Chapter 2:

Intro To Anti-Derivatives

Chapter 2 Overview: Anti-Derivatives

As noted in the introduction, Calculus is essentially comprised of four operations:

- Limits
- Derivatives
- Indefinite Integrals (Or Anti-Derivatives)
- Definite Integrals

As mentioned above, there are two types of integrals — the definite integral and the indefinite integral. The definite integral was explored first as a way to determine the area bounded by a curve, rather than bounded by a polygon. The summation of infinite rectangles is

$$A = \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x,$$

and the representation

$$\int_a^b f(x)\, dx$$

is the exact amount, with $\displaystyle\int$ being an elongated and stylized s for "sum".

Newton and Leibnitz made the connection between the definite integral and the antiderivative, showing that the process of reversing the derivative results in an infinite summation. The antiderivative and indefinite integral are inverses of each other, just as squares and square roots or exponential and log functions. In this chapter, we will consider how to reverse the differentiation process. In a later chapter, we will dive deeper into the definite integral. Let's start by reviewing our derivative rules, as they will be necessary for us to take the antiderivative.

You must know the derivative rules in order to know the antiderivative rules!

The Power Rule: $\dfrac{d}{dx}\left[u^n\right] = nu^{n-1}\dfrac{du}{dx}$

The Product Rule: $\dfrac{d}{dx}\left[u \cdot v\right] = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$

The Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{u(x)}{v(x)}\right] = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$

The Chain Rule: $\dfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx}$$

$$\frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right] = \left(e^u\right)\frac{du}{dx}$$

$$\frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\csc^{-1} u\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cot^{-1} u\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

2.1: The Anti-Power Rule

As we have seen, we can deduce things about a function if its derivative is known. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called antidifferentiation, or integration.

Symbol for the Integral

$\int f(x)\,dx$ "the integral of f of x, d-x"

The dx is called the differential. For now, we will treat it as part of the integral symbol. It tells us the independent variable of the function [usually, but not always, x]. It does have a meaning on its own, but we will explore that later.

Looking at the integral as an antiderivative, we should be able to figure out the basic process. Remember:

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

and

$$\frac{d}{dx}[\text{constant}] = 0$$

It follows that if we are starting with the derivative and want to reverse the process, the power must increase by one and we should divide by this new power. Formally,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

The $+C$ is to account for any constant that might've been in the equation before the derivative was taken. Note that $n = -1$ does not work with this rule because it results in a division by zero. However, we know from our derivative rules that the derivative of $\ln x$ yields x^{-1}. Therefore, we can append our anti-power rule.

The Complete Anti-Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Since $\dfrac{d}{dx}[f(x)+g(x)] = \dfrac{d}{dx}[f(x)] + [g(x)]$ and $\dfrac{d}{dx}[cx^n] = c\dfrac{d}{dx}[x^n]$, it follows that:

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c(f(x))\,dx = c\int f(x)\,dx$$

These allow us to integrate a polynomial by integrating each term separately.

OBJECTIVES

Find the Anti-Derivative of a Polynomial.
Integrate Functions Using Transcendental Operations
Use Integration to Solve Rectilinear Motion Problems

Ex 2.1.1: $\displaystyle\int \left(3x^2+4x+5\right)\,dx$

Sol 2.1.1:

$$\begin{aligned}\int \left(3x^2+4x+5\right)\,dx &= 3\frac{x^{2+1}}{2+1} + 4\frac{x^{1+1}}{1+1} + 5\frac{x^{0+1}}{0+1} + C \\ &= \frac{3x^3}{3} + \frac{4x^2}{2} + \frac{5x^1}{1} + C \\ &= \boxed{x^3+2x^2+5x+C}\end{aligned}$$

Ex 2.1.2: $\displaystyle\int \left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)\,dx$

Sol 2.1.2:

$$\begin{aligned}\int \left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)\,dx &= \frac{x^{4+1}}{4+1} + \frac{4x^{2+1}}{2+1} + \frac{5x^{0+1}}{0+1} + \ln|x| - \frac{x^{-5+1}}{-5+1} + C \\ &= \boxed{\frac{1}{5}x^5 + \frac{4}{3}x^3 + 5x + \ln|x| + \frac{1}{4x^4} + C}\end{aligned}$$

Ex 2.1.3: $\displaystyle\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right)\,dx$

Sol 2.1.3:

$$\begin{aligned}
\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right)\,dx &= \int \left(x^2 + x^{\frac{1}{3}} - \frac{4}{x}\right)\,dx \\
&= \frac{x^{2+1}}{2+1} + \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 4\ln|x| + C \\
&= \boxed{\frac{1}{3}x^3 - \frac{3}{4}x^{\frac{4}{3}} + 4\ln|x| + C}
\end{aligned}$$

Integrals of products and quotients can be done easily IF they can be turned into a polynomial.

