Chapter 3:

Integrals

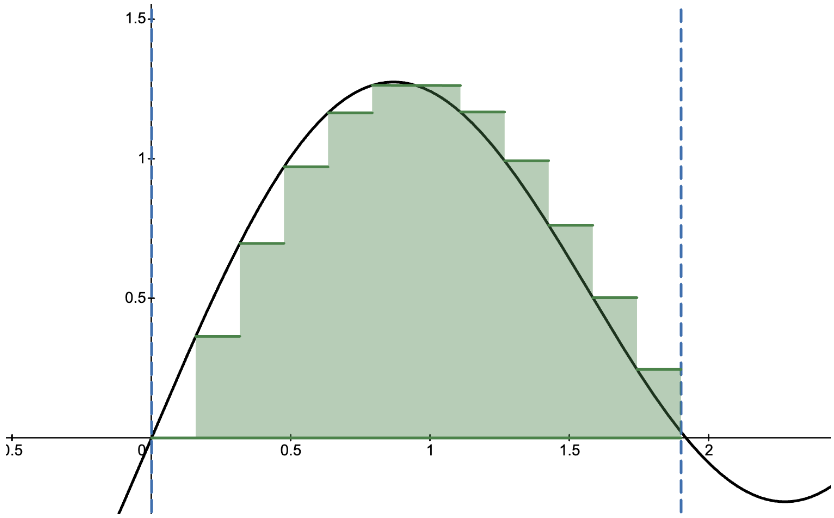
Chapter 3 Overview: Definite Integrals

In this chapter, we will study the Fundamental Theorem of Calculus, which establishes the link between the algebra and the geometry, with an emphasis on mechanics of how to find the definite integral. We will consider the differences implied between the context of the definite integral as an operation and as an area accumulator. We will learn some approximation techniques for definite integrals and see how they provide theoretical foundation for the integral. We will revisit graphical analysis in terms of the definite integral and view another typical AP context for it. Finally, we will consider what happens when trying to integrate at or near an asymptote.

As noted in the overview of the last chapter, antiderivatives are known as indefinite integrals because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an analytic-geometrical context of area. Though it is not necessary to know the theory behind this in order to calculate the integral, the theory is a major subject of integral calculus, so we will explore it briefly in here.

The Limit Definition of the Definite Integral

We know, from geometry, how to find the exact area of various polygons, but we never considered figures where one side is not made of a line segment. Here we want to consider an area bounded by some curve $y = f(x)$ on the top, the x-axis on the bottom, some arbitrary $x = a$ on the left, and $x = b$ on the right.



As we can see above, the area approximated by rectangles whose height is the y-value of the equation and whose width we will call Δx. The more rectangles we make, the better the approximation. For a good animation of this concept, consider the following video:

Riemann sum approximation animation

The area of each rectangle would be $f(x) \cdot \Delta x$, and the total area of n rectangles would be

$$A = \sum_{i=1}^{n} f(x_i) \cdot \Delta x.$$

This equation is known as the **Riemann summation**. Although this equation looks complicated, it represents a rather simple idea. We are adding up the areas of many thin rectangles to approximate the total area under the curve $y = f(x)$ between two points. As we increase the number of rectangles n, each rectangle becomes narrower, and thus our approximation becomes more accurate.

But, how can we find the exact area? With the Riemann sum, we are only coming up with better and better approximations right now. If we could make an infinite number of rectangles (which would be infinitely thin), we could potentially find the exact area under this curve. Luckily, we just so happen to have the mathematical tools to do this: we can take the limit as n approaches infinity.

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

This is where the definite integral comes in. The definite integral provides a precise way to calculate the exact area under a curve by taking the limit of the Riemann sum as the number of rectangles approaches infinity. In other words, instead of merely approximating the area with a finite number of rectangles, the definite integral captures what happens when the width of each rectangle becomes infinitesimally small. We write this limit in a compact form as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x,$$

where a is the "lower bound" and b is the "upper bound." Mathematicians sometimes nuance this statement as

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x, \text{ where } \Delta x = \frac{b-a}{n}}$$

3.1: The Fundamental Theorem of Calculus

Up to this point, we've seen how integration can be used to find the exact area under a curve by taking the limit of Riemann sums. But what's truly remarkable is how integration connects so deeply with differentiation. This connection is captured in one of the most important results in all of calculus: the Fundamental Theorem of Calculus (FTC).

The Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a, b]$, then:

1. $\dfrac{d}{dx}\displaystyle\int_c^x f(t)\,dt = f(x)$ or $\dfrac{d}{dx}\displaystyle\int_c^u f(t)\,dt = f(u)\cdot\dfrac{du}{dx}$

2. If $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

The first part of the Fundamental Theorem of Calculus simply says what we already know– that an integral is an anti-derivative. The second part of the Fundamental Theorem says that the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

The idea of the integral meaning the area may not make sense initially, mainly because we are used to geometry, where an area is always measured in square units. But, that is only because the length and width are always measured in the same kind of units, so multiplying length and width area always measured in the same kind of units, so multiplying length and width must yield square units. We are expanding our vision beyond that narrow view of things here. Consider a graph where the x-axis is time and the y-axis is velocity in feet per second. The area under the curve would be measured as seconds multiplied by feet per seconds, which is simply feet. So, the area under the curve is equal to the distance traveled in feet. In other words, the integral of velocity is distance.

OBJECTIVES

Evaluate Definite Integrals.
Find Average Value of a Continuous Function Over a Given Interval.
Differentiate Integral Expressions with the Variable in the Boundary.

Let us first consider part 2 of the Fundamental Theorem of Calculus, since it has a very practical application. This part of the Fundamental Theorem of Calculus gives us a clear method for evaluating definite integrals.

Ex 3.1.1: Evaluate $\displaystyle\int_2^8 (4x+3)\,dx$

Sol 3.1.1: First, let's start by treating this as a regular antiderivative

$$\int (4x+3)\,dx = 2x^2 + 3x$$

Note that our $+C$ will not be needed, as we will be taking a definite integral. Now, let's apply the Fundamental Theorem of Calculus.

$$\begin{aligned}\int_2^8 f(x)\,dx &= F(8) - F(2)\\ &= 2x^2 + 3x\Big|_2^8\\ &= 2(8)^2 + 3(8) - \Big(2(2)^2 + 3(2)\Big)\\ &= \boxed{138}\end{aligned}$$

The vertical bar that you see is called the evaluation bar, and it's used to indicate that we are evaluating the antiderivative at the upper and lower limits of integration.

Ex 3.1.2: Evaluate $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx$

Sol 3.1.2:

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}}\,dx &= 2\sqrt{x}\Big|_1^4\\ &= 2\sqrt{(4)} - 2\sqrt{(1)}\\ &= \boxed{2}\end{aligned}$$

Ex 3.1.3: Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\,dx$

Sol 3.1.3:

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = -\cos(x)\Big|_0^{\frac{\pi}{2}}$$

$$= -\cos\left(\frac{\pi}{2}\right) + \cos(0)$$

$$= \boxed{1}$$

Ex 3.1.4: Evaluate $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du$

Sol 3.1.4:

$$\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \left(4u^{-3} + u^{-1}\right)\,du$$

$$= \left(-2u^{-2} + \ln|u|\right)\Big|_1^2$$

$$= -2(2)^{-2} + \ln 2 - (-2(1)^{-2} + \ln 1)$$

$$= \boxed{\frac{3}{2} + \ln 2}$$

Ex 3.1.5: Evaluate $\displaystyle\int_{-5}^5 \frac{1}{x^3}\,dx$

Sol 3.1.5: When initially looking at the problem, one may simply proceed with finding the definite integral.

$$\int_{-5}^5 \frac{1}{x^3}\,dx = -\frac{1}{x^2}\Big|_{-5}^5$$

$$= -\frac{1}{(-5)^2} + \frac{1}{(5)^2}$$

$$= 0$$

But, this is a trap! We have to be careful here, because the function $\dfrac{1}{x^3}$ is *not defined* over the the interval $[-5, 5]$, since $\dfrac{1}{0^3}$ is undefined. Therefore, the Fundamental Theorem of Calculus $\boxed{\text{does not apply}}$.

