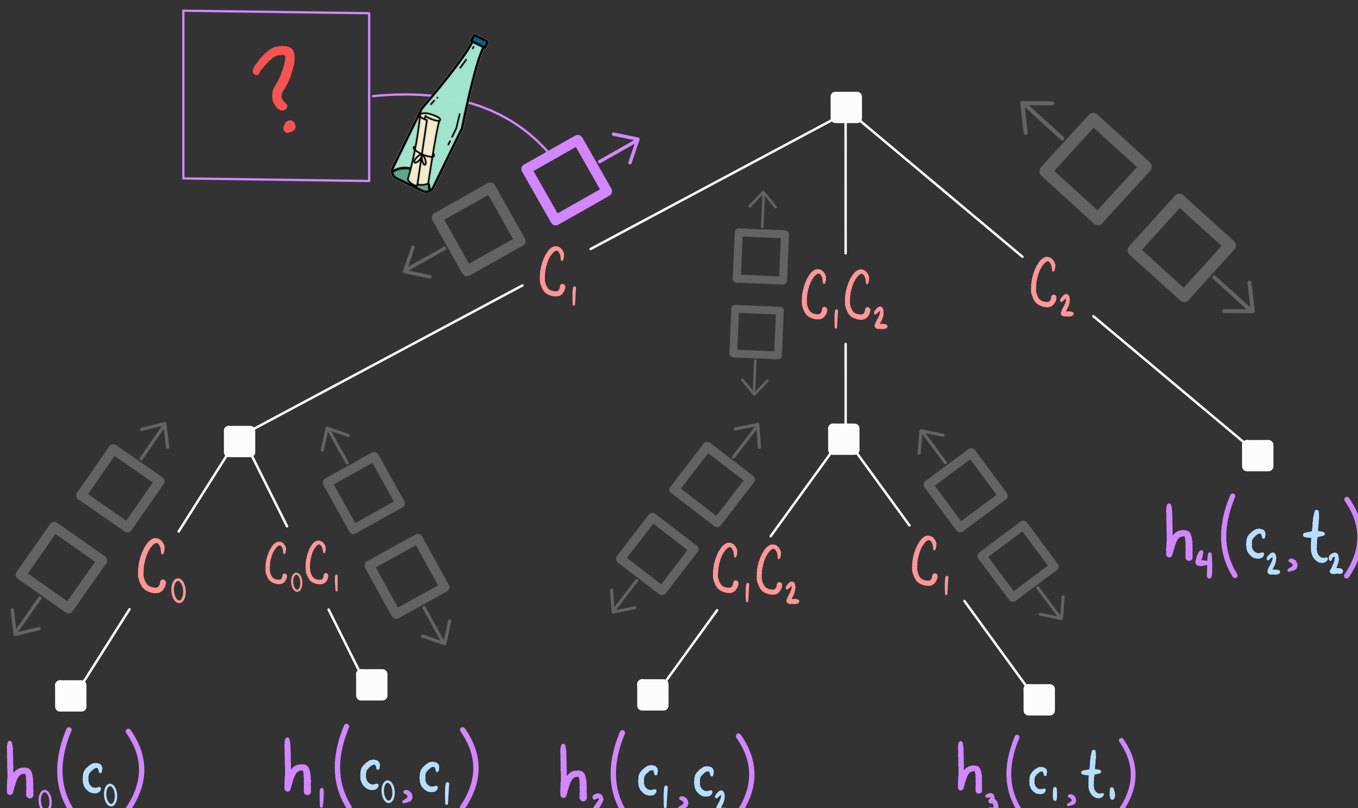
C_0	C_1	$h(c_0,c_1)$
0	0	0.9801
0	1	0.0099
1	0	0.001
1	1	0.009

C_1	$h(c_1)$
0	0.9811
1	0.0189

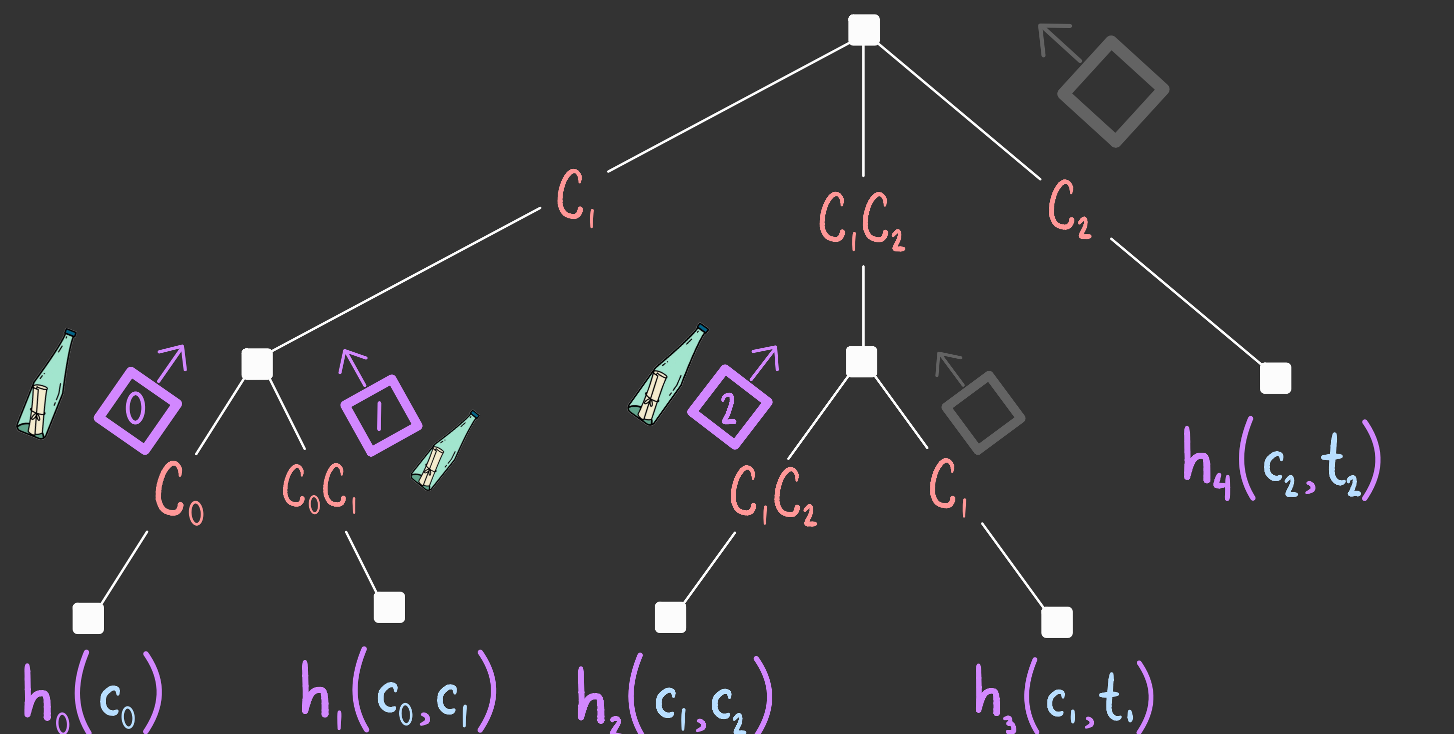
$$h(c_1) = \sum_{c_0} h(c_0, c_1)$$



step three: but what factor does each node send?

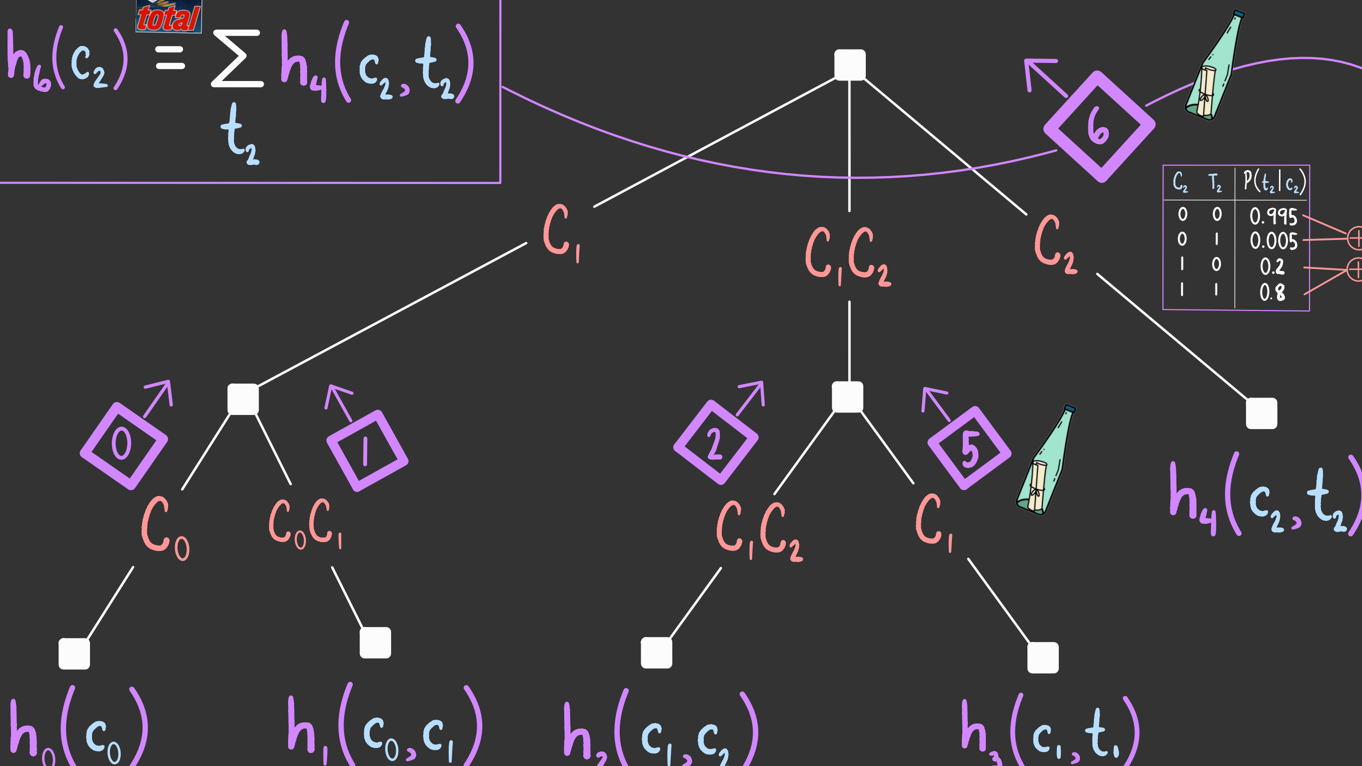


step three: leaf nodes send their factor to their only neighbor



step three: leaf nodes send their factor to their only neighbor, after marginalizing out non-separator variables

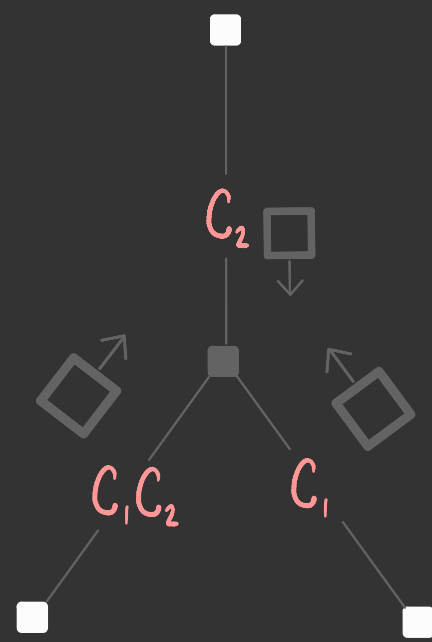
$$h_6(c_2) = \sum_{t_2} h_4(c_2, t_2)$$



C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

C_2	$h_6(c_2)$
0	1.0
1	1.0

step three: each internal node waits to receive factors from its neighbors



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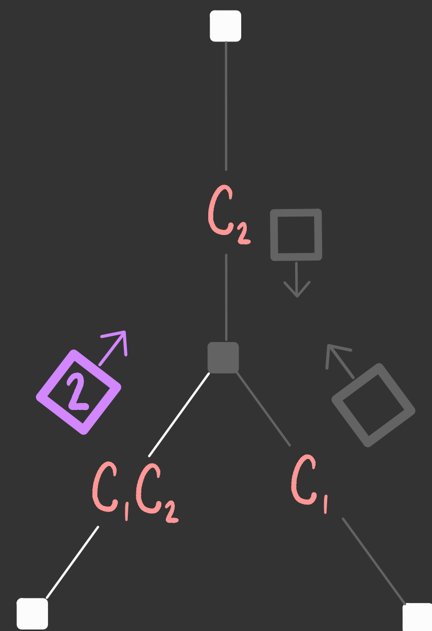

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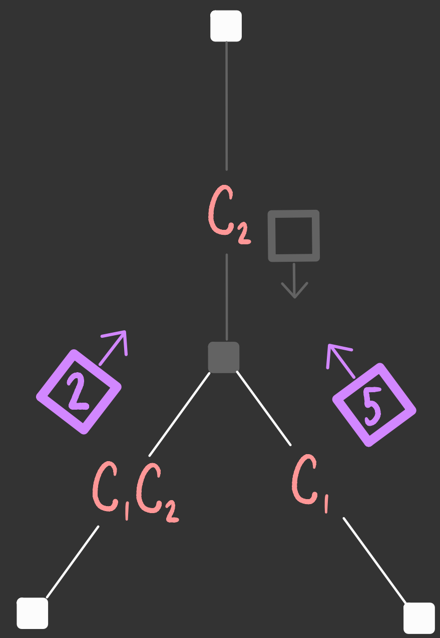
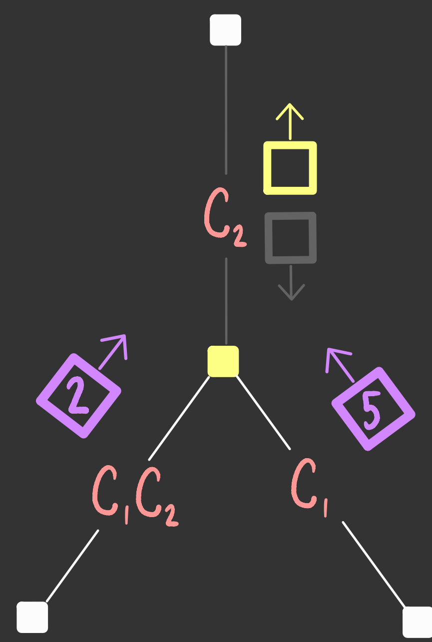
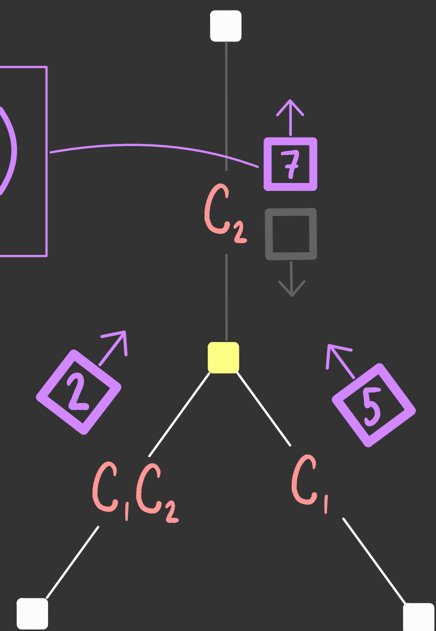


step three: once the node has received factors from all neighbors but one, it can send a factor to the remaining neighbor



step three: this new factor will be the product of the received factors...

