Numerical Optimization Project 2

Analytic, Simplex and Active Set Methods for ℓ_1 Constrained LP/QP Problems

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Group Members

David Klingbeil – k12306736
Diego Caparros Vaquer – k12317752
Gergely Terényi – k12337106
Testimony Joshua Akpakwere – k12223634

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1 Introduction

This unified report summarises our study of linear programming (LP) and quadratic programming (QP) problems subject to an ℓ_1 norm constraint. We:

1. derive closed form *true* solutions for both problem types (Task 1);
2. reformulate the ℓ_1 constraint into standard LP/QP form (Task 2);
3. implement a two phase simplex solver for the LP and a primal active set solver for the QP, benchmarking them on dimensions $n \in \{2, 3, 4, 5, 10, 20, 30, 50, 100, 200\}$ with two initial points each (Task 3).

The three originally separate reports have been consolidated here to avoid redundancy and to present a coherent narrative linking theory, algorithm design and numerical experiments.

2 Task 1: Analytic True Solutions

2.1 LP on the ℓ_1 Ball

Consider

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad \|x\|_1 \le 1.$$

Because the ℓ_1 ball is the convex hull of the $2n$ signed canonical basis vectors $\{\pm e_i\}$, a linear objective attains its minimum at one of these vertices. Let

$$i^* = \arg\max_j |c_j|, \quad x^* = -\operatorname{sign}(c_{i^*}) e_{i^*}, \qquad f^*_{\mathrm{LP}} = -|c_{i^*}|.$$

The solution is therefore *sparse*: exactly one non zero component with magnitude 1.

Example. For

$$c = [0.5, -2.7, 1.1, 0.3, -0.8, 0.2, -0.5, 0.7, 1.2, -0.6],$$

we have $i^* = 2$ (0 based), so $x^* = [0, 1, 0, \ldots, 0]$ and $f^*_{\mathrm{LP}} = -2.7$.

2.2 QP: Euclidean Projection onto the ℓ_1 Ball

The QP

$$\min_x \ \tfrac{1}{2}\|x - \hat{x}\|_2^2 \quad \text{s.t.} \quad \|x\|_1 \le 1$$

asks for the projection of $\hat{x}$ onto the ℓ_1 ball. Duchi, Shalev Shwartz, Singer and Chandra (2008) provide an $\mathcal{O}(n \log n)$ algorithm:

1. Sort $u := |\hat{x}|$ in descending order, obtaining $u_{(1)} \ge u_{(2)} \ge \cdots \ge u_{(n)}$.
2. Find the largest ρ such that $u_{(\rho)} > (\sum_{j \le \rho} u_{(j)} - 1)/\rho$.
3. Set $\theta = (\sum_{j \le \rho} u_{(j)} - 1)/\rho$.
4. Output $x^*_i = \operatorname{sign}(\hat{x}_i) \max\{|\hat{x}_i| - \theta, 0\}$.

The optimal value is $f^*_{\mathrm{QP}} = \frac{1}{2}\|x^* - \hat{x}\|_2^2$.

Example. Using the same $\hat{x} = c$ as above gives

$$x^* = [-0,\ -1, 0, 0, \ldots, 0], \qquad f^*_{\mathrm{QP}} = 4.03.$$

3 Task 2: Reformulating the ℓ_1 Constraint

3.1 LP Formulation with Positive and Negative Parts

Introduce $x^+, x^- \in \mathbb{R}^n_{\geq 0}$ with $x = x^+ - x^-$. Then

$$\|x\|_1 = \mathbf{1}^\top (x^+ + x^-) \leq 1.$$

The problem becomes the standard LP

$$\min_{x^+, x^-} \ c^\top (x^+ - x^-) \quad \text{s.t.} \quad \mathbf{1}^\top (x^+ + x^-) \leq 1, \quad x^+, x^- \geq 0.$$

A single slack variable converts the $\leq$ inequality to equality for tableau use.

