Chapter 3:

Integrals

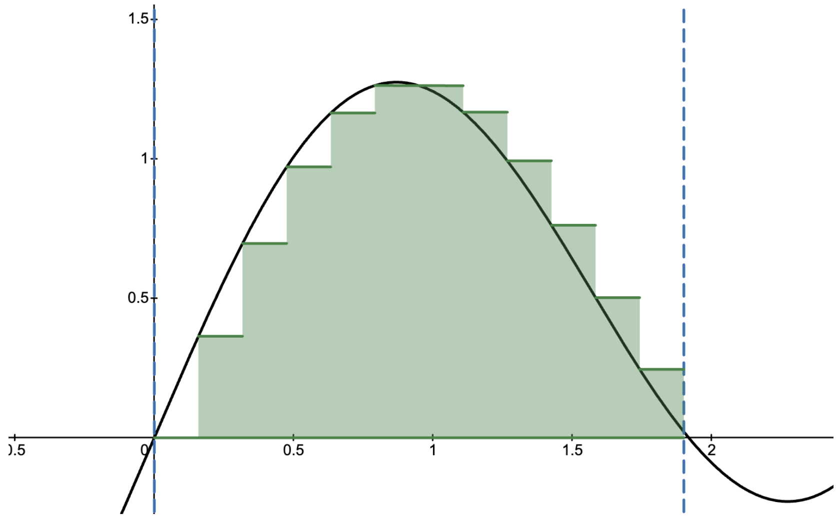
Chapter 3 Overview: Definite Integrals

In this chapter, we will study the Fundamental Theorem of Calculus, which establishes the link between the algebra and the geometry, with an emphasis on mechanics of how to find the definite integral. We will consider the differences implied between the context of the definite integral as an operation and as an area accumulator. We will learn some approximation techniques for definite integrals and see how they provide theoretical foundation for the integral. We will revisit graphical analysis in terms of the definite integral and view another typical AP context for it. Finally, we will consider what happens when trying to integrate at or near an asymptote.

As noted in the overview of the last chapter, antiderivatives are known as indefinite integrals because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an analytic-geometrical context of area. Though it is not necessary to know the theory behind this in order to calculate the integral, the theory is a major subject of integral calculus, so we will explore it briefly in here.

The Limit Definition of the Definite Integral

We know, from geometry, how to find the exact area of various polygons, but we never considered figures where one side is not made of a line segment. Here we want to consider an area bounded by some curve $y = f(x)$ on the top, the x-axis on the bottom, some arbitrary $x = a$ on the left, and $x = b$ on the right.



As we can see above, the area approximated by rectangles whose height is the y-value of the equation and whose width we will call Δx. The more rectangles we make, the better the approximation. For a good animation of this concept, consider the following video:

Riemann sum approximation animation

The area of each rectangle would be $f(x) \cdot \Delta x$, and the total area of n rectangles would be

$$A = \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x.$$

This equation is known as the **Riemann summation**. Although this equation looks complicated, it represents a rather simple idea. We are adding up the areas of many thin rectangles to approximate the total area under the curve $y = f(x)$ between two points. As we increase the number of rectangles n, each rectangle becomes narrower, and thus our approximation becomes more accurate.

But, how can we find the exact area? With the Riemann sum, we are only coming up with better and better approximations right now. If we could make an infinite number of rectangles (which would be infinitely thin), we could potentially find the exact area under this curve. Luckily, we just so happen to have the mathematical tools to do this: we can take the limit as n approaches infinity.

$$\lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x$$

This is where the definite integral comes in. The definite integral provides a precise way to calculate the exact area under a curve by taking the limit of the Riemann sum as the number of rectangles approaches infinity. In other words, instead of merely approximating the area with a finite number of rectangles, the definite integral captures what happens when the width of each rectangle becomes infinitesimally small. We write this limit in a compact form as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\cdot\Delta x,$$

where a is the "lower bound" and b is the "upper bound." Mathematicians sometimes nuance this statement as

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(a+k\Delta x)\cdot\Delta x,\text{ where }\Delta x = \frac{b-a}{n}}$$

3.1: The Fundamental Theorem of Calculus

Up to this point, we've seen how integration can be used to find the exact area under a curve by taking the limit of Riemann sums. But what's truly remarkable is how integration connects so deeply with differentiation. This connection is captured in one of the most important results in all of calculus: the Fundamental Theorem of Calculus (FTC).

The Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a, b]$, then:

1. $\dfrac{d}{dx}\displaystyle\int_c^x f(t)\,dt = f(x)$ or $\dfrac{d}{dx}\displaystyle\int_c^u f(t)\,dt = f(u)\cdot\dfrac{du}{dx}$

2. If $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

The first part of the Fundamental Theorem of Calculus simply says what we already know– that an integral is an anti-derivative. The second part of the Fundamental Theorem says that the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

The idea of the integral meaning the area may not make sense initially, mainly because we are used to geometry, where an area is always measured in square units. But, that is only because the length and width are always measured in the same kind of units, so multiplying length and width area always measured in the same kind of units, so multiplying length and width must yield square units. We are expanding our vision beyond that narrow view of things here. Consider a graph where the x-axis is time and the y-axis is velocity in feet per second. The area under the curve would be measured as seconds multiplied by feet per seconds, which is simply feet. So, the area under the curve is equal to the distance traveled in feet. In other words, the integral of velocity is distance.

OBJECTIVES

Evaluate Definite Integrals.
Find Average Value of a Continuous Function Over a Given Interval.
Differentiate Integral Expressions with the Variable in the Boundary.

Let us first consider part 2 of the Fundamental Theorem of Calculus, since it has a very practical application. This part of the Fundamental Theorem of Calculus gives us a clear method for evaluating definite integrals.

Ex 3.1.1: Evaluate $\displaystyle\int_2^8 (4x + 3)\,dx$

Sol 3.1.1: First, let's start by treating this as a regular antiderivative

$$\int (4x+3)\,dx = 2x^2 + 3x$$

Note that our $+C$ will not be needed, as we will be taking a definite integral. Now, let's apply the Fundamental Theorem of Calculus.

$$\begin{aligned}
\int_2^8 f(x)\,dx &= F(8) - F(2) \\
&= \left(2x^2 + 3x\right)\Big|_2^8 \\
&= 2(8)^2 + 3(8) - \left(2(2)^2 + 3(2)\right) \\
&= \boxed{138}
\end{aligned}$$

The vertical bar that you see is called the evaluation bar, and it's used to indicate that we are evaluating the antiderivative at the upper and lower limits of integration.

Ex 3.1.2: Evaluate $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx$

Sol 3.1.2:

$$\begin{aligned}
\int_1^4 \frac{1}{\sqrt{x}}\,dx &= 2\sqrt{x}\Big|_1^4 \\
&= 2\sqrt{(4)} - 2\sqrt{(1)} \\
&= \boxed{2}
\end{aligned}$$

Ex 3.1.3: Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\,dx$

Sol 3.1.3:

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = -\cos(x)\Big|_0^{\frac{\pi}{2}}$$

$$= -\cos\left(\frac{\pi}{2}\right) + \cos(0)$$

$$= \boxed{1}$$

Ex 3.1.4: Evaluate $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du$

Sol 3.1.4:

$$\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \left(4u^{-3} + u^{-1}\right)\,du$$

$$= \left(-2u^{-2} + \ln|u|\right)\Big|_1^2$$

$$= -2(2)^{-2} + \ln 2 - (-2(1)^{-2} + \ln 1)$$

$$= \boxed{\frac{3}{2} + \ln 2}$$

Ex 3.1.5: Evaluate $\displaystyle\int_{-5}^5 \frac{1}{x^3}\,dx$

Sol 3.1.5: When initially looking at the problem, one may simply proceed with finding the definite integral.

$$\int_{-5}^5 \frac{1}{x^3}\,dx = -\frac{1}{x^2}\Big|_{-5}^5$$

$$= -\frac{1}{(-5)^2} + \frac{1}{(5)^2}$$

$$= 0$$

But, this is a trap! We have to be careful here, because the function $\dfrac{1}{x^3}$ is *not defined* over the the interval $[-5, 5]$, since $\dfrac{1}{0^3}$ is undefined. Therefore, the Fundamental Theorem of Calculus $\boxed{\text{does not apply}}$.

Just as we had many properties for the indefinite integral, we have many properties for the definite integral. There are three main properties that are utilized often on the AP exam.

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2. $\int_a^a f(x)\,dx = 0$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

Ex 3.1.6: If $\int_{-5}^{2} f(x)\,dx = -17$ and $\int_{5}^{2} f(x)\,dx = -4$, find $\int_{-5}^{5} f(x)\,dx$.

Sol 3.1.6:

$$\begin{aligned}
\int_{-5}^{5} f(x)\,dx &= \int_{-5}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx \\
&= \int_{-5}^{2} f(x)\,dx - \int_{5}^{2} f(x)\,dx \\
&= -17 - (-4) \\
&= \boxed{-13}
\end{aligned}$$

Part I of the Fundamental Theorem of Calculus is very important for the **theory** of calculus, but is limited (hehe) in the context of this course to L'Hospital problems which will will explore in a later chapter. Here is how the formula may be applied.

Ex 3.1.7: Use the Fundamental Theorem of Calculus to find $f'(t)$ if $f(t) = \int_2^{3t^2} (4x + 3)\,dx$

Sol 3.1.7:

$$f'(t) = \frac{d}{dt}\int_2^{3t^2} (4x+3)\,dx$$

$$= \left(4\left(3t^2\right)+3\right)(6t)$$

$$= \boxed{72t^3+18t}$$

3.1 Free Response Homework

Use part II of the Fundamental Theorem of Calculus to evaluate the integral, or explain why the integral cannot be evaluated.

1. $\displaystyle\int_{-1}^{3} x^5\,dx$
2. $\displaystyle\int_{2}^{7} (5x-1)\,dx$
3. $\displaystyle\int_{-5}^{5} \frac{2}{x^3}\,dx$
4. $\displaystyle\int_{-3}^{-1} \frac{x^7-4x^3-3}{x}\,dx$
5. $\displaystyle\int_{1}^{2} \frac{3}{t^4}\,dt$
6. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc(y)\cot(y)\,dy$
7. $\displaystyle\int_{0}^{\frac{\pi}{4}} \sec^2(y)\,dy$
8. $\displaystyle\int_{1}^{9} \frac{3}{2z}\,dz$
9. $\displaystyle\int_{1}^{8} \frac{x^2-4}{\sqrt[3]{x}}\,dx$
10. $\displaystyle\int_{\pi}^{\frac{5\pi}{4}} \sin(y)\,dy$
11. $\displaystyle\int_{1}^{4} \frac{x^4-4x^2-5}{x^2}\,dx$
12. $\displaystyle\int_{3}^{5} \left(x^2+5x+6\right)\,dx$
13. $\displaystyle\int_{\pi}^{\frac{\pi}{4}} \cos(y)\,dy$
14. $\displaystyle\int_{1}^{4} \frac{x^3-2x^2-4x}{x^2}\,dx$
15. $\displaystyle\int_{1}^{2} \frac{x^2-4x+7}{x}\,dx$
16. $\displaystyle\int_{1}^{16} \frac{2x^2-1}{\sqrt[4]{x}}$

Use the following values for problems 17 - 27 to evaluate the given integrals.

$\displaystyle\int_{-2}^{5} f(x)\,dx = -2$	$\displaystyle\int_{1}^{5} f(x)\,dx = 3$
$\displaystyle\int_{-2}^{1} g(x)\,dx = 4$	$\displaystyle\int_{5}^{1} g(x)\,dx = 9$
$\displaystyle\int_{1}^{5} h(x)\,dx = 7$	$\displaystyle\int_{5}^{-2} h(x)\,dx = -6$

17. $\displaystyle\int_{-2}^{1} f(x)\,dx$
18. $\displaystyle\int_{-2}^{5} g(x)\,dx$
19. $\displaystyle\int_{-2}^{1} h(x)\,dx$
20. $\displaystyle\int_{1}^{5} (f(x)-g(x))\,dx$

21. $\int_{-2}^{5} (g(x) + h(x))\, dx$

22. $\int_{-2}^{1} (h(x) - f(x))\, dx$

23. $\int_{-2}^{5} (h(x) + f(x))\, dx$

24. $\int_{1}^{5} (2f(x) + 3h(x))\, dx$

25. $\int_{-2}^{1} (2f(x) - 3g(x))\, dx$

26. $\int_{-2}^{5} \left(\frac{1}{2}g(x) + 4h(x)\right)\, dx$

27. $\int_{1}^{5} \left(\frac{1}{3}h(x) + 2f(x)\right)\, dx$

28. $\int_{5}^{5} (f(x) + g(x) + h(x))\, dx$

Use part I of the Fundamental Theorem of Calculus to find the derivative of the function.

29. $g(y) = \int_{2}^{y} t^2 \sin(t)\, dt$

30. $g(x) = \int_{0}^{x} \sqrt{1+2t}\, dt$

31. $F(x) = \int_{x}^{2} \cos\left(t^2\right)\, dt$

32. $h(x) = \int_{2}^{\frac{1}{x}} \arctan(t)\, dt$

33. $y = \int_{3}^{\sqrt{x}} \frac{\cos(t)}{t}\, dt$

34. $f(x) = \int_{e}^{x^2} \ln\left(t^2+1\right)\, dt$

35. $f(x) = \int_{10}^{x^2} t \ln t\, dt$

36. $f(x) = \int_{e^x}^{5} \left(t^3 + t + 1\right)\, dt$

37. If $F(x) = \int_{1}^{t^2} \frac{\sqrt{1+u^4}}{u}\, du$, find $F'(t)$.

38. If $h(x) = \int_{\pi}^{\sqrt{x}} e^{5t}\, dt$, find $h'(x)$.

39. If $h(m) = \int_{5}^{\cos(m)} t^2 \cos^{-1}(t)\, dt$, find $h'(m)$.

40. If $h(y) = \int_{5}^{\ln y} \frac{e^t}{t^4}\, dt$, find $h'(y)$.

3.1 Multiple Choice Homework

1. If $\int_{-5}^{2} f(x)\, dx = -17$ and $\int_{5}^{2} f(x)\, dx = -4$, then $\int_{-5}^{5} f(x)\, dx =$

a) -21 b) -13 c) 0 d) 13 e) 21

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2. Let f and g be continuous functions such that $\int_{0}^{6} f(x)\, dx = 9$, $\int_{3}^{6} f(x)\, dx = 5$, and $\int_{3}^{0} g(x)\, dx = -7$. What is the value of $\int_{0}^{3} \left(\frac{1}{2}f(x) - 3g(x)\right)\, dx$.

a) -23 b) -19 c) $-\frac{17}{2}$ d) 19 e) 23

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3. Given that $\int_2^3 P(t)\,dt = 7$ and $\int_2^7 P(t)\,dt = -2$, what is $\int_7^3 P(t)\,dt$?

a) -9 b) -5 c) 5 d) 9 e) not enough information

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4. Based on the information below, find $\int_1^{-2} (g(x) + f(x))\,dx$

$\int_{-2}^{5} f(x)\,dx = -2$	$\int_1^5 f(x)\,dx = 3$
$\int_{-2}^{1} g(x)\,dx = 4$	$\int_5^1 g(x)\,dx = 9$

a) -9 b) -1 c) 0 d) 1 e) 9

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5. Based on the information below, find $\int_5^{-2} (g(x) - f(x))\,dx$

$\int_{-2}^{5} f(x)\,dx = -2$	$\int_1^5 f(x)\,dx = 3$
$\int_{-2}^{1} g(x)\,dx = 4$	$\int_5^1 g(x)\,dx = 9$

a) -3 b) 3 c) 6 d) -6 e) 14

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6. Using the table values from questions 5 and 6, which of the following cannot be determined?

a) $\int_5^1 (g(x) + f(x))\,dx$ b) $\int_1^{-2} (g(x) - f(x))\,dx$

c) $\int_{-2}^{5} 3g(x)(-4(f(x)))\,dx$ d) $\int_1^5 (3g(x) + 4f(x))\,dx$

3.2: Definite Integrals and the Substitution Rule

Now, it's time to revisit u-substitution within the context of definite integrals. Although the process is largely similar, there are some nuances that we must consider.

