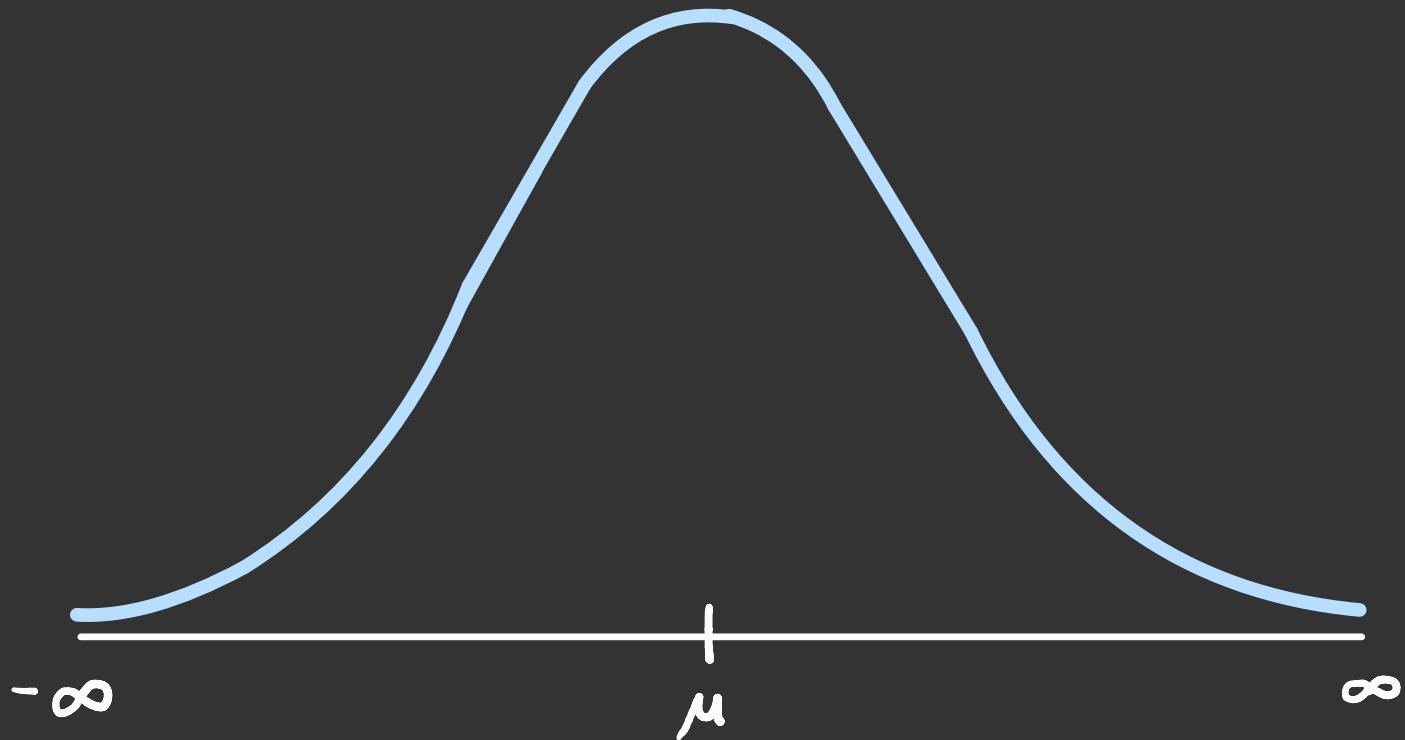


normal
distributions
24 oct
2022

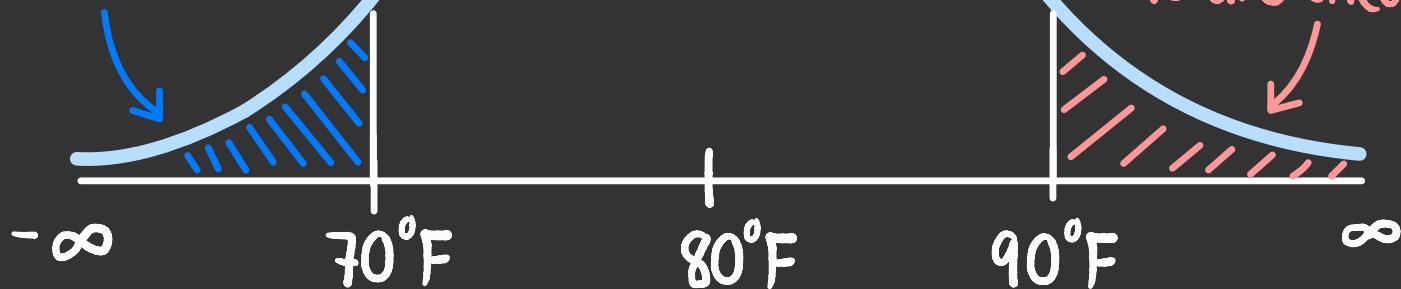
CSCI
373



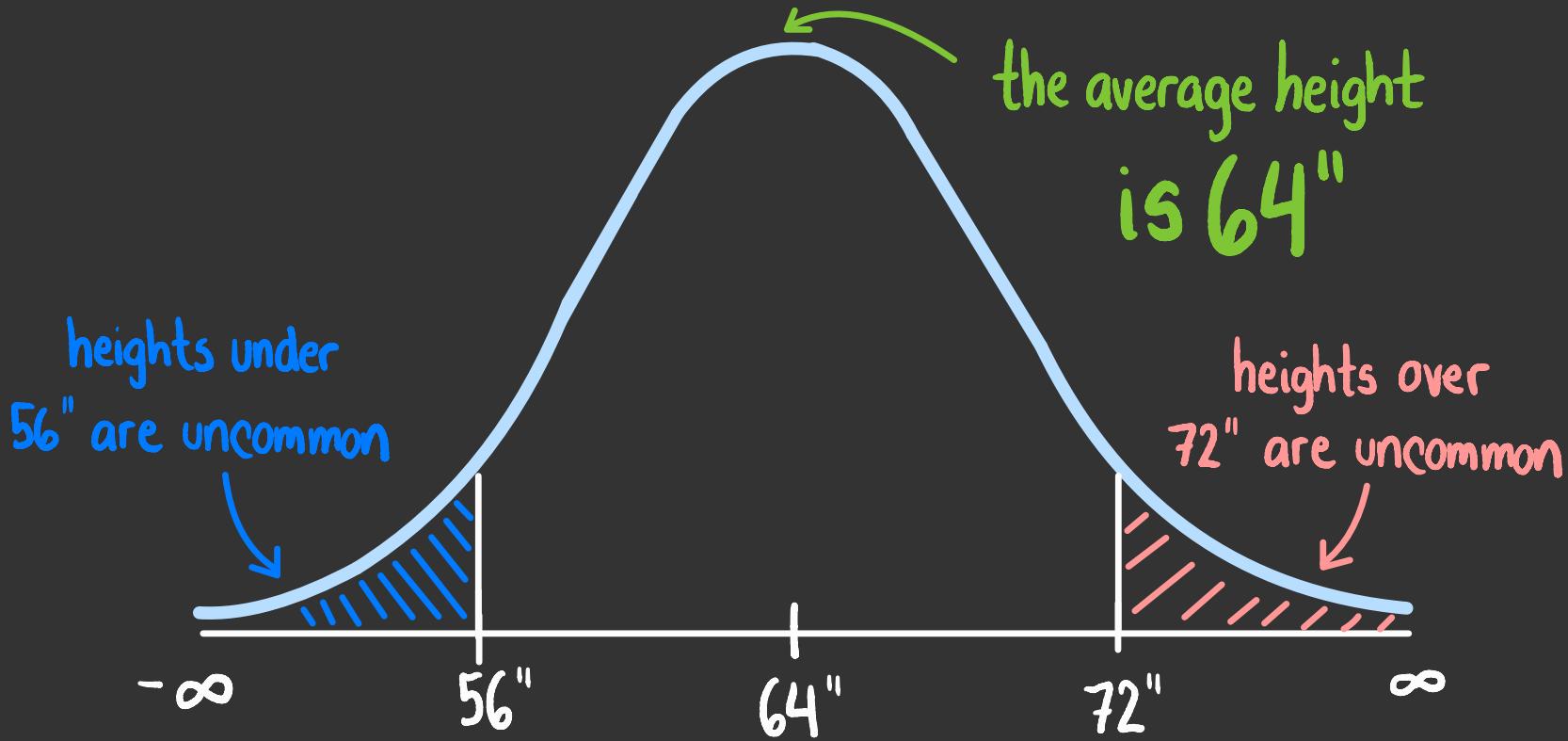
the normal distribution

the average temperature
is 80°F

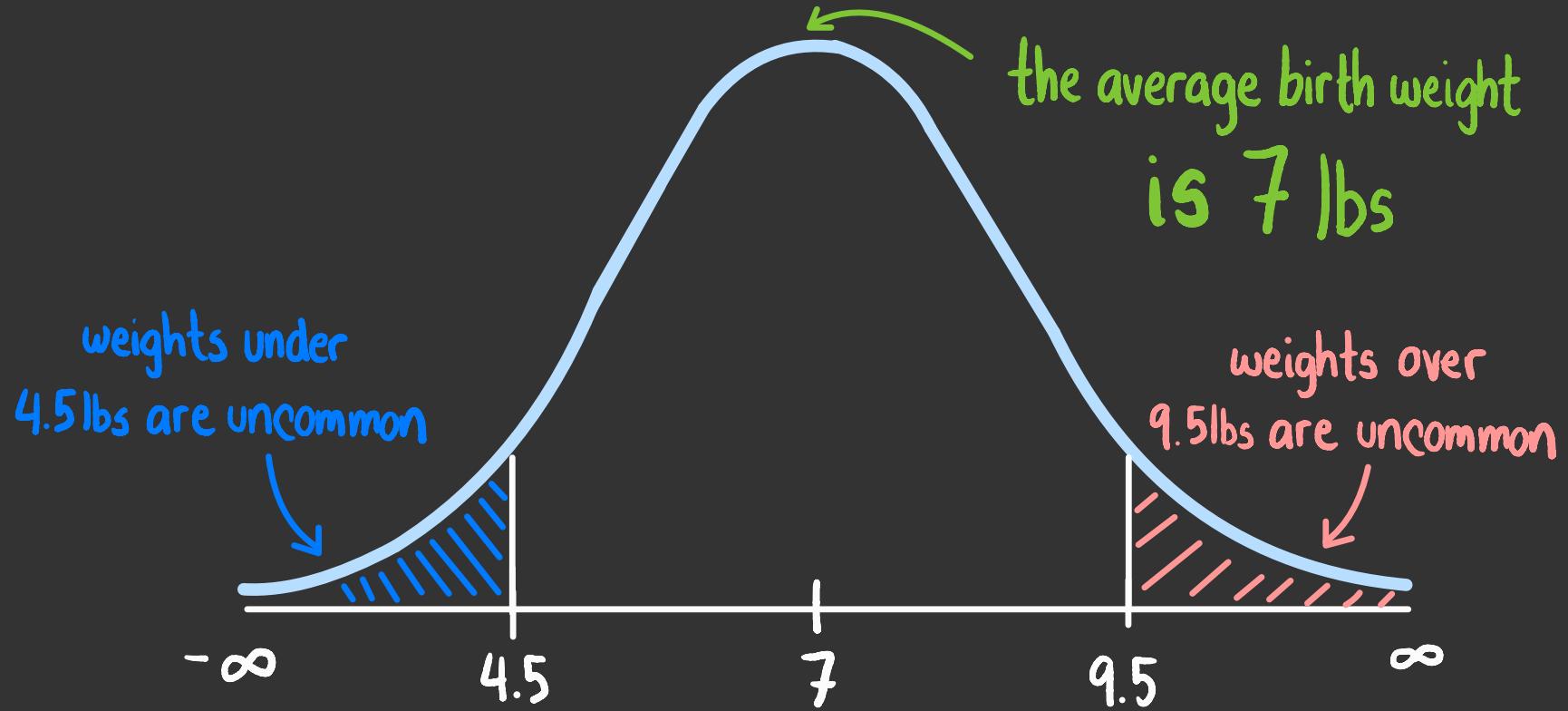
temperatures under
70 are uncommon



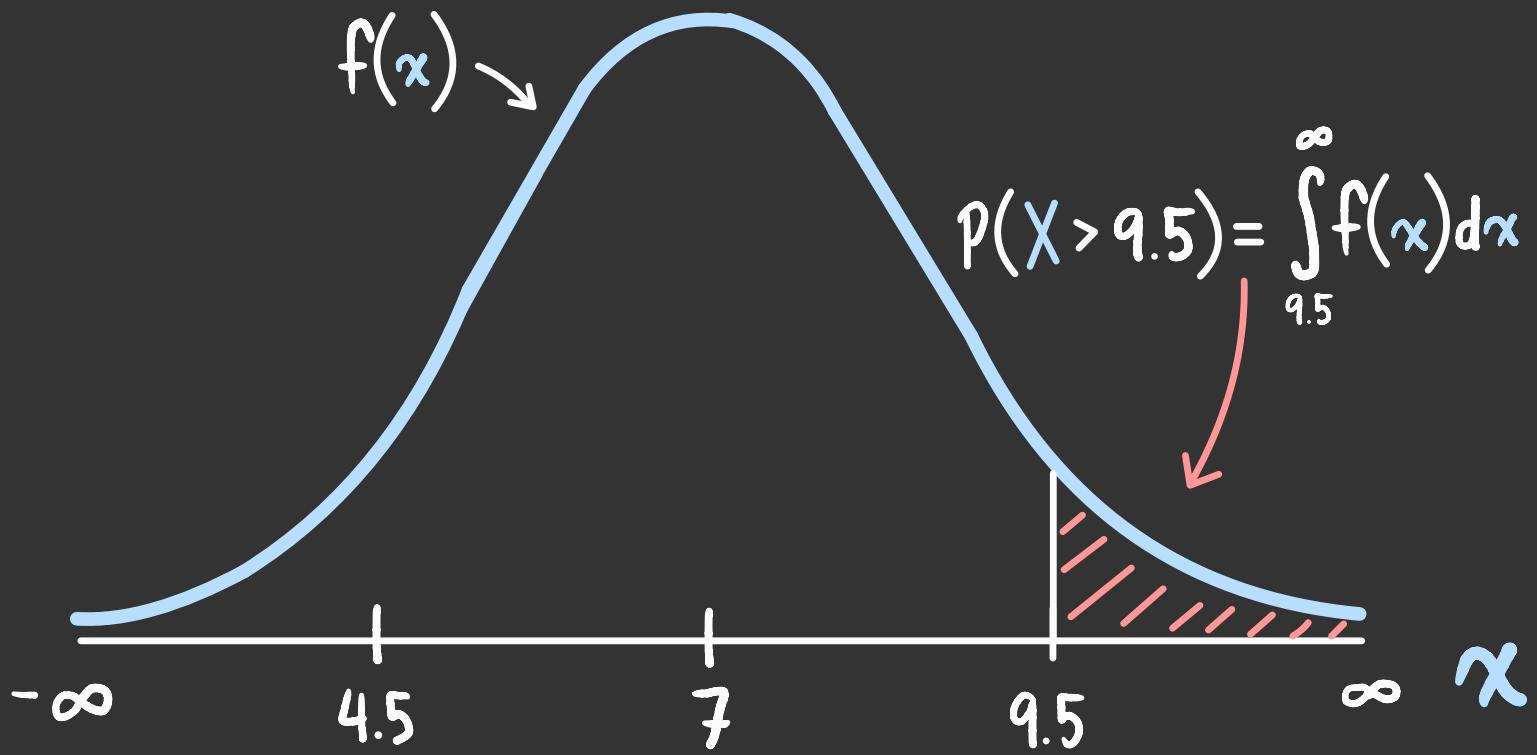
the average temperature in honolulu



