



Multiple View Geometry: Exercise Sheet 10

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Part I: Theory

1. Variational Calculus and Euler-Lagrange

- (a) Under the assumption that h vanishes at the boundary of Ω , prove that

$$\frac{dE(u)}{du} = \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} - \operatorname{div} \left(\frac{\partial \mathcal{L}(u, \nabla u)}{\partial (\nabla u)} \right).$$

We can expand $\mathcal{L}(u + \epsilon h, \nabla u + \epsilon \nabla h)$ in terms of ϵ :

$$\mathcal{L}(u + \epsilon h, \nabla u + \epsilon \nabla h) = \mathcal{L}(u, \nabla u) + \epsilon h \frac{\partial \mathcal{L}}{\partial u} + \epsilon \nabla h \frac{\partial \mathcal{L}}{\partial (\nabla u)} + \mathcal{O}(\epsilon^2)$$

Inserting into $\frac{\delta E(u)}{\delta u} \Big|_h$ gives

$$\frac{\delta E(u)}{\delta u} \Big|_h = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{\Omega} \left(\epsilon h(x) \frac{\partial \mathcal{L}}{\partial u} \Big|_{u(x)} + \epsilon \nabla h(x) \frac{\partial \mathcal{L}}{\partial (\nabla u)} \Big|_{\nabla u(x)} + \mathcal{O}(\epsilon^2) \right) dx$$

The ϵ in the first two terms cancels, and in the $\mathcal{O}(\epsilon^2)$ it will go to zero for $\epsilon \rightarrow 0$. Integration by parts of the second term yields

$$\begin{aligned} \int_{\Omega} \nabla h(x) \frac{\partial \mathcal{L}}{\partial (\nabla u)} dx &= \int_{\partial \Omega} h(x) \frac{\partial \mathcal{L}}{\partial (\nabla u)} ds - \int_{\Omega} h(x) \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial (\nabla u)} \right) dx = \\ &= - \int_{\Omega} h(x) \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial (\nabla u)} \right) dx. \end{aligned}$$

Thus,

$$\frac{\delta E(u)}{\delta u} \Big|_h = \int_{\Omega} h(x) \left(\frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial (\nabla u)} \right) \right) dx \Rightarrow \text{claim}.$$

- (b) Which condition must hold true for a minimizer u_0 of $E(u)$...

- ... in general?

$$\frac{dE(u)}{du} = 0 \Rightarrow \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} = \operatorname{div} \left(\frac{\partial \mathcal{L}(u, \nabla u)}{\partial (\nabla u)} \right).$$

- ... if $\mathcal{L}(u, \nabla u) = \mathcal{L}(u)$?

$$\frac{\partial \mathcal{L}(u)}{\partial u} = 0.$$

- ... if $\mathcal{L}(u, \nabla u) = \mathcal{L}(\nabla u)$?

$$\operatorname{div} \left(\frac{\partial \mathcal{L}(\nabla u)}{\partial(\nabla u)} \right) = 0 .$$

2. Multiview Reconstruction as Shape Optimization

- (a) Write down the Euler-Lagrange equation for the given energy $E(u)$.

The E-L equations are

$$\frac{dE(u)}{du} = -\operatorname{div} \left(\frac{\partial \mathcal{L}(\nabla u)}{\partial(\nabla u)} \right) = 0 \quad \text{with} \quad \mathcal{L}(\nabla u) = \rho |\nabla u|$$

Taking the derivative w.r.t. ∇u gives

$$0 = -\operatorname{div} \left(\rho(x) \frac{\nabla u(x)}{|\nabla u(x)|} \right) .$$

It is also possible (but not necessarily required) to expand this further using the product rule for divergence:

$$\operatorname{div} \left(\rho(x) \frac{\nabla u(x)}{|\nabla u(x)|} \right) = \nabla \rho(x)^\top \frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x) \operatorname{div} \left(\frac{\nabla u(x)}{|\nabla u(x)|} \right)$$

- (b) Write down one gradient descent iteration for $E(u)$.

$$u^{(k+1)}(x) = u^{(k)}(x) - \tau \frac{dE(u)}{du} = u^{(k)}(x) + \tau \left(\nabla \rho(x)^\top \frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x) \operatorname{div} \left(\frac{\nabla u(x)}{|\nabla u(x)|} \right) \right)$$