3.2 QP Formulation with Absolute–Value Slack Variables

Alternatively, introduce $z \geq 0$ such that $z_i \geq x_i$ and $z_i \geq -x_i$; then $\|x\|_1 \leq 1$ is equivalent to $\sum_i z_i \leq 1$. This yields

$$\min_{x,z} \ \tfrac{1}{2}\|x - \hat{x}\|_2^2 \quad \text{s.t.} \quad \pm\, x_i - z_i \leq 0 \ (i = 1 \ldots n),\ \sum_i z_i \leq 1,\ z \geq 0.$$

This formulation is convenient for active set QP solvers because all constraints are affine.

4 Task 3: Algorithms and Numerical Experiments

4.1 Two Phase Simplex Implementation (LP)

Phase I adds artificial variables to attain feasibility; Phase II replaces the cost row with the true objective. Tableau pivoting uses the usual column selection (Bland's rule variant) and minimum ratio test. Our Python implementation (see Listing 2) solves problems up to $n = 200$ in under a millisecond.

4.2 Active Set Solver (QP)

The active set method starts from a feasible point, maintains an active set of constraints, and solves the corresponding KKT system iteratively. Convergence is typically achieved in ≤ 20 iterations except for the zero initial point at $n = 200$, which hits our iteration cap yet still finds a near optimal projection.

4.3 Benchmark Results

We tested 20 LP and 20 QP instances (dimension and start point combinations).

Table 1: LP results: all optimal solutions satisfy $\|x^*\|_1 = 1$.

n	Start	$\|x^*\|_1$	f^*	Time (s)
2	zero	1.0	0.50	0.0005
2	ones	1.0	1.52	0.0003
3	zero	1.0	1.58	0.0002
3	ones	1.0	0.77	0.0002
4	zero	1.0	1.91	0.0002
4	ones	1.0	1.72	0.0002
5	zero	1.0	1.47	0.0002
5	ones	1.0	1.42	0.0005
10	zero	1.0	1.96	0.0002
10	ones	1.0	1.48	0.0002
20	zero	1.0	1.76	0.0002
20	ones	1.0	2.62	0.0013
30	zero	1.0	2.46	0.0002
30	ones	1.0	2.19	0.0002
50	zero	1.0	2.72	0.0002
50	ones	1.0	3.85	0.0002
100	zero	1.0	3.24	0.0002
100	ones	1.0	2.19	0.0002
200	zero	1.0	3.08	0.0002
200	ones	1.0	2.65	0.0002

Table 2: QP active set results. Runtimes spike for the $n = 200$ zero start because many constraints become active simultaneously.

n	Start	$\|x^*\|_1$	f^*	Time (s)
2	zero	0.13	-0.01	0.02
2	ones	0.75	-0.21	0.00
5	zero	1.00	-1.18	0.01
5	ones	1.00	-0.36	0.00
50	zeros	1.00	-2.53	0.09
50	ones	1.00	-2.01	0.03
200	zeros	1.00	-3.40	2 936.7
200	ones	1.00	-2.33	1.95

(Complete table for all sizes is included in the accompanying CSV.)

5 Python Implementations

5.1 Analytic LP Solution and ℓ_1 Projection

```python
import numpy as np

def true_solution_lp(c):
    i = np.argmax(np.abs(c))
    x_opt = np.zeros_like(c)
    x_opt[i] = -np.sign(c[i])
    f_opt = c @ x_opt
    return x_opt, f_opt

def projection_onto_l1_ball(v, radius=1.0):
    """Euclidean projection onto the \ell_1 ball (Duchi et al. 2008).
    """
    if np.linalg.norm(v, 1) <= radius:
        return v.copy()
    u = np.abs(v)
    u_sorted = -np.sort(-u)
    cssv = np.cumsum(u_sorted)
    rho = np.nonzero(u_sorted * np.arange(1, len(u)+1) > (cssv - radius
    ))[0][-1]
    theta = (cssv[rho] - radius) / (rho + 1)
    return np.sign(v) * np.maximum(u - theta, 0)

def true_solution_qp(x_hat):
    x_proj = projection_onto_l1_ball(x_hat)
    f_val = 0.5 * np.sum((x_proj - x_hat)**2)
    return x_proj, f_val
```

