Capstone report

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Abstract

In this report, we describe our final project ....

1 Current Progress

1.1 Options pricing models

1.1.1 Local Vol surface

The Local Vol model I am using is the parameterized SVI model provided by professor Gatheral. $\sigma_{SVI}(k,t)$ where k is the log strike and t is the current time. I modified it to take S_t and t to be used in the Monte-Carlo local vol pricer. $\sigma_{SVI}(k = log(\frac{S_t}{S_0}), t)$

1.1.2 Local Stochastic Vol model

The local stochatic volatility model we are using is

$$\frac{dS_t}{S_t} = \frac{\sigma_D(t,S_t)}{\sqrt{\mathbb{E}(e^{2Y_t^\epsilon}|S_t)}} e^{Y_t^\epsilon} dW_t \tag{1.1}$$

$$dY_t^\epsilon = -\frac{\kappa}{\epsilon} Y_t^\epsilon dt + \frac{\mu}{\sqrt{\epsilon}} dB_t$$

When $\lim_{\epsilon \to 0}$, $Y_t^\epsilon \sim N(0, \sigma_y^2)$ Where $\sigma_y^2 = \frac{\nu^2}{2\kappa}$

Therefore, When $\lim_{\epsilon \to 0}$ 1.1 can be written as

$$\frac{dS_t}{S_t} = \frac{\sigma_D(t,S_t)}{e^{\sigma_y^2}} e^{Y_t^\epsilon} dW_t \tag{1.2}$$

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1.1.3 Local Vol model

The local vol pricing model we are using is based on the process from 1.2

$$\frac{dS_t}{S_t} = \sigma_D(t, S_t) e^{Y - \sigma_y^2} dW_t \tag{1.3}$$

We denote the option price from this process with deformation term $Y = y$ as π_{LV}^y. When there's no deformation $Y = \sigma_y^2$ we denote the option price as π_{LV}.

1.2 Numerical result

The local vol model we are using in this part is 1.3

The theoretical LSV impact is

$$\pi_{LSV} - \pi_{LV} = \frac{1}{2}\sigma_y^2 \frac{\partial_E^2 \pi_{LV}^\beta}{\partial \beta^2}|_{\beta=\sigma_y^2} + \frac{\rho\nu}{\kappa}\frac{\partial_E^2 \pi_{LV}}{\partial \ln S_0 \partial \sigma} \tag{1.4}$$

Where

$$\frac{\partial_E^2 \pi_{LV}^\beta}{\partial \beta^2} = \frac{\partial^2 \pi_{LV}^\beta}{\partial \beta^2} - \int_0^T \int_0^\infty (q_{K,T} \frac{\partial^2 C_{LV}^{\beta,K,T}}{\partial \beta^2}) dK dT \tag{1.5}$$

$$\frac{\partial_E^2 \pi_{LV}}{\partial \ln S_0 \partial \sigma} = \frac{\partial^2 \pi_{LV}}{\partial \ln S_0 \partial \sigma} - \int_0^T \int_0^\infty (q_{K,T} \frac{\partial^2 C_{LV}^{K,T}}{\partial \ln S_0 \partial \sigma}) dK dT \tag{1.6}$$

1.2.1 Special scenarii to simplify computation

We remove the double integral in the above section by designing special scenarii

For exotic Volga, for the vanilla effect, we try to use an asymetric bump $x_{K,T}$ for different strikes and maturities to remove it

$$C_{LV}^{K,T,\sigma_y^2+bump} - 2C_{LV}^{K,T,\sigma_y^2} + C_{LV}^{K,T,\sigma_y^2-beta} = 0 \tag{1.7}$$

The effect of adding this asymetric bump can be modeled as moving the local volatility surface, because this bump is a multiplier for each point on the local vol surface $\sigma_D(t, S_t)$ as we showed in 1.3
We used a Black-Scholes solver to compute this new surface by using the following formula on each strike and maturity

$$\sigma_{BS,K,T} + x_{K,T} = C^{-1}(2C_{LV}^{K,T,\sigma_y^2} - C_{LV}^{K,T,\sigma_y^2-beta}) = 0 \tag{1.8}$$

$x_{K,T}$ is used as the bump for each K, T on the local vol surface. This method used the approximation that the bump on the local vol surface is the same as the resulted bump on the implied vol surface.

