inference of conditional probability

21 oct 2022

CSCI 373

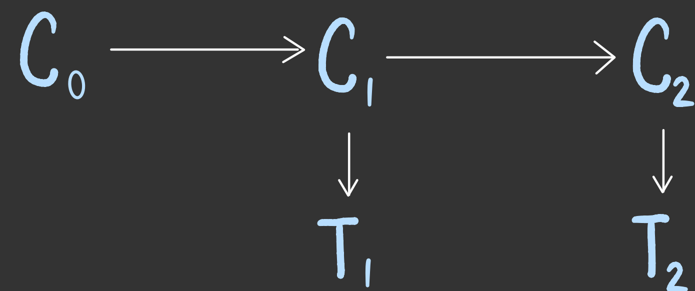
so far we've focused on computing marginal probabilities from a bayesian network

examples?

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9



C_1	T_1	$P(t_1 \mid c_1)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

so far we've focused on computing **marginal probabilities** from a bayesian network

C_0	$P(c_0)$
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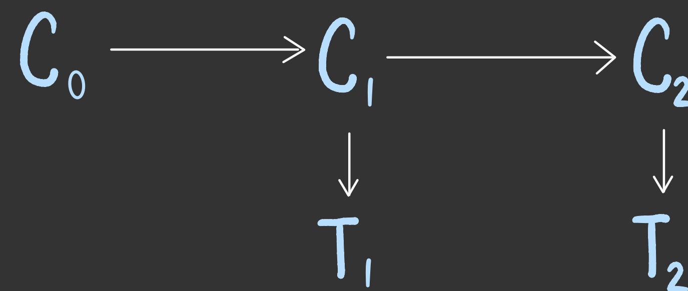

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0	0	0.995
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C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

what is $P(T_2 = 1)$?

what is $P(C_0 = 0, C_2 = 1)$?

what if we want to compute conditional probabilities from a bayesian network?

C_0	$P(c_0)$
0	0.99
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C_0	C_1	$P(c_1 \mid c_0)$
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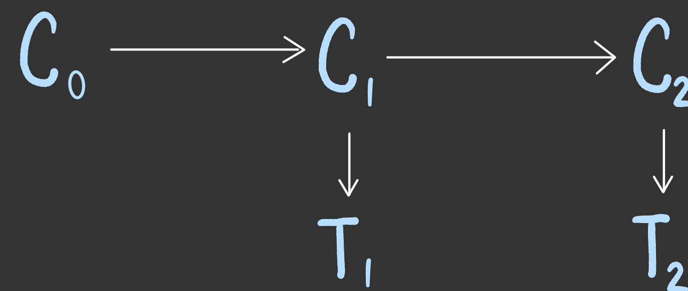

$$P(C_0=0 \mid T_2=1)$$

$$= \boxed{?}$$

how can we compute this using marginal probabilities?

C_1	T_1	$P(t_1 \mid c_1)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

what if we want to compute conditional probabilities from a bayesian network?

$$P(C_0 = 0 \mid T_2 = 1)$$

$$= \frac{P(C_0 = 0, T_2 = 1)}{P(T_2 = 1)}$$

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
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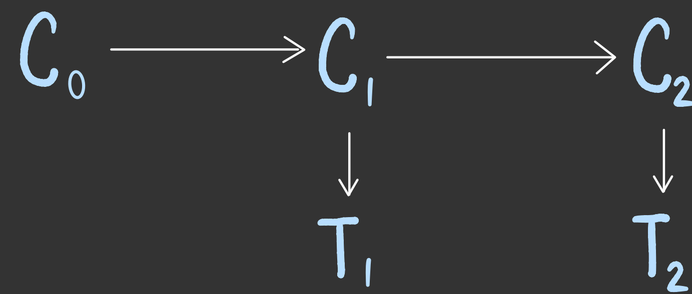

