A FIXED-PARAMETER ALGORITHM FOR THE MINIMUM MANHATTAN NETWORK PROBLEM

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ABSTRACT. A Manhattan network for a finite set P of n points in the plane is a geometric graph such that its vertex set contains P, its edges are axis-parallel and non-crossing and, for any two points p and q in P, there exists a path in the network connecting p and q whose length equals the l_1-distance between p and q. The problem of computing a Manhattan network of minimum total edge length for a given point set P has recently been shown to be NP-hard. In this paper, using as the parameter the minimum number h of horizontal straight lines that contain the points in P, we present a fixed-parameter algorithm for this problem running in $O^*(2^{14h})$ time (neglecting a factor that is polynomial in n) and note that, under the exponential time hypothesis for 3-SAT, a run time that is subexponential in h is impossible.

1 Introduction

A *Manhattan network* is a finite graph $\mathcal{N} = (V, E)$ whose vertex set $V \subseteq \mathbb{R}^2$ consists of a set of points in the plane such that, for every edge $e = \{p, q\}$ in E, the straight line segment $\overline{p, q}$ with endpoints p and q is either horizontal or vertical and, for any two distinct edges $e_1 = \{p_1, q_1\}$ and $e_2 = \{p_2, q_2\}$ in E, the straight line segments $\overline{p_1, q_1}$ and $\overline{p_2, q_2}$ do not *cross*, that is, $\overline{p_1, q_1} \cap \overline{p_2, q_2} \subseteq e_1 \cap e_2$ holds. Defining the *length* $\ell(e)$ of an edge $e = \{p, q\}$ in E to be the l_1-distance $l_1(p, q)$ between the points p and q and, as usual, the length $\ell(\mathbf{p})$ of a path $\mathbf{p} = p_0, p_1, \dots, p_k$ in $\mathcal{N}$ as the sum $\sum_{i=1}^{k} \ell(\{p_{i-1}, p_i\})$ of the lengths of the edges on $\mathbf{p}$, we call $\mathbf{p}$ a *monotone path* in $\mathcal{N}$ if $l_1(p_0, p_k) = \ell(\mathbf{p})$ holds. In addition, given a finite set of points $P \subseteq \mathbb{R}^2$, we define a *Manhattan network for* P as a Manhattan network $\mathcal{N} = (V, E)$ with $P \subseteq V$ such that for any two distinct $p, q \in P$ there exists a monotone path from p to q in $\mathcal{N}$ (an example of such a network is depicted in Figure 1(a)). Finally, such a network is called *minimum* if its total edge length $\lambda(\mathcal{N})$ is minimum among all Manhattan networks for P.

The problem of computing a minimum Manhattan network for a given point set P, known in the literature as the *minimum Manhattan network problem (MMN)*, has only recently been shown to be NP-hard and also the existence of an FPTAS for MMN would imply P=NP [8]. Most previous work focused on approximation algorithms for MMN [4, 5, 7, 13, 14, 15, 16, 19]. All these algorithms yield a constant-factor approximation. The best approximation factor so far, namely 2, is guaranteed for an algorithm based on rounding the solution of a suitable linear program [7], for an algorithm based on applying the

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primal-dual method to the same linear program [18], and for a greedy algorithm [14]. The latter two algorithms run in $O(n \log n)$ time for a set P containing n points. Other previous work on MMN includes the generalization of the problem to 3-dimensional space [17] and in [7] a variant of MMN was suggested where only a certain subset of the pairs of points in P must be connected by a monotone path in the Manhattan network. A generalization of this variant to 3-dimensional space is studied in [10].

Concerning exact algorithms for MMN, an approach based on a mixed-integer linear program formulation is presented in [5], where it is also posed as an open problem to design a fixed-parameter algorithm for MMN. In this paper we will present such an algorithm using as the parameter the minimum number $h = h(P)$ of horizontal straight lines such that every point in P is contained in one of these straight lines. This parameter has previously been used to design fixed-parameter algorithms for problems similar to MMN such as, for example, the *Minimum Rectilinear Steiner Tree problem (MRST)*. For MRST, an algorithm running in $O^*(16^h)$ time is outlined in [1] and in [6] the run time has been improved to $O^*(10^h)$ (the O^*-notation neglects polynomial factors). Interestingly, the parameter h has also been considered in the context of parameterizing problems that admit a polynomial time algorithm but are hard to parallelize (see e.g. [12]). The parameter h is also related to the concept of r-outerplanar graphs introduced in [3], which proved useful in the design of fixed-parameter algorithms on planar graphs (see e.g. [2]), and to the number of layers used as a parameter in layered graph drawing problems (see e.g. [9]).

The paper is structured as follows. In Section 2, after introducing some more notation, we present a fixed-parameter algorithm for MMN. While, in view of the existing fixed-parameter algorithms for MRST, it is probably not surprising that MMN is also fixed-parameter tractable with respect to the parameter h, it is not immediately clear that a run time in $O^*(c^h)$ for some constant c can be achieved. This is the main result of the paper and will be established in Section 3. We conclude in the last section, briefly observing that a run time that is subexponential in h is impossible under the exponential time hypothesis for 3-SAT. The reader not familiar with this hypothesis or parameterized complexity theory is referred to [11].

2 Description of the algorithm

As above, let P denote the given set of points in the plane. For any point $q \in \mathbb{R}^2$ we denote by $x(q)$ and $y(q)$ the x- and y-coordinate of q, respectively. Let $x_1, \dots, x_l$ denote the increasingly sorted sequence of x-coordinates in $X := \{x(p) : p \in P\}$ and define, for any $i \in \{1, \dots, l\}$, the set $P_i = \{p \in P : x(p) \leq x_i\}$. Similarly, let $y_1, \dots, y_h$ denote the increasingly sorted sequence of y-coordinates in $Y := \{y(p) : p \in P\}$. Note that, by definition, the y-coordinates in Y correspond to the h horizontal straight lines $L_1, \dots, L_h$, $L_j := \{q \in \mathbb{R}^2 : y(q) = y_j\}$, $j \in \{1, \dots, h\}$, that contain the point set P. In the following we will use the fact that there always exists a minimum Manhattan network $\mathcal{N} = (V, E)$ for P with $V \subseteq X \times Y$ (see e.g. [7, Lemma 2.1]), that is, $\mathcal{N}$ is a subgraph of the grid induced by the points in P (cf. Figure 1(b)).

Our algorithm sweeps over the point set P from left to right. For every $i \in \{1, \dots, l\}$

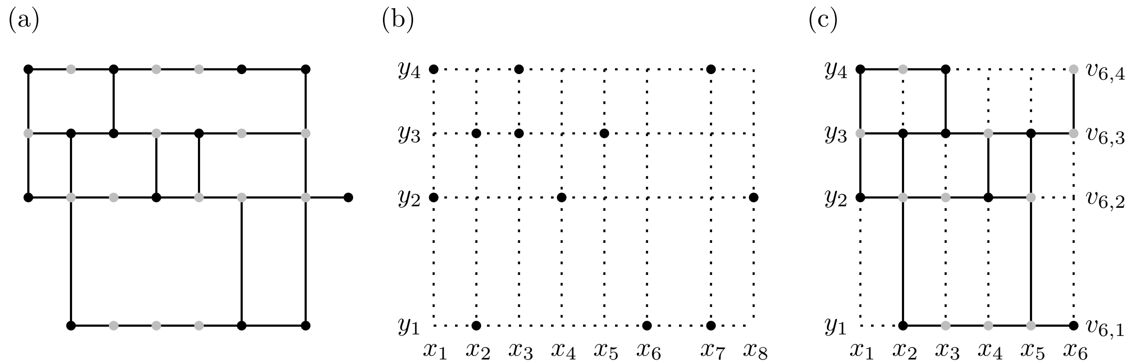

Figure 1: (a) An example of a point set P and a Manhattan network for P. The points in P are indicated by black dots. The other vertices of the Manhattan network are indicated by gray dots. (b) The grid induced by the points in P. (c) A Manhattan network for P_6 used to illustrate the idea underlying the construction of the collection $\mathcal{C}_6$.