OBJECTIVES

Evaluate Definite Integrals Using the Fundamental Theorem of Calculus.
Evaluate Definite Integrals Applying the Substitution Rule, When Appropriate.
Use Proper Notation When Evaluating These Integrals

Ex 3.2.1: Evaluate $\int_0^2 t^2\sqrt{t^3+1}\,dx$.

Sol 3.2.1: Now, this problem may look just like a regular u-substitution problem that we did in the previous chapter. However, when we switch our integration variable to u, we also need to make sure to switch our definite integration boundaries to match. Let's see what that means in the solution below.

$$\int_0^2 t^2\sqrt{t^3+1}\,dt = \frac{1}{3}\int_0^2 3t^2\sqrt{t^3+1}\,dt$$

$$\hookrightarrow\ u = t^3+1 \quad\Big|\quad \hookrightarrow\ du = 3t^2\,dt$$

Here is where we have to be careful. We cannot simply rewrite our definite integral in terms of u, since our upper and lower boundaries are in terms of t! Therefore, we need to also find our new boundaries for the u variables. Luckily, we can do this simply by plugging in our t values into our equation for u.

$$u(0) = (0)^3+1 = 1 \quad\Big|\quad u(2) = (2)^3+1 = 9$$

Now, we can continue integrating! (Note the boundary change in red.)

$$\begin{aligned}\frac{1}{3}\int_{0}^{2} 3t^2\sqrt{t^3+1}\,dt &= \frac{1}{3}\int_{1}^{9}\sqrt{u}\,du\\ &= \frac{1}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_1^9\\ &= \frac{2}{9}\left(9^{\frac{3}{2}}-1^{\frac{3}{2}}\right)\\ &= \boxed{\frac{52}{9}}\end{aligned}$$

Ex 3.2.2: Evaluate $\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)\,dx$.

Sol 3.2.2:

$$\hookrightarrow u = x^2 \quad \Big| \quad \hookrightarrow du = 2x\,dx$$

$$u(0) = (0)^2 = 0 \quad \Big| \quad u\left(\sqrt{\pi}\right) = \left(\sqrt{\pi}\right)^2 = \pi$$

$$= \frac{1}{2}\int_0^{\pi} \cos(u)\,du$$

$$= \frac{1}{2}\sin(u)\Big|_0^{\pi}$$

$$= \boxed{0}$$

Ex 3.2.3: Evaluate $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$.

Sol 3.2.3:

$$\hookrightarrow u = \frac{1}{x} \quad \Big| \quad \hookrightarrow du = -\frac{1}{x^2}$$

$$u(1) = \frac{1}{(1)} = 1 \quad \Big| \quad u(2) = \frac{1}{(2)} = \frac{1}{2}$$

$$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx = -\int_1^{\frac{1}{2}} e^u\,du$$

$$= -e^u\Big|_1^{\frac{1}{2}}$$

$$= \boxed{-e^{\frac{1}{2}} + e}$$

Ex 3.2.4: Evaluate $\int_1^{\sqrt{13}} \frac{x}{x^2+3}\,dx$.

Sol 3.2.4:

$$\hookrightarrow u = x^2 + 3 \quad \Big| \quad \hookrightarrow du = 2x\,dx$$

$$u(1) = (1)^2 + 3 = 4 \quad \Big| \quad u\left(\sqrt{13}\right)^2 + 3 = 16$$

$$\begin{aligned}
\int_1^{\sqrt{13}} \frac{x}{x^2+3}\,dx &= \frac{1}{2}\int_1^{\sqrt{13}} \frac{2x}{x^2+3}\,dx \\
&= \frac{1}{2}\int_4^{16} \frac{1}{u}\,du \\
&= \frac{1}{2}\ln|u|\Big|_4^{16} \\
&= \boxed{\frac{1}{2}\ln(16) - \frac{1}{2}\ln 4}
\end{aligned}$$

Now, technically, $\frac{1}{2}\ln(16) - \frac{1}{2}\ln 4$ is the correct answer. However, when you are taking the multiple choice portion of the AP Calc BC exam, the expectation is that trivial logarithms should be simplified, so this answer may not appear as a choice. So, we must utilize our log rules:

$$\begin{aligned}
\frac{1}{2}\ln(16) - \frac{1}{2}\ln 4 &= \frac{1}{2}\ln\left(\frac{16}{4}\right) \\
&= \frac{1}{2}\ln 4 \\
&= \ln 4^{\frac{1}{2}} \\
&= \boxed{\ln 2}
\end{aligned}$$

The Average Value

One simple application of the definite integral is the Average Value Theorem. Recall that the average of a finite set of numbers is the total of the numbers divided by how many numbers we are averaging. Or, in technical terms:

$$\text{avg} = \frac{\sum_{i=1}^{n} x_i}{n}$$

But, what does it mean to take the average value of a continuous function? Let's say you drive from home to school—what was your average velocity? What was the average temperature today? What was your average height for the first 15 years of your life? All these questions can be answered with the following formula:

The Average Value Formula

The average value of a function f on a closed interval $[a, b]$ is defined by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

If we look at this formula in the context of the Fundamental Theorem of Calculus, it will start to make a little more sense.

$$\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a)\\ \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{b-a}(F(b) - F(a))\\ &= \frac{F(b) - F(a)}{b-a}\end{aligned}$$

Notice that this is just the average slope for $F(x)$ on $x \in [a, b]$. Since the derivative $F'(x)$ gives the instantaneous rate of change of $F(x)$, the average slope of $F(x)$ is the same as average value of $F'(x)$. But, since the definition in the Fundamental Theorem of Calculus says that $F'(x) = f(x)$, this is actually just the average value of $f(x)$.

Ex 3.2.5: Find the average value of $f(x) = x^2 + 1$ on $[0, 5]$.

Sol 3.2.5: We first start with our formula:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Then, we can substitute in the function and the interval.

$$f_{avg} = \frac{1}{5-0}\int_0^5 \left(x^2+1\right)\,dx$$

Using either $\boxed{\texttt{MATH}}$ $\boxed{\texttt{9}}$ on our TI-84 or integrating analytically gives us $f_{avg} = \boxed{\frac{28}{3}}$.

Ex 3.2.6: Find the average value of $h(\theta) = \sec(\theta)\tan(\theta)$ on $\left[0, \frac{\pi}{4}\right]$.

Sol 3.2.6:

$$\begin{aligned} h_{avg} &= \frac{1}{b-a}\int_a^b h(\theta)\,d\theta \\ &= \frac{1}{\frac{\pi}{4}-0}\int_0^{\frac{\pi}{4}} \sec(\theta)\tan(\theta)d\theta \\ &= \boxed{\frac{4}{\pi}\left(\sqrt{2}-1\right) \approx 0.524} \end{aligned}$$

3.2 Free Response Homework

1. $\displaystyle\int_0^1 x^2 \left(1+2x^3\right)^5\, dx$

2. $\displaystyle\int_0^1 x^3 \left(x^4+5\right)^3\, dx$

3. $\displaystyle\int_{-1}^1 x\sqrt{4-x^2}\, dx$

4. $\displaystyle\int_1^2 \frac{x^2}{\sqrt[3]{9-x^3}}\, dx$

5. $\displaystyle\int_0^3 \frac{10t+15}{\sqrt[4]{t^2+3t+1}}\, dt$

6. $\displaystyle\int_1^2 \frac{x+1}{\sqrt{x^2+2x+4}}\, dx$

7. $\displaystyle\int_1^3 \frac{5t}{t^2+1}\, dt$

8. $\displaystyle\int_{-1}^2 \frac{1}{2x+5}\, dx$

9. $\displaystyle\int_{\sqrt{3}}^2 ye^{y^2-3}\, dy$

10. $\displaystyle\int_0^1 \frac{v^2}{8-v^3}\, dv$

11. $\displaystyle\int_3^{e^2+2} \frac{1}{x-2}\, dx$

12. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin(\theta)}{\cos^2(\theta)} d\theta$

13. $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx$

14. $\displaystyle\int_0^{\pi} \sec^2\left(\frac{t}{4}\right)\, dt$

15. $\displaystyle\int_0^{\pi} \frac{\sin(x)}{2-\cos(x)}\, dx$

16. $\displaystyle\int_2^4 \frac{1}{x\ln x}\, dx$

17. $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\, dx$

18. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^5(x)\sin(x)\, dx$

19. $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x)\, dx$

20. $\displaystyle\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2(\ln y)}{y}\, dy$

21. $\displaystyle\int_0^{\pi} \frac{\cos(x)}{2+\sin(x)}\, dx$

22. $\displaystyle\int_0^{\pi} \frac{\sin(y)}{2+\cos(y)}\, dy$

23. $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} m\sec\left(m^2\right)\tan\left(m^2\right)\, dm$

24. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2(x)\tan^3(x)\, dx$

25. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos^9(x)\sin(x)\, dx$

26. $\displaystyle\int_0^{\pi} \cos^6\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)\, dx$

27. $\displaystyle\int_{\pi}^{2\pi} \cos\left(\frac{1}{2}\theta\right)\, d\theta$

28. $\displaystyle\int_2^{e^3+1} \frac{(\ln(x-1))^4}{x-1}\, dx$

29. $\displaystyle\int_0^{e^2-1} \frac{1}{x+1}\, dx$

30. $\displaystyle\int_5^{e^3+4} \frac{1}{x-4}\, dx$

Find the average value of each of the following functions over the given interval.

31. $F(x) = (x-3)^2$ on $x \in [3,\ 7]$

32. $H(x) = \sqrt{x}$ on $x \in [0,\ 3]$

33. $F(x) = \sec^2(x)$ on $x \in \left[0,\ \dfrac{\pi}{4}\right]$

34. $F(x) = \dfrac{1}{x}$ on $x \in [1\,3]$

35. $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [1,\ 4]$

36. $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [4,\ 9]$

37. $f(x) = \cos(x)\sin^4(x)$ on $x \in [0,\ \pi]$

38. $g(x) = xe^{-x^2}$ on $x \in [1,\ 5]$

39. $G(x) = \dfrac{x}{(1+x^2)^3}$ on $x \in [0,\ 2]$

40. $h(x) = \dfrac{x}{(1+x^2)^2}$ on $x \in [0,\ 4]$

41. If a cookie taken out of a 450° F oven cools in a 60° F room, then according to Newton's Law of Cooling, the temperature of the cookie t minutes after it has been taken out of the oven is given by

$$T(t) = 60 + 390e^{-0.205t}.$$

What is the average value of the cookie's temperature during its first 10 minutes out of the oven?

42. We know as the seasons change so do the length of the days. Suppose the length of the day varies sinusoidally with time by the equation

$$L(t) = 10 - 3\cos\left(\frac{\pi t}{182}\right),$$

where t is the number of days after the winter solstice (December 22, 2007). What was the average day length from January 1, 2008 to March 31, 2008?

43. During one summer in the Sunset, the temperature is modeled by the function

$$T(t) = 50 + 15\sin\left(\frac{\pi}{12}t\right),$$

where T is measured in F° and t is measured in hours after 7 a.m. What is the average temperature in the Sunset during the six-hour chemistry class that runs from 9 a.m. to 3 p.m.?

3.2 Multiple Choice Homework

1. $\displaystyle\int_1^4 \frac{1}{(1+\sqrt{x})^2\sqrt{x}}\,dx$

a) $\dfrac{6}{5}$ b) $\dfrac{1}{3}$ c) $\dfrac{2}{3}$ d) $\dfrac{4}{9}$ e) $\dfrac{3}{2}$

2. If $\int_1^4 h(x)\,dx = 6$, then $\int_1^4 h(5-x)\,dx =$

a) -6 b) -1 c) 0 d) 3 e) 6

3. $\int_e^{e^2} \frac{1}{x\ln x}\,dx =$

a) $\ln(\ln 2)$ b) $\frac{2}{e^2}$ c) $\ln 2$ d) $\frac{1-2e}{2e^2}$ e) DNE

4. Determine the average value of $y = e^{6x}$ on $x \in [0,\, 4]$.

a) $\frac{e^{24}-1}{4}$ b) $\frac{e^{24}-1}{6}$ c) $\frac{e^{24}}{24}$ d) $\frac{e^{24}}{6}$ e) $\frac{e^{24}-1}{24}$

5. Determine the average value of $g(x) = (2x+3)^2$ on $x \in [-3,\, -1]$.

a) $\frac{7}{3}$ b) -4 c) 5 d) $\frac{14}{3}$ e) 3

6. Determine the average value of $g(x) = e^{7x}$ on $x \in [0,\, 4]$.

a) $\frac{1}{14}e^{14}$ b) $\frac{1}{7}\left(e^{14}-1\right)$ c) $\frac{1}{14}\left(e^{14}-1\right)$ d) $\frac{1}{2}\left(e^{14}-1\right)$ e) $\frac{1}{7}e^{14}$

7. If the function $y = x^3$ has an average value of 9 on $x \in [0,\, k]$, then $k =$.

a) 3 b) $\sqrt{3}$ c) $\sqrt[3]{18}$ d) $\sqrt[4]{36}$ e) $\sqrt[3]{36}$

8. Find the average rate of change of $y = x^2 + 5x + 14$ on $x \in [-1\, 2]$

a) 3 b) 6 c) 9 d) $\frac{65}{6}$ e) 18

9. If the average of the function $f(x) = |x - a|$ on $[-1\,1]$ is $\dfrac{5}{4}$, what is/are the values of a?

a) ± 1 b) $\pm\dfrac{1}{2}$ c) $\pm\dfrac{1}{4}$ d) 0 e) None of these

---

10. What is the average rate of change of the function $f(x) = x^4 - 5x$ on the closed interval $[0,\ 3]$?

a) 8.5 b) 8.7 c) 22 d) 23 e) 66

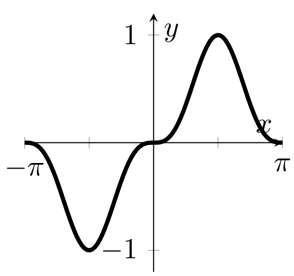
---

11. Determine the average value of $y = e^x \cos(x)$ on $x \in \left[0,\ \dfrac{\pi}{2}\right]$

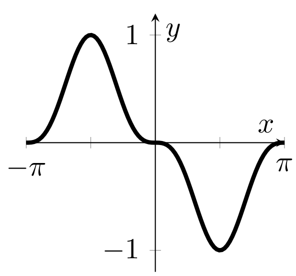
a) 0 b) 1.213 c) 1.905 d) 2.425 e) 3.810

---

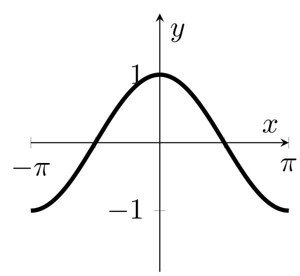
12. The graphs of five functions are shown below. Which function has a nonzero average value over the closed interval $x = [-\pi,\ \pi]$.



