Chapter 2:

Intro To Anti-Derivatives

Chapter 2 Overview: Anti-Derivatives

As noted in the introduction, Calculus is essentially comprised of four operations:

- Limits
- Derivatives
- Indefinite Integrals (Or Anti-Derivatives)
- Definite Integrals

As mentioned above, there are two types of integrals — the definite integral and the indefinite integral. The definite integral was explored first as a way to determine the area bounded by a curve, rather than bounded by a polygon. The summation of infinite rectangles is

$$A = \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x,$$

and the representation

$$\int_a^b f(x)\, dx$$

is the exact amount, with $\int$ being an elongated and stylized s for "sum".

Newton and Leibnitz made the connection between the definite integral and the antiderivative, showing that the process of reversing the derivative results in an infinite summation. The antiderivative and indefinite integral are inverses of each other, just as squares and square roots or exponential and log functions. In this chapter, we will consider how to reverse the differentiation process. In a later chapter, we will dive deeper into the definite integral. Let's start by reviewing our derivative rules, as they will be necessary for us to take the antiderivative.

You must know the derivative rules in order to know the antiderivative rules!

The Power Rule: $\dfrac{d}{dx}\left[u^n\right] = nu^{n-1}\dfrac{du}{dx}$

The Product Rule: $\dfrac{d}{dx}\left[u \cdot v\right] = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$

The Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{u(x)}{v(x)}\right] = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$

The Chain Rule: $\dfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}[\sin(u)] = (\cos(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\csc(u)] = (-\csc(u)\cot(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\cos(u)] = (-\sin(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\sec(u)] = (\sec(u)\tan(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2(u)\right)\frac{du}{dx}$$

$$\frac{d}{dx}[\cot(u)] = \left(-\csc^2(u)\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right] = \left(e^u\right)\frac{du}{dx}$$

$$\frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1}(u)\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\csc^{-1}(u)\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1}(u)\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sec^{-1}(u)\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1}(u)\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cot^{-1}(u)\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

2.1: The Anti-Power Rule

As we have seen, we can deduce things about a function if its derivative is known. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called antidifferentiation, or integration.

Symbol for the Integral

$\int f(x)\,dx$ "the integral of f of x, d-x"

The dx is called the differential. For now, we will treat it as part of the integral symbol. It tells us the independent variable of the function [usually, but not always, x]. It does have a meaning on its own, but we will explore that later.

Looking at the integral as an antiderivative, we should be able to figure out the basic process. Remember:

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

and

$$\frac{d}{dx}[\text{constant}] = 0$$

It follows that if we are starting with the derivative and want to reverse the process, the power must increase by one and we should divide by this new power. Formally,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

The $+C$ is to account for any constant that might've been in the equation before the derivative was taken. Note that $n = -1$ does not work with this rule because it results in a division by zero. However, we know from our derivative rules that the derivative of $\ln x$ yields x^{-1}. Therefore, we can append our anti-power rule.

The Complete Anti-Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Since $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + [g(x)]$ and $\frac{d}{dx}[cx^n] = c\frac{d}{dx}[x^n]$, it follows that:

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c(f(x))\,dx = c\int f(x)\,dx$$

These allow us to integrate a polynomial by integrating each term separately.

OBJECTIVES

Find the Anti-Derivative of a Polynomial.
Integrate Functions Using Transcendental Operations
Use Integration to Solve Rectilinear Motion Problems

Ex 2.1.1: $\int \left(3x^2+4x+5\right)\,dx$

Sol 2.1.1:

$$\begin{aligned}\int \left(3x^2+4x+5\right)\,dx &= 3\frac{x^{2+1}}{2+1} + 4\frac{x^{1+1}}{1+1} + 5\frac{x^{0+1}}{0+1} + C \\ &= \frac{3x^3}{3} + \frac{4x^2}{2} + \frac{5x^1}{1} + C \\ &= \boxed{x^3+2x^2+5x+C}\end{aligned}$$

Ex 2.1.2: $\int \left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)\,dx$

Sol 2.1.2:

$$\begin{aligned}\int \left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)\,dx &= \frac{x^{4+1}}{4+1} + \frac{4x^{2+1}}{2+1} + \frac{5x^{0+1}}{0+1} + \ln|x| - \frac{x^{-5+1}}{-5+1} + C \\ &= \boxed{\frac{1}{5}x^5 + \frac{4}{3}x^3 + 5x + \ln|x| + \frac{1}{4x^4} + C}\end{aligned}$$

Ex 2.1.3: $\displaystyle\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right)\, dx$

Sol 2.1.3:

$$\begin{aligned}
\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right)\, dx &= \int \left(x^2 + x^{\frac{1}{3}} - \frac{4}{x}\right)\, dx \\
&= \frac{x^{2+1}}{2+1} + \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 4\ln|x| + C \\
&= \boxed{\frac{1}{3}x^3 - \frac{3}{4}x^{\frac{4}{3}} + 4\ln|x| + C}
\end{aligned}$$

Integrals of products and quotients can be done easily IF they can be turned into a polynomial.

Ex 2.1.4: $\displaystyle\int \left(x^2 + \sqrt[3]{x}\right)(2x+1)\, dx$

Sol 2.1.4:

$$\begin{aligned}
\int \left(x^2 + \sqrt[3]{x}\right)(2x+1)\, dx &= \int \left(2x^3 + 2x^{\frac{4}{3}} + x^2 + x^{\frac{1}{3}}\right)\, dx \\
&= \frac{2x^4}{4} + \frac{2x^{\frac{7}{3}}}{\frac{7}{3}} + \frac{x^3}{3} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} \\
&= \boxed{\frac{1}{2}x^4 + \frac{6}{7}x^{\frac{7}{3}} + \frac{1}{3}x^3 + \frac{3}{4}x^{\frac{4}{3}} + C}
\end{aligned}$$

The next example is called an initial value problem. It has an ordered pair (or initial value pair) that allows us to solve for C.

Ex 2.1.5: $f'(x) = 4x^3 - 6x + 3$. Find $f(x)$ if $f(0) = 13$.

Sol 2.1.5:

$$\begin{aligned}
f(x) &= \int \left(4x^3 - 6x + 3\right)\, dx \\
&= x^4 - 3x^2 + 3x + C
\end{aligned}$$

$$f(0) = 0^4 - 3(0)^2 + 3(0) + C$$

$$= 13 \therefore C = 13$$

$$\therefore \boxed{f(x) = x^4 - 3x^2 + 3x + 13}$$

Now, let's take a look at a type of problem called a *rectilinear motion* problem. In these problems, we study the motion of an object moving along a straight line—its position, velocity, and acceleration.

Ex 2.1.6: The acceleration of particle is described by $a(t) = 3t^2 + 8t + 1$. Find the distance equation for $x(t)$ if $v(0) = 3$ and $a(0) = 1$.

Sol 2.1.6:

$$v(t) = \int a(t)\,dt$$

$$= \int \left(3t^2 + 8t + 1\right)\,dt$$

$$= t^3 + 4t^2 + t + C_1$$

$$3 = (0)^3 + 4(0)^2 + (0) + C_1 \therefore 3 = C_1$$

$$v(t) = t^3 + 4t^2 + t + 3$$

$$x(t) = \int v(t)\,dt$$

$$= \int \left(t^3 + 4t^2 + t + 3\right)\,dt$$

$$= \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + C_2$$

$$1 = \frac{1}{4}(0)^4 + \frac{4}{3}(0)^3 + \frac{1}{2}(0)^2 + 3(0) + C_2 \therefore 1 = C_2$$

$$\boxed{x(t) = \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + 1}$$

Ex 2.1.7: The acceleration of a particle is described by $a(t) = 12t^2 - 6t + 4$. Find the distance equation for $x(t)$ if $v(1) = 0$ and $x(1) = 3$.

Sol 2.1.7:

$$\begin{aligned}
v(t) &= \int a(t)\,dt \\
&= \int \left(12t^2 - 6t + 4\right) \\
&= 4t^3 - 3t^2 - 4t + C_1
\end{aligned}$$

$$0 = 4(1)^3 - 3(1)^2 + 4(1) + C_1 \therefore -5 = C_1$$

$$v(t) = 4t^3 - 3t^2 - 4t - 5$$

$$\begin{aligned}
x(t) &= \int v(t)\,dt \\
&= \int \left(4t^3 - 3t^2 - 4t - 5\right)\,dt \\
&= t^4 - t^3 - 2t^2 - 5t + C_2
\end{aligned}$$

$$3 = (1)^4 - (1)^3 - 2(1)^2 - 5(1) + C_2 \therefore 6 = C_2$$

$$\boxed{x(t) = t^4 - t^3 - 2t^2 - 5t + 6}$$

The proof of all the transcendental integral rules can be left to a more formal Calculus course. But, since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

Transcendental Integral Rules

$$\int \cos(u)\,du = \sin(u) + C \qquad \int \csc(u)\cot(u)\,du = -\csc(u) + C$$

$$\int \sin(u)\,du = -\cos(u) + C \qquad \int \sec(u)\tan(u)\,du = \sec(u) + C$$

$$\int \sec^2(u)\,du = \tan(u) + C \qquad \int \csc^2(u)\,du = -\cot(u) + C$$

$$\int e^u\,du = e^u + C \qquad \int \frac{1}{u}\,du = \ln|u| + C$$

$$\int a^u\,du = \frac{a^u}{\ln|a|} + C \qquad \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$$

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C \qquad \int \frac{1}{u\sqrt{u^2-1}}\,du = sec^{-1} + C$$

Note that there are only three integrals that yield inverse trig functions, but there were six inverse trig derivatives. This is because the other three derivative rules are just the negatives of the first three.

Ex 2.1.8: $\int (\sin(x) + 3\cos(x))\ dx$

Sol 2.1.8:

$$\begin{aligned}\int (\sin(x) + 3\cos(x))\ dx &= \int \sin(x)\,dx + 3\int \cos(x)\,dx \\ &= \boxed{-\cos(x) + 3\sin(x) + C}\end{aligned}$$

Ex 2.1.9: $\int \left(e^x + 4 + 3\csc^2(x)\right)\ dx$

Sol 2.1.9:

$$\int \left(e^x + 4 + 3\csc^2(x)\right)\ dx = \int e^x\,dx + 4\int\ dx + 3\int \csc^2(x)\,dx$$

$$= \boxed{e^x + 4x - 3\cot(x) + C}$$

Now, let's take a look at some more complex integrals that yield inverse trig functions. These more general forms extend the earlier rules by introducing a constant a, and they are especially useful when working with substitutions or integrals that don't simplify neatly to the unit case.

