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MAST30025

Linear Statistical Models Assignment 1

Due: 28 Mar 2021 5:00 pm

1. Idempotent : $(\underline{I} - kA)^2 = \underline{I} - kA$ ①

Case 1 : $A = 0$

$$(\underline{I} - kA)^2 = (\underline{I} - 0k)^2 = (\underline{I})^2 = \underline{I} \quad \forall k \in \mathbb{R}$$

$\therefore$ $\forall k \in \mathbb{R}$ $(\underline{I} - kA)^2$ is idempotent

Case 2: $A \in \mathbb{R}$, $A \neq 0$

$$(1 - kA)(1 - kA) = 1 - kA$$
$$1 - 2kA + k^2A^2 = 1 - kA$$
$$1 - 2kA + k^2A = 1 - kA \quad \text{(by } A \text{ idempotent)}$$
$$k^2A - kA = 0$$
$$(k^2 - k)A = 0$$

$\Rightarrow k^2 - k = 0$

$\Rightarrow k(k-1) = 0$

$\Rightarrow k = 0,\ k = 1$ by NFL

$\therefore$ When $k \in \{0, 1\}$, $1 - kA$ is idempotent

2. Proof: (2)

Proof	Justification
1) A, B sym	Premise
2) $A+B$ is idempotent $\Rightarrow (A+B)^2 = A+B$	Premise
3) $AB = BA = 0$	Premise
4) $\exists P'$ that diagonalizes A: $P'^T A P' = \Lambda_1 = \begin{bmatrix} \lambda_1' & 0 & \cdots & 0 \\ 0 & \lambda_2' & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda_n' \end{bmatrix}$ $\exists P''$ that diagonalizes B: $P''^T A P'' = \Lambda_2 = \begin{bmatrix} \lambda_1'' & 0 & \cdots & 0 \\ 0 & \lambda_2'' & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda_n'' \end{bmatrix}$	By 2.1 Since A, B sym (1)
5) $P' = P'' = P$	By 2.4 Since $AB = BA$ $(=0)$ (3)
6) $(\Lambda_1 + \Lambda_2) = P^T A P + P^T B P = P^T (A+B) P$ $\forall i \in \mathbb{Z}\ 1 \le i \le k \quad (\Lambda_1 + \Lambda_2)_{ii} \in \{0, 1\}$	by 5) + by 2.2 since $A+B$ idempotent
7) $\Lambda_1 \Lambda_2 = P^T A P P^T B P = P^T A B P = 0$	by $AB = 0$, P orthonormal so $P^T P = I$
8) $\Rightarrow \forall i \in \mathbb{Z}\ 1 \le i \le k \quad \Lambda_{1\,ii} \Lambda_{2\,ii} = 0$ $\Rightarrow \forall i \in \mathbb{Z}\ 1 \le i \le k \quad \Lambda_{1\,ii} = 0 \vee \Lambda_{2\,ii} = 0$	by Null Factor law
9) $\forall i \in \mathbb{Z}\ 1 \le i \le k, \quad \Lambda_{1\,ii}, \Lambda_{2\,ii} \in \{0, 1\}$ where $\Lambda_{1\,ii}, \Lambda_{2\,ii}$ are eigvals of A and B respectively	by 8), 6)
Case 1: $(\Lambda_1 + \Lambda_2)_i = 1$, $(\Lambda_1)_i = 0$ $\Rightarrow (\Lambda_2)_i = 1 - 0 = 1$	
Case 2: $(\Lambda_1 + \Lambda_2)_i = 1$, $(\Lambda_2)_i = 0$ $\Rightarrow (\Lambda_1)_i = 1 - 0 = 1$	
Case 3: $(\Lambda_1 + \Lambda_2)_i = 0$, $(\Lambda_1)_i = 0$ $\Rightarrow (\Lambda_2)_i = 0 - 0 = 0$	
Case 4: $(\Lambda_1 + \Lambda_2)_i = 0$, $(\Lambda_2)_i = 0$ $\Rightarrow (\Lambda_1)_i = 0 - 0 = 0$	
10) A, B is idempotent $\square$	see next pg by 2.2's $E^\#$ version by 1) A, B symmetric

(2)

Proof of 2.2 $\Leftarrow$ version:

"If a symmetric matrix has eigenvalues of only 0 or 1 then it is idempotent"