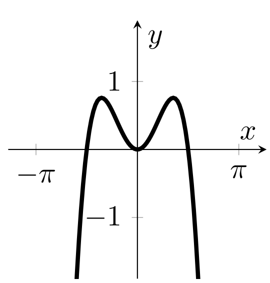
a)



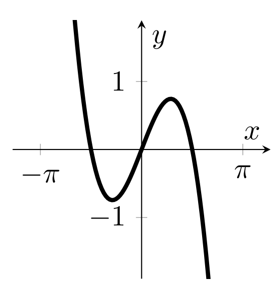
b)



c)



d)

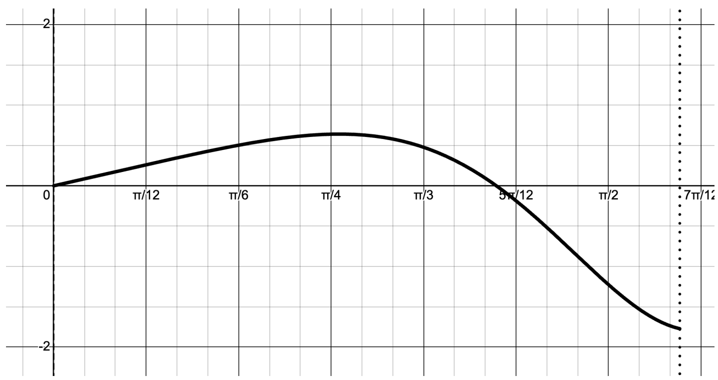


e)

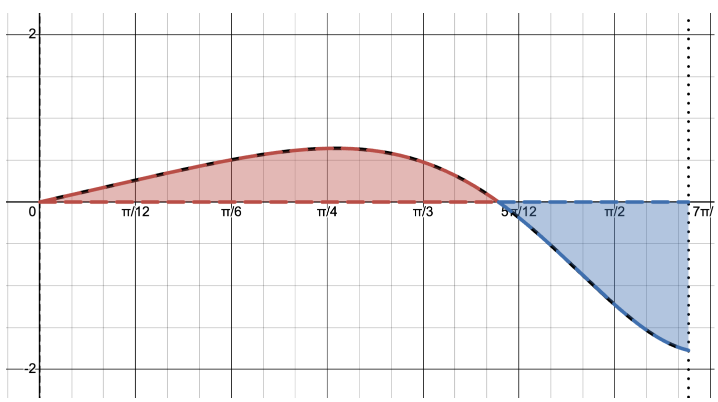
3.3: Context For Definite Integrals: Area, Displacement, and Net Change

Since we originally defined the definite integral in terms of "area under a curve," we need to consider what this idea of "area" really means in relation to the definite integral.

Recall **Ex 3.2.2**, where we had a function $y = x\cos\left(x^2\right)$ on $x \in [0\,\sqrt{\pi}]$. The graph looks like this:



In **Ex 3.2.2**, we found that $\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)\,dx = 0$. But, there's clearly area under the curve, so how can the integral equal both the area and 0? Well, as it turns out, because the integral was created from rectangles with width dx and height $f(x)$, a negative $f(x)$ will result in a rectangle with "negative area." Take a look at the following graph:



It's clear that by our definition of the definite integral, the "area" in red would cancel out the "area" in blue. So, how would we find the actual, positive area? That is what we are going to talk about in this section.

OBJECTIVES

Relate Definite Integrals to Area Under a Curve.
Understand the Difference Between Displacement and Distance.
Understand Displacement and Distance in Other Contexts.

Ex 3.3.1: What is the area under $y = x\cos\left(x^2\right)$ on $x \in [0, \sqrt{x}]$?

Sol 3.3.1: First, let's make sure we're clear on terminology. In this context, "area under" means "area between the graph and x-axis." Now, to find the area under this graph, we have two methods.

The first method is to split the graph into two parts: one part above the x-axis and one part below the x-axis. We can integrate the part above the x-axis as usual, and for the part below the x-axis, we simply negate the result of the definite integral. The sum of these two parts should equal the area under the curve. We can begin by finding the point where the graph passes the x-axis.

$$x\cos\left(x^2\right) = 0$$

$$x = 1.253$$

Now, by the rule $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$, we can split our integral into two parts. Remember that we are negating the second term to compensate for the area being below the x-axis.

$$\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)\,dx \to \int_0^{1.253} x\cos\left(x^2\right)\,dx + \left(-\int_{1.253}^{\sqrt{\pi}} x\cos\left(x^2\right)\,dx\right)$$

Finally, using $\boxed{\texttt{MATH}}$ $\boxed{9}$, we arrive at the answer $\boxed{1}$.

However, there is a much cleaner way to do this problem. As it turns out, if we simply take the integral of the absolute value of the integrand (the function that is being integrated), we arrive at the same answer. This is because the absolute value automatically accounts for both the parts of the graph that are above and below the x-axis, turning any negative "signed area" into positive area. Therefore, we can write

$$\int_0^{\sqrt{\pi}} \left|x\cos\left(x^2\right)\right|\,dx,$$

which turns out to also evaluate to $\boxed{1}$.

Now that we've seen an example, let's generalize some steps to finding the total area described by a definite integral.

Steps to Finding Total Area

1. Draw the function.

2. Find the zeroes between $x = a$ and $x = b$.

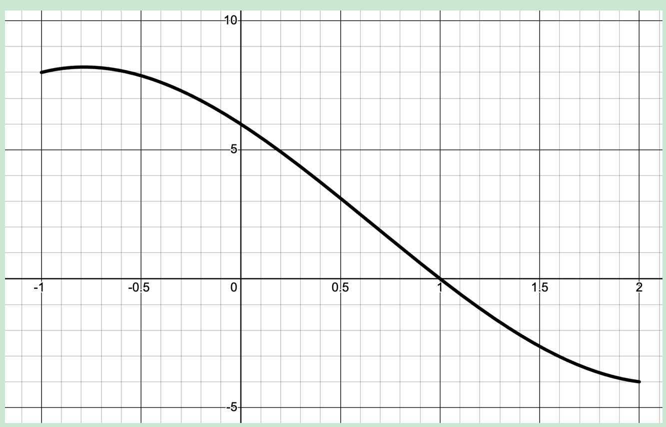
3. Set up separate integrals representing the area above and below the x-axis.

4. Change the sign on those integrals which represent the negative values (i.e., those where the curve is below the x-axis).

5. Solve the integral expression.

Ex 3.3.2: Find the area under $y = x^3 - 2x^2 - 5x + 6$ on $x \in [-1, 2]$.

Sol 3.3.2: Let's take a quick look at the graph:



It's clear that the graph crosses the x-axis at $x = 1$. Therefore, to get the total area, let's set up our two-term integration and solve.

$$\int_{-1}^{1} \left(x^3 - 2x^2 + 5x + 6\right)\, dx + \left(-\int_{1}^{2} \left(x^3 - 2x^2 + 5x + 6\right)\, dx\right)$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{-1}^{1} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{1}^{2}$$

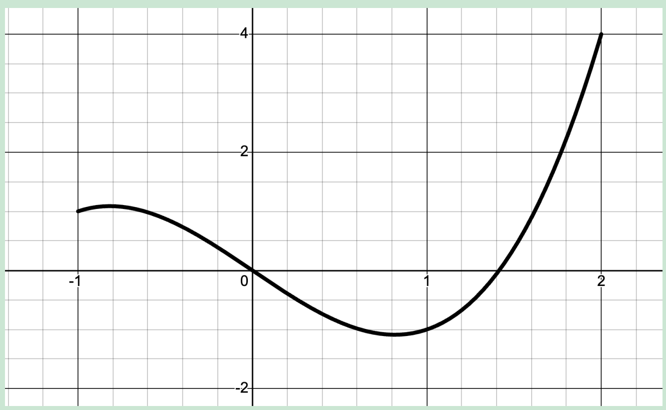
$$= \frac{32}{3} + \frac{29}{12}$$

$$= \boxed{\frac{157}{12}}$$

Now, two important things about this specific example. Instead of using the evaluation bar to denote evaluating the antiderivative, we instead can just use standard square brackets. These two notations can be used interchangeably. The second point is that the reason why we didn't use the quicker, absolute value method was because that method cannot be efficiently done by hand. On the AP exam, one can expect this problem to be on the no-calculator portion, and without a calculator, the absolute value method cannot be done.

Ex 3.3.3: Find the area under $y = x^3 - 2x$ on $x \in [-1, 2]$.

Sol 3.3.3: Once again, we begin with a sketch of the function:



Note that there are three regions here, so we will need to separate our integral into three terms. Our zeroes between $[-1, 2]$ are 0 and $\sqrt{2}$.

$$\int_{-1}^{0} \left(x^3 - 2x\right)\, dx + \left(-\int_{0}^{\sqrt{2}} \left(x^3 - 2x\right)\, dx\right) + \int_{\sqrt{2}}^{2} \left(x^3 - 2x\right)\, dx$$

$$= \left[\frac{x^4}{4} - x^2\right]_{-1}^{0} - \left[\frac{x^4}{4} - x^2\right]_{0}^{\sqrt{2}} + \left[\frac{x^4}{4} - x^2\right]_{\sqrt{2}}^{2}$$

$$= \frac{3}{4} - (-1) + 1$$

$$= \boxed{\frac{11}{4}}$$

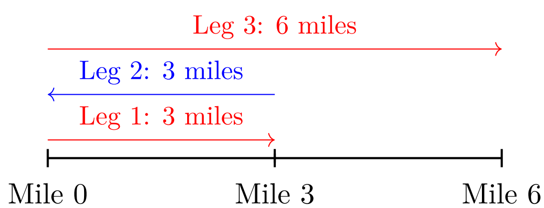
Distance and Displacement

Imagine you were leaving your house to go to school, and that school is 6 miles away. You leave your house and halfway to school you realize you have forgotten your AP Calc BC homework. You head back home, grab your assignment (that you just finished in the morning, of course), and then head to school.

There are two different questions that can be asked here. How far are you from where you started? And how far have you actually traveled? Although these questions seem extremely similar, their distinction is very important.

The first question is rather easy to answer. School is 6 miles from home, so we are 6 miles from where we started.

The second question is a little more difficult to answer. We first travel 3 miles to the the halfway point, but end up traveling another 3 miles back to home to grab the homework. Finally, you travel the 6 miles to school. We can see this in the visual below.



Therefore, the total mileage that you've covered is $3 + 3 + 6 = 12$ miles. These two questions are the root behind the concepts of **displacement** and **distance**.

Displacement $\rightarrow$ Definition: How far apart the starting position and ending position are. Note that this value can be positive or negative.

Distance $\rightarrow$ Definition: How far you travel in total. This value can only be positive.

Given v is the velocity function and x is the position function,

$$\text{Displacement} \rightarrow \int_a^b v\,dt \qquad \text{Total Distance} \rightarrow \int_a^b |v|\,dt$$

$$\text{Position at } t = a \rightarrow x(a) + \int_a^b v\,dt$$

Ex 3.3.4: A particle moves along a line so that its velocity at any time t is $v(t) = t^2 + t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during the time period $1 \leq t \leq 4$.

Sol 3.3.4:
a)

$$\int_a^b v\,dt = \int_1^4 \left(t^2 + t - 6\right)\,dt$$

$$= \left[\frac{t^3}{3} + \frac{t^2}{2} - 6t\right]_1^4$$

$$= \boxed{-\frac{21}{2}\text{ meters}}$$

b) $v(t) = 0$ when $t = 2$.

$$\int_a^b |v|\,dt = -\int_1^2 \left(t^2 + t - 6\right)\,dt + \int_2^4 \left(t^2 + t - 6\right)\,dt$$

$$= -\left[\frac{t^3}{3} + \frac{t^2}{2} - 6t\right]_1^2 + \left[\frac{t^3}{3} + \frac{t^2}{2} - 6t\right]_2^4$$

$$= \boxed{\frac{89}{6}\text{ meters}}$$

Make sure to remember your units in your answer! The AP exam will dock points for unitless answers.

3.3 Free Response Homework

Find the area between the curve of the given equation and the x-axis on the given interval.

1a. $y = x^3$ on $x \in [0,\, 2]$

1b. $y = x^3$ on $xin[-1,\, 2]$

2a. $y = 4x - x^3$ on $x \in [0,\, 2]$

2b. $y = 4x - x^3$ on $x \in [-1,\, 2]$

3a. $y = \sin(x)$ on $x \in [0,\, \pi]$

3b. $y = \sin(x)$ on $x \in [-\pi,\, \pi]$

4a. $y = 2x^2 - x^3$ on $x \in [0,\, 2]$

4b. $2x^2 - x^3$ on $x \in [-1,\, 2]$

5. $y = x^3 - 2x^2 - 3x$ on $x = [-2,\, 2]$

6. $y = x^3 - 4x^2 + 4x$ on $x \in [-1, 2]$

7. $y = x^3 - 2x^2 - x + 2$ on $x \in [-3,\, 3]$

8. $y = \dfrac{\pi}{2}\cos(x)\sin(\pi + \pi\sin(x))$ on $x \in \left[-\dfrac{\pi}{2},\, \pi\right]$

9. $y = -\dfrac{x}{x^2+4}$ on $x \in [-2,\, 2]$

10. $y = \dfrac{4-x^2}{x^2+4}$ on $x \in [-3,\, 3]$

11. $y = \dfrac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}$ on $x \in [0.01,\, \pi^2]$

12. $y = x\sqrt{18-2x^2}$ on $x \in [-2,\, 1]$

13. $y = 3\sin(x)\sqrt{1-\cos(x)}$ on $x \in \left[-\dfrac{\pi}{2},\, \dfrac{\pi}{3}\right]$

14. $y = x^2 e^{x^3}$ on $x \in [0,\, 1.5]$

Answer the following questions regarding distance and displacement.

15. The velocity function (in meters per second) for a particle moving along a line is $v(t) = 3t - 5$.

(a) Find the displacement of the particle during the time period $0 \le t \le 3$.

(b) Find the distance traveled during the time period $0 \le t \le 3$.

16. The velocity function (in meters per second) for a particle moving along a line is $v(t) = t^2 - 2t - 8$.

(a) Find the displacement of the particle during the time period $1 \le t \le 6$.

(b) Find the distance traveled during the time period $1 \le t \le 6$.

For problems #17-20, show the setup to determine the area and solve the integral. Do not use the absolute value method.

17. Find the area under the curve $f(x) = e^{-x^2} - x$ on $x \in [-1,\, 2]$.

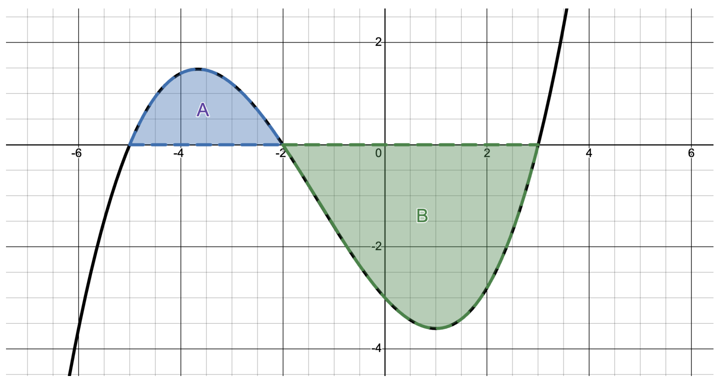
18. Find the area under the curve $f(x) = e^{-x^2} - 2x$ on $x \in [-1, 2]$.

19. Find the area under the curve $f(x) = \dfrac{x}{x^2+1} + \cos(x)$ on $x \in [0, \pi]$.

20. Find the area under the curve $g(x) = -1 - x\sin(x)$ on $x \in [0, 2\pi]$.

3.3 Multiple Choice Homework

1. The graph of $y = f(x)$ is shown below. A and B are positive numbers that represent the areas between the curve and the x-axis.