Trig Inverse Integral Rules

$$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u^2 + a^2}\, du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$

Ex 2.1.10: Find $\displaystyle\int \frac{1}{x^2 + 4}\, dx$

Sol 2.1.10: All we need to do is apply our formula above.

$$\int \frac{1}{x^2 + 4}\, dx = \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

Ex 2.1.11: If $\dfrac{dy}{dx} = \sec(x)(\sec(x)\tan(x))$, find $y(x)$ if $y(0) = 0$.

Sol 2.1.11:

$$\begin{aligned} y &= \int (\sec(x)(\sec(x)\tan(x)))\, dx \\ &= \int \left(\sec^2(x)\right)\, dx + \int (\sec(x)\tan(x))\, dx \\ &= \tan(x) + \sec(x) + C \end{aligned}$$

$$0 = \tan(0) + \sec(0) + C$$

$$0 = 0 + 1 + C \therefore C = -1$$

$$\boxed{y = \tan(x) + \sec(x) - 1}$$

2.1 Free Response Homework

Perform the antidifferentiation.

1. $\displaystyle\int \left(6x^2 - 2x + 3\right) dx$

2. $\displaystyle\int \left(x^3 + 3x^2 - 2x + 4\right) dx$

3. $\displaystyle\int \frac{2}{\sqrt[3]{x}}\, dx$

4. $\displaystyle\int \left(8x^4 - 4x^3 + 9x^2 + 2x + 1\right) dx$

5. $\displaystyle\int x^3 \left(4x^2 + 5\right) dx$

6. $\displaystyle\int (4x-1)(3x+8)\, dx$

7. $\displaystyle\int \left(\sqrt{x} - \frac{6}{\sqrt{x}}\right) dx$

8. $\displaystyle\int \frac{x^2 + \sqrt{x} + 3}{x}\, dx$

9. $\displaystyle\int (x+1)^3\, dx$

10. $\displaystyle\int (4x-3)^2\, dx$

11. $\displaystyle\int \left(\sqrt{x} + 3\sqrt{3} - \frac{6}{\sqrt{x}}\right) dx$

12. $\displaystyle\int \frac{4x^3 + \sqrt{x} + 3}{x^2}\, dx$

13. $\displaystyle\int \left(x^2 + 5x + 6\right) dx$

14. $\displaystyle\int \frac{x^2 - 4x + 7}{x}\, dx$

15. $\displaystyle\int \frac{x^5 - 7x^3 + 2x - 9}{2x}\, dx$

16. $\displaystyle\int \frac{x^3 + 3x^2 + 3x + 1}{x+1}\, dx$

17. $\displaystyle\int \left(y^2 + 5\right)^2 dy$

18. $\displaystyle\int \left(4t^2 + 1\right)\left(3t^3 + 7\right) dt$

Complete the following problems.

19. $f'(x) = 3x^2 - 6x + 3$. Find $f(x)$ if $f(0) = 2$.

20. $f'(x) = x^3 + x^2 - x + 3$. Find $f(x)$ if $f(1) = 0$.

21. $f'(x) = (\sqrt{x} - 2)(3\sqrt{x} + 1)$. Find $f(x)$ if $f(4) = 1$.

22. The acceleration of a particle is described by $a(t) = 36t^2 - 12t + 8$. Find the distance equation for $x(t)$ if $v(1) = 1$ and $x(1) = 3$.

23. The acceleration of a particle is described by $a(t) = t^2 - 2t + 4$. Find the distance equation for $x(t)$ if $v(0) = 2$ and $x(0) = 4$.

2.1 Multiple Choice Homework

1. $\displaystyle\int \frac{1}{x^2}\,dx =$

a) $\ln\left(x^2\right) + C$ b) $-\ln\left(x^2\right) + C$ c) $\dfrac{1}{x} + C$ d) $-\dfrac{1}{x} + C$ e) $-\dfrac{2}{x^3} + C$

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2. $\displaystyle\int x\left(10 + 8x^4\right)\,dx =$

a) $5x^2 + \dfrac{4}{3}x^6 + C$ b) $5x^2 + \dfrac{8}{5}x^5 + C$ c) $10x + \dfrac{4}{3}x^6 + C$

d) $5x^2 + 8x^6 + C$ e) $5x^2 + \dfrac{8}{7}x^6 + C$

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3. $\displaystyle\int x\sqrt{3x}\,dx$

a) $\dfrac{2\sqrt{3}}{5}x^{\frac{5}{2}} + C$ b) $\dfrac{5\sqrt{3}}{2}x^{\frac{5}{2}} + C$ c) $\dfrac{\sqrt{3}}{2}x^{\frac{1}{2}} + C$ d) $2\sqrt{3x} + C$ e) $\dfrac{5\sqrt{3}}{2}x^{\frac{3}{2}} + C$

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4. $\displaystyle\int (x-1)\sqrt{x}\,dx =$

a) $\dfrac{3}{2}\sqrt{x} - \dfrac{1}{\sqrt{x}} + C$ b) $\dfrac{2}{3}x^{\frac{2}{3}} + \dfrac{1}{2}x^{\frac{1}{2}} + C$ c) $\dfrac{1}{2}x^2 - x + C$

d) $\dfrac{2}{5}x^{\frac{5}{2}} - \dfrac{2}{3}x^{\frac{3}{2}} + C$ e) $\dfrac{1}{2}x^2 + 2x^{\frac{2}{3}} + C$

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5. A particle is moving upward along the y-axis until it reaches the origin and then it moves downward such that $v(t) = 8 - 2t$ for $t \geq 0$. The position of the particle at time t is given by

a) $y(t) = -t^2 + 8t - 16$ b) $y(t) = -t^2 + 8t + 16$ c) $y(t) = 2t^2 - 8t - 16$

d) $8t - t^2$ e) $8t - 2t^2$

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6. If a particle's acceleration is given by $a(t) = 12t + 4$, and $v(1) = 5$ and $y(0) = 2$, then

$y(2) =$

a) 20 b) 10 c) 4 d) 16 e) 12

2.2: Integration by U-Substitution

Reversing the Power Rule was fairly easy. The other three core derivative rules—the Product Rule, the Quotient Rule, and the Chain Rule—are a little more complicated to undo. This is because they yield a more complicated function as a derivative, one which has several algebraic simplification steps. The integral of a rational function is particularly difficult to unravel because, as we have seen, rational derivatives can be obtained by differentiating a composite function with a log or a radical, or by differentiating another rational function. The same goes for reversing the Product Rule.

Key Idea: There is no single Product or Quotient Rule for integrals.

Instead, there are several techniques that apply in different situations, and it is not always obvious at the outset which one will be most effective. The choice depends on the algebraic manipulations that produced the product or quotient in the first place.

Products can be a result of:	Quotients can be the result of:
• The Chain Rule • Differentiating a product • Differentiating some trig functions	• Common denominators • Differentiating a quotient • Differentiating a log with a composite function • Differentiating some trig inverse functions

Composite functions are among the most pervasive functions in math. Therefore, we will start with undoing products and quotients that involve composites.

Remember:

$$\text{The Chain Rule: } \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

The derivative of a composite function often becomes a product of two functions: one part still composite and the other not. So, when we see a product in an integral, it may have originated from the Chain Rule. Unlike differentiation, however, integration in this case does not follow a fixed formula. Instead, it involves a process of substitution that sometimes works and sometimes does not. We make an informed guess and check whether it simplifies the integral. In later parts of Calculus, you will learn additional techniques to use when this approach is not successful.

Steps to Integration by U-Substitution

1. Make sure that you are integrating a product or quotient.

2. Identify the inner function of the composite and set u equal to it.

3. Differentiate u to find du in terms of dx.

4. Adjust the integral by multiplying and dividing by a constant if needed so a factor matches du. [See **Ex 2.2.2**]

5. Rewrite the integral entirely in terms of u and du.

6. Integrate using the power rule or other appropriate rules.

7. Substitute back the original x-expression for u.

This is one of those mathematical processes that makes little sense when first seen. But, after seeing several examples, the meaning should become clear. *Be patient!*

OBJECTIVES

Use U-Substitution to Integrate Composite Expressions

Ex 2.2.1: $\int 3x^2 \left(x^3+5\right)^{10} dx$

Sol 2.2.1:

$\left(x^3+5\right)$ is the inner function.

$\hookrightarrow u = x^3 + 5$

$\hookrightarrow du = 3x^2\,dx$

$$\begin{aligned}\int 3x^2 \left(x^3+5\right)^{10} dx &= \int u^{10}\,du \\ &= \frac{u^{11}}{11} + C \\ &= \boxed{\frac{1}{11}\left(x^3+5\right)^{11}}\end{aligned}$$

Ex 2.2.2: $\int x \left(x^2+5\right)^{3} dx$

Sol 2.2.2:

$\left(x^2+5\right)$ is the inner function.

$\hookrightarrow u = x^2 + 5$

$\hookrightarrow du = 2x\,dx$

$$\begin{aligned}\int x\left(x^2+5\right)^3\,dx &= \frac{1}{2}\int (2x)\left(x^2+5\right)^3\,dx\\ &= \frac{1}{2}\int u^3\,du\\ &= \frac{1}{2}\cdot\frac{u^4}{4}+C\\ &= \boxed{\frac{1}{8}\left(x^2+5\right)^4+C}\end{aligned}$$

Notice how the factor of 2 from $du = 2x\,dx$ is accounted for by multiplying by $\frac{1}{2}$ when substituting. This ensures the integral is correctly expressed in terms of u.

Ex 2.2.3: $\displaystyle\int \left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\,dx$

Sol 2.2.3:

$\sqrt[4]{x^4+2x^2-5}$ is the inner function.

