Chapter 2:

Intro To Anti-Derivatives

Chapter 2 Overview: Anti-Derivatives

As noted in the introduction, Calculus is essentially comprised of four operations:

- Limits
- Derivatives
- Indefinite Integrals (Or Anti-Derivatives)
- Definite Integrals

As mentioned above, there are two types of integrals — the definite integral and the indefinite integral. The definite integral was explored first as a way to determine the area bounded by a curve, rather than bounded by a polygon. The summation of infinite rectangles is

$$A = \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x,$$

and the representation

$$\int_a^b f(x)\,dx$$

is the exact amount, with $\displaystyle\int$ being an elongated and stylized s for "sum".

Newton and Leibnitz made the connection between the definite integral and the antiderivative, showing that the process of reversing the derivative results in an infinite summation. The antiderivative and indefinite integral are inverses of each other, just as squares and square roots or exponential and log functions. In this chapter, we will consider how to reverse the differentiation process. In a later chapter, we will dive deeper into the definite integral. Let's start by reviewing our derivative rules, as they will be necessary for us to take the antiderivative.

You must know the derivative rules in order to know the antiderivative rules!

The Power Rule: $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

The Product Rule: $\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

The Quotient Rule: $\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx}$$

$$\frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}[e^u] = (e^u)\frac{du}{dx}$$

$$\frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}[a^u] = (a^u \cdot \ln u)\frac{du}{dx}$$

$$\frac{d}{dx}[\log_a u] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\csc^{-1} u\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cot^{-1} u\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

2.1: The Anti-Power Rule

As we have seen, we can deduce things about a function if its derivative is known. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called antidifferentiation, or integration.

Symbol for the Integral

$$\int f(x)\,dx \qquad \text{“the integral of f of x, d-x”}$$

The dx is called the differential. For now, we will treat it as part of the integral symbol. It tells us the independent variable of the function [usually, but not always, x]. It does have a meaning on its own, but we will explore that later.

Looking at the integral as an antiderivative, we should be able to figure out the basic process. Remember:

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

and

$$\frac{d}{dx}[\text{constant}] = 0$$

It follows that if we are starting with the derivative and want to reverse the process, the power must increase by one and we should divide by this new power. Formally,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

The $+C$ is to account for any constant that might've been in the equation before the derivative was taken. Note that $n = -1$ does not work with this rule because it results in a division by zero. However, we know from our derivative rules that the derivative of $\ln x$ yields x^{-1}. Therefore, we can append our anti-power rule.

The Complete Anti-Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Since $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + [g(x)]$ and $\frac{d}{dx}[cx^n] = c\frac{d}{dx}[x^n]$, it follows that:

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c(f(x))\,dx = c\int f(x)\,dx$$

These allow us to integrate a polynomial by integrating each term separately.

OBJECTIVES

Find the Anti-Derivative of a Polynomial.
Integrate Functions Using Transcendental Operations
Use Integration to Solve Rectilinear Motion Problems

Ex 2.1.1: $\int \left(3x^2+4x+5\right)\,dx$

Sol 2.1.1:

$$\int \left(3x^2+4x+5\right)\,dx = 3\frac{x^{2+1}}{2+1} + 4\frac{x^{1+1}}{1+1} + 5\frac{x^{0+1}}{0+1} + C$$

$$= \frac{3x^3}{3} + \frac{4x^2}{2} + \frac{5x^1}{1} + C$$

$$= \boxed{x^3+2x^2+5x+C}$$

Ex 2.1.2: $\int \left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)\,dx$

Sol 2.1.2:

$$\int \left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)\,dx = \frac{x^{4+1}}{4+1} + \frac{4x^{2+1}}{2+1} + \frac{5x^{0+1}}{0+1} + \ln|x| - \frac{x^{-5+1}}{-5+1} + C$$

$$= \boxed{\frac{1}{5}x^5 + \frac{4}{3}x^3 + 5x + \ln|x| + \frac{1}{4x^4} + C}$$

Ex 2.1.3: $\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right)\, dx$

Sol 2.1.3:

$$\begin{aligned}\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right) dx &= \int \left(x^2 + x^{\frac{1}{3}} - \frac{4}{x}\right) dx \\ &= \frac{x^{2+1}}{2+1} + \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 4\ln|x| + C \\ &= \boxed{\frac{1}{3}x^3 - \frac{3}{4}x^{\frac{4}{3}} + 4\ln|x| + C}\end{aligned}$$

Integrals of products and quotients can be done easily IF they can be turned into a polynomial.

Ex 2.1.4: $\int \left(x^2 + \sqrt[3]{x}\right)(2x+1)\, dx$

Sol 2.1.4:

$$\begin{aligned}\int \left(x^2 + \sqrt[3]{x}\right)(2x+1)\, dx &= \int \left(2x^3 + 2x^{\frac{4}{3}} + x^2 + x^{\frac{1}{3}}\right) dx \\ &= \frac{2x^4}{4} + \frac{2x^{\frac{7}{3}}}{\frac{7}{3}} + \frac{x^3}{3} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} \\ &= \boxed{\frac{1}{2}x^4 + \frac{6}{7}x^{\frac{7}{3}} + \frac{1}{3}x^3 + \frac{3}{4}x^{\frac{4}{3}} + C}\end{aligned}$$

The next example is called an initial value problem. It has an ordered pair (or initial value pair) that allows us to solve for C.