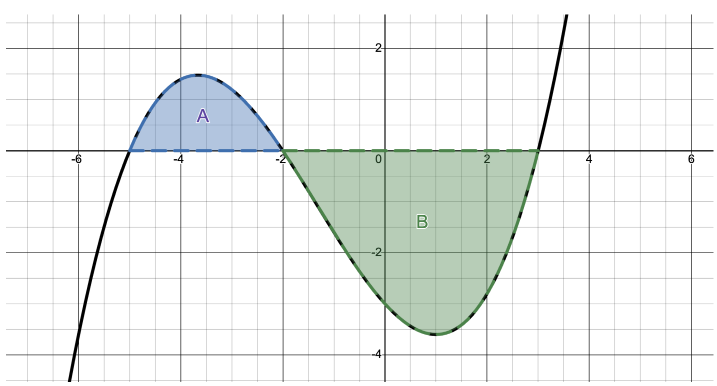


In terms of A and B, $\displaystyle\int_{-5}^{3} f(x)\,dx + \int_{-2}^{3} f(x)\,dx =$

a) A b) $A - B$ c) $2A - B$ d) $A + B$ e) $A - 2B$

---

2. The graph of $y = f(x)$ is shown below. A and B are positive numbers that represent the areas between the curve and the x-axis.

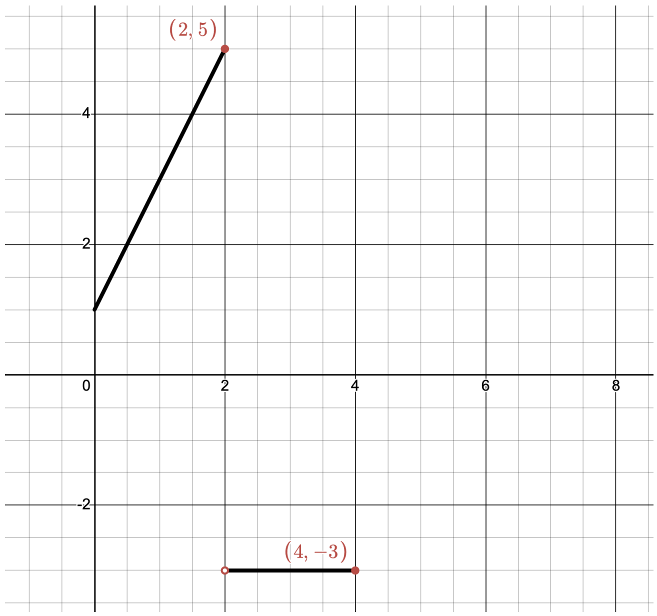


In terms of A and B, $2\int_{-5}^{3} f(x)\,dx + 3\int_{-2}^{3} f(x)\,dx =$

a) A b) $A - B$ c) $2A - B$ d) $A + B$ e) $A - 2B$

---

3. The graph of $f(x)$ on $0 \leq x \leq 4$ is shown.



a) -1 b) 0 c) 2 d) 6 e) 12

---

4. A particle moves along the x-axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t+1) - 2t + 1$. What is the total **distance**, in meters, traveled by the particle during the time interval $0 \leq t \leq 3$?

a) -3.455 b) 0.704 c) 1.540 d) 2.667 e) 4.291

---

5. A particle travels along a straight line with a velocity of $v(t) = 3e^{-t^2}\sin(2t)$ meters per second. What is the total **distance**, in meters, traveled by the particle during the time interval $0 \leq t \leq 2$?

a) 0.835 b) 1.625 c) 1.661 d) 2.261 e) 5.350

---

6. A particle moves along the x-axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t+1) - 2t + 1$. What is the total **displacement**, in meters, of the particle during the time interval $0 \leq t \leq 3$ seconds?

a) -3.455 b) 0.704 c) 1.540 d) 2.667 e) 4.291

---

7. A particle travels along a straight line with a velocity of $v(t) = 3e^{-t^2}\sin(2t)$ meters per second. What is the total **distance**, in meters, traveled by the particle during the time interval $0 \leq t \leq 2$?

a) 0.835 b) 1.625 c) 1.661 d) 2.261 e) 5.350

3.4: Accumulation of Rates

As we saw with the Riemann sums, in $\int f(x)\,dx$, $f(x_i)\cdot dx$ is the area of a rectangle (height times base). The $\int$ is the sum of the areas. The main concept in these "accumulation of rates" problem is that since a definite integral is a sum of values, the integral of a rate of change over some time t equals the total change over that time t.

OBJECTIVES

Analyze the Relationship Between Rates of Change and Integrals.

Beginning in 2002, CollegeBoard shifted emphasis on understanding of the accumulation aspect of the Fundamental Theorem to a new kind of rate problem. Previously, accumulation of rates problems were mostly in context of velocity and distance, though the 1996 Cola Consumption problem hinted at a new direction that these problems would take. The 2002 Amusement Park problem caught many students and teachers off guard, though. Almost every year since then, the test has included this kind of problem. Here is an example similar to the 2002 Amusement Park problem:

Ex 3.4.1:

The Amusement Park Problem (AP 2002)

The rate at which people enter a park is given by the function

$$E(t) = \frac{15600}{t^2 - 24t + 160},$$

and the rate at which they are leaving is given by

$$L(t) = \frac{9890}{t^2 - 38t + 370} - 76.$$

Both $E(t)$ and $L(t)$ are measured in people per hour where t is the number of hours past midnight. The functions are valid for when the park is open, $8 \le t \le 24$. At $t = 8$ there are no people in the park.

(a) How many people have entered the park at 4 pm ($t = 16$)? Round your answer to the nearest whole number.

(b) The price of admission is \$36 until 4 pm ($t = 16$). After that, the price drops to \$20. How much money is collected from admissions that day? Round your answer to the nearest whole number.

(c) Let $H(t) = \int_8^t E(x) - L(x)\, dx$ for $8 \le t \le 24$. The value of $H(16)$ to the nearest whole number is 5023. Find the value of $H'(16)$ and explain the meaning of $H(16)$ and $H'(16)$ in the context of the amusement park.

(d) At what time t, for $8 \le t \le 24$, does the model predict the number of people in the park is at at maximum?

Now, before we attempt this problem, there are many key phrases and concepts that we must understand. Take a look at the following table.

Key Phrases for Interpreting Accumulation of Rates Problems

Total change:	$\int_a^b R(t)\, dt$ or $\int_a^t (\text{incoming rate} - \text{outgoing rate})\, dx$
Total rate of change:	incoming rate − outgoing rate
Total amount:	initial value $+ \int_a^t (\text{incoming rate} - \text{outgoing rate})\, dx$
Instantaneous rate of change:	$\frac{dx}{dt}$ or $R(t)$
Average rate of change:	$\frac{f(b) - f(a)}{b - a}$
Average value of $f(x)$:	$\frac{1}{b-a} \int_a^b f(x)\, dx$
Amount increase/decrease:	rate of change positive/negative
Rate of change inc/dec:	$\frac{d}{dt}$[rate of change] positive/negative

Note that there is a difference between a function that is increasing or decreasing and its derivative that is increasing or decreasing.

One technique that is super useful in accumulation of rates problem is unit analysis. By analyzing the units of the problem, we are able to interpret what the question is asking, and provide an appropriate answer. For instance, if $R(t)$ is measured in miles per hour,

$$\int_a^b R(t)\, dt = \int_a^b \frac{\text{miles}}{\text{hour}} (\text{hours}) = \text{miles}$$

To that end, let's take a look at the above chart, but with the intended units of the answer.

Units for Accumulation of Rates Problems

Total change:	units of accumulated quantity (e.g., miles, people)
Total rate of change:	$\dfrac{\text{units of quantity}}{\text{units of time}}$ (e.g., miles/hour, people/day)
Total amount:	units of quantity (e.g., total miles, total gallons)
Instantaneous rate of change:	$\dfrac{\text{units of quantity}}{\text{units of time}}$
Average rate of change:	$\dfrac{\text{units of quantity}}{\text{units of time}}$
Average value of $f(x)$:	same units as $f(x)$
Amount increase/decrease:	units of quantity (increase or decrease)
Rate of change inc/dec:	$\dfrac{\text{units of rate}}{\text{units of time}}$ (e.g., acceleration: miles/hour^2)

Finally, let's go through some tips for answering these questions. There are four key parts to a good explanation:

1. The meaning of the equation.
2. The description of what is being measured.
3. The units.
4. The time frame involved.

Tips for Explaining Answers

1. **Echo the question in your answer.**
   Restate what the problem is asking so your explanation directly addresses it.

2. **Reason from the given information.**
   Base your explanation on the data, equations, or graphs provided—not on assumptions.

3. **Use and check units.**
   Always include proper units (e.g., miles, hours, people/day) and make sure they match the context.

4. **Specify the time frame.**
   When discussing change, clearly state *when* it occurs and include correct time units.

5. **Refer to functions, not numbers or graphs.**
   Describe how functions behave—whether they are positive/negative or increasing/decreasing—even if you are using a table or graph.

   - Instead of "the slope is positive," say "the function is increasing."
   - Instead of "the slope is increasing," say "the rate is increasing."
   - Avoid mentioning slope, second derivatives, or concavity unless the question specifically asks for them.

6. **Be concise and precise.**
   Express your idea in one clear sentence whenever possible. If your explanation turns into a paragraph, you are probably being unclear or unfocused.

With those instructions out the way, we can finally look at the solution for **Ex 3.4.1**.

Sol 3.4.1:
a) Since $E(t)$ is a rate in people per hour, the number of people who have entered the park will be an integral from $t = 8$ to $t = 16$. Therefore,

$$\text{Total entered} = \int_8^{16} E(t)\,dt$$

$$= 6126.105$$

$$\approx \boxed{6126 \text{ people}}$$

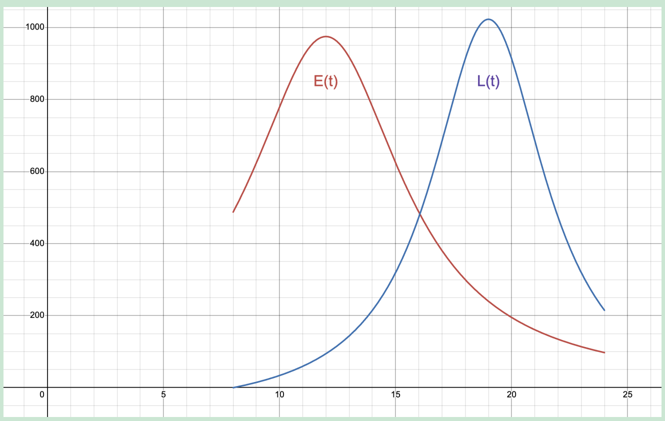
b) Since there are different entry fees for different times of day, we need to determine how many people paid each fee. So, we will have to do $\int_8^{16} E(t)\,dt$ and $\int_{16}^{24} E(t)\,dt$. Conveniently, we know that $\int_8^{16} E(t)\,dt = 6126$ people from the previous problem, so we can just find $\int_{16}^{24} E(t)\,dt$.

$$\int_{16}^{24} E(t)\,dt \approx 1808 \text{ people}$$

The total revenue that the park gets from admissions means multiplying the number of people by the admission charge. Therefore, we have

$$\begin{aligned}\text{Total revenue} &= \$36 \cdot 6126 + \$20 \cdot 1808 \\ &= \boxed{\$256696}\end{aligned}$$

c) While a graph is not necessary for solving this problem, sometimes it helps to visualize the situation. Below are the graphs of $E(x)$ and $L(x)$ on $8 \le t \le 24$:



Note how each function increases and then decreases. Since $H(t) = \int_8^t E(x) - L(x)\,dx$, we can use the Fundamental Theorem of Calculus to determine the derivative.

$$\begin{aligned}H'(t) &= \frac{d}{dt}\int_8^t E(x) - L(x)\,dx \\ &= E(t) - L(t) \\ &= E(16) - L(16) \\ &= 14\end{aligned}$$

Now let's interpret what these numbers mean. We know that $H(16) = 5023$ people. Since the derivative is a rate of change, we know that $H'(16) = 14$ means 14 people per hour. Because integrating a rate gives a total change, and $H(t)$ represents the integral of the difference between the entry and leaving rates, $H(t)$ tells us the *net number of people in the park* at 4 pm—that is, how many have entered minus how many have left since 8 am.

Meanwhile, $H'(t)$ represents the *instantaneous rate of change* of the park's population. This $H'(16) = 14$ means that **at 4 pm, the number of people in the park is increasing at a rate of 14 people per hour**.

In other words, more people are entering than leaving at that moment.

d) This question requires a little more advanced technique, so we will return to this problem in Chapter 4.

Ex 3.4.2:

The Cat Population Problem

The Peninsula Humane Society (PHS) is dedicated to the care and adoption of as many animals who they receive as possible. Since cats breed seasonally, the number of cats and kittens they receive into their facility in a given year varies roughly sinusoidally with time. The data available from 2019 show the rate $R(t)$, measured in healthy cats per month, varies with time, measured in months after New Years Day, according to the equation

$$R(t) = 120 - 88\cos\left(\frac{\pi}{6}(t-2)\right).$$

The rate $A(t)$ at which adoption occur, measured in cats per month, varies with time, measured in months after New Year's Day, according to the equation

$$A(t) = 125 - 85\cos\left(\frac{\pi}{6}(t-3)\right).$$

On New Years' Day ($t = 0$), there were 131 cats in the PHS nursery waiting to be adopted.

(a) $\displaystyle\int_0^{12} R(t)\,dt = 1440$ cats. Using the correct units, explain the meaning of this result in context of the problem.

(b) Assume $A'(10.3) = -28.008$. Using the correct units, explain the meaning of $A'(10.3)$ in context of the problem.

(c) $C(t) = 131 + \displaystyle\int_0^{12} R(t) - A(t)\,dt = 71$. Using the correct units, explain the meaning of this result in context of the problem.

(d) Using the correct units, explain the meaning of $\frac{1}{t}\int_0^t R(x)\,dx$ in the context of the problem.

(e) Suppose $R(7.4) = 185.4$ and $R'(7.4) = -30.8$. Using the correct units, explain the meaning of this information in context of the problem.

Sol 3.4.2:
a) $R(t)$ is the rate at which healthy cats per month are received at PHS. Therefore, $\int_0^{12} R(t)\,dt$ would be the total number of cats received over this time period. So, **1440 healthy cats were received at PHS between $t = 0$ and $t = 12$ (or, during the 12 months of 2019).**

b) $A(t)$ is the rate at which adoptions occur, measured in cats per month. Therefore, **the rate at which adoptions are occurring at $t = 10.3$ is decreasing by 28.008 cats per month, per month.**

c) $C(t) = 131 + \int_0^t R(x) - A(x)\,dx$ represents the total number of cats and kittens sheltered at PHS at any time t. So, we can say that **the number of cats and kittens sheltered at PHS at the end of the 12 months of 2019 is 71 cats and kittens.**

d) $\frac{1}{t}\int_0^t R(x)\,dx$ is the average number of cats and kittens being received at any time t. Therefore, $\frac{1}{t}\int_0^t R(x)\,dx$ **represents the average number of cats and kittens, in cats and kittens per month, that have been received at PHS during the interval $[0, t]$ of 2019.**

e) $R(7.4) > 0$ and $R'(7.4) < 0$. So, we can say that **at $t = 7.4$, the total number of cats and kittens that have been received at PHS is increasing at a decreasing rate of -30.8 cats and kittens per month, per month.**

Phew, that was a lot of words!! However, notice that each of the problems can be answered simply by looking back at the “Key Phrases” chart and matching the expression to the concept.

Ex 3.4.3:

The Synthetic Oil Problem

A certain industrial chemical reaction produces synthetic oil at a rate of

$$S(t) = \frac{15t}{1+3t}.$$

At the same time, the oil is removed from the reaction vessel by a skimmer that has a rate of

$$R(t) = 2 + 5\sin\left(\frac{4\pi}{25}t\right).$$

Both functions have units of gallons per hour, and the reaction runs from $t = 0$ to $t = 6$. At time $t = 0$, the reaction vessel contains 2500 gallons of oil.