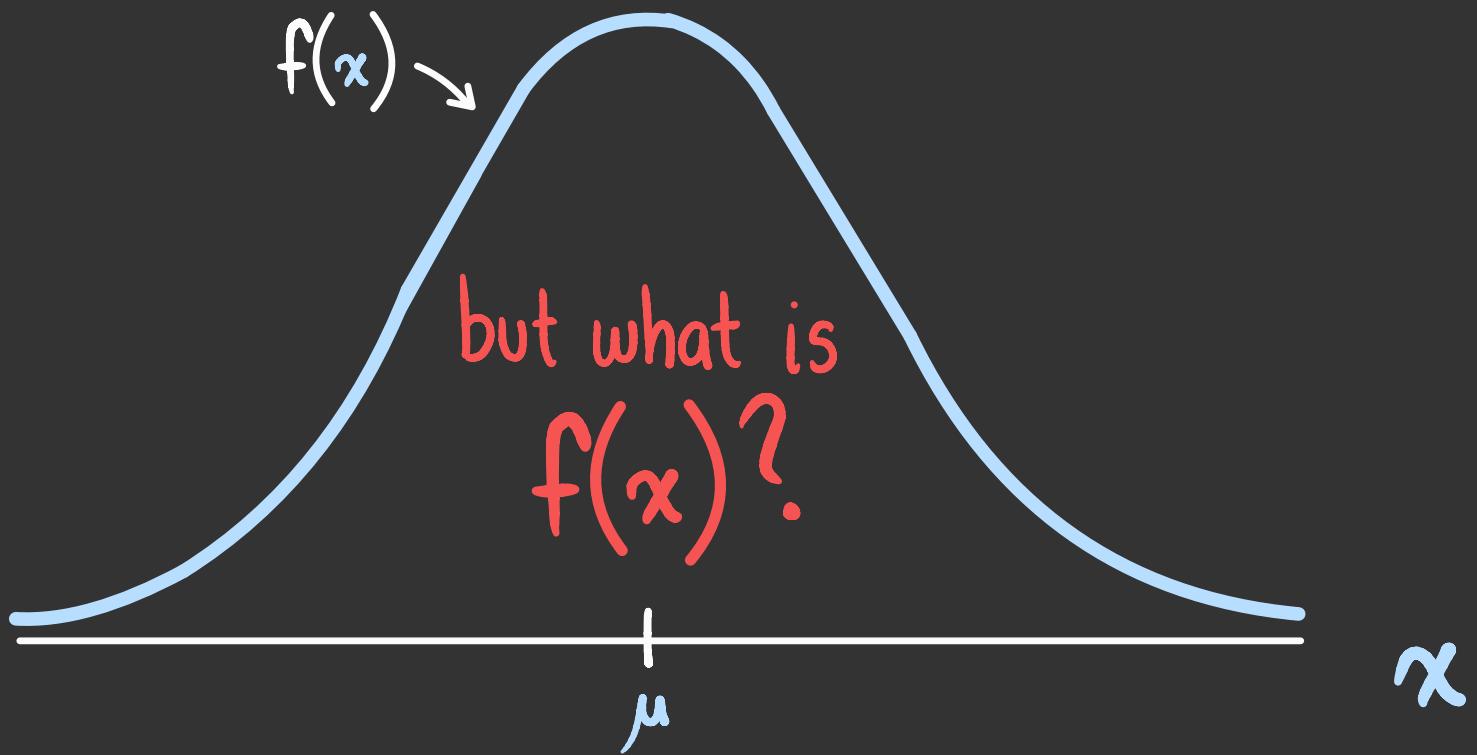
average height of american females

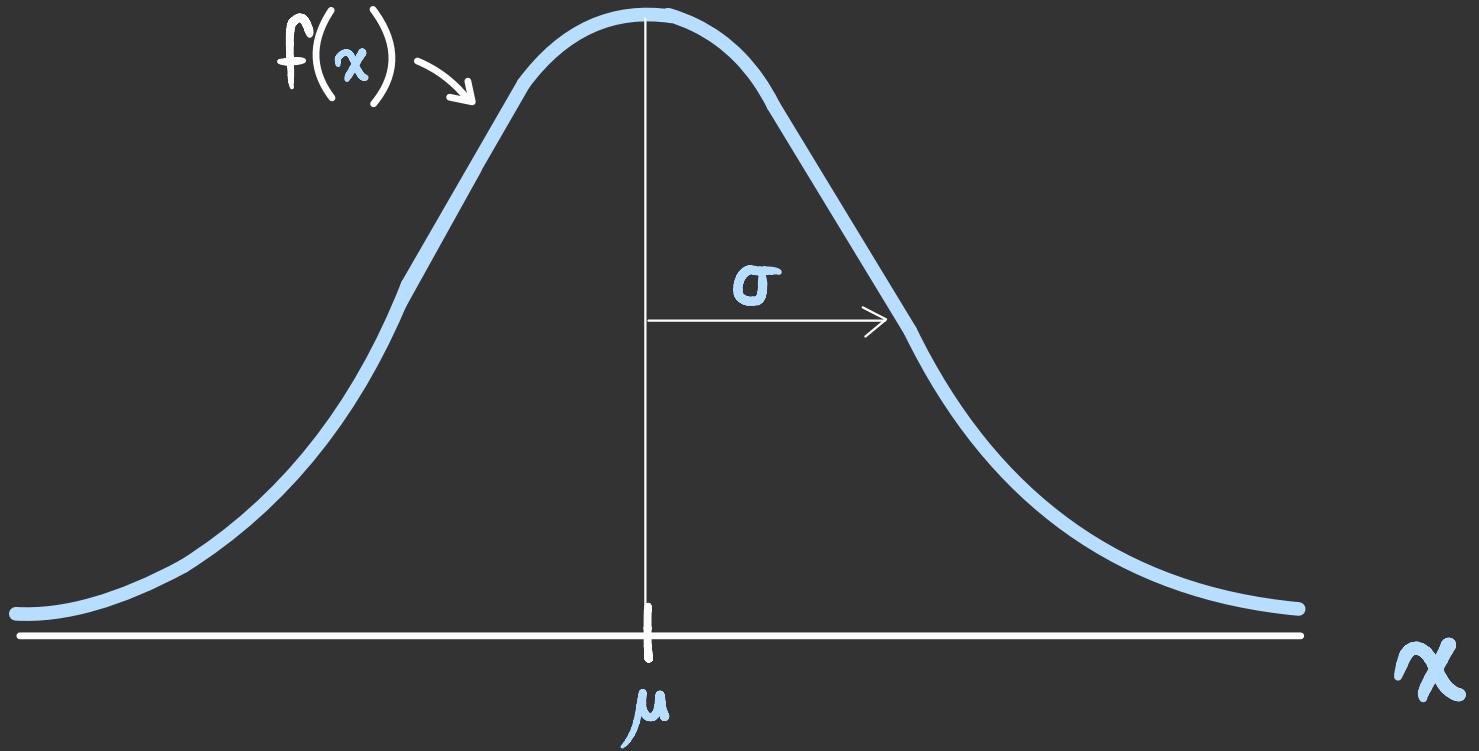


average birth weight in the united states

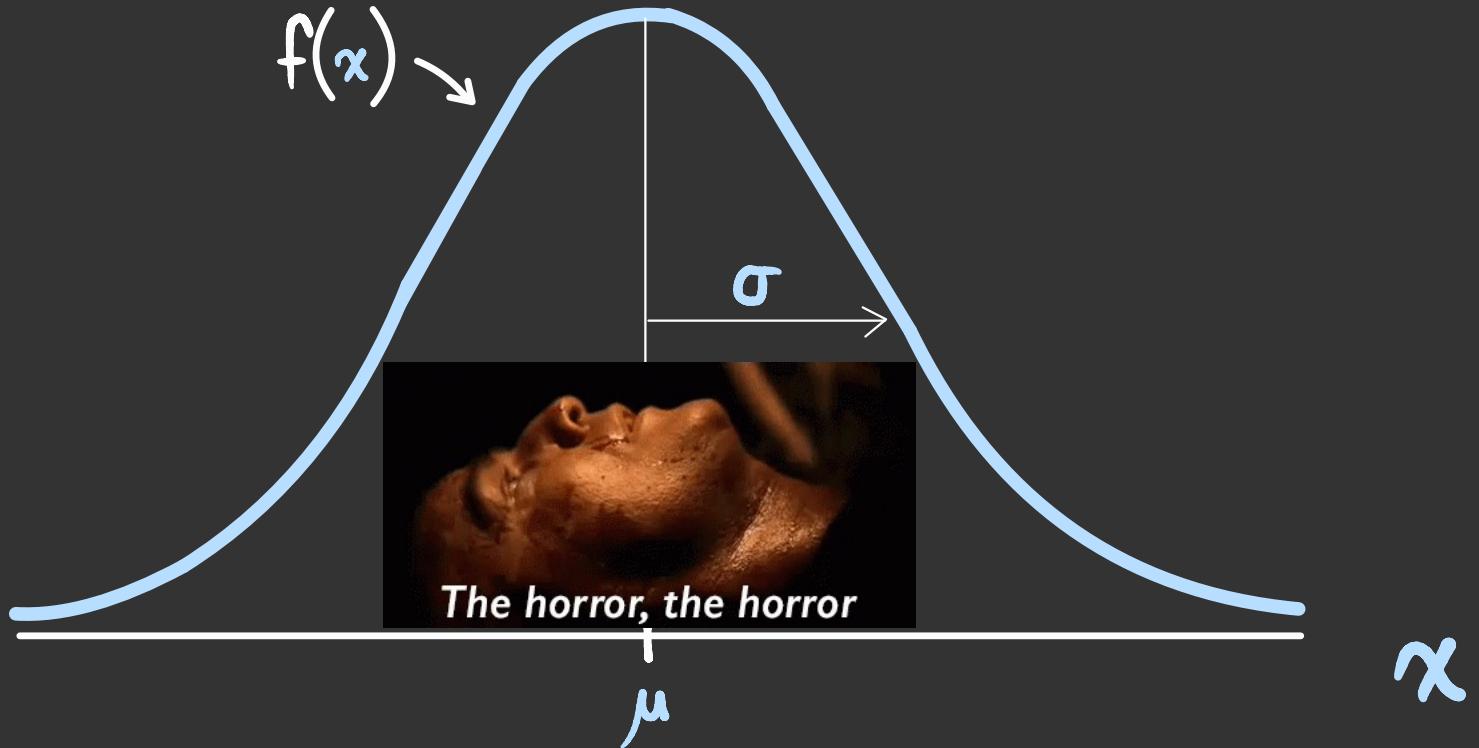


average birth weight in the united states





$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



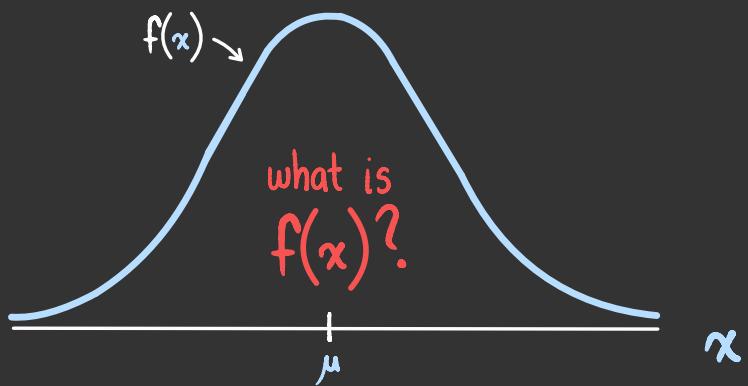
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