Ex 2.1.4: $\displaystyle\int \left(x^2 + \sqrt[3]{x}\right)(2x+1)\,dx$

Sol 2.1.4:

$$\begin{aligned}
\int \left(x^2 + \sqrt[3]{x}\right)(2x+1)\,dx &= \int \left(2x^3 + 2x^{\frac{4}{3}} + x^2 + x^{\frac{1}{3}}\right)\,dx \\
&= \frac{2x^4}{4} + \frac{2x^{\frac{7}{3}}}{\frac{7}{3}} + \frac{x^3}{3} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} \\
&= \boxed{\frac{1}{2}x^4 + \frac{6}{7}x^{\frac{7}{3}} + \frac{1}{3}x^3 + \frac{3}{4}x^{\frac{4}{3}} + C}
\end{aligned}$$

The next example is called an initial value problem. It has an ordered pair (or initial value pair) that allows us to solve for C.

Ex 2.1.5: $f'(x) = 4x^3 - 6x + 3$. Find $f(x)$ if $f(0) = 13$.

Sol 2.1.5:

$$\begin{aligned}
f(x) &= \int \left(4x^3 - 6x + 3\right)\,dx \\
&= x^4 - 3x^2 + 3x + C
\end{aligned}$$

$$f(0) = 0^4 - 3(0)^2 + 3(0) + C$$

$$= 13 \therefore C = 13$$

$$\therefore \boxed{f(x) = x^4 - 3x^2 + 3x + 13}$$

Now, let's take a look at a type of problem called a *rectilinear motion* problem. In these problems, we study the motion of an object moving along a straight line—its position, velocity, and acceleration.

Ex 2.1.6: The acceleration of particle is described by $a(t) = 3t^2 + 8t + 1$. Find the distance equation for $x(t)$ if $v(0) = 3$ and $a(0) = 1$.

Sol 2.1.6:

$$v(t) = \int a(t)\, dt$$

$$= \int \left(3t^2 + 8t + 1\right)\, dt$$

$$= t^3 + 4t^2 + t + C_1$$

$$3 = (0)^3 + 4(0)^2 + (0) + C_1 \therefore 3 = C_1$$

$$v(t) = t^3 + 4t^2 + t + 3$$

$$x(t) = \int v(t)\, dt$$

$$= \int \left(t^3 + 4t^2 + t + 3\right)\, dt$$

$$= \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + C_2$$

$$1 = \frac{1}{4}(0)^4 + \frac{4}{3}(0)^3 + \frac{1}{2}(0)^2 + 3(0) + C_2 \therefore 1 = C_2$$

$$\boxed{x(t) = \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + 1}$$

Ex 2.1.7: The acceleration of a particle is described by $a(t) = 12t^2 - 6t + 4$. Find the distance equation for $x(t)$ if $v(1) = 0$ and $x(1) = 3$.

Sol 2.1.7:

$$
\begin{aligned}
v(t) &= \int a(t)\, dt \\
&= \int \left(12t^2 - 6t + 4\right) \\
&= 4t^3 - 3t^2 - 4t + C_1
\end{aligned}
$$

$$0 = 4(1)^3 - 3(1)^2 + 4(1) + C_1 \therefore -5 = C_1$$

$$v(t) = 4t^3 - 3t^2 - 4t - 5$$

$$
\begin{aligned}
x(t) &= \int v(t)\, dt \\
&= \int \left(4t^3 - 3t^2 - 4t - 5\right)\, dt \\
&= t^4 - t^3 - 2t^2 - 5t + C_2
\end{aligned}
$$

$$3 = (1)^4 - (1)^3 - 2(1)^2 - 5(1) + C_2 \therefore 6 = C_2$$

$$\boxed{x(t) = t^4 - t^3 - 2t^2 - 5t + 6}$$

The proof of all the transcendental integral rules can be left to a more formal Calculus course. But, since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

