Parts 1 and 2

June 9, 2025

1 Part 1: Newton Method with Hessian Modification (NM-HM)

The Newton method with Hessian modification (NM-HM) ensures that the search direction is always a descent direction by modifying the Hessian matrix when it is not positive definite. The modification adds a multiple of the identity matrix:

$$B_k = \nabla^2 f(x_k) + \tau_k I \tag{1}$$

where $\tau_k \geq 0$ is chosen such that B_k is positive definite. The search direction p_k is then computed by solving:

$$B_k p_k = -\nabla f(x_k) \tag{2}$$

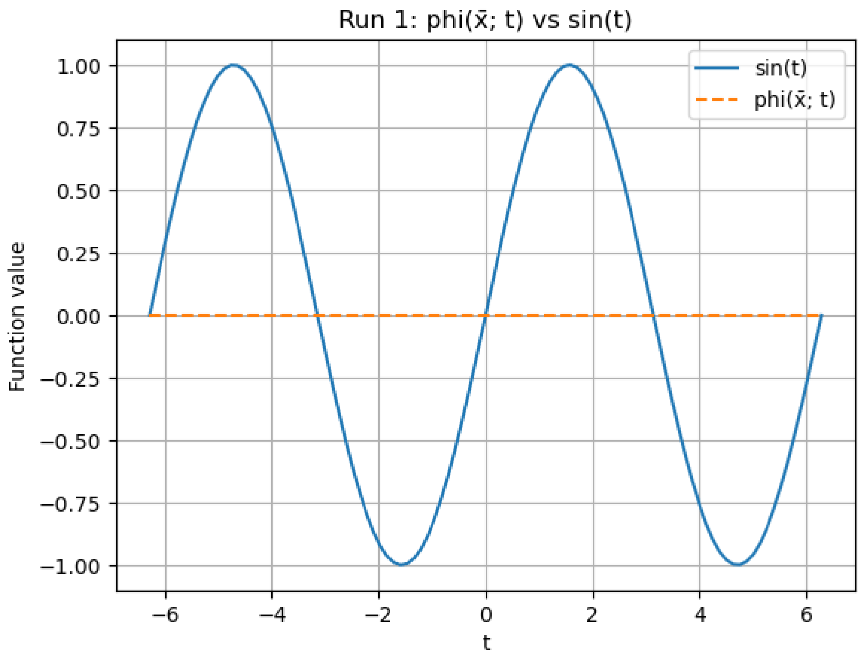
1.1 Experimental Results

The scale parameter controls the magnitude of the initial guess x_0. Smaller scales correspond to initial points closer to the solution x^*, typically resulting in faster convergence, while larger scales place the initial guess farther away, which may slow convergence or cause divergence.