we compute a collection $\mathcal{C}_i$ of Manhattan networks for P_i. Define the points $v_{i,j} := (x_i, y_j)$, $j \in \{1, \ldots, h\}$, and put $V_i := \{v_{i,j} : 1 \leq j \leq h\}$. For each Manhattan network $\mathcal{N}$ in C_i we keep track of those points v in V_i that are vertices of $\mathcal{N}$ and, in case they are, for which points q in P_i there exists a monotone path from v to q in $\mathcal{N}$. This information will be used in the computation of the collection $\mathcal{C}_{i+1}$ from the collection $\mathcal{C}_i$. For example, in Figure 1(c) a Manhattan network $\mathcal{N}$ for P_6 is depicted that contains, in addition to the point $v_{6,1}$ that must be a vertex of $\mathcal{N}$ by definition, the points $v_{6,3}$ and $v_{6,4}$ as vertices but not the point $v_{6,2}$. The points q in P_6 for which there exists a monotone path in $\mathcal{N}$ from $v_{6,4}$ to q are precisely those with $x(q) \leq x_5$ and $y(q) \leq y_3$.

To describe the collection $\mathcal{C}_i$ more formally, define, for all $i \in \{1, \ldots, l\}$ and all $j \in \{1, \ldots, h\}$, the set $R_{i,j}$ containing the rightmost point in P_i that lies on line L_j, denoted by $r_{i,j}$, in case such a point exists. Otherwise the set $R_{i,j}$ is empty. So, the set $R_{i,j}$ is either empty or contains a single point. If it is not empty the point $r_{i,j}$ represents all points q in P_i with $x(q) \leq x(r_{i,j})$ and $y(q) = y(r_{i,j})$ in the following sense: For any point $q' \in \mathbb{R}^2$ with $x(q') \geq x(r_{i,j})$, a Manhattan network for P_i that contains a monotone path from q' to $r_{i,j}$ also contains a monotone path from q' to any point q represented by $r_{i,j}$. In particular, this implies the following fact that will be used in our algorithm.

Observation 1. *When describing for a Manhattan network $\mathcal{N}$ in $\mathcal{C}_i$ the points q in P_i for which there exists a monotone path from $v_{i,j}$ to q in $\mathcal{N}$, $j \in \{1, \ldots, h\}$, it suffices to keep track of those points q in $\cup_{k=1}^{h} R_{i,k}$ for which such a path exists.*

Next, motivated by Observation 1, we introduce some notation to capture for which points q in $\cup_{k=1}^{h} R_{i,k}$ there exists a monotone path from $v_{i,j}$, $j \in \{1, \ldots, h\}$, to q in a Manhattan network $\mathcal{N}$ for P_i. It will be convenient to further distinguish, among the points q in $\cup_{k=1}^{h} R_{i,k}$ for which such a path exists, between those with $y(q) \geq y(v_{i,j})$ and those with $y(q) < y(v_{i,j})$. To this end, we put, for every $i \in \{1, \ldots, l\}$, $R_i := \{j \in \{1, \ldots, h\} : R_{i,j} \neq \emptyset\}$ and define a pair $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ in which, for all $j \in \{1, \ldots, h\}$, A_j is a subset of $\{j, j+1, \ldots, h\} \cap R_i$ and B_j is a subset of $\{1, 2, \ldots, j-$

ComputeMMN(P)

Input:	a set $P \subseteq \mathbb{R}^2$ of n points
Output:	a minimum Manhattan network for P

1. Compute $X = \{x_1, \ldots, x_l\}$ and $Y = \{y_1, \ldots, y_h\}$.
2. Initialize an empty collection $\mathcal{C}_1$.
3. **for each** pair Π that is admissible for P_1 **do**
4.   Compute a minimum Manhattan network $\mathcal{N}$ for P_1 and Π.
5.   Add $\mathcal{N}$ to $\mathcal{C}_1$.
6. **for** $i = 1$ to $l - 1$ **do**
7.   Initialize an empty collection $\mathcal{C}_{i+1}$.
8.   **for each** $\mathcal{N}$ in $\mathcal{C}_i$ **do**
9.   **for each** $H \subseteq E_{i+1}$ **do**
10.   Form the Manhattan network $\mathcal{N}'$ by adding the edges in H to $\mathcal{N}$.
11.   **if** $\mathcal{N}'$ is a Manhattan network for P_{i+1} **then**
12.   **if** $\mathcal{C}_{i+1}$ contains a Manhattan network $\mathcal{N}''$ with $\pi(\mathcal{N}') = \pi(\mathcal{N}'')$ **then**
13.   **if** $\lambda(\mathcal{N}'') > \lambda(\mathcal{N}')$ **then**
14.   Remove $\mathcal{N}''$ from $\mathcal{C}_{i+1}$ and add $\mathcal{N}'$ to $\mathcal{C}_{i+1}$.
15.   **else**
16.   Add $\mathcal{N}'$ to $\mathcal{C}_{i+1}$.
17. **return** a Manhattan network $\mathcal{N}$ in $\mathcal{C}_l$ with $\lambda(\mathcal{N})$ minimum.

Figure 2: Pseudocode for our algorithm for computing a minimum Manhattan network.

$1\} \cap R_i$ to be *admissible for* P_i. The collection $\mathcal{C}_i$ contains, for a set of certain pairs $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ that are admissible for P_i and that we will describe below, a Manhattan network that has minimum total edge length among those Manhattan networks for P_i that contain, for all $j \in \{1, \ldots, h\}$ and all $k \in A_j \cup B_j$, a monotone path from $v_{i,j}$ to $r_{i,k}$. We call such a network a *minimum Manhattan network for* P_i *and* Π. Note that, conversely, one can associate with every Manhattan network $\mathcal{N}$ for P_i a *canonical* pair $\pi(\mathcal{N}) = ((A_1(\mathcal{N}), \ldots, A_h(\mathcal{N})), (B_1(\mathcal{N}), \ldots, B_h(\mathcal{N})))$ that is admissible for P_i by putting $A_j(\mathcal{N})$ to be the set of those $k \in R_i$ for which there exists a monotone path in $\mathcal{N}$ from $v_{i,j}$ to $r_{i,k}$ and $y(r_{i,k}) \geq y(v_{i,j})$ holds, and, similarly, putting $B_j(\mathcal{N})$ to be the set of those $k \in R_i$ for which there exists a monotone path in $\mathcal{N}$ from $v_{i,j}$ to $r_{i,k}$ and $y(r_{i,k}) < y(v_{i,j})$ holds, $j \in \{1, \ldots, h\}$. For example, with the Manhattan network for P_6 depicted in Figure 1(c), is associated the canonical admissible pair $((\{1,2,3,4\}, \emptyset, \{3,4\}, \emptyset), (\emptyset, \emptyset, \{2\}, \{2,3\}))$.