Just as we had many properties for the indefinite integral, we have many properties for the definite integral. There are three main properties that are utilized often on the AP exam.

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2. $\int_a^a f(x)\,dx = 0$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

Ex 3.1.6: If $\int_{-5}^{2} f(x)\,dx = -17$ and $\int_{5}^{2} f(x)\,dx = -4$, find $\int_{-5}^{5} f(x)\,dx$.

Sol 3.1.6:

$$\begin{aligned}
\int_{-5}^{5} f(x)\,dx &= \int_{-5}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx \\
&= \int_{-5}^{2} f(x)\,dx - \int_{5}^{2} f(x)\,dx \\
&= -17 - (-4) \\
&= \boxed{-13}
\end{aligned}$$

Part I of the Fundamental Theorem of Calculus is very important for the **theory** of calculus, but is limited (hehe) in the context of this course to L'Hospital problems which will will explore in a later chapter. Here is how the formula may be applied.

Ex 3.1.7: Use the Fundamental Theorem of Calculus to find $f'(t)$ if $f(t) = \int_2^{3t^2} (4x + 3)\,dx$

Sol 3.1.7:

$$f'(t) = \frac{d}{dt}\int_2^{3t^2} (4x+3)\,dx$$

$$= \left(4\left(3t^2\right)+3\right)(6t)$$

$$= \boxed{72t^3+18t}$$

3.1 Free Response Homework

Use part II of the Fundamental Theorem of Calculus to evaluate the integral, or explain why the integral cannot be evaluated.

1. $\int_{-1}^{3} x^5 \, dx$
2. $\int_{2}^{7} (5x-1) \, dx$
3. $\int_{-5}^{5} \frac{2}{x^3} \, dx$
4. $\int_{-3}^{-1} \frac{x^7 - 4x^3 - 3}{x} \, dx$
5. $\int_{1}^{2} \frac{3}{t^4} \, dt$
6. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc(y)\cot(y) \, dy$
7. $\int_{0}^{\frac{\pi}{4}} \sec^2(y) \, dy$
8. $\int_{1}^{9} \frac{3}{2z} \, dz$
9. $\int_{1}^{8} \frac{x^2 - 4}{\sqrt[3]{x}} \, dx$
10. $\int_{\pi}^{\frac{5\pi}{4}} \sin(y) \, dy$
11. $\int_{1}^{4} \frac{x^4 - 4x^2 - 5}{x^2} \, dx$
12. $\int_{3}^{5} \left(x^2 + 5x + 6\right) \, dx$
13. $\int_{\pi}^{\frac{\pi}{4}} \cos(y) \, dy$
14. $\int_{1}^{4} \frac{x^3 - 2x^2 - 4x}{x^2} \, dx$
15. $\int_{1}^{2} \frac{x^2 - 4x + 7}{x} \, dx$
16. $\int_{1}^{16} \frac{2x^2 - 1}{\sqrt[4]{x}}$

Use the following values for problems 17 - 27 to evaluate the given integrals.

$\int_{-2}^{5} f(x)\,dx = -2$	$\int_{1}^{5} f(x)\,dx = 3$
$\int_{-2}^{1} g(x)\,dx = 4$	$\int_{5}^{1} g(x)\,dx = 9$
$\int_{1}^{5} h(x)\,dx = 7$	$\int_{5}^{-2} h(x)\,dx = -6$

17. $\int_{-2}^{1} f(x) \, dx$
18. $\int_{-2}^{5} g(x) \, dx$
19. $\int_{-2}^{1} h(x) \, dx$
20. $\int_{1}^{5} (f(x) - g(x)) \, dx$