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0	0	0.995
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C_2	T_2	$P(t_2 \mid c_2)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

what if we want to compute **conditional probabilities** from a bayesian network?

$$P(C_0=0 \mid T_2=1)$$

$$= \frac{P(C_0=0, T_2=1)}{P(T_2=1)}$$

but these are 2 $O(nd^{w})$ queries

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
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C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$\begin{tikzcd} C_0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[d] \\ & T_1 & T_2 \end{tikzcd}$$

C_1	T_1	$P(t_1 \mid c_1)$
0	0	0.995
0	1	0.005
1	0	0.2
1	1	0.8

C_2	T_2	$P(t_2 \mid c_2)$
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0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

C_0	C_1	C_2	$P(c_2 \mid c_1)\, P(c_1 \mid c_0)\, P(c_0)$
0	0	0	$0.99 \cdot 0.99 \cdot 0.99$
0	0	1	$0.01 \cdot 0.99 \cdot 0.99$
0	1	0	$0.1 \cdot 0.01 \cdot 0.99$
0	1	1	$0.9 \cdot 0.01 \cdot 0.99$
1	0	0	$0.99 \cdot 0.1 \cdot 0.01$
1	0	1	$0.01 \cdot 0.1 \cdot 0.01$
1	1	0	$0.1 \cdot 0.9 \cdot 0.01$
1	1	1	$0.9 \cdot 0.9 \cdot 0.01$

expand joint

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

C_0	C_1	C_2	$P(c_0, c_1, c_2)$
0	0	0	0.970299
0	0	1	0.009801
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0.00099
1	0	1	0.00001
1	1	0	0.0009
1	1	1	0.0081

expand joint

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
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0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

expand joint

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0	0	0	0.970299
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0	1	1	0.00891
1	0	0	0.00099
1	0	1	0.00001
1	1	0	0.0009
1	1	1	0.0081

observe evidence

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

expand joint + observe evidence

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
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compute $P(c_0, c_2 \mid C_1 = 1)$

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0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

$$P(c_0, c_2 \mid C_1 = 1)$$

$y \mid x$

$$= \; ?$$

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
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$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

expand joint + observe evidence

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

$$P(c_0, c_2 \mid C_1 = 1)$$

$$= \frac{P(c_0, C_1 = 1, c_2)}{P(C_1 = 1)}$$

C_0	$P(c_0)$
0	0.99
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C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
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C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
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$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

expand joint + observe evidence

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
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0	1	0	0.00099
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$$P(c_0, c_2 \mid C_1 = 1)$$

$$= \frac{P(c_0, C_1 = 1, c_2)}{P(C_1 = 1)}$$

$$= \frac{P(c_0, C_1 = 1, c_2)}{\sum_{c_0} \sum_{c_2} P(c_0, C_1 = 1, c_2)}$$

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

expand joint + observe evidence

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

the sum

$$P(c_0, c_2 \mid C_1 = 1)$$

$$= \frac{P(c_0, C_1 = 1, c_2)}{P(C_1 = 1)}$$

$$= \frac{P(c_0, C_1 = 1, c_2)}{\sum_{c_0} \sum_{c_2} P(c_0, C_1 = 1, c_2)}$$

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

$$\boxed{\textbf{goal:} \quad \text{compute } P(c_0, c_2 \mid C_1 = 1)}$$

expand joint + observe evidence

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

divide each **value** by the **sum** of the values $\longrightarrow$

C_0	C_1	C_2	$P(c_0, c_2 \mid C_1 = 1)$
0	1	0	0.05238
0	1	1	0.47143
1	1	0	0.04762
1	1	1	0.42857

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

expand joint + reduce ~~observe evidence~~

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

normalize

~~divide each value by the sum of the values~~

C_0	C_1	C_2	$P(c_0, c_2 \mid C_1 = 1)$
0	1	0	0.05238
0	1	1	0.47143
1	1	0	0.04762
1	1	1	0.42857