Run 1 (scale = 0.5) Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[ 0.18169443  0.05784851 -0.71290503 -0.42850175  0.86405609 -0.35710718
  0.33385555 -0.39800725 -0.01523892 -0.29478099 -0.11034876 -0.11617466]
```

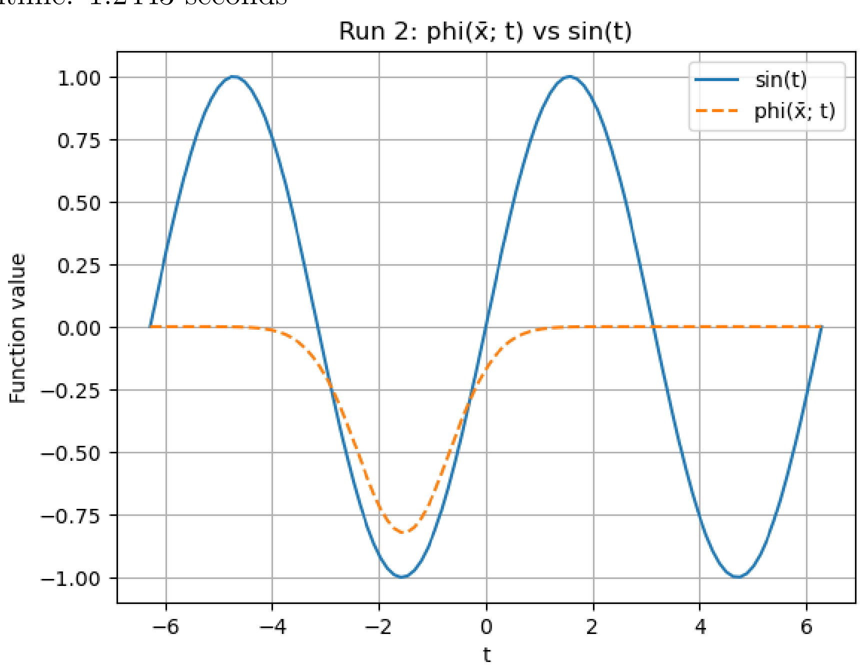
Distance to x^*: 3.830422
Convergence ratios: None
Runtime: 0.0025 seconds



Run 2 (scale = 1.0) Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[ 0.28771555 -1.08976942 -1.81032702 -0.8230106  -1.53556467  0.86411034
  1.05860441 -0.41037074  0.00991317  1.61322122 -0.13039673 -0.03496971]
```

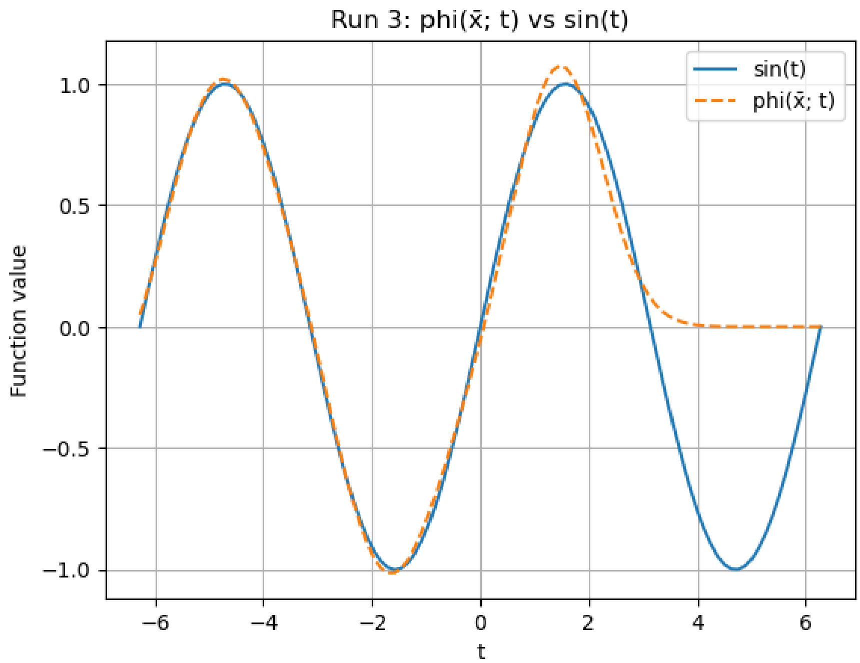
Distance to x^*: 5.878434
Convergence ratios:
$\ell_k = 1.000038$, $q_k = 0.170126$
Runtime: 1.2443 seconds