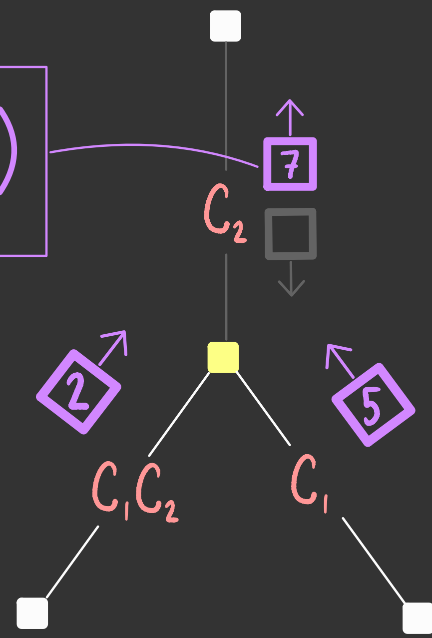
$$\boxed{h_7(c_1, c_2) = h_2(c_1, c_2)\, h_5(c_1)}$$



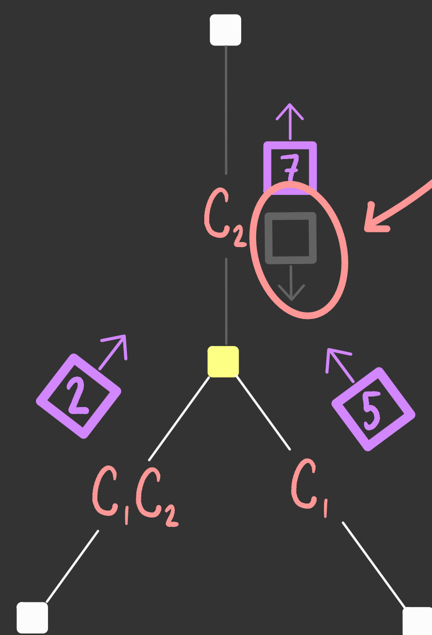
step three: this new factor will be the product of the received factors...

after **marginalizing** out variables not in the separator with that neighbor

$$h_7(c_2) = \sum_{c_1} h_2(c_1, c_2)\, h_5(c_1)$$

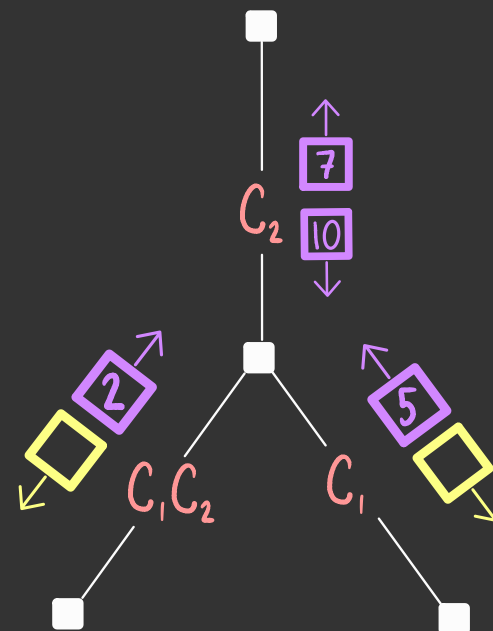


step three: once we receive the factor from the final neighbor...



step three: once we receive the factor from the final neighbor...

we can send analogous factors to our other neighbors



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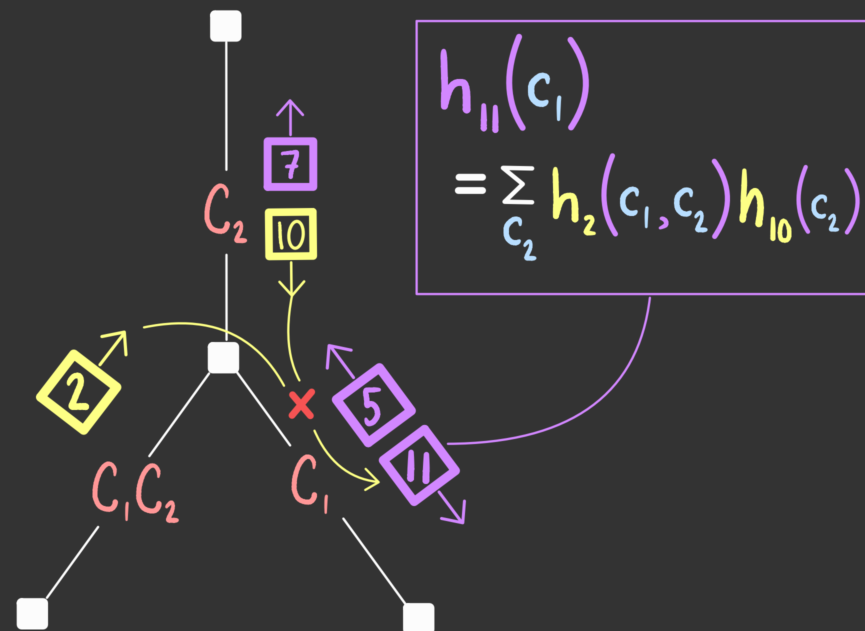
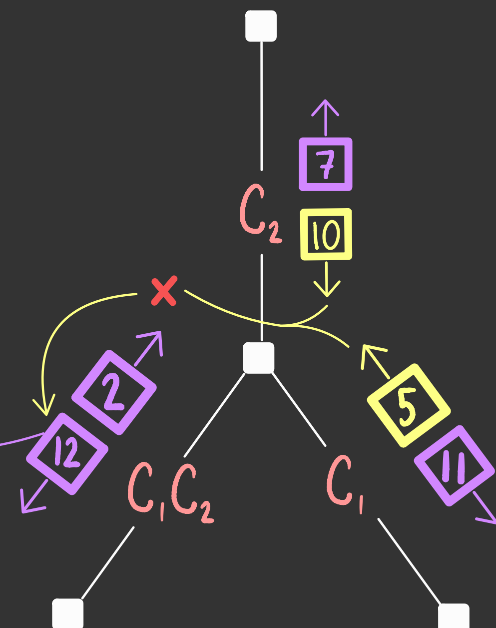


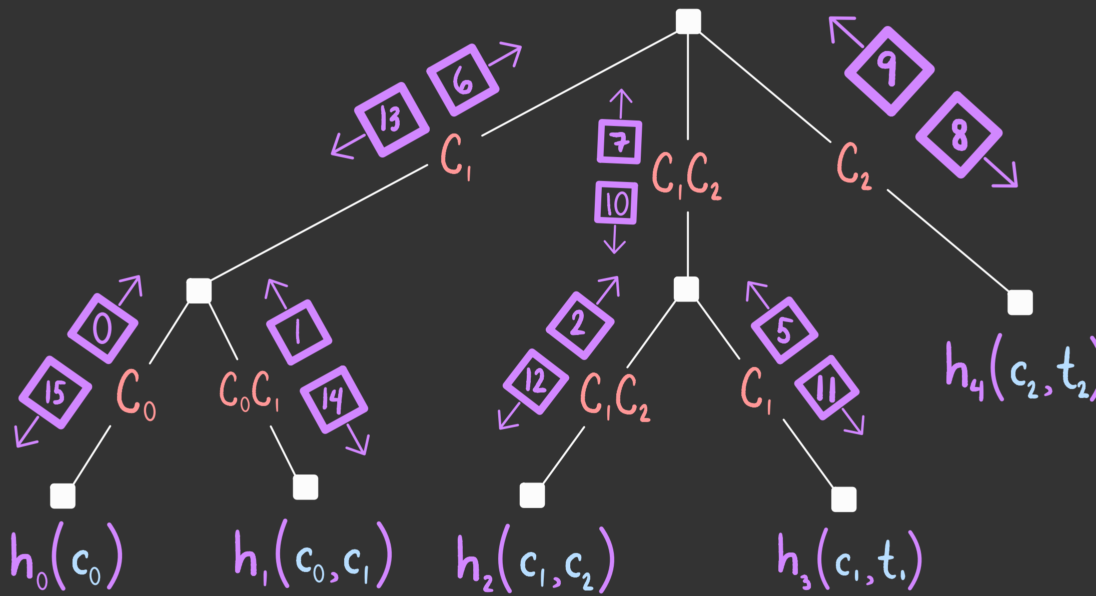
step three: once we receive the factor from the final neighbor...

we can send analogous factors to our other neighbors

$$h_{12}(c_1, c_2) = h_5(c_1)\, h_{10}(c_2)$$



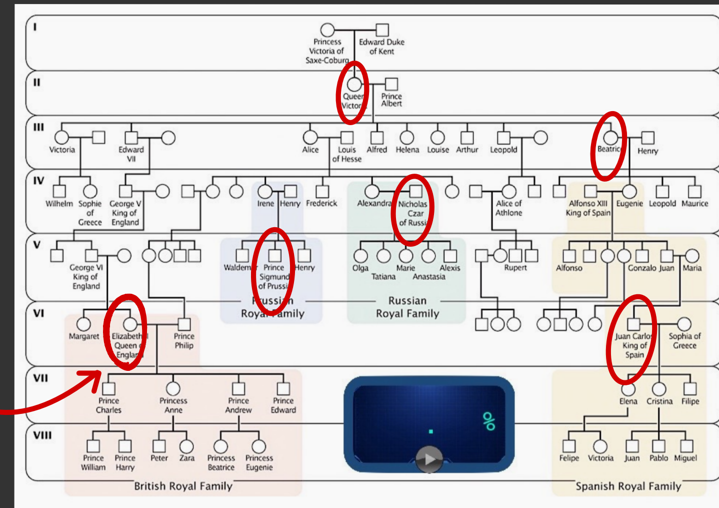
$\boxed{5}$	$\sum_{t_1} h_3(c_1, t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)$
$\boxed{7}$	$h_2(c_1, c_2)\, h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1)\, h_7(c_1, c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2, t_2)$
$\boxed{10}$	$h_9(c_2)\, h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1, c_2)\, h_{10}(c_1, c_2)$
$\boxed{12}$	$h_5(c_1)\, h_{10}(c_1, c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1, c_2)\, h_9(c_2)$
$\boxed{14}$	$h_0(c_0)\, h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0, c_1)\, h_{13}(c_1)$



running this algorithm (the junction tree algorithm) gives us every factor we need to compute all the single-variable marginals

5	$\sum_{t_1} h_3(c_1, t_1)$
6	$\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$
7	$h_2(c_1, c_2) h_5(c_1)$
8	$\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$
9	$\sum_{t_2} h_4(c_2, t_2)$
10	$h_9(c_2)\, h_6(c_1)$
11	$\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$
12	$h_5(c_1) h_{10}(c_1, c_2)$
13	$\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$
14	$h_0(c_0) h_{13}(c_1)$
15	$\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

probability that PERSON is a carrier for hemophilia



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$\boxed{9}$	$\sum_{t_2} h_4(c_2, t_2)$
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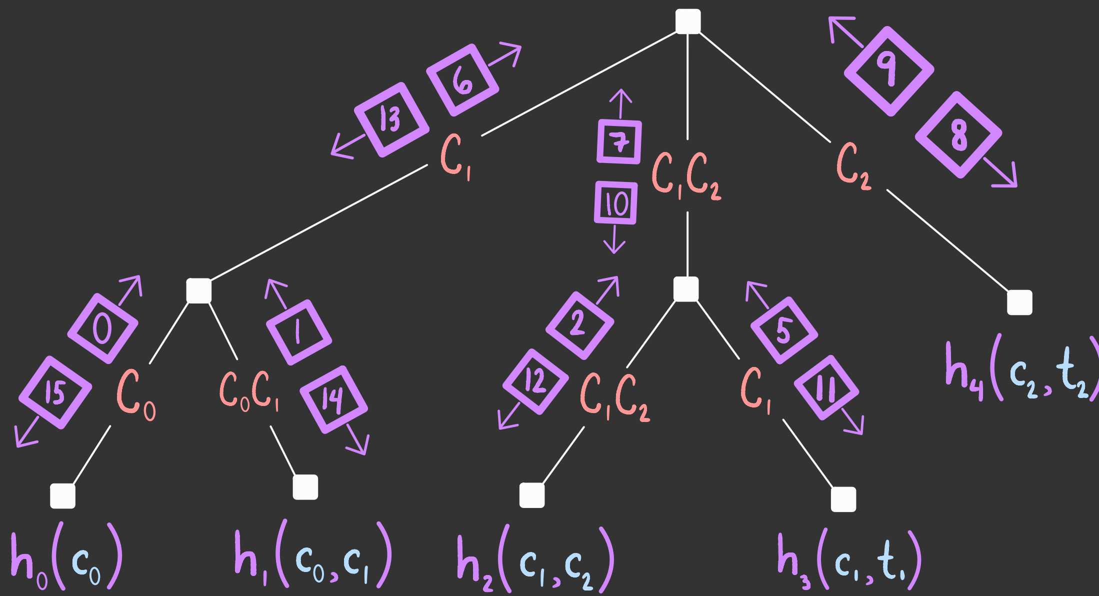

to compute a single-variable marginal:

5	$\sum_{t_1} h_3(c_1, t_1)$
6	$\sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)$
7	$h_2(c_1, c_2)\, h_5(c_1)$
8	$\sum_{c_1} h_6(c_1)\, h_7(c_1, c_2)$
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12	$h_5(c_1)\, h_{10}(c_1, c_2)$
13	$\sum_{c_2} h_7(c_1, c_2)\, h_9(c_2)$
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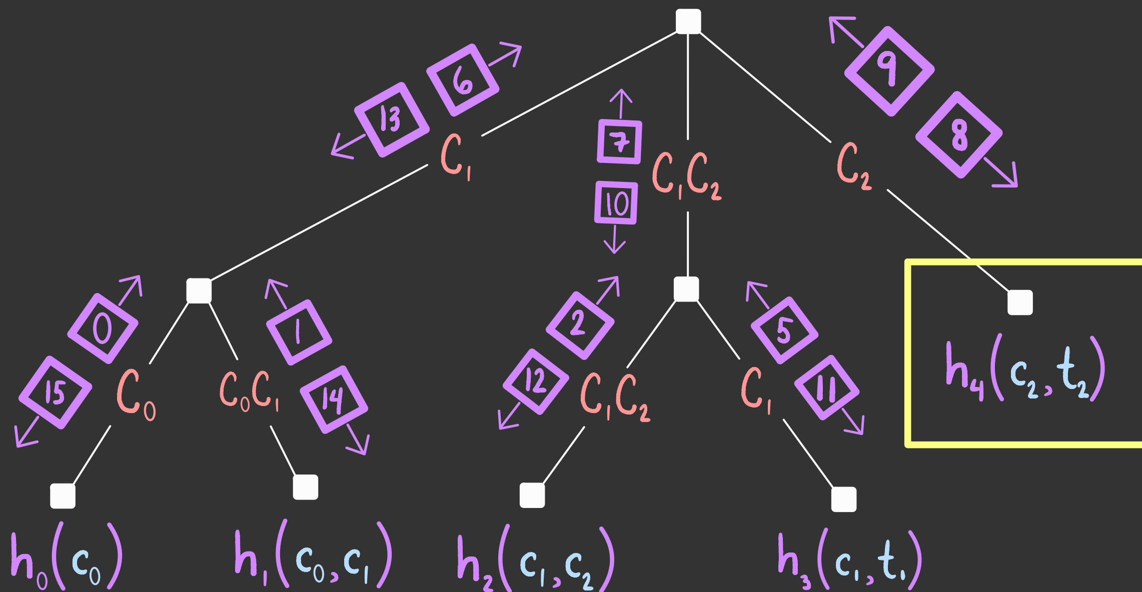
15	$\sum_{c_1} h_1(c_0, c_1)\, h_{13}(c_1)$



to compute a single-variable marginal:

- choose a leaf containing the variable

$\boxed{5}$	$\sum_{t_1} h_3(c_1, t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$
$\boxed{7}$	$h_2(c_1, c_2) h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2, t_2)$
$\boxed{10}$	$h_9(c_2)\, h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$
$\boxed{12}$	$h_5(c_1) h_{10}(c_1, c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$
$\boxed{14}$	$h_0(c_0) h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

$$h(c_2, t_2) = h_8(c_2)\, h_4(c_2, t_2)$$

8

■

$h_4(c_2, t_2)$

to compute a single-variable marginal:

- choose a leaf containing the variable
- multiply its factor with its received "message"

5	$\sum_{t_1} h_3(c_1, t_1)$
6	$\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$
7	$h_2(c_1, c_2) h_5(c_1)$
8	$\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$
9	$\sum_{t_2} h_4(c_2, t_2)$
10	$h_9(c_2)\, h_6(c_1)$
11	$\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$
12	$h_5(c_1) h_{10}(c_1, c_2)$
13	$\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$
14	$h_0(c_0) h_{13}(c_1)$
15	$\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

we obtain the marginal

$$p(t_2) = \sum_{c_2} h_8(c_2) h_4(c_2, t_2)$$

8

$$\boxed{h_4(c_2, t_2)}$$

to compute a single-variable marginal:

- choose a leaf containing the variable
- multiply its factor with its received "message"
- marginalize out other variables

$\boxed{5}$ $\sum_{t_1} h_3(c_1, t_1)$

$\boxed{6}$ $\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$

$\boxed{7}$ $h_2(c_1, c_2) h_5(c_1)$

$\boxed{8}$ $\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$

$\boxed{9}$ $\sum_{t_2} h_4(c_2, t_2)$

$\boxed{10}$ $h_9(c_2)\, h_6(c_1)$

$\boxed{11}$ $\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$

$\boxed{12}$ $h_5(c_1) h_{10}(c_1, c_2)$

$\boxed{13}$ $\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$

$\boxed{14}$ $h_0(c_0) h_{13}(c_1)$

$\boxed{15}$ $\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

why does this work?

to compute a single-variable marginal:

- choose a leaf containing the variable
- multiply its factor with its received "message"
- marginalize out other variables

$\boxed{5}$	$\sum_{t_1} h_3(c_1,t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0)h_1(c_0,c_1)$
$\boxed{7}$	$h_2(c_1,c_2)h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1)h_7(c_1,c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2,t_2)$
$\boxed{10}$	$h_9(c_2)\,h_6(c_1)$
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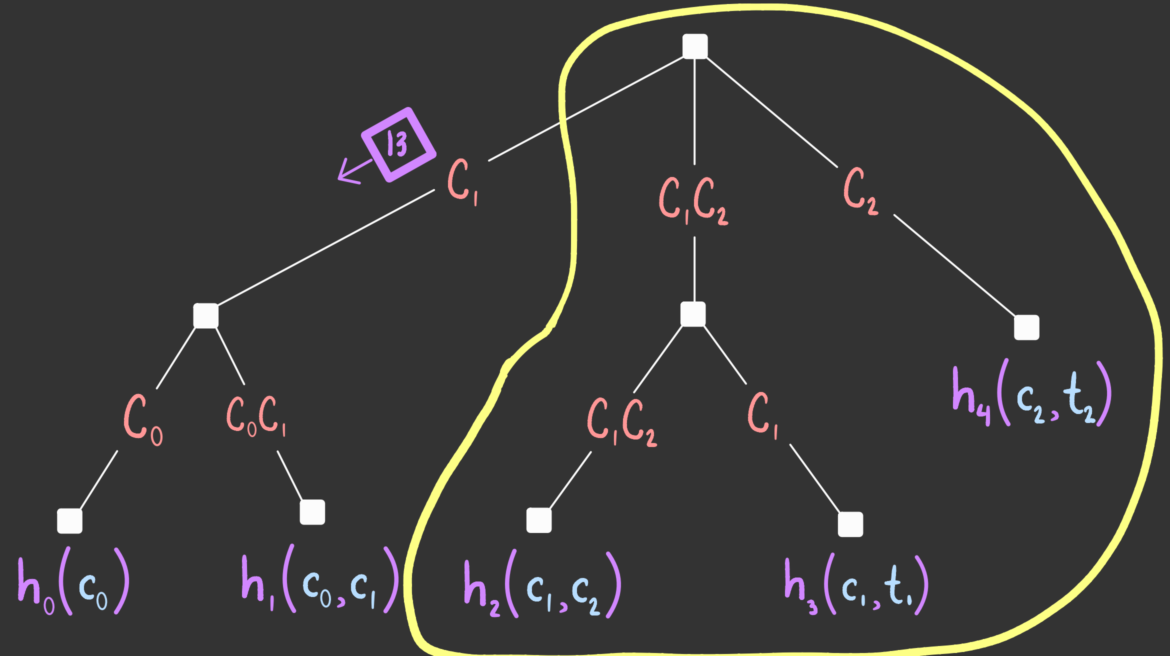


the message that a node sends its neighbor is the product of the factors on its side of the edge...

$\boxed{5}$ $\sum_{t_1} h_3(c_1, t_1)$

$\boxed{6}$ $\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$

$\boxed{7}$ $h_2(c_1, c_2) h_5(c_1)$

$\boxed{8}$ $\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$

$\boxed{9}$ $\sum_{t_2} h_4(c_2, t_2)$

$\boxed{10}$ $h_9(c_2)\, h_6(c_1)$

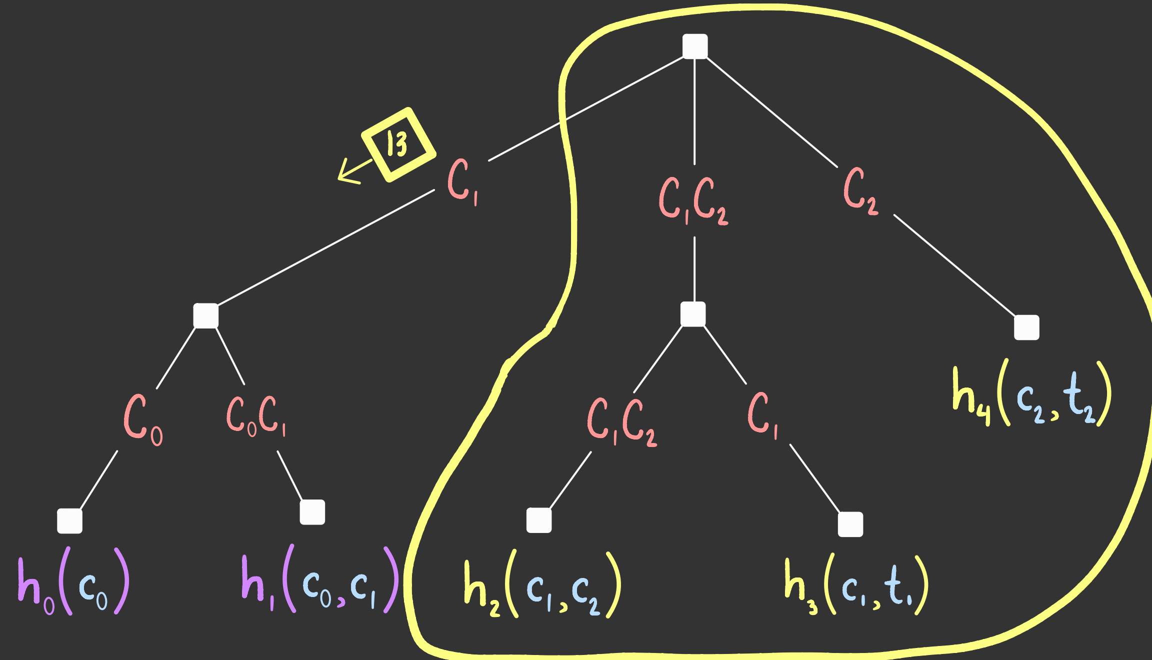
$\boxed{11}$ $\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$

$\boxed{12}$ $h_5(c_1) h_{10}(c_1, c_2)$

$\boxed{13}$ $\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$

$\boxed{14}$ $h_0(c_0) h_{13}(c_1)$

$\boxed{15}$ $\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$



the message that a node sends its neighbor is the product of the factors on its side of the edge...

$\boxed{13}$ $h_2(c_1, c_2)\, h_3(c_1, t_1)\, h_4(c_2, t_2)$

$\boxed{5}$ $\sum_{t_1} h_3(c_1, t_1)$

$\boxed{6}$ $\sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)$

$\boxed{7}$ $h_2(c_1, c_2)\, h_5(c_1)$

$\boxed{8}$ $\sum_{c_1} h_6(c_1)\, h_7(c_1, c_2)$

$\boxed{9}$ $\sum_{t_2} h_4(c_2, t_2)$

$\boxed{10}$ $h_9(c_2)\, h_6(c_1)$

$\boxed{11}$ $\sum_{c_2} h_2(c_1, c_2)\, h_{10}(c_1, c_2)$

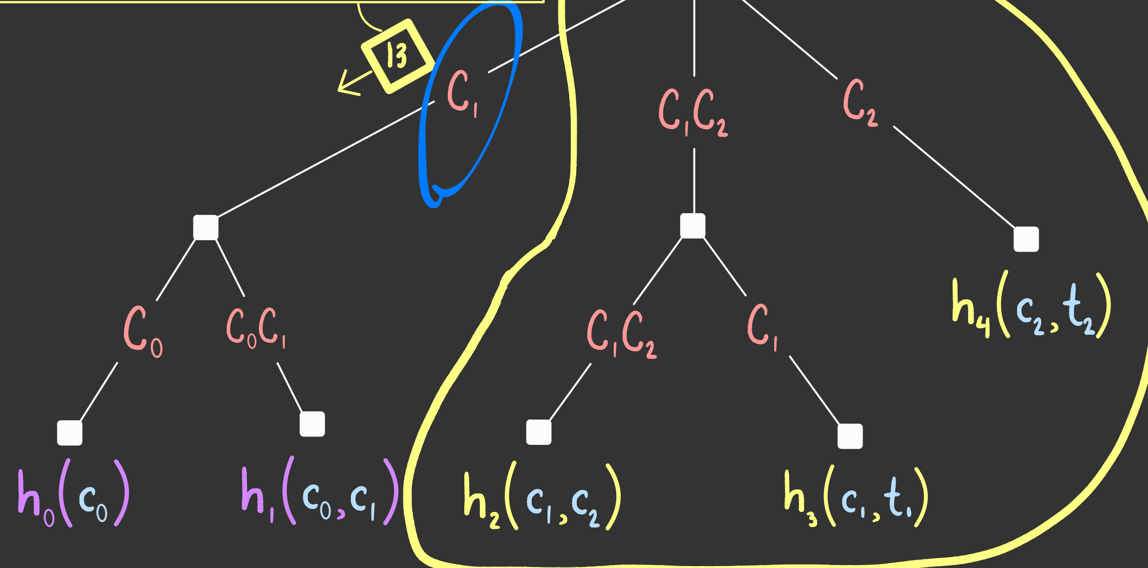
$\boxed{12}$ $h_5(c_1)\, h_{10}(c_1, c_2)$

$\boxed{13}$ $\sum_{c_2} h_7(c_1, c_2)\, h_9(c_2)$

$\boxed{14}$ $h_0(c_0)\, h_{13}(c_1)$

$\boxed{15}$ $\sum_{c_1} h_1(c_0, c_1)\, h_{13}(c_1)$

claim: $\sum_{c_2} \sum_{t_1} \sum_{t_2} h_2(c_1, c_2)\, h_3(c_1, t_1)\, h_4(c_2, t_2)$

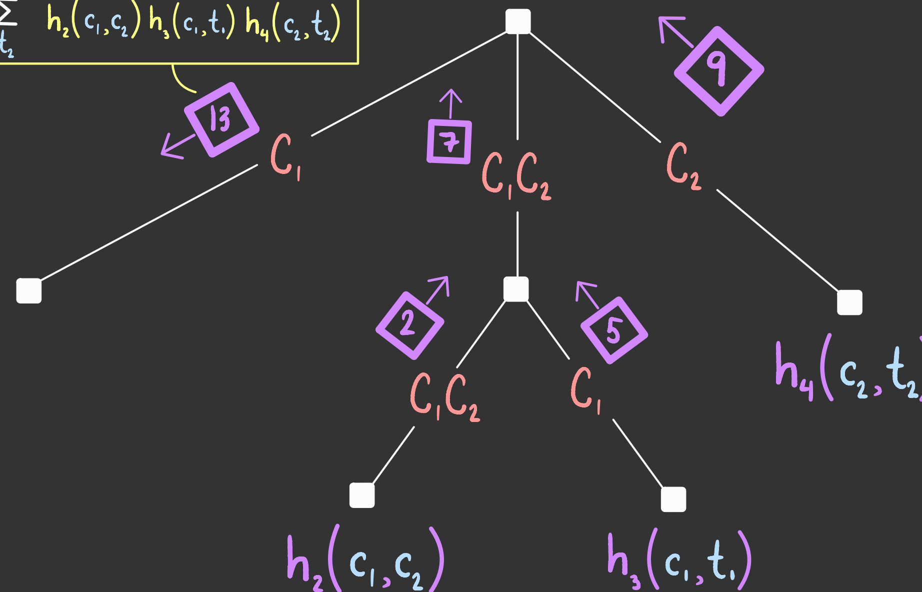


the message that a node sends its neighbor is the product of the factors on its side of the edge...
with the non-separator variables marginalized out

$\boxed{13}$ $\sum_{c_2} \sum_{t_1} \sum_{t_2} h_2(c_1, c_2)\, h_3(c_1, t_1)\, h_4(c_2, t_2)$

$\boxed{5}$	$\sum_{t_1} h_3(c_1, t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$
$\boxed{7}$	$h_2(c_1, c_2) h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2, t_2)$
$\boxed{10}$	$h_9(c_2) h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$
$\boxed{12}$	$h_5(c_1) h_{10}(c_1, c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$
$\boxed{14}$	$h_0(c_0) h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

Claim: $\sum_{c_2} \sum_{t_1} \sum_{t_2} h_2(c_1, c_2) h_3(c_1, t_1) h_4(c_2, t_2)$



$\boxed{13}$ $\sum_{c_2} h_7(c_1, c_2) h_9(c_2) =$

$\boxed{5}$ $\sum_{t_1} h_3(c_1, t_1)$

$\boxed{6}$ $\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$

$\boxed{7}$ $h_2(c_1, c_2) h_5(c_1)$

$\boxed{8}$ $\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$

$\boxed{9}$ $\sum_{t_2} h_4(c_2, t_2)$

$\boxed{10}$ $h_9(c_2) h_6(c_1)$

$\boxed{11}$ $\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$

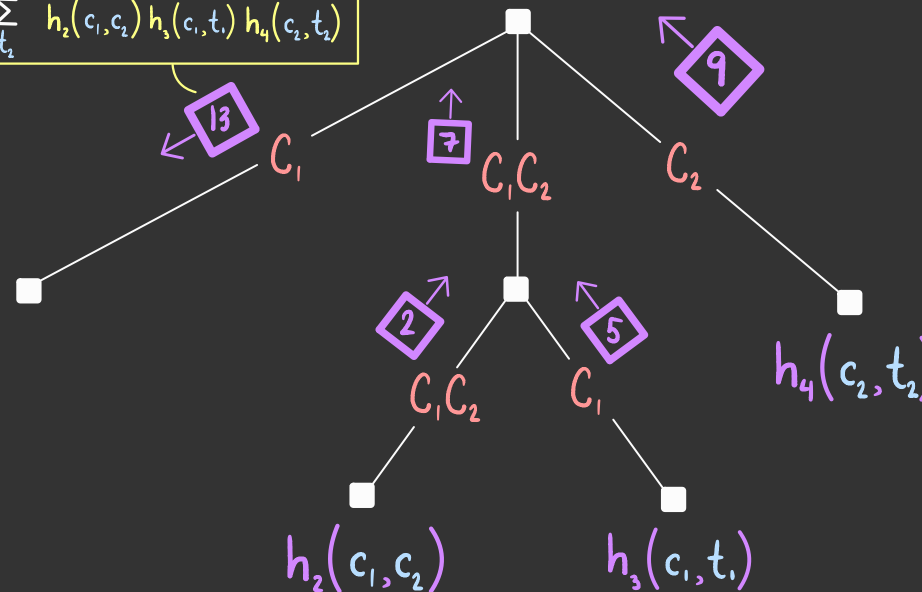
$\boxed{12}$ $h_5(c_1) h_{10}(c_1, c_2)$