(a) How much oil will the skimmer remove from the reaction vessel in this six hour period? Indicate units of measurement.

(b) Write an expression for $P(t)$, the total number of gallons of oil in the reaction vessel at time t.

(c) Find the rate at which the total amount of oil is changing at $t = 4$.

Sol 3.4.3:
a) To find the total amount removed over the time interval, we integrate the rate $R(t)$ over $0 \le t \le 6$.

$$\text{Amount removed} = \int_0^6 \left(2 + 5\sin\left(\frac{4\pi}{25}t\right)\right) dt$$

$$= \int_0^6 R(t)\, dt$$

$$= \boxed{31.816 \text{ gallons}}$$

b) Let $P(t)$ represent the total amount of oil in the reaction vessel at time t. Initially, $P(0) = 2500$ gallons. The net change of oil is production minus removal, which is $S(t) - R(t)$. Since the variable t is the upper boundary, we need to use a dummy variable x in the integrand. This gives us the expression

$$\boxed{P(t) = 2500 + \int_0^t (S(x) - R(x))\, dx}.$$

c) The rate of change of P is $P'(t) = S(t) - R(t)$. Therefore, at $t = 4$,

$$P'(4) = S(4) - R(4)$$

$$= \boxed{-1.909 \text{ gal/hr}}$$

3.4 Free Response Homework

1. The Puffin Problem

The puffin population on the Skellig Islands off the coast of County Kerry, Ireland, can be modeled by a differentiable function P in terms of time t, where $P(t)$ is the number of puffins and t is measured in years, for $0 \leq t \leq 50$. There are 10,000 puffins on the island at time $t = 0$. The birth rate for the penguins on the island is modeled by

$$B(t) = 500e^{0.05t} \text{ puffins per year}$$

and the death rate for the penguins on the island is modeled by

$$D(t) = 110e^{0.09t} \text{ puffins per year.}$$

(a) Using the correct units, explain the meaning of $\displaystyle\int_0^{50} (B(t) - D(t))\, dt$.

(b) Using the correct units, explain the meaning of $\displaystyle\frac{1}{25}\int_0^{25} B(t)\, dt$.

(c) Using the correct units, explain the meaning of $D'(35)$,

(d) Using the correct units, explain the meaning of $\displaystyle\frac{B(35) - B(10)}{35 - 10}$.

(e) Suppose $D(35) = 2567$ and $D'(35) = 143.8$. Using the correct units, explain the meaning of these data in context of the number of puffin deaths.

2. The Alcohol Metabolization Problem

Metabolism is the body's process of converting ingested substances to other compounds. Alcohol is absorbed into the blood stream, then, through oxidation, it is detoxified and removed from the blood. Research shows that, after consuming three shots of alcohol in rapid succession, an average (180 lb) adult, fasting male has the alcohol metabolized at a rate modeled by $R(t) = 0.04\left(t^2 - 8t + 6\right)e^{-t}$, where $R(t)$ is the measured in percentage of alcohol in the blood stream per hour, t is measured in hours after consuming the third drink and is valid for $0 \leq t \leq 6$.

(a) $\displaystyle\int_0^{2.5} R(t)\, dt = 0.029$. Using the correct units, explain the meaning of this result.

(b) At $t = 2$ hours, $R(t)$ is negative and $R'(2) = 0.011$. Using the correct units, explain the meaning of this result in terms of the amount of alcohol in the blood stream.

(c) Using the correct units, explain the meaning of $\displaystyle\frac{2}{5}\int_0^{2.5} R(t)\, dt$.

(d) Using the correct units, explain the meaning of $\dfrac{R(5) - R(1)}{5 - 1}$.

3. The Novel-for-November Problem

A young novelist enters the Novel-for-November contest, where amateur writers attempt to write a full-length novel over the course of the month. After the months ends, she realizes that the rate at which she wrote varied over the month. She determines that a good model of the daily rate of her writing would be

$$W(t) = 7.5 + 7.5\cos\left(\frac{\pi}{15}(t-1)\right),$$

where t is measured in days and $t = 1$ is the morning of November 1st and $t = 31$ is the end of the day on November 30th. Furthermore, let

$$E(t) = 6 + 8\cos\left(\frac{\pi}{30}(t-10)\right)$$

model the rate at which her editor goes through the manuscript, beginning on November 10th ($t = 10$).

(a) How many pages does the writer complete during the month of November?

(b) Find the value of $W(17)$ and $W'(17)$. Using the correct units, explain the meaning of both.

(c) Find the number of pages which still need to be edited at the end of the day on November 30th.

(d) Set up, but do not solve, an integral equation that would determine how many more days would be needed to finish editing the manuscript.

4. The Tombstone Mine Problem

In 1881, the silver mines in Tombstone, Arizona, struck the local aquifer at 520 feet and began to flood. The owners of the Grand Central mine bought the Cornish engines from the Comstock mines to pump the water out. On a given day, the water was seeping into the mine at a constant rate of 100 gal/hr, and the pumps could drain the water at a rate described by the equation

$$D(t) = 414 + 375\sin\left(\frac{x^2}{72}\right) \text{ gal/hr.}$$

When the pumps start, there are 10,000 gallons of water in the mine.

(a) How many gallons of water were pumped out of the mine during the time interval $0 \le t \le 24$ hours?

(b) Is the level of water rising or falling at $t = 6$? Explain your reasoning.

(c) How many gallons of water are in the mine at $t = 14$ hours?

<u>5. The Ellis Island Problem</u>

In 1920, Dr. Quattrin's grandfather Andrea returned to America from Italy after fighting in World War I. He arrived in New York Harbor on the *SS Pannonia* and, despite having established residency in 1913, had to be processed through the Immigration Center at Ellis Island. There were 1123 non-citizen, third-class passengers on the *Pannonia* that had to go through processing. Immigrants entered the processing line at a rate modeled by the function

$$E(t) = 8843\left(\frac{t}{5}\right)^4\left(1 - \frac{t}{10}\right)^5.$$

where t, in $0 \leq t \leq 10$, is measured in hours after the ship began offloading immigrants. The new arrivals were processed out a rate of 250 people per hour. The *Pannonia* was the third ship in port, so there were already 2500 people in line when the *Pannonia* passenger got into line.

(a) How many passengers from the *Pannonia* had gotten in line for processing in the first 6.2 hours?

(b) Is the rate of change of people entering the processing line increasing or decreasing at $t = 6.2$?

(c) How many people were in line at $t = 6.2$?

<u>6. The Trinary Star Problem</u>

More than 30% of observed star systems have multiple stars, and 70% of those have more than two stars. When stars are closed together, they exchange mass in a process known as accretion. Consider a trinary system where S_1 is larger than S_2, and S_2 is larger than S_3. S_3 will lose mass to S_2, and S_2 will lose mass to S_1. While scientific readings are not available because of the time scale, let us suppose that S_2 loses mass to the larger S_1 at a rate of

$$L(t) = 1 + (0.01t)^2 + 0.23\sin\left(\frac{\pi}{25}t\right)$$

and gains mass from the smaller S_3 at a rate of

$$G(t) = 0.2 + 0.15\sqrt{t}$$

where $0 \leq t \leq 100$ years. $L(t)$ and $G(t)$ are measured in yottatons per year $\left(\frac{Y}{yr}\right)$. (A yottaton is 10^{26} tons, or 10^{-7} solar masses.)

(a) How much mass does S_2 lose to S_1 on $0 \leq t \leq 100$? State the units.

(b) At $t = 50$, is the mass S_2 is gaining from S_3 increasing at an increasing rate? Using the correct units, justify your answer.

(c) At what times on $0 \leq t \leq 100$ is S_2 losing as much mass to S_1 as it is gaining from S_3?

7. The Metformin Problem

A diabetic patient takes Metformin twice a day to control her blood sugar. The medication enters the bloodstream at a rate expressed by

$$M(t) = 8 - \frac{e^{0.47t}}{t+6},$$

where $M(t)$ is measured in centigrams per hour cg/hr and t is measured in hours for $0 \leq t \leq 12$. The liver cleans the medication out of the bloodstream at a rate of $L(t) = 7 - 0.46t\cos(t)$ cg/hr.

(a) How much Metformin enters the bloodstream during this 12-hour time period?

(b) After 9 hours, how much Metformin is still in her bloodstream?

(c) Find $L'(6)$ and explain the meaning of the answer, using the correct units.

(d) Set up, but do not solve, an integral equation that would determine the time when the dose of Metformin has been completely cleaned out of the bloodstream.

8. The San Francisco Intersection Problem

At an intersection in San Francisco, cars turn left at the rate

$$L(t) = 50\sqrt{t}\sin^2\left(\frac{t}{3}\right)$$

cars per hour for the time interval $0 \leq t \leq 18$.

(a) To the nearest whole number, find the total number of cars turning left on the time interval given above.

(b) Traffic engineers will consider turn restrictions if $L(t)$ equals or exceeds 125 cars per hour. Find the time interval where $L(t) \geq 125$, and find the average value of $L(t)$ for this time interval. Indicate units of measurement.

(c) San Francisco will install a traffic light if there is a two-hour time interval in which the product of the number of cars turning left and the number of cars traveling through the intersection exceeds 160,000. In every two-hour interval, 480 cars travel straight through the intersection. Does this intersection need a traffic light? Explain your reasoning.

9. The Post Office Letter Problem

Letters arrive at a post office at a rate of

$$P(t) = 8 + t\sin\left(\frac{t^3}{90}\right)$$

hundred letters per hour over the course of a workday. The day begins at 9 am ($t = 0$) and ends at 5 pm ($t = 8$). There are three hundred letters in the office at 9 am. Workers send letters out of the office at a constant rate of 5 hundred letters per hour.

(a) Find $P'(2)$. Using correct units, interpret the meaning of $P'(2)$ in the context of this problem.

(b) Find the total number of letters that arrive at the office between 9 am and noon ($t = 3$). Round to the nearest whole number of letters.

(c) Write an expression for $L(t)$, the total number of letters in the post office at time t.

10. The Flooded Basement Problem

The basement of a house is flooded, and water keeps pouring in at a rate of

$$w(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right)$$

gallons per hour. At the same time, water is being pumped out at a rate of

$$r(t) = 275\sin^2\left(\frac{t}{3}\right).$$

When the pump is started, at time $t = 0$, there is 1200 gallons of water in the basement. Water continues to pour in and be pumped out for the interval $0 \le t \le 18$.

(a) Is the amount of water increasing at $t = 15$? Why or why not?

(b) To the nearest whole number, how many gallons are in the basement at the time $t = 18$?

(c) For $t > 18$, the water stops pouring into the basement, but the pump continues to remove water until all of the water is pumped out of the basement. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find a value of k.

11. The Sewage Processing Problem

A tank at a sewage processing plant contains 125 gallons of raw sewage at time $t = 0$. During the time interval $0 \leq t \leq 12$ hours, sewage is pumped into the tank at the rate

$$E(t) = 2 + \frac{10}{1 + \ln(t+1)}.$$

During the same time interval, sewage is pumped out at a rate of

$$L(t) = 12 \sin\left(\frac{t^2}{47}\right).$$

(a) How many gallons of sewage are pumped into the tank during the time interval $0 \leq t \leq 12$ hours?

(b) Is the level of sewage rising or falling at $t = 6$? Explain your reasoning.

(c) How many gallons of sewage are in the tank at $t = 12$.

12. The Carbon Sequestration Problem

One innovative approach to global warming is to capture carbon dioxide that is created as a byproduct of producing concrete and storing the CO_2 in the sandstone in depleted natural gas fields. At a particular site on a particular day, the CO_2 is injected into the sandstone at a rate of

$$I(t) = 121 \sin\left(\frac{\pi}{65} t^2\right).$$

The CO_2 stabilizes the ground and forces remaining natural gas upward where it can be extracted at a rate of

$$E(t) = 50 = 50 \cos\left(\frac{\pi}{8} t\right).$$

$I(t)$ and $E(t)$ are measured in metric tons per hour and t is measured in hours where $0 \leq t \leq 8$.

(a) How many metric tons of CO_2 is injected into the field over $0 \leq t \leq 8$?

(b) Find the value of $I(4)$ and $I'(4)$. Using the correct units, explain the meaning of both.

(c) Find the time if any, when the rate of injection of CO_2 is equal the rate of extraction of natural gas.

(d) Find the total change of gasses in the sandstone during this 8-hour day. Using the correct units, explain the results.

3.4 Multiple Choice Homework

1. For $t \geq 0$ hours, H is a differentiable function of t that gives the temperature, in degrees Celsius, at an Arctic weather station. Which of the following is the best interpretation of $H'(24)$?

   a) The change in temperature during the first day.

   b) The change in temperature during the 24th hour.

   c) The average rate at which the temperature changed during the 24th hour.

   d) The rate at which the temperature is changing during the first day.

   e) The rate at which the temperature is changing at the end of the 24th day.

---

2. For $t \geq 0$ hours, H is a differentiable function of t that gives the temperature, in degrees Celsius, at an Arctic weather station. Which of the following is the best interpretation of $\int_0^t H(x)\,dx$.

   a) The change in temperature during the first day.

   b) The change in temperature during the 24th hour.

   c) The average rate at which the temperature changed during the 24th hour.

   d) The rate at which the temperature is changing during the first day.

   e) The rate at which the temperature is changing at the end of the 24th day.

---

3. In the classic 2002 Amusement Park Problem, equations $E(t)$ and $L(t)$ were given, representing the rate at which people were entering and leaving the park respectively, for time $9 \leq t \leq 23$, the hours during which the park was open, with $t = 9$ corresponding to 9 am. Let us assume that $F(t) = E(t) - L(t)$. Which of the following is the best interpretation of $F(16)$?

   a) The number of people in the park at 4 pm.

   b) The number of people entering and leaving the park before 4 pm.

   c) The average number of people in the park between 9 am and 4 pm.

   d) The rate at which the number of people in the park is changing at 4pm.

e) The rate of change of how quickly the number of people in the park is changing at 4pm.

---

4. The cost, in dollars, to shred the confidential documents of a company is modeled by C, a differentiable function of the weights of documents in pounds. Of the following, which is the best interpretation of $C'(500) = 80$?

a) The cost to shred 500 pounds of documents is \$80

b) The average cost to shred documents is $\dfrac{80}{500}$ dollar per pound.

c) Increasing the weight of documents by 500 pounds will increase the cost to shred the documents by approximately \$80.

d) The cost to shred documents is increasing at a rate of \$80 per pound when the weight of the documents is 500 pounds.