Our algorithm for computing a minimum Manhattan network for a given point set P is summarized in Figure 2. After computing the sets of coordinates X and Y (line 1), which provides the information needed to perform the sweep, the collection $\mathcal{C}_1$ is computed (lines 2-5). For $\mathcal{C}_1$ we consider every pair $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ that is admissible for P_1. To compute a minimum Manhattan network for P_1 and Π we first construct the set $U(\Pi) := \{v_{1,j} \in V_1 : A_j \cup B_j \neq \emptyset\}$, that is, the set of those points in V_1 that must be

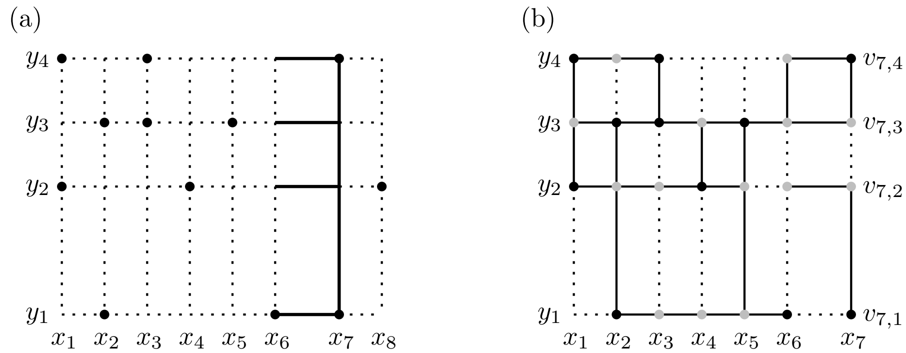

Figure 3: (a) The edges of the grid induced by P that are contained in E_7 are drawn bold. (b) The Manhattan network obtained by adding a specific subset $H \subseteq E_7$ to the Manhattan network in Figure 1(c). In this case, the resulting network is not a Manhattan network for P_7.

connected to some point in P_1 in the Manhattan network. Then we compute the points v_{min} and v_{max} in $P_1 \cup U(\Pi)$ with minimum and maximum y-coordinate, respectively. It is not hard to see that the minimum Manhattan network $\mathcal{N} = (V, E)$ for P_1 and Π is unique with $V := \{v \in V_1 : y(v_{min}) \leq y(v) \leq y(v_{max})\}$ and E containing the edges of the grid induced by P connecting the vertices in V.

The actual sweep is described in lines 6-16. The basic idea is to consider each Manhattan network in the collection $\mathcal{C}_i$ and extend it by adding edges from the set (cf. Figure 3(a))

$$E_{i+1} := \{\{v_{i,j}, v_{i+1,j}\} : j \in \{1, \ldots, h\}\} \cup \{\{v_{i+1,j}, v_{i+1,j+1}\} : j \in \{1, \ldots, h-1\}\}.$$

So, for each $\mathcal{N}$ in $\mathcal{C}_i$ and each subset $H \subseteq E_{i+1}$ we obtain a Manhattan network $\mathcal{N}'$ that is a subgraph of the grid induced by the points in P_{i+1} (cf. Figure 3(b)). Among the Manhattan networks $\mathcal{N}'$ obtained by this construction, we are only interested in those that are a Manhattan network for P_{i+1}. For these we distinguish two cases. If the collection $\mathcal{C}_{i+1}$ contains already a Manhattan network $\mathcal{N}''$ for P_{i+1} whose associated canonical pair $\pi(\mathcal{N}'')$ equals $\pi(\mathcal{N}')$ and, in addition, the total edge length $\lambda(\mathcal{N}')$ of $\mathcal{N}'$ is smaller than the total edge length $\lambda(\mathcal{N}'')$ of $\mathcal{N}''$, then we replace $\mathcal{N}''$ by $\mathcal{N}'$ in $\mathcal{C}_{i+1}$. Otherwise we simply add $\mathcal{N}'$ to $\mathcal{C}_{i+1}$. As can be seen, the set of admissible pairs Π for P_{i+1} for which $\mathcal{C}_{i+1}$ contains a Manhattan network $\mathcal{N}$ with $\Pi = \pi(\mathcal{N})$ is implicitly determined by our algorithm. When the sweep has finished, we return a Manhattan network of minimum total length in $\mathcal{C}_l$ (line 17). This concludes the description of our algorithm.

3 Analysis of the algorithm

Before we start the analysis, we introduce some more notation that is motivated by the fact that, as we will see below, not all admissible pairs are actually relevant. To make this more precise, let $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ and $\Pi' = ((A'_1, \ldots, A'_h), (B'_1, \ldots, B'_h))$ be two admissible pairs for P_i, $i \in \{1, \ldots, l\}$. We say that Π *makes* Π' *redundant* if $A'_j \subseteq A_j$ and

$B'_j \subseteq B_j$ holds for all $j \in \{1, \ldots, h\}$ and every minimum Manhattan network for P_i and Π' is also a minimum Manhattan network for P_i and Π. Intuitively, viewing Π' as being formed by removing elements from some of the sets A_j or B_j in Π, the total edge length of a minimum Manhattan network for P_i and Π' might be strictly smaller than the total edge length of a minimum Manhattan network for P_i and Π. However, if it stays the same then considering Π' is not really necessary because with Π even more connections via monotone paths from points in V_i to points in P_i are guaranteed without increasing the total edge length.

The following lemma forms the basis of the correctness proof for our algorithm. In it we refer to the concept of a set $\mathcal{A}$ of admissible pairs for P_i, $i \in \{1, \ldots, l\}$, to be a *cover*, that is, for every pair Π' that is admissible for P_i there exists some pair Π in $\mathcal{A}$ that makes Π' redundant.

Lemma 1. *Let $i \in \{1, 2, \ldots, l-1\}$. Assume that $\mathcal{C}_i$ is such that $\mathcal{A}_i := \{\pi(\mathcal{N}) : \mathcal{N} \in \mathcal{C}_i\}$ is a cover and for every $\Pi \in \mathcal{A}_i$ there is a minimum Manhattan network for P_i and Π in $\mathcal{C}_i$. Then there exists, for every pair $\widetilde{\Pi} = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ that is admissible for P_{i+1}, a Manhattan network $\mathcal{N}$ in $\mathcal{C}_i$ and a subset $H \subseteq E_{i+1}$ such that the Manhattan network $\mathcal{N}'$ formed by adding the edges in H to $\mathcal{N}$ is a minimum Manhattan network for P_{i+1} and $\widetilde{\Pi}$.*

Proof: Consider an arbitrary minimum Manhattan network $\widetilde{\mathcal{N}} = (\widetilde{V}, \widetilde{E})$ for P_{i+1} and $\widetilde{\Pi} = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$. Let $\mathcal{N}^* = (V^*, E^*)$ denote the subnetwork of $\widetilde{\mathcal{N}}$ induced by those vertices $v \in \widetilde{V}$ with $x(v) \leq x_i$. Since $\widetilde{\mathcal{N}}$ is a Manhattan network for P_{i+1}, the network $\mathcal{N}^*$ is a Manhattan network for P_i.

Now, consider the canonical pair $\Pi^* = ((A^*_1, \ldots, A^*_h), (B^*_1, \ldots, B^*_h)) := \pi(\mathcal{N}^*)$ associated to $\mathcal{N}^*$ that is admissible for P_i. Since, by assumption, $\mathcal{A}_i$ is a cover, Π^* is made redundant by some $\Pi \in \mathcal{A}_i$. Moreover, also by assumption, the collection $\mathcal{C}_i$ contains a minimum Manhattan network $\mathcal{N}$ for P_i and Π and, thus, $\lambda(\mathcal{N}^*) \geq \lambda(\mathcal{N})$ must hold.

Next recall that our algorithm will consider the network $\mathcal{N}'$ formed by adding to $\mathcal{N}$ the edges in the set $H := E_{i+1} \cap \widetilde{E}$ consisting of those edges in E_{i+1} that are edges of the network $\widetilde{\mathcal{N}}$. We claim that $\mathcal{N}'$ is a minimum Manhattan network for P_{i+1} and $\widetilde{\Pi}$.

First we show that $\mathcal{N}'$ is a Manhattan network for P_{i+1}: Since $\mathcal{N}$ is a Manhattan network for the points in P_i and both $\widetilde{\mathcal{N}}$ and $\mathcal{N}'$ contain the same subset H of edges in E_{i+1}, the only possibility for $\mathcal{N}'$ to fail to be a Manhattan network for P_{i+1} is that there exists some $j \in \{1, \ldots, h\}$ and a point $q \in P_i$ such that $v_{i+1,j}$ is a point in P_{i+1} and there is no monotone path from $v_{i+1,j}$ to q in $\mathcal{N}'$. To see that this situation cannot occur, consider an arbitrary $q \in P_i$ and an arbitrary $v \in V_{i+1} \cap P_{i+1}$. Let k be such that $y(q) = y_k$. Since $\widetilde{\mathcal{N}}$ is a Manhattan network for P_{i+1}, it contains a monotone path $\mathbf{p}$ from v to $r_{i,k}$. Let u denote the first vertex on $\mathbf{p}$ that we meet when we walk along $\mathbf{p}$ from v to $r_{i,k}$ which belongs to V_i. Then the network $\mathcal{N}^*$ contains a monotone path from u to $r_{i,k}$. Thus, by the definition of the canonical pair Π^* and the fact that Π^* is made redundant by Π, it follows that the network $\mathcal{N}$ contains a monotone path from u to $r_{i,k}$. Furthermore, since $\mathcal{N}$ is a Manhattan network for P_i, there exists a monotone path from $r_{i,k}$ to q in $\mathcal{N}$. Thus, $\mathcal{N}$ contains a monotone path from u to q and, therefore, $\mathcal{N}'$ contains a monotone path from v to q. Hence $\mathcal{N}'$ is indeed a Manhattan network for P_{i+1}.