$\boxed{13}$ $\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$

$\boxed{14}$ $h_0(c_0) h_{13}(c_1)$

$\boxed{15}$ $\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

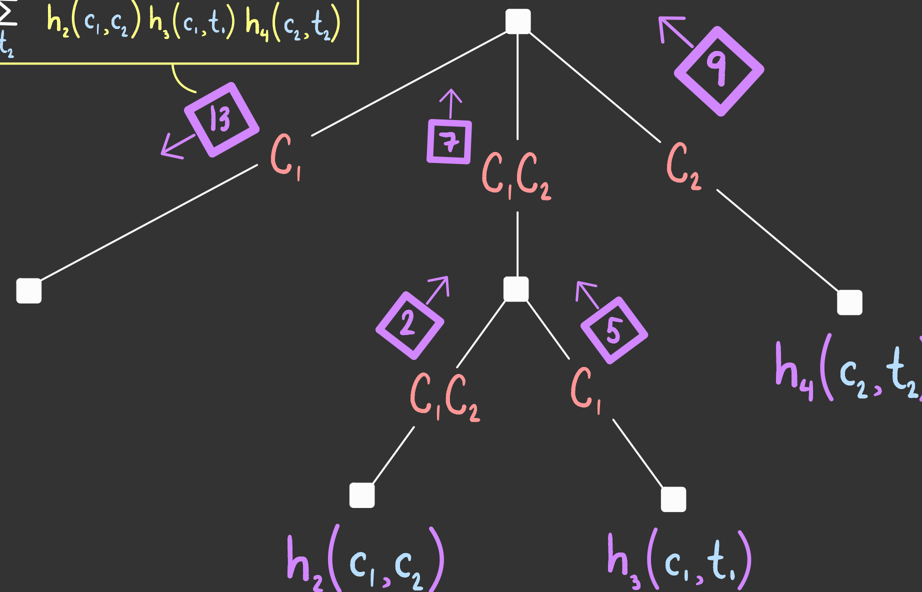
Claim: $\sum_{c_2} \sum_{t_1} \sum_{t_2} h_2(c_1, c_2) h_3(c_1, t_1) h_4(c_2, t_2)$



$\boxed{13}$ $\sum_{c_2} h_7(c_1, c_2) h_9(c_2) = \sum_{c_2} h_2(c_1, c_2) h_5(c_1) h_9(c_2)$

$\boxed{5}$	$\sum_{t_1} h_3(c_1, t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)$
$\boxed{7}$	$h_2(c_1, c_2)\, h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1)\, h_7(c_1, c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2, t_2)$
$\boxed{10}$	$h_9(c_2)\, h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1, c_2)\, h_{10}(c_1, c_2)$
$\boxed{12}$	$h_5(c_1)\, h_{10}(c_1, c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1, c_2)\, h_9(c_2)$
$\boxed{14}$	$h_0(c_0)\, h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0, c_1)\, h_{13}(c_1)$

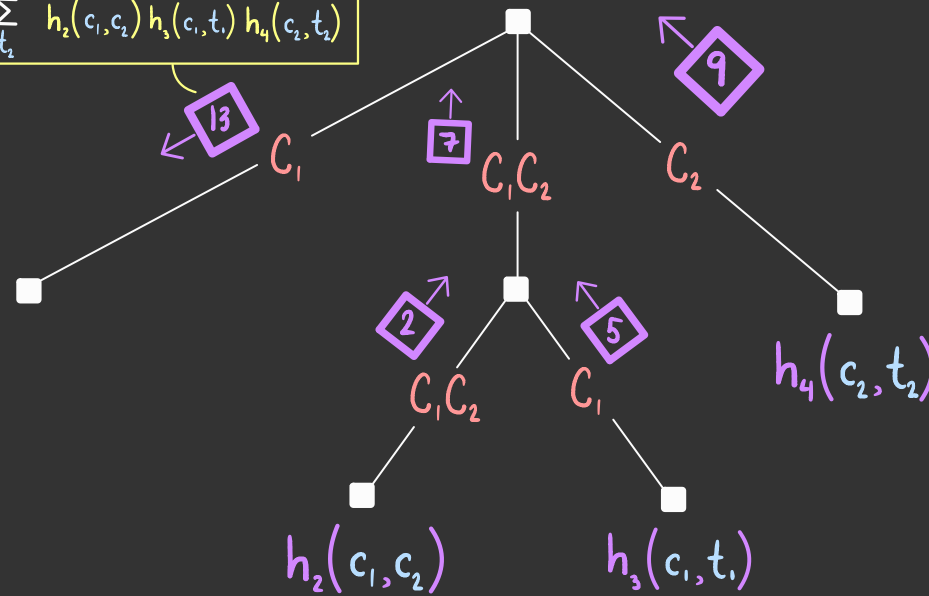
claim: $\sum_{c_2}\sum_{t_1}\sum_{t_2} h_2(c_1, c_2)\, h_3(c_1, t_1)\, h_4(c_2, t_2)$



$$\boxed{13} \quad \sum_{c_2} h_7(c_1, c_2)\, h_9(c_2) = \sum_{c_2} h_2(c_1, c_2)\, h_5(c_1)\, h_9(c_2)$$

$\boxed{5}$	$\sum_{t_1} h_3(c_1, t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$
$\boxed{7}$	$h_2(c_1, c_2) h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2, t_2)$
$\boxed{10}$	$h_9(c_2) h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$
$\boxed{12}$	$h_5(c_1) h_{10}(c_1, c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$
$\boxed{14}$	$h_0(c_0) h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

Claim: $\sum_{c_2} \sum_{t_1} \sum_{t_2} h_2(c_1, c_2) h_3(c_1, t_1) h_4(c_2, t_2)$

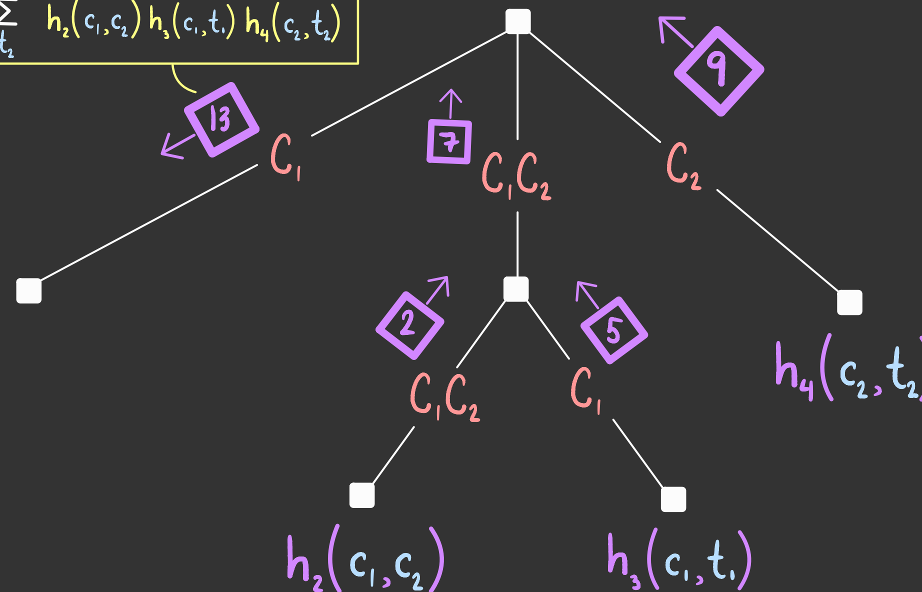


$$\boxed{13} \quad \sum_{c_2} h_7(c_1, c_2) h_9(c_2) = \sum_{c_2} h_2(c_1, c_2) h_5(c_1) h_9(c_2)$$

$$= \sum_{c_2} h_2(c_1, c_2) \sum_{t_1} h_3(c_1, t_1) h_9(c_2)$$

$\boxed{5}$	$\sum_{t_1} h_3(c_1,t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0)\, h_1(c_0,c_1)$
$\boxed{7}$	$h_2(c_1,c_2)\, h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1)\, h_7(c_1,c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2,t_2)$
$\boxed{10}$	$h_9(c_2)\, h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1,c_2)\, h_{10}(c_1,c_2)$
$\boxed{12}$	$h_5(c_1)\, h_{10}(c_1,c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1,c_2)\, h_9(c_2)$
$\boxed{14}$	$h_0(c_0)\, h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0,c_1)\, h_{13}(c_1)$

Claim: $\sum_{c_2}\sum_{t_1}\sum_{t_2} h_2(c_1,c_2)\, h_3(c_1,t_1)\, h_4(c_2,t_2)$



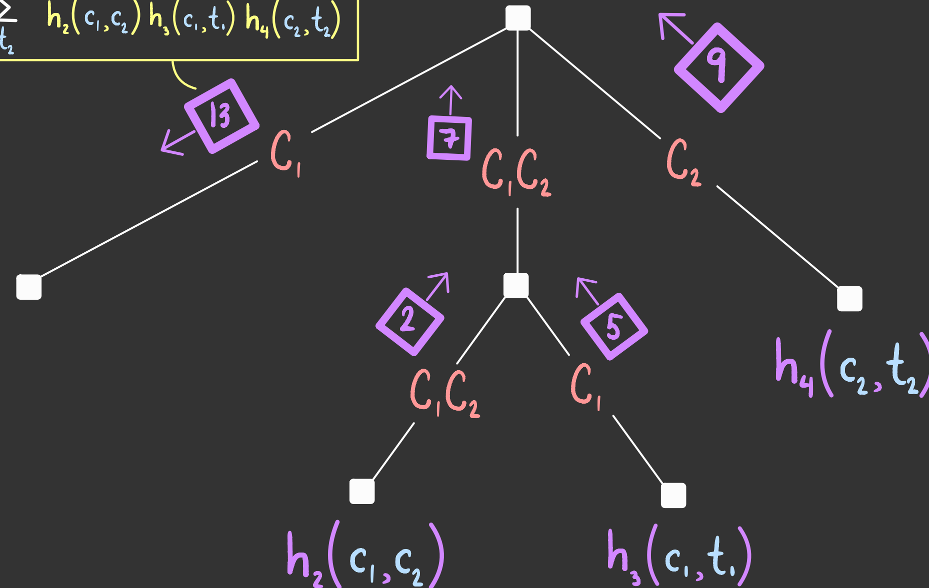
$\boxed{13}$

$$\sum_{c_2} h_7(c_1,c_2)\, h_9(c_2) = \sum_{c_2} h_2(c_1,c_2)\, h_5(c_1)\, h_9(c_2)$$

$$= \sum_{c_2} h_2(c_1,c_2) \sum_{t_1} h_3(c_1,t_1) \sum_{t_2} h_4(c_2,t_2)$$

$\boxed{5}$	$\sum_{t_1} h_3(c_1,t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0) h_1(c_0,c_1)$
$\boxed{7}$	$h_2(c_1,c_2) h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1) h_7(c_1,c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2,t_2)$
$\boxed{10}$	$h_9(c_2)\, h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1,c_2) h_{10}(c_1,c_2)$
$\boxed{12}$	$h_5(c_1) h_{10}(c_1,c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1,c_2) h_9(c_2)$
$\boxed{14}$	$h_0(c_0) h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0,c_1) h_{13}(c_1)$

claim: $\sum_{c_2}\sum_{t_1}\sum_{t_2} h_2(c_1,c_2)\, h_3(c_1,t_1)\, h_4(c_2,t_2)$



$\boxed{13}$

$$\begin{aligned}
\sum_{c_2} h_7(c_1,c_2) h_9(c_2) &= \sum_{c_2} h_2(c_1,c_2) h_5(c_1) h_9(c_2) \\
&= \sum_{c_2} h_2(c_1,c_2) \sum_{t_1} h_3(c_1,t_1) \sum_{t_2} h_4(c_2,t_2) \\
&= \sum_{c_2}\sum_{t_1}\sum_{t_2} h_2(c_1,c_2) h_3(c_1,t_1) h_4(c_2,t_2)
\end{aligned}$$

✓

5 $\sum_{t_1} h_3(c_1, t_1)$

6 $\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$

7 $h_2(c_1, c_2) h_5(c_1)$

8 $\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$

9 $\sum_{t_2} h_4(c_2, t_2)$

10 $h_9(c_2)\, h_6(c_1)$

11 $\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$

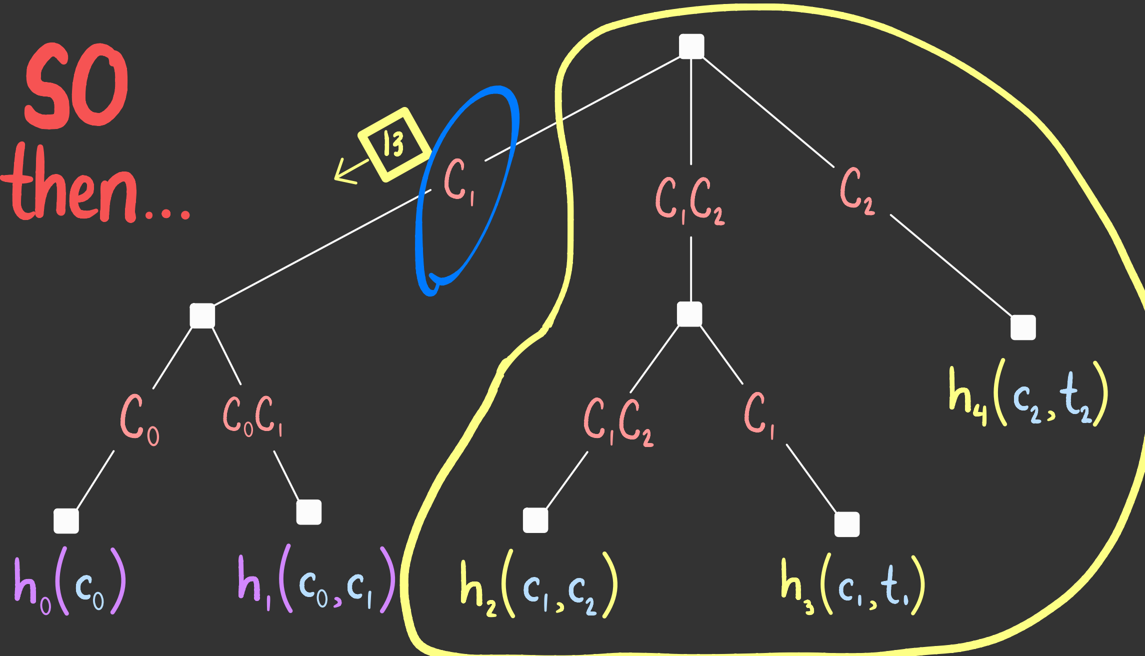
12 $h_5(c_1) h_{10}(c_1, c_2)$

13 $\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$

14 $h_0(c_0) h_{13}(c_1)$

15 $\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

SO then...