21. $\int_{-2}^{5} (g(x) + h(x))\, dx$

22. $\int_{-2}^{1} (h(x) - f(x))\, dx$

23. $\int_{-2}^{5} (h(x) + f(x))\, dx$

24. $\int_{1}^{5} (2f(x) + 3h(x))\, dx$

25. $\int_{-2}^{1} (2f(x) - 3g(x))\, dx$

26. $\int_{-2}^{5} \left(\frac{1}{2}g(x) + 4h(x)\right)\, dx$

27. $\int_{1}^{5} \left(\frac{1}{3}h(x) + 2f(x)\right)\, dx$

28. $\int_{5}^{5} (f(x) + g(x) + h(x))\, dx$

Use part I of the Fundamental Theorem of Calculus to find the derivative of the function.

29. $g(y) = \int_{2}^{y} t^2 \sin(t)\, dt$

30. $g(x) = \int_{0}^{x} \sqrt{1+2t}\, dt$

31. $F(x) = \int_{x}^{2} \cos\left(t^2\right)\, dt$

32. $h(x) = \int_{2}^{\frac{1}{x}} \arctan(t)\, dt$

33. $y = \int_{3}^{\sqrt{x}} \frac{\cos(t)}{t}\, dt$

34. $f(x) = \int_{e}^{x^2} \ln\left(t^2+1\right)\, dt$

35. $f(x) = \int_{10}^{x^2} t \ln t\, dt$

36. $f(x) = \int_{e^x}^{5} \left(t^3 + t + 1\right)\, dt$

37. If $F(x) = \int_{1}^{t^2} \frac{\sqrt{1+u^4}}{u}\, du$, find $F'(t)$.

38. If $h(x) = \int_{\pi}^{\sqrt{x}} e^{5t}\, dt$, find $h'(x)$.

39. If $h(m) = \int_{5}^{\cos(m)} t^2 \cos^{-1}(t)\, dt$, find $h'(m)$.

40. If $h(y) = \int_{5}^{\ln y} \frac{e^t}{t^4}\, dt$, find $h'(y)$.

3.1 Multiple Choice Homework

1. If $\int_{-5}^{2} f(x)\, dx = -17$ and $\int_{5}^{2} f(x)\, dx = -4$, then $\int_{-5}^{5} f(x)\, dx =$

a) -21 b) -13 c) 0 d) 13 e) 21

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2. Let f and g be continuous functions such that $\int_{0}^{6} f(x)\, dx = 9$, $\int_{3}^{6} f(x)\, dx = 5$, and $\int_{3}^{0} g(x)\, dx = -7$. What is the value of $\int_{0}^{3} \left(\frac{1}{2}f(x) - 3g(x)\right)\, dx$.

a) -23 b) -19 c) $-\frac{17}{2}$ d) 19 e) 23

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3. Given that $\int_2^3 P(t)\,dt = 7$ and $\int_2^7 P(t)\,dt = -2$, what is $\int_7^3 P(t)\,dt$?

a) -9 b) -5 c) 5 d) 9 e) not enough information

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4. Based on the information below, find $\int_1^{-2} (g(x) + f(x))\,dx$

$\int_{-2}^{5} f(x)\,dx = -2$	$\int_1^5 f(x)\,dx = 3$
$\int_{-2}^{1} g(x)\,dx = 4$	$\int_5^1 g(x)\,dx = 9$

a) -9 b) -1 c) 0 d) 1 e) 9

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5. Based on the information below, find $\int_5^{-2} (g(x) - f(x))\,dx$

$\int_{-2}^{5} f(x)\,dx = -2$	$\int_1^5 f(x)\,dx = 3$
$\int_{-2}^{1} g(x)\,dx = 4$	$\int_5^1 g(x)\,dx = 9$

a) -3 b) 3 c) 6 d) -6 e) 14

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6. Using the table values from questions 5 and 6, which of the following cannot be determined?

a) $\int_5^1 (g(x) + f(x))\,dx$ b) $\int_1^{-2} (g(x) - f(x))\,dx$

c) $\int_{-2}^{5} 3g(x)(-4(f(x)))\,dx$ d) $\int_1^5 (3g(x) + 4f(x))\,dx$

3.2: Definite Integrals and the Substitution Rule

Now, it's time to revisit u-substitution within the context of definite integrals. Although the process is largely similar, there are some nuances that we must consider.