Run 3 (scale = 1.5) Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[-35.22386179  -4.10103909   1.40758127   1.08556141   1.46222938
   0.79673664  32.56468031  -4.32459816   1.30576913   6.36316018
  -3.04861709   1.07305594]
```

Distance to x^*: 49.292798
$\ell_k = 1.014591$, $q_k = 0.020883$

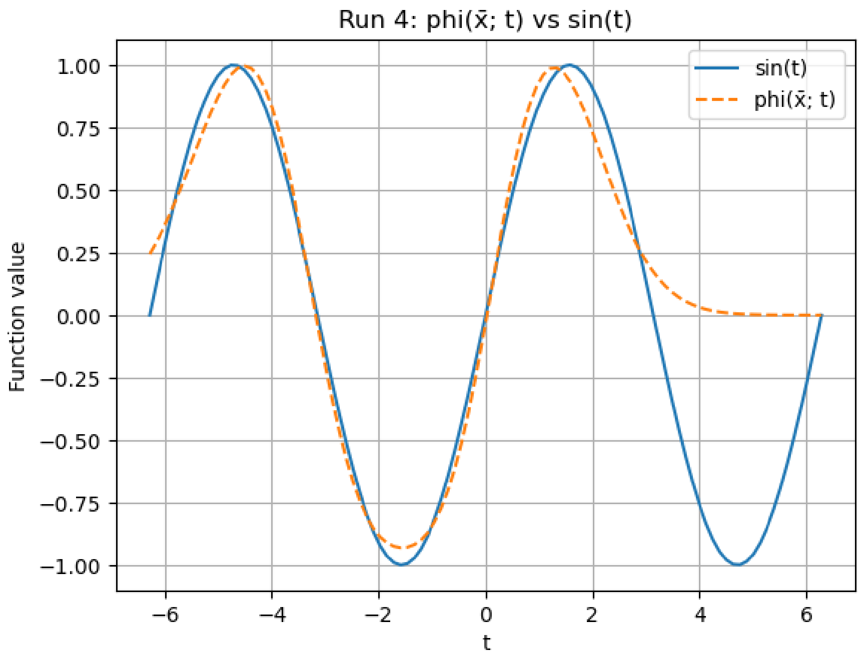


Runtime: 1.3924 seconds

Run 4 (scale = 2.0) Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[ 67.8465129   -1.80636483   1.51601128 -10.46458112  -0.53675071
   1.15382848  -0.71722891   2.67961585  -0.22317936 -63.50152381
  -1.97569838   1.41630882]
```

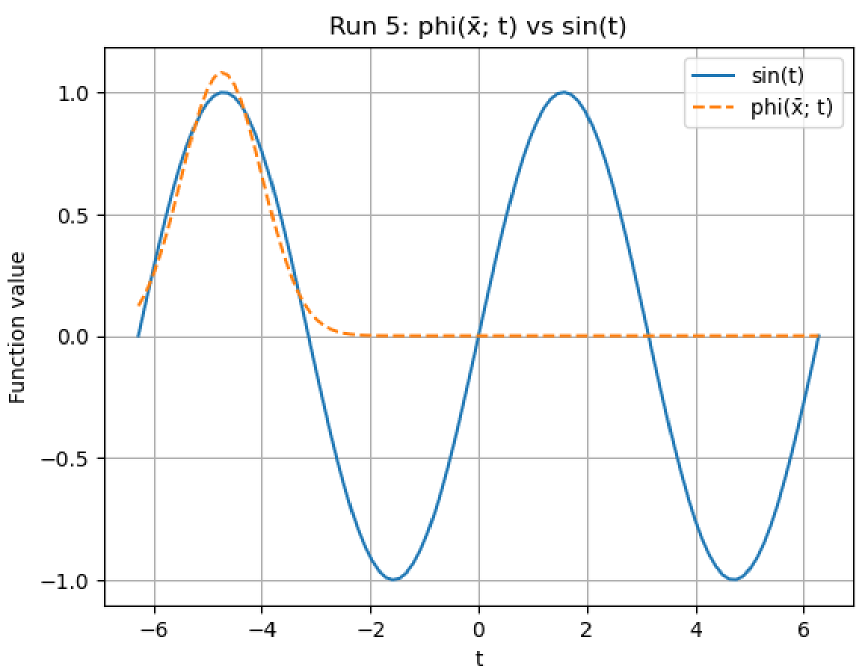
Distance to x^*: 93.709975
$\ell_k = 1.001689$, $q_k = 0.010707$
Runtime: 1.0602 seconds