$\hookrightarrow u = x^4+2x^2-5$

$\hookrightarrow du = \left(4x^3+4x\right)\,dx = 4\left(x^3+x\right)\,dx$

$$\int \left(x^3 + x\right) \sqrt[4]{x^4 + 2x^2 - 5}\, dx = \frac{1}{4} \int 4 \left(x^3 + x\right) \sqrt[4]{x^4 + 2x^2 - 5}\, dx$$

$$= \frac{1}{4} \int \sqrt[4]{u}\, du$$

$$= \frac{1}{4} \cdot \frac{4u^{\frac{5}{4}}}{5} + C$$

$$= \boxed{\frac{1}{5} \left(x^4 + 2x^2 - 5\right)^{\frac{5}{4}} + C}$$

Ex 2.2.4: $\displaystyle\int \frac{3x^2 + 4x - 5}{\left(x^3 + 2x^2 - 5x + 2\right)^3}\, dx$

Sol 2.2.4:

$x^3 + 2x^2 - 5x + 2$ is the inner function.

$\hookrightarrow u = x^3 + 2x^2 - 5x + 2$

$\hookrightarrow du = \left(3x^2 + 4x - 5\right)\, dx$

$$\int \frac{3x^2 + 4x - 5}{\left(x^3 + 2x^2 - 5x + 2\right)^3}\, dx = \int \frac{1}{u^3}\, du$$

$$= -\frac{1}{2} u^{-2} + C$$

$$= \boxed{-\frac{1}{2} \left(x^3 + 2x^2 - 5x + 2\right)^{-2} + C}$$

Of course, u-substitution will apply to the transcendental functions as well.

Ex 2.2.5: $\displaystyle\int \sin(5x)\, dx$

Sol 2.2.5:

$\hookrightarrow u = 5x$

$\hookrightarrow du = 5\, dx$

$$\begin{aligned}
\int \sin(5x)\,dx &= \frac{1}{5}\int 5\sin(5x)\,dx \\
&= \frac{1}{5}\int \sin(u)\,du \\
&= \frac{1}{5}\cdot(-\cos(u)) + C \\
&= \boxed{-\frac{1}{5}\cos(5x) + C}
\end{aligned}$$

Ex 2.2.6: $\displaystyle\int \sin^6(x)\cos(x)\,dx$

Sol 2.2.6:

$$\hookrightarrow u = \sin(x)$$

$$\hookrightarrow du = \cos(x)\,dx$$

$$\begin{aligned}
\int \sin^6(x)\cos(x)\,dx &= \int u^6\,du \\
&= \frac{1}{7}u^7 + C \\
&= \boxed{\frac{1}{7}\sin^7(x) + C}
\end{aligned}$$

Ex 2.2.7: $\displaystyle\int x^5\sin\left(x^6\right)\,dx$

Sol 2.2.7:

$$\hookrightarrow u = x^6$$

$$\hookrightarrow du = 6x^5\,dx$$

$$\begin{aligned}\int x^5 \sin\left(x^6\right)\,dx &= \frac{1}{6}\int 6x^5\sin\left(x^6\right)\,dx \\ &= \frac{1}{6}\int \sin(u)\,du \\ &= -\frac{1}{6}\cos(u) + C \\ &= \boxed{-\frac{1}{6}\cos\left(x^6\right) + C}\end{aligned}$$

Ex 2.2.8: $\int \cot^3(x)\csc^2(x)\,dx$

Sol 2.2.8

$\hookrightarrow u = \cot(x)$

$\hookrightarrow du = -\csc^2(x)\,dx$

$$\begin{aligned}\int \cot^3(x)\csc^2(x)\,dx &= -\int \cot^3(x)\left(-\csc^2(x)\right)\,dx \\ &= -\int u^3\,du \\ &= -\frac{1}{4}u^4\,du + C \\ &= \boxed{-\frac{1}{4}\cot^4(x) + C}\end{aligned}$$

Ex 2.2.9: $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

Ex 2.2.9:

$\hookrightarrow u = \sqrt{x}$

$\hookrightarrow du = \frac{1}{2}x^{-\frac{1}{2}}$

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx = 2\int (\cos(\sqrt{x}))\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\,dx$$

$$= 2\int \cos(u)\,du$$

$$= 2\sin(u) + C$$

$$= \boxed{2\sin(\sqrt{x}) + C}$$

Ex 2.2.10: $\int xe^{x^2+1}\,dx$

Sol 2.2.10:

$$\hookrightarrow u = x^2 + 1$$

$$\hookrightarrow du = 2x\,dx$$

$$\int xe^{x^2+1}\,dx = \frac{1}{2}\int (2x)e^{x^2+1}\,dx$$

$$= \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C$$

$$= \boxed{\frac{1}{2}e^{x^2+1} + C}$$

Ex 2.2.11: $\int \frac{x}{\sqrt{1-x^4}}\,dx$

Sol 2.2.11:

$$\hookrightarrow u = x^2$$

$$\hookrightarrow du = 2x\,dx$$

$$
\begin{aligned}
\int \frac{x}{\sqrt{1-x^4}}\,dx &= \frac{1}{2}\int (2x)\frac{1}{\sqrt{1-(x^2)^2}}\,dx \\
&= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du \\
&= \frac{1}{2}\sin^{-1}(u)+C \\
&= \boxed{\frac{1}{2}\sin^{-1}\left(x^2\right)+C}
\end{aligned}
$$

Ex 2.2.12: $\displaystyle\int \left(xe^{x^2}+4x^2-3\sin(5x)\right)\,dx$

Ex 2.2.12:

$$
\int \left(xe^{x^2}+4x^2-3\sin(5x)\right)\,dx = \int xe^{x^2}\,dx+\int 4x^2\,dx-\int 3\sin(5x)\,dx
$$

$\hookrightarrow u_1 = x^2$ $\qquad$ $\hookrightarrow u_2 = 5x$

$\hookrightarrow du_1 = 2x\,dx$ $\qquad$ $\hookrightarrow du_2 = 5\,dx$

$$
\begin{aligned}
\int xe^{x^2}\,dx+\int 4x^2\,dx-\int 3\sin(5x)\,dx &= \frac{1}{2}\int (2x)e^{x^2}\,dx+4\int x^2\,dx-\frac{3}{5}\int 5\sin(5x)\,dx \\
&= \frac{1}{2}\int e^{u_1}\,du_1+4\int x^2\,dx-\frac{3}{5}\int \sin\left(u_2\right)\,du_2 \\
&= \frac{1}{2}e^{u_1}+4\frac{x^3}{3}-\frac{3}{5}(-\cos(u_2))+C \\
&= \boxed{\frac{1}{2}e^{x^2}+\frac{4}{3}x^3+\frac{3}{5}\cos(5x)+C}
\end{aligned}
$$

2.2 Free Response Homework Set A

Perform the antidifferentiation.

1. $\int (5x+3)^3 \, dx$

2. $\int \left(x^3\left(x^4+5\right)\right)^{24} \, dx$

3. $\int \left(1+x^3\right)^2 \, dx$

4. $\int (2-x)^{\frac{2}{3}} \, dx$

5. $\int x\sqrt{2x^2+3} \, dx$

6. $\int \frac{1}{(5x+2)^3} \, dx$

7. $\int \frac{x^3}{\sqrt{1+x^4}} \, dx$

8. $\int \frac{x+1}{\sqrt[3]{x^2+2x+3}} \, dx$

9. $\int \left(x^5 - \sin(3x) + xe^{x^2}\right) \, dx$

10. $\int \left(x^2\sec^2\left(x^3\right) + \frac{\ln^3 x}{x}\right) \, dx$

11. $\int x^4 \cos\left(x^5\right) \, dx$

12. $\int \sin(7x+1) \, dx$

13. $\int \sec^2(3x-1) \, dx$

14. $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} \, dx$

15. $\int \tan^4(x)\sec^2(x) \, dx$

16. $\int \frac{\ln x}{x} \, dx$

17. $\int e^{6x} \, dx$

18. $\int \frac{\cos(2x)}{\sin^3(2x)} \, dx$

19. $\int \frac{x\ln\left(x^2+1\right)}{x^2+1} \, dx$

20. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

21. $\int \sqrt{\cot(x)}\csc^2(x) \, dx$

22. $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right) \, dx$

23. $\int \frac{x}{1+x^4} \, dx$

24. $\int \frac{\cos(x)}{\sqrt{1-\sin^2(x)}} \, dx$

2.2 Free Response Homework Set B

Perform the antidifferentiation.

1. $\int (2x+5)\left(x^2+5x+6\right)^6 \, dx$

2. $\int 3t^2\left(t^3+1\right)^5 \, dt$

3. $\int \frac{10m+15}{\sqrt[4]{m^2+3m+1}} \, dm$

4. $\int \frac{3x^2}{(1+x^3)^5} \, dx$

5. $\int (4s+1)^5 \, ds$

6. $\int \frac{5t}{t^2+1} \, dt$

7. $\int \frac{3m^2}{m^3+8} \, dm$

8. $\int (181x+1)^5 \, dx$

9. $\int \frac{v^2}{5-v^3} \, dv$

10. $\int \left(x^7 - \cot(5x) + xe^{x^2}\right) \, dx$

11. $\int \frac{\cos(x)}{1+\sin(x)} \, dx$

12. $\int \left(x^2 \sec^2\left(4x^3\right) + 2xe^{x^2}\right) \, dx$

13. $\int \sec^2(2x) \, dx$

14. $\int \frac{\csc^2\left(e^{-x}\right)}{e^x} \, dx$

15. $\int \frac{\sec(\ln x)\tan(\ln x)}{3x} \, dx$

16. $\int \left(x^5 + \frac{7}{x^2} - e^{2x} + \sec^2(x)\right) \, dx$

17. $\int e^x \csc\left(e^x\right) \cot\left(e^x\right) \, dx$

18. $\int \left(e^x - 2\right)\left(e^x - 1\right) \, dx$

19. $\int x^2 \sin\left(x^3\right) \, dx$

20. $\int te^{5t^2+1} \, dt$

21. $\int \left(e^y + 1\right)^2 \, dy$

22. $\int x \sec^2\left(x^2\right) \sqrt{\tan\left(x^2\right)} \, dx$

23. $\int \sin(3t) \cos^5(3t) \, dt$

24. $\int x \cos\left(x^2\right) e^{\sin\left(x^2\right)} \, dx$

25. $\int \tan(\theta) \ln(\sec(\theta)) \, d\theta$

26. $\int \left(e^{4y} + 2y^2 - 7\cos(3y)\right) \, dy$

27. $\int \frac{\sin(x+4)}{\cos^7(x+4)} \, dx$

28. $\int \left(\frac{2x}{x^2+5} - \sec^2(3x) + xe^{x^2} - \pi\right) \, dx$

29. $\int e^{2t} \sec^2\left(e^{2t}\right) \, dt$

30. $\int \frac{18 \ln m}{m} \, dm$

31. $\int \sec^2(y) \tan^5(y)$. Verify your answer by taking the derivative.