A similar argument establishes that the network $\mathcal{N}'$ satisfies the additional requirements imposed by the admissible pair $\widetilde{\Pi}$: Consider any $j \in \{1, \ldots, h\}$ and any $k \in A_j \cup B_j$. The network $\widetilde{\mathcal{N}}$ contains a monotone path $\mathbf{p}$ from $v_{i+1,j}$ to $r_{i+1,k}$. If this path $\mathbf{p}$ uses only edges in E_{i+1} it is, by construction, also contained in $\mathcal{N}'$. Otherwise let u denote the first vertex on $\mathbf{p}$ that we meet when we walk along $\mathbf{p}$ from $v_{i+1,j}$ to $r_{i+1,k}$ which belongs to V_i. There is a monotone path from u to $r_{i+1,k}$ in $\mathcal{N}^*$ and, therefore, by construction, also in $\mathcal{N}$. This yields a monotone path from $v_{i+1,j}$ to $r_{i+1,k}$ in $\mathcal{N}'$, as required.

Finally, as noted above, $\lambda(\mathcal{N}^*) \geq \lambda(\mathcal{N})$ must hold. Therefore, in view of the facts that (i) $\mathcal{N}^*$ is by construction a subnetwork of $\widetilde{\mathcal{N}}$ that does not contain any edge in E_{i+1} and (ii) $\mathcal{N}'$ and $\widetilde{\mathcal{N}}$ contain the same subset H of edges in E_{i+1}, we must also have $\lambda(\widetilde{\mathcal{N}}) \geq \lambda(\mathcal{N}')$. Since $\widetilde{\mathcal{N}}$ is a minimum Manhattan network for P_{i+1} and $\widetilde{\Pi}$, this immediately implies that also $\mathcal{N}'$ is a minimum Manhattan network for P_{i+1} and $\widetilde{\Pi}$, as claimed. □

We continue with a simple observation that will be used later on.

Lemma 2. *Let $i \in \{1, \ldots, l\}$, let $\Pi' = ((A'_1, \ldots, A'_h), (B'_1, \ldots, B'_h))$ be a pair that is admissible for P_i and let $\mathcal{N}$ be a minimum Manhattan network for P_i and Π'. Then Π' is made redundant by $\pi(\mathcal{N})$.*

Proof: By definition of a minimum Manhattan network for P_i and Π', the network $\mathcal{N}$ must contain a monotone path from $v_{i,j}$ to $r_{i,k}$ for all $k \in A'_j \cup B'_j$ and all $j \in \{1, \ldots, h\}$. But this immediately implies that $A'_j \subseteq A_j(\mathcal{N})$ and $B'_j \subseteq B_j(\mathcal{N})$ holds for all $j \in \{1, \ldots, h\}$. This implies that $\pi(\mathcal{N})$ makes Π' redundant in view of the fact that $\mathcal{N}$ is a minimum Manhattan network for P_i and Π' as well as for P_i and $\pi(\mathcal{N})$. □

Now we are in a position to establish the correctness of our algorithm.

Corollary 1. *The collection $\mathcal{C}_i$ constructed by our algorithm, $i \in \{1, \ldots, l\}$, is such that*

(*) $\mathcal{A}_i := \{\pi(\mathcal{N}) : \mathcal{N} \in \mathcal{C}_i\}$ is a cover, and, for every $\Pi \in \mathcal{A}_i$, there is a minimum Manhattan network for P_i and Π in $\mathcal{C}_i$.

Proof: We use induction on i. For $i = 1$ our algorithm considers all pairs Π that are admissible for P_1 and computes a minimum Manhattan network $\mathcal{N}$ for P_1 and Π. This implies that (*) holds for $\mathcal{C}_1$.

So, assume that (*) has been established for some $\mathcal{C}_i$, $i \in \{1, \ldots, l-1\}$, and consider $\mathcal{C}_{i+1}$. Let $\widetilde{\Pi}$ be an arbitrary pair that is admissible for P_{i+1}. Using induction and Lemma 1, it follows that there exists a Manhattan network $\mathcal{N}$ in $\mathcal{C}_i$ and a subset $H \subseteq E_{i+1}$ such that the Manhattan network $\mathcal{N}'$ obtained by adding the edges in H to $\mathcal{N}$ is a minimum Manhattan network for P_{i+1} and $\widetilde{\Pi}$. By Lemma 2, $\widetilde{\Pi}$ is made redundant by $\pi(\mathcal{N}')$. Since $\widetilde{\Pi}$ was chosen arbitrarily, this immediately implies that $\mathcal{A}_{i+1}$ is a cover. Moreover, as the above argument includes the special case where $\widetilde{\Pi}$ is an element of $\mathcal{A}_{i+1}$, it also follows that, for every $\Pi \in \mathcal{A}_{i+1}$, there is a minimum Manhattan network for P_i and Π in $\mathcal{C}_{i+1}$ □

Corollary 2. *The collection $\mathcal{C}_l$ contains a minimum Manhattan network for P.*

Proof: Let $\widetilde{\mathcal{N}}$ be a minimum Manhattan network for P. Consider the canonical pair $\widetilde{\Pi} := \pi(\widetilde{\mathcal{N}})$ associated to $\widetilde{\mathcal{N}}$. By Corollary 1 and Lemma 1, there exists a Manhattan network $\mathcal{N}$ in $\mathcal{C}_{l-1}$ and a subset $H \subseteq E_l$ such that the network $\mathcal{N}'$ obtained by adding the edges in H to $\mathcal{N}$ is a minimum Manhattan network for $P_l = P$ and $\widetilde{\Pi}$. But then $\mathcal{N}'$ is also a minimum Manhattan network for P, as required. □

In the remainder of this section we focus on bounding the run time and space for our algorithm. The key ingredient will be to describe the admissible pairs that occur in $\mathcal{A}_i = \{\pi(\mathcal{N}) : \mathcal{N} \in \mathcal{C}_i\}$ in such a way that a suitable upper bound on $|\mathcal{A}_i|$ can be derived. Intuitively, for every admissible pair $\Pi = ((A_1, \dots, A_h), (B_1, \dots, B_h))$ in $\mathcal{A}_i$, we associate with each non-empty set A_j, $j \in \{1, \dots, h\}$, an axis-parallel rectangle K_j whose upper left corner is at (x_1, y_h) and that contains precisely those among the points in $\{r_{i,k} : k \in \bigcup_{s=1}^{h} A_s\}$ for which $k \in A_j$. Similarly, we associate with each non-empty set B_j, $j \in \{1, \dots, h\}$, an axis-parallel rectangle K'_j with lower left corner at (x_1, y_1) such that K'_j contains precisely those among the points in $\{r_{i,k} : k \in \bigcup_{s=1}^{h} B_s\}$ for which $k \in B_j$.