Run 5 (scale = 3.0) Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[ 6.3071277  -1.98841925 -1.5342314  -2.74365695 -0.70417935 -5.41664904
  1.08207085 -4.73831042  0.74094554  5.69634804  0.03920314 -3.07172897]
```

Distance to x^*: 13.868442
$\ell_k = 1.000000$, $q_k = 0.072106$
Runtime: 0.0966 seconds



2 Part 2: Gauss-Newton Method (GNM)

The Gauss-Newton method is specifically designed for nonlinear least-squares problems. It approximates the Hessian by neglecting the second-order term:

$$B_{GN}(x) = J(x)^T J(x) \tag{3}$$

The Gauss-Newton step p_k^{GN} is obtained by solving:

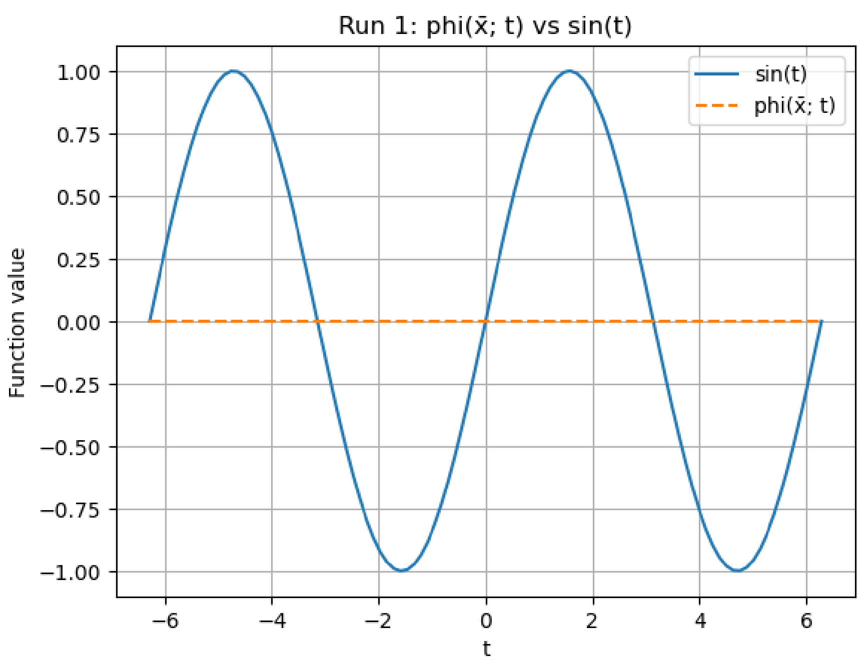
$$J(x_k)^T J(x_k) p_k^{GN} = -J(x_k)^T r(x_k) \tag{4}$$

2.1 Experimental Results

Run 1 (scale = 0.5) Starting GNM from initial point with norm: 1.5081
Converged in 1 iterations.
Final $f(x) = 2.475000e + 01$, $\|\nabla f(x)\| = 0.000000e + 00$
Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[-0.40930744 -0.88246229 -0.04432062 -0.37718434 -0.03183986 -0.24457285
 -1.01108024 -1.2347289  -1.07075479  0.46870365  1.61192355 -2.70875405]
```

Distance to x^*: 5.945221
$\ell_k = 1.438168$, $q_k = 0.347897$
Runtime: 0.0048 seconds