OBJECTIVES

Evaluate Definite Integrals Using the Fundamental Theorem of Calculus.
Evaluate Definite Integrals Applying the Substitution Rule, When Appropriate.
Use Proper Notation When Evaluating These Integrals

Ex 3.2.1: Evaluate $\int_0^2 t^2\sqrt{t^3+1}\,dx$.

Sol 3.2.1: Now, this problem may look just like a regular u-substitution problem that we did in the previous chapter. However, when we switch our integration variable to u, we also need to make sure to switch our definite integration boundaries to match. Let's see what that means in the solution below.

$$\int_0^2 t^2\sqrt{t^3+1}\,dt = \frac{1}{3}\int_0^2 3t^2\sqrt{t^3+1}\,dt$$

$$\hookrightarrow\ u = t^3+1 \quad\Big|\quad \hookrightarrow\ du = 3t^2\,dt$$

Here is where we have to be careful. We cannot simply rewrite our definite integral in terms of u, since our upper and lower boundaries are in terms of t! Therefore, we need to also find our new boundaries for the u variables. Luckily, we can do this simply by plugging in our t values into our equation for u.

$$u(0) = (0)^3+1 = 1 \quad\Big|\quad u(2) = (2)^3+1 = 9$$

Now, we can continue integrating! (Note the boundary change in red.)

$$\begin{aligned}\frac{1}{3}\int_{0}^{2} 3t^2\sqrt{t^3+1}\,dt &= \frac{1}{3}\int_{1}^{9}\sqrt{u}\,du\\ &= \frac{1}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^9\\ &= \frac{2}{9}\left(9^{\frac{3}{2}}-1^{\frac{3}{2}}\right)\\ &= \boxed{\frac{52}{9}}\end{aligned}$$

Ex 3.2.2: Evaluate $\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)\,dx$.

Sol 3.2.2:

$$\hookrightarrow\ u = x^2 \quad\Bigg|\quad \hookrightarrow\ du = 2x\,dx$$

$$u(0) = (0)^2 = 0 \quad\Bigg|\quad u\left(\sqrt{\pi}\right) = \left(\sqrt{\pi}\right)^2 = \pi$$

$$\begin{aligned}
&= \frac{1}{2}\int_0^{\pi} \cos(u)\,du \\
&= \frac{1}{2}\sin(u)\Big|_0^{\pi} \\
&= \boxed{0}
\end{aligned}$$

Ex 3.2.3: Evaluate $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$.

Sol 3.2.3:

$$\hookrightarrow\ u = \frac{1}{x} \quad\Bigg|\quad \hookrightarrow\ du = -\frac{1}{x^2}$$

$$u(1) = \frac{1}{(1)} = 1 \quad\Bigg|\quad u(2) = \frac{1}{(2)} = \frac{1}{2}$$

$$\begin{aligned}
\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx &= -\int_1^{\frac{1}{2}} e^u\,du \\
&= -e^u\Big|_1^{\frac{1}{2}} \\
&= \boxed{-e^{\frac{1}{2}} + e}
\end{aligned}$$

Ex 3.2.4: Evaluate $\int_1^{\sqrt{13}} \frac{x}{x^2+3}\,dx$.