---

5. An ice field is melting at the rate $M(t) = 4 - \sin^3(t)$ acre-feet per day, where t is measured in days. How many acre-feet of this field will melt from the beginning of day 1 ($t = 0$) to the beginning of day 4 ($t = 3$)?

a) 6.846 b) 10.667 c) 10.951 d) 11.544 e) 11.999

---

6. Let $R(t)$ represent the rate in gal/hr at which water is leaking out of a tank, where t is measured in hours. Which of the following expressions represents the average rate of change of gallons of water per hour that leaks out in the first three hours?

a) $\int_0^3 R(t)\,dt$ b) $\frac{1}{3}\int_0^3 R(t)\,dt$ c) $\int_0^3 R'(t)\,dt$ d) $R(3) - R(0)$ e) $\dfrac{R(3) - R(0)}{3 - 0}$

---

7. The rate of natural gas sales for the year 1993 at a certain gas company is given by $P(t) = t^2 - 400t + 160000$, where $P(t)$ is measured in gallons/day and t is the number of days in 1993 from day 0 to 365. To the nearest gallon, what is the average rate of natural gas sales at this company for the 31 days of January 1993?

a) $4,777,730$ b) $4,617,930$ c) $154,120$ d) $148,965$ e) $148,561$

---

8. The rate at which ice is melting in a pond is given by $\frac{dV}{dt} = \sqrt{1+2^t}$, where V is the volume of the ice in cubic feet and t is the time in minutes. The amount of ice which has melted in the first five minutes is

a) 14.49 ft^3 b) 14.51 ft^3 c) 14.53 ft^3 d) 14.55 ft^3 e) 14.57 ft^3

---

9. The number of parts per million (ppm), $C(t)$, of chlorine in a pool changes at the rate of $C'(t) = 1 - 3e^{-0.2\sqrt{t}}$ ounces per day, where t is measured in days. There are 10 ppm of chlorine in the pool at time $t = 0$. How many ounces of chlorine are in the pool when $t = 9$?

a) -0.646 b) 9.354 c) -9.285 d) 9.285 e) 0.715

---

10. The amount of money in a bank account is increasing at the rate of $R(t) = 10000e^{0.06t}$ dollars per year, where t is measured in years. If $t = 0$ corresponds to the year 2005, then what is the approximate total amount of increase from 2005 to 2007?

a) $\$21,250$ b) $\$4,500$ c) $\$18,350$ d) $\$32,560$ e) $\$16,250$

---

11. The rate at which water is pumped into a tank is $r(t) = 20e^{0.02t}$, where t is in minutes and $r(t)$ in gallons per minute. Approximately how many gallons of water are pumped into the tank during the first five minutes?

a) 20 b) 22 c) 85 d) 105 e) 150

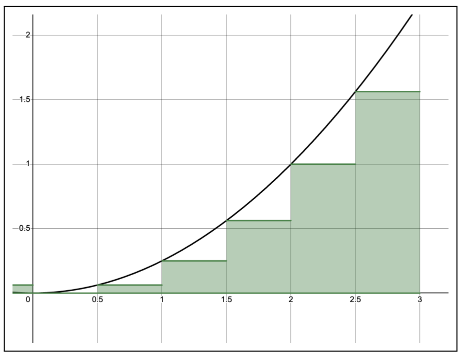
---

12. Oil is leaking from a tanker at the rate of $R(t) = 2000e^{-0.2t}$ gallons per hour, where t is measured in hours. How much oil, in gallons, leaks out of the tanker from $t = 0$ to $t = 10$?

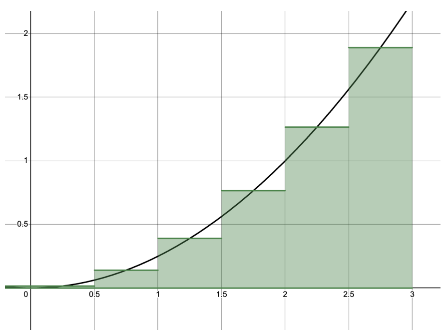
a) 54 b) 271 c) 865 d) 8,647 e) 14,778

3.5: Integral Approximations: Riemann Rectangles and Trapezoidal Sums

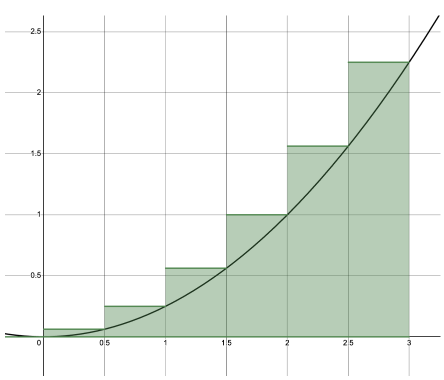
We have been focusing on anti-derivatives of functions were the equation is known. But let us suppose we need to evaluate an integral where either the function is unknown or cannot be anti-differentiated—such as $\displaystyle\int_{-2}^{3} e^{x^2}\,dx$. This is where the concepts that we touched on in the overview of the chapter come into play. If we have some exact y-values, we could approximate the area geometrically. This can be done by dividing the area in question into rectangles, and then finding the area of each rectangle. There are three way to draw the rectangles:



Left-Hand Rectangles



Midpoint Rectangles



Right-Hand Rectangles

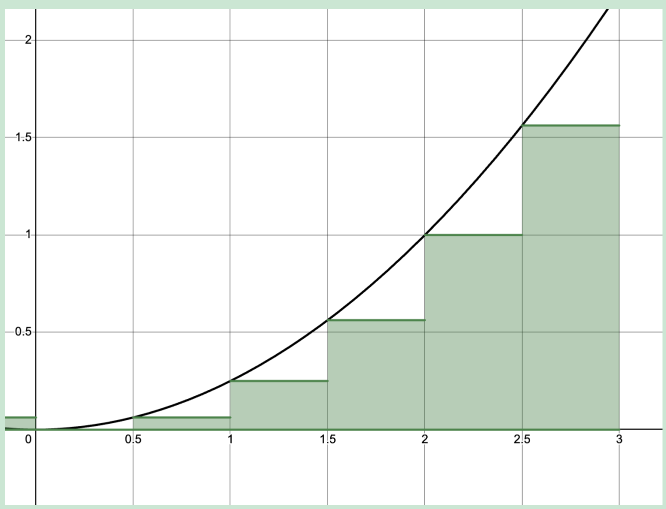
OBJECTIVES

Find Approximations of Integrals Using Different Rectangles.
Use Proper Notation When Dealing with Integral Approximation.

Ex 3.5.1: Use a left hand Riemann sum with four equal sub-intervals to approximate $\int_1^5 f(x)\,dx$ given the table of values below.

x	1	2	3	4	5
$f(x)$	1	4	9	16	25

Sol 3.5.1: Although this problem may seem a little difficult, it's actually quite simple. Let's first recall what a left hand Riemann summation should look like:



So, to find the areas of the rectangle, we multiply the width of the rectangle by the height. The width of the rectangle is the difference between adjacent x-values. Because we are asked to find a left hand Riemann sum, we are gonna start with the leftmost height. Therefore, we have

$$\int_1^5 x^2\,dx \approx (2-1)\cdot 1 + (3-2)\cdot 4 + (4-3)\cdot 9 + (5-4)\cdot 16 = \boxed{30}$$

Notice how we never use the $f(x)$ value of 25. This is because when we are using left-hand rectangles, we will never make a rectangle that

With this example, we can create some steps for approximation an integral with rectangles:

Steps to Approximating an Integral with Rectangles

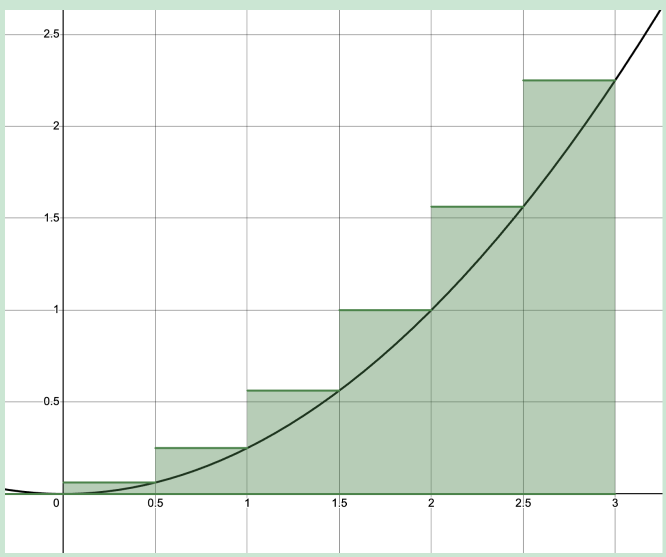
1. Read the question and determine which type of rectangles to calculate
2. Write out the integral as a sum of lengths times widths.
3. Calculate the areas of each rectangle.
4. Add the areas together for your approximation.
5. State answer using proper notation.

Although **Ex 3.5.1** had rectangles of equal width, this does not necessarily have to be true

Ex 3.5.2: Use a right hand Riemann sum with the sub-intervals defined below to approximate $\int_2^{14} f(x)\,dx$ given the table of values below.

x	2	5	10	14
$f(x)$	4	10	7	12

Sol 3.5.2: Let's remind ourselves what a right-hand rectangle approximation looks like:



Now, as this time we are asked to find a right hand Riemann sum, we start from the right. So, we have

$$\int_2^{14} f(x)\,dx \approx (14-10)\cdot 12 + (10-5)\cdot 7 + (5-2)\cdot 10 = \boxed{113}$$

The third kind of Riemann rectangle is where the height comes from the midpoints of the rectangles. Two conditions are needed for midpoint rectangles:

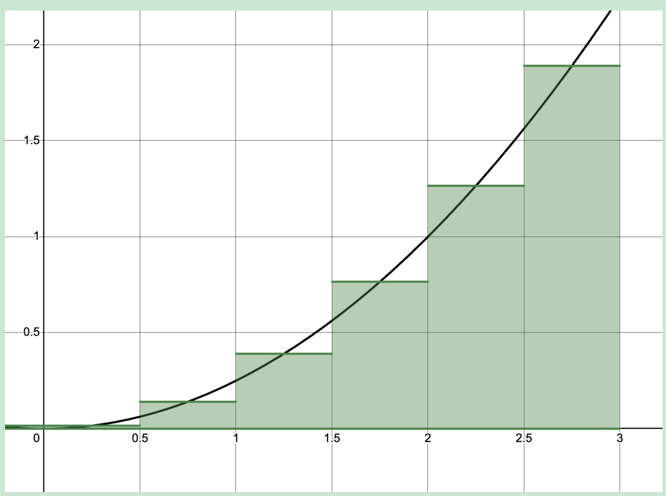
1. There needs to be an odd number of x values.
2. The even-numbered (i.e. 2nd, 4th) x-values must be midpoints of adjacent $x - values$.

Ex 3.5.3: The rate of consumption, in gallons per minute, recorded during an airplane flight is given by a twice differentiable and strictly increasing function $R(t)$. A table of selected values for $R(t)$ for the time interval $0 \leq t \leq 90$ is shown below.

t (minutes)	0	20	40	50	60	90	120
$R(t)$ (gallons per minute)	20	30	40	55	65	70	95

Use the Riemann sum with midpoint rectangles and the sub-intervals given by the table to approximate the value of $\int_0^{120} R(t)\,dt$.

Sol 3.5.3: Recall the structure of midpoint rectangles:



To find the Riemann sum with midpoint rectangles, we need to take the difference between the odd-numbered x-values as our width, and use the $R(t)$ values in between those x-values as our height. Therefore, we get

$$\int_0^{120} R(t) \approx (40 - 0) \cdot 30 + (60 - 40) \cdot 55 + (120 - 60) \cdot 70 = \boxed{6500}$$

Ex 3.5.4: Use the midpoint rule and the given data to approximate the value of

$$\int_0^{2.6} f(x)\,dx.$$

x	0	0.4	0.8	1.2	1.6	2.1	2.6
$f(x)$	3.5	2.3	3.2	4.3	4.7	5.9	4.1

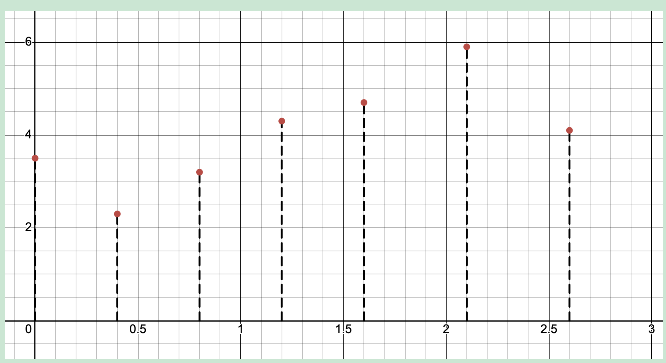
Sol 3.5.4: For this example, we simply follow the steps in **Ex 3.5.3**. This gives us

$$\int_0^{2.6} f(x)\,dx \approx (0.8-0)\cdot 2.3 + (1.6-0.8)\cdot 4.3 + (2.6-1.6)\cdot 5.9 = \boxed{11.18}$$

The left, right, and midpoint rectangle methods estimate the area under a curve by using rectangles with constant heights. The trapezoidal rule builds on this idea by connecting the tops of the rectangles with straight lines, creating shapes that better match the curve. It can be seen as the average of the left and right rectangle methods and serves as a simple step toward more accurate ways to approximate areas.

Ex 3.5.5: Using the table of data from **Ex 3.5.4** approximate $\int_0^{2.6} f(x)\,dx$ with 6 trapezoids.

Sol 3.5.5: Since this is a new method that we are approaching, let's try visualizing it. We can first start by drawing the heights.



Next, we are gonna connect the points to draw out our trapezoids.

Now, it becomes pretty clear how we can find the Riemann sum with this method. We use the difference between adjacent x-values as the height of the trapezoid, and adjacent pairs of $f(x)$-values as the two bases of the trapezoid. Therefore, let's use the area formula for a trapezoid,

$$A_{\text{trapezoid}} = \left(\frac{b_1 + b_2}{2}\right) \cdot h,$$

to calculate this Riemann sum:

$$\begin{aligned}\int_0^{2.6} f(x)\,dx &\approx \frac{3.5+2.3}{2}\cdot(0.4-0) + \frac{2.3+3.2}{2}\cdot(0.8-0.4) + \frac{3.2+4.3}{2}\cdot(1.2-0.8)\\ &+ \frac{4.3+4.7}{2}\cdot(1.6-1.2) + \frac{4.7+5.9}{2}\cdot(2.1-1.6) + \frac{5.9+4.1}{2}\cdot(2.6-2.1)\\ &= \boxed{8.15}\end{aligned}$$

However, one may notice that this process is quite cumbersome. We have to do multiple computations involving decimals just to get our answer, and sometimes this could lead to arithmetic errors. Luckily, there is a formula that could simplify this process a lot. However, note that this formula could only apply **if the heights of the trapezoid are all the same**. So, what exactly is this formula? Well, let's take a look.

Let's say we have points $x_0,\ x_1,\ x_2, \ldots, x_n$ and $f(x)$ values $f(x_0),\ f(x_1),\ f(x_2), \ldots, f(x_n)$. We know that we are integrating from x_0 to x_n, we can call those a and b, respectively. Since we know that the height of the trapezoids are the same, we know that the height of each trapezoid is the total sum of the heights divided by the number of trapezoids. Therefore, each trapezoid has a height of

$$\frac{b-a}{n}.$$

Now, let's start writing out the Riemann sum in its complete form. We do the method as we described in **Ex 3.5.5**, taking adjacent $x - values$ as the heights and adjacent pairs of $f(x)$-values as the bases. Therefore, we get

$$\int_a^b f(x)\,dx \approx \frac{f(x_0)+f(x_1)}{2}\left(\frac{b-a}{n}\right) + \frac{f(x_1)+f(x_2)}{2}\left(\frac{b-a}{n}\right) + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\left(\frac{b-a}{n}\right).$$

Factoring out $\frac{b-a}{n}$ and simplifying the fraction gives us

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)\left(\frac{f(x_0)+f(x_1)+f(x_1)+f(x_2)+\cdots f(x_{n-1})+f(x_n)}{2}\right).$$

We can simplify this even further to

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{2n}\right)(f(x_0)+2f(x_1)+2(fx_2)+\cdots+2f(x_{n-1})+f(x_n)).$$

This gives us the Trapezoidal Rule for equal sub-intervals.

The Trapezoidal Rule for Riemann summations

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{2n}\right)(f(x_0)+2f(x_1)+2(fx_2)+\cdots+2f(x_{n-1})+f(x_n)).$$

Let's see this formula in use with an example.