Want to show: $A^2 = A$	Justification
1) Let A be symmetric	Premise
2) Let eigval of $A \in \{0,1\}$	Premise
3) $\exists\ \Lambda = P^T A P$ where Λ is diagonal matrix with Λ_{ii} = eigenvalues of A	2.1 1) A is symmetric
4) $\Lambda_{ii} \in \{0,1\}$	3) 2)
5) $\Lambda_{ii}^2 = \Lambda_{ii} \Rightarrow \Lambda^2 = \Lambda$	Property of diagonal matrix: Λ^2 is diagonal matrix w/ $\Lambda_{ii}^2 = (\Lambda_{ii})^2$
6) $\Lambda = \Lambda^2 = P^T A P P^T A P = P^T A^2 P$ $\Rightarrow P^T A^2 P = \Lambda = P^T A P$	
7) $A^2 = A$	
8) A is idempotent $\square$	

$\therefore$ If A is symmetric w/ eigenvalues $\in \{0,1\}$ then A is idempotent.

3. R code attached on next pg after Q3 (3)

a. $A\vec{y} \sim MVN(AE(\vec{y}), AVA^T)$

$$AE(\vec{y}) = A\vec{\mu} = \begin{bmatrix} \frac{2}{3} \\ -1 \\ -\frac{1}{3} \end{bmatrix}$$

$$AVA^T = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

$$A\vec{y} \sim MVN\left(\begin{bmatrix} \frac{2}{3} \\ -1 \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \right)$$

b. $E(\vec{y}^T A\vec{y}) = tr(AV) + \vec{\mu}^T A\vec{\mu}$ by 3.2

$= tr\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) + \frac{5}{3}$

$= 3 + \frac{5}{3} = \frac{14}{3}$

$\Rightarrow E(\vec{y}^T A\vec{y}) = \frac{14}{3}$

c. $\vec{y}^T A\vec{y} \sim \chi^2_{k,\lambda}$
where

$k = 3$ from part B: $AV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda = \frac{1}{2}\vec{\mu}^T A\vec{\mu} = \frac{5}{6}$

$\Rightarrow \vec{y}^T A\vec{y} \sim \chi^2_{3,\frac{5}{6}}$

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d. By 3.11 to be indep, require $AVB = 0$.

$$AVB = I_3 B = 0$$
$$\Rightarrow B \Rightarrow I_3 0$$
$$\Rightarrow B = 0.$$

$\Rightarrow$ $B=0$ (the zero matrix) is the only matrix that satisfies $\vec{y}^T B \vec{y}$ indep $\vec{y}^T A \vec{y}$.

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R code for Question 3

Setup

```
A= 1/3*matrix(c(2, 0, -1, 0, 3, 0, -1, 0, 2), c(3, 3))
mu = c(1, -1, 0)
V = matrix(c(2, 0, 1, 0, 1, 0, 1, 0, 2), c(3, 3))
A

##            [,1] [,2]       [,3]
## [1,]  0.6666667    0 -0.3333333
## [2,]  0.0000000    1  0.0000000
## [3,] -0.3333333    0  0.6666667

mu

## [1]  1 -1  0

V

##      [,1] [,2] [,3]
## [1,]    2    0    1
## [2,]    0    1    0
## [3,]    1    0    2
```

a)

```
mean3a = A %*% mu
var3a = A %*% V %*% t(A)

mean3a

##            [,1]
## [1,]  0.6666667
## [2,] -1.0000000
## [3,] -0.3333333

var3a

##            [,1] [,2]       [,3]
## [1,]  0.6666667    0 -0.3333333
## [2,]  0.0000000    1  0.0000000
## [3,] -0.3333333    0  0.6666667
```

b)

```
A %*% V
```

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```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

```
t(mu) %*% A %*% mu
```

```
##          [,1]
## [1,] 1.666667
```

c)

```
lambda3c = 0.5 * t(mu) %*% A %*% mu
lambda3c
```

```
##           [,1]
## [1,] 0.8333333
```

4.