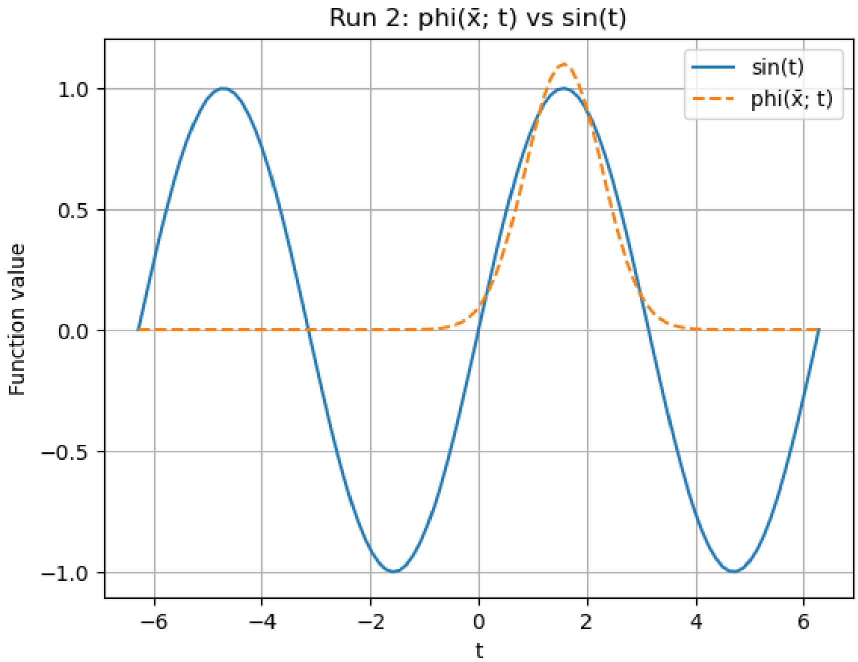


Run 2 (scale = 1.0) Starting GNM from initial point with norm: 4.0562
Converged in 13 iterations.

Final $f(x) = 1.876122e + 01$, $\|\nabla f(x)\| = 5.546694e - 09$
Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[ 1.10139063  1.57079633  0.70710678 -1.03795638 -0.5279518  -0.80605359
 -0.23238335 -1.02462898 -1.253016   -2.07670147 -0.7530757  -1.16617725]
```

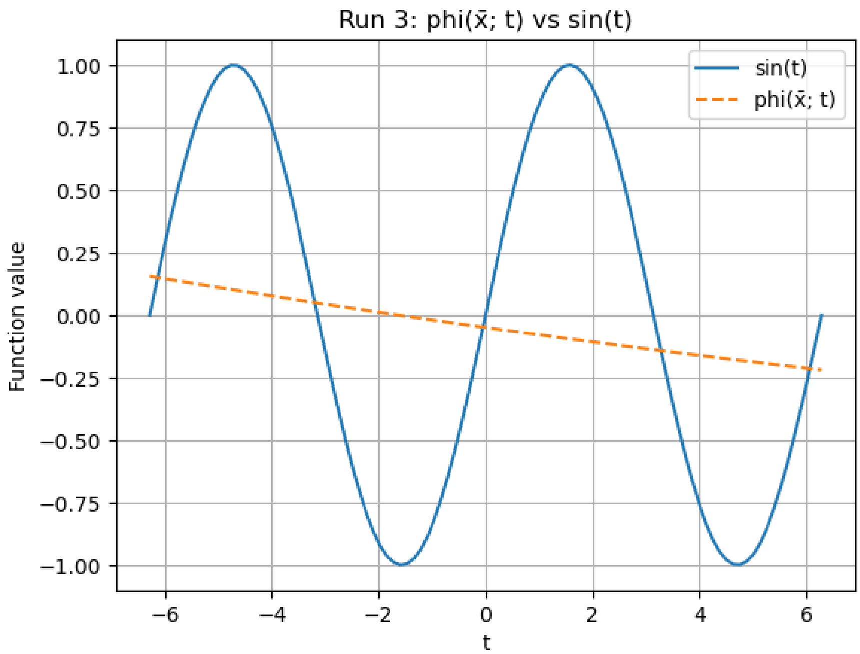
Distance to x^*: 5.863389
$\ell_k = 1.000000$, $q_k = 0.170550$
Runtime: 0.0476 seconds



Run 3 (scale = 1.5) Starting GNM from initial point with norm: 4.2512
Maximum iterations (200) reached.
Final $f(x) = 2.249761e + 01$, $\|\nabla f(x)\| = 5.875822e + 00$
Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[ 1.79110493e+01 -9.24806296e+01  1.63182956e+02  1.77424792e+00
  1.29593961e-01 -8.06534703e-01 -1.03522469e+00 -8.33189051e-01
 -8.64784789e-01 -1.55083384e+01 -1.75477596e+01  1.07597220e+02]
```