the message that a node sends its neighbor is the product of the factors on its side of the edge...
with the non-separator variables marginalized out

$$\boxed{13} \quad \sum_{c_2} \sum_{t_1} \sum_{t_2} h_2(c_1, c_2)\, h_3(c_1, t_1)\, h_4(c_2, t_2)$$

$\boxed{5}$ $\sum_{t_1} h_3(c_1, t_1)$

$\boxed{6}$ $\sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)$

$\boxed{7}$ $h_2(c_1, c_2)\, h_5(c_1)$

$\boxed{8}$ $\sum_{c_1} h_6(c_1)\, h_7(c_1, c_2)$

$\boxed{9}$ $\sum_{t_2} h_4(c_2, t_2)$

$\boxed{10}$ $h_9(c_2)\, h_6(c_1)$

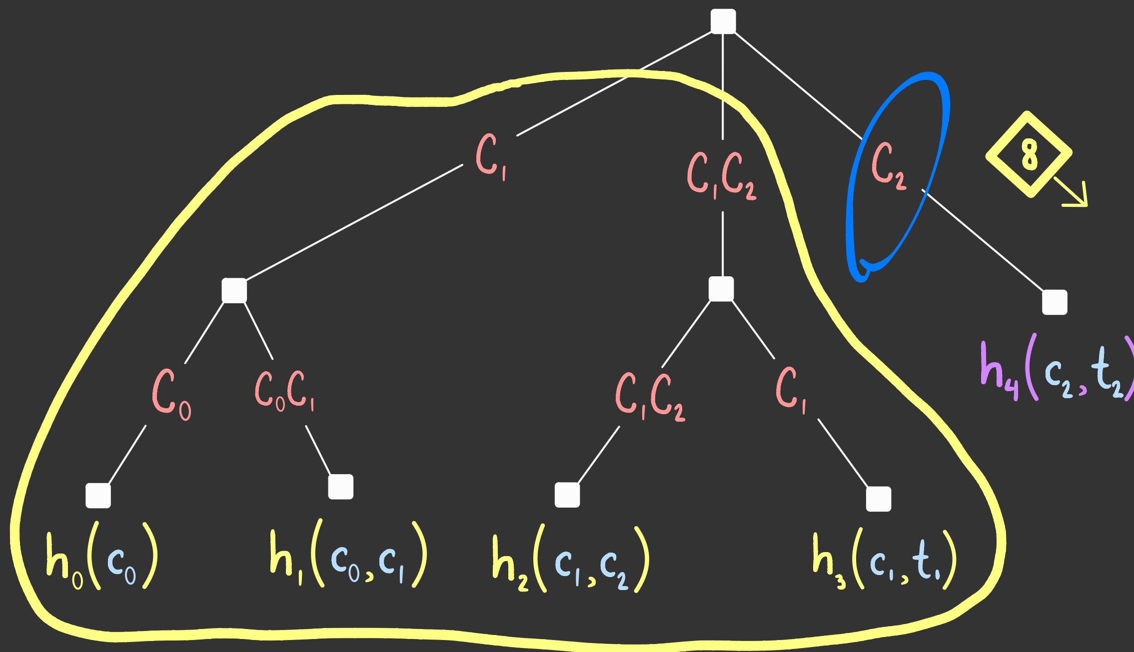
$\boxed{11}$ $\sum_{c_2} h_2(c_1, c_2)\, h_{10}(c_1, c_2)$

$\boxed{12}$ $h_5(c_1)\, h_{10}(c_1, c_2)$

$\boxed{13}$ $\sum_{c_2} h_7(c_1, c_2)\, h_9(c_2)$

$\boxed{14}$ $h_0(c_0)\, h_{13}(c_1)$

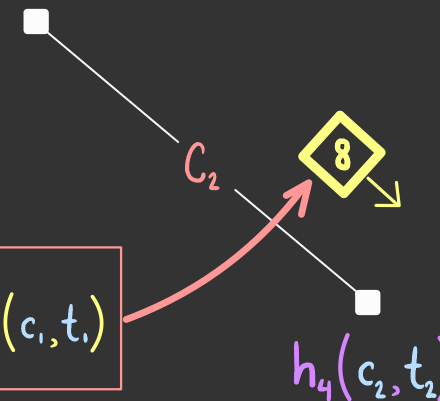
$\boxed{15}$ $\sum_{c_1} h_1(c_0, c_1)\, h_{13}(c_1)$



the message that a node sends its neighbor is the product of the factors on its side of the edge...
with the non-separator variables marginalized out

$\boxed{8}$ $\sum_{t_1}\sum_{c_1}\sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)\, h_2(c_1, c_2)\, h_3(c_1, t_1)$

$\boxed{5}$	$\sum_{t_1} h_3(c_1,t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0)\,h_1(c_0,c_1)$
$\boxed{7}$	$h_2(c_1,c_2)\,h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1)\,h_7(c_1,c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2,t_2)$
$\boxed{10}$	$h_9(c_2)\,h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1,c_2)\,h_{10}(c_1,c_2)$
$\boxed{12}$	$h_5(c_1)\,h_{10}(c_1,c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1,c_2)\,h_9(c_2)$
$\boxed{14}$	$h_0(c_0)\,h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0,c_1)\,h_{13}(c_1)$

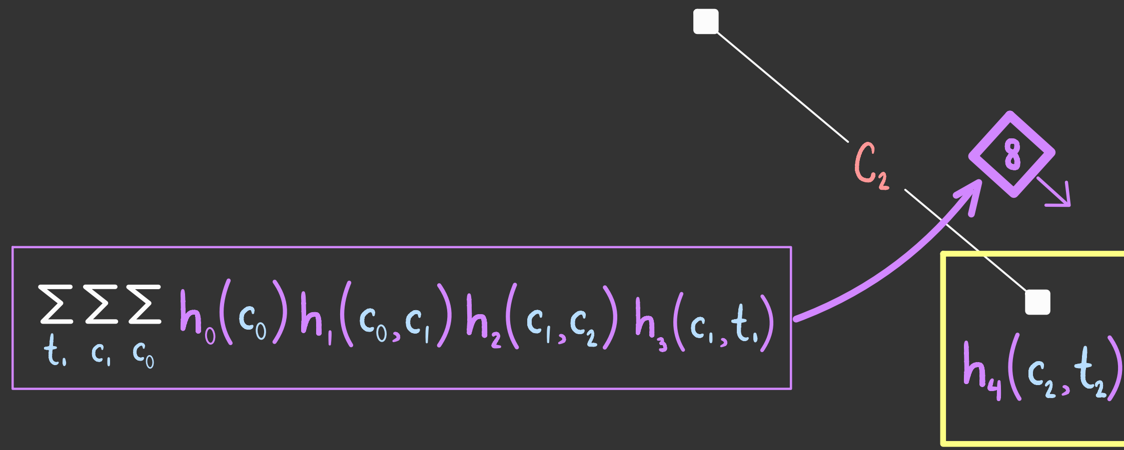


$$\boxed{\sum_{t_1}\sum_{c_1}\sum_{c_0} h_0(c_0)\,h_1(c_0,c_1)\,h_2(c_1,c_2)\,h_3(c_1,t_1)}$$

the message that a node sends its neighbor is the product of the factors on its side of the edge...
with the non-separator variables marginalized out

$$\boxed{8}\quad \sum_{t_1}\sum_{c_1}\sum_{c_0} h_0(c_0)\,h_1(c_0,c_1)\,h_2(c_1,c_2)\,h_3(c_1,t_1)$$

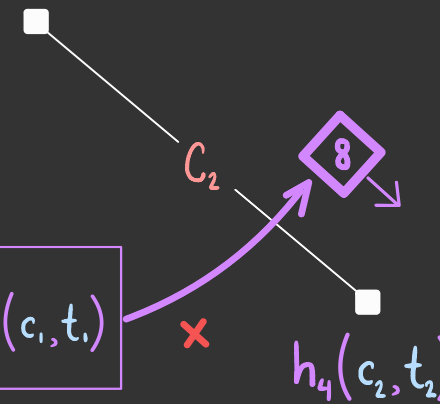
$\boxed{5}$	$\sum_{t_1} h_3(c_1,t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0)\,h_1(c_0,c_1)$
$\boxed{7}$	$h_2(c_1,c_2)\,h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1)\,h_7(c_1,c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2,t_2)$
$\boxed{10}$	$h_9(c_2)\,h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1,c_2)\,h_{10}(c_1,c_2)$
$\boxed{12}$	$h_5(c_1)\,h_{10}(c_1,c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1,c_2)\,h_9(c_2)$
$\boxed{14}$	$h_0(c_0)\,h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0,c_1)\,h_{13}(c_1)$



to compute single-variable marginal $P(t_2)$:

- choose a leaf containing variable T_2
- multiply its factor with its received "message"
- marginalize out other variables

$\boxed{5}$	$\sum_{t_1} h_3(c_1, t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$
$\boxed{7}$	$h_2(c_1, c_2) h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2, t_2)$
$\boxed{10}$	$h_9(c_2)\, h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$
$\boxed{12}$	$h_5(c_1) h_{10}(c_1, c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$
$\boxed{14}$	$h_0(c_0) h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$



$$\boxed{\sum_{t_1}\sum_{c_1}\sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)\, h_2(c_1, c_2)\, h_3(c_1, t_1)}$$

$$= \quad \sum_{t_1}\sum_{c_1}\sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)\, h_2(c_1, c_2)\, h_3(c_1, t_1)\, h_4(c_2, t_2)$$

to compute single-variable marginal $P(t_2)$:

- choose a leaf containing variable T_2
- multiply its factor with its received "message"
- marginalize out other variables

$\boxed{5}$ $\sum_{t_1} h_3(c_1, t_1)$

$\boxed{6}$ $\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$

$\boxed{7}$ $h_2(c_1, c_2) h_5(c_1)$

$\boxed{8}$ $\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$

$\boxed{9}$ $\sum_{t_2} h_4(c_2, t_2)$

$\boxed{10}$ $h_9(c_2) h_6(c_1)$

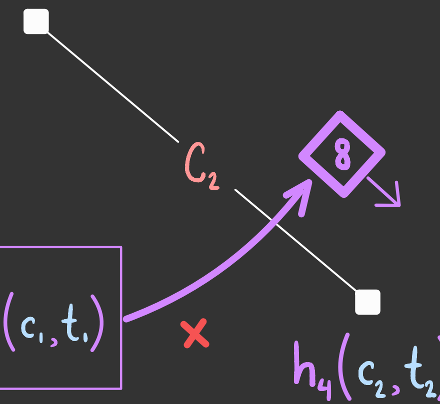
$\boxed{11}$ $\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$

$\boxed{12}$ $h_5(c_1) h_{10}(c_1, c_2)$

$\boxed{13}$ $\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$

$\boxed{14}$ $h_0(c_0) h_{13}(c_1)$

$\boxed{15}$ $\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$



$$\boxed{\sum_{t_1} \sum_{c_1} \sum_{c_0} h_0(c_0) h_1(c_0, c_1) h_2(c_1, c_2) h_3(c_1, t_1)} \times$$

$$= \sum_{c_2} \sum_{t_1} \sum_{c_1} \sum_{c_0} h_0(c_0) h_1(c_0, c_1) h_2(c_1, c_2) h_3(c_1, t_1) h_4(c_2, t_2)$$

to compute single-variable marginal $P(t_2)$:

- choose a leaf containing variable T_2
- multiply its factor with its received "message"
- marginalize out other variables

$\boxed{5}$	$\sum_{t_1} h_3(c_1, t_1)$
$\boxed{6}$	$\sum_{c_0} h_0(c_0) h_1(c_0, c_1)$
$\boxed{7}$	$h_2(c_1, c_2) h_5(c_1)$
$\boxed{8}$	$\sum_{c_1} h_6(c_1) h_7(c_1, c_2)$
$\boxed{9}$	$\sum_{t_2} h_4(c_2, t_2)$
$\boxed{10}$	$h_9(c_2)\, h_6(c_1)$
$\boxed{11}$	$\sum_{c_2} h_2(c_1, c_2) h_{10}(c_1, c_2)$
$\boxed{12}$	$h_5(c_1) h_{10}(c_1, c_2)$
$\boxed{13}$	$\sum_{c_2} h_7(c_1, c_2) h_9(c_2)$
$\boxed{14}$	$h_0(c_0) h_{13}(c_1)$
$\boxed{15}$	$\sum_{c_1} h_1(c_0, c_1) h_{13}(c_1)$

$$P(t_2)$$

$$= \sum_{c_2} \sum_{t_1} \sum_{c_1} \sum_{c_0} h_0(c_0)\, h_1(c_0, c_1)\, h_2(c_1, c_2)\, h_3(c_1, t_1)\, h_4(c_2, t_2)$$

to compute single-variable marginal $P(t_2)$:

- choose a leaf containing variable T_2
- multiply its factor with its received "message"
- marginalize out other variables

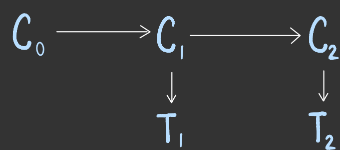
it's as if we took a bayesian network...

intuition

C_0	$P(c_0)$
0	0.99
1	0.01

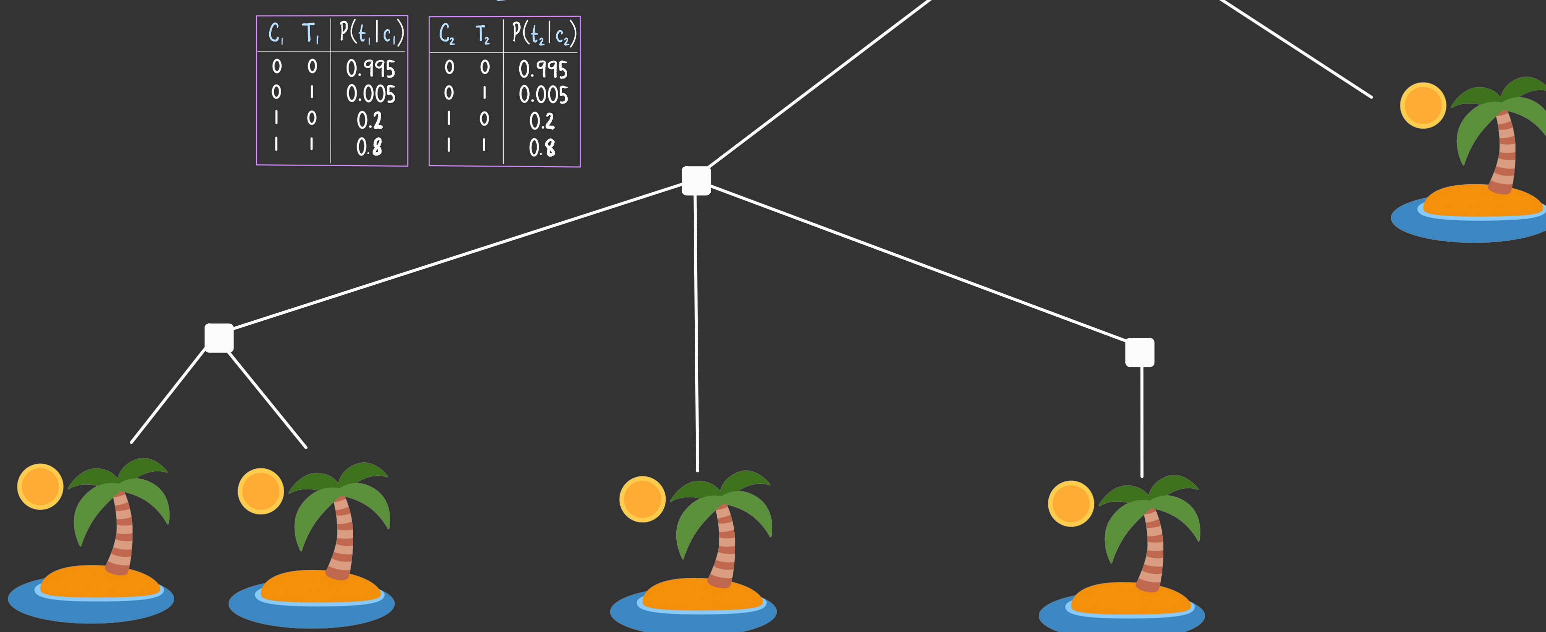
C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