32. $\int \left(\cos(\theta) e^{\sin(\theta)} + \frac{\theta}{\theta^2+1}\right) \, d\theta$. Verify your answer by taking the derivative.

33. $\int t \sec^2\left(4t^2\right) \sqrt{\tan\left(4t^2\right)} \, dt$. Verify your answer by taking the derivative.

34. $\int \frac{2y \cos\left(y^2\right)}{\sin^4\left(y^2\right)} \, dy$. Verify your answer by taking the derivative.

2.2 Multiple Choice Homework

1. $\displaystyle\int \frac{x}{x^2-4}\,dx =$

a) $-\dfrac{1}{4\left(x^2-4\right)^2}+C$ b) $\dfrac{1}{2\left(x^2-4\right)}+C$ c) $\dfrac{1}{2}\ln\left|x^2-4\right|+C$

d) $2\ln\left|x^2-4\right|+C$ e) $\dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right)+C$

---

2. $\displaystyle\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx =$

a) $\ln\left(\sqrt{x}\right)+C$ b) $x+C$ c) e^x+C d) $\dfrac{1}{2}e^{2\sqrt{x}}+C$ e) $e^{\sqrt{x}}+C$

---

3. When using the substitution $u=\sqrt{1+x}$, an antiderivative of $\displaystyle\int 60x\sqrt{1+x}\,dx$ is.

a) $20u^3-60u+C$ b) $15u^4-30u^2+C$ c) $30u^4-60u^2+C$

d) $24u^5-40u^3+C$ e) $12u^6-20u^4+C$

---

4. $\displaystyle\int \frac{3x^2}{\sqrt{x^3+3}}\,dx =$

a) $2\sqrt{x^3+3}+C$ b) $\dfrac{3}{2}\sqrt{x^3+3}+C$ c) $\sqrt{x^3+3}+C$

d) $\ln\left(\sqrt{x^3+3}\right)+C$ e) $\ln\left(x^3+3\right)+C$

---

5. $\displaystyle\int x\left(x^2-1\right)^4\,dx =$

a) $\dfrac{1}{10}x^2\left(x^2-1\right)^5+C$ b) $\dfrac{1}{10}\left(x^2-1\right)^5+C$ c) $\dfrac{1}{5}\left(x^3-x\right)^5+C$

d) $\dfrac{1}{5}\left(x^2-1\right)^5+C$ e) $\dfrac{1}{5}\left(x^2-x\right)^5+C$

---

6. $\displaystyle\int 4x^2\sqrt{3+x^3}\,dx =$

a) $\dfrac{16\,(3+x^3)^{\frac{3}{2}}}{9}+C$ b) $\dfrac{8\,(3+x^3)^{\frac{3}{2}}}{9}+C$ c) $\dfrac{8\,(3+x^3)^{\frac{3}{2}}}{3}+C$

d) $\dfrac{4}{3\,(3+x^3)^{\frac{1}{2}}}+C$ e) $\dfrac{8}{3\,(3+x^3)^{\frac{1}{2}}}+C$

---

7. $\displaystyle\int \left(x^3+2+\frac{1}{x^2+1}\right)dx =$

a) $\dfrac{x^4}{4}+2x+\tan^{-1}(x)+C$ b) $x^4+2+\tan^{-1}(x)+C$ c) $\dfrac{x^4}{4}+2x+\dfrac{3}{x^3+3}+C$

d) $\dfrac{x^4}{4}+2x+\tan^{-1}\left(2x^2\right)+C$ e) $4+2x+\tan^{-1}(x)+C$

---

8. $\displaystyle\int \cos(3-2x)\,dx =$

a) $\sin(3-2x)+C$ b) $-\sin(3-2x)+C$ c) $\dfrac{1}{2}\sin(3-2x)+C$

d) $-\sin(3-2x)+C$ e) $-\dfrac{1}{5}\sin(3-2x)+C$

---

9. $\displaystyle\int \frac{x-2}{x-1}\,dx =$

a) $-\ln|x-1|+C$ b) $x+\ln|x-1|+C$ c) $x-\ln|x-1|+C$

d) $x+\sqrt{x-1}+C$ e) $x-\sqrt{x-1}+C$

---

10. $\displaystyle\int \frac{e^{x^2}-2x}{e^{x^2}}\,dx =$

a) $x-e^{x^2}+C$ b) $x-e^{-x^2}+C$ c) $x+e^{-x^2}+C$ d) $-e^{x^2}+C$ e) $-e^{-x^2}+C$

---

11. $\displaystyle\int 6\sin(x)\cos^2(x)\,dx =$

a) $2\sin^3(x) + C$

b) $-2\sin^3(x) + C$

c) $2\cos^3(x) + C$

d) $-2\cos^3(x) + C$

e) $3\sin^2(x)\cos^2(x) + C$

---

12. $\displaystyle\int \frac{4x}{1+x^2}\,dx =$

a) $4\arctan(x) + C$

b) $\dfrac{4}{x}\arctan(x) + C$

c) $\dfrac{1}{2}\ln\left(1+x^2\right) + C$

d) $2\ln\left(1+x^2\right) + C$

e) $2x^2 + 4\ln|x| + C$

---

13. $\displaystyle\int \frac{x}{4+x^2}\,dx =$

a) $\tan^{-1}\left(\dfrac{x}{2}\right) + C$

b) $\ln\left(4+x^2\right) + C$

c) $\tan^{-1}(x) + C$

d) $\dfrac{1}{2}\ln\left(4+x^2\right) + C$

e) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + C$

---

14. $\displaystyle\int \left(2^x - 4e^{2\ln x}\right)\,dx =$

a) $2^x\ln 2 - \dfrac{4}{3}e^{2\ln x} + C$

b) $x2^{x-1} - \dfrac{4}{3}x^3 + C$

c) $\dfrac{2^x}{\ln 2} - \dfrac{4}{3}e^{2\ln x} + C$

d) $x2^{x-1} - \dfrac{4}{3}e^{2\ln x} + C$

e) $\dfrac{2^x}{\ln 2} - \dfrac{4}{3}x^3 + C$

---

15. The antiderivative of $2\tan(x)$ is:

a) $2\ln|\sec(x)| + C$

b) $2\sec^2(x) + C$

c) $\ln|\sec^2(x)| + C$

d) $2\ln|\cos(x)| + C$

e) $\ln|2\sec(x)| + C$

---

16. Which of the following statements are true?

I. $\displaystyle\int x^4\sin\left(x^5\right)\,dx = -\frac{1}{5}\cos\left(x^5\right) + C$

II. $\displaystyle\int \tan(x)\,dx = \sec^2(x) + C$

III. $\displaystyle\int \left(x^3 + x\right) \sqrt[4]{x^4 + 2x^2 - 5}\,dx = \frac{1}{5}\left(x^4 + 2x^2 - 5\right)^{\frac{5}{4}} + C$

a) I only b) II only c) III only d) I and II only

e) II and III only f) I and III only g) I, II, and III h) None of these

---

17. If $x'(t) = 2t\cos\left(t^2\right)$, find $x(t)$ when $x\left(\sqrt{\frac{\pi}{2}}\right) = 3$

a) $x(t) = -4t^2\sin\left(t^2\right)$ b) $x(t) = -4t^2\sin\left(t^2\right) + 2\cos\left(t^2\right)$ c) $x(t) = \sin\left(t^2\right) + 3$

d) $x(t) = -\sin\left(t^2\right) + 4$ e) $x(t) = \sin\left(t^2\right) + 2$

---

18. A particle moves along the y-axis so that at any time $t \geq 0$, its velocity is given $v(t) = \sin(2t)$. If the position of the particle at time $t = \frac{\pi}{2}$ is $y = 3$. What is the particle's position at time $t = 0$?

a) -4 b) 2 c) 3 d) 4 e) 6

2.3 Separable Differential Equations

Differential Equation → Definition: An equation that contains a derivative.

General Solution → Definition: All of the y-equations that would have the given equation as their derivative. Note the $+C$ which gives multiple equations.

Initial Condition → Definition: Constraint placed on a differential equation; sometimes called an initial value.

Particular Solution → Definition: Solution obtained from solving a differential equation when an initial condition allows you to solve for C.

Separable Differential Equation → Definition: A differential equation in which all terms with y's can be moved to the left side of an equals sign (=), and in which all terms with x's can be moved to the right side of an equals sign (=), by multiplication and division only.

Let's take a look at some examples of separable differential equations.

Ex 2.3.1: Separate the variables in the following differential equations:

$$\text{a) } \frac{dy}{dx} = -\frac{x}{y}$$

$$\text{b) } \frac{dy}{dx} = x\sec(y)$$

$$\text{c) } y' = 2xy - 3y$$

Sol 2.3.1:
a)

$$(y)\frac{dy}{dx} = -\frac{x}{y}(y)$$

$$(dx)\frac{y\,dy}{dx} = -x(dx)$$

$$\boxed{y\,dy = -x\,dx}$$

b)

$$\left(\frac{1}{\sec(y)}\right)\frac{dy}{dx} = x\sec(y)\left(\frac{1}{\sec(y)}\right)$$

$$(dx)\cos(y)\frac{dy}{dx} = x(dx)$$

$$\boxed{\cos(y)\,dy = x\,dx}$$

c)

$$\frac{dy}{dx} = (2x-3)y$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = (2x-3)y\left(\frac{1}{y}\right)$$

$$(dx)\left(\frac{1}{y}\right)\frac{dy}{dx} = (2x-3)(dx)$$

$$\boxed{\frac{1}{y}\,dy = (2x-3)\,dx}$$

OBJECTIVES

Given a Separable Differential Equation, Find the General Solution.
Given a Separable Differential Equation and an Initial Condition, Find a Particular Solution.

Steps to Solving Differential Equations

1. Separate the variables. Move all terms involving y (and dy) to one side and all terms involving x (and dx) to the other. Keep any constants on the right side of the equation.

2. Integrate both sides. Keep $+C$ only on the right side of the equation.

3. Solve for y, if possible. If the integration produces a natural log, isolate y. If not, solve for C. Note: $e^{\ln|y|} = y$, as e raised to any power is positive.

