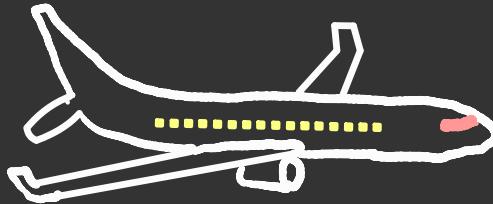
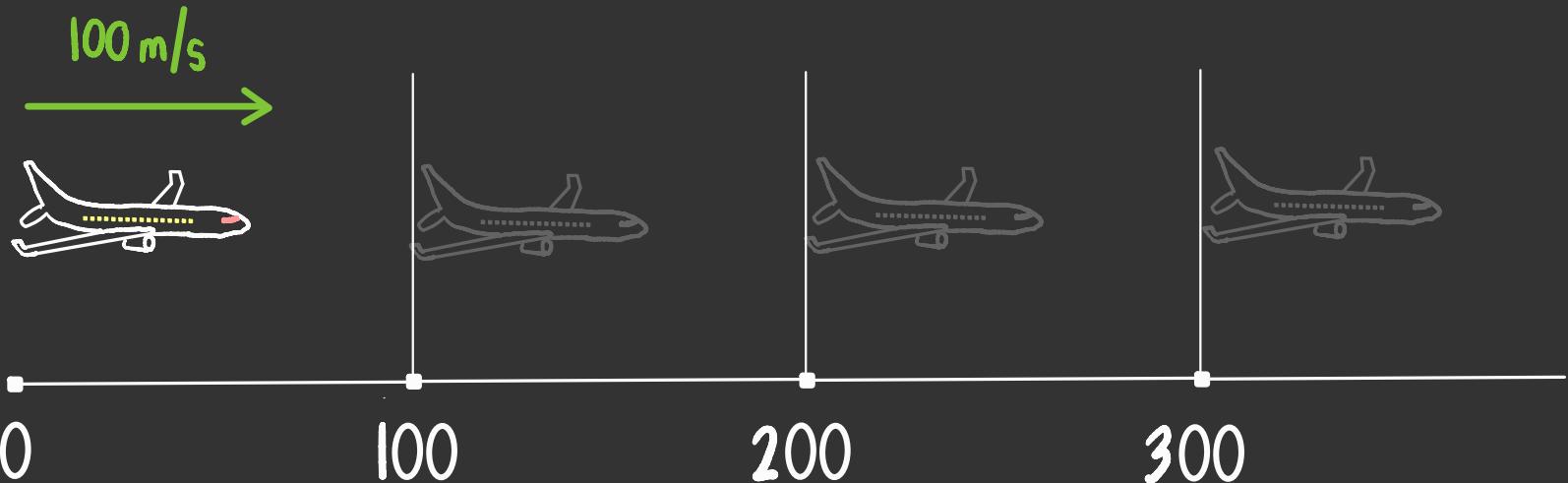


kalman  
filters  
26 oct  
2022

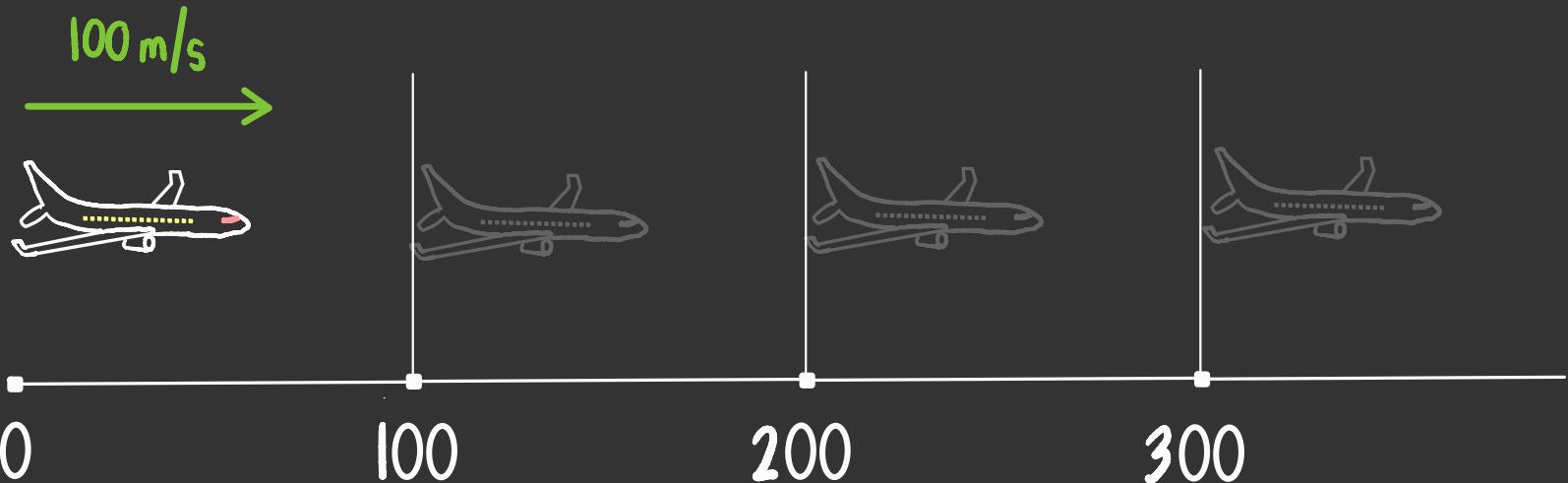
CSCI  
373

Consider the problem  
of tracking an  
airplane  
at cruising altitude

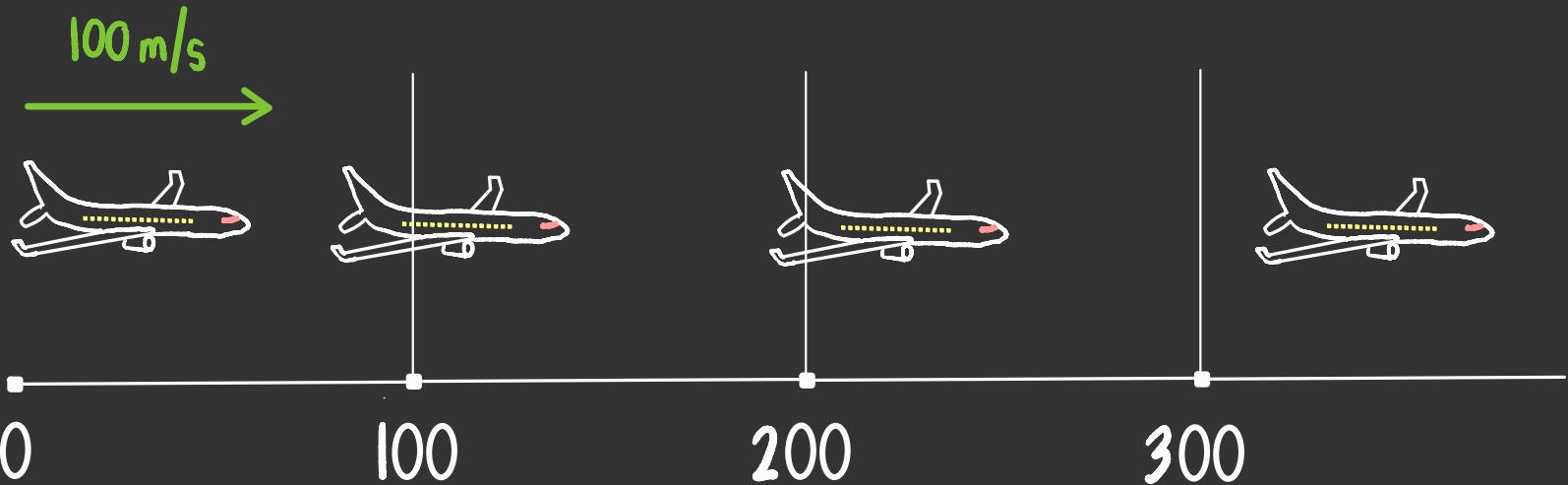




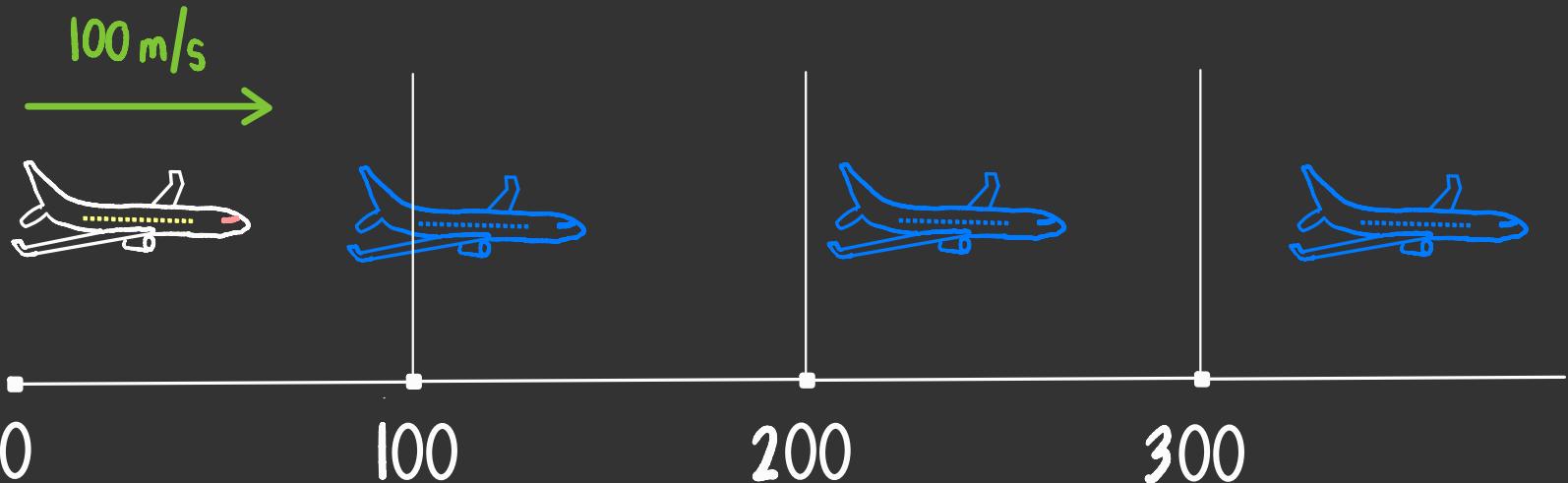
strategy 1: we know this plane flies at an average velocity of 100 meters per second, so we predict that the plane is at position  $100t$  after  $t$  seconds



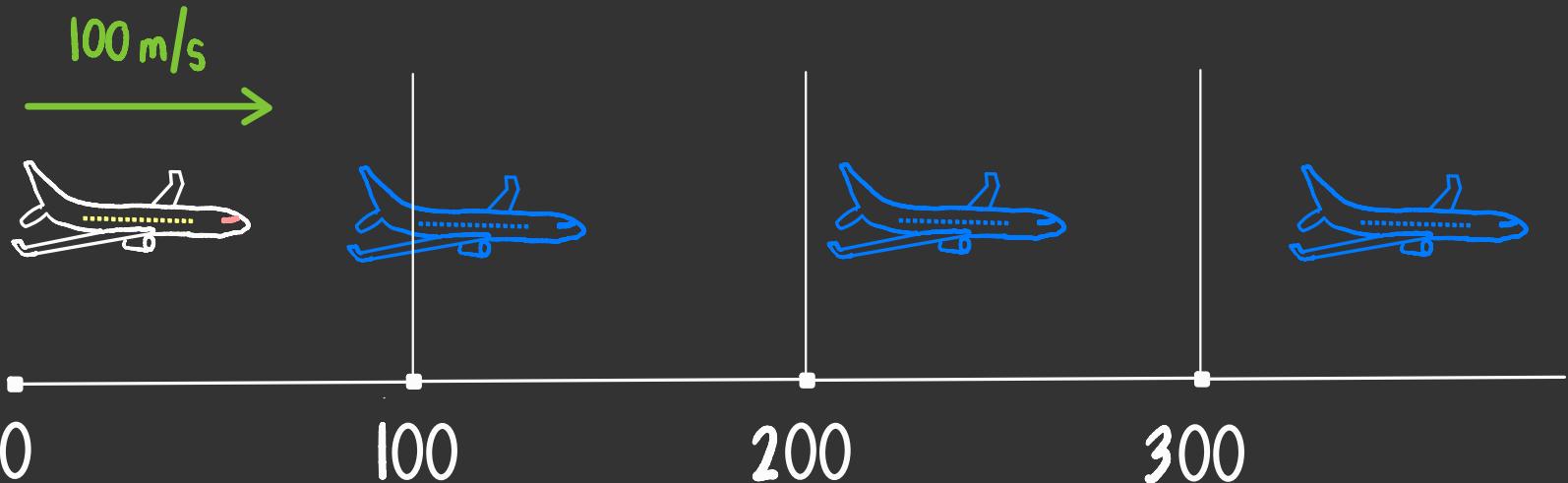
what is a drawback of this strategy?



an average velocity of 100 meters per second  
does not mean  
a constant velocity of 100 meters per second



strategy 2: use **radar** to  
determine the plane's position



what is a drawback of this strategy?