Transcendental Integral Rules

$$\int \cos(u)\,du = \sin(u) + C \qquad \int \csc(u)\cot(u)\,du = -\csc(u) + C$$

$$\int \sin(u)\,du = -\cos(u) + C \qquad \int \sec(u)\tan(u)\,du = \sec(u) + C$$

$$\int \sec^2(u)\,du = \tan(u) + C \qquad \int \csc^2(u)\,du = -\cot(u) + C$$

$$\int e^u\,du = e^u + C \qquad \int \frac{1}{u}\,du = \ln|u| + C$$

$$\int a^u\,du = \frac{a^u}{\ln|a|} + C \qquad \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$$

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C \qquad \int \frac{1}{u\sqrt{u^2-1}}\,du = sec^{-1} + C$$

Note that there are only three integrals that yield inverse trig functions, but there were six inverse trig derivatives. This is because the other three derivative rules are just the negatives of the first three.

Ex 2.1.8: $\int (\sin(x) + 3\cos(x))\;dx$

Sol 2.1.8:

$$\begin{aligned}\int (\sin(x) + 3\cos(x))\;dx &= \int \sin(x)\,dx + 3\int \cos(x)\,dx \\ &= \boxed{-\cos(x) + 3\sin(x) + C}\end{aligned}$$

Ex 2.1.9: $\int \left(e^x + 4 + 3\csc^2(x)\right)\;dx$

Sol 2.1.9:

$$\int \left(e^x + 4 + 3\csc^2(x)\right)\;dx = \int e^x\,dx + 4\int\;dx + 3\int \csc^2(x)\,dx$$

$$= \boxed{e^x + 4x - 3\cot(x) + C}$$

Now, let's take a look at some more complex integrals that yield inverse trig functions. These more general forms extend the earlier rules by introducing a constant a, and they are especially useful when working with substitutions or integrals that don't simplify neatly to the unit case.

Trig Inverse Integral Rules

$$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u^2 + a^2}\, du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$

Ex 2.1.10: Find $\displaystyle\int \frac{1}{x^2+4}\, dx$

Sol 2.1.10: All we need to do is apply our formula above.

$$\int \frac{1}{x^2+4}\, dx = \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

Ex 2.1.11: If $\dfrac{dy}{dx} = \sec(x)(\sec(x)\tan(x))$, find $y(x)$ if $y(0) = 0$.

Sol 2.1.11:

$$\begin{aligned} y &= \int (\sec(x)(\sec(x)\tan(x)))\, dx \\ &= \int \left(\sec^2(x)\right)\, dx + \int (\sec(x)\tan(x))\, dx \\ &= \tan(x) + \sec(x) + C \end{aligned}$$

$$0 = \tan(0) + \sec(0) + C$$

$$0 = 0 + 1 + C \therefore C = -1$$

$$\boxed{y = \tan(x) + \sec(x) - 1}$$

2.1 Free Response Homework

Perform the antidifferentiation.

1. $\int \left(6x^2 - 2x + 3\right)\, dx$

2. $\int \left(x^3 + 3x^2 - 2x + 4\right)\, dx$

3. $\int \frac{2}{\sqrt[3]{x}}\, dx$

4. $\int \left(8x^4 - 4x^3 + 9x^2 + 2x + 1\right)\, dx$

5. $\int x^3 \left(4x^2 + 5\right)\, dx$

6. $\int (4x - 1)(3x + 8)\, dx$

7. $\int \left(\sqrt{x} - \frac{6}{\sqrt{x}}\right)\, dx$

8. $\int \frac{x^2 + \sqrt{x} + 3}{x}\, dx$

9. $\int (x + 1)^3\, dx$

10. $\int (4x - 3)^2\, dx$

11. $\int \left(\sqrt{x} + 3\sqrt{3} - \frac{6}{\sqrt{x}}\right)\, dx$

12. $\int \frac{4x^3 + \sqrt{x} + 3}{x^2}\, dx$

13. $\int \left(x^2 + 5x + 6\right)\, dx$

14. $\int \frac{x^2 - 4x + 7}{x}\, dx$

15. $\int \frac{x^5 - 7x^3 + 2x - 9}{2x}\, dx$

16. $\int \frac{x^3 + 3x^2 + 3x + 1}{x + 1}\, dx$

17. $\int \left(y^2 + 5\right)^2\, dy$

18. $\int \left(4t^2 + 1\right)\left(3t^3 + 7\right)\, dt$

Complete the following problems.

19. $f'(x) = 3x^2 - 6x + 3$. Find $f(x)$ if $f(0) = 2$.