4. Apply initial conditions (if given). Substitute initial values to solve for C.

Ex 2.3.2: Find the general solution to the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

Sol 2.3.2:

$\frac{dy}{dx} = -\frac{x}{y}$ Start here.

$y\,dy = -x\,dx$ Separate all the y terms to the left side of the equation and all of the x terms to the right side.

$\int y\,dy = \int -x\,dx$ Integrate both sides.

$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ You only need C on one side of the equation.

$y^2 = -x^2 + C$ Multiply both sides by 2. Note that $2C$ is still a constant, so we'll continue to denote it as C.

$x^2 + y^2 = C$ This is the family of circles with radius $\sqrt{C}$ centered at the origin.

$y = \pm\sqrt{C - x^2}$ Isolate y.

Since we usually solve our equation for y, our solution will be $\boxed{y = \pm\sqrt{C - x^2}}$.

Also, note that we can check our solution by taking its derivative.

$$x^2 + y^2 = C$$

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[C]$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \checkmark$$

Ex 2.3.3: Find the general solution to the differential equation $\frac{dm}{dt} = mt$

Sol 2.3.2:

$\frac{dm}{dt} = mt$ Start here.

$\frac{1}{m}\,dm = t\,dt$	Separate all the m terms to the left side of the equation and all of the t terms to the right side of the equation.
$\int \frac{1}{m}\,dm = \int t\,dt$	Integrate both sides.
$\ln\lvert m\rvert = \frac{1}{2}t^2 + C$	You only need C on one side of the equation.
$e^{\ln\lvert m\rvert} = e^{\frac{1}{2}t^2+C}$	e both sides of the equation to solve for y.
$m = e^{\frac{1}{2}t^2}e^C$	Pull out the constants from the equation.
$m = Ke^{\frac{1}{2}t^2}$	e^C is still a constant, which we will just denote as K.

Ex 2.3.4: Find the particular solution to $y' = 2xy - 3y$, given $y(3) = 2$.

Sol 2.3.4: To find the particular solution, recall that we first need to find the general solution.

$$\frac{dy}{dx} = (2x-3)y$$

$$\int \frac{1}{y}\,dy = \int (2x-3)\,dx$$

$$\ln\lvert y\rvert = x^2 - 3x + C$$

$$y = e^{x^2-3x+C}$$

$$= e^{x^2-3x}e^C$$

$$= Ke^{x^2-3x}$$

Now, let's plug in our initial value to solve for K.

$$y(3) = 2 \therefore Ke^{3^2-3(3)} = 2 \therefore Ke^0 = 2 \therefore K = 2$$

So, our particular solution is $\boxed{y = 2e^{x^2-3x}}$.

Ex 2.3.5: Find the particular solution to $\dfrac{dy}{dx} = x^2 y$, given $y(0) = -2$.

Sol 2.3.5: Once again, let's find the general solution first.

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{1}{y} = \int x^2\, dx$$

$$\ln|y| = \frac{1}{3}x^3 + C$$

$$y = e^{\frac{1}{3}x^3 + C}$$

$$= e^{\frac{1}{3}}x^3 e^C$$

$$= Ke^{\frac{1}{3}}$$

Now, let's find K.

$$y(0) = -2 \therefore Ke^{\frac{1}{3}(0)} = -2 \therefore Ke^0 = -2 \therefore K = -2$$

So, our particular solution is $\boxed{y = -2e^{\frac{1}{3}x^3}}$.

Ex 2.3.6: Find the particular solution to $\dfrac{dy}{dx} = x^2 y^3$, given $y(0) = 1$.

Sol 2.3.6:

$$\frac{dy}{dx} = x^2 y^3$$

$$\int \frac{1}{y^3}\, dy = \int x^2\, dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + C$$

Usually, at this step, we would isolate y to find the general solution. However, notice that it's easier to solve for C here, because solving for y will likely result in nested fractions, which can be difficult to work with.

$$y(0) = 1 \therefore -\frac{1}{2(1)^2} = \frac{0^2}{2} + C \therefore C = -\frac{1}{2}$$

Now, we can continue solving for the particular solution.

$$-\frac{1}{2y^2} = \frac{x^2}{2} - \frac{1}{2}$$

$$\frac{1}{y^2} = -x^2 + 1$$

$$y^2 = \frac{1}{1 - x^2}$$

$$y = \frac{1}{\pm\sqrt{1 - x^2}}$$

Since $x = 0$ gave us $y = +1$, our particular solution must be:

$$\boxed{y = \frac{1}{\sqrt{1 - x^2}}}$$

Ex 2.3.7: Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = \dfrac{y+1}{x^2+9}$ and suppose the point $(0, -3)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Determine if the point $(0, -3)$ is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = \dfrac{y+1}{x^2+9}$ at $(0, -3)$.

Sol 2.3.7:
a)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{y+1}{x^2+9}\right]$$

$$= \frac{(x^2+9)\dfrac{dy}{dx} - (y+1)(2x)}{(x^2+9)^2}$$

$$= \frac{(x^2+9)\left(\frac{y+1}{x^2+9}\right) - (y+1)(2x)}{(x^2+9)}$$

$$= \frac{(y+1) - (y+1)(2x)}{(x^2+9)^2}$$

$$= \boxed{\frac{(y+1)(1-2x)}{(x^2+9)}}$$

b)

$$\left.\frac{d^2y}{dx^2}\right|_{(0,-3)} = \frac{(-3)+1}{(0)^2+9}$$

$$= -\frac{2}{9}$$

Because the derivative at the point $(0, -3)$ is not equal to zero, the point is $\boxed{\text{neither a maximum nor a minimum}}$.

c)

$$\frac{dy}{dx} = \frac{y+1}{x^2+9}$$

$$\int \frac{1}{y+1}\,dy = \int \frac{1}{x^2+9}\,dx$$

$$\ln|y+1| = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$y+1 = e^{\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C}$$

$$y = Ke^{\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)} - 1$$

$$y(0) = -3 \therefore -3 = Ke^{\frac{1}{3}\tan^{-1}\left(\frac{(0)}{3}\right)} - 1 \therefore K = -2$$

$$\boxed{y = -2e^{\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)} - 1}$$

2.3 Free Response Homework

Separate the variables for the following differential equations.

1. $\dfrac{dy}{dx} = \dfrac{y}{x}$

2. $\dfrac{dy}{dx} = xy^2$

3. $\left(x^2+1\right)\dfrac{dy}{dx} = xy$

4. $\dfrac{dy}{dt} = \dfrac{\sec^2(t)}{\sec(y)\tan(y)}$

5. $\dfrac{dv}{dt} = 2 + 2v + t + tv$

6. $\dfrac{dy}{dx} = \dfrac{x^2+1}{\sec(y)\tan(y)}$

7. $\dfrac{dy}{dt} = \dfrac{t}{y\sqrt{y^2+1}}$

8. $\dfrac{dy}{dx} = \dfrac{y^2+1}{xy}$

Find the particular solutions to these differential equations with the given initial conditions.

9. $\dfrac{dy}{dx} = \dfrac{3y^2}{1+x^2} \;\Big|\; y(1) = 5$

10. $\dfrac{dy}{dx} = \dfrac{x^2\sqrt{x^3-2}}{y^2} \;\Big|\; y(3) = 0$

11. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{4y^3} \;\Big|\; y(0) = 1$

12. $\dfrac{dy}{dx} = y^2\cos(x) \;\Big|\; y\left(\dfrac{\pi}{2}\right) = 1$

13. $\dfrac{dy}{dx} = 4xy^3 \;\Big|\; y(0) = -2$

14. $\dfrac{d\theta}{dr} = \dfrac{1+\sqrt{r}}{\sqrt{\theta}} \;\Big|\; \theta(4) = 9$

15. $\dfrac{dy}{dx} = 8xy \;\Big|\; y(0) = 5$

16. $\dfrac{dy}{dx} = 3y(2x+1) \;\Big|\; y(0) = 1$

17. $\dfrac{dy}{dx} = 6x^2y^2 \;\Big|\; y(1) = 3$

18. $\dfrac{dy}{dx} = y^2+1 \;\Big|\; y(1) = 0$

19. $\dfrac{dy}{dx} = \dfrac{2x^3}{3y^2} \;\Big|\; y\left(\sqrt{2}\right) = 0$

20. $\dfrac{dy}{dx} = \dfrac{\cos(x)}{y} \;\Big|\; y\left(\dfrac{\pi}{2}\right) = 3$

21. $\dfrac{dy}{dx} = xy^2 \;\Big|\; y(0) = 5$

22. $\dfrac{dy}{dx} = \dfrac{\sec(y)}{x} \;\Big|\; y(1) = \dfrac{\pi}{2}$

23. $\dfrac{dy}{dx} = y^2+1 \;\Big|\; y(1) = 0$

24. $\dfrac{dy}{dx} = \dfrac{2x}{y} \;\Big|\; y(0) = 1$

25. $\dfrac{du}{dt} = \dfrac{2t+\sec^2(t)}{2u} \;\Big|\; u(0) = -5$

26. $\dfrac{dy}{dx}\left(x^2+1\right)(2-y) \;\Big|\; y(1) = 3$

27. $\dfrac{dy}{dx} = \dfrac{y^2+1}{xy} \;\Big|\; y(0) = -1$

28. $\dfrac{dy}{dx} = \dfrac{xy^2+x}{y} \;\Big|\; y(0) = -1$

29. $\dfrac{dy}{dx} = \dfrac{\sin(x)}{\sin(y)}$ $\quad\Big|\quad$ $y(0) = \dfrac{\pi}{2}$

30. $\dfrac{dy}{dx} = yx - y\sin(x)$ $\quad\Big|\quad$ $y(0) = 5e$

31. Solve the equation $e^{-y}\dfrac{dy}{dx} + \cos(x) = 0$.

32. Find an equation of the curve that satisfies $\dfrac{dy}{dx} = 4x^3y$ and whose y-intercept is 7.

33. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = y^2(6-2x)$, and suppose the point $\left(3, -\dfrac{1}{3}\right)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Use the solution to a) to determine if the point $\left(3, -\dfrac{1}{3}\right)$ is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = y^2(6-2x)$ at $\left(3\ -\dfrac{1}{3}\right)$.

34. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = xy + y$, and suppose the point $(-1, 2)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Use the solution to a) to determine if the point $(-1,\ 2)$ is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = xy + y$ at $(-1,\ 2)$.

35. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = \dfrac{3x^2}{y+2}$, and suppose the point $(-1, 2)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Use the solution to a) to determine if the point $(0,\ 1)$ is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = \dfrac{3x^2}{y+2}$ at $(0,\ 1)$.

36. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = (x-1)(y+2)$, and suppose the point $(1,0)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Use the solution to a) to determine if the point (1, 3) is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = (x-1)(y+2)$ at (1, 3).

2.3 Multiple Choice Homework

1. If $\dfrac{dy}{dx} = \sin(x)\cos^3(x)$ and if $y = 1$ when $x = \pi$, what is the value of y when $x = 0$?

a) 0   b) 1   c) 1.5   d) 2   e) 2.5

---

2. If $\dfrac{dy}{dx} = \cos(x)\sin^2(x)$ and if $y = 0$ when $x = \pi$, what is the value of y when $x = 0$?

a) -1 b) $-\dfrac{1}{3}$ c) 0   d) $\dfrac{1}{3}$ e) 1

---

3. The solution to the differential equation $\dfrac{dy}{dx} = 8xy$ with initial condition $y(0) = 5$ is

a) $\ln\left(4x^2+5\right)$ b) $e^{4x^2}+5$ c) $e^{4x^2}+4$ d) $5\ln\left(4x^2\right)$ e) $5e^{4x^2}$

---

4. Identify the first mistake (if any) in this process:

Problem:	$\dfrac{dy}{dx} = xy + x$
Step 1:	$\dfrac{1}{y+1}\,dy = x\,dx$
Step 2:	$\ln\lvert y+1\rvert = x^2 + C$
Step 3:	$y + 1 = e^{x^2} + C$
Step 4:	$y = e^{x^2} + C$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

---

5. Identify the first mistake (if any) in this process:

Problem:	$\dfrac{dy}{dx} = 6x^2y^2$
Step 1:	$\dfrac{1}{y^2}\,dy = 6x^2\,dx$
Step 2:	$\ln\lvert y^2\rvert = 2x^3 + C$
Step 3:	$y^2 = e^{2x^3+C}$
Step 4:	$y = \pm\sqrt{Ke^{2x^3}}$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

2.4 Integration by Back-Substitution

Sometimes when applying the Chain Rule, the other factor is not the du, or there are extra $x's$ that must be replaced with some form of u. The method of choosing u to equal the inside of the composite function remains the same, but there is more substitution necessary.

Steps to Integration by Back-Substitution

1. Find u and du, just as with u-substitution.
2. Handle extra x's. Identify any remaining "extra" x terms, and express x in terms of u.
3. Replace all x-expressions in the integral with u and du.
4. Integrate appropriately, and replace u with the original x-expression to get the final answer

All of this is best understood with some examples.

Ex 2.4.1: $\int x^3 \left(x^2+4\right)^{\frac{3}{2}}\, dx$

Sol 2.4.1:

$$\hookrightarrow u = x^2 + 4 \therefore x^2 = u - 4$$

$$\hookrightarrow du = 2x\, dx$$

$$\begin{aligned}
\int x^3 \left(x^2+4\right)^{\frac{3}{2}}\, dx &= \frac{1}{2}\int (2x)x^2 \left(x^2+4\right)^{\frac{3}{2}}\, dx \\
&= \frac{1}{2}\int (u-4)u^{\frac{3}{2}}\, du \\
&= \frac{1}{2}\int \left(u^{\frac{5}{2}} - 4u^{\frac{3}{2}}\right)\, du \\
&= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{8}{5}u^{\frac{5}{2}}\right) + C \\
&= \boxed{\frac{1}{7}\left(x^2+4\right)^{\frac{7}{2}} - \frac{4}{5}\left(x^2+4\right)^{\frac{5}{2}} + C}
\end{aligned}$$

Notice how when we attempted u-substitution, an x^2 remained in the equation. That is the reason why we expressed x^2 in terms of u.

Ex 2.4.2: $\int (x+1)\sqrt{x-1}\,dx$

Sol 2.4.2:

$\hookrightarrow u = x - 1 \therefore x = u + 1$

$\hookrightarrow du = dx$

$$\begin{aligned}
\int (x+1)\sqrt{x-1}\,dx &= \int ((u+1)+1)\sqrt{u}\,du \\
&= \int (u+2)\sqrt{u}\,du \\
&= \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right)\,du \\
&= \frac{2}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + C \\
&= \boxed{\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + C}
\end{aligned}$$

Ex 2.4.3: $\int (x+2)(x-3)^4\,dx$

Ex 2.4.3:

$\hookrightarrow u = x - 3 \therefore x = u + 3$

$\hookrightarrow du = dx$

$$\begin{aligned}
\int (x+2)(x-3)^4\, dx &= ((u+3)+2)u^4\, du \\
&= \int (u+5)u^4\, du \\
&= \int u^5 + 5u^4\, du \\
&= \frac{1}{6}u^6 + u^5 + C \\
&= \boxed{\frac{1}{6}(x-3)^6 + (x-3)^5 + C}
\end{aligned}$$

Ex 2.4.4: $\displaystyle\int \frac{x^2+4}{x+2}\, dx$

Sol 2.4.4: There are two ways to approach this problem. One could use polynomial long division to simplify before integrating:

$$\int \frac{x^2+4}{x+2}\, dx = \int \left(x-2+\frac{8}{x+2}\right)\, dx$$

Then

$$\hookrightarrow u = x+2$$
$$\hookrightarrow du = dx$$

$$\begin{aligned}
\int \left(x-2+\frac{8}{x+2}\right)\, dx &= \int (x-2)\, dx + 8\int \frac{1}{u}\, du \\
&= \frac{1}{2}x^2 - 2x + 8\ln|u| + C \\
&= \boxed{\frac{1}{2}x^2 - 2x + 8\ln|x+2| + C}
\end{aligned}$$

An alternative method to solving this problem would be to make the denominator u and use back-substitution:

$$\hookrightarrow u = x+2 \therefore x = u = 2$$
$$\hookrightarrow du = dx$$

$$
\begin{aligned}
\int \frac{x^2+4}{x+2}\,dx &= \int \frac{(u-2)^2+4}{u}\,du \\
&= \int \frac{u^2-4u+4+4}{u}\,du \\
&= \int \frac{u^2-4u+8}{u}\,du \\
&= \int \left(u-4+\frac{8}{u}\right)\,du \\
&= \frac{1}{2}u^2-4u+8\ln|u|+C \\
&= \boxed{\frac{1}{2}(x+2)^2-4(x+2)+8\ln|x+2|+C}
\end{aligned}
$$

From visual inspection, these answers may appear different. However, with FOILing and adding like terms, it can be shown that these are in fact the same answers.

As a rule of thumb, doing algebraic simplification before calculus will generally make the problem shorter and simpler. In this case, polynomial long division made the problem easier than back-substitution.

2.4 Free Response Homework

Perform the antidifferentiation.

1. $\int x\sqrt{4-x}\,dx$

2. $\int x^5\sqrt{x^3+4}\,dx$

3. $\int \frac{x+5}{2x+3}\,dx$

4. $\int x^3\left(x^2+1\right)^{12}$

5. $\int \frac{(3+\ln x)^2(2-\ln x)}{x}\,dx$

6. $\int \sqrt{4-\sqrt{x}}\,dx$

7. $\int x^5\left(x^2+4\right)^2\,dx$

8. $\int \sqrt{x+3}(x+1)^2\,dx$

9. $\int (t-1)(2t+4)^5\,dt$

10. $\int (z-3)(3z-1)^3\,dz$

11. $\int \frac{y^5}{\sqrt{y^3+5}}\,dy$

12. $\int \frac{w^5}{w^2+4}\,dw$

13. $\int \frac{x^5}{(x^2-1)^{\frac{5}{2}}}\,dx$

14. $\int \frac{x^7}{(x^4+4)^{\frac{3}{2}}}\,dx$

15. $\int (x+2)\sqrt[3]{x-1}\,dx$

16. $\int \sqrt{4-x}(2x+5)\,dx$

2.4 Multiple Choice Homework

1. $\int x^3\sqrt{1+x^2}\,dx =$

a) $\frac{x^4}{2}\cdot\frac{(1+x^2)^{\frac{3}{2}}}{3}+C$

b) $\frac{1}{2}\left(1+x^2\right)^{\frac{1}{2}}+\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

c) $-\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+\frac{1}{5}\left(1+x^2\right)^{\frac{5}{2}}+C$

d) $\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}-\frac{1}{5}\left(1+x^2\right)^{\frac{5}{2}}+C$

e) $\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

---

2. $\int x^5\sqrt{1+x^2}\,dx =$

a) $\frac{x^6\,(1+x^2)^{\frac{3}{2}}}{18}+C$

b) $\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+\frac{2}{7}\left(1+x^2\right)^{\frac{7}{2}}+C$

c) $\frac{1}{7}\left(1+x^2\right)^{\frac{7}{2}}-\frac{2}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

d) $\frac{2}{7}\left(1+x^2\right)^{\frac{7}{2}}-\frac{4}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

e) $\frac{1}{7}\left(1+x^2\right)^{\frac{7}{2}}+\frac{2}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