100 m/s



0      100      200      300

radar is a noisy sensor

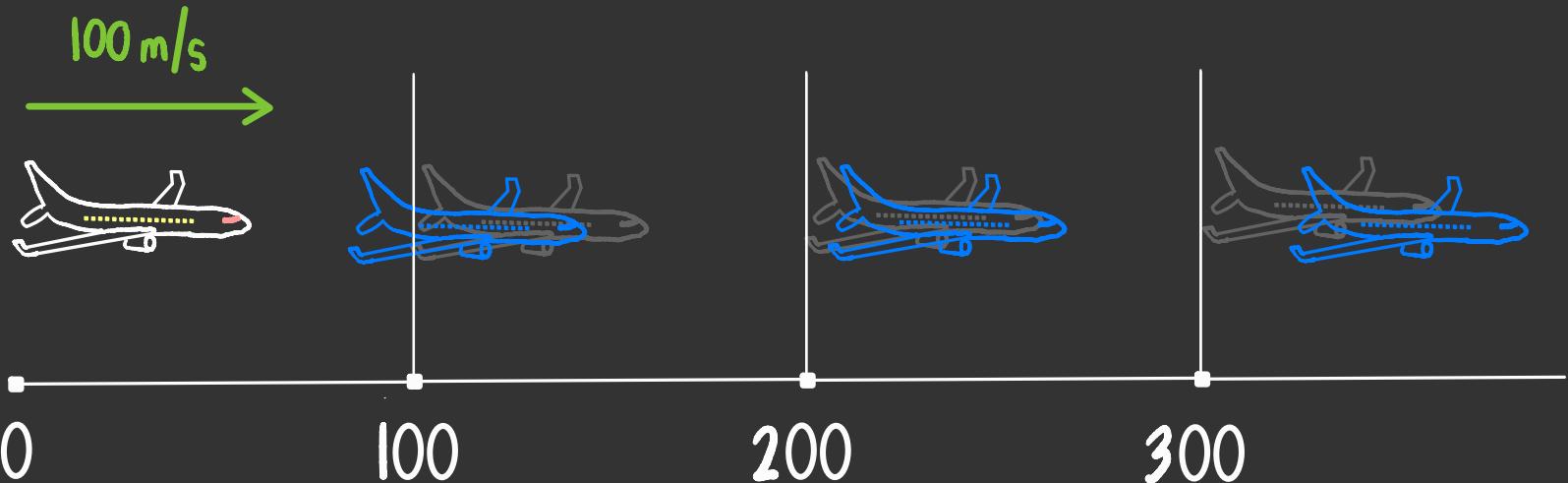
100 m/s



0                  100                  200                  300



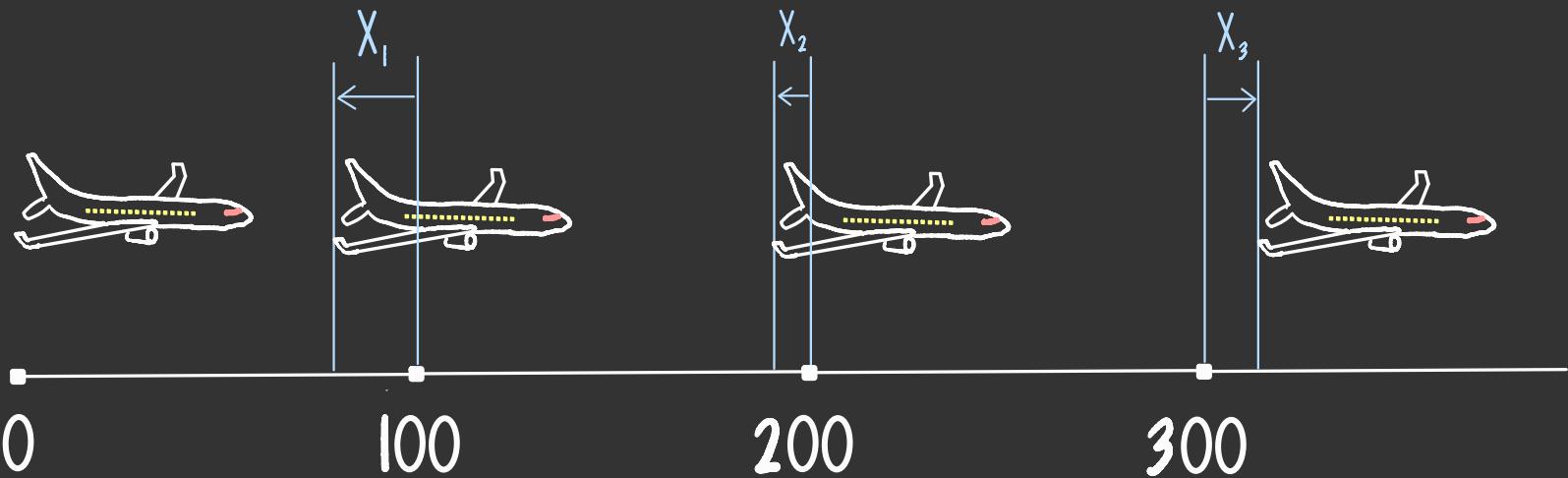
strategy 3: use both  
strategies 1 and 2!



strategy 3: use both  
strategies 1 and 2!  
but how can we fuse the two?

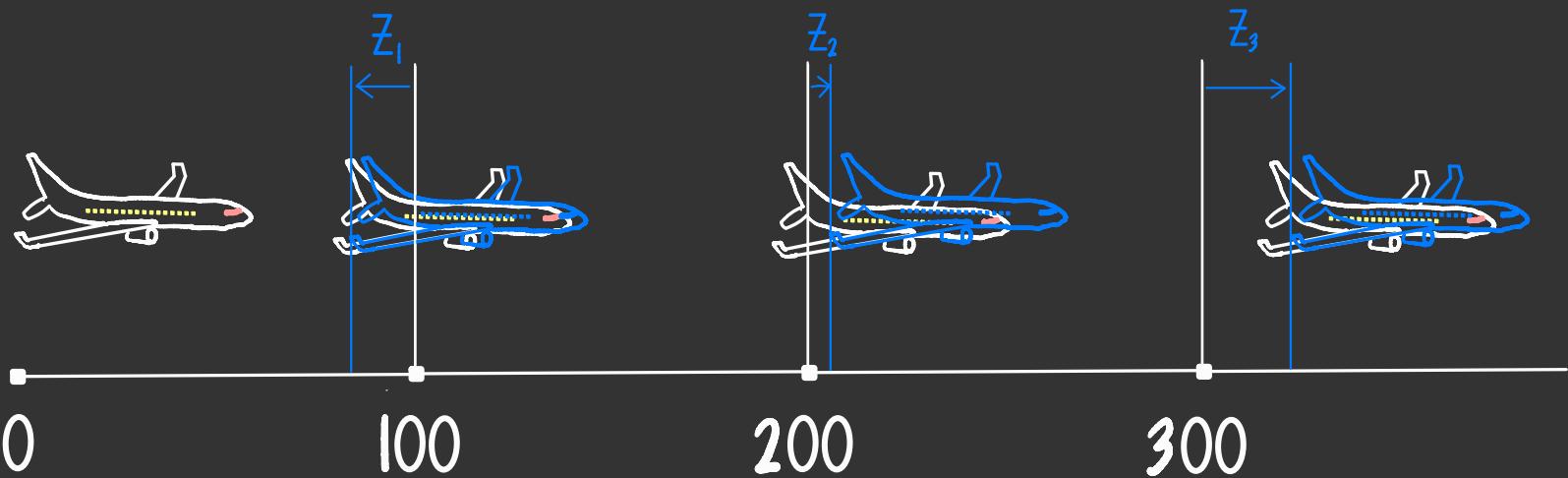


let  $X_0$  be the starting position of the plane



let  $X_0$  be the starting position of the plane

let  $X_t$  be the position of the plane after  $t$  seconds  
minus the expected position after  $t$  seconds



let  $X_0$  be the starting position of the plane

let  $X_t$  be the position of the plane after  $t$  seconds  
minus the expected position after  $t$  seconds

let  $Z_t$  be the radar reading at time  $t$  minus the expected position

create a bayesian network  
using these variables

let  $X_0$  be the starting position of the plane

let  $X_t$  be the plane's position after  $t$  seconds relative to the expected position

let  $Z_t$  be the radar reading at time  $t$  relative to the expected position

$$\begin{array}{cccc} X_0 & \rightarrow & X_1 & \rightarrow \\ \downarrow & & \downarrow & \downarrow \\ Z_1 & & Z_2 & & Z_3 \end{array}$$

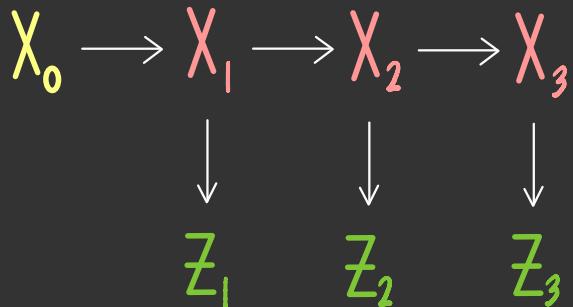
let  $X_0$  be the starting position of the plane

let  $X_t$  be the plane's position after  $t$  seconds relative to the expected position

let  $Z_t$  be the radar reading at time  $t$  relative to the expected position

want to compute:

$$P(X_3 \mid Z_1, Z_2, Z_3)$$



maybe  
observed

let  $X_0$  be the starting position of the plane

unobserved

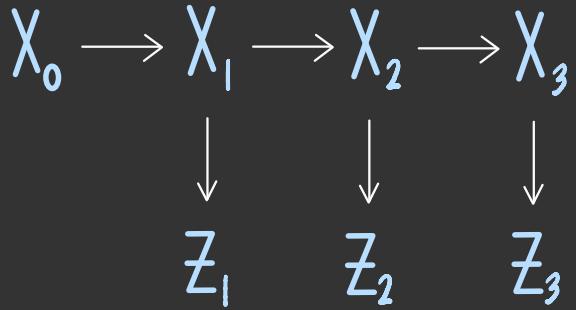
let  $X_t$  be the plane's position after  $t$  seconds relative to the expected position

observed

let  $Z_t$  be the radar reading at time  $t$  relative to the expected position

want to compute:

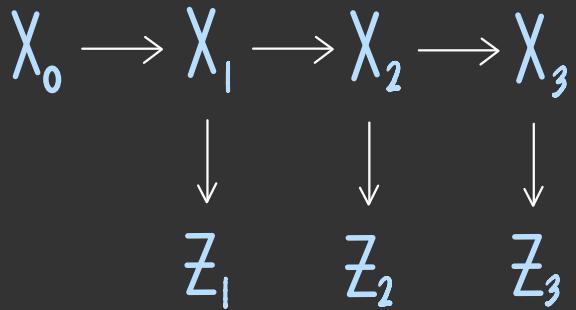
$$P(x_3 | z_1, z_2, z_3)$$



what kind of  
bayesian network  
does this look like?

want to compute:

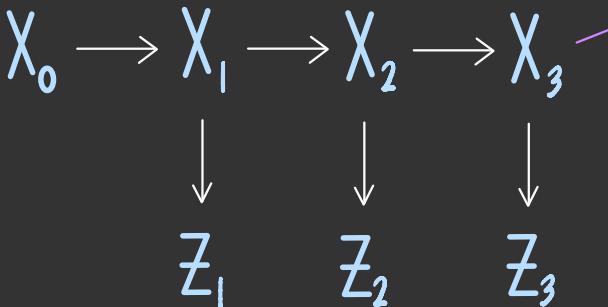
$$P(x_3 | z_1, z_2, z_3)$$



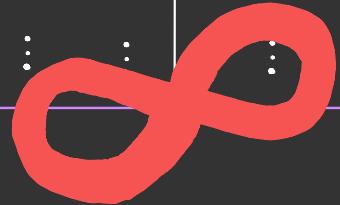
this has the same graph as a hidden markov model, but the variables are continuous not discrete

want to compute:

$$P(x_3 \mid z_1, z_2, z_3)$$



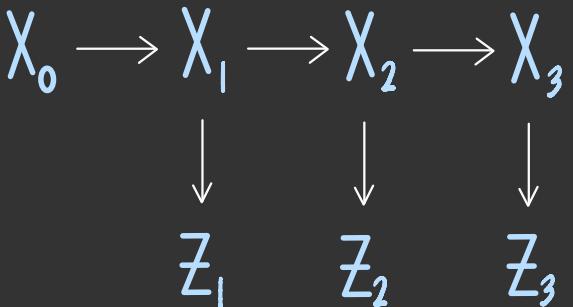
$x_2$	$x_3$	$P(x_3 \mid x_2)$
0	0	•
0	0.17	•
0.29	0	•
⋮	⋮	⋮



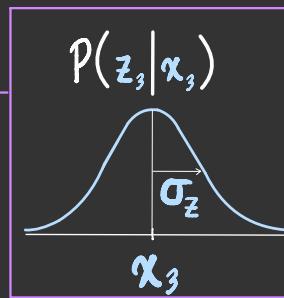
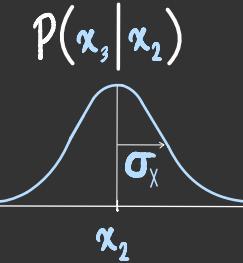
so we can't represent the conditional  
probabilities as tables

want to compute:

$$P(x_3 | z_1, z_2, z_3)$$



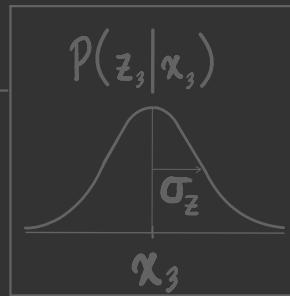
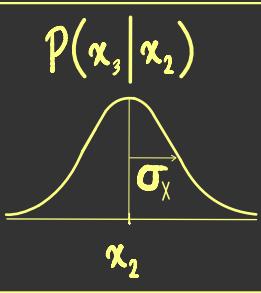
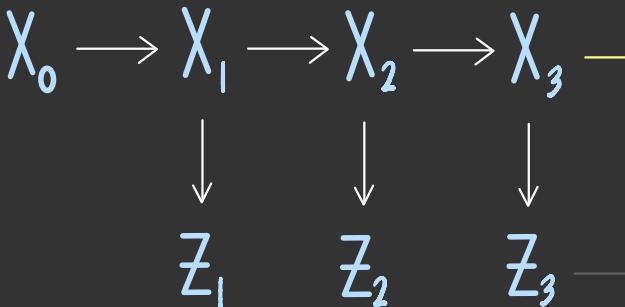
$$z_1 \quad z_2 \quad z_3$$



let's try this instead

want to compute:

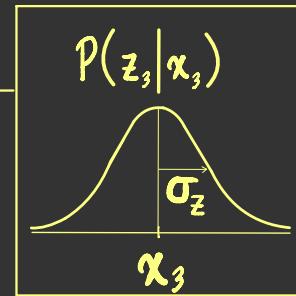
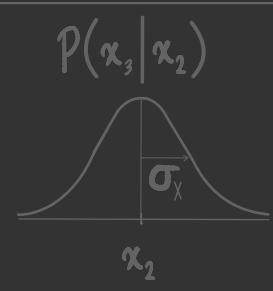
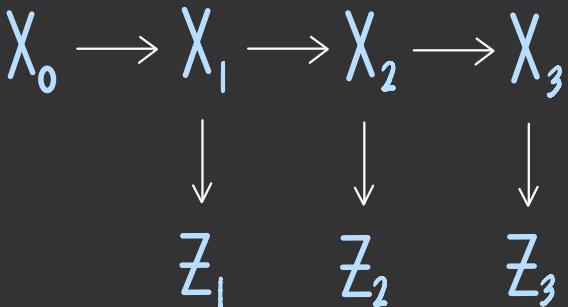
$$P(x_3 | z_1, z_2, z_3)$$



the plane's position at time 3  
relative to its expected position is  
normally distributed with mean  $x_2$   
and standard deviation  $\sigma_x$

want to compute:

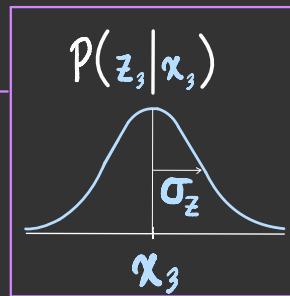
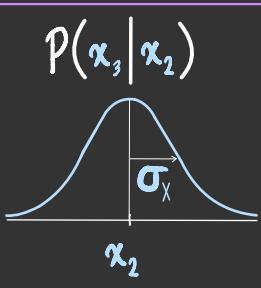
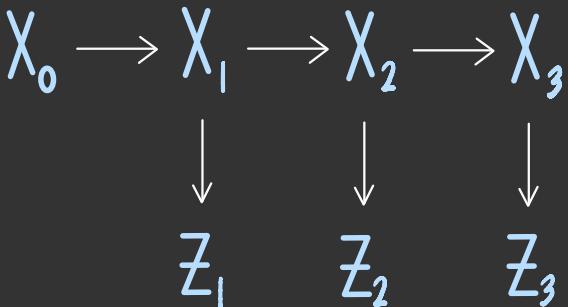
$$P(x_3 | z_1, z_2, z_3)$$



the radar reading at time 3 is  
normally distributed with mean  $x_3$   
and standard deviation  $\sigma_z$

want to compute:

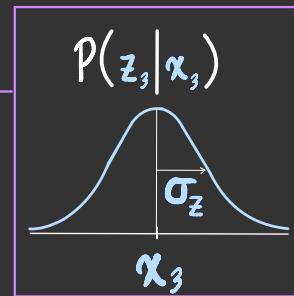
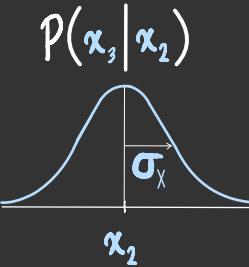
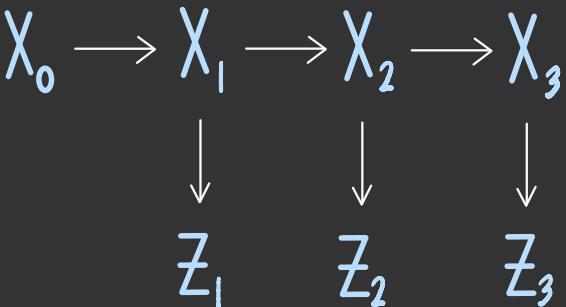
$$P(x_3 | z_1, z_2, z_3)$$



but without probability tables, we  
can't compute probabilities  
using variable elimination or  
the junction tree algorithm

want to compute:

$$P(x_3 | z_1, z_2, z_3)$$



we need to use **calculus**

interlude :

the rules of continuous probability

# how do these translate to continuous variables?

given variables  $X_1, \dots, X_n$  with domains  $D(X_i)$   
and joint probability  $P(x_1, \dots, x_n)$

$$P(x_1, \dots, x_m) = \sum_{x_{m+1} \in D(X_{m+1})} \cdots \sum_{x_n \in D(X_n)} P(x_1, \dots, x_n)$$



$$P(x_{m+1}, \dots, x_n | x_1, \dots, x_m) = \frac{P(x_1, \dots, x_n)}{P(x_1, \dots, x_m)}$$

$y | x$

# how do these translate to continuous variables?

given variables  $X_1, \dots, X_n$  with domains  $D(X_i)$

and joint probability  $P(x_1, \dots, x_n)$

$$P(x_1, \dots, x_m) = \sum_{x_{m+1} \in D(X_{m+1})} \cdots \sum_{x_n \in D(X_n)} P(x_1, \dots, x_n)$$



# how do these translate to continuous variables?

given variables  $X_1, \dots, X_n$  with domains  $D(X_i)$

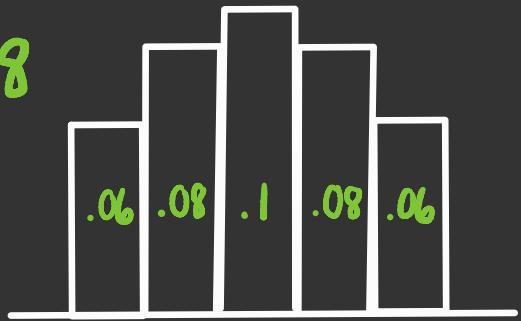
and joint probability  $P(x_1, \dots, x_n)$

how do we sum  
over all real numbers?

$$P(x_1, \dots, x_m) = \sum_{x_{m+1} \in D(X_{m+1})} \dots \sum_{x_n \in D(X_n)} P(x_1, \dots, x_n)$$



$$\sum = .38$$

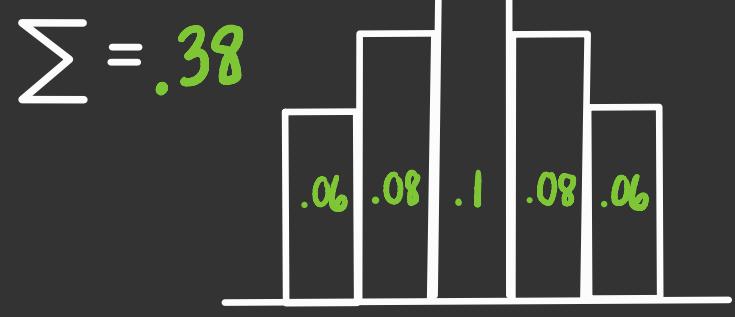


# how do these translate to continuous variables?

given variables  $X_1, \dots, X_n$  with domains  $D(X_i)$

and joint probability  $P(x_1, \dots, x_n)$

$$P(x_1, \dots, x_m) = \int \cdots \int P(x_1, \dots, x_n) dx_{m+1} \cdots dx_n$$