Ex 3.5.6: The following table gives values of a continuous function. Approximate the average value of the function using the Trapezoidal Rule.

x	10	20	30	40	50	60	70
$f(x)$	3.649	4.718	6.482	9.389	14.182	22.086	35.115

Sol 3.5.6: Note that this question is asking for the *average value* of the function. Therefore, we will have to use our average value formula.

$$\begin{aligned}
f_{avg} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{70-10}\int_{10}^{70} f(x)\,dx \\
&\approx \frac{1}{70-10}\left(\frac{70-10}{2(6)}\right)(3.649+2\cdot 4.718+2\cdot 6.482+2\cdot 9.389 \\
&\qquad +2\cdot 14.182+2\cdot 22.086+35.115) \\
&= \boxed{12.707}
\end{aligned}$$

Over and Underestimations

Now that we've learned about the left, right, midpoint, and trapezoidal Riemann sum methods, it's time to explore how these approximations can overestimate or underestimate the true value of an integral. Depending on whether the function is increasing or decreasing, the rectangles or trapezoids we draw may cover too much or too little area under the curve. First, though, let's recall what we know about tangent line approximations:

→ Your approximation will be an **overestimate** if the curve is **concave down** (since your "tangent lines" will be above the curve).

→ Your approximation will be an **underestimate** if the curve is **concave up** (since your "tangent lines" will be below the curve).

Let's extend that concept to Riemann sums with the following statements.

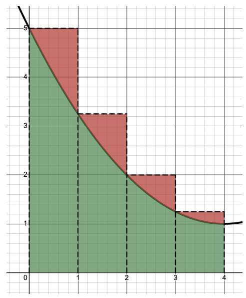
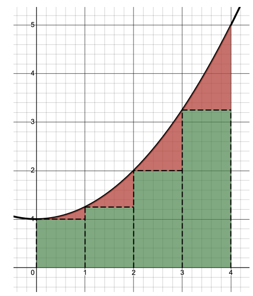
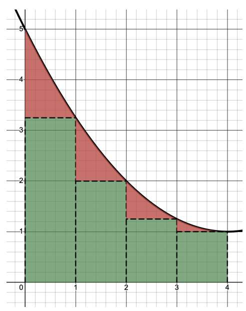
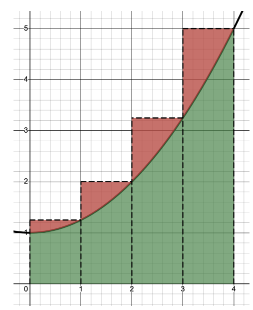
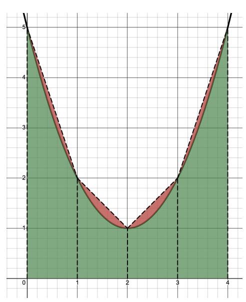
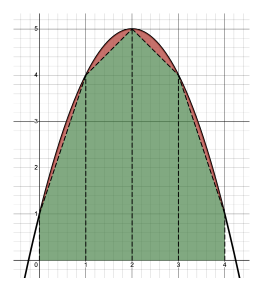
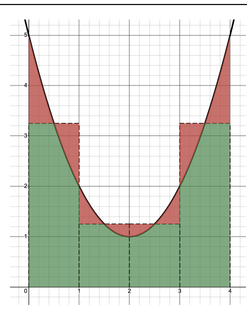
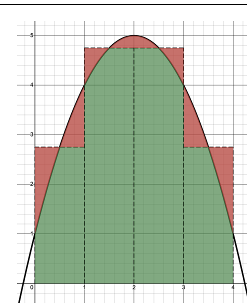
→ Left-hand rectangles are an **overestimate** if the curve is **decreasing** and an **underestimate** if the curve is **increasing**.

→ Right-hand rectangles are in **overestimate** if the curve is **increasing** and an **underestimate** if the curve is **decreasing**.

→ Midpoint Rectangles are an **overestimate** if the curve is **concave down** and an **underestimate** if the curve is **concave down**.

→ Trapezoids are an **overestimate** if the curve is **concave up** and an **underestimate** if the curve is **concave down**.

We can further illustrate this concept with the table in the following page:

	Decreasing	**Increasing**
Left Hand	 *Overestimate*	 *Underestimate*
Right Hand	 *Underestimate*	 *Overestimate*
	Concave Up	**Concave Down**
Trapezoid	 *Overestimate*	 *Underestimate*
Midpoint	 *Underestimate*	 *Overestimate*

3.5 Free Response Homework

1. The following table gives values of a continuous function:

x	0	1	2	3	4	5	6	7	8
$F(x)$	10	15	17	12	3	-5	8	-2	10

Estimate the average value of the function on $x \in [0,\ 8]$ using:

(a) right-hand Riemann rectangles.

(b) left-hand Riemann rectangles.

(c) midpoint Riemann rectangles.

2. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the midpoint rule to estimate the distance traveled by the car.

t (seconds)	0	10	20	30	40	50
$v(t)$ (mi/h)	0	38	52	58	55	51
t (seconds)	60	70	80	90	100	
$v(t)$ (mi/h)	56	53	50	47	45	

3. Below is a chart showing the rate of a rocket flying according to time in minutes:

t (minutes)	0	10	20	30	40	50	60
$v(t)$ (km/h)	30	28	32	18	52	48	28

Use this information to answer each of the questions below. Make sure you express your answer in correct units.

(a) Find an approximation for $\displaystyle\int_0^{60} v(t)\,dt$ using midpoint rectangles.

(b) Find an approximation for $\displaystyle\int_0^{30} v(t)\,dt$ using trapezoids.

(c) Find an approximation for $\displaystyle\int_{30}^{60} v(t)\,dt$ using left-hand rectangles.

(d) Find an approximation for $\int_0^{40} v(t)\,dt$ using right-hand rectangles.

4. Below is a chart showing the rate of water flowing through a pipeline according to time in minutes:

t (minutes)	0	8	16	24	32	40	48
$V(t)$ (m^3/min)	26	32	43	24	19	24	26

Use this information to answer each of the questions below. Make sure you express your answer in correct units.

(a) Find an approximation for $\int_0^{48} V(t)\,dt$ using midpoint rectangles.

(b) Find an approximation for $\int_0^{16} V(t)\,dt$ using right-hand rectangles.

5. Below is a chart of your speed driving to school in meters per second:

t (seconds)	0	30	90	120	220	300	360
$v(t)$ (m/sec)	0	21	43	38	30	24	0

Use this information to answer each of the questions below. Make sure you express your answer in correct units.

(a) Find an approximation for $\int_0^{360} v(t)\,dt$ using left-hand rectangles.

(b) Find an approximation for $\int_0^{220} v(t)\,dt$ using trapezoids.

6. Below is a chart showing the velocity of the Flash is he runs across the country:

t (seconds)	0	4	8	12	16	20	24
$v(t)$ (km/sec)	10	12	15	19	24	18	7

Use this information to answer each of the questions below. Make sure you express your answer in correct units.

(a) Find an approximation for $\int_0^{24} v(t)\,dt$ using midpoint rectangles.

(b) Find an approximation for $\int_0^{16} v(t)\,dt$ using trapezoids.

7. Below is a chart showing the rate of sewage flowing through a pipeline according to time in minutes:

t (minutes)	0	4	6	10	13	15	20
$V(t)$ (gallons/min)	83	68	82	40	38	30	68

Use this information to answer each of the questions below. Make sure you express your answer in correct units.

(a) Find an approximation for $\int_0^{20} V(t)\,dt$ using trapezoids.

(b) Find an approximation for $\int_0^{20} V(t)\,dt$ using left-hand rectangles.

8. Star Formation Rate (SFR) observations of red-shift allow scientists to track the total mass gained in a galaxy by the making of new stars. Below is a table of such data:

t	0	1	2	3	4	5	6	7	8
SFR	0.0029	0.0051	0.0055	0.0049	0.0042	0.0035	0.0029	0.0025	0.0021

SFR is measured in solar masses per cubic parsec per gigayear (millions of years), and t is measured in gigayears.

(a) Use midpoint rectangles to approximate the total mass of stars formed from $t = 0$ to $t = 8$.

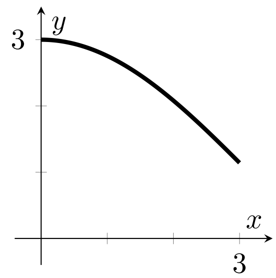
(b) Use right-hand rectangles to approximate the average solar masses per cubic parsec per gigayear.

9. Use (a) the trapezoidal rule and (b) the midpoint rule to approximate $\int_0^2 \sqrt[4]{1+x^2}\,dx$ with the specified value of $n = 8$ (8 intervals).

10. Use (a) the trapezoidal rule and (b) the midpoint rule to approximate $\int_1^2 \frac{\ln x}{1+x}\,dx$ with the specified value of $n = 10$ (10 intervals).

3.5 Multiple Choice Homework

1. The graph of the function f is shown below for $0 \le x \le 3$.



Of the following, which has the smallest value? Assume each approximation is done with 6 equal subintervals.

a) $\displaystyle\int_1^3 f(x)\,dx$

b) Left-hand Riemann approximation of $\displaystyle\int_1^3 f(x)\,dx$.

c) Right-hand Riemann approximation of $\displaystyle\int_1^3 f(x)\,dx$.

d) Midpoint Riemann approximation of $\displaystyle\int_1^3 f(x)\,dx$.

e) Trapezoidal approximation of $\displaystyle\int_1^3 f(x)\,dx$.

---

2. The table below gives the values for the rate at which water flowed into a lake, with readings taken at specific times.

Time (sec)	0	10	25	37	46	60
Rate (gal/sec)	500	400	350	280	200	180

A right-hand Riemann sum, with the five subintervals indicated by the data in the table, is used to estimate the total amount of water that flowed into the lake during the time period $0 \le t \le 60$. What is this estimate?

a) 1910 gal b) 14100 gal c) 16930 gal d) 18725 gal e) 20520 gal

---

3. A car is traveling on a straight road such that selected measures of the velocity have

values given on the table below.

t (sec)	10	20	40	70	80
$v(t)$ (m/sec)	90	88	100	90	85

Using four left-hand Riemann rectangles based on this table, the estimated distance traveled by the car between $t = 10$ and $t = 80$ seconds is

a) 6125 m b) 6380 m c) 6430 m d) 6495 m e) 6560 m

---

4. The function f, with some values represented in the table below, is continuous and differentiable on the closed interval [3, 12].

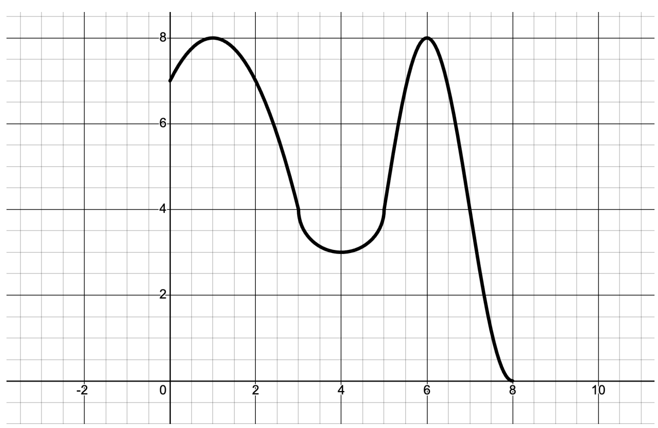
x	3	6	9	12
$f(x)$	12	8	7	5

What is the right Riemann approximation of $\int_3^{12} f(x)\,dx$?

a) 69 b) 90 c) 111 d) 126 e) 201

---

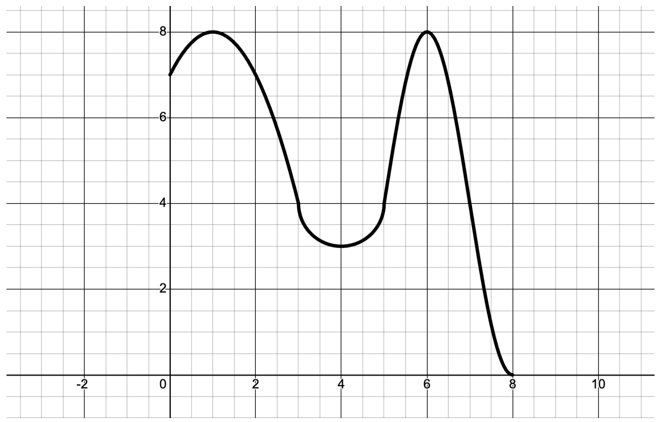
5. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using eight right-hand rectangles with equal widths is

a) 18.5 b) 37 c) 40 d) 40.5 e) 44

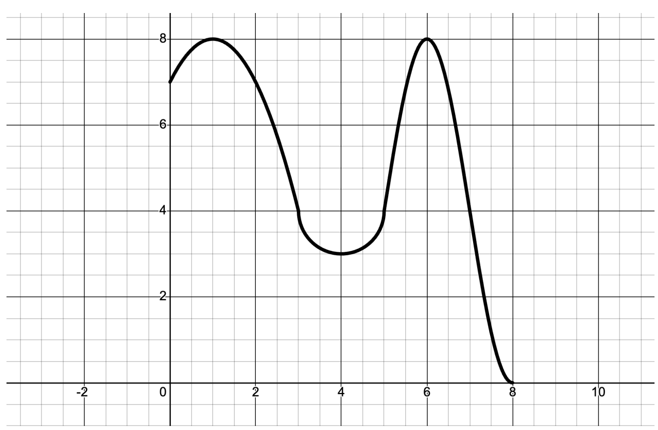
6. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using eight left-hand rectangles with equal widths, is

a) 23 b) 37 c) 40 d) 40.5 e) 44

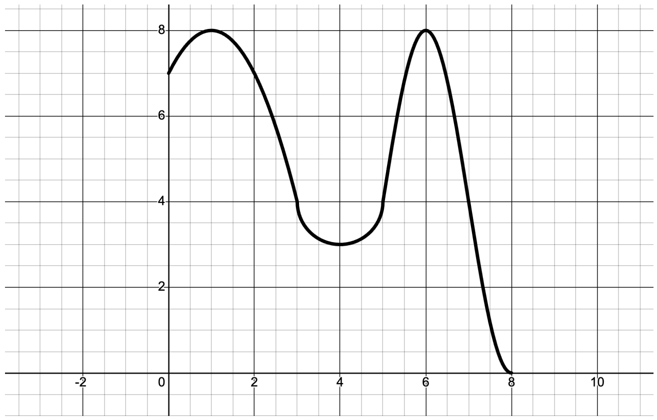
7. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using eight trapezoids with equal widths, is

a) 37 b) 40 c) 40.5 d) 44 e) 48

8. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using four midpoint rectangles with equal widths, is

a) 20 b) 37 c) 40 d) 40.5 e) 44

---

9. Let f be a differentiable function on the closed interval [2, 14] and which has values shown on the table below.

x	2	5	10	14
$f(x)$	12	28	34	30

Using the sub-intervals defined by the table, estimate $\int_2^{14} f(x)\,dx$ using a right-hand Riemann approximation.

a) 296 b) 312 c) 343 d) 374 e) 390

---

10. The following table lists the known values of a function $f(x)$.

x	1	2	3	4	5
$f(x)$	0	1.1	1.4	1.2	1.5

Use the trapezoidal rule to approximate $\int_1^5 f(x)\,dx$.

a) 4.1 b) 4.3 c) 4.5 d) 4.7 e) 4.9

---

11. A small plant is purchased from a nursery and the change in height of the plant is measured at the end of the each day for four days. The data, where $H(t)$ is measured in inches per day and t is measured in days, are listed below.

x	0	1	2	3	4
$f(x)$	0	1.3	1.5	2.1	2.6

Using the trapezoidal rule, which of the following represents an estimate of the average rate of growth of the plant over the four-day period?

a) $\frac{1}{4}(0 + 1.3 + 1.5 + 2.1 + 2.6)$

b) $\frac{1}{4}\left[\frac{1}{2}(0 + 1.3 + 1.5 + 2.1 + 2.6)\right]$

c) $\frac{1}{4}\left[\frac{1}{2}(0 + 2(1.3) + 2(1.5) + 2(2.1) + 2.6)\right]$

d) $\frac{1}{4}\left[\frac{1}{2}(0 + 2(1.3) + 2(1.5) + 2(2.1) + 2(2.6))\right]$

e) $\frac{1}{4}\left[\frac{1}{4}(0 + 2(1.3) + 2(1.5) + 2(2.1) + 2.6)\right]$

3.6: Intro to AP: Reasoning with Tabular Data

Reasoning with tabular data (aka Table Problems) has been one of the most commonly recurring topics on the AP exam. It is often the first free-response question on the exam.

There are two main kinds of table problems. The first kind appears in multiple choice questions that include tables of values meant to be plugged into the Chain, Product, or Quotient Rule. These tables often contain distractor values, so the challenge is identifying only the information you actually need. The second and more common type is a word problem involving an unknown function paired with specific data points. These problems closely connect to Accumulation of Rates and Rectilinear Motion topics, since they require interpreting how a function behaves based on limited numerical information. It is this second kind of problems which we will investigate here.

An analysis of the past ten years of AP exams reveals the most common subtopics on the test. They are, in order:

- Riemann and trapezoidal sums.
- Overestimate vs underestimate.
- Tangent approximations using secant lines.
- Graphic problems with the First Derivative Test.
- Interpretation of units.
- Mean value, intermediate value theorem.

For this section, we need to recall our key phrases and units for accumulation of rates problems, along with our tips for explaining answers.

OBJECTIVES

Analyze the Relationship Between Rates of Change and Integrals with Tabular Data.

Let us consider a problem similar to the ones in the Accumulation of Rates section. but with a table of data as part of the problem.

Ex 3.6.1: At 6am at the *Popular Potatoes* potato chip factory, there are already 5 tons of potatoes in the factory. More potatoes are delivered from 6am ($t = 6$) to noon ($t = 12$) at a rate modeled by

$$P(t) = 9 - \frac{9\sin(x-2)}{x-2}$$

tons of potatoes per hour. Workers arrive at 6am and begin to process the potatoes to turn them into potato chips. Their supervisor measures their rate of output every hour and records her findings in the chart below,

t	6	9	13	14	16
$C(t)$	7.9	6.5	3.9	3.1	1.3

where t represents the time after midnight (in hours) and $C(t)$ represents the rate of potatoes processed in tons per hour. The supervisor determines that the workers' rate of processing is a decreasing function throughout the day.

(a) How many tons of potatoes arrive at the *Popular Potatoes* factory between 6am and noon?

(b) Use a left Riemann sum with sub-intervals indicated by the table to approximate $\int_6^{16} C(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(c) Is your approximation in part (b) an under or over-approximation? Explain.

(d) Approximate $C'(11.5)$. Explain the meaning of your answer

(e) The workers end their shift at 4pm. At that time, are there still potatoes in the factory left to process? Explain your reasoning.

Sol 3.6.1:
a) Since this question asks for a total amount, we know that we must find the integral of the rate of potatoes that are being delivered to the factory. Therefore, we have

$$\text{Total tons} = \int_6^{12} P(t)\,dt = \int_6^{12} 9 - \frac{9\sin(x-2)}{x-2} = \boxed{54.899 \text{ tons of potatoes}}$$

b) Using our setup for approximation with left-hand rectangles, we have

$$\int_6^{16} C(t)\,dt \approx (9-6)\cdot 7.9 + (13-9)\cdot 6.5 + (14-13)\cdot 3.9 + (16-14)\cdot 3.1 = 59.8$$

Therefore, **approximately 59.8 tons of potatoes were processed into chips between 6am and 4pm**.

Note the use of the word "approximately." Because the Riemann sum is an *approximation*, we must use "approximately" in our answer.

c) For this question, let's simply refer to our chart for over and underestimations. From the table, we know that function $C(t)$ is a decreasing function, and we also know that we are creating left-hand Riemann rectangles. Therefore, we know that our Riemann approximation must be an $\boxed{\text{overestimation}}$.

d) The first thing we notice about the problem is that $t = 11.5$ is not a value on our table. So, to approximate $C'(11.5)$, let's utilize the average rate of change formula, using the closest t values greater than and less than 11.5.

$$C'(11.5) \approx \frac{f(b) - f(a)}{b - a} = \frac{3.9 - 6.5}{13 - 9} = -0.65$$

Therefore, we can say that **at $t = 11.5$, the rate of potatoes processed is decreasing by approximately 0.65 tons per hour per hour**.

Once again, notice how we must use "approximately" in our answer!

e) To determine if there were still potatoes left to process, we need to take the total number of potatoes that have been processed and subtract that from the total number of potatoes there were to process.

From the problem, we know that we start with 5 tons of potatoes, and we found from part (a) that we accumulate an extra 54.899 tons of potatoes. We also know from part (b) that approximately 59.8 tons of potatoes were processed. Therefore, we have

$$\text{Potatoes left} = 5 + 54.899 - 59.8 = 0.099 \text{ tons}$$

Therefore, **there are still potatoes in the factory left to process**.

Some astute students may notice that our value of 59.8 was an estimate, so how do we know for certain that not all the potatoes were processed? Well, in part (c), we determined that this value was an overestimate. This means it represents the maximum possible processed amount. Since we are subtracting this maximum from $5 + 54.899$, even this largest possible subtraction still leaves a positive amount. Any more accurate value (which would be smaller) would leave even more potatoes remaining. Therefore, we can conclude with certainty that there are still potatoes left to process.

Ex 3.6.2: Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (minutes)	0	4	6	10	13	15	20
$C(t)$ (gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function.

(a) Estimate $V'(7)$. Show the work that leads to your answer. Indicate the units.

(b) Use a trapezoidal sum with subintervals indicated by the table to approximate $\int_0^{20} V(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(c) Find the value of $\int_0^{20} V'(t)\,dt$ and explain the meaning of this value in the context of the problem.

Sol 3.6.2:
a) Just like in part (d) in **Ex 3.6.1**, we will utilize the average rate of change formula with the closest t values great than and less than 7.

$$V'(7) \approx \frac{f(b)-f(a)}{b-a} = \frac{48-83}{10-6} = \boxed{-8.75 \text{ gal/min}^2}$$

b) Using our setup for the trapezoidal rule, we get

$$\begin{aligned}\int_0^{20} V(t)\,dt &\approx \frac{83+68}{2}\cdot(4-0) + \frac{68+83}{2}\cdot(6-4) + \frac{83+48}{2}\cdot(10-6)\\ &+ \frac{48+38}{2}\cdot(13-10) + \frac{38+30}{2}\cdot(15-13) + \frac{30+38}{2}\cdot(20-15)\\ &= 1082\end{aligned}$$

Therefore, we can say that **approximately 1082 gallons of sewage flowed through the pipeline between $t = 0$ minutes and $t = 20$ minutes**.

c) For this problem, since we see an integral and a derivative, we know that we must utilize the Fundamental Theorem of Calculus. Therefore, we have

$$\int_0^{20} V'(t)\,dt = V(20) - V(0) = 38 - 83 = -45 \text{ gal/min}$$

From this result, we can say that **the total change of the rate of flow between $t = 0$ and $t = 20$ minutes is -45 gallons per minutes**.

3.6 Free Response Homework

1. Below is a chart showing the rate of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (minutes)	0	8	16	24	32	40	48
$V(t)$ (m^3/min)	26	32	43	24	19	24	26

(a) Estimate $V'(7)$. Show the work that leads to your answer. Indicate the units.

(b) Find $\int_8^{40} V'(t)\,dt$.

(c) Use a trapezoidal sum with sub-intervals indicated by the table to approximate $\int_0^{48} V(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(d) Using correct units, explain the meaning of $\frac{1}{48}\int_0^{48} V(t)\,dt$ in the context of the problem.

2. A small plant is purchased from a nursery and the change in height of the plant is measured at the end of each day for four days. The data, where $H(t)$ is measured in millimeters per day and t is measured in days, are listed below.

t (days)	0	8	16	24	32	40	48
$H(t)$ (mm per day)	26	32	43	24	19	24	26

(a) Estimate $H'(3)$. Show the work that leads to your answer. Indicate the units.

(b) Explain how one would know that the plant's growth is not increasing at a decreasing rate.

(c) Use right-hand rectangles with sub-intervals indicated by the table to approximate $\int_0^{4} H(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(d) Using correct units, explain the meaning of $\frac{1}{4}\int_0^{4} H(t)\,dt$ in the context of the problem.

3. The rate of consumption of fuel, in gallons per minute, recorded during an airplane flight is given by a twice differentiable and strictly increasing function $R(t)$. A table of selected values of $R(t)$ for the time interval $0 \le t \le 90$ is shown below.

t (minutes)	0	20	40	50	60	90
$H(t)$ (gallons/min)	20	30	40	55	65	70

(a) Estimate $R'(30)$. Show the work that leads to your answer. Indicate the units.

(b) Use right-hand Riemann rectangles to approximate $\int_0^{90} R(t)\,dt$ and indicate units of measure. Explain the meaning of $\int_0^{90} R(t)\,dt$ in terms of the fuel consumption.

(c) Use left-hand rectangles to find $\frac{1}{70}\int_{20}^{90} R(t)\,dt$. Using the correct units, explain the meaning of $\frac{1}{70}\int_{20}^{90} R(t)\,dt$ in terms of the fuel consumption.

4. A diabetic patient tests his blood glucose level every morning. After being put on insulin, the data below shows the glucose levels, $G(t)$, in milligrams per deciliter (mg/dL) over one week.

t (days)	1	2	3	4	5	6	7
$G(t)$ (mg/dL)	233	198	185	168	147	130	147

(a) Estimate $G'(3.7)$. Using the correct units, explain the meaning of the result.

(b) Use midpoint Riemann rectangles to approximate $\int_1^7 G(t)\,dt$. Using the correct units, explain the meaning of $\frac{1}{7}\int_1^7 G(t)\,dt$ in terms of the patient's glucose levels.

(c) Ignoring the last data point, $M(t) = 237.6e^{-0.082t}$ is a model of $G(t)$. Find $M'(3.7)$. Is $M(t)$ decreasing at an increasing rate? Show the work that leads to your conclusion.

5. Diabetic patients take a test called A1c every three months, which measures the three-month average percentage of glycated hemoglobin (that is, hemoglobin covered in glucose.) Let $A(t)$ represent the A1c score, measured as a percentage. The following data shows a patient's A1c score over the course of 21 months.

t (months)	0	3	6	9	12	15	18	21
$A(t)$ (%)	10.2	10.0	10.5	9.1	8.0	8.9	8.3	8.6

(a) Find $\int_0^{21} A'(t)\,dt$. Show the work that leads to your answer. Explain the meaning of $\int_0^{21} A'(t)\,dt$ in terms of A1c scores.

(b) Use a right-hand Riemann approximation to approximate $\int_0^{21} A(t)\,dt$. Using the correct units, explain the meaning of $\frac{1}{21}\int_0^{21} A(t)\,dt$ in terms of the patient's A1c score.

(c) A model of $A(t)$ is $B(t) = -0.305x + 10.571 + 0.1\sin(\pi x)$. Find $\frac{1}{21}\int_0^{21} B(t)\,dt$.

6. A family leases solar panels on their house. At the end of the year, they receieve a report, including the table below, which shows the monthly production $P(t)$ of electricity, in kilowatts per month (kW/month), from the panels.

t (months)	0	1	2	3	4	5	6
$P(t)$ (kW/month)	160.3	192.8	345.7	746.1	944.2	873.0	1128.6
t (months)	7	8	9	10	11	12	
$P(t)$ (kW/month)	928.3	851.3	751.3	535.5	216.4	150.7	

(a) Use a right-hand Riemann approximation to approximate $\int_0^{12} P(t)\,dt$. Indicate the units.

(b) A model of $P(t)$ is $k(t) = 660 - 489\cos\left(\frac{\pi}{6}t\right)$ is a model of $P(t)$. Find $\int_0^{12} k(t)\,dt$.

(c) Using the model $k(t) = 660 - 489\cos\left(\frac{\pi}{6}t\right)$, show that the production is decreasing at $t = 9$. Is the production decreasing at an increasing rate?

7. Dr. Quattrin analyzes his PG&E bill to track his consumption of both electricity ($C_e(t)$) and gas ($C_g(t)$) over the course of a year. The tables below are the result. $C_e(t)$ is measured in kilowatts (kW) and $C_g(t)$ is measured in therms (thm).

t (months)	0	1	2	3	4	5	6
$C_e(t)$ (kW)	390.7	660	667.1	538.4	420.5	412.1	347.8
$C_g(t)$ (thm)	87.6	84.6	109	116	79.8	53.9	42.9
t (months)	7	8	9	10	11	12	
$C_e(t)$ (kW)	287.5	303.1	322.4	342.5	390.3	384.2	
$C_g(t)$ (thm)	24.9	25.6	18	20.3	48.9	91.8	

(a) Approximate $C_e'(3.4)$ and $C_g'(3.4)$. Using correct units, explain the meaning of these estimations in terms of increasing and/or decreasing consumption of each commodity at $t = 3.4$

(b) Use right-hand Riemann rectangles to approximate $\int_0^{12} C_e(t)\,dt$. Indicate the units.

(c) Use midpoint Riemann rectangles to approximate $\int_0^{12} C_g(t)\,dt$. Indicate the units.

(d) Using the correct units, explain the meaning of $\frac{1}{12}\int_0^{12} C_g(t)\,dt$.

8. Dr. Quattrin decides to lease solar panels from Sunrun Solar. After a year, he reanalyzes his PG&E bill to track both his consumption of electricity ($C_e(t)$) and his production of electricity ($P_e(t)$) over the course of a year. The tables below show the consumption of electricity, measured in kilowatts (kWs).

t (months)	0	1	2	3	4	5	6
$C_e(t)$ (kW)	326.5	660.0	667.1	538.4	420.5	412.1	347.8
t (months)	7	8	9	10	11	12	
$C_e(t)$ (kW)	287.5	303.1	322.4	342.5	390.3	384.2	

The equation below is a model for the production in kW per month that PG&E buys back.

$$P_e(t) = 407 - 374.2\cos\left(\frac{\pi}{6}t\right)$$

(a) How much power does PG&E buy back from the Quattrins over the course of the year? Indicate the units.

(b) Using a trapezoidal approximation, approximate the amount of power the Quattrins consume over the course of the year. Based on this estimate and your answer for part (a), does Dr. Quattrin owe PG&E for electricity at the end of the year, or does PG&E owe Dr. Quattrin a refund?

(c) Electricity costs \$0.28 per kW. Write an expression for amount due on the PG&E bill at time t months.

9. Dr. Quattrin's paternal grandmother's family originated in the alpine town of Sauris, Italy, where the temperature in January changes at a rate of $W(t)$ degrees Celsius per hour. $W(t)$ is a twice-differentiable, increasing, and concave up function with selected values in the table below. At midnight ($t = 0$), the temperature in Sauris is $-8°$ C.

t (hours after midnight)	0	1	3	6	8
$W(t)$ (deg Celsius per hr)	-2.6	-3.1	-1.2	1.9	2.5

(a) At approximately what rate is the rate of change of the temperature changing at 2am ($t = 2$)? Indicate the units.

(b) Use a right Riemann sum with sub-intervals indicated by the table to approximate $\int_0^8 W(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(c) Set up, but do not solve, an integral equation which would determine the temperature in Sauris at 1pm.

10. 1998 AP Calculus AB #3

11. 2001 AP Calculus AB #2

12. 2007 AP Calculus BC #5