<u>MAST30027 Assignment 1</u>

Name: Lang (Ron) Chen

Student ID: 1181506

Subject: MAST30027

Assignment number: 1

Tutorial time and tutor: Tues 11:00 Yidi Deng

QUESTION 1

Setup

```
library(faraway)
data(orings)
str(orings)

## 'data.frame':    23 obs. of  2 variables:
##  $ temp  : num  53 57 58 63 66 67 67 67 68 69 ...
##  $ damage: num  5 1 1 1 0 0 0 0 0 0 ...
```

a)

$Y_i \sim Bi(6, p_i)$ $\quad \forall i \quad 1 \le i \le n \quad$ where Y_i are independent

$p_i = 1 - e^{-e^{\beta_0 + \beta_1 t_i}}$

$$l = \sum \log\left(\binom{6}{y} p_i^{y_i} (1-p_i)^{m_i - y_i}\right)$$

$$= c + \sum\left(y_i \log p_i + (m_i - y_i)\log(1-p_i)\right)$$

$$= c + \sum\left(y_i \log\left(1 - e^{-e^{\beta_0+\beta_1 t_i}}\right) + (m_i - y_i)\log\left(e^{-e^{\beta_0+\beta_1 t_i}}\right)\right)$$

$$= c + \sum\left(y_i \log\left(1 - e^{-e^{\beta_0+\beta_1 t_i}}\right) - (m_i - y_i)e^{\beta_0+\beta_1 t_i}\right)$$

where c is constant term

```
logL = function(beta, orings){
  eta = cbind(1, orings$temp) %*% beta
  return( sum( orings$damage * log((1-exp(-exp(eta)))) + (6-orings$damage)*lo
g(exp(-exp(eta))) ))
}

(betahat = optim(c(10, -.1), logL, orings=orings, control=list(fnscale=-1))$p
ar)

## [1] 10.8585961 -0.2054664
```

Soln: the Maximum Likelihood Estimates for beta0 = 10.859, beta1 = -0.205

b)

```r
# Fisher Information
phat = iloglog(betahat[1]+betahat[2]*orings$temp)

I11 = -sum( 6 * phat * (log(1-phat)*(1-phat)*(phat+log(1-phat)))/(-phat**2) +
 (6 - 6 * phat) * (log(1-phat)) )

I12 = -sum( orings$temp * 6 * phat * (log(1-phat)*(1-phat)*(phat+log(1-pha
t)))/(-phat**2) + orings$temp * (6 - 6 * phat) * (log(1-phat)) )

I22 = -sum( orings$temp**2 * 6 * phat * (log(1-phat)*(1-phat)*(phat+log(1-pha
t)))/(-phat**2) + orings$temp**2 * (6 - 6 * phat) * (log(1-phat)) )

(Iinv = solve(matrix(c(I11, I12, I12, I22), 2, 2)))

##           [,1]         [,2]
## [1,]  7.497555 -0.124187977
## [2,] -0.124188  0.002082332
```

so confidence_interval_of betahat[i] = (betahat[i] - 1.96 * se(betahat[i]), betahat[i] + 1.96 * se(betahat[i])),

where se(betahat[i]) = sqrt(fisher_information^{-1} [i,i]),

i = {1, 2} (for betahat0, betahat1 respectively)

```r
paste("betahat_0:")

## [1] "betahat_0:"

(betahat[1] + c(-1, 1) * qnorm(0.975) * sqrt(Iinv[1,1]))

## [1]  5.491889 16.225304

paste("betahat_1:")

## [1] "betahat_1:"

(betahat[2] + c(-1, 1) * qnorm(0.975) * sqrt(Iinv[2,2]))

## [1] -0.2949046 -0.1160282
```

The 95% confidence interval for betahat_0 is (5.49, 16.23); the 95% confidence interval for betahat_1 is (-0.295, -0.116).