# how do these translate to continuous variables?

given variables  $X_1, \dots, X_n$  with domains  $D(X_i)$   
and joint probability  $P(x_1, \dots, x_n)$

$$P(x_1, \dots, x_m) = \int \cdots \int P(x_1, \dots, x_n) dx_{m+1} \cdots dx_n$$



$$P(x_{m+1}, \dots, x_n | x_1, \dots, x_m) = \frac{P(x_1, \dots, x_n)}{P(x_1, \dots, x_m)}$$

$y | x$

# how do these translate to continuous variables?

given variables  $X_1, \dots, X_n$  with domains  $D(X_i)$   
and joint probability  $P(x_1, \dots, x_n)$

$$P(x_1, \dots, x_m) = \int \dots \int P(x_1, \dots, x_n) dx_{m+1} \dots dx_n$$



$$P(x_{m+1}, \dots, x_n | x_1, \dots, x_m) = \frac{P(x_1, \dots, x_n)}{P(x_1, \dots, x_m)}$$

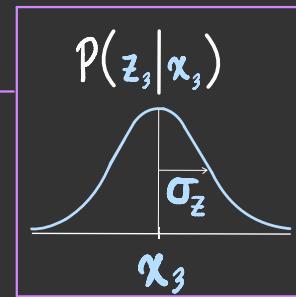
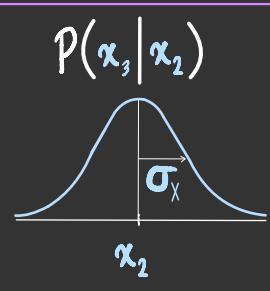
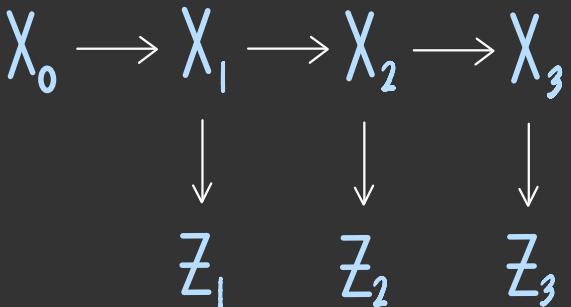
$y | x$



end of  
interlude

want to compute:

$$P(x_3 | x_0, z_1, z_2, z_3)$$



we need to use **calculus**

$$P(x_3 | x_0, z_1, z_2, z_3) = \boxed{?}$$

$$P(\alpha_3 \left| \alpha_0, z_1, z_2, z_3 \right) \overset{y|x}{=} \frac{P(\alpha_3, \alpha_0, z_1, z_2, z_3)}{P(\alpha_0, z_1, z_2, z_3)}$$

$$\begin{aligned}
 P(x_3 | x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
 &\stackrel{\text{total}}{=} \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3) dx_3}
 \end{aligned}$$

$$\begin{aligned}
 P(x_3 | x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
 &\stackrel{\text{total}}{=} \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3) dx_3} \\
 &\stackrel{\text{exp}}{=} \frac{P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0)}{\int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) dx_3}
 \end{aligned}$$

$$\begin{aligned}
P(\alpha_3 | \alpha_0, z_1, z_2, z_3) & \stackrel{y|x}{=} \frac{P(\alpha_3, \alpha_0, z_1, z_2, z_3)}{P(\alpha_0, z_1, z_2, z_3)} \\
& \stackrel{\text{total}}{=} \frac{P(\alpha_3, \alpha_0, z_1, z_2, z_3)}{\int P(\alpha_3, \alpha_0, z_1, z_2, z_3) d\alpha_3} \\
& \stackrel{\text{exp}}{=} \frac{P(z_3 | \alpha_3, \alpha_0, z_1, z_2) P(\alpha_3 | \alpha_0, z_1, z_2) P(z_2 | z_1, \alpha_0) P(z_1 | \alpha_0) P(\alpha_0)}{\int P(z_3 | \alpha_3, \alpha_0, z_1, z_2) P(\alpha_3 | \alpha_0, z_1, z_2) P(z_2 | z_1, \alpha_0) P(z_1 | \alpha_0) P(\alpha_0) d\alpha_3} \\
& = \frac{P(z_2 | z_1, \alpha_0) P(z_1 | \alpha_0) P(\alpha_0) P(z_3 | \alpha_3, \alpha_0, z_1, z_2) P(\alpha_3 | \alpha_0, z_1, z_2)}{P(z_2 | z_1, \alpha_0) P(z_1 | \alpha_0) P(\alpha_0) \int P(z_3 | \alpha_3, \alpha_0, z_1, z_2) P(\alpha_3 | \alpha_0, z_1, z_2) d\alpha_3}
\end{aligned}$$

$$\begin{aligned}
P(x_3 | x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
&\stackrel{\text{total}}{=} \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3) d\alpha_3} \\
&\stackrel{\text{def}}{=} \frac{P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0)}{\int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) d\alpha_3} \\
&= \frac{P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)}{P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) \int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) d\alpha_3} \\
&= \frac{P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)}{\int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) d\alpha_3}
\end{aligned}$$

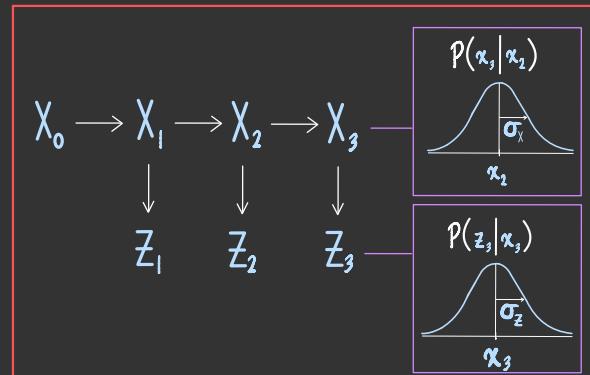
$$\begin{aligned}
P(x_3 | x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
&\stackrel{\text{total}}{=} \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3) dx_3} \\
&\stackrel{\text{def}}{=} \frac{P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0)}{\int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) dx_3} \\
f(x) &= \frac{g(x)}{\int_{-\infty}^{\infty} g(x) dx} = \frac{P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)}{P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) \int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) dx_3} \\
&\stackrel{\text{def}}{=} \frac{P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)}{\int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) dx_3}
\end{aligned}$$

$$\begin{aligned}
P(x_3 | x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
&\stackrel{\text{total}}{=} \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3) d\alpha_3} \\
&\stackrel{\text{def}}{=} \frac{P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0)}{\int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) d\alpha_3} \\
&= \frac{P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)}{P(z_2 | z_1, x_0) P(z_1 | x_0) P(x_0) \int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) d\alpha_3} \\
&= \frac{P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)}{\int P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) d\alpha_3} \\
&= P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)
\end{aligned}$$

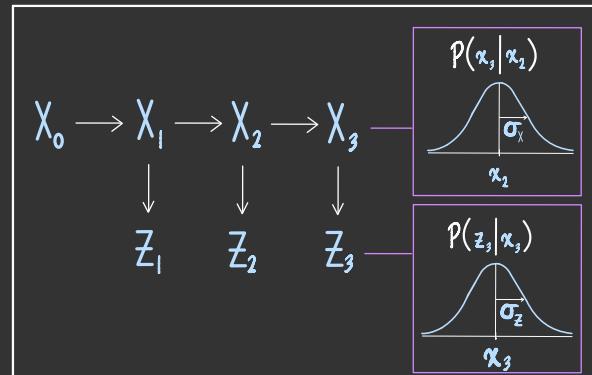
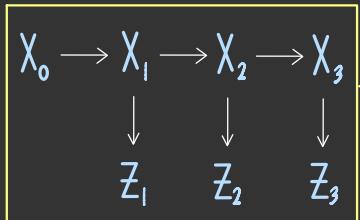
$$P(x_3 | x_0, z_1, z_2, z_3) = \underbrace{\alpha P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)}$$



what does the  
bayesian network  
tell us?



$$\begin{aligned}
 P(x_3 | x_0, z_1, z_2, z_3) & \propto P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) \\
 & \propto P(z_3 | x_3) P(x_3 | x_0, z_1, z_2)
 \end{aligned}$$



$$P(x_3 | x_0, z_1, z_2, z_3) = \propto P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2)$$

$$= \propto P(z_3 | x_3) P(x_3 | x_0, z_1, z_2)$$

a conditional  
distribution is  
still a distribution

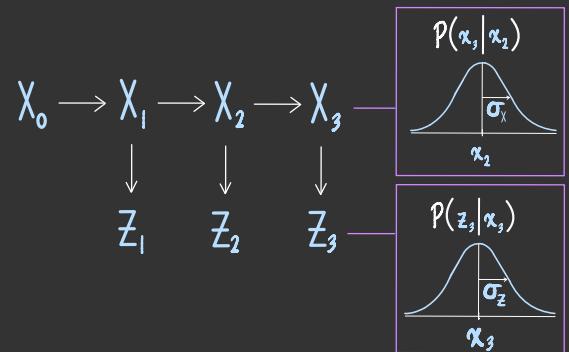
$$= \propto P(z_3 | x_3) \int P(x_3 | x_2, x_0, z_1, z_2) P(x_2 | x_0, z_1, z_2) dx_2$$

$$P(x_3 | x_0, z_1, z_2) = Q(x_3)$$

$$\stackrel{\text{total}}{=} \int Q(x_2, x_3) dx_2$$

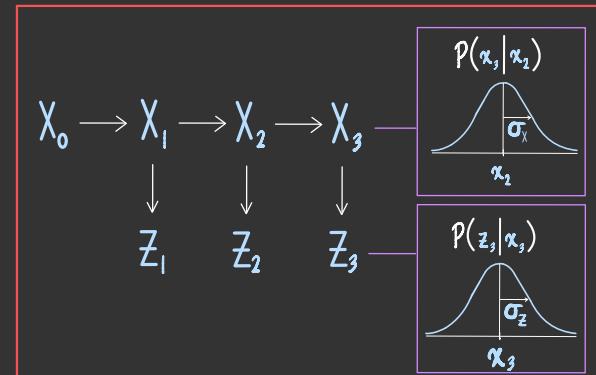
$$\stackrel{y|x}{=} \int Q(x_3 | x_2) Q(x_2) dx_2$$

$$= \int P(x_3 | x_2, x_0, z_1, z_2) P(x_2 | x_0, z_1, z_2) dx_2$$



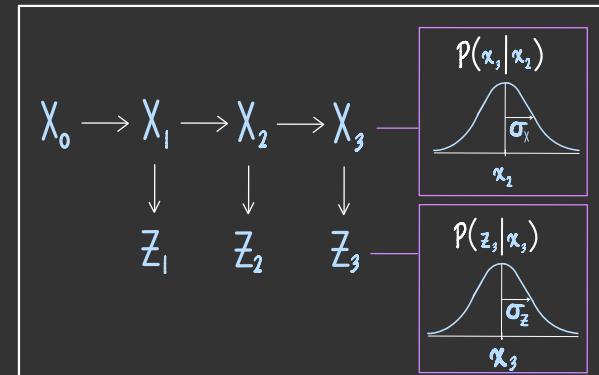
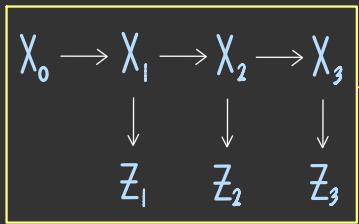
$$\begin{aligned}
 P(x_3 | x_0, z_1, z_2, z_3) &= \propto P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) \\
 &= \propto P(z_3 | x_3) P(x_3 | x_0, z_1, z_2) \\
 &= \propto P(z_3 | x_3) \underbrace{\int P(x_3 | x_2, x_0, z_1, z_2) P(x_2 | x_0, z_1, z_2) dx_2}_{\text{---}}
 \end{aligned}$$

what does the  
bayesian network  
tell us?