if god
did not exist,
it would be necessary
to invent him

- voltaire

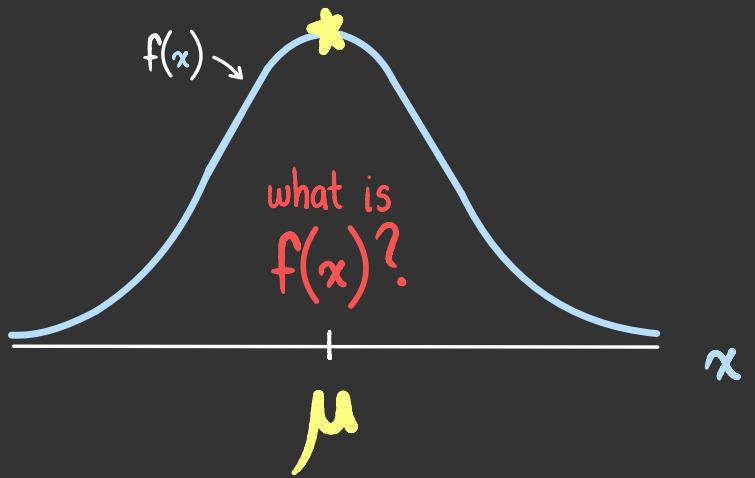
if the normal distribution
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- voltaire



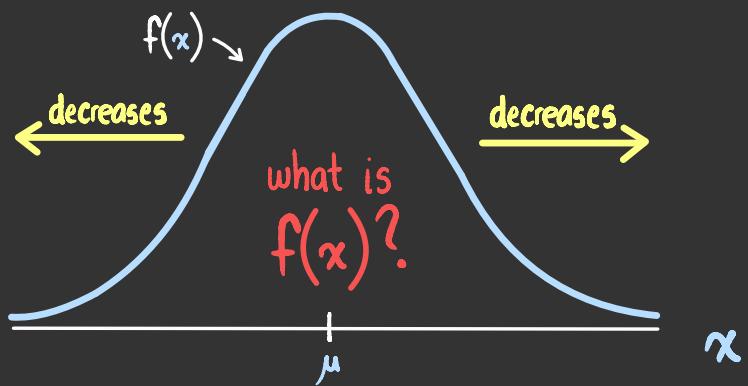
we want a function $f(x)$
with the following properties:

- ▶ f has a unique maximum μ
- ▶ f decreases as we get further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} f(x) dx = 1$



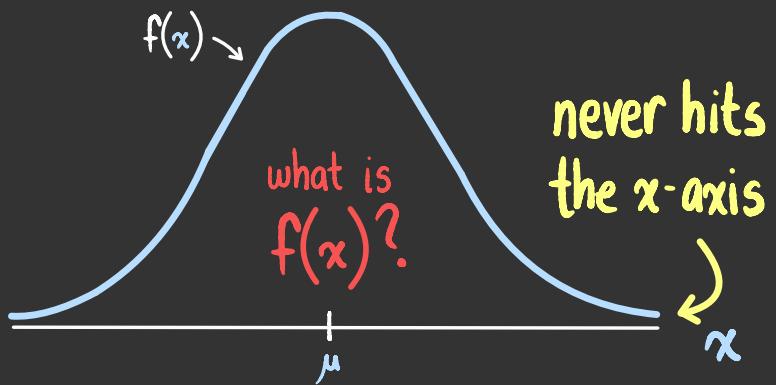
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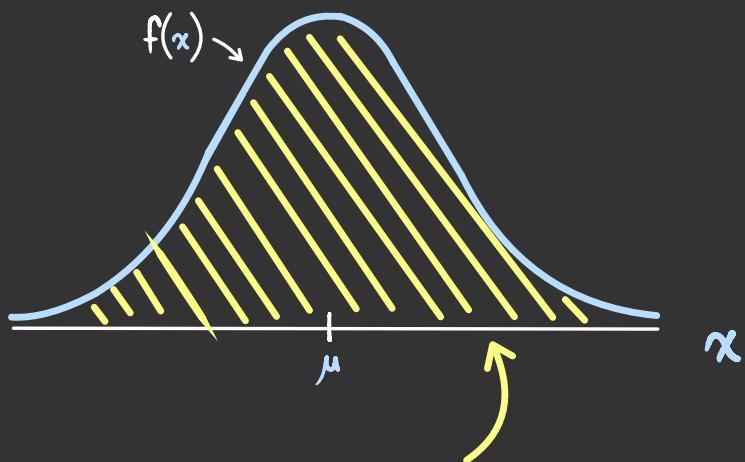
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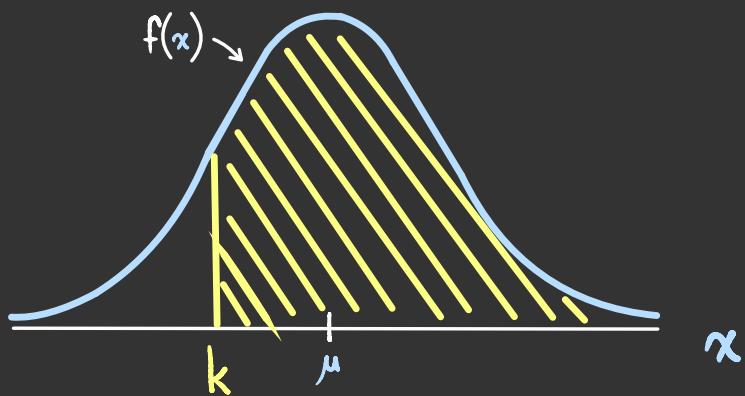


area under the
curve equals one

why is this
important?

we want a function $f(x)$
with the following properties:

- ▶ f has a unique maximum μ
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further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$
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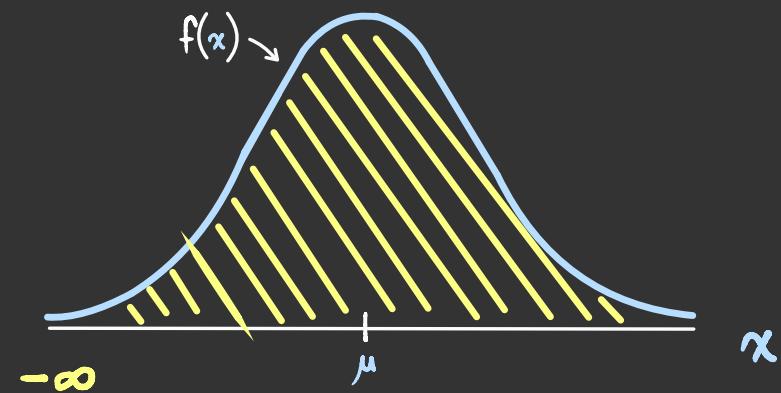


$$P(X > k) = \int_k^{\infty} f(x) dx$$

a probability must
be at most one

we want a function $f(x)$
with the following properties:

- ▶ f has a unique maximum μ
- ▶ f decreases as we get further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} f(x) dx = 1$

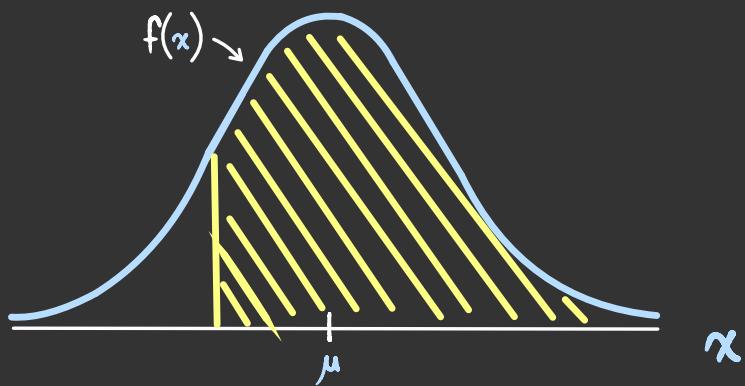


$$P(X > -\infty) = \int_{-\infty}^{\infty} f(x) dx = 1$$

the probability over all values of x must be exactly one

we want a function $f(x)$ with the following properties:

- ▶ f has a unique maximum μ
- ▶ f decreases as we get further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} f(x) dx = 1$



let's consider this
property first

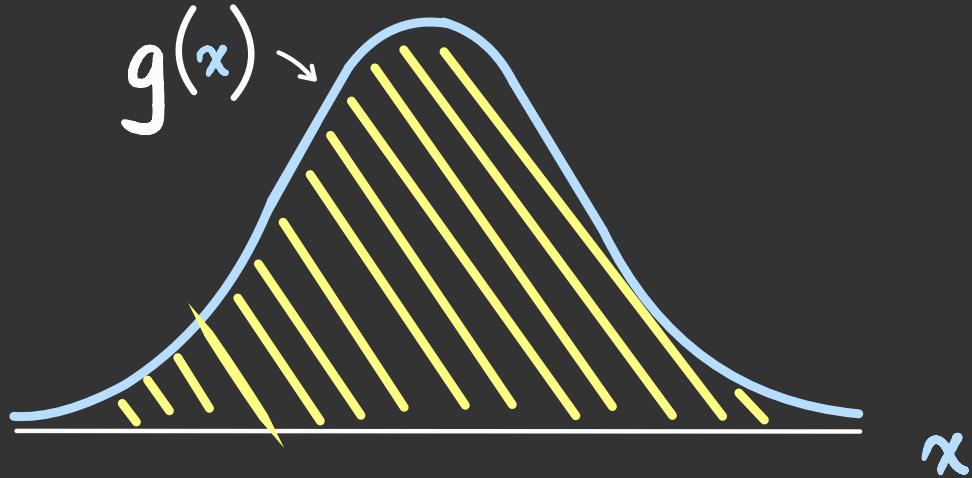
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- ▶ f has a unique maximum μ
- ▶ f decreases as we get further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$

▶ $\int_{-\infty}^{\infty} f(x) dx = 1$

how do we create
a function f such that:

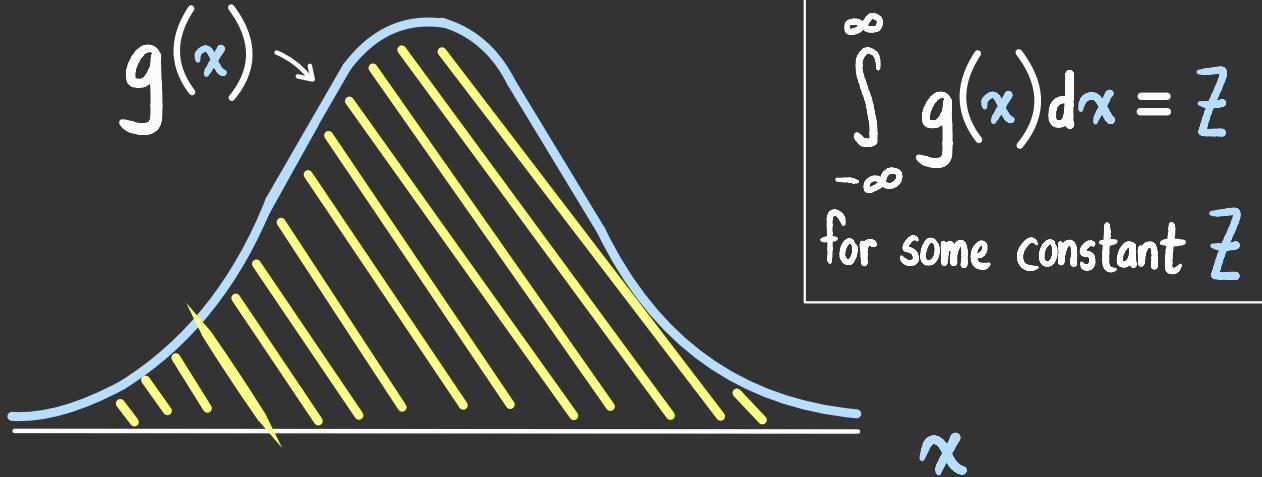
$$\int_{-\infty}^{\infty} f(x) dx = | \quad ?$$