PLEASE SEE NEXT PAGE FOR DERIVATIONS

b. $V(\vec{\theta}^*) = I(\theta^*)^{-1}$

$$I(\vec{\theta}) = E(J(\vec{\theta})) \quad \text{where} \quad J(\vec{\theta}) = -\frac{\partial^2 l(\vec{\theta})}{\partial\vec{\theta}\,\partial\vec{\theta}^T}$$

$$I(\vec{\theta}) = \begin{bmatrix} -E\left(\frac{\partial^2 l(\vec{\beta})}{\partial \beta_0^2}\right) & -E\left(\frac{\partial^2 l(\vec{\beta})}{\partial \beta_0 \partial \beta_1}\right) \\ -E\left(\frac{\partial^2 l(\vec{\beta})}{\partial \beta_0 \partial \beta_1}\right) & -E\left(\frac{\partial^2 l(\vec{\beta})}{\partial \beta_1^2}\right) \end{bmatrix}$$

$$l(\beta) = \sum_{i=1}^{n} \log \mathbb{P}(Y = y_i)$$

$$= c + \sum_{i=1}^{n} \left(y_i \log(g^{-1}(\eta_i)) + (m_i - y_i) \log(1 - g^{-1}(\eta_i)) \right)$$

$$= c + \sum_{i=1}^{n} \left(y_i \log(1 - e^{-e^{\beta_0+\beta_1 t}}) + (m_i - y_i)(-e^{\beta_0+\beta_1 t}) \right)$$

Note:

$$\log(-\log(1-p_i)) = \beta_0 + \beta_1 t_i$$

$$1 - e(-e^{\eta}) = \vec{p}$$

$$1 - p_i = e(-e^{\eta})$$

$$\log(1-p_i) = -e^{\eta}$$

$$\frac{d}{dx} e^{ax+b} = a e^{ax+b}$$

$$\frac{d}{dx} -e^{-e^{ax+b}} = -(-a e^{ax+b} e^{-e^{ax+b}}) = a e^{-e^{ax+b} + ax + b}$$

$$\frac{d}{dx} \log(f(x)) = \frac{f'(x)}{f(x)}$$

$$\frac{\partial l(\beta_0, \beta_1)}{\partial \beta_0} = \sum_{i=1}^{n} \left(y_i \frac{e^{-e^{\beta_0+\beta_1 t}} e^{\beta_0+\beta_1 t}}{1 - e^{-e^{\beta_0+\beta_1 t_i}}} - (m_i - y_i) e^{\beta_0+\beta_1 t_i} \right)$$

$$= \sum_{i=1}^{n} \left(y_i \frac{e^{\beta_0+\beta_1 t_i}}{e^{e^{\beta_0+\beta_1 t_i}} - 1} - (m_i - y_i) e^{\beta_0+\beta_1 t_i} \right)$$

$$\frac{\partial L(\beta_0,\beta_1)}{\partial \beta_1} = \sum_{i=1}^{n}\left(y_i \frac{t_i e^{-e^{\beta_0+\beta_1 t_i}+\beta_0+\beta_1 t_i}}{1-e^{-e^{\beta_0+\beta_1 t_i}}} - t_i(m_i-y_i)e^{\beta_0+\beta_1 t_i}\right)$$

$$= \sum_{i=1}^{n}\left(y_i \frac{t_i e^{\beta_0+\beta_1 t_i}}{e^{e^{\beta_0+\beta_1 t_i}}-1} - t_i(m_i-y_i)e^{\beta_0+\beta_1 t_i}\right)$$

$$\frac{\partial L(\beta_0,\beta_1)}{\partial \beta_0^2} : \quad \frac{\partial}{\partial \beta_0}\, y_i\, e^{\beta_0+\beta_1 t_i}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}$$

$$u = e^{\beta_0+\beta_1 t_i} \qquad v = \left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}$$

$$u' = e^{\beta_0+\beta_1 t_i} \qquad v' = -e^{e^{\beta_0+\beta_1 t_i}+\beta_0+\beta_1 t_i}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-2}$$

$$uv' = -e^{e^{\beta_0+\beta_1 t_i}+2(\beta_0+\beta_1 t_i)}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-2} + e^{\beta_0+\beta_1 t_i}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}$$

$$= \left(e^{\beta_0+\beta_1 t_i}\right)\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}\left(1-e^{\beta_0+\beta_1 t_i+e^{\beta_0+\beta_1 t_i}}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}\right)$$

$$\frac{\partial L(\beta_0,\beta_1)}{\partial \beta_0^2} =$$

$$\sum_i\left[y_i\left(e^{\beta_0+\beta_1 t_i}\right)\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}\left(1-e^{\beta_0+\beta_1 t_i+e^{\beta_0+\beta_1 t_i}}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}\right)-(m_i-y_i)e^{\beta_0+\beta_1 t_i}\right]$$

$$= \sum_i\left[y_i \underbrace{(-\log(1-p_i))\left((1-p_i)^{-1}-1\right)^{-1}\left(1+\log(1-p_i)\,(1-p_i)^{-1}\left((1-p_i)^{-1}-1\right)^{-1}\right)} + (m_i-y_i)(\log(1-p_i))\right]$$