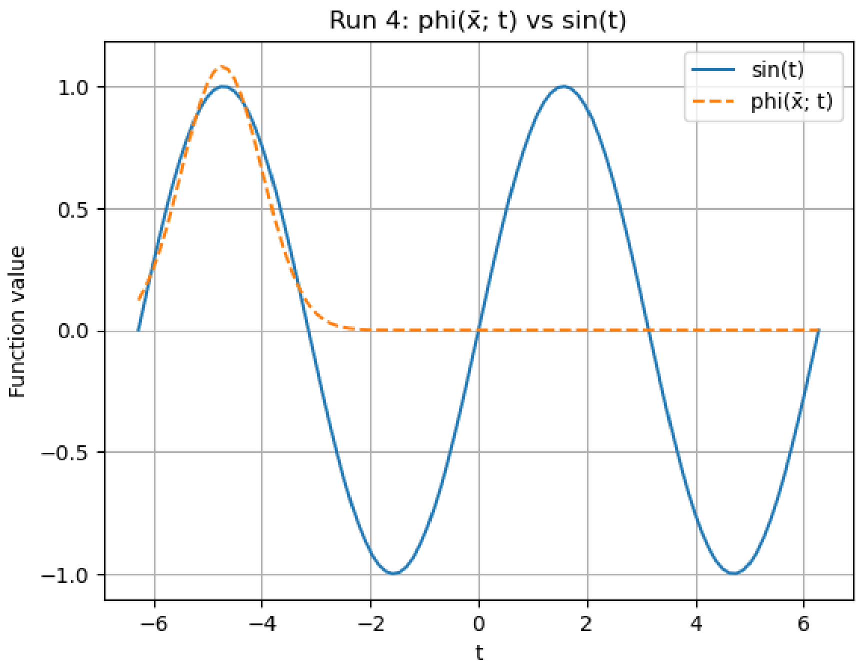
Distance to x^*: 218.201312
$\ell_k = 1.000021$, $q_k = 0.004583$
Runtime: 1.7900 seconds