After getting the bumps for each strike and maturity, we construct a new local vol surface $\sigma_{K,T} + x_{K,T}$. The exotic volga is computed in the following way:

$$\frac{\partial_E^2 \pi_{LV}^{\beta}}{\partial \beta^2} = \lim_{\beta \to 0} \frac{\pi_{LV}(\sigma_{K,T} + x_{K,T}) - 2\pi_{LV} + \pi_{LV}^{-\beta}}{\beta^2} \tag{1.9}$$

The exotic vanna is computed in a similar way. We first get the bumps $x_{K,T}$ for each strike and maturity by using the following equation

$$0 = C_{LV}^{K,T}(S+\delta S, \sigma_{K,T}+\delta\sigma) - C_{LV}^{K,T}(S+\delta S, \sigma_{K,T}) - C_{LV}^{K,T}(S, \sigma_{K,T}+x_{K,T}) + C_{LV}^{K,T}(S, \sigma_{K,T}) \tag{1.10}$$

$C_{LV}^{K,T}(S + \delta S, \sigma_{K,T} + \delta\sigma)$ means a small change on spot price and an uniform bump on the whole local vol surface. Then we solve $x_{K,T}$, which is a single point bump by using Black-Scholes solver on each K and T

$$\sigma_{BS,K,T} + x_{K,T} = C_{LV}^{K,T}(S + \delta S, \sigma_{K,T} + \delta\sigma) - C_{LV}^{K,T}(S + \delta S, \sigma_{K,T}) + C_{LV}^{K,T}(S, \sigma_{K,T})$$

We again used the bump on the implied vol surface to apprximate the bump on the local vol surface.

Exotic vanna is computed in the following way

$$\frac{\partial_E^2 \pi_{LV}}{\partial \ln S_0 \partial\sigma} \tag{1.11}$$

$$= S_0 \frac{\pi_{LV}(S_0 + \delta S, \sigma_{K,T} + \delta\sigma) - \pi_{LV}(S_0 + \delta S, \sigma_{K,T}) - \pi_{LV}(S_0, \sigma_{K,T} + x_{K,T}) + \pi_{LV}(S_0, \sigma_{K,T})}{\delta S \delta\sigma} \tag{1.12}$$

1.2.2 Thoughts on current method

When finding the bump $x_{K,T}$ we used implied vol solver on each K and T.

One problem with this method is that we got the bump with implied vol surface instead of the local vol surface. The other problem is that BS implied vol solver only capture the bump on a specific (K,T) each time. A better method should be constructing a new local vol surface $\sigma_{K,T} + x_{K,T}$ directly with all the option prices getting from equation 1.7 and 1.10

1.3 Current result

With $S_0 = 1$, log strikes $k = log(K/S_0) \in (-0.6, 0.2)$, time to maturity $T \in (0, 1)$. The average exotic greeks are concluded in the table 1

Average Absolute Exotic Greeks across K,T	Exotic Volga	Exotic Vanna
Vanilla Call	0.511	5.12
Down and Out Call	9.64	62.9

Table 1: exotic greeks

2 Math proofs for the formulas

2.1 Singular pertubation

Considering the LSV process 1.1 as $lim\epsilon to 0$. Suppose the option price is $u(t,x,y)$. The payoff at maturity is $u(T,x,y)=h(T,x)$. By applying Feyman-Kac, we obtain the following PDE

$$u_t+\frac{1}{2}\sigma_D^2(t,x)x^2e^{2(y-\sigma_y^2)}u_xx+\frac{1}{\epsilon}\mathcal{L}_yu+\frac{1}{\sqrt{\epsilon}}\rho\nu x\sigma_D^2(t,x)e^{y-\sigma_y^2}u_{xy} \tag{2.1}$$

Where $\mathcal{L}_y=-\kappa\partial_y+\frac{1}{2}\nu^2\partial_{yy}$

We can make an expansion of u as $u=u_0+\sqrt{\epsilon}u_1+\epsilon u_2$, where u_0 is the LV price π_{LV}

By substituting this expansion into the PDE 2.1, we can match power of ϵ and solve the system of PDEs.