Note that, at first glance, the smallest axis-parallel rectangle with upper left corner at (x_1, y_h) that contains the point set $\{r_{i,k} : k \in A_j\}$ might appear to be a good choice for K_j (cf. Lemma 3 below). Similarly, for K'_j one might choose the smallest axis-parallel rectangle with lower left corner at (x_1, y_1) that contains the point set $\{r_{i,k} : k \in B_j\}$. With a more careful choice of these rectangles, however, we can achieve that the x- and y-coordinates of the lower right corners of the rectangles describing the non-empty sets A_j, $j \in \{1, \dots, h\}$, as well as the x- and y-coordinates of the upper right corners of the rectangles describing the non-empty sets B_j, $j \in \{1, \dots, h\}$, form monotone sequences, a fact that is subsequently used to establish an upper bound on $|\mathcal{A}_i|$. To present the construction of these sequences, next we introduce some more notation where we use some special element $\perp$ not contained in $\mathbb{R}$ to indicate that the corresponding sets A_j and B_j, $j \in \{1, \dots, h\}$, are empty and, therefore, have no rectangle associated with it.

So, put $X_i := \{x(r_{i,j}) : j \in R_i\}$, that is, the set of the x-coordinates of the rightmost point in P_i on each line L_j. A sequence $z_1, z_2, \dots, z_h$ of elements in $X_i \cup \{\perp\}$ is called *increasing on* X_i if, for all $j_1, j_2 \in \{1, \dots, h\}$ with $j_1 \leq j_2$, $z_{j_1} \neq \perp$ and $z_{j_2} \neq \perp$, we have $z_{j_1} \leq z_{j_2}$. Similarly, a sequence $z'_1, z'_2, \dots, z'_h$ of elements in $Y \cup \{\perp\}$ is called *increasing* (*decreasing*) *on* Y if, for all $j_1, j_2 \in \{1, \dots, h\}$ with $j_1 \leq j_2$, $z'_{j_1} \neq \perp$ and $z'_{j_2} \neq \perp$, we have $z'_{j_1} \leq z'_{j_2}$ ($z'_{j_1} \geq z'_{j_2}$). Consider, for example, the point set P_5 depicted in Figure 4(a). We have $X_5 = \{x_2, x_3, x_4, x_5\}$ and $\perp, x_4, x_5, \perp$ is an increasing sequence on it.

In addition, we define a 6-tuple $\Phi = (S_1, S'_1, S_2, S'_2, A, B)$ *on* X_i *and* Y, consisting of two increasing sequences $S_1 = a_1, \dots, a_h$ and $S_2 = b_1, \dots, b_h$ on X_i, an increasing sequence $S'_1 = a'_1, \dots, a'_h$ on Y, a decreasing sequence $S'_2 = b'_1, \dots, b'_h$ on Y and two subsets A and B of R_i, to be *compatible* if $a_j = \perp \Leftrightarrow a'_j = \perp$ and $b_j = \perp \Leftrightarrow b'_j = \perp$ hold for all $j \in \{1, \dots, h\}$. Note that the purpose of the sequences S_1 and S'_1 is to list the x- and y-coordinates of the lower right corner of the rectangle K_j associated with any non-empty set A_j, $j \in \{1, \dots, h\}$ in a pair $\Pi = ((A_1, \dots, A_h), (B_1, \dots, B_h))$ that is admissible for P_i. Similarly, S_2 and S'_2 list the x- and y-coordinates of the upper right corner of the rectangle K'_j associated with any non-empty set B_j, $j \in \{1, \dots, h\}$, in Π. Moreover, A and B represent the sets $\bigcup_{k=1}^{h} A_k$ and $\bigcup_{k=1}^{h} B_k$, respectively. Referring again to Figure 4(a), putting $S_1 = \perp, x_4, x_5, \perp$, $S'_1 = \perp, y_2, y_3, \perp$, $S_2 = x_2, x_4, x_5, \perp$, $S'_2 = y_4, y_3, y_3, \perp$, $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, we obtain an example of a compatible 6-tuple Φ on $\{x_2, x_3, x_4, x_5\}$ and $\{y_1, y_2, y_3, y_4\}$.

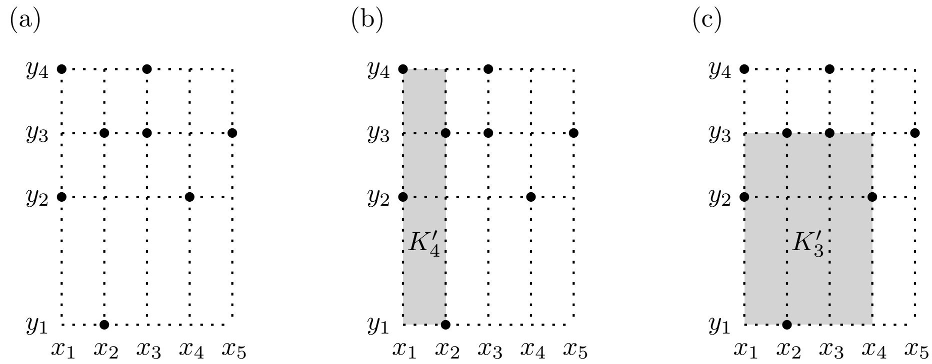

Figure 4: (a) The grid induced by P_5, where P is the point set already used in Figure 1. (b) The rectangle K'_4 describing the set B_4. (c) The rectangle K'_3 describing the set B_3.

Next, while we are actually interested in constructing a compatible 6-tuple that captures the information in a given admissible pair, it might be helpful to first describe how, conversely, compatible 6-tuples on X_i and Y give rise to pairs that are admissible for P_i. More specifically, for any compatible 6-tuple Φ on X_i and Y, define the pair $\pi(\Phi) = ((A_1, \dots, A_h), (B_1, \dots, B_h))$ by putting, for all $j \in \{1, \dots, h\}$, $A_j := \emptyset$ in case $a_j = a'_j = \perp$ and $A_j := \{k \in A : x(r_{i,k}) \leq a_j,\ y(r_{i,k}) \geq a'_j,\ k \geq j\}$ otherwise and, similarly, $B_{h-j+1} := \emptyset$ in case $b_j = b'_j = \perp$ and $B_{h-j+1} := \{k \in B : x(r_{i,k}) \leq b_j,\ y(r_{i,k}) \leq b'_j,\ k < h-j+1\}$ otherwise. It follows immediately from the definition that $\pi(\Phi)$ is admissible for P_i. To illustrate the definition of $\pi(\Phi)$, consider the compatible 6-tuple Φ on $\{x_2, x_3, x_4, x_5\}$ and $\{y_1, y_2, y_3, y_4\}$ constructed at the end of the previous paragraph. For this Φ we obtain $\pi(\Phi) = ((\emptyset, \{2\}, \{3\}, \emptyset), (\emptyset, \emptyset, \{2\}, \emptyset))$. As illustrated in Figure 4(b) and (c), the sets A_j and B_j, $j \in \{1, \dots, 4\}$, are indeed represented by axis-parallel rectangles. Consider, for example, the set B_4. The first elements of S_2 and S'_2 determine the x- and y-coordinate, respectively, of the upper right corner of the corresponding rectangle K'_4. As mentioned above, the lower left corner of K'_4 is at (x_1, y_1). Then (cf. Figure 4(b)), among the points $r_{5,k}$, $k \in R_5$, only $r_{5,1}$ lies in K'_4 and we even have $1 < 4$. But we have $1 \notin B = \{2, 4\}$. Therefore, we obtain $B_4 = \emptyset$. Similarly, we obtain $B_3 = \{2\}$ (cf. Figure 4(c)).

In the next lemma, using the construction alluded to above, we establish that, for any Manhattan network $\mathcal{N}$ for P_i, $i \in \{1, \dots, l\}$, and any non-empty set A_j, $j \in \{1, \dots, h\}$, in the canonical admissible pair $((A_1, \dots, A_h), (B_1, \dots, B_h)) = \pi(\mathcal{N})$, the elements in A_j can be described by some rectangle. The construction of some rectangle describing the elements in any non-empty set B_j, $j \in \{1, \dots, h\}$, in $\pi(\mathcal{N})$ is completely analogous.

Lemma 3. *Let $\mathcal{N}$ be a Manhattan network for P_i and $((A_1, \dots, A_h), (B_1, \dots, B_h)) = \pi(\mathcal{N})$ be the canonical pair that is admissible for P_i. Put $A := \cup_{k=1}^{h} A_k$ and let $j \in \{1, \dots, h\}$ be such that $A_j \neq \emptyset$. In addition, let $k_1 \in A_j$ be such that $x(r_{i,k_1}) = \max\{x(r_{i,k}) : k \in A_j\}$ and put $a_j := x(r_{i,k_1})$. Similarly, let $k_2 \in A_j$ be such that $y(r_{i,k_2}) := \min\{y(r_{i,k}) : k \in A_j\}$ and put $a'_j := y(r_{i,k_2})$. Then, for every $k \in A$ with $x(r_{i,k}) \leq a_j$ and $y(r_{i,k}) \geq a'_j$, a monotone path in $\mathcal{N}$ from $v_{i,j}$ to $r_{i,k}$ exists, that is, $k \in A_j$.*

Proof: To illustrate the construction described in the lemma, consider, for example, the

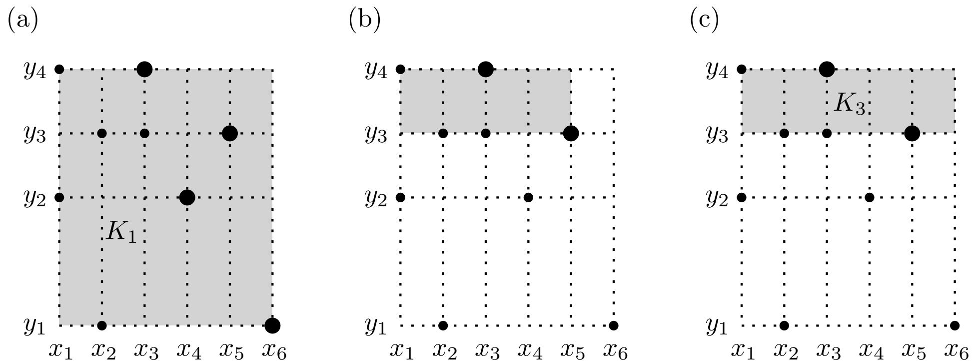

Figure 5: (a) The rectangle K_1 describing the set A_1 constructed in the proof of Lemma 3 for the Manhattan network in Figure 1(c). The points $r_{6,k}$ with $k \in A_1$ are marked as large black dots. (b) A possible rectangle to describe the set A_3, but it does not yield an increasing sequence S_1. The points $r_{6,k}$ with $k \in A_3$ are again marked as large black dots. (c) Stretching the rectangle in (b) to the right, we obtain the rectangle K_3 which also describes the set A_3 and, in addition, yields an increasing sequence S_1.

Manhattan network for P_6 depicted in Figure 1(c). We have $A = \{1, 2, 3, 4\}$ and, for $j = 1$, we obtain $k_1 = 1$ and $k_2 = 1$. This yields (x_6, y_1) as the lower right corner of the rectangle K_1 in this case, as depicted in Figure 5(a).

Now, to prove the lemma, assume for a contradiction that there exists some $k \in A$ with $x(r_{i,k}) \leq a_j$ and $y(r_{i,k}) \geq a'_j$ but $k \notin A_j$. We cannot have $x(r_{i,k}) \leq x(r_{i,k_2})$ in view of the fact that there must exist a monotone path from r_{i,k_2} to $r_{i,k}$ in $\mathcal{N}$, since $\mathcal{N}$ is a Manhattan network for P_i, and the concatenation of a monotone path from $v_{i,j}$ to r_{i,k_2} and a monotone path from r_{i,k_2} to $r_{i,k}$ yields a monotone path from $v_{i,j}$ to $r_{i,k}$ in $\mathcal{N}$. Similarly, we obtain a monotone path from $v_{i,j}$ to $r_{i,k}$ in case $y(r_{i,k}) \geq y(r_{i,k_1})$. So it remains to consider the case that $r_{i,k}$ is contained in the axis-parallel rectangle with lower left corner at r_{i,k_2} and upper right corner at r_{i,k_1}. Then, since $k \in A \setminus A_j$, there must exist some $j^* \in \{1, \dots, h\}$ such that $k \in A_{j^*}$. Hence, $\mathcal{N}$ must contain a monotone path $\mathbf{p}$ from v_{i,j^*} to $r_{i,k}$. We distinguish two cases:

Case 1: $j^* < j$. The situation is depicted in Figure 6(a). The path $\mathbf{p}$ has at least one vertex in common with any monotone path from $v_{i,j}$ to r_{i,k_2}. Therefore, $\mathcal{N}$ must also contain a monotone path from $v_{i,j}$ to $r_{i,k}$, a contradiction.

Case 2: $j^* > j$. The situation is depicted in Figure 6(b). Similarly to the previous case, the path $\mathbf{p}$ has at least one vertex in common with any monotone path from $v_{i,j}$ to r_{i,k_1}. Therefore, also in this case $\mathcal{N}$ must contain a monotone path from $v_{i,j}$ to $r_{i,k}$, a contradiction. $\square$

The following lemma establishes that the number of compatible 6-tuples on X_i and Y is an upper bound on $|\mathcal{C}_i|$.

Lemma 4. *There is an injective map φ from the set $\mathcal{A}_i$ into the set of compatible 6-tuples on X_i and Y such that $\pi(\varphi(\Pi)) = \Pi$ holds for all $\Pi \in \mathcal{A}_i$.*

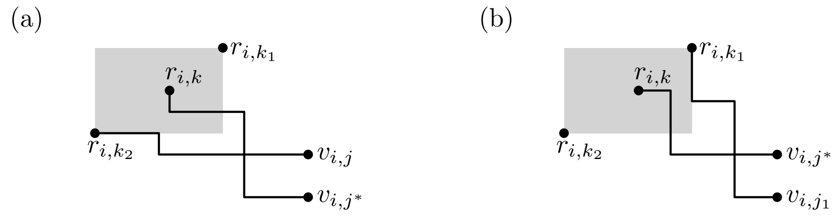

Figure 6: This illustrates the two cases considered in the proof of Lemma 3. In both cases, the two monotone paths depicted induce a monotone path from $v_{i,j}$ to $r_{i,k}$. (a) Case 1 (b) Case 2.

Proof: To describe the map φ, we will only show how, for an arbitrary pair $\Pi \in \mathcal{A}_i$, a suitable increasing sequence $S_1 = a_1, \dots, a_h$ on X_i, an increasing sequence $S'_1 = a'_1, \dots, a'_h$ on Y and a set $A \subseteq R_i$ can be constructed. Suitable sequences S_2 and S'_2 as well as a set B can then be constructed in a completely analogous way, yielding together the compatible 6-tuple $\Phi = (S_1, S'_1, S_2, S'_2, A, B) = \varphi(\Pi)$.

Let $\mathcal{N}$ denote the Manhattan network in $\mathcal{C}_i$ with $\Pi = ((A_1, \dots, A_h), (B_1, \dots, B_h)) := \pi(\mathcal{N})$. We put, for every $j \in \{1, \dots, h\}$ with $A_j = \emptyset$, $a_j = a'_j := \perp$ and put $A := \bigcup_{j=1}^{h} A_j$. If $A = \emptyset$, this finishes our construction. Otherwise, consider the smallest index $j_1 \in \{1, \dots, h\}$ with $A_{j_1} \neq \emptyset$. To construct a suitable axis-parallel rectangle K_{j_1} with upper left corner at (x_1, y_h) that contains precisely those points in $\{r_{i,k} : k \in A\}$ for which $k \in A_{j_1}$ holds, we use the construction described in Lemma 3. This will place the lower right corner of K_{j_1} at a suitable point. Recall that, to construct the coordinates of this point, we let $k_1 \in A_{j_1}$ be such that $x(r_{i,k_1}) = \max\{x(r_{i,k}) : k \in A_{j_1}\}$ and put $a_{j_1} := x(r_{i,k_1})$. Similarly, we let $k_2 \in A_{j_1}$ be such that $y(r_{i,k_2}) := \min\{y(r_{i,k}) : k \in A_{j_1}\}$ and put $a'_{j_1} := y(r_{i,k_2})$.

The basic idea for continuing our construction of S_1 and S'_1 is to next consider the smallest index $j_2 \in \{j_1 + 1, \dots, h\}$ with $A_{j_2} \neq \emptyset$, if such an index exists. Ideally, we would like to use again Lemma 3 to construct a suitable rectangle K_{j_2} describing the set A_{j_2}. This is, however, not always possible as illustrated in Figure 5(b). More specifically, the x- and y-coordinate of the lower right corner of the so constructed rectangle need not yield sequences S_1 and S'_1 that are increasing. In the remainder of the proof we will show that it is always possible to find some suitable rectangle K_{j_2} such that the resulting sequences S_1 and S'_1 are guaranteed to be increasing, as indicated in Figure 5(c).

To describe the construction in general, assume that we have already found values $a_{j_1}, a_{j_2}, \dots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \dots, a'_{j_s}$, $j_1 < j_2 < \dots < j_s$, up to some index $j_s \in \{1, \dots, h\}$, where $\{j_1, j_2, \dots, j_s\}$ consists of those indices $j \in \{1, 2, \dots, j_s\}$ with $A_j \neq \emptyset$, that satisfy the following properties:

(a) $a_{j_1} \leq a_{j_2} \leq \dots \leq a_{j_s}$ and $a'_{j_1} \leq a'_{j_2} \leq \dots \leq a'_{j_s}$, that is, we have increasing prefixes of S_1 and S'_1.

(b) The set A_j equals, for every $j \in \{j_1, j_2, \dots, j_s\}$, the set of those indices $k \in A$ with $x(r_{i,k}) \leq a_j$ and $y(r_{i,k}) \geq a'_j$, that is, we have found a suitable rectangle K_j with upper left corner at (x_1, y_h) and lower right corner at (a_j, a'_j) that describes A_j.

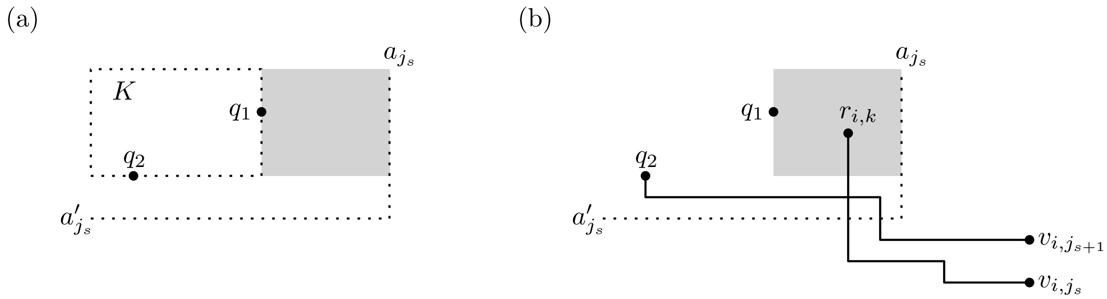

Figure 7: An example illustrating the argument in Case 2 in the proof of Lemma 4.

(c) For every $j \in \{j_1, j_2, \dots, j_s\}$ there exists some $j^* \in \{j_1, j_2, \dots, j_s\}$, $j^* \leq j$, such that $a_j = \max\{x(r_{i,k}) : k \in A_{j^*}\}$. This is a technical condition used in our construction.

Note that, for $s = 1$, Lemma 3 implies that Properties (a)-(c) hold. It is also not hard to check that, in case $A_j = \emptyset$ for every $j \in \{j_s{+}1, j_s{+}2, \dots, h\}$, Properties (a)-(c) imply that we have found suitable sequences $S_1 = a_1, \dots, a_h$ and $S'_1 = a'_1, \dots, a'_h$. So assume that there exists some $j \in \{j_s + 1, j_s + 2, \dots, h\}$ with $A_j \neq \emptyset$ and let j_{s+1} denote the smallest index in $\{j_s{+}1, j_s{+}2, \dots, h\}$ with this property. It suffices to show how suitable values $a_{j_{s+1}}$ and $a'_{j_{s+1}}$ can be found so that, after adjusting some of the values $a_{j_1}, a_{j_2}, \dots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \dots, a'_{j_s}$, if necessary, the resulting sequences $a_{j_1}, a_{j_2}, \dots, a_{j_s}, a_{j_{s+1}}$ and $a'_{j_1}, a'_{j_2}, \dots, a'_{j_s}, a'_{j_{s+1}}$ again satisfy Properties (a)-(c) (with s replaced by $s + 1$). To this end, similarly as for j_1, let $k_1 \in A_{j_{s+1}}$ be such that $x(r_{i,k_1}) = \max\{x(r_{i,k}) : k \in A_{j_{s+1}}\}$ and put $q_1 := r_{i,k_1}$. Similarly, let $k_2 \in A_{j_{s+1}}$ be such that $y(r_{i,k_2}) = \min\{y(r_{i,k}) : k \in A_{j_{s+1}}\}$ and put $q_2 := r_{i,k_2}$. By Lemma 3 the rectangle K with upper left corner at (x_1, y_h) and lower right corner at $(x(q_1), y(q_2))$ describes the set $A_{j_{s+1}}$. We divide our argument into four cases.

Case 1: $a_{j_s} \leq x(q_1)$ and $a'_{j_s} \leq y(q_2)$. Then, putting $a_{j_{s+1}} := x(q_1)$ and $a'_{j_{s+1}} := y(q_2)$, it is not hard to check that Properties (a)-(c) hold.

Case 2: $a_{j_s} > x(q_1)$ and $a'_{j_s} \leq y(q_2)$. The situation is depicted in Figure 7(a). We claim that there is no $k \in A$ with $x(q_1) < x(r_{i,k}) \leq a_{j_s}$ and $y(q_2) \leq y(r_{i,k})$, that is, no $r_{i,k}$ with $k \in A$ lies in the region indicated by shading in Figure 7(a). To show this, assume for a contradiction that there is such a $k \in A$. Note that, since Property (b) holds for $a_{j_1}, a_{j_2}, \dots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \dots, a'_{j_s}$, there exists a monotone path from v_{i,j_s} to $r_{i,k}$ in $\mathcal{N}$. This path has at least one vertex in common with any monotone path in $\mathcal{N}$ from $v_{i,j_{s+1}}$ to q_2, implying that there also exists a monotone path from $v_{i,j_{s+1}}$ to $r_{i,k}$ (cf. Figure 7(b)). But this contradicts the choice of k_1 above. Hence, putting $a_{j_{s+1}} := a_{j_s}$ and $a'_{j_{s+1}} := y(q_2)$, that is, stretching the rectangle K to the right, it is not hard to check that Properties (a)-(c) hold.