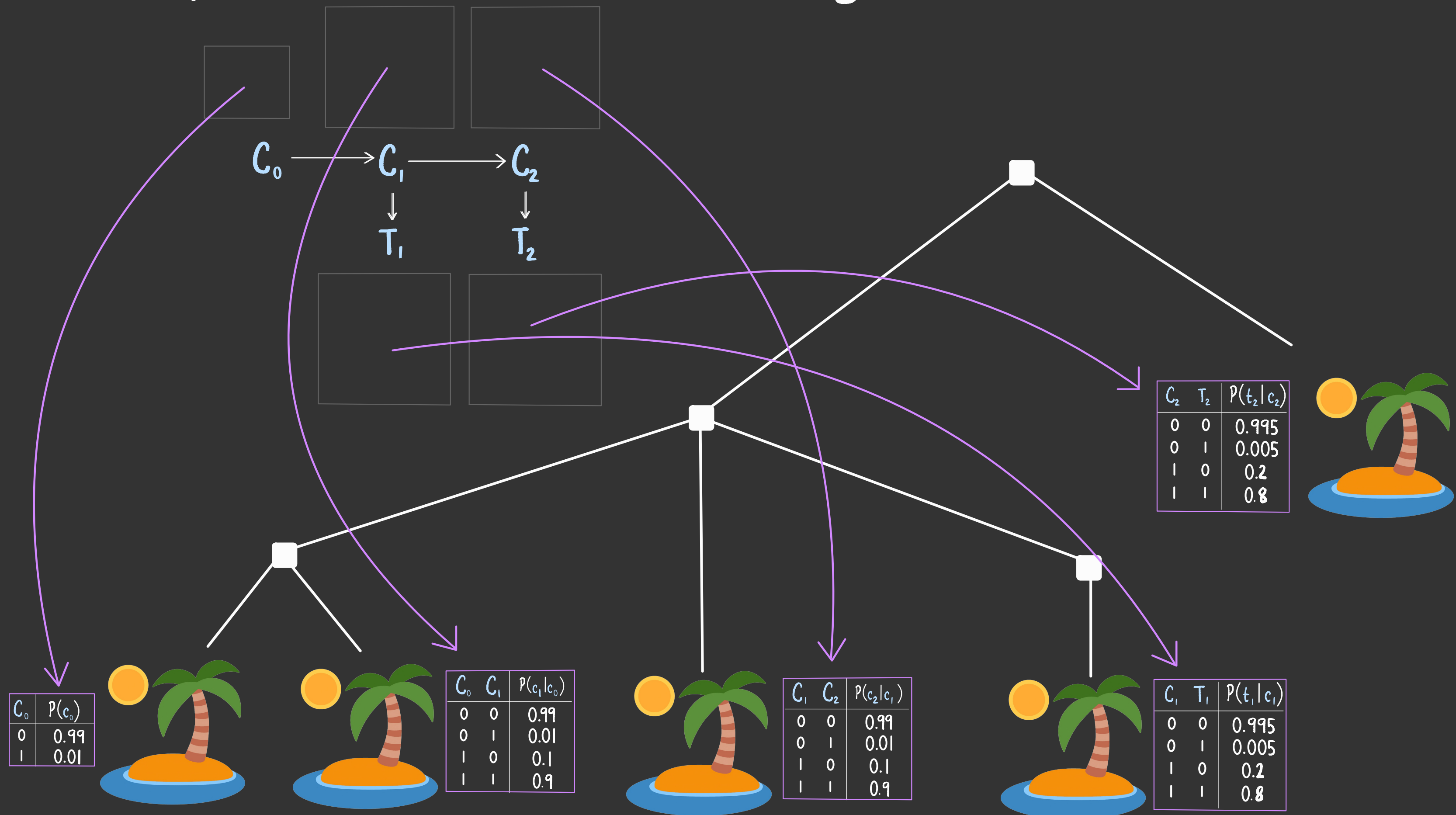


C_1	T_1	$P(t_1 \mid c_1)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8



…and dispersed its conditional probability tables to remote islands



C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

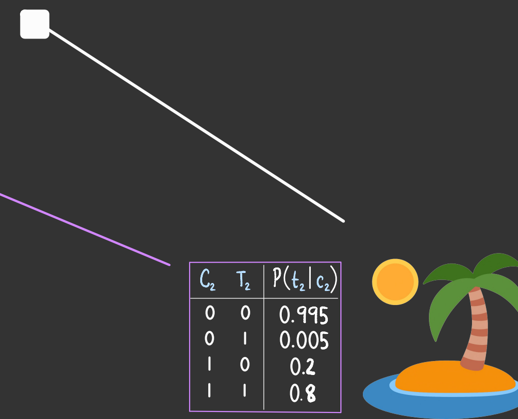
C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	T_1	$P(t_1 \mid c_1)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

on our island, we know conditional probabilities $P(t_2 | c_2)$
but we have dreams... dreams of things beyond our ken...

C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

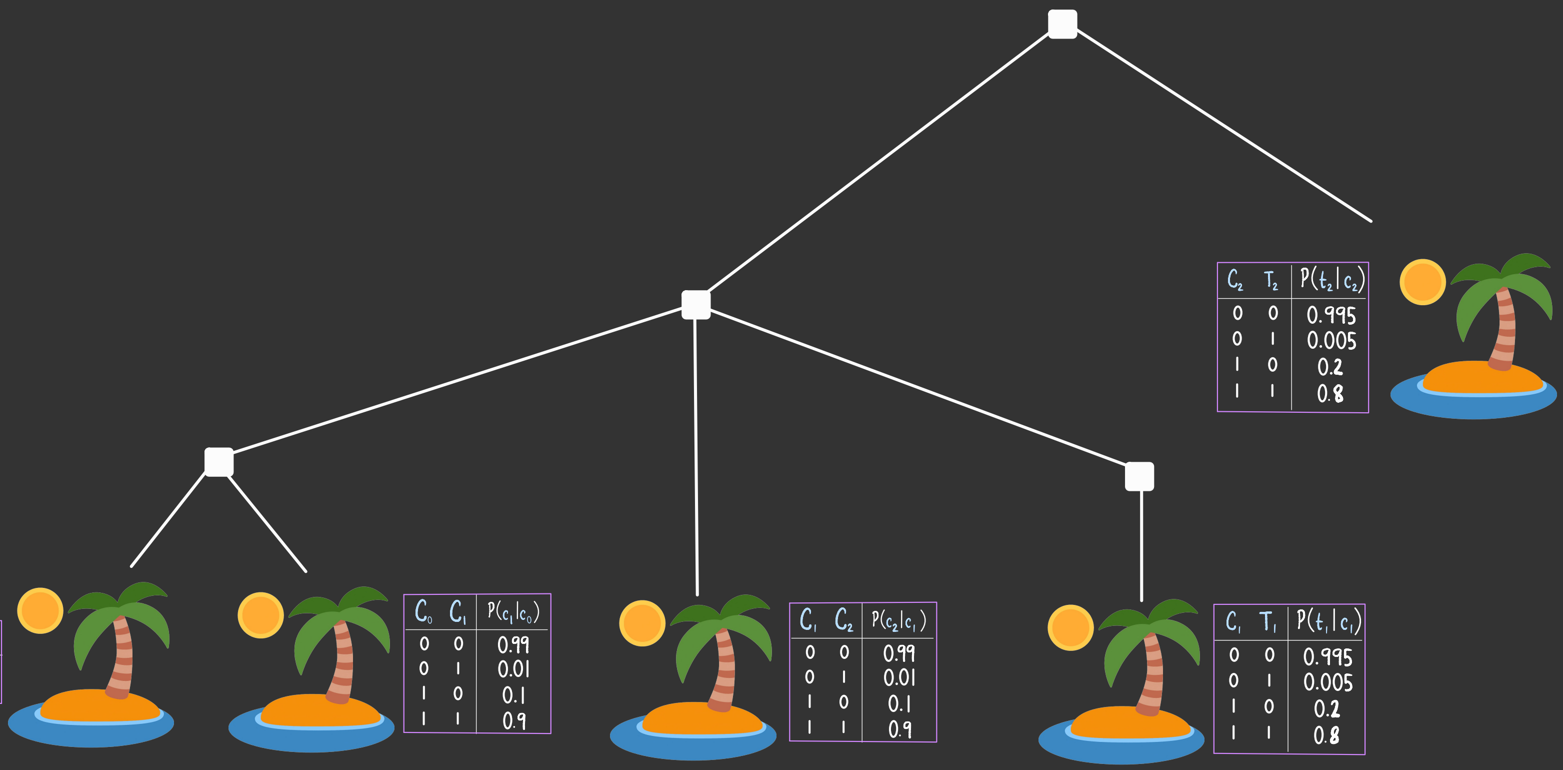


...dreams of knowing the marginal probabilities $P(c_2, t_2)$

C_2	T_2	$P(c_2, t_2)$
0	0	?
0	1	?
1	0	?
1	1	?

C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

but computing the marginal probabilities $P(c_2, t_2)$ requires information possessed by other islands



C_0	$P(c_0)$
0	0.99
1	0.01

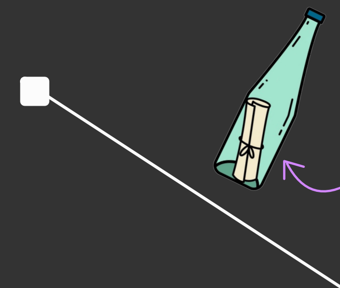
C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

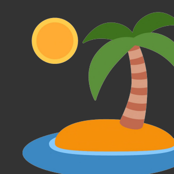
C_1	T_1	$P(t_1 \mid c_1)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

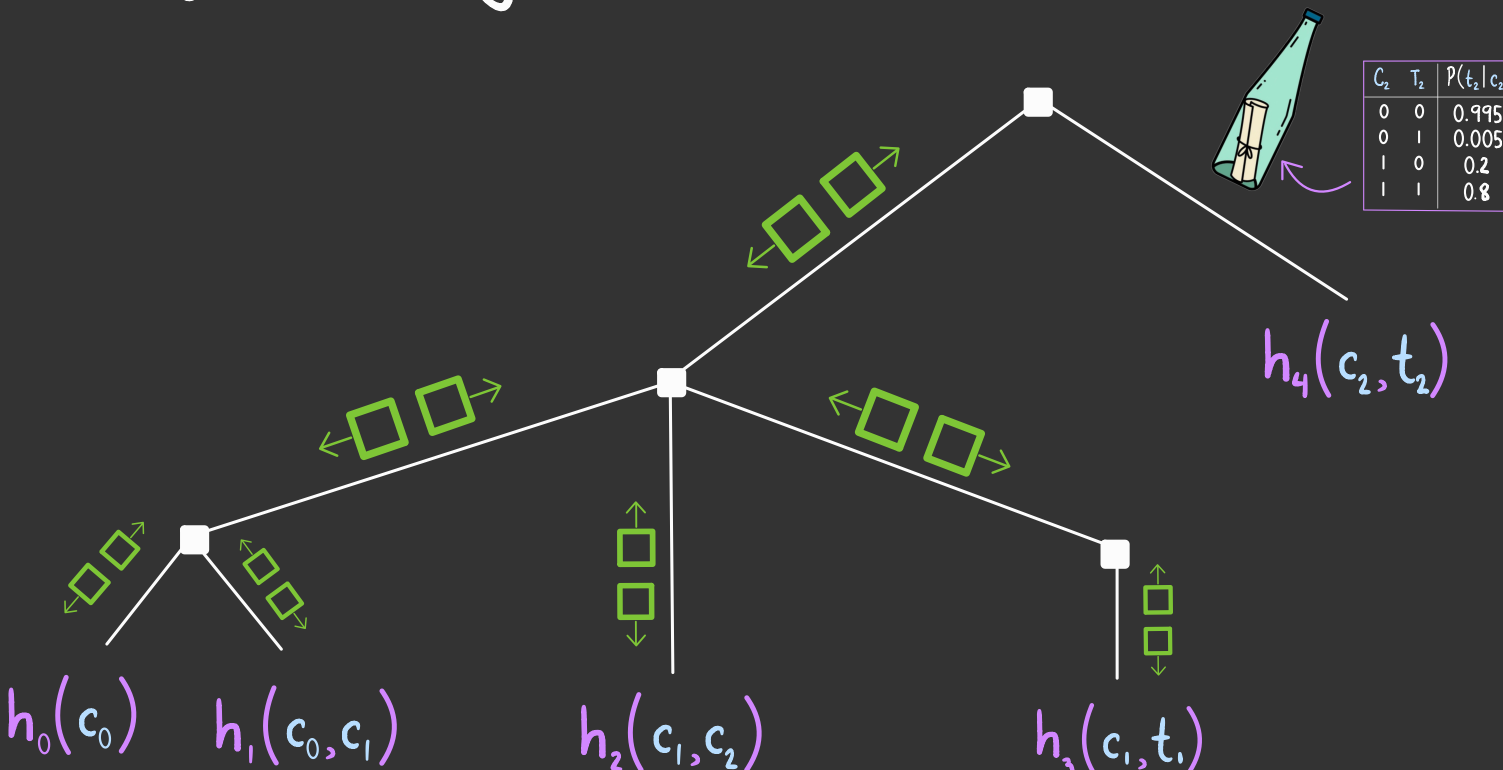
so we send our conditional probability table adrift on the waves as a message in a bottle



C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

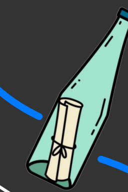


the other islands all cooperate, sending their own messages according to the belief propagation step