20. $f'(x) = x^3 + x^2 - x + 3$. Find $f(x)$ if $f(1) = 0$.

21. $f'(x) = (\sqrt{x} - 2)(3\sqrt{x} + 1)$. Find $f(x)$ if $f(4) = 1$.

22. The acceleration of a particle is described by $a(t) = 36t^2 - 12t + 8$. Find the distance equation for $x(t)$ if $v(1) = 1$ and $x(1) = 3$.

23. The acceleration of a particle is described by $a(t) = t^2 - 2t + 4$. Find the distance equation for $x(t)$ if $v(0) = 2$ and $x(0) = 4$.

2.1 Multiple Choice Homework

1. $\displaystyle\int \frac{1}{x^2}\,dx =$

a) $\ln\left(x^2\right)+C$ b) $-\ln\left(x^2\right)+C$ c) $\dfrac{1}{x}+C$ d) $-\dfrac{1}{x}+C$ e) $-\dfrac{2}{x^3}+C$

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2. $\displaystyle\int x\left(10+8x^4\right)\,dx =$

a) $5x^2+\dfrac{4}{3}x^6+C$ b) $5x^2+\dfrac{8}{5}x^5+C$ c) $10x+\dfrac{4}{3}x^6+C$

d) $5x^2+8x^6+C$ e) $5x^2+\dfrac{8}{7}x^6+C$

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3. $\displaystyle\int x\sqrt{3x}\,dx$

a) $\dfrac{2\sqrt{3}}{5}x^{\frac{5}{2}}+C$ b) $\dfrac{5\sqrt{3}}{2}x^{\frac{5}{2}}+C$ c) $\dfrac{\sqrt{3}}{2}x^{\frac{1}{2}}+C$ d) $2\sqrt{3x}+C$ e) $\dfrac{5\sqrt{3}}{2}x^{\frac{3}{2}}+C$

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4. $\displaystyle\int (x-1)\sqrt{x}\,dx =$

a) $\dfrac{3}{2}\sqrt{x}-\dfrac{1}{\sqrt{x}}+C$ b) $\dfrac{2}{3}x^{\frac{2}{3}}+\dfrac{1}{2}x^{\frac{1}{2}}+C$ c) $\dfrac{1}{2}x^2-x+C$

d) $\dfrac{2}{5}x^{\frac{5}{2}}-\dfrac{2}{3}x^{\frac{3}{2}}+C$ e) $\dfrac{1}{2}x^2+2x^{\frac{2}{3}}+C$

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5. A particle is moving upward along the y-axis until it reaches the origin and then it moves downward such that $v(t)=8-2t$ for $t\geq 0$. The position of the particle at time t is given by

a) $y(t)=-t^2+8t-16$ b) $y(t)=-t^2+8t+16$ c) $y(t)=2t^2-8t-16$

d) $8t-t^2$ e) $8t-2t^2$

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6. If a particle's acceleration is given by $a(t)=12t+4$, and $v(1)=5$ and $y(0)=2$, then

$y(2) =$

a) 20   b) 10   c) 4   d) 16   e) 12

2.2: Integration by U-Substitution

Reversing the Power Rule was fairly easy. The other three core derivative rules—the Product Rule, the Quotient Rule, and the Chain Rule—are a little more complicated to undo. This is because they yield a more complicated function as a derivative, one which has several algebraic simplification steps. The integral of a rational function is particularly difficult to unravel because, as we have seen, rational derivatives can be obtained by differentiating a composite function with a log or a radical, or by differentiating another rational function. The same goes for reversing the Product Rule.

Key Idea: There is no single Product or Quotient Rule for integrals.

Instead, there are several techniques that apply in different situations, and it is not always obvious at the outset which one will be most effective. The choice depends on the algebraic manipulations that produced the product or quotient in the first place.

Products can be a result of:	Quotients can be the result of:
• The Chain Rule • Differentiating a product • Differentiating some trig functions	• Common denominators • Differentiating a quotient • Differentiating a log with a composite function • Differentiating some trig inverse functions

Composite functions are among the most pervasive functions in math. Therefore, we will start with undoing products and quotients that involve composites.

Remember:

$$\text{The Chain Rule: } \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

The derivative of a composite function often becomes a product of two functions: one part still composite and the other not. So, when we see a product in an integral, it may have originated from the Chain Rule. Unlike differentiation, however, integration in this case does not follow a fixed formula. Instead, it involves a process of substitution that sometimes works and sometimes does not. We make an informed guess and check whether it simplifies the integral. In later parts of Calculus, you will learn additional techniques to use when this approach is not successful.