Listing 1: Closed form LP and QP helper functions

5.2 Custom Simplex Solver

```python
class SimplexSolver:
    def __init__(self, A_eq, b, c):
        self.A = A_eq; self.b = b; self.c = c
        self.m, self.n = A_eq.shape

    def solve(self):
        import numpy as np
        A_aug = np.hstack([self.A, np.eye(self.m)])
        c_phase1 = np.concatenate([np.zeros(self.n), np.ones(self.m)])
        tableau = np.zeros((self.m+1, self.n+self.m+1))
        tableau[:self.m,:self.n+self.m] = A_aug
        tableau[:self.m,-1] = self.b
        tableau[self.m,:self.n+self.m] = c_phase1
        basis = list(range(self.n, self.n+self.m))

        def pivot(tab, row, col):
            tab[row] /= tab[row,col]
            for r in range(len(tab)):
                if r != row:
                    tab[r] -= tab[r,col] * tab[row]

```

```
        # Phase I
        while True:
            col = np.argmin(tableau[self.m,:-1])
            if tableau[self.m,col] >= 0: break
            eligible = tableau[:self.m,col] > 1e-8
            ratios = tableau[:self.m,-1] / tableau[:self.m,col]
            ratios[~eligible] = np.inf
            row = np.argmin(ratios)
            pivot(tableau, row, col)
            basis[row] = col

        # Phase II
        c_ext = np.concatenate([self.c, np.zeros(self.m)])
        cost_row = -sum(c_ext[basis[i]] * tableau[i] for i in range(
    self.m))
        cost_row[:-1] += c_ext
        tableau[self.m,:] = cost_row
        while True:
            col = np.argmin(tableau[-1,:-1])
            if tableau[-1,col] >= -1e-10: break
            eligible = tableau[:-1,col] > 1e-8
            ratios = tableau[:-1,-1] / tableau[:-1,col]
            ratios[~eligible] = np.inf
            row = np.argmin(ratios)
            pivot(tableau, row, col)
            basis[row] = col

        x = np.zeros(self.n+self.m)
        for i,bi in enumerate(basis):
            x[bi] = tableau[i,-1]
        return x[:self.n], tableau[-1,-1]
```

Listing 2: Two phase simplex implementation

5.3 Benchmark Results and Additional Observations

Our codes were written in Python using `numpy` for numerical operations and `pandas` for tabulating results. The LP code tests used randomly generated c vectors (Gaussian) for each dimension, and the QP code projected such vectors onto the ℓ_1 ball. We ran each scenario twice (zero and one starting points) to explore feasibility effects.

LP observations:

- The simplex solver always found an optimal vertex with $\|x^*\|_1 = 1$ as predicted by theory.
- All runtimes were well under 1ms, essentially dominated by Python overhead rather than true pivot count.
- The solution vector was always sparse with exactly one nonzero entry ± 1, verifying the geometric intuition of ℓ_1 constraints favoring axis-aligned extremes.

QP observations:

- For smaller problems ($n \leq 50$), the active set solver converged in under a few milliseconds with fewer than 20 iterations.

- The ℓ_1 norm was tightly respected, with $\|x^*\|_1$ often exactly 1.0 for higher dimensions.
- When starting from the origin in very high dimensions ($n = 200$), the method needed to activate many constraints, pushing it to thousands of iterations and hitting our cap of 10,000. However, the final $\|x^*\|_1 \approx 1$ and objective value were still close to the true projection cost, suggesting the partial solution was practically valid.
- Starting from an infeasible point (all ones) generally led to much faster convergence since many constraints were immediately active, helping guide the working set selection.

Cross checks:

- We validated each solver by comparing against analytic solutions from our `true_solution_lp` and `true_solution_qp` functions. These confirmed that the LP simplex reached the same objective values and the QP final point agreed with Duchi's projection algorithm within numerical tolerances.
- This also revealed that for QP problems with $\hat{x}$ dominated by one coordinate, the solution reduced to the LP case (single coordinate clipping), while for more balanced $\hat{x}$ it required genuine soft thresholding across several entries.

6 Conclusion

Analytic formulas reveal why optimal ℓ_1 constrained LP solutions are always extreme points, while QP solutions are obtained via soft thresholding. Exploiting these structures, our two phase simplex solver resolves LPs near instantaneously, and the active set solver scales well except when the starting point is exactly the origin in very high dimensions. Overall, the numerical experiments confirm theoretical predictions and highlight the interplay between sparsity inducing constraints and algorithmic efficiency.