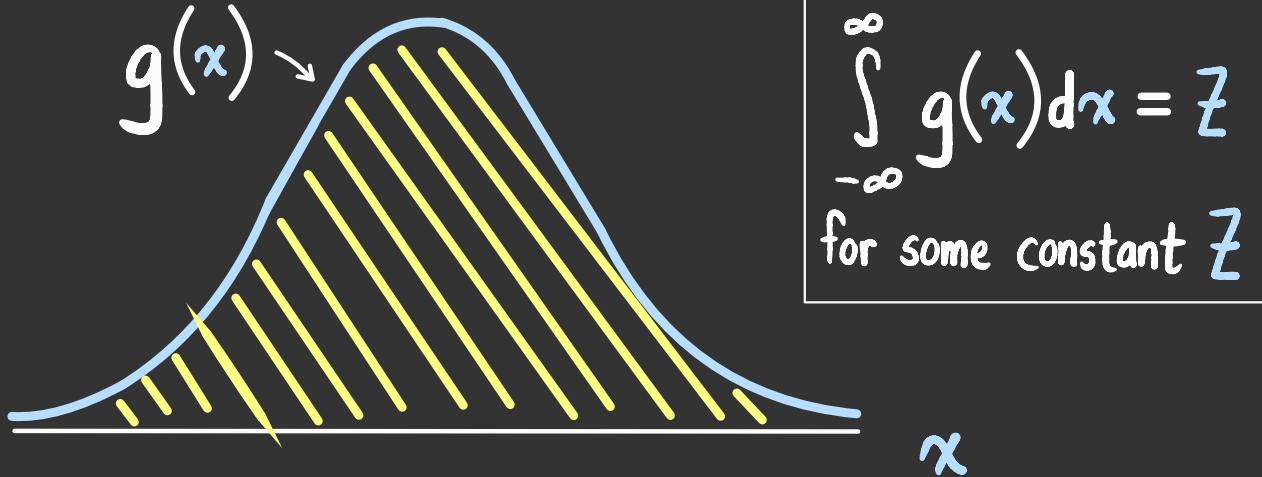
Suppose we have a positive function $g(x)$ such that

$$\int_{-\infty}^{\infty} g(x) dx = Z$$

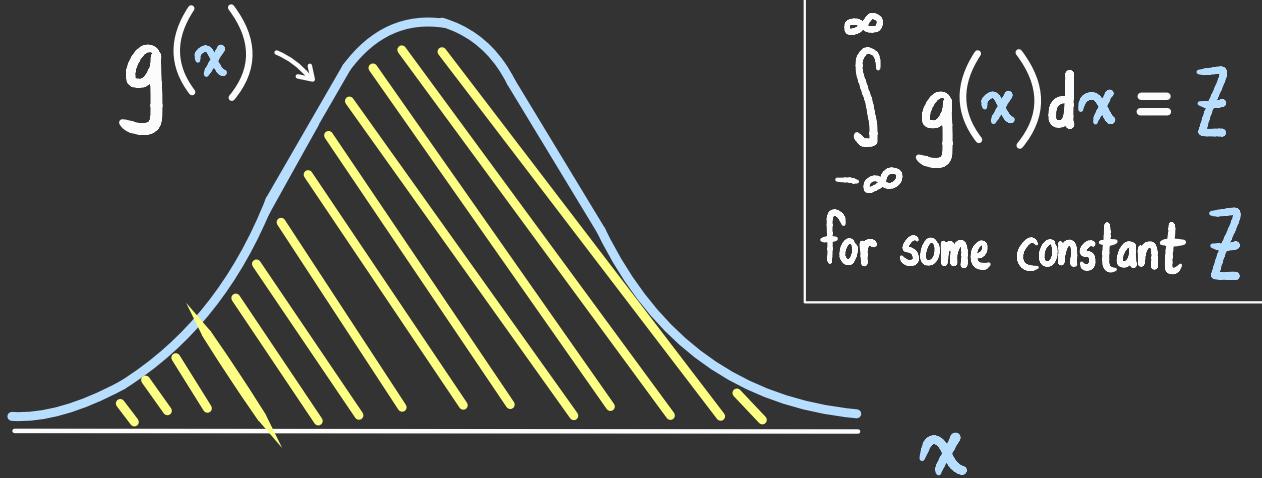
for some constant Z



if $\int_{-\infty}^{\infty} g(x) dx = Z$, then $\frac{1}{Z} \int_{-\infty}^{\infty} g(x) dx = 1$

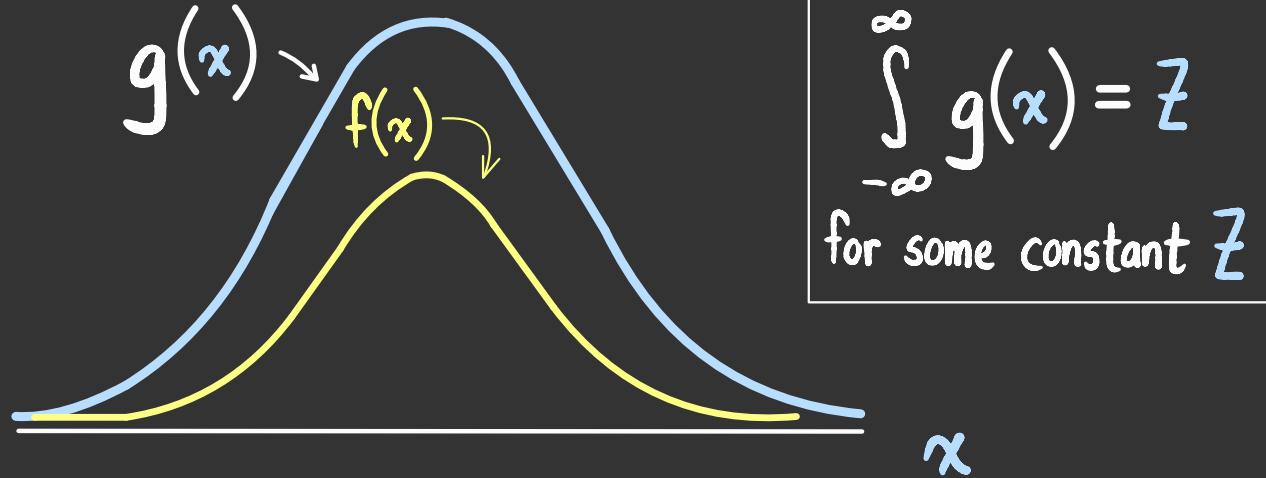


if $\int_{-\infty}^{\infty} g(x) dx = Z$, then $\int_{-\infty}^{\infty} \frac{g(x)}{Z} dx = 1$



if $\int_{-\infty}^{\infty} g(x) dx = Z$, then $\int_{-\infty}^{\infty} \frac{g(x)}{Z} dx = 1$

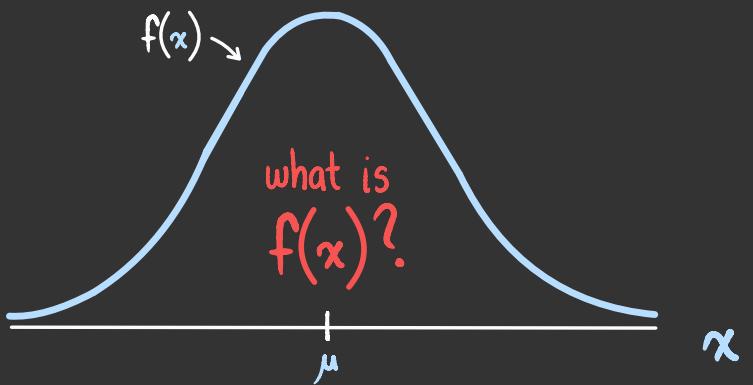
$f(x)$



let $f(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x) dx}$. then $\int_{-\infty}^{\infty} f(x) dx = 1$



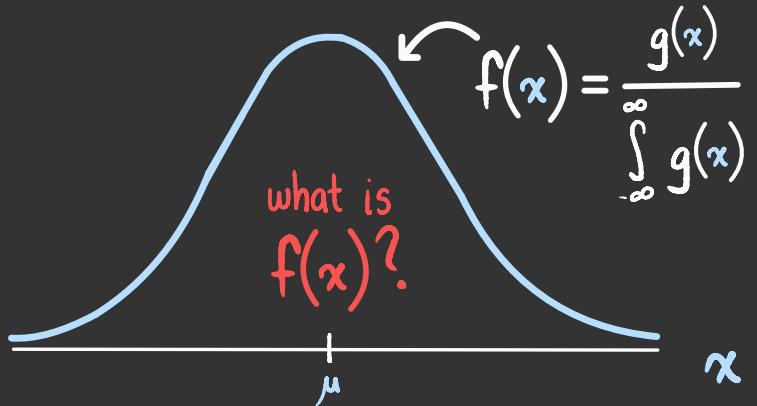
let $f(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x) dx}$. then $\int_{-\infty}^{\infty} f(x) dx = 1$



so let's revisit
our desired
properties

we want a function $f(x)$
with the following properties:

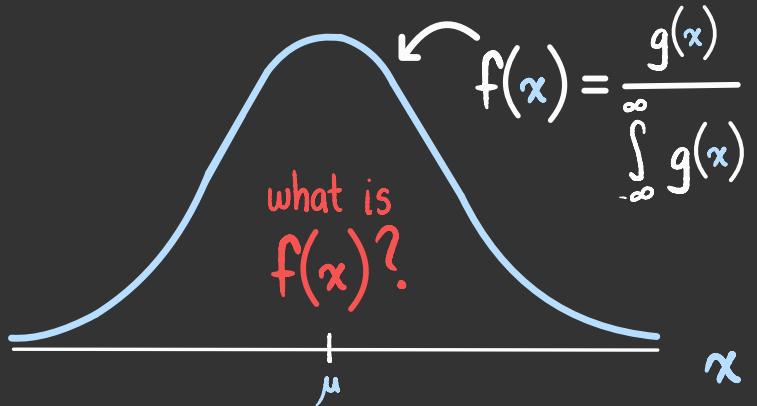
- ▶ f has a unique maximum μ
- ▶ f decreases as we get further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$
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so let's revisit
our desired
properties

we want a function ~~$f(x)$~~ $g(x)$
with the following properties:

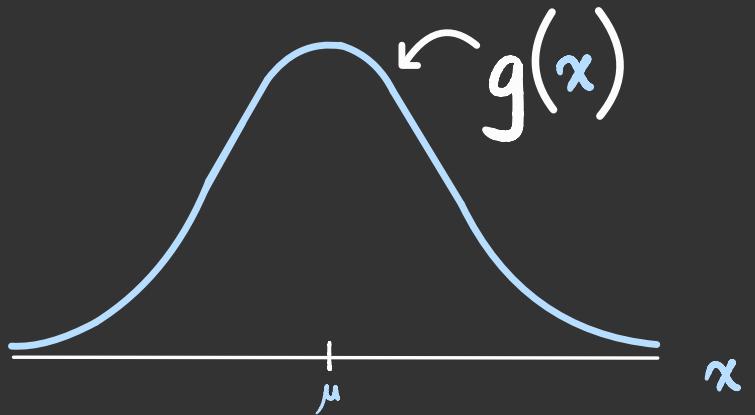
- ▶ g has a unique maximum μ
- ▶ g decreases as we get further from μ
- ▶ $g(x) > 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} g(x) dx = Z$ for some constant Z



so let's revisit
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we want a function $g(x)$
with the following properties:

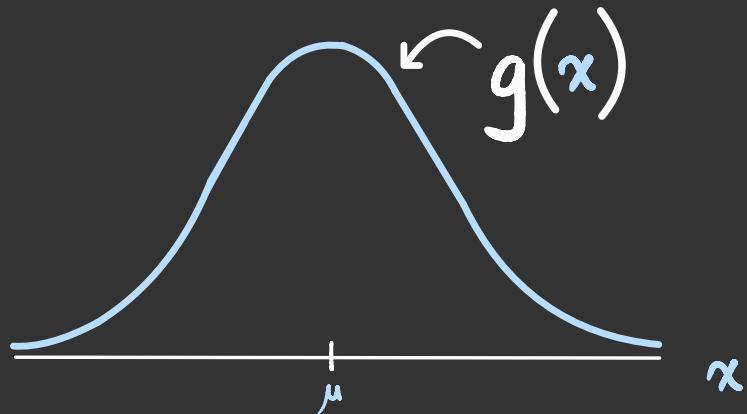
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- ▶ $\int_{-\infty}^{\infty} g(x) dx = Z$ for some constant Z



now let's consider
the first three
properties

we want a function $g(x)$
with the following properties:

- ▶ g has a unique maximum μ
- ▶ g decreases as we get further from μ
- ▶ $g(x) > 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} g(x) dx = Z$ for some constant Z

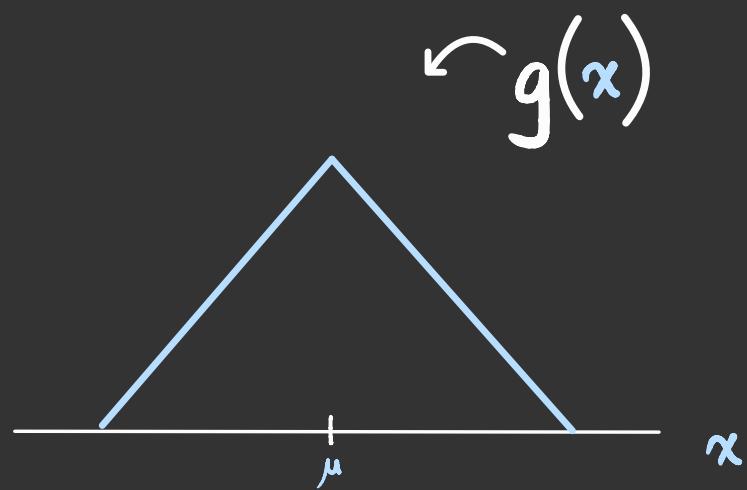
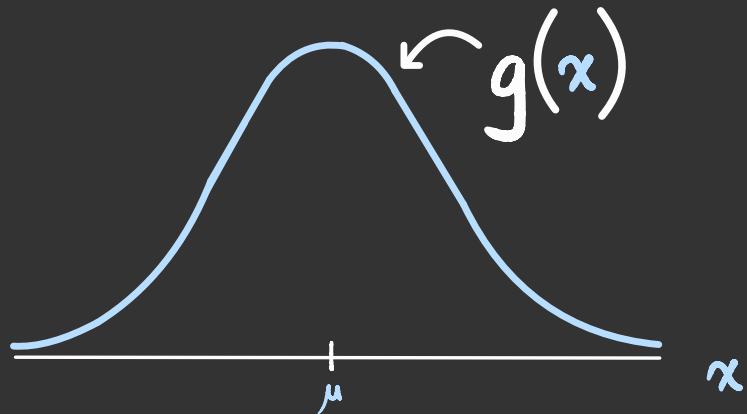


what other curves might
satisfy these properties?