---

3. $\int \frac{x^3}{9-x^2}\,dx =$

a) $\frac{x^3}{3}\cdot\frac{(9-x^2)^{\frac{3}{2}}}{9}+C$

b) $\frac{9}{2}\ln|9-x^2|+\frac{1}{2}\left(9-x^2\right)+C$

c) $\frac{9}{2}\left(9-x^2\right)^2-\frac{1}{2}\left(9-x^2\right)+C$

d) $\frac{9}{2}\left(9-x^2\right)+\frac{1}{2}\left(9-x^2\right)^2+C$

e) $\frac{x^4}{36}-\frac{x^2}{2}+C$

---

4. $\int \frac{x^5}{x^2+5}\,dx =$

a) $\frac{1}{4}\left(x^2+5\right)^2-5\left(x^2+5\right)+\frac{25}{2}\ln|x^2+5|+C$

b) $\frac{1}{4}\left(x^2+5\right)^2-5\left(x^2+5\right)+\frac{25}{2}\tan^{-1}\left(x^2+5\right)+C$

c) $\frac{1}{4}\left(x^2+5\right)^2+5\left(x^2+5\right)+\frac{25}{2}\ln|x^2+5|+C$

d) $\frac{1}{4}\left(x^2+5\right)^2+5\left(x^2+5\right)+\frac{25}{2}\tan^{-1}\left(x^2+5\right)+C$

e) None of the above

---

5. $\int e^{2x}\sqrt{e^x+1}\,dx =$

a) $e^{2x}\left(e^x+1\right)^{\frac{3}{2}}+C$

b) $\frac{2}{5}\left(e^x+1\right)^{\frac{5}{2}}-3\left(e^x+1\right)^{\frac{3}{2}}+C$

c) $\dfrac{2}{5}\left(e^{x}+1\right)^{\frac{5}{2}}-\dfrac{2}{3}\left(e^{x}+1\right)^{\frac{3}{2}}+C$

d) $\dfrac{2}{5}\left(e^{x}+1\right)^{\frac{5}{2}}+3\left(e^{x}+1\right)^{\frac{3}{2}}+C$

e) $\dfrac{2}{5}e^{\frac{5x}{2}}-5e^{\frac{3x}{2}}+C$

---

6. $\dfrac{x^{3}}{\sqrt{4-x^{2}}}\,dx=$

a) $\dfrac{1}{3}\left(4-x^{2}\right)^{\frac{3}{2}}-4\left(4-x^{2}\right)^{\frac{1}{2}}+C$

b) $\dfrac{2}{3}\left(4-x^{2}\right)^{\frac{3}{2}}-2\left(4-x^{2}\right)^{\frac{1}{2}}+C$

c) $\dfrac{1}{3}\left(4-x^{2}\right)^{\frac{3}{2}}-4\sin^{-1}\left(\dfrac{x}{2}\right)+C$

d) $\dfrac{2}{3}\left(4-x^{2}\right)^{\frac{3}{2}}-2\sin^{-1}\left(\dfrac{x}{2}\right)+C$

e) $\dfrac{1}{3}\left(4-x^{2}\right)^{\frac{3}{2}}-\sin^{-1}\left(\dfrac{x}{2}\right)+C$

---

7. $\dfrac{4-x}{\sqrt{4-x^{2}}}\,dx=$

a) $4\sin^{-1}\left(\dfrac{x}{2}\right)+\left(4-x^{2}\right)^{\frac{1}{2}}+C$

b) $2\sin^{-1}\left(\dfrac{x}{2}\right)+\left(4-x^{2}\right)^{\frac{1}{2}}+C$

c) $4\left(4-x^{2}\right)^{\frac{1}{2}}+C$

d) $\sin^{-1}\left(\dfrac{x}{2}\right)-\left(4-x^{2}\right)^{\frac{1}{2}}+C$

e) $\dfrac{2}{3}\left(4-x^{2}\right)^{\frac{3}{2}}+C$

2.5 Powers of Trig Functions: Sine and Cosine

Another instance of back-substitution involves the trig functions and the Pythagorean Identities. As we saw in the previous section, since

$$\frac{d}{dx}[\cos(x)] = -\sin(x) \text{ and } \frac{d}{dx}[\sin(x)] = \cos(x),$$

one of these functions can serve as the du while the other serves as u. But, what about when higher exponents are involved? In general, what about integrals in the form

$$\int \sin^m(x)\cos^n(x)\,dx?$$

Remember:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

OBJECTIVES

Use Integration by Substitution to Integrate Integrands Involving Sine and Cosine.

There are two cases of integration of this kind of integrand, depending on the powers m and n.

Case 1

The simpler (and more common on the AP Test) case is when either m, n, or both m and n are odd numbers. One of whichever function has the odd power will be the du and the rest of those functions can convert to the other trig function by means of the Pythagorean Identities.

Ex 2.5.1: $\int \sin^4(x)\cos^3(x)\,dx$

Sol 2.5.1: Since $\cos(x)$ has the odd power, we will make that our du.

$\hookrightarrow u = \sin(x)$

$\hookrightarrow du = \cos(x)\,dx$

$$
\begin{aligned}
\int \sin^4(x)\cos^3(x)\,dx &= \int \sin^4(x)\cos^2(x)\cos(x)\,dx \\
&= \int \sin^4(x)\left(1-\sin^2(x)\right)\cos(x)\,dx \\
&= \int u^4\left(1-u^2\right)\,du \\
&= \int \left(u^4-u^6\right)\,du \\
&= \frac{1}{5}u^5-\frac{1}{7}u^7+C \\
&= \boxed{\frac{1}{5}\sin^5(x)-\frac{1}{7}\sin^7(x)+C}
\end{aligned}
$$

Ex 2.5.2: $\int \sin^5(x)\cos^2(x)\,dx$

Sol 2.5.2: Since $\sin(x)$ has the odd power, we will make that our du.

$\hookrightarrow$ $u = \cos(x)$

$\hookrightarrow$ $du = -\sin(x)\,dx$

$$
\begin{aligned}
\int \sin^5(x)\cos^2(x) &= -\int \sin^4(x)(-\sin(x))\cos^2(x)\,dx \\
&= -\int \left(\sin^2(x)\right)^2 (-\sin(x))\cos^2(x)\,dx \\
&= -\int \left(1-\cos^2(x)\right)^2 (-\sin(x))\cos^2(x)\,dx \\
&= -\int \left(1-u^2\right)^2 u^2\,du \\
&= -\int \left(1-2u^2+u^4\right) u^2\,du \\
&= -\int \left(u^2-2u^4+u^6\right)\,du \\
&= -\left(\frac{1}{3}u^3-\frac{2}{5}u^5+\frac{1}{7}u^7\right)+C \\
&= \boxed{-\frac{1}{3}\cos^3(x)+\frac{2}{5}\cos^5(x)-\frac{1}{7}\cos^7(x)+C}
\end{aligned}
$$

If both powers are odd, either function can serve as the u. However, it is generally easier to choose $u = \sin(x)$ as there is no negative sign to deal with.

Ex 2.5.3: $\displaystyle\int \tan(x)\,dx$

Sol 2.5.3: At first, this does not appear to be a sine or cosine integral, but a basic substitution reveals that it is.

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$$\hookrightarrow u = \cos(x)$$

$$\hookrightarrow du = -\sin(x)\,dx$$

$$\int \frac{\sin(x)}{\cos(x)}\,dx = -\int(-\sin(x))\frac{1}{\cos(x)}\,dx$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos(x)| + C$$

$$= \boxed{\ln|\sec(x)| + C}$$

It may not be immediately apparent why $-\ln|\cos(x)|$ can be rewritten as $\ln|\sec(x)|$. The reason lies within the log rule $\log\left(a^b\right) = b\log a$:

$$-\ln|\cos(x)| = -\ln\left|\left(\frac{1}{\cos(x)}\right)^{-1}\right|$$

$$= -1\cdot -\ln\left|\frac{1}{\cos(x)}\right|$$

$$= \ln\left|\frac{1}{\cos(x)}\right|$$

$$= \ln|\sec(x)|$$

This gives us two more integral rules:

$$\int \tan(u)\,du = \ln|\sec(u)| + C \qquad \int \cot(u)\,du = \ln|\sin(u)| + C$$

Case 2

The more difficult situation is when both powers are even. In this case, variations on the half angle argument rules come into play.

Remember:

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \qquad\qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

Ex 2.5.4: $\displaystyle\int \cos^2(x)\,dx$

Sol 2.5.4:

$$\int \cos^2(x) = \int \frac{1}{2}(1 + \cos(x))\,dx$$

$\hookrightarrow$ $u = 2x$

$\hookrightarrow$ $du = 2\,dx$

$$\int \frac{1}{2}(1 + \cos(x))\,dx = \frac{1}{2} \cdot \frac{1}{2} \int (1 + \cos(x))2\,dx$$

$$= \frac{1}{4} \cdot (1 + \cos(u))\,du$$

$$= \frac{1}{4}u + \frac{1}{4}\sin(u) + C$$

$$= \boxed{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C}$$

This example alludes to two more integral equations that are helpful to know:

$$\int \cos^2(u)\,du = \frac{1}{2}u + \frac{1}{4}\sin(2u) + C \qquad\qquad \int \sin^2(u)\,du = \frac{1}{2}u - \frac{1}{4}\sin(2u) + C$$

Ex 2.5.5: $\displaystyle\int \sin^4(x)\cos^2(x)\,dx$

Sol 2.5.5:

$$\int \sin^4(x)\cos^2(x)\,dx = \int \left(\frac{1}{2}(1 - \cos(2x))\right)^2 \left(\frac{1}{2}(1 + \cos(2x))\right)\,dx$$

$$= \frac{1}{8}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right)(1 + \cos(2x))\,dx$$

$$= \frac{1}{8}\int \Big(1 - \cos(2x) - \cos^2(2x) + \cos^3(2x)\Big)\ dx$$

$\hookrightarrow u = 2x$

$\hookrightarrow du = 2\,dx$

$$= \frac{1}{2}\cdot\frac{1}{8}\int \Big(1 - \cos(2x) - \cos^2(2x) + \cos^3(2x)\Big)\,2\,dx$$

$$= \frac{1}{16}\int \Big(1 - \cos(u) - \cos^2(u) + \cos^3(u)\Big)\ du$$

$$= \frac{1}{16}\int\ du - \frac{1}{16}\int \cos(u)\,du - \frac{1}{16}\int \cos^2(u)\,du + \frac{1}{16}\int \cos^3(u)\,du$$

$$= \frac{1}{16}u - \frac{1}{16}\sin(u) - \frac{1}{16}\left(\frac{1}{2}u - \frac{1}{4}\sin(2u)\right)$$
$$+ \frac{1}{16}\int \cos^2(u)\cos(u)\,du + C$$

$\hookrightarrow v = \sin(u)$

$\hookrightarrow dv = \cos(u)\,dx$

$$= \frac{1}{16}u - \frac{1}{16}\sin(u) - \frac{1}{16}\left(\frac{1}{2}u - \frac{1}{4}\sin(2u)\right)$$
$$+ \frac{1}{16}\int \Big(1 - \sin^2(u)\Big)\cos(u)\,du + C$$

$$= \frac{1}{16}u - \frac{1}{16}\sin(u) - \frac{1}{32}u + \frac{1}{64}\sin(2u) + \frac{1}{16}\int \Big(1 - v^2\Big)\ dv + C$$

$$= \frac{1}{16}u - \frac{1}{16}\sin(u) - \frac{1}{32}u + \frac{1}{64}\sin(2u) + \frac{1}{16}\left(v - \frac{1}{3}v^3\right) + C$$

$$= \frac{1}{16}(2x) - \frac{1}{16}\sin(2x) - \frac{1}{32}(2x) + \frac{1}{64}\sin(2(2x))$$
$$+ \frac{1}{16}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right) + C$$

$$= \frac{1}{8}x - \frac{1}{16}\sin(2x) - \frac{1}{16}x + \frac{1}{64}\sin(4x) + \frac{1}{16}\sin(2x)$$
$$- \frac{1}{48}\sin^3(2x) + C$$

$$= \boxed{\frac{1}{16}x + \frac{1}{64}\sin(4x) - \frac{1}{48}\sin^3(2x) + C}$$

2.5 Free Response Homework

Perform the antidifferentiation.