C_0	$P(c_0)$
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C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

expand joint + reduce

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

normalize

C_0	C_1	C_2	$P(c_0, c_2 \mid C_1 = 1)$
0	1	0	0.05238
0	1	1	0.47143
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0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

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0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

goal:

compute $P(c_0, c_2 \mid C_1 = 1)$

!!!

expand joint

+

reduce

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

normalize

C_0	C_1	C_2	$P(c_0, c_2 \mid C_1 = 1)$
0	1	0	0.05238
0	1	1	0.47143
1	1	0	0.04762
1	1	1	0.42857

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

but we can't expand the bayesian network into an explicit joint!

expand joint

C_0	C_1	C_2	$P(c_0, c_1, c_2)$
0	0	0	0.970299
0	0	1	0.009801
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0.00099
1	0	1	0.00001
1	1	0	0.0009
1	1	1	0.0081

reduce

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	0
0	0	1	0
0	1	0	0.00099
0	1	1	0.00891
1	0	0	0
1	0	1	0
1	1	0	0.0009
1	1	1	0.0081

normalize

C_0	C_1	C_2	$P(c_0, c_2 \mid C_1 = 1)$
0	1	0	0.05238
0	1	1	0.47143
1	1	0	0.04762
1	1	1	0.42857

C_0	C_1	C_2	$P(c_0, c_1, c_2)$
0	0	0	$0.99 \cdot 0.99 \cdot 0.99$
0	0	1	$0.01 \cdot 0.99 \cdot 0.99$
0	1	0	$0.1 \cdot 0.01 \cdot 0.99$
0	1	1	$0.9 \cdot 0.01 \cdot 0.99$
1	0	0	$0.99 \cdot 0.1 \cdot 0.01$
1	0	1	$0.01 \cdot 0.1 \cdot 0.01$
1	1	0	$0.1 \cdot 0.9 \cdot 0.01$
1	1	1	$0.9 \cdot 0.9 \cdot 0.01$

reduce

C_0	C_1	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0	$0.99 \cdot 0 \cdot 0.99$
0	0	1	$0.01 \cdot 0 \cdot 0.99$
0	1	0	$0.1 \cdot 0.01 \cdot 0.99$
0	1	1	$0.9 \cdot 0.01 \cdot 0.99$
1	0	0	$0.99 \cdot 0 \cdot 0.01$
1	0	1	$0.01 \cdot 0 \cdot 0.01$
1	1	0	$0.1 \cdot 0.9 \cdot 0.01$
1	1	1	$0.9 \cdot 0.9 \cdot 0.01$

variable elimination

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$C_0 \longrightarrow C_1 \longrightarrow C_2$

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0
0	1	0.01
1	0	0
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$C_0 \longrightarrow C_1 \longrightarrow C_2$

interestingly:

we can get the same result by just reducing the factor associated with the evidence var

1 to compute $P(c_0, c_2 \mid C_1 = 1)$ for this bayesian network

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

2 reduce the factors associated with the evidence variables

C_0	$P(c_0)$
0	0.99
1	0.01

C_0	C_1	$P(c_1 \mid c_0)$
0	0	0
0	1	0.01
1	0	0
1	1	0.9

C_1	C_2	$P(c_2 \mid c_1)$
0	0	0.99
0	1	0.01
1	0	0.1
1	1	0.9

$$C_0 \longrightarrow C_1 \longrightarrow C_2$$

3 run variable elimination to compute marginal $P(c_0, C_1 = 1, c_2)$

C_0	C_2	$P(c_0, C_1 = 1, c_2)$
0	0	0.00099
0	1	0.00891
1	0	0.0009
1	1	0.0081

4 normalize the marginal to obtain the conditional $P(c_0, c_2 \mid C_1 = 1)$

C_0	C_2	$P(c_0, c_2 \mid C_1 = 1)$
0	0	0.05238
0	1	0.47143
1	0	0.04762
1	1	0.42857