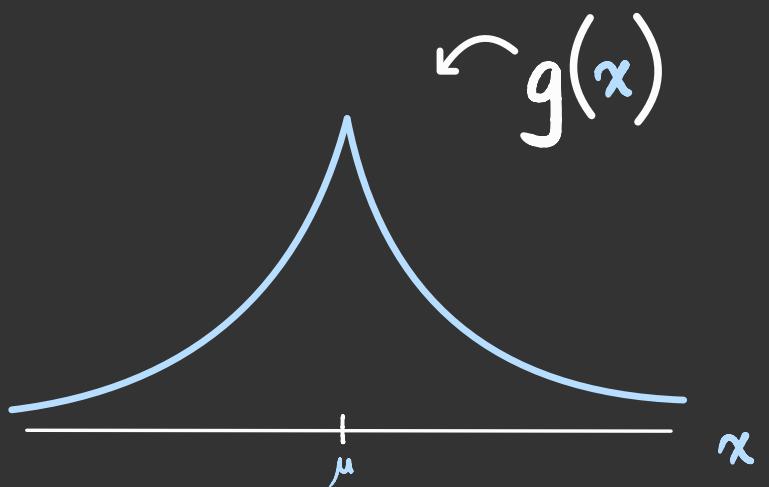
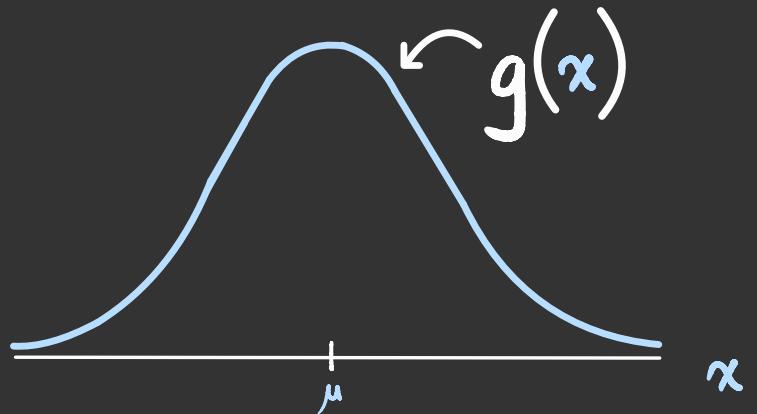
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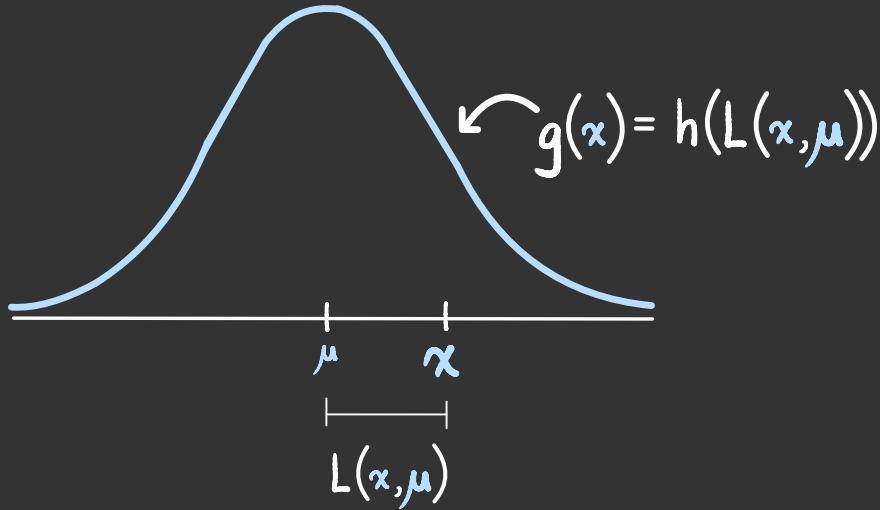
we want a function $g(x)$ with the following properties:

- ▶ g has a unique maximum μ
- ▶ g decreases as we get further from μ
- ▶ $g(x) \geq 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} g(x) dx = Z$ for some constant Z



we want a function $g(x)$
with the following properties:

- ▶ g has a unique maximum μ
- ▶ g decreases as we get further from μ
- ▶ $g(x) > 0$ for all $x \in \mathbb{R}$
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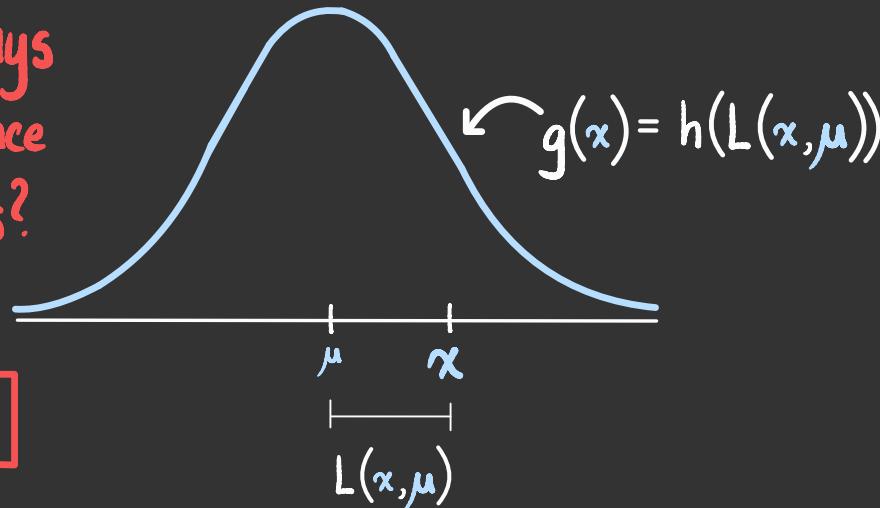
we want a function $g(x)$
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- ▶ g has a unique maximum μ
- ▶ g decreases as we get
further from μ

One way to satisfy these conditions:
define $g(x)$ as a function h of
the distance $L(x, \mu)$ between x and μ

what are some ways
to measure the distance
between two numbers?

$$L(x, \mu) = \boxed{?}$$



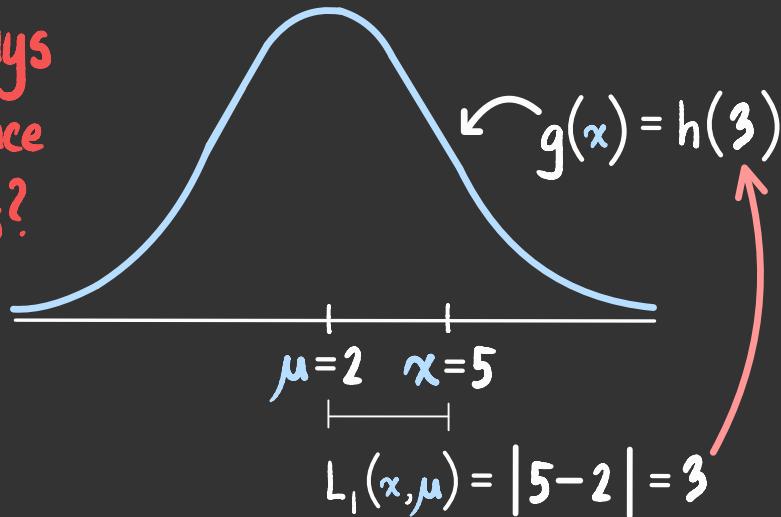
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$$L_1(x, \mu) = |x - \mu|$$



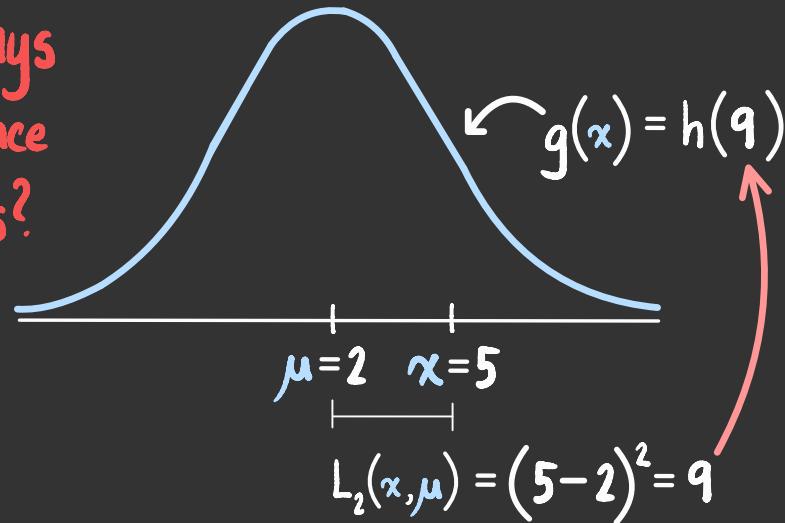
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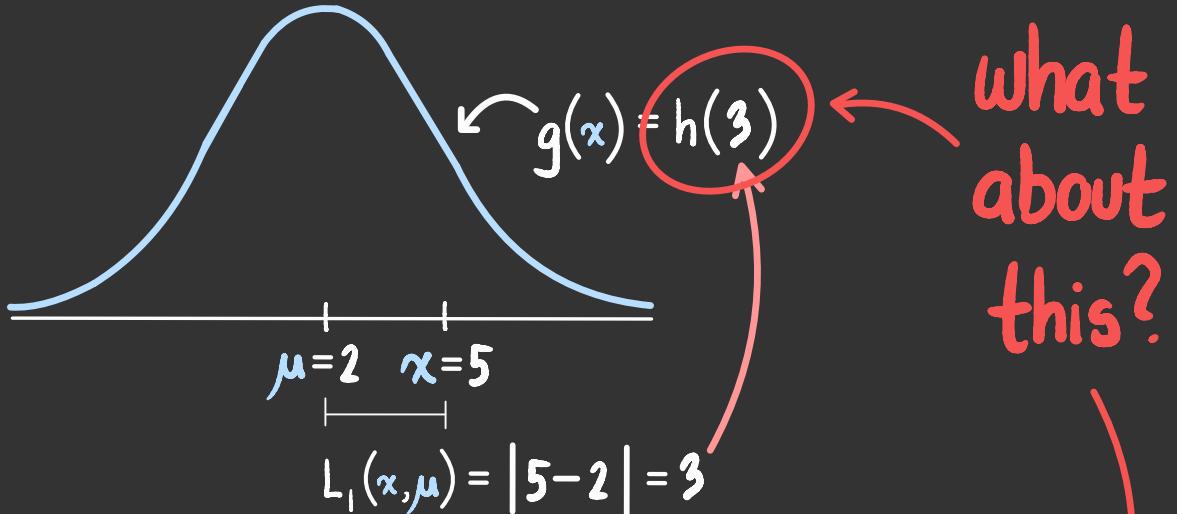
$$L_2(x, \mu) = (x - \mu)^2$$



we want a function $g(x)$
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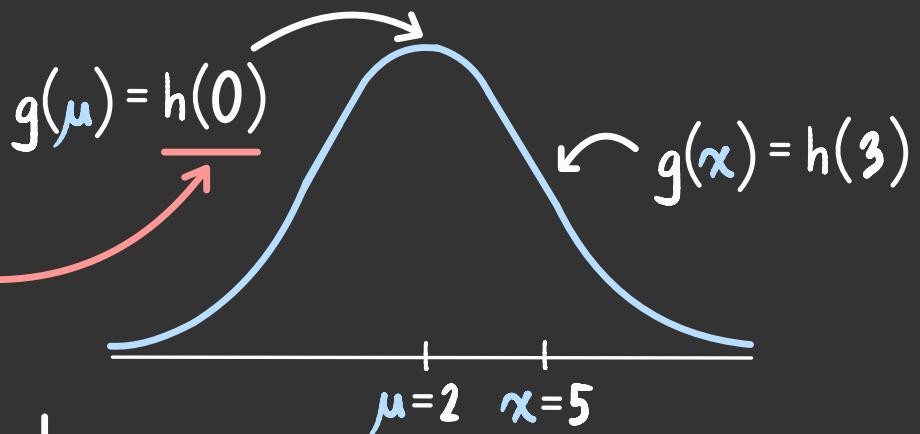


we want a function $g(x)$ with the following properties:

- ▶ g has a unique maximum μ
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One way to satisfy these conditions:
define $g(x)$ as a function h of the distance $L(x, \mu)$ between x and μ

want $h(0)$ to
be finite



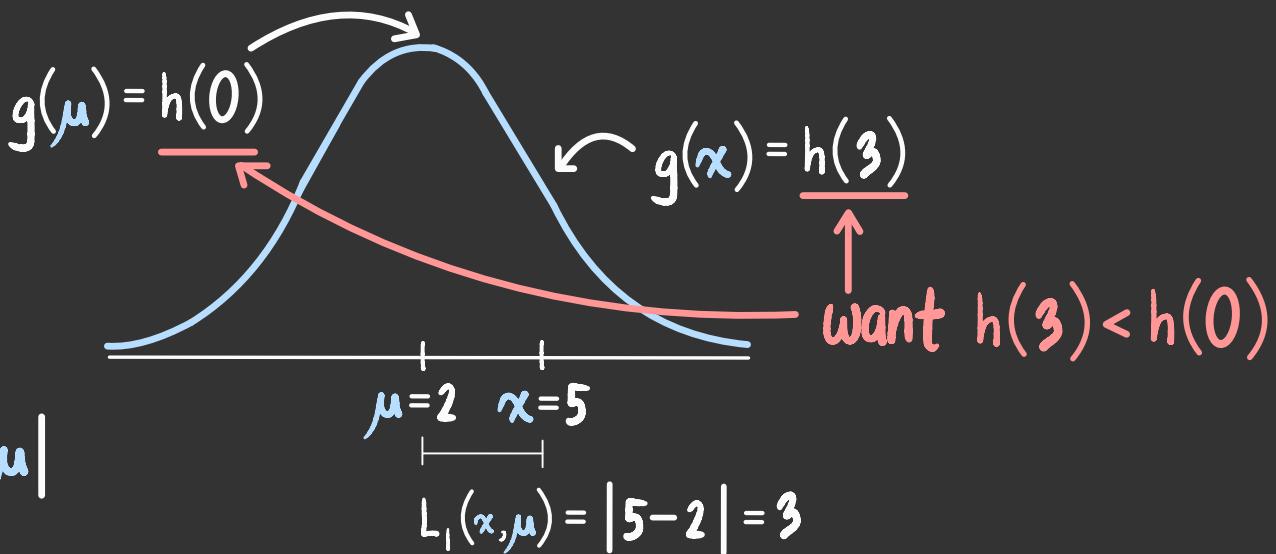
$$L_1(x, \mu) = |x - \mu|$$

$$\begin{array}{c} \mu=2 \quad x=5 \\ \hline L_1(x, \mu) = |5 - 2| = 3 \end{array}$$

we want a function $g(x)$
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- g has a unique maximum μ
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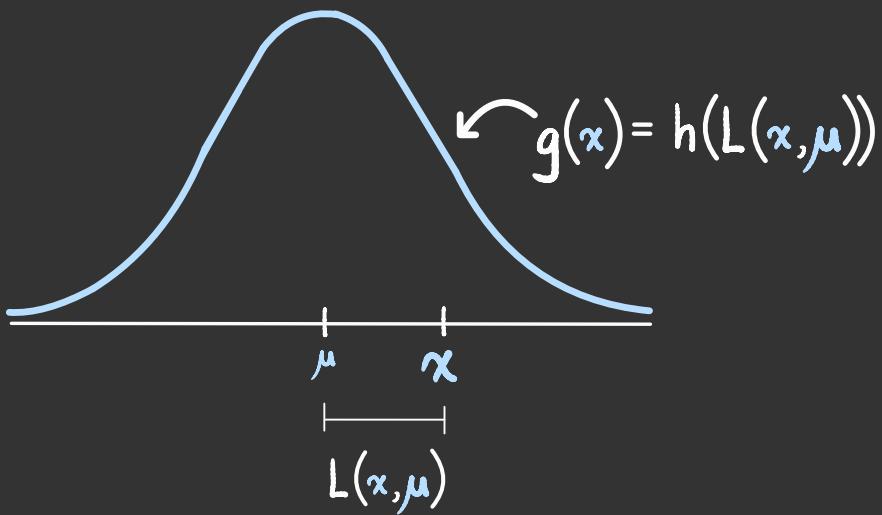
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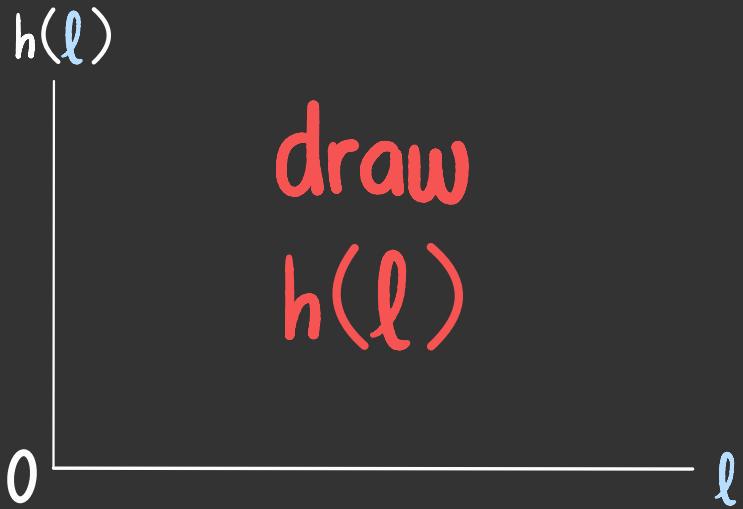
so we want to define h as a positive function such that:

- its domain is $(0, \infty)$
- $h(0)$ is finite
- as ℓ increases,
 $h(\ell)$ decreases



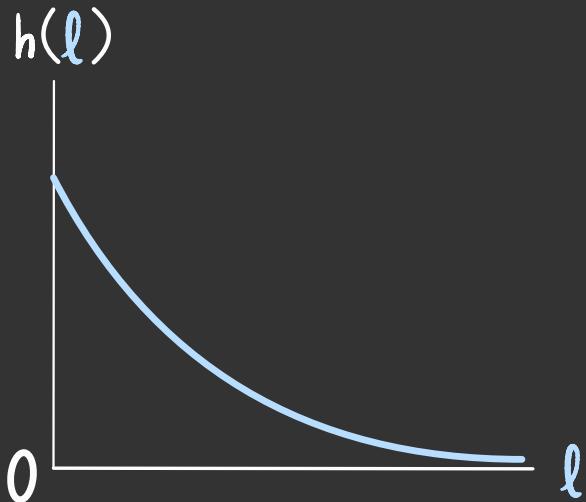
so we want to define h as a positive function such that:

- its domain is $(0, \infty)$
- $h(0)$ is finite
- as l increases,
 $h(l)$ decreases



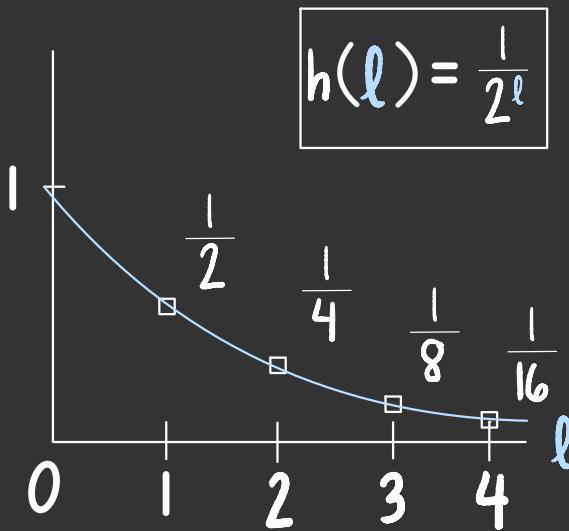
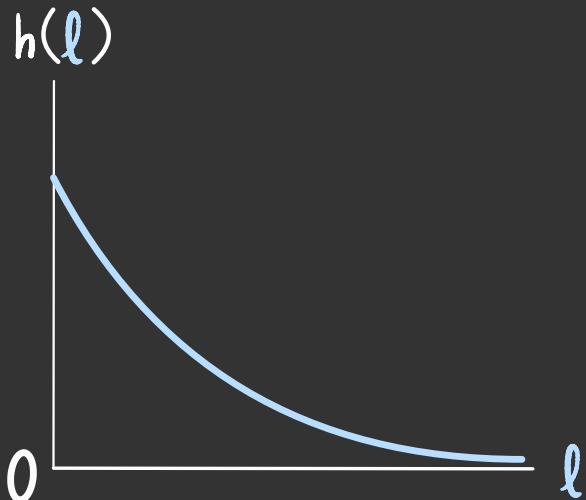
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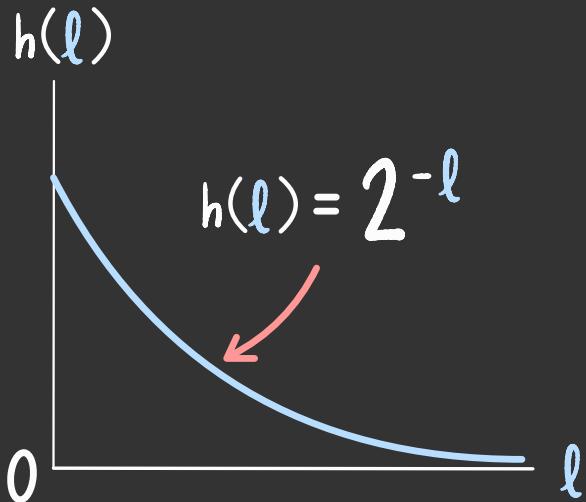
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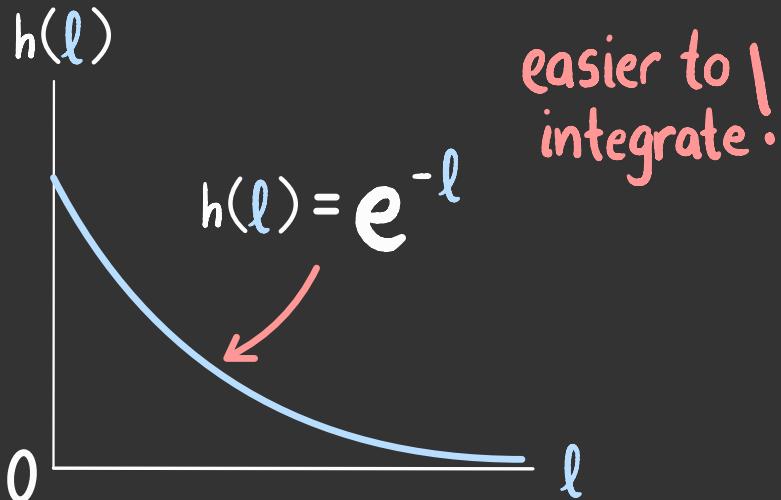
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so we want to define h as a positive function such that:

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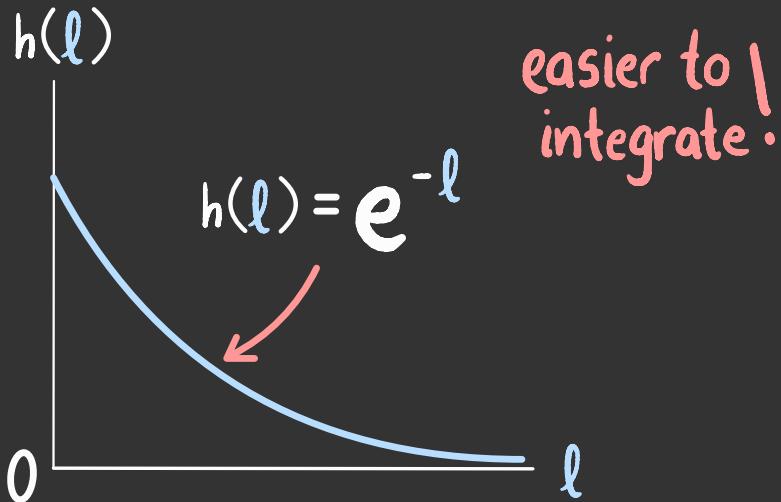


$$\int e^{-l} dl = ?$$

easier to integrate.