Steps to Integration by U-Substitution

1. Make sure that you are integrating a product or quotient.

2. Identify the inner function of the composite and set u equal to it.

3. Differentiate u to find du in terms of dx.
4. Adjust the integral by multiplying and dividing by a constant if needed so a factor matches du. [See **Ex 2.2.2**]
5. Rewrite the integral entirely in terms of u and du.
6. Integrate using the power rule or other appropriate rules.
7. Substitute back the original x-expression for u.

This is one of those mathematical processes that makes little sense when first seen. But, after seeing several examples, the meaning should become clear. *Be patient!*

OBJECTIVES

Use U-Substitution to Integrate Composite Expressions

Ex 2.2.1: $\displaystyle\int 3x^2\left(x^3+5\right)^{10}\,dx$

Sol 2.2.1:

$\left(x^3+5\right)$ is the inner function.

$\hookrightarrow u = x^3+5$

$\hookrightarrow du = 3x^2\,dx$

$$\begin{aligned}\int 3x^2\left(x^3+5\right)^{10}\,dx &= \int u^{10}\,du\\ &= \frac{u^{11}}{11}+C\\ &= \boxed{\frac{1}{11}\left(x^3+5\right)^{11}}\end{aligned}$$

Ex 2.2.2: $\displaystyle\int x\left(x^2+5\right)^{3}\,dx$

Sol 2.2.2:

$$\left(x^2+5\right) \text{ is the inner function.}$$

$$\hookrightarrow\ u = x^2+5$$

$$\hookrightarrow\ du = 2x\,dx$$

$$\begin{aligned}
\int x\left(x^2+5\right)^3\,dx &= \frac{1}{2}\int (2x)\left(x^2+5\right)^3\,dx \\
&= \frac{1}{2}\int u^3\,du \\
&= \frac{1}{2}\cdot\frac{u^4}{4}+C \\
&= \boxed{\frac{1}{8}\left(x^2+5\right)^4+C}
\end{aligned}$$

Notice how the factor of 2 from $du = 2x\,dx$ is accounted for by multiplying by $\frac{1}{2}$ when substituting. This ensures the integral is correctly expressed in terms of u.

Ex 2.2.3: $\displaystyle\int \left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\,dx$

Sol 2.2.3:

$$\sqrt[4]{x^4+2x^2-5} \text{ is the inner function.}$$

$$\hookrightarrow\ u = x^4+2x^2-5$$

$$\hookrightarrow\ du = \left(4x^3+4x\right)\,dx = 4\left(x^3+x\right)\,dx$$

$$\int \left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\,dx = \frac{1}{4}\int 4\left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\,dx$$
$$= \frac{1}{4}\int \sqrt[4]{u}\,du$$
$$= \frac{1}{4}\cdot\frac{4u^{\frac{5}{4}}}{5}+C$$
$$= \boxed{\frac{1}{5}\left(x^4+2x^2-5\right)^{\frac{5}{4}}+C}$$

Ex 2.2.4: $\displaystyle\int \frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3}\,dx$

Sol 2.2.4:

x^3+2x^2-5x+2 is the inner function.

$\hookrightarrow u = x^3+2x^2-5x+2$

$\hookrightarrow du = \left(3x^2+4x-5\right)\,dx$

$$\int \frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3}\,dx = \int \frac{1}{u^3}\,du$$
$$= -\frac{1}{2}u^{-2}+C$$
$$= \boxed{-\frac{1}{2}\left(x^3+2x^2-5x+2\right)^{-2}+C}$$

Of course, u-substitution will apply to the transcendental functions as well.

Ex 2.2.5: $\displaystyle\int \sin(5x)\,dx$

Sol 2.2.5:

$\hookrightarrow u = 5x$

$\hookrightarrow du = 5\,dx$

$$\int \sin(5x)\,dx = \frac{1}{5}\int 5\sin(5x)\,dx$$

$$= \frac{1}{5}\int \sin(u)\,du$$

$$= \frac{1}{5}\cdot(-\cos(u)) + C$$

$$= \boxed{-\frac{1}{5}\cos(5x) + C}$$

Ex 2.2.6: $\int \sin^6(x)\cos(x)\,dx$

Sol 2.2.6:

$\hookrightarrow u = \sin(x)$

$\hookrightarrow du = \cos(x)\,dx$

$$\int \sin^6(x)\cos(x)\,dx = \int u^6\,du$$

$$= \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{7}\sin^7(x) + C}$$

Ex 2.2.7: $\int x^5\sin\left(x^6\right)\,dx$

Sol 2.2.7:

$\hookrightarrow u = x^6$

$\hookrightarrow du = 6x^5\,dx$

$$\int x^5 \sin\left(x^6\right)\,dx = \frac{1}{6}\int 6x^5 \sin\left(x^6\right)\,dx$$

$$= \frac{1}{6}\int \sin(u)\,du$$

$$= -\frac{1}{6}\cos(u) + C$$

$$= \boxed{-\frac{1}{6}\cos\left(x^6\right) + C}$$

Ex 2.2.8: $\int \cot^3(x)\csc^2(x)\,dx$

Sol 2.2.8

$\hookrightarrow u = \cot(x)$

$\hookrightarrow du = -\csc^2(x)\,dx$

$$\int \cot^3(x)\csc^2(x)\,dx = -\int \cot^3(x)\left(-\csc^2(x)\right)\,dx$$

$$= -\int u^3\,du$$

$$= -\frac{1}{4}u^4\,du + C$$

$$= \boxed{-\frac{1}{4}\cot^4(x) + C}$$

Ex 2.2.9: $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

Ex 2.2.9:

$\hookrightarrow u = \sqrt{x}$

$\hookrightarrow du = \frac{1}{2}x^{-\frac{1}{2}}$

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx = 2\int (\cos(\sqrt{x}))\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\,dx$$

$$= 2\int \cos(u)\,du$$

$$= 2\sin(u) + C$$

$$= \boxed{2\sin(\sqrt{x}) + C}$$

Ex 2.2.10: $\int xe^{x^2+1}\,dx$

Sol 2.2.10:

$$\hookrightarrow\ u = x^2 + 1$$

$$\hookrightarrow\ du = 2x\,dx$$

$$\int xe^{x^2+1}\,dx = \frac{1}{2}\int (2x)e^{x^2+1}\,dx$$

$$= \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C$$

$$= \boxed{\frac{1}{2}e^{x^2+1} + C}$$

Ex 2.2.11: $\int \frac{x}{\sqrt{1-x^4}}\,dx$

Sol 2.2.11:

$$\hookrightarrow\ u = x^2$$

$$\hookrightarrow\ du = 2x\,dx$$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int (2x)\frac{1}{\sqrt{1-(x^2)^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\sin^{-1}(u) + C$$

$$= \boxed{\frac{1}{2}\sin^{-1}\left(x^2\right) + C}$$

Ex 2.2.12: $\int \left(xe^{x^2} + 4x^2 - 3\sin(5x)\right)\,dx$

Ex 2.2.12:

$$\int \left(xe^{x^2} + 4x^2 - 3\sin(5x)\right)\,dx = \int xe^{x^2}\,dx + \int 4x^2\,dx - \int 3\sin(5x)\,dx$$

$\hookrightarrow u_1 = x^2$ $\qquad$ $\hookrightarrow u_2 = 5x$

$\hookrightarrow du_1 = 2x\,dx$ $\qquad$ $\hookrightarrow du_2 = 5\,dx$

$$\int xe^{x^2}\,dx + \int 4x^2\,dx - \int 3\sin(5x)\,dx = \frac{1}{2}\int (2x)e^{x^2}\,dx + 4\int x^2\,dx - \frac{3}{5}\int 5\sin(5x)\,dx$$

$$= \frac{1}{2}\int e^{u_1}\,du_1 + 4\int x^2\,dx - \frac{3}{5}\int \sin\left(u_2\right)\,du_2$$

$$= \frac{1}{2}e^{u_1} + 4\frac{x^3}{3} - \frac{3}{5}(-\cos(u_2)) + C$$

$$= \boxed{\frac{1}{2}e^{x^2} + \frac{4}{3}x^3 + \frac{3}{5}\cos(5x) + C}$$

2.2 Free Response Homework Set A

Perform the anti-differentiation.

1. $\int (5x+3)^3 \, dx$
2. $\int \left(x^3\left(x^4+5\right)\right)^{24} \, dx$
3. $\int \left(1+x^3\right)^2 \, dx$
4. $\int (2-x)^{\frac{2}{3}} \, dx$
5. $x\sqrt{2x^2+3}\, dx$
6. $\int \frac{1}{(5x+2)^3} \, dx$