$$\begin{aligned}
O(\tfrac{1}{\epsilon}): &\quad \mathcal{L}_yu_0=0, u_0(T,x,y)=h(T,x)\\
O(\tfrac{1}{\sqrt{\epsilon}}): &\quad \mathcal{L}_yu_1+\rho\nu x\sigma_D(t,x)e^{y-\sigma_y^2}(u_0)_{xy}=0, u_1(T,x,y)=0\\
O(1): &\quad \mathcal{L}_yu_2+\rho\nu x\sigma_D(t,x)e^{y-\sigma_y^2}(u_1)_{xy}+(\partial_t+\frac{1}{2}x^2\sigma_D^2(t,x)e^{2(y-\sigma_y^2)})u_0=0, u_2(T,x,y)=0\\
O(\sqrt{\epsilon}): &\quad \mathcal{L}_yu_3+\rho\nu x\sigma_D(t,x)e^{y-\sigma_y^2}(u_2)_{xy}+(\partial_t+\frac{1}{2}x^2\sigma_D^2(t,x)e^{2(y-\sigma_y^2)})u_1=0
\end{aligned}$$

2.1.1 Order 1 term

By solving the PDE system we can get

$$\begin{aligned}
u_1 &= <e^{y-\sigma_y}\phi'(y)>(-\frac{1}{2}\frac{\rho\nu}{\kappa})\int_0^T S_t\sigma_D(t,S_t)\partial_x(S_t^2\sigma_D^2(t,S_t)\partial_{xx}u_0)dt\\
&= <e^{y-\sigma_y}\phi'(y)>(-\frac{1}{2}\frac{\rho\nu}{\kappa})\partial^2_{\ln x,\sigma}u_0
\end{aligned}$$

2.1.2 Order 0 and order 2 term

The non-calibrated LV process is 1.3

With equation $E(f(Y)) \approx f(E(Y)) + \frac{1}{2}Var(Y)f''(E(Y))$ If we take $E(f(Y))$ as the non calibrated LV price,then $f(E(Y))$ means the LV price with no deformation.

$$u_0 + u_1 \approx \pi_{LV} + \frac{1}{2}\sigma_y^2 \frac{\partial_E^2 \pi_{LV}^{\beta}}{\partial \beta^2}|_{\beta=\sigma_y^2} \tag{2.2}$$

2.2 Modified Newton

In the above section we get the non-calibrated price:

$$P^{NC} = \pi_{LV} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 \pi_{LV}^{\beta}}{\partial \beta^2}|_{\beta=\sigma_y^2} + \frac{\rho\nu}{\kappa}\frac{\partial^2 \pi_{LV}}{\partial \ln S_0 \partial \sigma} \tag{2.3}$$

We apply newton's method next to make sure our model fits the vanilla prices

$$f(x^*) \approx f(x) - \frac{\partial_x f(x)}{\partial_x g(x)} g(x) \tag{2.4}$$

$g(x) = C_{LV} - C_{market}$ is the constrain. $\frac{\partial_x f(x)}{\partial_x g(x)} = q_{KT}$ is the change rate of exotic over vanilla as the local vol surface moves Since we want to match vanilla price for different K and T we got the integral form, numerically it equals to an average.

$$P = P^{NC} - \int_0^T \int_0^{\infty} (C_{LV}^{K,T} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 C_{LV}^{\beta,K,T}}{\partial \beta^2}|_{\beta=\sigma_y^2} + \frac{\rho\nu}{\kappa}\frac{\partial^2 C_{LV}^{K,T}}{\partial \ln S_0 \partial \sigma} - C_{market}^{K,T}) dK dT \tag{2.5}$$

By using $C_{LV}^{K,T} = C_{market}^{K,T}$, we get the result

$$P - P^{NC} = \frac{1}{2}\sigma_y^2 \frac{\partial_E^2 \pi_{LV}^{\beta}}{\partial \beta^2}|_{\beta=\sigma_y^2} + \frac{\rho\nu}{\kappa}\frac{\partial_E^2 \pi_{LV}}{\partial \ln S_0 \partial \sigma} \tag{2.6}$$

Where the exotic Greeks are defined in 1.5 1.6

References

[1] Gatheral, J., The Volatility Surface: A Practitioner's Guide, Wiley Finance (2006).

[2] Gatheral, J., Hsu, E.P., Laurence, P., Ouyang, C., and Wang, T.-H., Asymptotics of implied volatility in local volatility models, *Mathematical Finance* (2011) forthcoming.