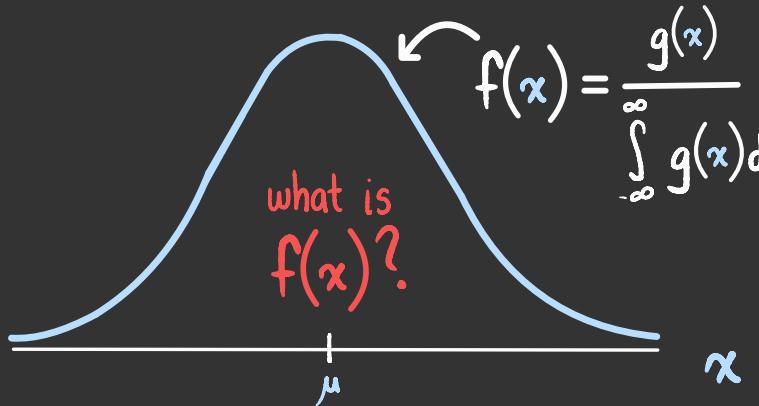
so we want to define h as a positive function such that:

- its domain is $(0, \infty)$
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- as l increases, $h(l)$ decreases



$$\int e^{-l} dl = -e^{-l}$$

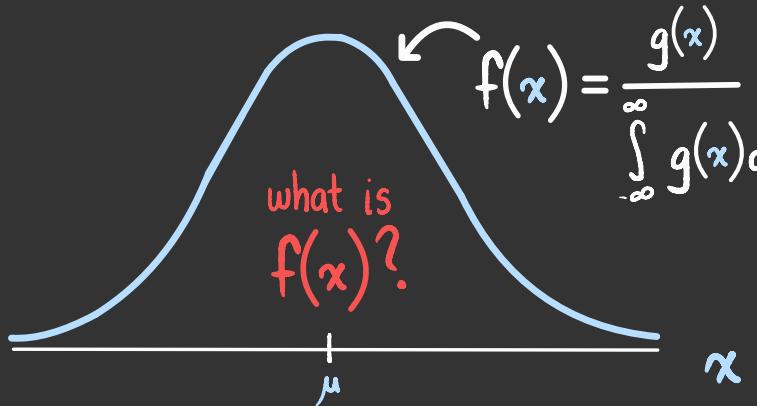
easier to integrate.



what have we learned?

we want a function $g(x)$ with the following properties:

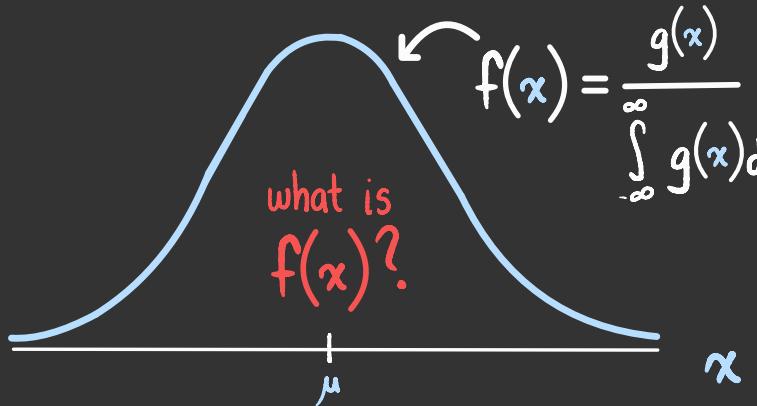
- ▶ g has a unique maximum μ
- ▶ g decreases as we get further from μ
- ▶ $g(x) > 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} g(x) dx = Z$ for some constant Z



what have we learned?

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what have we learned?

- $f(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x) dx}$
- we want a function $g(x)$ with the following properties:
- ▶ $g(x) = h(L(x, \mu))$ where
 - ▶ $L(x, \mu)$ is the distance between x and μ
 - ▶ $h(l) = e^{-l}$
 - ▶ $\int_{-\infty}^{\infty} g(x) dx = Z$ for some constant Z

the only decision we haven't fully settled on is the distance function $L(x, \mu)$

we want a function $g(x)$ with the following properties:

- ▶ $g(x) = h(L(x, \mu))$ where
- ▶ $L(x, \mu)$ is the distance between x and μ
- ▶ $h(l) = e^{-l}$
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distance functions

option one

$$L_1(x, \mu) = |x - \mu|$$



$$L_1(x, \mu) = |5 - 2| = 3$$

option two

$$L_2(x, \mu) = (x - \mu)^2$$



$$L_2(x, \mu) = (5 - 2)^2 = 9$$

so far we've seen two options

distance functions

option one

$$L_1(x, \mu) = |x - \mu|$$



$$L_1(x, \mu) = |5 - 2| = 3$$

option two

$$L_2(x, \mu) = (x - \mu)^2$$



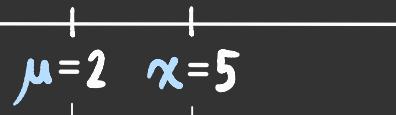
$$L_2(x, \mu) = (5 - 2)^2 = 9$$

neither option addresses the matter of scale

distance functions

option one

$$L_1(x, \mu) = |x - \mu|$$



$$L_1(x, \mu) = |5 - 2| = 3$$

{ 3 km?
3 cm?
3 km?

neither option addresses the matter of scale

distance functions

option one

$$L_1(x, \mu) = \frac{|x - \mu|}{b}$$

$$\begin{array}{c} \mu=2 \quad x=5 \\ \hline \end{array}$$

$$L_1(x, \mu) = \frac{|5 - 2|}{.001} = 3000$$

3 km?

3 cm?

3 km?

to specify scale, we will introduce a scaling factor

distance functions

option one

$$L_1(x, \mu) = \frac{|x - \mu|}{b}$$

$$\begin{array}{c} \mu=2 \quad x=5 \\ \hline \end{array}$$

$$L_1(x, \mu) = \frac{|5 - 2|}{100} = .03$$

3 km?

3 cm?

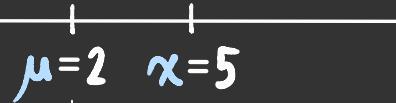
3 mm?

to specify scale, we will introduce a scaling factor

distance functions

option one

$$L_1(x, \mu) = \frac{|x - \mu|}{b}$$



$$L_1(x, \mu) = \frac{|5 - 2|}{1000} = .003$$

3 km?

3 cm?

3 mm?

to specify scale, we will introduce a scaling factor

distance functions

option one

$$L_1(x, \mu) = \frac{|x - \mu|}{b}$$

option two

$$L_2(x, \mu) = \frac{(x - \mu)^2}{b}$$

scaling factor

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distance functions

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let's try option one

if we use distance $L_1(x, \mu) = \frac{|x - \mu|}{b}$,

what do we get for $f(x)$?

first, compute $g(x)$:

$$\begin{aligned} g(x) &= h(L(x, \mu)) \\ &= h\left(\frac{|x - \mu|}{b}\right) \\ &= e^{-\frac{|x - \mu|}{b}} \end{aligned}$$

- $g(x) = h(L(x, \mu))$ where
 - $L(x, \mu)$ is the distance between x and μ
 - $h(l) = e^{-l}$
- $\int_{-\infty}^{\infty} g(x) dx = Z$ for some constant Z
- $f(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x) dx}$

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to compute $f(x)$, we need $\int_{-\infty}^{\infty} g(x) dx$

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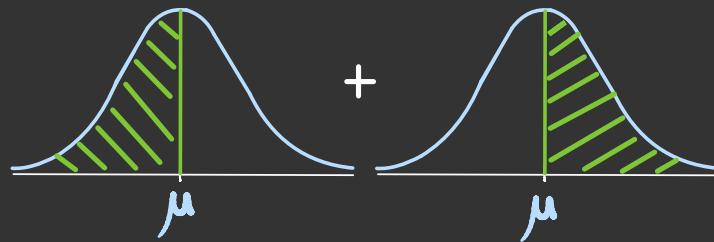
next, compute $\int_{-\infty}^{\infty} g(x) dx$:

$$\begin{aligned} \int_{-\infty}^{\infty} g(x) dx &= \int_{-\infty}^{\infty} e^{-\frac{|x - \mu|}{b}} dx \\ &= ? \end{aligned}$$

$$\int_{-\infty}^{\infty} g(x) dx = \int_{-\infty}^{\infty} e^{-\frac{|x-\mu|}{b}} dx$$

$$\int_{-\infty}^{\infty} g(x) dx = \int_{-\infty}^{\infty} e^{-\frac{|x-\mu|}{b}} dx$$

$$= \int_{-\infty}^{\mu} e^{-\frac{|x-\mu|}{b}} dx + \int_{\mu}^{\infty} e^{-\frac{|x-\mu|}{b}} dx$$



$x < \mu$

$x > \mu$

$$\text{so } |x - \mu| = ?$$

$$\text{so } |x - \mu| = ?$$

$$\int_{-\infty}^{\infty} g(x) dx = \int_{-\infty}^{\infty} e^{-\frac{|x-\mu|}{b}} dx$$

$$= \int_{-\infty}^{\mu} e^{-\frac{\mu-x}{b}} dx + \int_{\mu}^{\infty} e^{-\frac{x-\mu}{b}} dx$$



$$x < \mu$$

$$\text{so } |x - \mu| = \mu - x$$

$$x > \mu$$

$$\text{so } |x - \mu| = x - \mu$$

$$\begin{aligned}\int_{-\infty}^{\infty} g(x) dx &= \int_{-\infty}^{\infty} e^{-\frac{|x-\mu|}{b}} dx \\&= \int_{-\infty}^{\mu} e^{\frac{x-\mu}{b}} dx + \int_{\mu}^{\infty} e^{\frac{\mu-x}{b}} dx\end{aligned}$$

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 \int_{-\infty}^{\infty} g(x) dx &= \int_{-\infty}^{\infty} e^{-\frac{|x-\mu|}{b}} dx \\
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&= \lim_{k \rightarrow -\infty} \left[be^{\frac{x-\mu}{b}} \right]_k^{\mu} + \lim_{k \rightarrow \infty} \left[-be^{\frac{\mu-x}{b}} \right]_{\mu}^k \\
&= \lim_{k \rightarrow -\infty} \left(be^{\frac{\mu-\mu}{b}} - be^{\frac{k-\mu}{b}} \right) + \lim_{k \rightarrow \infty} \left(-be^{\frac{\mu-k}{b}} + be^{\frac{\mu-\mu}{b}} \right)
\end{aligned}$$

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&= \lim_{k \rightarrow -\infty} \left[be^{\frac{x-\mu}{b}} \right]_k^{\mu} + \lim_{k \rightarrow \infty} \left[-be^{\frac{\mu-x}{b}} \right]_{\mu}^k \\
&= \lim_{k \rightarrow -\infty} \left(be^0 - be^{\frac{k-\mu}{b}} \right) + \lim_{k \rightarrow \infty} \left(-be^{\frac{\mu-k}{b}} + be^0 \right)
\end{aligned}$$

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&= \lim_{k \rightarrow -\infty} \left(b - b e^{\frac{k-\mu}{b}} \right) + \lim_{k \rightarrow \infty} \left(-b e^{\frac{\mu-k}{b}} + b \right)
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&= b - 0 + b - 0
\end{aligned}$$

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&= 2b
\end{aligned}$$

if we use distance $L_1(x, \mu) = \frac{|x - \mu|}{b}$,

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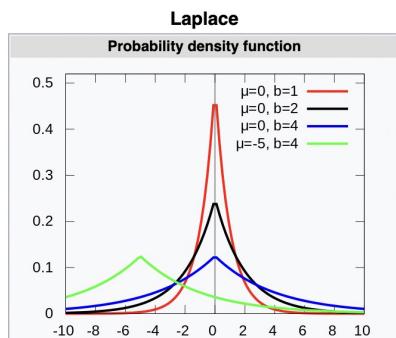
$$f(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x) dx} = \frac{e^{-\frac{|x - \mu|}{b}}}{2b}$$

this is called the
laplace distribution

Laplace distribution

From Wikipedia, the free encyclopedia

In probability theory and statistics, the **Laplace distribution** is a continuous probability distribution named after Pierre-Simon Laplace. It is also sometimes called the **double exponential distribution**, because it can be thought of as two **exponential distributions** (with an additional location parameter) spliced together along the **abscissa**, although the term is also sometimes used to refer to the **Gumbel distribution**. The difference between



next, compute $\int_{-\infty}^{\infty} g(x) dx :$

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finally, compute $f(x) :$

$$f(x) = \frac{e^{-\frac{|x-\mu|}{b}}}{2b}$$

distance functions

option one

$$L_1(x, \mu) = \frac{|x - \mu|}{b}$$

option two

$$L_2(x, \mu) = \frac{(x - \mu)^2}{b}$$



if we use distance $L_2(x, \mu) = \frac{(x-\mu)^2}{b}$,

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next, compute $\int_{-\infty}^{\infty} g(x) dx$:

$$\int_{-\infty}^{\infty} g(x) dx = \sqrt{b\pi}$$

we'll just
look this one
up online

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finally, compute $f(x)$:

$$f(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x) dx} = \frac{e^{-\frac{(x-\mu)^2}{b}}}{\sqrt{b\pi}}$$

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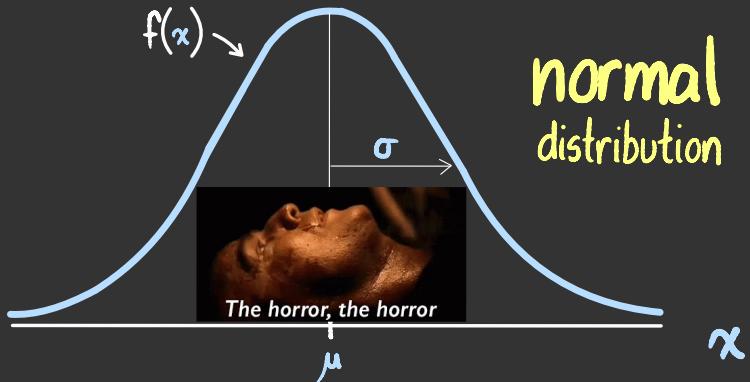
what distribution
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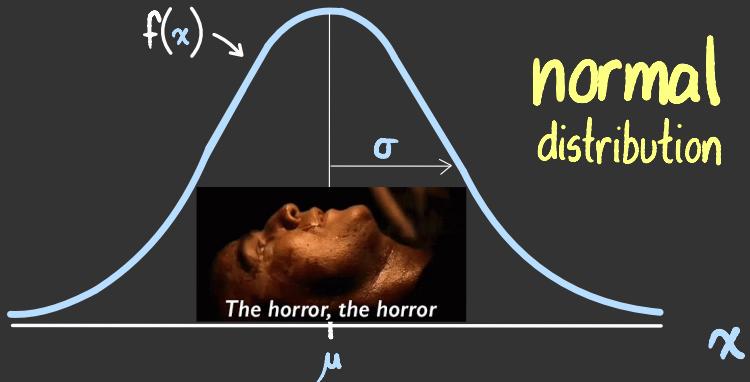
$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{b}}}{\sqrt{b\pi}}$$



what distribution
is this?