1st term simplified:

$$(-\log(1-p_i))\left(\frac{1-(1-p_i)}{(1-p_i)}\right)^{-1}\left(1+\frac{\log(1-p_i)}{(1-p_i)}\left(\frac{1-(1-p_i)}{1-p_i}\right)^{-1}\right)$$

$$= (-\log(1-p_i))\left(\frac{1-p_i}{p_i}\right)\left(1+\frac{\log(1-p_i)}{p_i}\right)$$

$$= (-\log(1-p_i))\left(\frac{1-p_i}{p_i}\right)\left(\frac{p_i+\log(1-p_i)}{p_i}\right)$$

$$= \frac{-\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{p_i^2}$$

$$= \sum_i\left[y_i \frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i-y_i)(\log(1-p_i))\right]$$

$$-E\left(\frac{\partial L(\beta_0,\beta_1)}{\partial \beta_0^2}\right)$$

$$= -E\left(\sum_i\left[y_i \frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i-y_i)(\log(1-p_i))\right]\right)$$

$$= -\sum_i\left[m_i p_i \frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i-m_i p_i)(\log(1-p_i))\right]$$

$$\frac{\partial L(\beta_0,\beta_1)}{\partial \beta_1^2} : \frac{\partial}{\partial \beta_1} y_i t_i e^{\beta_0+\beta_1 t_i} \left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-1}$$

$$u : e^{\beta_0+\beta_1 t_i} \qquad v : \left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-1}$$

$$u' : t_i e^{\beta_0+\beta_1 t_i} \qquad v' : -t_i e^{e^{\beta_0+\beta_1 t_i} + \beta_0+\beta_1 t_i}\left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-2}$$

$$uv' = -t_i e^{e^{\beta_0+\beta_1 t_i} + 2(\beta_0+\beta_1 t_i)}\left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-2} + t_i e^{\beta_0+\beta_1 t_i}\left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-1}$$

$$= t_i\left(e^{\beta_0+\beta_1 t_i}\right)\left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-1}\left(1 - e^{\beta_0+\beta_1 t_i + e^{\beta_0+\beta_1 t_i}}\left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-1}\right)$$

$$\frac{\partial L(\beta_0,\beta_1)}{\partial \beta_1^2} =$$

$$\Sigma\left[y_i t_i^2\left(e^{\beta_0+\beta_1 t_i}\right)\left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-1}\left(1 - e^{\beta_0+\beta_1 t_i + e^{\beta_0+\beta_1 t_i}}\left(e^{e^{\beta_0+\beta_1 t_i}} - 1\right)^{-1}\right) - t_i^2 (m_i - y_i) e^{\beta_0+\beta_1 t}\right]$$

$$= \Sigma_i\left[y_i t_i^2(-\log(1-p_i))\left((1-p_i)^{-1}-1\right)^{-1}\left(1 + (\log(1-p_i)\,(1-p_i)^{-1})\left((1-p_i)^{-1}-1\right)^{-1}\right) + (m_i - y_i)\, t_i^2(\log(1-p_i))\right]$$

$$= \Sigma_i\left[y_i t_i^2 \frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i-y_i)t_i^2(\log(1-p_i))\right]$$

$$-E\left(\frac{\partial L(\beta_0,\beta_1)}{\partial \beta_1^2}\right)$$

$$= -E\left(\Sigma_i\left[y_i t_i^2 \frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i-y_i)t_i^2(\log(1-p_i))\right]\right)$$

$$= -\Sigma_i\left[m_i p_i t_i^2 \frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i - m_i p_i)t_i^2(\log(1-p_i))\right]$$

$$\frac{\partial l(\beta_0,\beta_1)}{\partial\beta_0\partial\beta_1} : \quad \frac{\partial}{\partial\beta_1} y_i e^{\beta_0+\beta_1 t_i} \left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}$$

$$u = e^{\beta_0+\beta_1 t_i} \qquad v = \left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}$$

$$u' = t_i e^{\beta_0} \qquad v' = -t_i e^{e^{\beta_0+\beta_1 t_i}+\beta_0+\beta_1 t_i}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-2}$$