Case 3: $a_{j_s} \leq x(q_1)$ and $a'_{j_s} > y(q_2)$. The situation is depicted in Figure 8(a). Let $j^* \in \{j_1, j_2, \dots, j_s\}$ be such that $a_{j_s} = \max\{x(r_{i,k}) : k \in A_{j^*}\}$. Such an index j^* must exist, since Property (c) holds for $a_{j_1}, a_{j_2}, \dots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \dots, a'_{j_s}$. First note that there is no $k \in A$ with $x(r_{i,k}) \leq a_{j_s}$ and $y(q_2) \leq y(r_{i,k}) < a'_{j^*}$. Assume for a contradiction that such a k exists. Then, in view of the fact that $\mathcal{N}$ must contain a monotone path from v_{i,j^*} to a point $r_{i,t}$, $t \in A_{j^*}$, with maximum x-coordinate as well as a monotone path from $v_{i,j_{s+1}}$

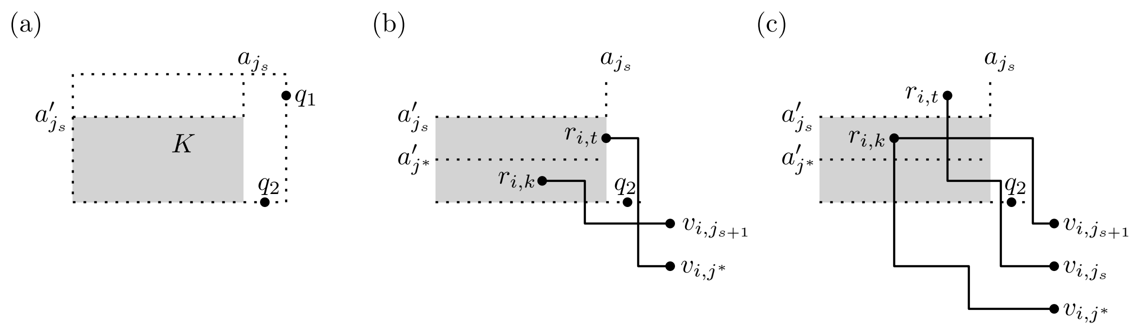

Figure 8: An example illustrating the argument in Case 3 in the proof of Lemma 4.

to $r_{i,k}$, it follows that $\mathcal{N}$ also contains a monotone path from v_{i,j^*} to $r_{i,k}$ (cf. Figure 8(b)), contradicting that Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$.

Similarly, there is no $k \in A$ with $x(r_{i,k}) \leq a_{j_s}$ and $\max\{y(q_2), a'_{j^*}\} \leq y(r_{i,k}) < a'_{j_s}$. Assume again for a contradiction that such a k exists. Then, in view of the fact that $\mathcal{N}$ must contain monotone paths from v_{i,j^*} and $v_{i,j_{s+1}}$ to $r_{i,k}$ as well as a monotone path from v_{i,j_s} to some point $r_{i,t}$ with $t \in A_{j_s}$, it follows that $\mathcal{N}$ also contains a monotone path from v_{i,j_s} to $r_{i,k}$ (cf. Figure 8(c)), again contradicting that Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$. As a consequence, putting $a_{j_{s+1}} := x(q_1)$, $a'_{j_{s+1}} := y(q_2)$ and, for every $j \in \{j_1, j_2, \ldots, j_s\}$ with $y(q_2) < a'_j$, $a'_j := y(q_2)$, that is, stretching the rectangle K_j downwards, it is again not hard to check that Properties (a)-(c) hold.

Case 4: $a_{j_s} > x(q_1)$ and $a'_{j_s} > y(q_2)$. The situation is depicted in Figure 9(a). As in Case 3, consider an index $j^* \in \{j_1, j_2, \ldots, j_s\}$ such that $a_{j_s} = \max\{x(r_{i,k}) : k \in A_{j^*}\}$. Then, in case $a'_{j^*} > y(q_2)$, there would be a monotone path from v_{i,j^*} to q_2 in $\mathcal{N}$, in view of the fact that $\mathcal{N}$ must contain a monotone path from v_{i,j^*} to a point $r_{i,t}$, $t \in A_{j^*}$, with maximum x-coordinate and also a monotone path from $v_{i,j_{s+1}}$ to q_2 (cf. Figure 9(b)). But this contradicts the fact that Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$.

And in case $a'_{j^*} \leq y(q_2)$, there would be a monotone path from v_{i,j_s} to q_2 in $\mathcal{N}$ in view of the fact that $\mathcal{N}$ must contain monotone paths from v_{i,j^*} and $v_{i,j_{s+1}}$ to q_2 and also a monotone path from v_{i,j_s} to some point $r_{i,t}$ with $t \in A_{j_s}$ (cf. Figure 9(c)). This also contradicts the fact that Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$. Hence Case 4 never applies. This finishes the case analysis.

To conclude the proof of the lemma, note that by construction we have indeed $\pi(\varphi(\Pi)) = \Pi$, implying that φ is injective. $\square$

We conclude this section summarizing the main result of this paper.

Theorem 1. *A minimum Manhattan network for P can be computed in $O^*(2^{14h})$ time and $O^*(2^{12h})$ space.*

Proof: The correctness of our algorithm was established in Corollary 2. To bound the space used by our algorithm, it suffices to give an upper bound on $|\mathcal{C}_i| = |\mathcal{A}_i|$, $i \in \{1, \ldots, l\}$. For $i = 1$, while it is convenient in the pseudocode of the algorithm to write that we loop through all pairs Π that are admissible for P_1, we have already observed in the description

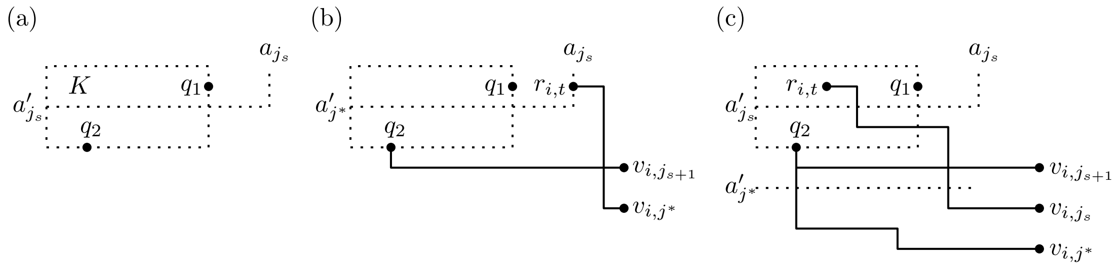

Figure 9: An example illustrating the argument in Case 4 in the proof of Lemma 4.

of the algorithm in the text that a minimum Manhattan network for P_1 and Π is uniquely determined and, therefore, the number of Manhattan networks that it suffices to consider for $i = 1$ can be bounded by $O(h^2)$: we simply choose the vertex with minimum and maximum y-coordinate in the network.

For $i > 1$, we rely on Lemma 4 and give an upper bound on the number of compatible 6-tuples on X_i and Y. We claim that the number of these 6-tuples is in $O^*(2^{12h})$. To show the claim, we first bound the number of pairs of sequences S_1 and S'_1 that can occur in a compatible 6-tuple $T = (S_1, S'_1, S_2, S'_2, A, B)$ on X_i and Y. Suppose there are precisely r entries in S_1 and S'_1 which equal $\perp$. The remaining entries of S_1 and S'_1, since they are sorted, correspond to a submultiset of X_i and Y, respectively, of size $h - r$. In view of $|X_i| \leq |Y| = h$, this implies that there are no more than $h \cdot 2^h \cdot 2^{2h} \cdot 2^{2h} = h \cdot 2^{5h}$ pairs S_1 and S'_1. An analogous argument yields the same bound on the number of pairs S_2 and S'_2. Finally, the number of subsets A and B is clearly bounded by 2^h each, implying that there are $O^*(2^{12h})$ compatible 6-tuples on X_i and Y, as claimed.

Finally, to bound the run time of our algorithm, note that there are at most 2^{2h} subsets H of E_{i+1} considered by our algorithm. Hence, using the fact that there are $O^*(2^{12h})$ Manhattan networks in $\mathcal{C}_i$ established above, it follows that the run time is in $O^*(2^{14h})$. $\square$

4 Concluding remarks

While here we were mainly concerned with establishing the existence of a fixed-parameter algorithm for MMN with a run time in $O^*(c^h)$ for some constant c, it could be interesting to try and refine the approach in order to design a fixed-parameter algorithm for MMN that is competitive to the existing exact algorithm for MMN based on integer linear programming. As for many other natural parameterizations, however, a subexponential run time in terms of h is ruled out under the exponential time hypothesis. This follows from the fact that in the reduction from 3-SAT to MMN presented in [8] a Boolean formula F with m literals is transformed into an input point set $P(F)$ for MMN that is contained in $O(m)$ horizontal straight lines. Hence, any algorithm for MMN running in subexponential time with respect to the parameter h would yield a subexponential time algorithm for 3SAT with respect to the parameter m.

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