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{b}}}{\sqrt{b\pi}}$$



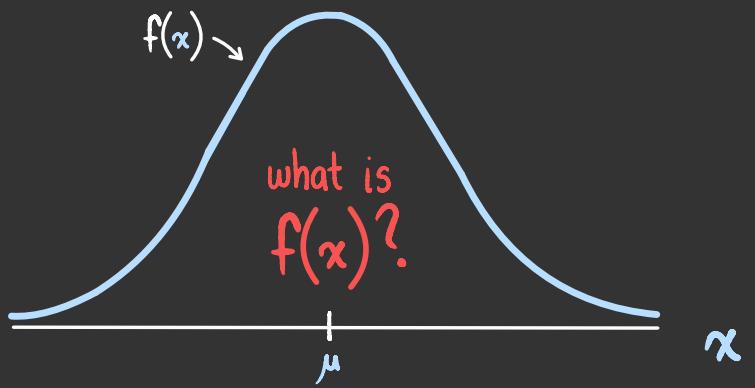
normal
distribution

it's the
normal
distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

rewrite b
as $2\sigma^2$

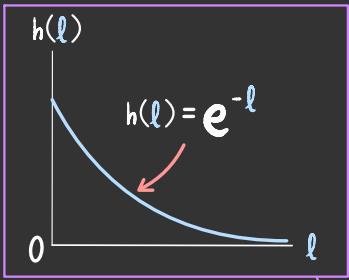
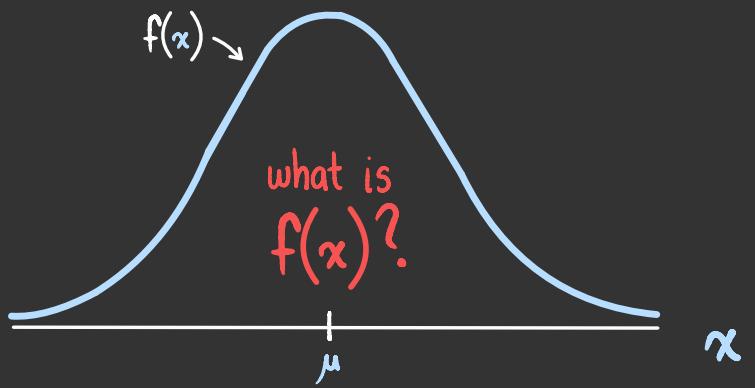
$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{b}}}{\sqrt{b\pi}}$$



$$f(x) =$$

we want a function $f(x)$ with the following properties:

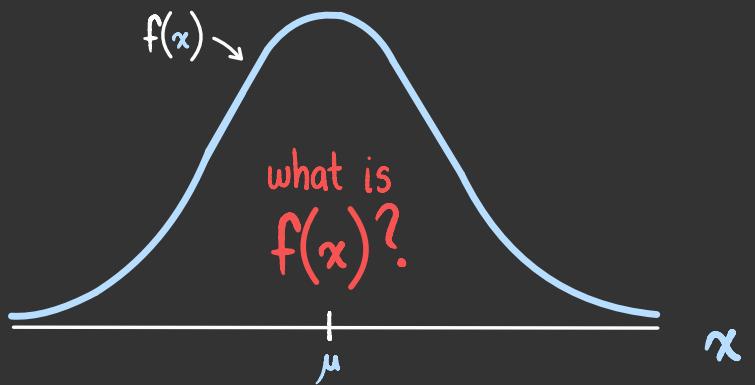
- ▶ f has a unique maximum μ
- ▶ f decreases as we get further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} f(x) dx = 1$



$$f(x) = e^{-L(x, \mu)}$$

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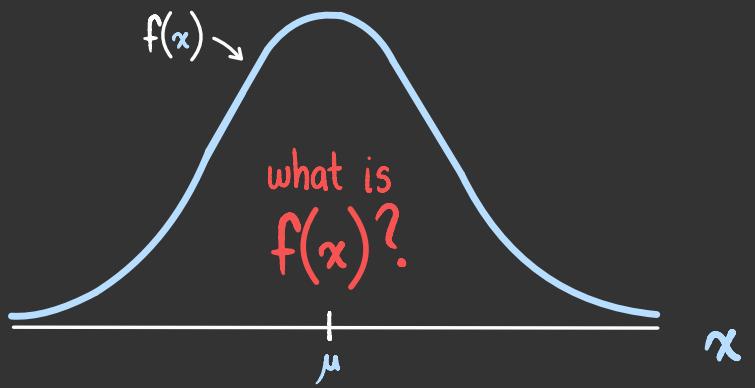


$$\boxed{\begin{array}{c} \mu=2 \quad x=5 \\ | \qquad | \\ L_2(x,\mu) = (5-2)^2 = 9 \end{array}}$$

$$f(x) = e^{-\frac{(x-\mu)^2}{b}}$$

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- ▶ f has a unique maximum μ
- ▶ f decreases as we get further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$
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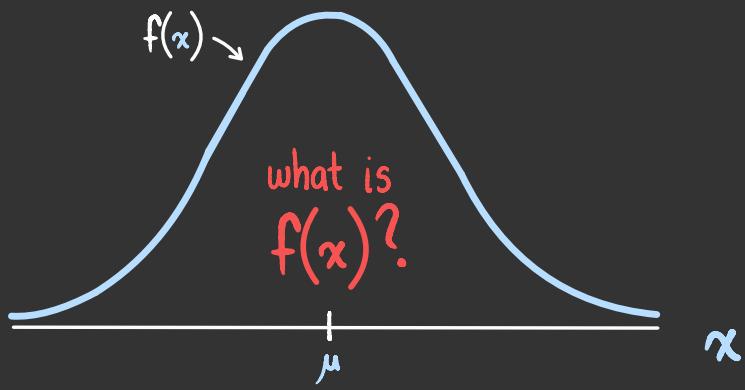
$$f(x) = \frac{1}{\sqrt{b\pi}} e^{-\frac{(x-\mu)^2}{b}}$$

Diagram illustrating the components of the function:

- $\frac{g(x)}{\int_{-\infty}^{\infty} g(x) dx}$: The fraction where the numerator is $g(x)$ and the denominator is the integral from $-\infty$ to ∞ of $g(x)$.
- $\sqrt{b\pi}$: The denominator of the entire function.
- $e^{-\frac{(x-\mu)^2}{b}}$: The exponential term.

we want a function $f(x)$ with the following properties:

- ▶ f has a unique maximum μ
- ▶ f decreases as we get further from μ
- ▶ $f(x) > 0$ for all $x \in \mathbb{R}$
- ▶ $\int_{-\infty}^{\infty} f(x) dx = 1$



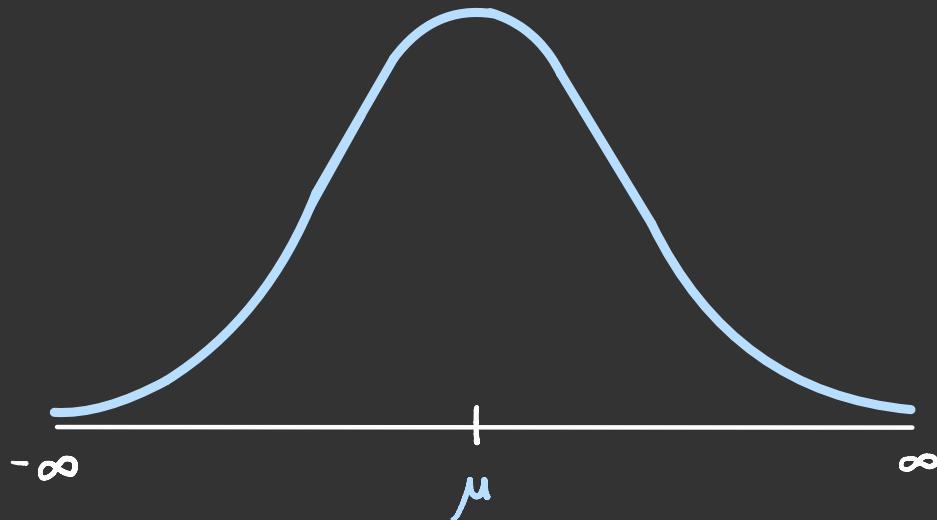
rewrite b
as $2\sigma^2$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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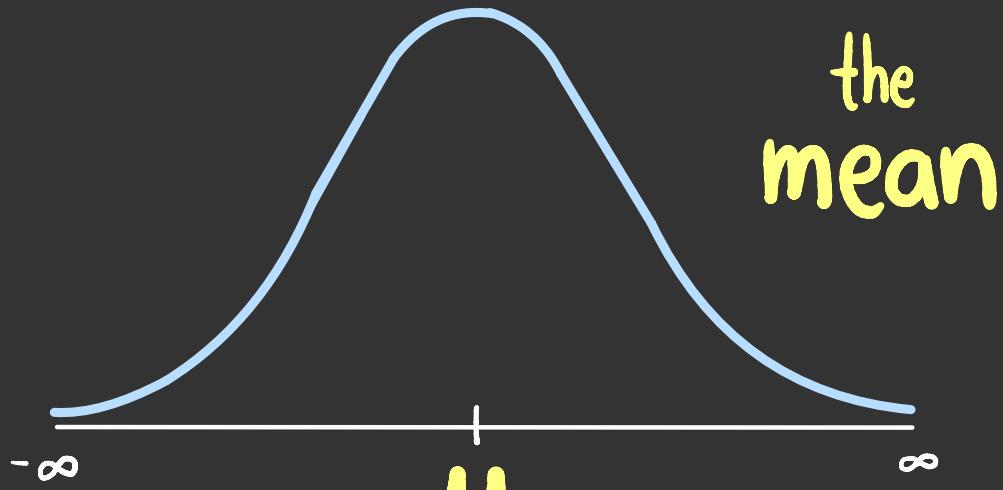
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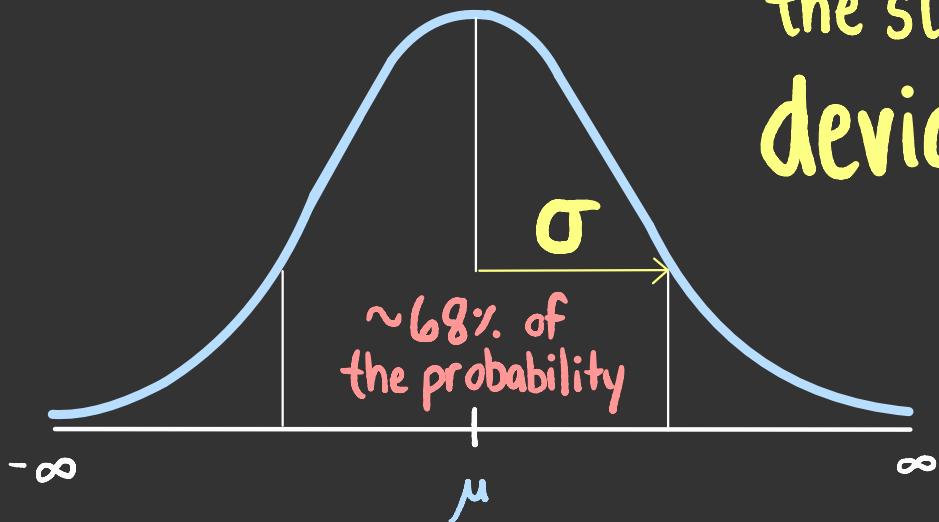
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the normal distribution has two parameters:

the standard deviation



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$