1. $\int \sin^3(x)\cos^2(x)\,dx$
2. $\int \sin^4(x)\cos^5(x)\,dx$
3. $\int \sin^2(x)\cos^7(x)\,dx$
4. $\int \sin^5(x)\cos^6(x)\,dx$
5. $\int \sin(x)\cos^5(x)\,dx$
6. $\int \sin^5(x)\cos^5(x)\,dx$
7. $\int \sin^2(x)\cos^2(x)\,dx$
8. $\int \sin^2(x)\cos^4(x)\,dx$

2.5 Multiple Choice Homework

1. For $\int \sin^3(x)\cos^5(x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

2. For $\int \sin^3(5x)\cos^2(5x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

3. For $\int \sin^4(4x)\cos^5(4x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

4. For $\int \sin^2(x)\cos^4(x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

5. $\int \cos^2(2x)\,dx =$

a) $\sin(4x) + C$ b) $\frac{1}{2}x + \frac{1}{8}\sin(4x) + C$ c) $\frac{1}{2}x - \frac{1}{8}\sin(4x) + C$

d) $x + \frac{1}{4}\sin(4x) + C$ e) $x + \frac{1}{8}\cos(4x) + C$

---

6. $\int \cos^2\left(\frac{1}{2}x\right)\, dx =$

a) $\sin(4x) + C$ b) $\frac{1}{2}x + \frac{1}{4}\sin(x) + C$ c) $\frac{1}{2}x - \frac{1}{4}\sin(x) + C$

d) $\frac{1}{4}x + \frac{1}{2}\sin(x) + C$ e) $\frac{1}{4}x + \frac{1}{2}\cos(x) + C$

---

7. Identify the first mistake (if any) in this process:

Problem: $\int \sin^3(2x)\cos^4(2x) =$

Step 1: $= -\frac{1}{2}\int \sin^2(2x)\cos^4(2x)(-\sin(2x))2\, dx$

Step 2: $= -\frac{1}{2}\int \left(1 - u^2\right) u^4\, du$

Step 3: $= -\frac{1}{2}\int \left(u^4 - u^6\right)\, du$

Step 4: $= -\frac{1}{2}\left(\frac{1}{5}u^5 - \frac{1}{7}u^7 + C\right)$

Step 5: $= -\frac{1}{10}\sin^5(4x) + \frac{1}{14}\sin^7(4x) + C$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

2.6 Powers of Trig Functions: Secant, Tangent, and Beyond

As with sine and cosine, secant and tangent work together in a Pythagorean Identity, as well as cosecant and cotangent. So, we will be considering integrals of the form

$$\int \sec^m(x)\tan^n(x)\,dx \quad \text{and} \quad \int \csc^m(x)\cot^n(x)\,dx$$

to be cases of integration by u-substitution.

Remember:

$$\tan^2(\theta)+1=\sec^2(\theta) \qquad \cot^2(\theta)+1=\csc^2(\theta)$$

$$\frac{d}{dx}[\tan(u)]=\left(\sec^2(u)\right)\frac{du}{dx} \qquad \frac{d}{dx}[\sec(u)]=(\sec(u)\tan(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\cot(u)]=\left(-\csc^2(u)\right)\frac{du}{dx} \qquad \frac{d}{dx}[\csc(u)]=(-\csc(u)\cot(u))\frac{du}{dx}$$

OBJECTIVES

Use Integration by Substitution to Integrate Integrands Involving Secant, Tangent, Cosecant, and Cotangent.

There are three cases of integration of this kind of integrand, depending on the powers m and n.

Case 1

The first case is when the secant's (or cosecant's) power is even. In this case, our u would be $\tan(x)$ or $\cot(x)$ and our du would be $\sec^2(x)\,dx$ or $-\csc^2(x)\,dx$.

Ex 2.6.1: $\int \sec^4(x)\tan^2(x)\,dx$

Sol 2.6.1:

$\hookrightarrow u=\tan(x)$

$\hookrightarrow du=\sec^2(x)$

$$
\begin{aligned}
\int \sec^4(x)\tan^2(x) &= \int \sec^2(x)\tan^2(x)\sec^2(x)\,dx \\
&= \int \left(1+\tan^2(x)\right)\tan^2(x)\sec^2(x)\,dx \\
&= \int \left(1+u^2\right)u^2\,du \\
&= \int \left(u^2+u^4\right)\,du \\
&= \frac{1}{3}u^3+\frac{1}{5}u^5+C \\
&= \boxed{\frac{1}{3}\tan^3(x)+\frac{1}{5}\tan^5(x)+C}
\end{aligned}
$$

Case 2

The second case is when the tangent's (or cotangent's) power is odd. In this case, our u would be $\sec(x)$ or $\csc(x)$ and our du would be $\sec(x)\tan(x)\,dx$ or $-\csc(x)\cot(x)\,dx$.

Ex 2.6.2: $\displaystyle\int \csc^3(x)\cot^3(x)\,dx$

Sol 2.6.2:

$\hookrightarrow$ $u=\csc(x)$

$\hookrightarrow$ $du=-\csc(x)\cot(x)\,dx$

$$
\begin{aligned}
\int \csc^3(x)\cot^3(x)\,dx &= -\int \csc^2(x)\cot^2(x)(-\csc(x)\cot(x))\,dx \\
&= -\int u^2\left(u^2-1\right)\,du \\
&= -\int \left(u^4-u^2\right)\,du \\
&= -\frac{1}{5}u^5+\frac{1}{3}u^3+C \\
&= \boxed{-\frac{1}{5}\csc^5(x)+\frac{1}{3}\csc^3(x)+C}
\end{aligned}
$$

If both cases are present (that is, the tangent's/cotangent's power is odd AND the secant's/cosecant's power is even), then any of the functions can serve as u.

Case 3

The third and final case is when the tangent's/cotangent's power is even AND the secant's/cosecant's power is odd. This case is no longer doable by integration by substitution, and requires a technique known as *integration by parts*. We will need to wait until Chapter 8 to approach this case.

Quick Summary of 2.5-2.6:

I. $\int \sin^m(x)\cos^n(x)\,dx$

  a) The odd power determines du. The other function is u.

  b) If both powers are even, use the half-angle formulas and simplify.

II. $\int \sec^m(x)\tan^n(x)\,dx$ or $\int \csc^n(x)\cot^m(x)\,dx$

  a) If both powers are even, $u = \tan(x)$ or $\cot(x)$ and $du = \sec^2(x)\,dx$ or $-\csc^2(x)\,dx$.

  b) If both powers are odd, $u = \sec(x)$ or $\csc(x)$ and $du = \sec(x)\tan(x)\,dx$ or $-\csc(x)\cot(x)\,dx$.

  c) If n is even and m is odd, either (a) or (b) will work.

  d) If n is odd and m is even, neither (a) nor (b) will work.

III. For any other mix of trig functions, convert all to sine and cosine and use I. above.

2.6 Free Response Homework

Perform the antidifferentiation.

1. $\int \sec^2(x)\tan^5(x)\,dx$
2. $\int \sec^6(x)\tan^4(x)\,dx$
3. $\int \sec^5(x)\tan^7(x)\,dx$
4. $\int \sec^2(x)\tan^6(x)\,dx$
5. $\int \sec^6(x)\tan^3(x)\,dx$
6. $\int \csc^2(x)\cot^5(x)\,dx$
7. $\int \csc^4(x)\cot(x)\,dx$
8. $\int \csc^7(x)\cot^5(x)\,dx$

2.6. Multiple Choice Homework

1. For $\int \csc^3(x)\cot^5(x)\,dx$, the correct u-substitution is

   a) $u = \csc(x)$ b) $u = \cot(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

2. For $\int \csc^4(x)\cot^4(x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

3. For $\int \sec^4(x)\tan^5(x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

4. For $\int \sec^5(x)\tan^4(x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

5. Which of the following statements are true?

$$\text{I. } \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

II. $\displaystyle\int \tan(x)\,dx = \sec^2(x) + C$

III. $\displaystyle\int x^2 \cot\left(x^3\right)\,dx = \frac{1}{3}\ln\left|\sin\left(x^3\right)\right| + C$

a) I only b) II only c) III only d) I and II only e) I and III only

---

6. Which of the following statements are **false**?

I. $\displaystyle\int x^5 \sec\left(x^6\right) = \frac{1}{6}\ln\left|\sec\left(x^6\right) + \tan\left(x^6\right)\right| + C$

II. $\displaystyle\int \cot(x)\,dx = -\csc^2(x) + C$

III. $\displaystyle\int \csc(x)\,dx = \ln|\csc(x) - \cot(x)| + C$

a) I only b) II only c) III only d) I and II only e) I and III only

---

7. Identify the first mistake (if any) in this process:

Problem: $\displaystyle\int \sec^4(2x)\tan^3(2x) =$

Step 1: $\displaystyle= \frac{1}{2}\int \sec^2(2x)\tan^3(2x)\sec^2(2x)2\,dx$

Step 2: $\displaystyle= \frac{1}{2}\int \left(1-u^2\right)u^3\,du$

Step 3: $\displaystyle= \frac{1}{2}\int \left(u^3-u^5\right)\,du$

Step 4: $\displaystyle= \frac{1}{2}\left(\frac{1}{4}u^4 - \frac{1}{6}u^6 + C\right)$

Step 5: $\displaystyle= -\frac{1}{8}\tan^4(2x) - \frac{1}{12}\tan^6(2x) + C$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake