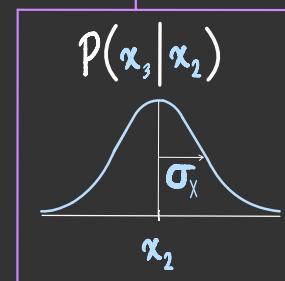
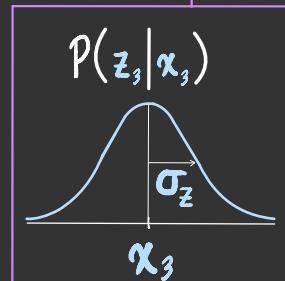


$$\begin{aligned}
P(x_3 | x_0, z_1, z_2, z_3) &= \propto P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) \\
&= \propto P(z_3 | x_3) P(x_3 | x_0, z_1, z_2) \\
&= \propto P(z_3 | x_3) \int P(x_3 | x_2, x_0, z_1, z_2) P(x_2 | x_0, z_1, z_2) dx_2 \\
&= \propto P(z_3 | x_3) \int P(x_3 | x_2) \underline{P(x_2 | x_0, z_1, z_2)} dx_2
\end{aligned}$$

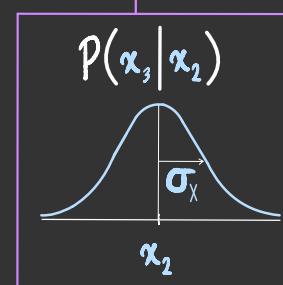
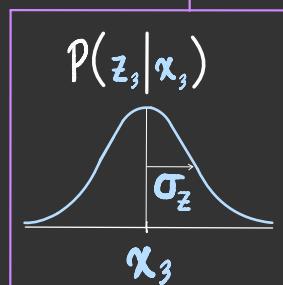
can't simplify further



$$\begin{aligned}
P(x_3 | x_0, z_1, z_2, z_3) &= \propto P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) \\
&= \propto P(z_3 | x_3) P(x_3 | x_0, z_1, z_2) \\
&= \propto P(z_3 | x_3) \int P(x_3 | x_2, x_0, z_1, z_2) P(x_2 | x_0, z_1, z_2) dx_2 \\
&= \propto P(z_3 | x_3) \int P(x_3 | x_2) P(x_2 | x_0, z_1, z_2) dx_2
\end{aligned}$$



$$\begin{aligned}
P(x_3 | x_0, z_1, z_2, z_3) &= \propto P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) \\
&= \propto P(z_3 | x_3) P(x_3 | x_0, z_1, z_2) \\
&= \propto P(z_3 | x_3) \int P(x_3 | x_2, x_0, z_1, z_2) P(x_2 | x_0, z_1, z_2) dx_2 \\
&= \propto P(z_3 | x_3) \int P(x_3 | x_2) P(x_2 | x_0, z_1, z_2) dx_2
\end{aligned}$$

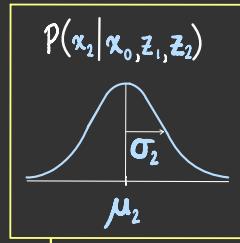
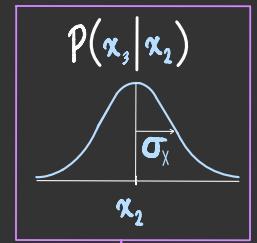
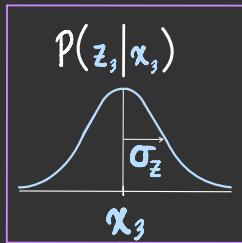


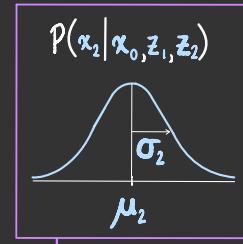
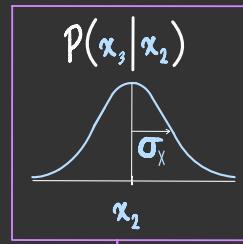
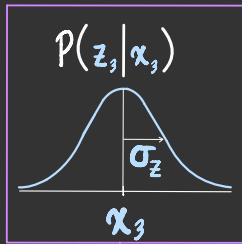
$$\begin{aligned}
P(x_3 | x_0, z_1, z_2, z_3) &= \propto P(z_3 | x_3, x_0, z_1, z_2) P(x_3 | x_0, z_1, z_2) \\
&= \propto P(z_3 | x_3) P(x_3 | x_0, z_1, z_2) \\
&= \propto P(z_3 | x_3) \int P(x_3 | x_2, x_0, z_1, z_2) P(x_2 | x_0, z_1, z_2) dx_2 \\
&= \propto P(z_3 | x_3) \int P(x_3 | x_2) P(x_2 | x_0, z_1, z_2) dx_2
\end{aligned}$$

\$P(z\_3 | x\_3)\$
\$P(x\_3 | x\_2)\$
\$P(x\_2 | x\_0, z\_1, z\_2)\$

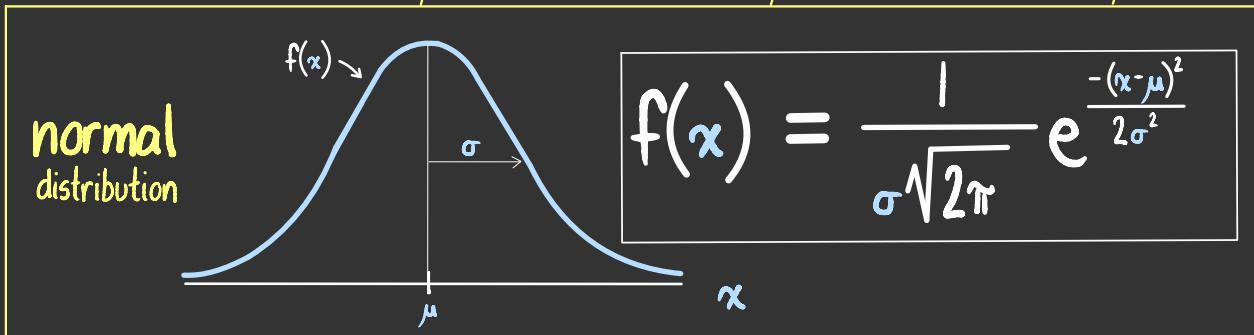
let's assume this is normally distributed with mean  $\mu_2$  and standard deviation  $\sigma_2$

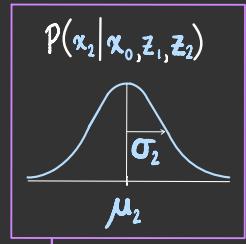
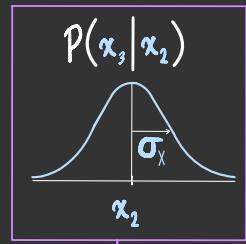
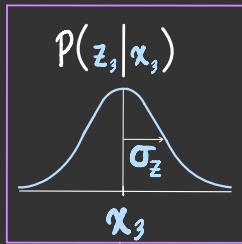
$$P(x_3 | x_0, z_1, z_2, z_3) = \propto \cdot P(z_3 | x_3) \int P(x_3 | x_2) \cdot P(x_2 | x_0, z_1, z_2) dx_2$$





$$\begin{aligned}
 P(\alpha_3 | \alpha_0, z_1, z_2, z_3) &= \alpha \cdot P(z_3 | \alpha_3) \int P(\alpha_3 | \alpha_2) \cdot P(\alpha_2 | \alpha_0, z_1, z_2) d\alpha_2 \\
 &= \alpha \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{(z_3 - \alpha_3)^2}{2\sigma_z^2}} \int \frac{1}{\sigma_\alpha \sqrt{2\pi}} e^{-\frac{(\alpha_3 - \alpha_2)^2}{2\sigma_\alpha^2}} \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(\alpha_2 - \mu_2)^2}{2\sigma_2^2}} d\alpha_2
 \end{aligned}$$





$$P(\alpha_3 | \alpha_0, z_1, z_2, z_3) = \alpha \cdot P(z_3 | \alpha_3) \int P(\alpha_3 | \alpha_2) \cdot P(\alpha_2 | \alpha_0, z_1, z_2) d\alpha_2$$

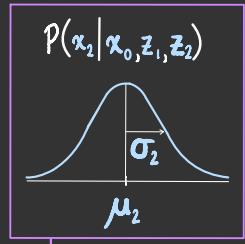
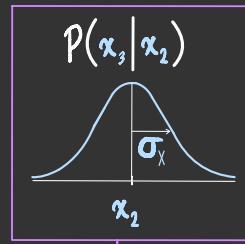
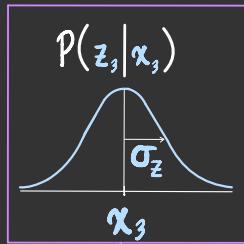
$$= \alpha \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{(z_3 - \alpha_3)^2}{2\sigma_z^2}} \int \frac{1}{\sigma_\alpha \sqrt{2\pi}} e^{-\frac{(\alpha_3 - \alpha_2)^2}{2\sigma_\alpha^2}} \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(\alpha_2 - \mu_2)^2}{2\sigma_2^2}} d\alpha_2$$



$$= \frac{1}{\sigma_3 \sqrt{2\pi}} e^{-\frac{(\alpha_3 - \mu_3)^2}{2\sigma_3^2}}$$

where

$$\left[ \begin{array}{l} \mu_3 = \frac{(\sigma_2^2 + \sigma_\alpha^2)z_3 + \sigma_z^2 \mu_2}{\sigma_2^2 + \sigma_\alpha^2 + \sigma_z^2} \\ \sigma_3^2 = \frac{(\sigma_2^2 + \sigma_\alpha^2)\sigma_z^2}{\sigma_2^2 + \sigma_\alpha^2 + \sigma_z^2} \end{array} \right]$$



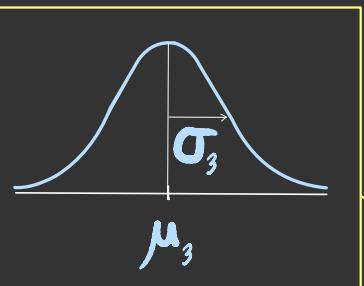
$$P(\alpha_3 | \alpha_0, z_1, z_2, z_3) = \alpha \cdot P(z_3 | \alpha_3) \int P(\alpha_3 | \alpha_2) \cdot P(\alpha_2 | \alpha_0, z_1, z_2) d\alpha_2$$

$$= \alpha \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{(z_3 - \alpha_3)^2}{2\sigma_z^2}} \int \frac{1}{\sigma_\alpha \sqrt{2\pi}} e^{-\frac{(\alpha_3 - \alpha_2)^2}{2\sigma_\alpha^2}} \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(\alpha_2 - \mu_2)^2}{2\sigma_2^2}} d\alpha_2$$



$$= \frac{1}{\sigma_3 \sqrt{2\pi}} e^{-\frac{(\alpha_3 - \mu_3)^2}{2\sigma_3^2}}$$

where

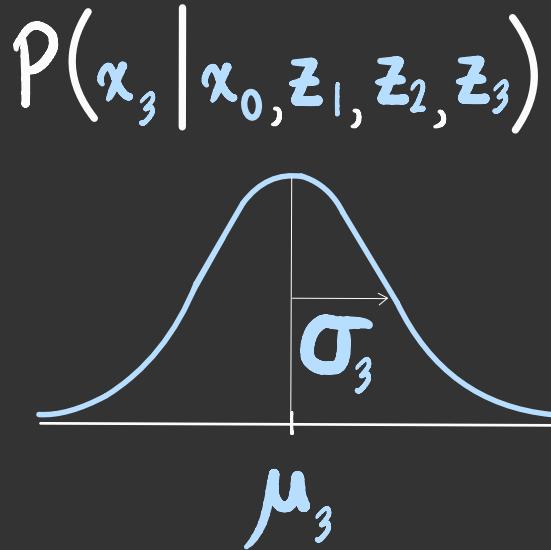


another  
normal distribution!