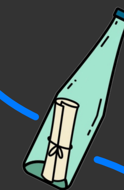
C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

eventually a factor washes up on our shore



C_2	T_2	$h(c_2, t_2)$
0	0	•
0	1	•
1	0	•
1	1	•

multiplying it with our original factor gives us a marginal



C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

×

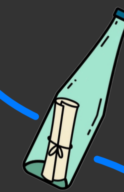
C_2	T_2	$h(c_2, t_2)$
0	0	·
0	1	·
1	0	·
1	1	·

multiplying it with our original factor gives us a marginal

C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

×

C_2	T_2	$h(c_2, t_2)$
0	0	·
0	1	·
1	0	·
1	1	·



=

C_2	T_2	$P(c_2, t_2)$
0	0	·
0	1	·
1	0	·
1	1	·

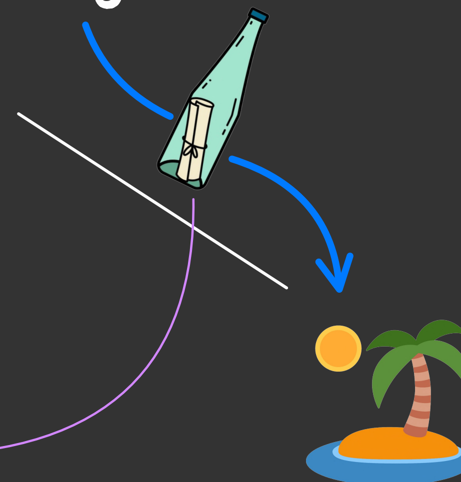
with this, we can compute single variable marginals for any variable that appeared in our original factor

$$P(c_2) = \sum_{t_2} P(c_2, t_2)$$

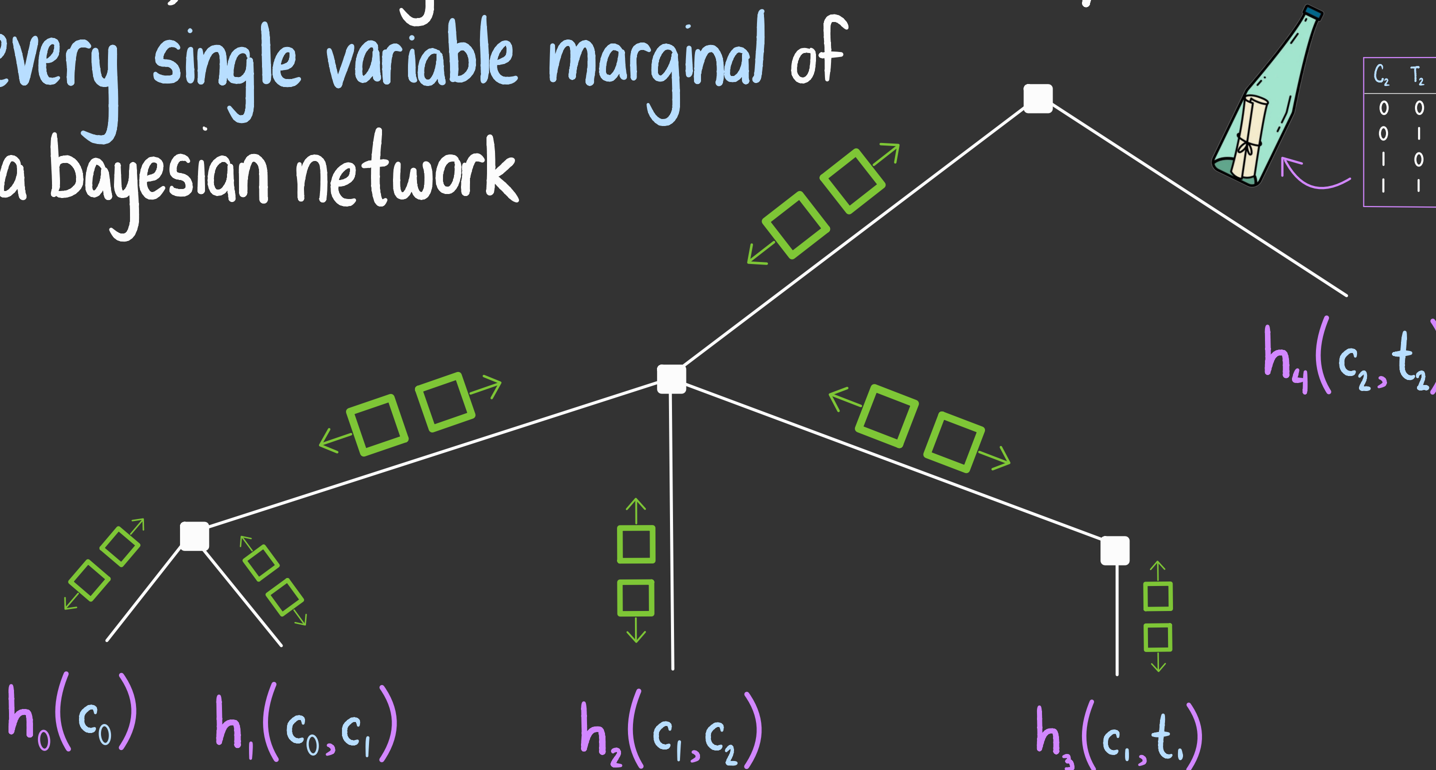
or

$$P(t_2) = \sum_{c_2} P(c_2, t_2)$$

C_2	T_2	$P(c_2, t_2)$
0	0	•
0	1	•
1	0	•
1	1	•



because every variable appears on at least one island, this algorithm allows us to compute every single variable marginal of a bayesian network

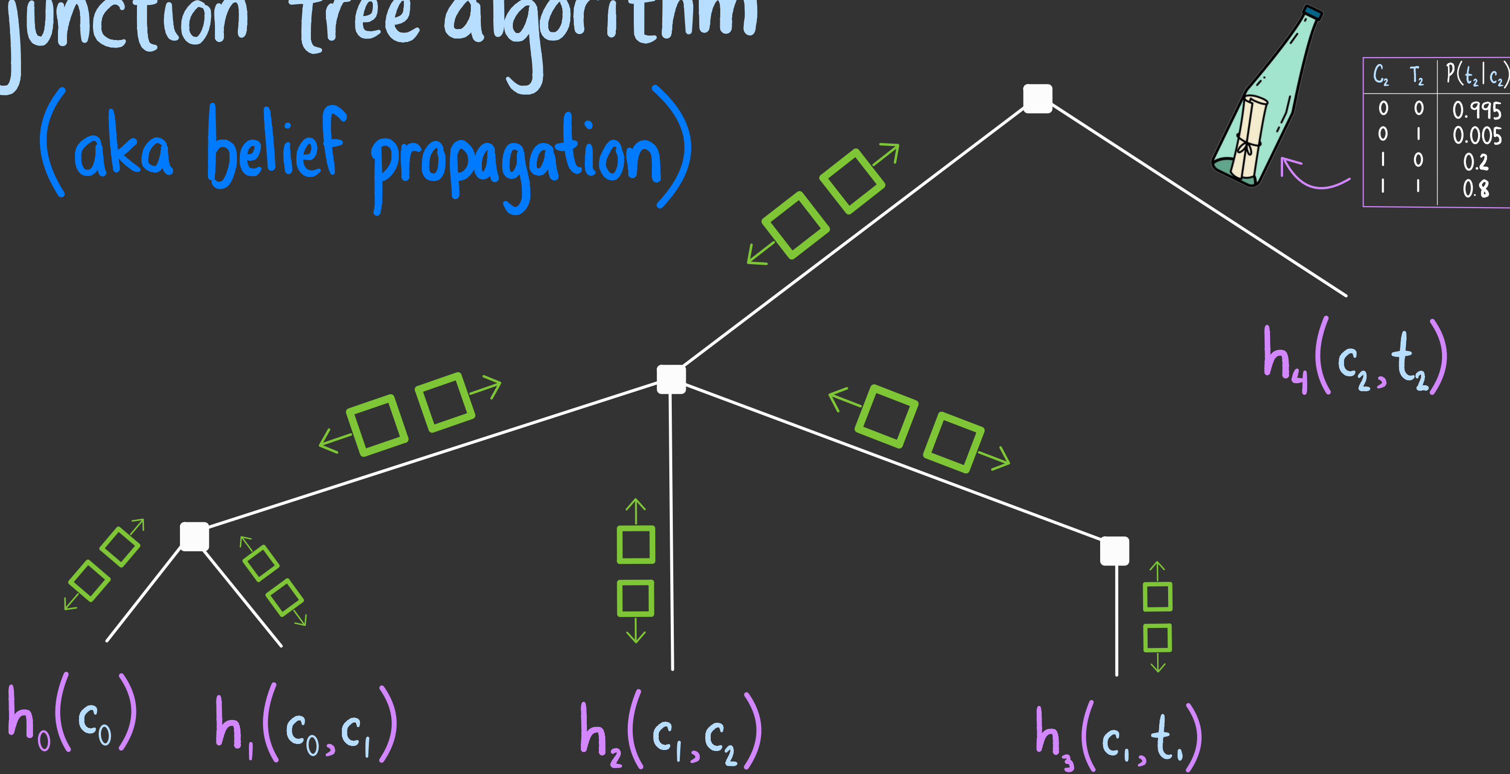


C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

this algorithm is called the

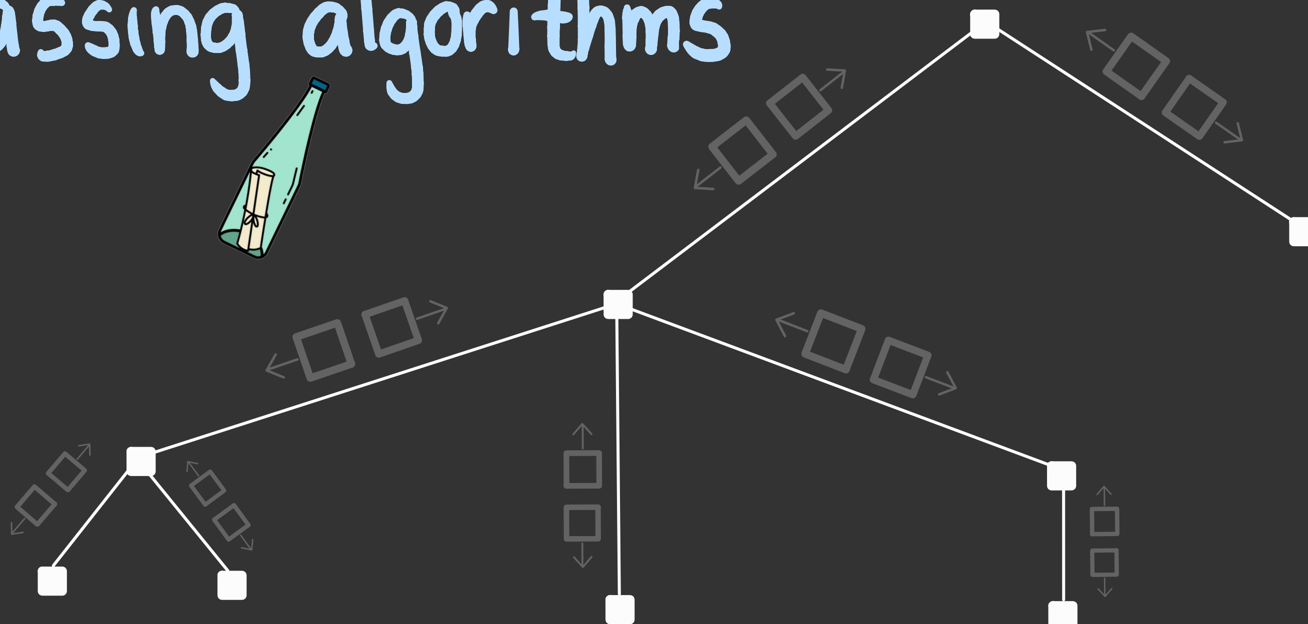
junction tree algorithm

(aka belief propagation)

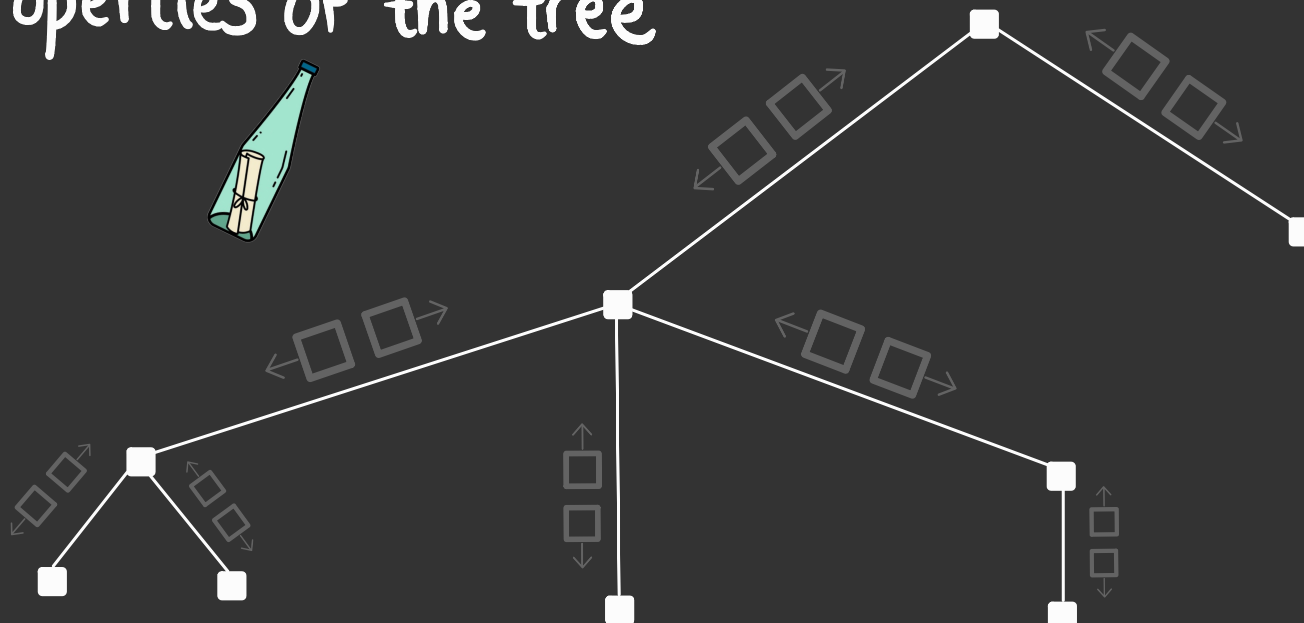


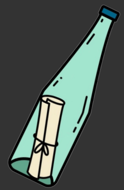
C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