$$uv' = -t_i e^{e^{\beta_0+\beta_1 t_i}+2(\beta_0+\beta_1 t_i)}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-2} + t_i e^{\beta_0+\beta_1 t_i}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}$$

$$= t_i\left(e^{\beta_0+\beta_1 t_i}\right)\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}\left(1 - e^{\beta_0+\beta_1 t_i + e^{\beta_0+\beta_1 t_i}}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}\right)$$

$$\frac{\partial l(\beta_0,\beta_1)}{\partial\beta_0\beta_1} =$$

$$\sum_i\left[y_i t_i\left(e^{\beta_0+\beta_1 t_i}\right)\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}\left(1 - e^{\beta_0+\beta_1 t_i + e^{\beta_0+\beta_1 t_i}}\left(e^{e^{\beta_0+\beta_1 t_i}}-1\right)^{-1}\right) - t_i(m_i-y_i)e^{\beta_0+\beta_1 t_i}\right]$$

$$= \sum_i\left[y_i t_i(-\log(1-p_i))\left((1-p_i)^{-1}-1\right)^{-1}\left(1+\log(1-p_i)(1-p_i)^{-1}\left((1-p_i)^{-1}-1\right)^{-1}\right) + t_i(m_i-y_i)(\log(1-p_i))\right]$$

$$= \sum_i\left[y_i t_i\frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i-y_i)t_i(\log(1-p_i))\right]$$

$$-E\left(\frac{\partial l(\beta_0,\beta_1)}{\partial\beta_0\beta_1}\right)$$

$$= -E\left(\sum_i\left[y_i t_i\frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i-y_i)t_i(\log(1-p_i))\right]\right)$$

$$= -\sum_i\left[m_i p_i t_i\frac{\log(1-p_i)(1-p_i)(p_i+\log(1-p_i))}{-p_i^2} + (m_i-m_i p_i)t_i(\log(1-p_i))\right]$$

$$\text{so} \quad I(\vec{\theta}) = \begin{bmatrix} -E\left(\frac{\partial^2 l(\vec{\beta})}{\partial\beta_0^2}\right) & -E\left(\frac{\partial^2 l(\vec{\beta})}{\partial\beta_0\partial\beta_1}\right) \\ -E\left(\frac{\partial^2 l(\vec{\beta})}{\partial\beta_0\partial\beta_1}\right) & -E\left(\frac{\partial^2 l(\vec{\beta})}{\partial\beta_1^2}\right) \end{bmatrix}$$

as per above derivation

$$se(\hat{\beta}_0) = \sqrt{I(\vec{\theta})^{-1}_{11}}$$
$$se(\hat{\beta}_1) = \sqrt{I(\vec{\theta})^{-1}_{22}} \qquad \text{as } \hat{\vec{\theta}} \overset{d}{\approx} N(\vec{\theta}^*, I(\vec{\theta})^{-1})$$

c)

Set model with only beta0 as our reduced model and model with beta0 and beta1 as full model.

-2*(ll_reduced - ll_full) ~ chisq_df=(2-1)=1.

H0: beta1 = 0; H1: beta1 != 0

```
paste('Full Model Log Likelihood')

## [1] "Full Model Log Likelihood"

(MaxlogL.F = logL(betahat,orings))

## [1] -26.93787

y <- orings$damage
n <- rep(6, length(y))
phatN <- sum(y)/sum(n)
paste('Reduced Model Log Likelihood')

## [1] "Reduced Model Log Likelihood"

(MaxlogL.R = sum(orings$damage)*log(phatN) + sum(6-orings$damage)*log(1-phat
N))

## [1] -38.3724

paste('Chisq statistic')

## [1] "Chisq statistic"

(LR = -2*(MaxlogL.R - MaxlogL.F))

## [1] 22.86905

paste('pvalue:')

## [1] "pvalue:"

pchisq(LR, df=1,lower=FALSE)

## [1] 1.734217e-06
```

p-value < 0.05, so reject H0 and hence conclude the temperature coefficient beta1 is significant according to likelihood ratio test.

d)