$$\mu_3 = \frac{(\sigma_2^2 + \sigma_\alpha^2)z_3 + \sigma_z^2\mu_2}{\sigma_2^2 + \sigma_\alpha^2 + \sigma_z^2}$$

$$\sigma_3^2 = \frac{(\sigma_2^2 + \sigma_\alpha^2)\sigma_z^2}{\sigma_2^2 + \sigma_\alpha^2 + \sigma_z^2}$$

so if  $P(x_2 | x_0, z_1, z_2)$  is a normal distribution,  
then so is  $P(x_3 | x_0, z_1, z_2, z_3)$

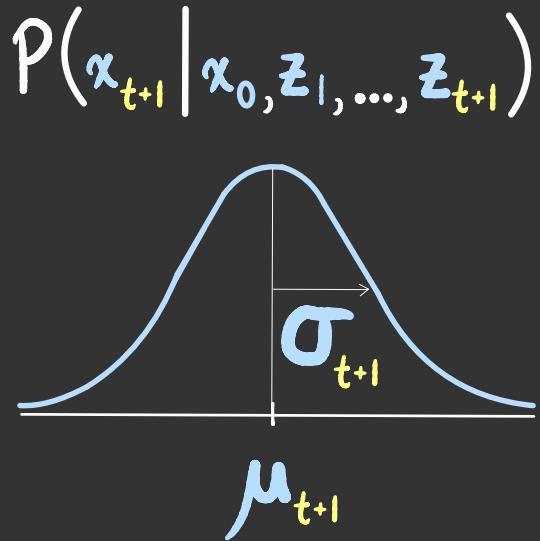


where

$$\mu_3 = \frac{(\sigma_z^2 + \sigma_x^2)z_3 + \sigma_z^2\mu_2}{\sigma_z^2 + \sigma_x^2 + \sigma_z^2}$$

$$\sigma_3^2 = \frac{(\sigma_z^2 + \sigma_x^2)\sigma_z^2}{\sigma_z^2 + \sigma_x^2 + \sigma_z^2}$$

so if  $P(x_t | x_0, z_1, \dots, z_t)$  is a normal distribution,  
then so is  $P(x_{t+1} | x_0, z_1, \dots, z_{t+1})$



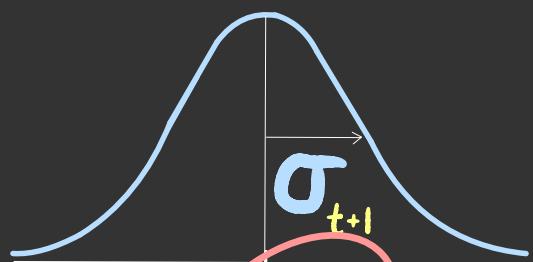
where

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2)z_{t+1} + \sigma_z^2\mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

$$\sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2)\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

and the most likely relative position of the plane  
is just the **mean** of the distribution

$$P(x_{t+1} | x_0, z_1, \dots, z_{t+1})$$



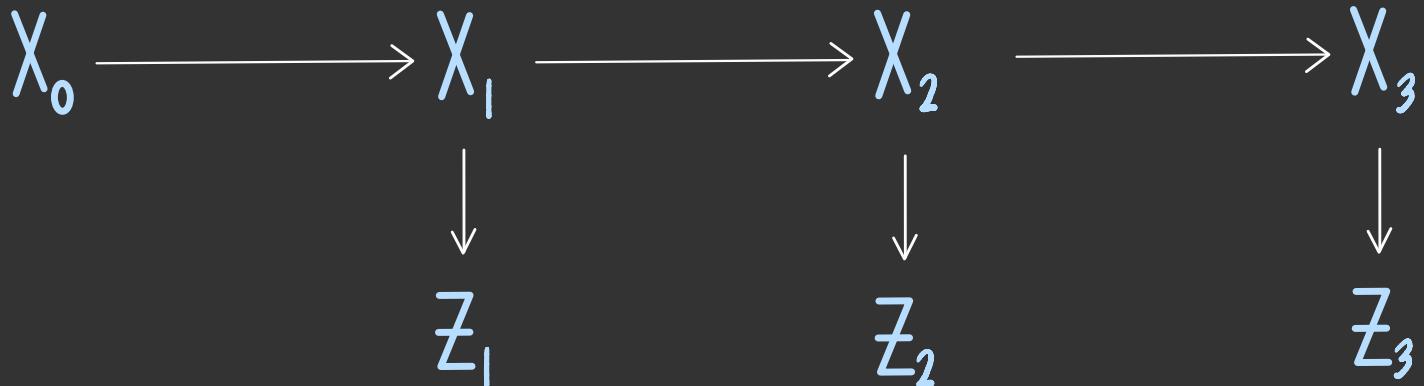
where

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2)z_{t+1} + \sigma_z^2\mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

$$\sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2)\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

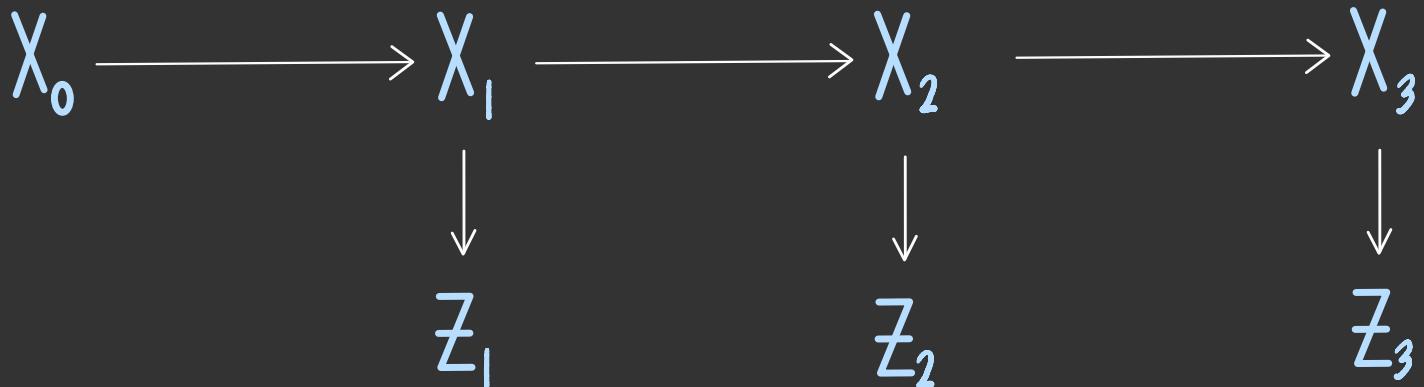
if  $P(x_t | x_0, z_1, \dots, z_t)$  is a normal distribution,  
then so is  $P(x_{t+1} | x_0, z_1, \dots, z_{t+1})$

if  $P(x_2 | x_0, z_1, z_2)$  is normal  
then  $P(x_3 | x_0, z_1, z_2, z_3)$  is normal

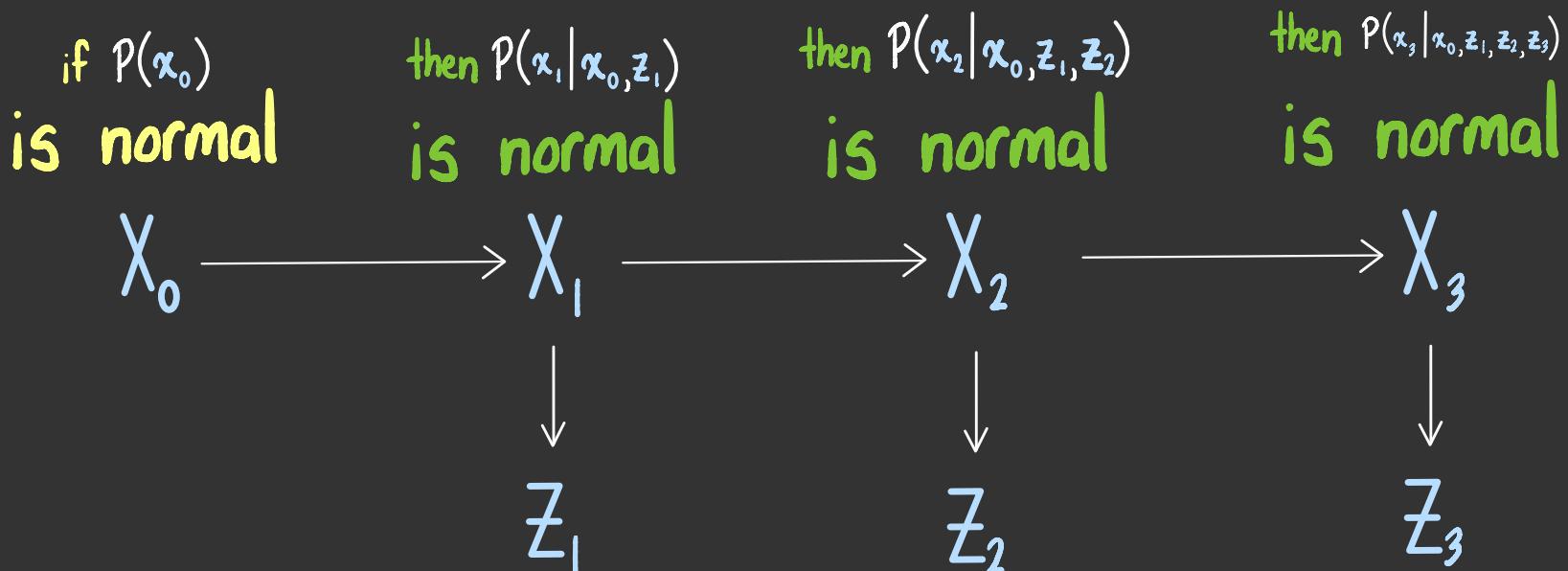


if  $P(x_t | x_0, z_1, \dots, z_t)$  is a normal distribution,  
then so is  $P(x_{t+1} | x_0, z_1, \dots, z_{t+1})$

if  $P(x_1 | x_0, z_1)$       then  $P(x_2 | x_0, z_1, z_2)$       then  $P(x_3 | x_0, z_1, z_2, z_3)$   
is normal                  is normal                  is normal