it is an example of a family of algorithms called message passing algorithms

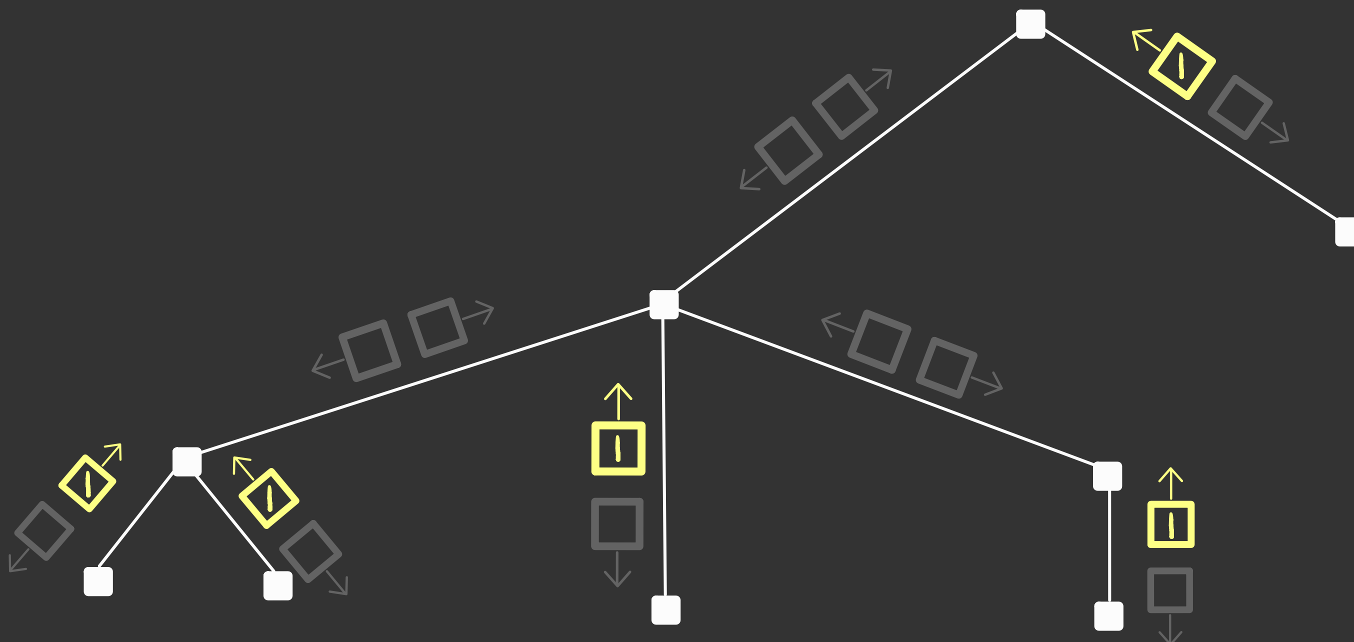


by changing the messages, we can compute other properties of the tree



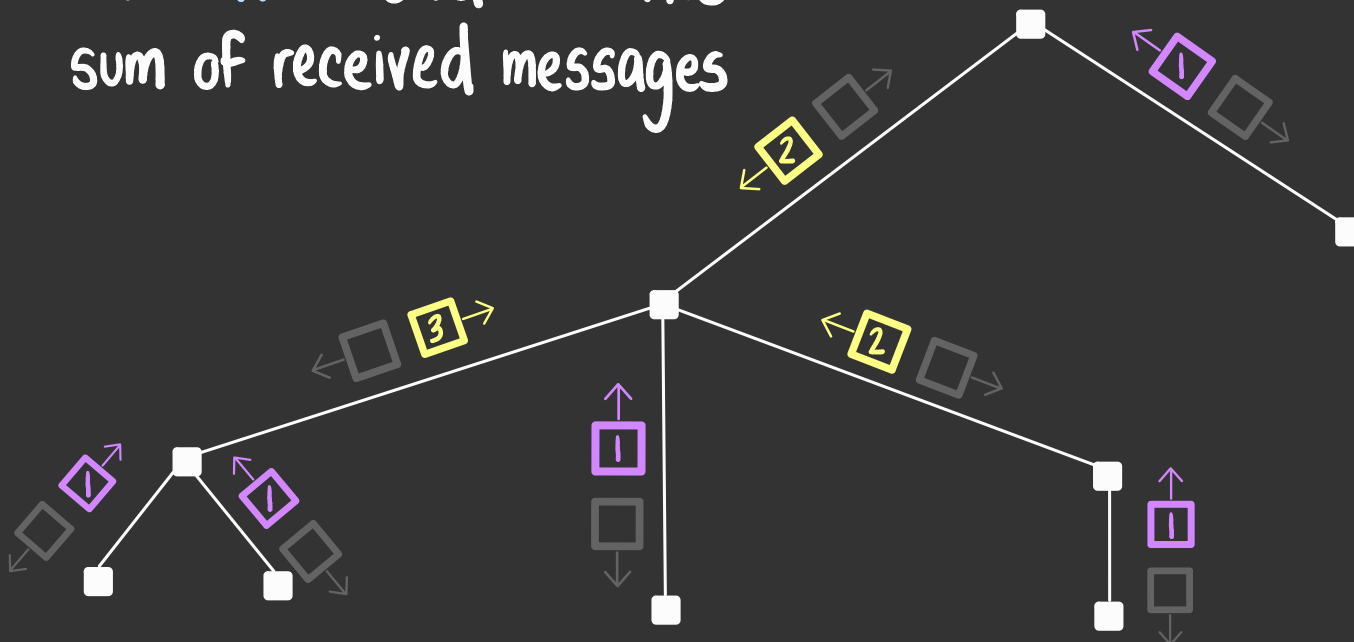


leaf : send the number 1



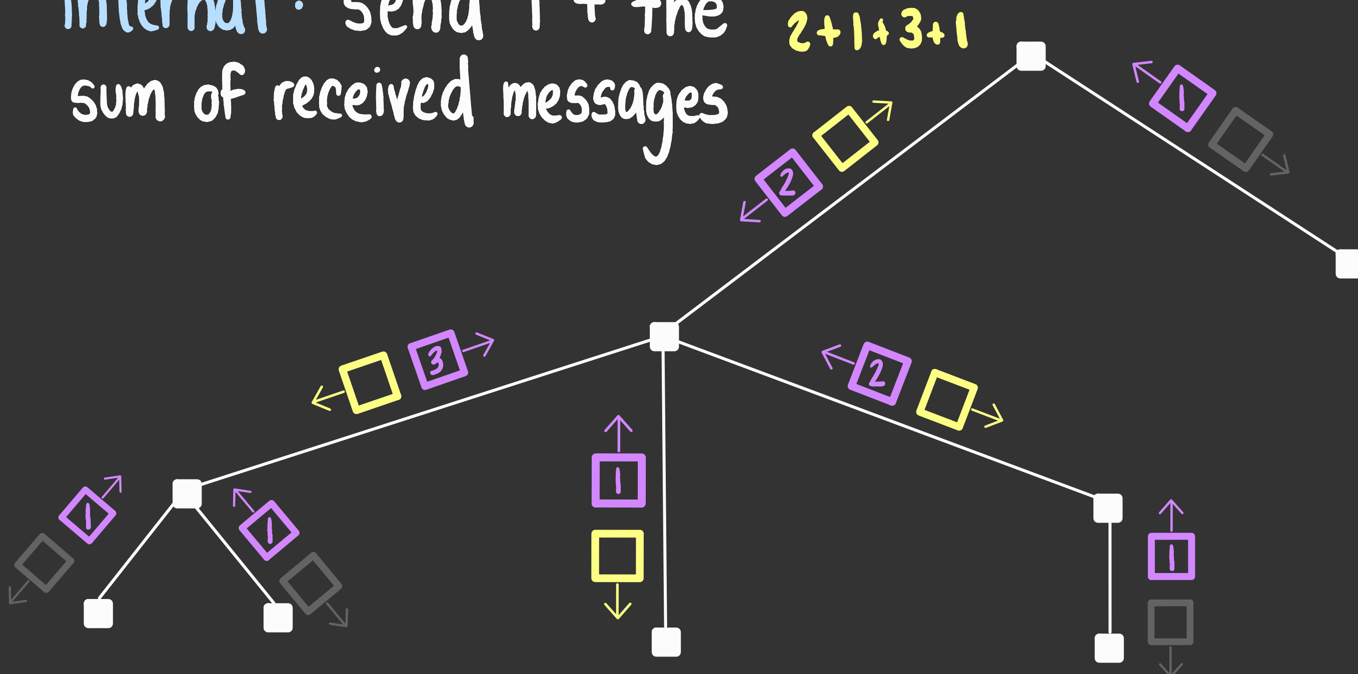
leaf: send the number 1

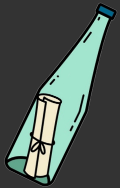
internal: send 1 + the sum of received messages



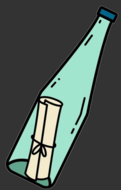
leaf: send the number 1

internal: send 1 + the sum of received messages

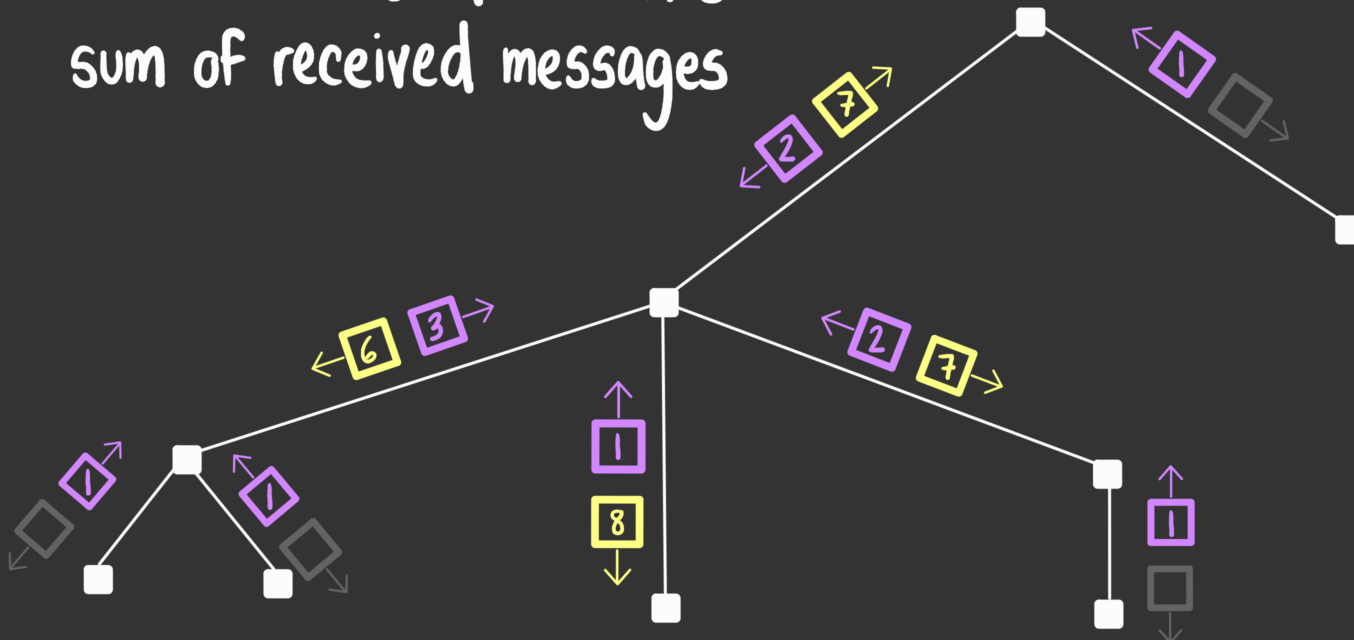




leaf: send the number 1

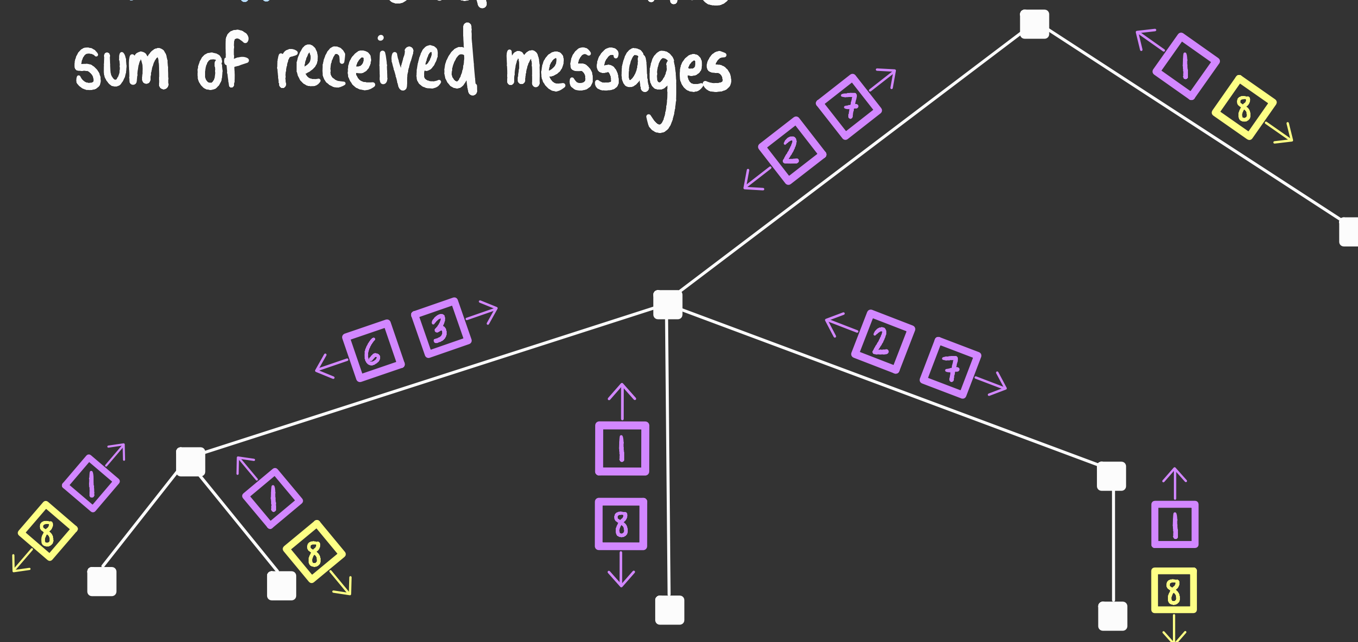


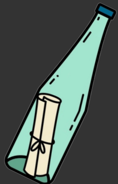
internal: send 1 + the sum of received messages



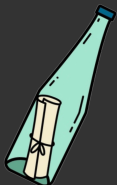
leaf: send the number 1

internal: send 1 + the sum of received messages



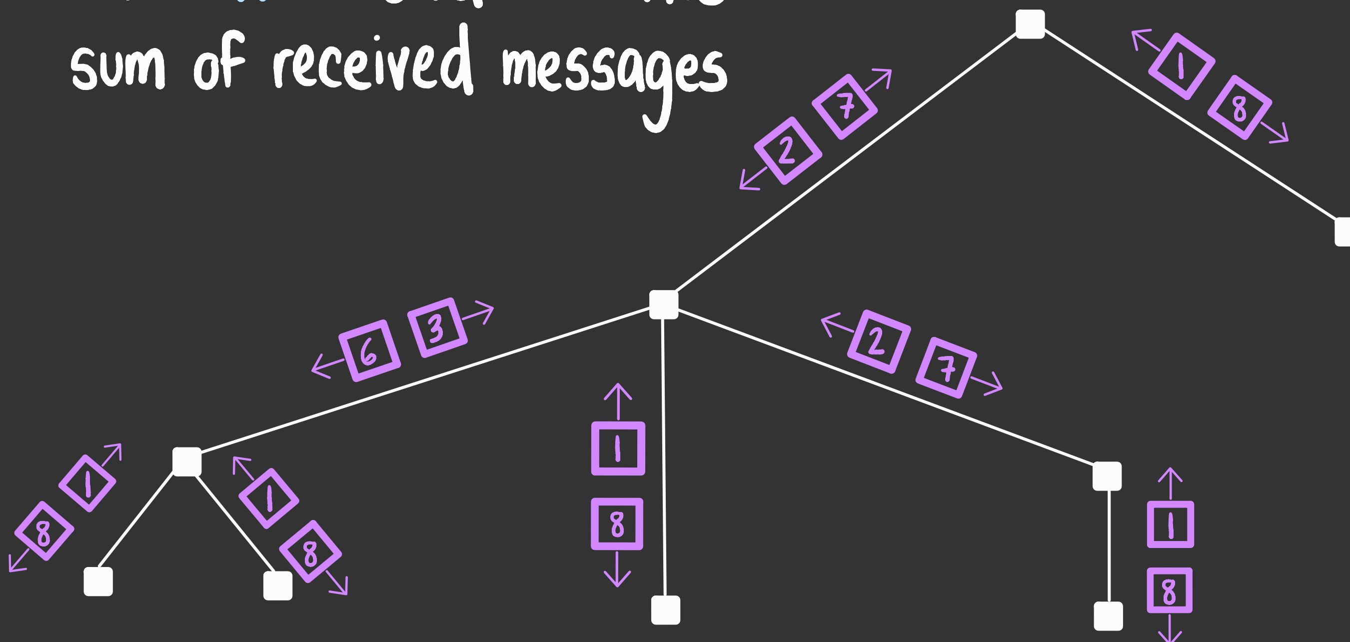


leaf : send the number 1



internal : send 1 + the sum of received messages

what is this computing?



leaf : send the number 1

internal : send 1 + the sum of received messages

the number of nodes in the tree