1.	BA is symmetric	Premise
2.	$AB = I_m$	Premise
3.	$r(AB) = tr(AB) = tr(I_m) = m$	By 2) + 2.3: I_m is idemp & sym.
4.	$tr(AB) = tr(BA) = m$	Trace rule 3 since BA exists
5.	$(BA)(BA) = B(AB)A = BA$ $\Rightarrow BA$ is idempotent	By 2): $AB = I_m$
6.	$r(BA) = tr(BA) = m$	2.3 by 1) BA sym $\wedge$ 5) BA idem
7.	$\vec{y}^T BA \vec{y} \sim \chi_{m, \frac{1}{2}\vec{\mu}^T BA \vec{\mu}}$	3.5 as BA is idempotent and has rank m

5. R code attached on next page

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a. $$\begin{bmatrix} 227 \\ 354 \\ 373 \\ 512 \\ 537 \\ 328 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 200 \\ 1 & 4 & 250 \\ 1 & 5 & 200 \\ 1 & 6 & 400 \\ 1 & 8 & 150 \\ 1 & 4 & 220 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

$$(\vec{y} = X \vec{\beta} + \vec{\varepsilon})$$

b. $$\vec{b} = (X^T X)^{-1} X^T \vec{y}$$

$$= \begin{bmatrix} 38.327 \\ 54.84 \\ 0.3596 \end{bmatrix}$$

$\hat{\beta}_0 = b_0 = 38.327$; $\hat{\beta}_1 = b_1 = 54.84$; $\hat{\beta}_2 = b_2 = 0.3596$

c. $$SS_{res} = (\vec{y} - X\vec{\beta})^T (\vec{y} - X\vec{\beta}) = 336.39$$

$$s^2 = \frac{SS_{res}}{6-(2+1)} = 112.1316$$

d. $$y^* = 38.327 + 3 \times 54.84 + 350 \times 0.3596$$

$$\vec{x}^* = [1, 3, 350]$$

$$y^* = \vec{x}^* \vec{b}$$

$$y^* = 328.717$$

Predicted avg sales figures for this store = \$328,717

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R code for Question 5

b)

```
X = matrix(c(1, 1, 1, 1, 1, 1, 2, 4, 5, 6, 8, 4, 200, 250, 200, 400, 150, 220
), c(6,3))
y = c(227, 354, 373, 512, 537, 328)

b = solve(t(X) %*% X ) %*% t(X) %*% y
b

##             [,1]
## [1,] 38.3273312
## [2,] 54.8403001
## [3,]  0.3596249
```

c)

```
n = length(y)
p = 2+1

SSres = t(y-X %*% b) %*% (y - X %*% b)
SSres

##           [,1]
## [1,] 336.3947

s2 = SSres/(n-p)
s2

##           [,1]
## [1,] 112.1316
```

d)

```
x. = c(1, 3, 350)
y. = x. %*% b

y.

##           [,1]
## [1,] 328.7169
```

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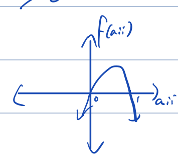
6. PART I

Proof:

		Justification
1)	Let A be symmetric and Idempotent (and real)	
2)	$A^T = A$	1) A is sym
3)	$A^2 = A$.	1) A is idempt
4)	$a_{ii} \in \mathbb{R}$	1) A is real
5)	$AA^T = A^TA = A$.	2) and 3)
6)	$a_{ii} = \sum_j a_{ij}^2$ $= a_{ii}^2 + \sum_{j \neq i} a_{ij}^2$	4) $AA^T = A$
7)	$\sum_{j \neq i} a_{ij}^2 \geq 0$	as $x^2 \geq 0 \ \forall x$
8)	$a_{ii} - a_{ii}^2 = \sum_{j \neq i} a_{ij}^2 \geq 0$	7)
9)	$a_{ii} - a_{ii}^2 \geq 0$	
10)	$a_{ii}(1 - a_{ii}) \geq 0$	
11)	$a_{ii} \in [0, 1]$ $(0 \leq a_{ii} \leq 1)$	



$\square$

PART II

Leverage $= H_{ii}$ for $H = X(X^TX)^{-1}X^T$

H is idempotent as $H^2 = X(X^TX)^{-1}\underbrace{X^TX(X^TX)^{-1}}_{I}X^T = X(X^TX)^{-1}X^T$
$= H$

H is symmetric as $H^T = (X(X^TX)^{-1}X^T)^T = X(X(X^TX)^{-1})^T$
$= X((X^TX)^{-1})^TX^T$
$= X((X^TX)^T)^{-1}X^T$
$= X(X^TX)^{-1}X^T$
$= H$

Thus $0 \leq H_{ii} \leq 1$, and so the limits of the leverage are 0 and 1