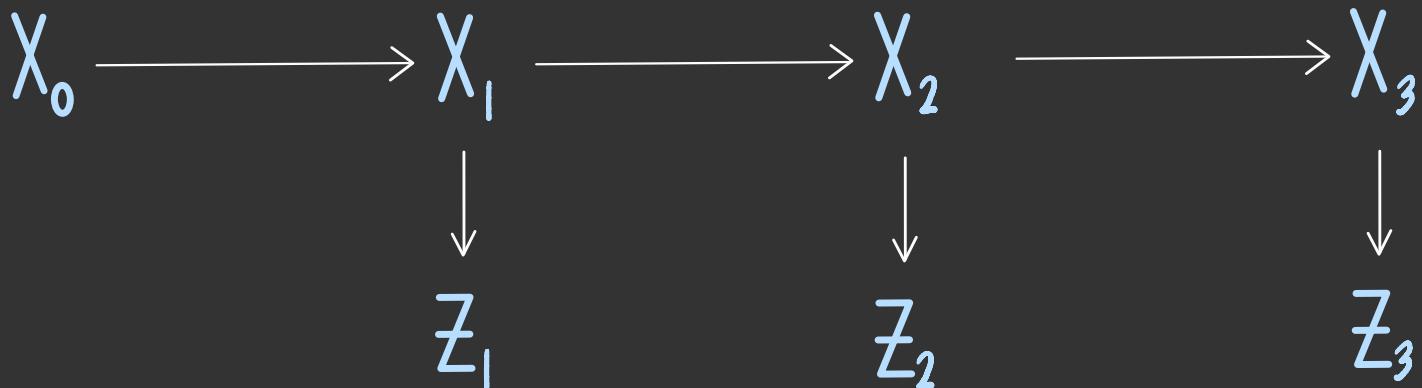


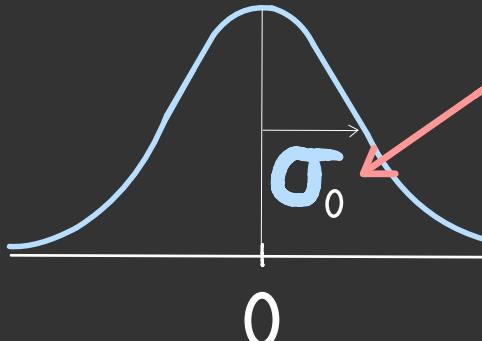
if  $P(x_t | x_0, z_1, \dots, z_t)$  is a normal distribution,  
then so is  $P(x_{t+1} | x_0, z_1, \dots, z_{t+1})$



if  $P(x_t | x_0, z_1, \dots, z_t)$  is a normal distribution,  
then so is  $P(x_{t+1} | x_0, z_1, \dots, z_{t+1})$

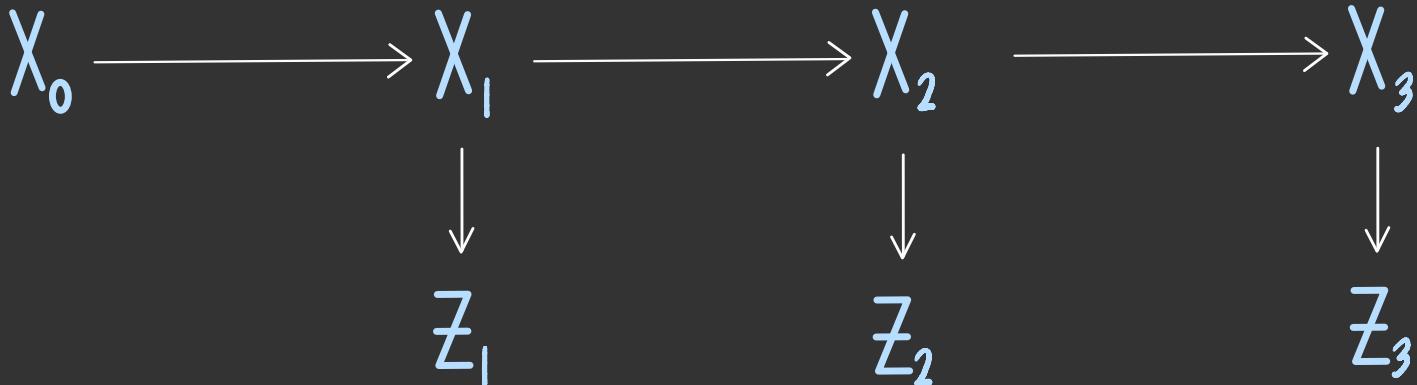
is  $P(x_0)$  normal?



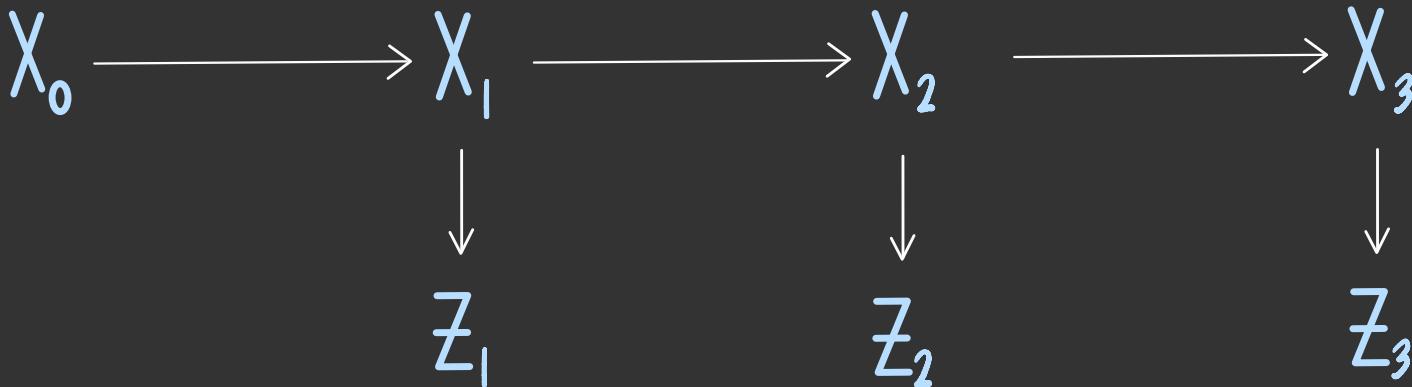
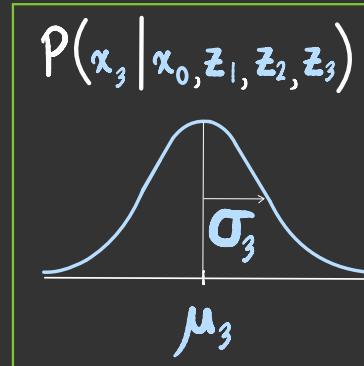
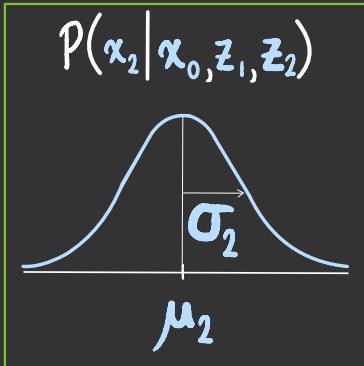
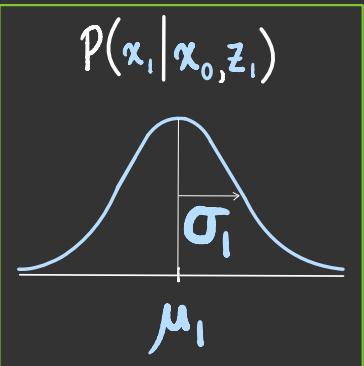
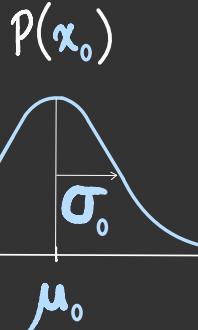


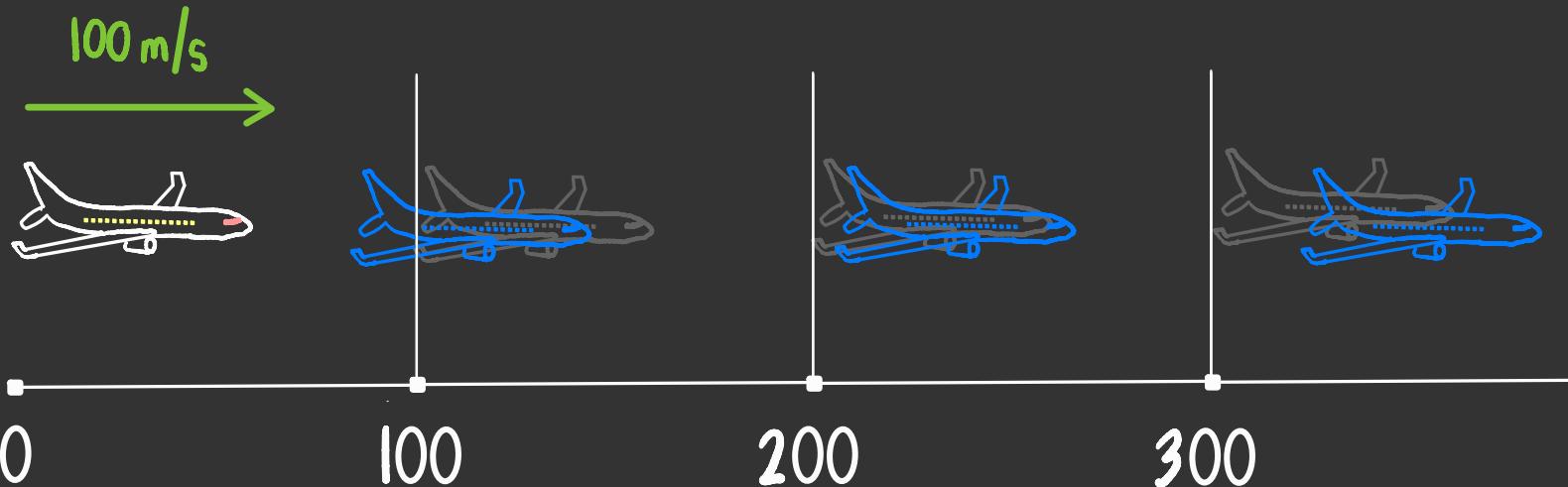
expresses  
our uncertainty  
about the  
starting position

sure, why not?



We can keep efficiently updating our estimate of the plane's position at each time step

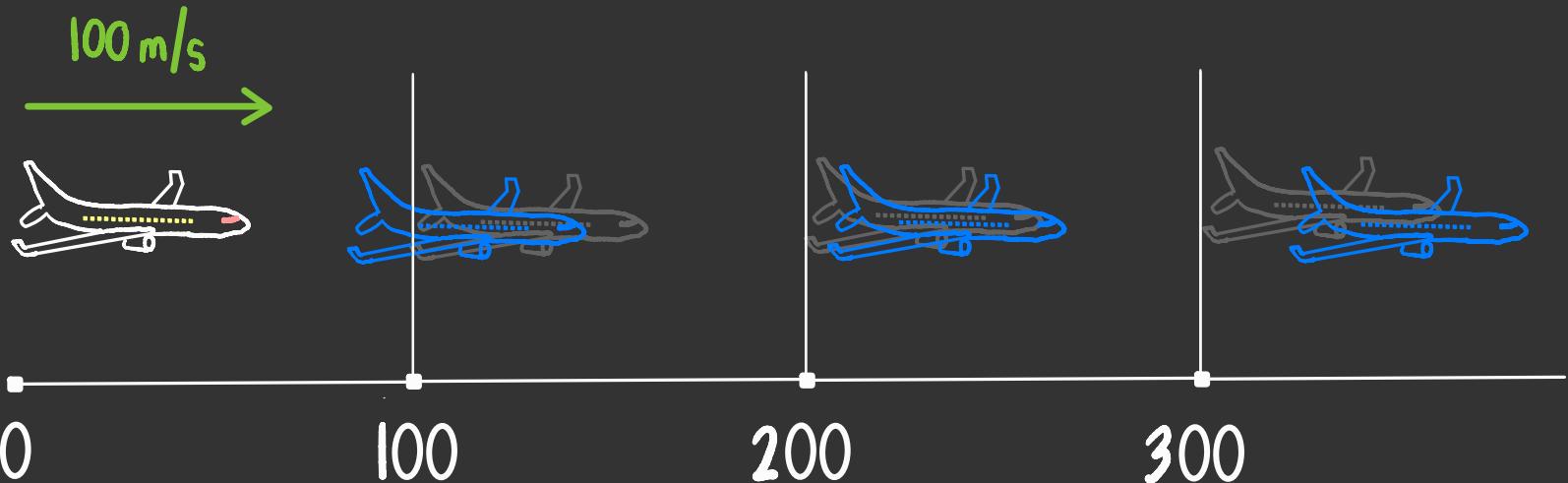




this approach fuses:

- our prior knowledge of the velocity
- our noisy radar observations

to produce a probabilistically optimal prediction



$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2)z_{t+1} + \sigma_z^2\mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

our previous estimate of the relative position

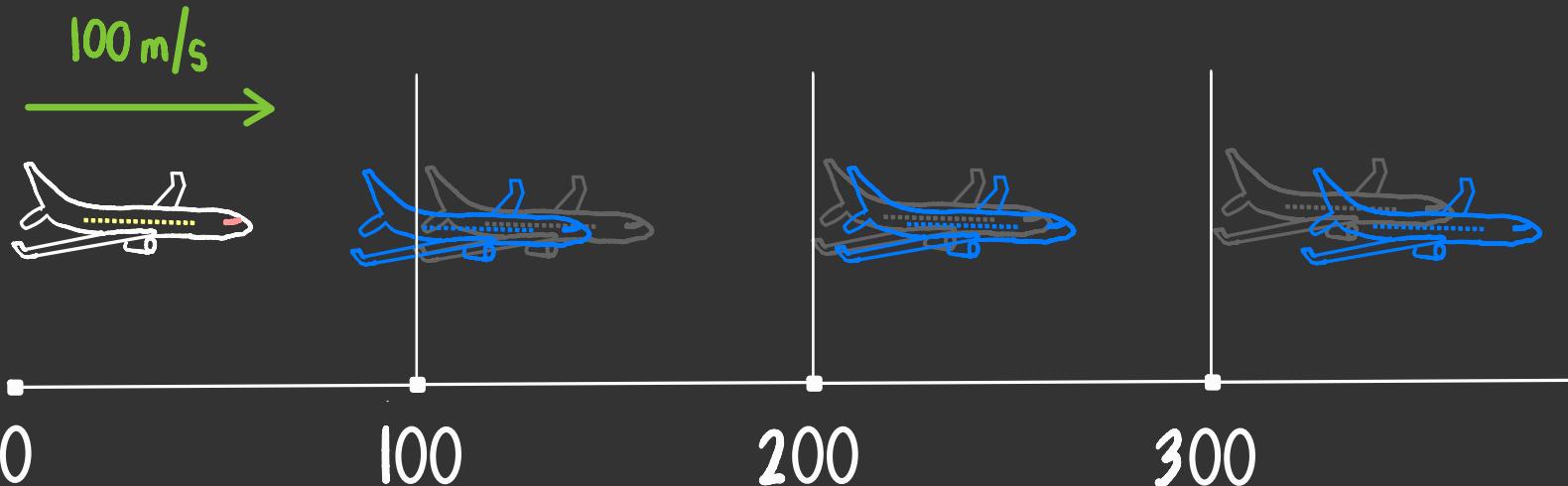
100 m/s



0      100      200      300

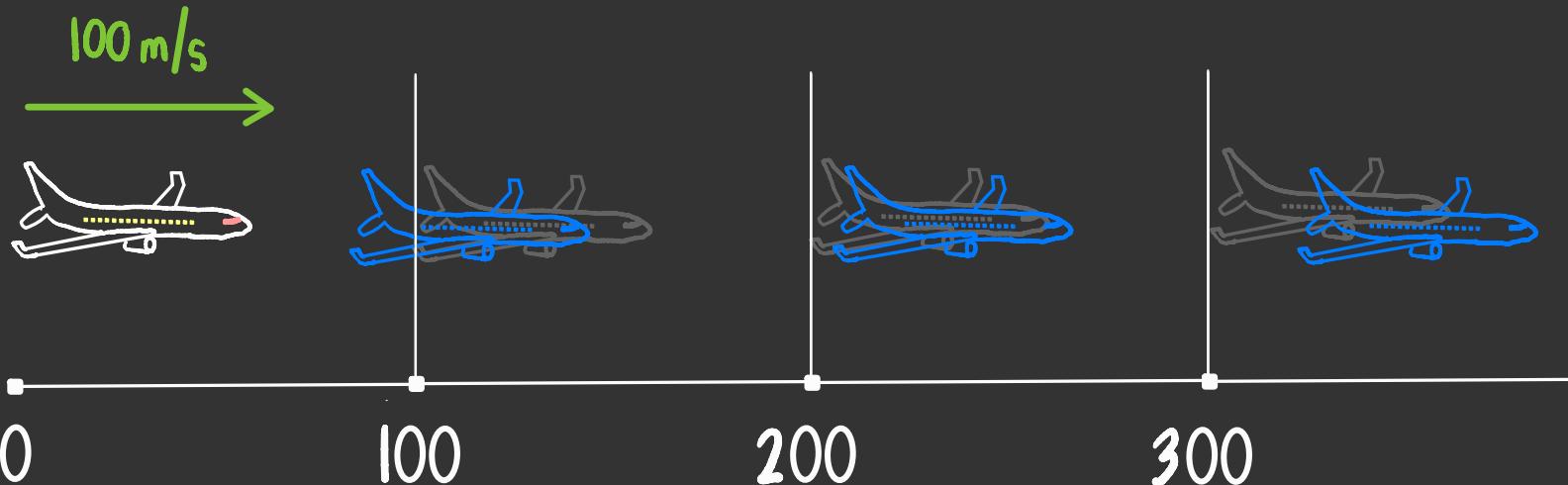
$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2)z_{t+1} + \sigma_z^2\mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

our radar observation



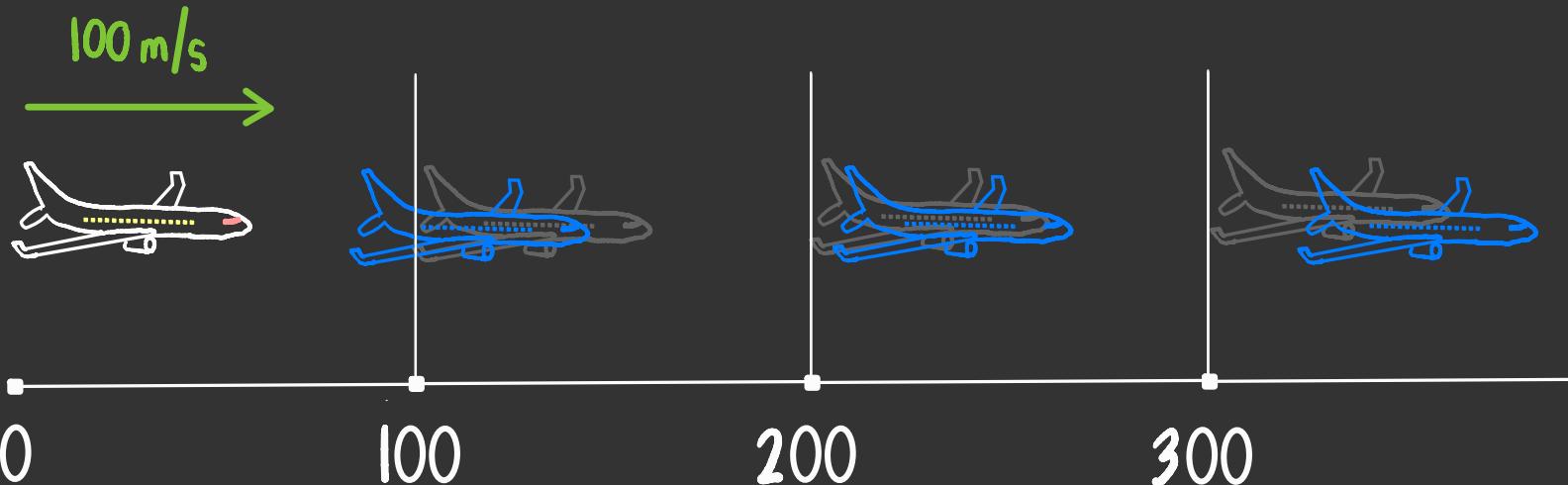
$$\mu_{t+1} = \frac{\sigma_t^2 + \sigma_x^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} z_{t+1} + \frac{\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \mu_t$$

this information is weighted by the different "noises"



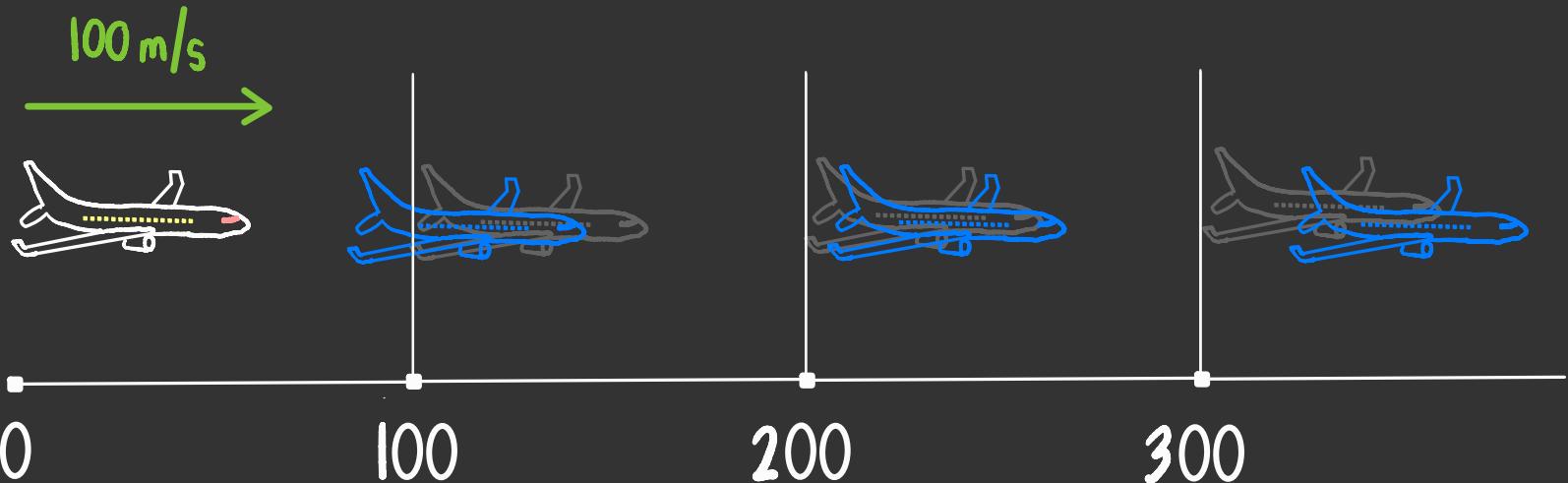
$$\mu_{t+1} = \frac{\sigma_t^2 + \sigma_x^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} z_{t+1} + \frac{\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \mu_t$$

the less variance in velocity, the less we rely on the radar observations



$$\mu_{t+1} = \frac{\sigma_t^2 + \sigma_x^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} z_{t+1} + \frac{\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \mu_t$$

the less noisy the radar, the less we rely  
on the estimate of the previous position



$$\mu_{t+1} = \frac{\sigma_t^2 + \sigma_x^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} z_{t+1} + \frac{\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \mu_t$$

this model is called a **kalman filter**