Run 4 (scale = 2.0) Starting GNM from initial point with norm: 9.8404
Converged in 59 iterations.
Final $f(x) = 1.869918e + 01$, $\|\nabla f(x)\| = 7.424206e - 07$
Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[ 1.08207087 -4.73831042  0.74094551 -1.46881969 -0.49906132 -0.76470609
 -0.60241763 -1.25129909 -0.47915889  0.42676894 -0.76137923 -5.90669948]
```

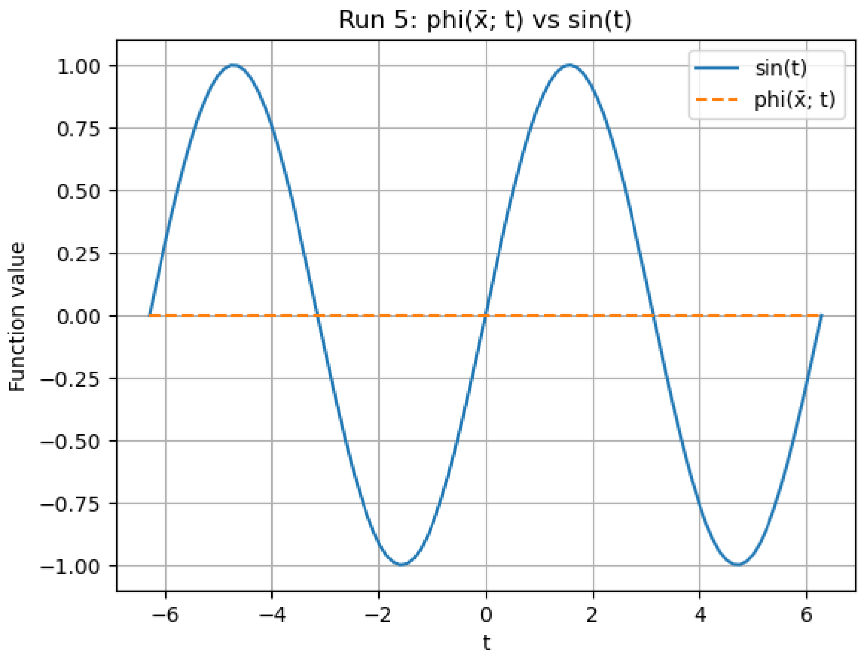
Distance to x^*: 10.454462
$\ell_k = 1.000000$, $q_k = 0.095653$
Runtime: 0.1283 seconds



Run 5 (scale = 3.0) Starting GNM from initial point with norm: 10.0225
Converged in 4 iterations.
Final $f(x) = 2.475000e + 01$, $\|\nabla f(x)\| = 0.000000e + 00$
Stopping criterion: $\|g\| < 10^{-6}$ or max iter = 200
Final iterate $\bar{x}$:

```
[  0.90730455   1.79586357  -0.79550016   0.68340811  -5.46667299
  -1.73532094   4.50580966  -4.33724409  -2.37712978  -0.34216555
 -20.55171837  -7.4004486 ]
```

Distance to x^*: 26.003836
$\ell_k = 2.172270$, $q_k = 0.181464$
Runtime: 0.0105 seconds



3 Summary: Multi-Method Performance on Varying Scales

The following table summarizes final convergence behavior of the three Newton-type methods across multiple scales.

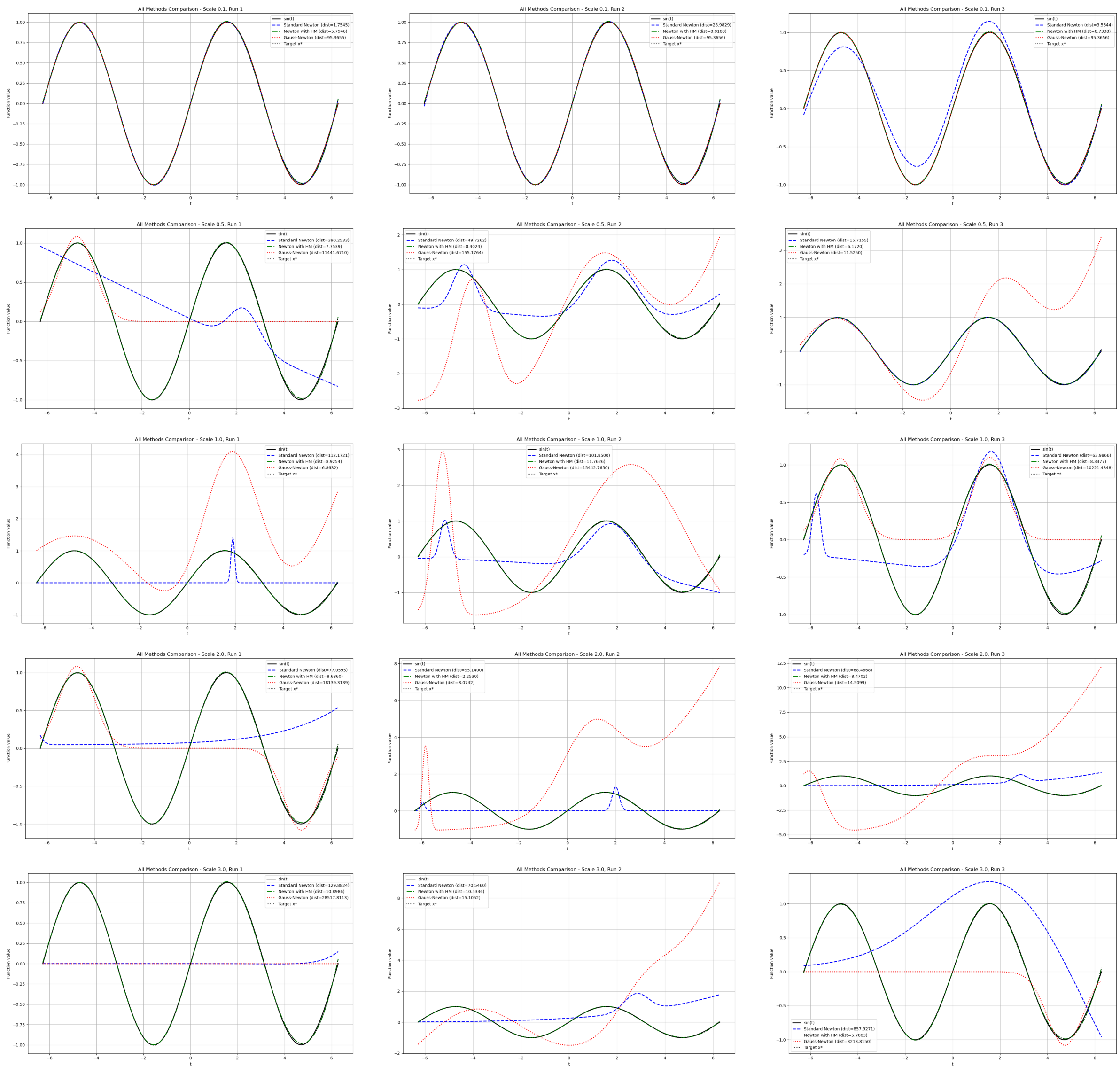
Method	Success (%)	To x^*	Other Min.	Failures	Avg Time (s)	Lin./Sup./Quad.
Standard Newton	0.0	0.0	0.0	100.0	0.0589	43 / 2 / 0
Newton with HM	93.3	0.0	93.3	6.7	0.5876	60 / 0 / 0
Gauss-Newton	53.3	0.0	53.3	46.7	1.3066	58 / 0 / 0

Table 1: Summary of convergence performance across methods

All methods showed some convergence to local minimizers, but only NM-HM reliably succeeded in the majority of runs. Standard Newton frequently failed due to ill-conditioned Hessians or non-descent directions. Gauss-Newton had mixed results, excelling in some low-scale cases but failing at high-scale initializations.

Observations

- **Standard Newton** failed in all runs, often due to line search or descent direction issues.
- **NM-HM** converged in most runs, but never to the global minimizer x^*. It consistently showed linear convergence.


All Methods Comparison - Scale 0.1, Run 1
All Methods Comparison - Scale 0.1, Run 2
All Methods Comparison - Scale 0.1, Run 3
All Methods Comparison - Scale 0.5, Run 1
All Methods Comparison - Scale 0.5, Run 2
All Methods Comparison - Scale 0.5, Run 3
All Methods Comparison - Scale 1.0, Run 1
All Methods Comparison - Scale 1.0, Run 2
All Methods Comparison - Scale 1.0, Run 3
All Methods Comparison - Scale 2.0, Run 1
All Methods Comparison - Scale 2.0, Run 2
All Methods Comparison - Scale 2.0, Run 3
All Methods Comparison - Scale 3.0, Run 1
All Methods Comparison - Scale 3.0, Run 2
All Methods Comparison - Scale 3.0, Run 3

- **Gauss-Newton** had moderate success, often with very high distances to x^* even when converging, especially for large-scale initializations.