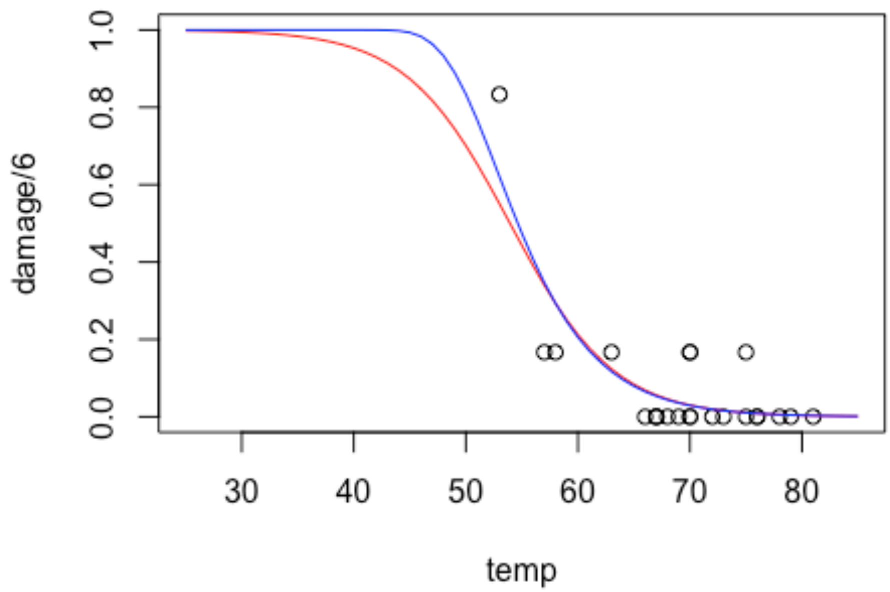
First create estimate and confidence interval for eta_hat, then transform the confidence interval using inverse of log-log function.

```
si2 <- matrix(c(1, 31), 1, 2) %*% Iinv %*% matrix(c(1, 31), 2, 1)
etahat = betahat[1] + betahat[2]*31
eta_l = etahat - 2*sqrt(si2)
eta_r = etahat + 2*sqrt(si2)
paste('Estimate of probability:')

## [1] "Estimate of probability:"

(iloglog(etahat))

## [1] 1

paste('Confidence interval of estimate')

## [1] "Confidence interval of estimate"

c(iloglog(eta_l), iloglog(eta_r))

## [1] 0.9977337 1.0000000
```

The estimated probability for damage when temperature = 31 degF is 1, with confidence interval (0.998, 1).

e)

```
logitmod = glm(cbind(damage, 6-damage)~temp, family=binomial, orings)
glm_betahat = logitmod$coefficients
ilogit = function(x) exp(x)/(1+exp(x))
plot(damage/6~temp, orings, xlim=c(25, 85), ylim = c(0,1))
lines(x, ilogit(glm_betahat[1]+glm_betahat[2]*x), col='red')
lines(x, iloglog(betahat[1]+betahat[2]*x), col='blue')
```



The logit model's probabilities (red line) increases slower as temp decreases compared to log-log (blue line), which quickly rises to prob=1. However the two lines crosses over around temp = 58F. Thus log-log will predict higher probabilities for failure for lower temperatures compared to logit (more conservative) in the lower temperatures, but when temperature > 58F the logit model will predict slightly higher probabilities until both lines converge to predicting 0.

QUESTION 2

Setup

```
library(faraway)
missing = with(pima, missing <- glucose==0 | diastolic==0 | triceps==0 | bmi=
=0)
pima_subset = pima[!missing, c(6,9)]
str(pima_subset)
```

```
## 'data.frame':    532 obs. of  2 variables:
##  $ bmi : num  33.6 26.6 28.1 43.1 31 30.5 30.1 25.8 45.8 43.3 ...
##  $ test: int  1 0 0 1 1 1 1 1 1 0 ...
```

a)

log(o) = log(p/(1-p)) = eta = beta0 + beta1*bmi because we are using logit link.

```
logit_model = glm(cbind(test, 1-test) ~ bmi, family=binomial, pima_subset)

summary(logit_model)
```

```
##
## Call:
## glm(formula = cbind(test, 1 - test) ~ bmi, family = binomial,
##     data = pima_subset)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -1.9227  -0.8920  -0.6568   1.2559   1.9560
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept) -4.03681    0.52783  -7.648 2.04e-14 ***
## bmi          0.09972    0.01528   6.524 6.84e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 676.79  on 531  degrees of freedom
## Residual deviance: 627.46  on 530  degrees of freedom
## AIC: 631.46
##
## Number of Fisher Scoring iterations: 4
```

beta0 = intercept = -4.03357392 beta1 = bmi coefficient = 0.09963413

```r
5*logit_model$coefficients[2]

##       bmi
## 0.4985842
```

Thus o increases by (5*bmi coefficient = 0.49858) when bmi increases by 5.

b)

```r
5*(logit_model$coefficients[2] + c(-1, 1) * 0.01528 * qnorm(0.975))

## [1] 0.3488430 0.6483255
```

Hence, the 95% Confidence Interval for the estimate of change_of_log(odds) = (0.3488, 0.648) when bmi increases by 5.