Ex 2.1.5: $f'(x) = 4x^3 - 6x + 3$. Find $f(x)$ if $f(0) = 13$.

Sol 2.1.5:

$$\begin{aligned}f(x) &= \int \left(4x^3 - 6x + 3\right) dx \\ &= x^4 - 3x^2 + 3x + C\end{aligned}$$

$$f(0) = 0^4 - 3(0)^2 + 3(0) + C$$

$$= 13 \therefore C = 13$$

$$\therefore \boxed{f(x) = x^4 - 3x^2 + 3x + 13}$$

Now, let's take a look at a type of problem called a *rectilinear motion* problem. In these problems, we study the motion of an object moving along a straight line—its position, velocity, and acceleration.

Ex 2.1.6: The acceleration of particle is described by $a(t) = 3t^2 + 8t + 1$. Find the distance equation for $x(t)$ if $v(0) = 3$ and $a(0) = 1$.

Sol 2.1.6:

$$v(t) = \int a(t)\,dt$$

$$= \int \left(3t^2 + 8t + 1\right)\,dt$$

$$= t^3 + 4t^2 + t + C_1$$

$$3 = (0)^3 + 4(0)^2 + (0) + C_1 \therefore 3 = C_1$$

$$v(t) = t^3 + 4t^2 + t + 3$$

$$x(t) = \int v(t)\,dt$$

$$= \int \left(t^3 + 4t^2 + t + 3\right)\,dt$$

$$= \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + C_2$$

$$1 = \frac{1}{4}(0)^4 + \frac{4}{3}(0)^3 + \frac{1}{2}(0)^2 + 3(0) + C_2 \therefore 1 = C_2$$

$$\boxed{x(t) = \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + 1}$$

Ex 2.1.7: The acceleration of a particle is described by $a(t) = 12t^2 - 6t + 4$. Find the distance equation for $x(t)$ if $v(1) = 0$ and $x(1) = 3$.

Sol 2.1.7:

$$\begin{aligned}
v(t) &= \int a(t)\,dt \\
&= \int \left(12t^2 - 6t + 4\right) \\
&= 4t^3 - 3t^2 - 4t + C_1
\end{aligned}$$

$$0 = 4(1)^3 - 3(1)^2 + 4(1) + C_1 \therefore -5 = C_1$$

$$v(t) = 4t^3 - 3t^2 - 4t - 5$$

$$\begin{aligned}
x(t) &= \int v(t)\,dt \\
&= \int \left(4t^3 - 3t^2 - 4t - 5\right)\,dt \\
&= t^4 - t^3 - 2t^2 - 5t + C_2
\end{aligned}$$

$$3 = (1)^4 - (1)^3 - 2(1)^2 - 5(1) + C_2 \therefore 6 = C_2$$

$$\boxed{x(t) = t^4 - t^3 - 2t^2 - 5t + 6}$$

The proof of all the transcendental integral rules can be left to a more formal Calculus course. But, since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

Transcendental Integral Rules

$$\int \cos(u)\,du = \sin(u) + C \qquad \int \csc(u)\cot(u)\,du = -\csc(u) + C$$

$$\int \sin(u)\,du = -\cos(u) + C \qquad \int \sec(u)\tan(u)\,du = \sec(u) + C$$

$$\int \sec^2(u)\,du = \tan(u) + C \qquad \int \csc^2(u)\,du = -\cot(u) + C$$

$$\int e^u\,du = e^u + C \qquad \int \frac{1}{u}\,du = \ln|u| + C$$

$$\int a^u\,du = \frac{a^u}{\ln|a|} + C \qquad \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$$

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C \qquad \int \frac{1}{u\sqrt{u^2-1}}\,du = sec^{-1} + C$$

Note that there are only three integrals that yield inverse trig functions, but there were six inverse trig derivatives. This is because the other three derivative rules are just the negatives of the first three.

Ex 2.1.8: $\int (\sin(x) + 3\cos(x))\ dx$

Sol 2.1.8:

$$\begin{aligned}\int (\sin(x) + 3\cos(x))\ dx &= \int \sin(x)\,dx + 3\int \cos(x)\,dx \\ &= \boxed{-\cos(x) + 3\sin(x) + C}\end{aligned}$$

Ex 2.1.9: $\int \left(e^x + 4 + 3\csc^2(x)\right)\ dx$

Sol 2.1.9:

$$\int \left(e^x + 4 + 3\csc^2(x)\right)\ dx = \int e^x\,dx + 4\int\ dx + 3\int \csc^2(x)\,dx$$

$$= \boxed{e^x + 4x - 3\cot(x) + C}$$

Now, let's take a look at some more complex integrals that yield inverse trig functions. These more general forms extend the earlier rules by introducing a constant a, and they are especially useful when working with substitutions or integrals that don't simplify neatly to the unit case.

Trig Inverse Integral Rules

$$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u^2 + a^2}\, du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$