QUESTION 3

3. The inverse Gaussian distribution has p.d.f.

$$f(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

for $x > 0$, where $\mu > 0$ and $\lambda > 0$.

a. $f(x;\theta,\phi) = \exp\left[\frac{x\theta - b(\theta)}{a(\phi)} + c(x,\phi)\right]$

$$f(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

$$= \exp\left(\log\left(\left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}}\right)\right)\exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

$$= \exp\left(\frac{1}{2}\log\left(\frac{\lambda}{2\pi x^3}\right) - \frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

$$= \exp\left(\frac{1}{2}\log\lambda - \frac{1}{2}\log(2\pi x^3) - \frac{\lambda(x^2-2x\mu+\mu^2)}{2\mu^2 x}\right)$$

$$= \exp\left(\frac{1}{2}\log\lambda - \frac{1}{2}\log(2\pi x^3) - \frac{\lambda x^2}{2\mu^2 x} + \frac{2\lambda x\mu}{2\mu^2 x} - \frac{\mu^2\lambda}{2\mu^2 x}\right)$$

$$= \exp\left(\frac{1}{2}\log\lambda - \frac{1}{2}\log(2\pi x^3) - \frac{\lambda x}{2\mu^2} + \frac{\lambda}{\mu} - \frac{\lambda}{2x}\right)$$

$$= \exp\left(-\frac{\lambda x}{2\mu^2} - \left(-\frac{\lambda}{\mu}\right) + \left(\frac{1}{2}\log\lambda - \frac{\lambda}{2x} - \frac{1}{2}\log(2\pi x^3)\right)\right)$$

$$= \exp\left(\frac{-\frac{x}{2\mu^2} - \left(-\frac{1}{\mu}\right)}{\lambda^{-1}} + \left(\frac{1}{2}\log\lambda - \frac{\lambda}{2x} - \frac{1}{2}\log(2\pi x^3)\right)\right)$$

$$= \exp\left(\frac{-\frac{x}{\mu^2} - \left(-\frac{2}{\mu}\right)}{2\lambda^{-1}} + \left(\frac{1}{2}\log\left(\frac{\lambda}{2\pi x^3}\right) - \frac{\lambda}{2x}\right)\right)$$

$\theta = -\frac{1}{\mu^2}$ $\quad\quad$ $\phi = \lambda$

$b(\theta) = -\frac{2}{\mu} = -2\sqrt{-\theta}$ $\quad\quad$ $a(\phi) = 2\lambda^{-1} = \frac{2}{\phi}$

$c(x,\phi) = \frac{1}{2}\log\left(\frac{\lambda}{2\pi x^3}\right) - \frac{\lambda}{2x} = \frac{1}{2}\log\left(\frac{\phi}{2\pi x^3}\right) - \frac{\phi}{2x}$

So Inverse Gaussian belongs to the exponential family

b. $\theta = -\frac{1}{\mu^2}$

Variance function:

$b(\theta) = -2(-\theta)^{\frac{1}{2}}$

$\mu = b'(\theta) = (-\theta)^{-\frac{1}{2}}$ $\quad\quad$ $\mu = (-\theta)^{-\frac{1}{2}}$

$\mu^{-2} = (-\theta) \Rightarrow -\mu^{-2} = \theta \Rightarrow b'^{-1}(\mu) = -\mu^{-2}$

$b''(\theta) = \frac{1}{2}(-\theta)^{-\frac{3}{2}}$

$v(\mu) = b''(b'^{-1}(\mu)) = \frac{1}{2}(-(-\mu^{-2}))^{-\frac{3}{2}} = \frac{1}{2}(\mu^{-2})^{-\frac{3}{2}} = \frac{1}{2}\mu^3$

Canonic Link

$g(\mu) = \theta \Rightarrow g$ is canonical link

$b'(\theta) = (-\theta)^{-\frac{1}{2}} = \mu$

$\mu^{-2} = -\theta$

$-\mu^{-2} = \theta$

$\Rightarrow g(\mu) = -\mu^{-2}$