Sol 3.2.4:

$$\hookrightarrow u = x^2 + 3 \quad \Big| \quad \hookrightarrow du = 2x\,dx$$

$$u(1) = (1)^2 + 3 = 4 \quad \Big| \quad u\left(\sqrt{13}\right)^2 + 3 = 16$$

$$\begin{aligned}\int_1^{\sqrt{13}} \frac{x}{x^2+3}\,dx &= \frac{1}{2}\int_1^{\sqrt{13}} \frac{2x}{x^2+3}\,dx \\ &= \frac{1}{2}\int_4^{16} \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|u|\Big|_4^{16} \\ &= \boxed{\frac{1}{2}\ln(16) - \frac{1}{2}\ln 4}\end{aligned}$$

Now, technically, $\frac{1}{2}\ln(16) - \frac{1}{2}\ln 4$ is the correct answer. However, when you are taking the multiple choice portion of the AP Calc BC exam, the expectation is that trivial logarithms should be simplified, so this answer may not appear as a choice. So, we must utilize our log rules:

$$\begin{aligned}\frac{1}{2}\ln(16) - \frac{1}{2}\ln 4 &= \frac{1}{2}\ln\left(\frac{16}{4}\right) \\ &= \frac{1}{2}\ln 4 \\ &= \ln 4^{\frac{1}{2}} \\ &= \boxed{\ln 2}\end{aligned}$$

The Average Value

One simple application of the definite integral is the Average Value Theorem. Recall that the average of a finite set of numbers is the total of the numbers divided by how many numbers we are averaging. Or, in technical terms:

$$\text{avg} = \frac{\sum_{i=1}^{n} x_i}{n}$$

But, what does it mean to take the average value of a continuous function? Let's say you drive from home to school—what was your average velocity? What was the average temperature today? What was your average height for the first 15 years of your life? All these questions can be answered with the following formula:

The Average Value Formula

The average value of a function f on a closed interval $[a, b]$ is defined by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

If we look at this formula in the context of the Fundamental Theorem of Calculus, it will start to make a little more sense.

$$\begin{aligned} \int_a^b f(x)\,dx &= F(b) - F(a) \\ \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{b-a}(F(b) - F(a)) \\ &= \frac{F(b) - F(a)}{b-a} \end{aligned}$$

Notice that this is just the average slope for $F(x)$ on $x \in [a, b]$. Since the derivative $F'(x)$ gives the instantaneous rate of change of $F(x)$, the average slope of $F(x)$ is the same as average value of $F'(x)$. But, since the definition in the Fundamental Theorem of Calculus says that $F'(x) = f(x)$, this is actually just the average value of $f(x)$.

Ex 3.2.5: Find the average value of $f(x) = x^2 + 1$ on $[0, 5]$.

Sol 3.2.5: We first start with our formula:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Then, we can substitute in the function and the interval.

$$f_{avg} = \frac{1}{5-0} \int_0^5 \left(x^2 + 1\right)\, dx$$

Using either $\boxed{\texttt{MATH}}$ $\boxed{\texttt{9}}$ on our TI-84 or integrating analytically gives us $f_{avg} = \boxed{\frac{28}{3}}$.

Ex 3.2.6: Find the average value of $h(\theta) = \sec(\theta)\tan(\theta)$ on $\left[0, \frac{\pi}{4}\right]$.

Sol 3.2.6:

$$\begin{aligned} h_{avg} &= \frac{1}{b-a} \int_a^b h(\theta)\, d\theta \\ &= \frac{1}{\frac{\pi}{4} - 0} \int_0^{\frac{\pi}{4}} \sec(\theta)\tan(\theta) d\theta \\ &= \boxed{\frac{4}{\pi}\left(\sqrt{2} - 1\right) \approx 0.524} \end{aligned}$$

3.2 Free Response Homework

1. $\displaystyle\int_0^1 x^2\left(1+2x^3\right)^5\,dx$

2. $\displaystyle\int_0^1 x^3\left(x^4+5\right)^3\,dx$

3. $\displaystyle\int_{-1}^1 x\sqrt{4-x^2}\,dx$

4. $\displaystyle\int_1^2 \frac{x^2}{\sqrt[3]{9-x^3}}\,dx$

5. $\displaystyle\int_0^3 \frac{10t+15}{\sqrt[4]{t^2+3t+1}}\,dt$

6. $\displaystyle\int_1^2 \frac{x+1}{\sqrt{x^2+2x+4}}\,dx$

7. $\displaystyle\int_1^3 \frac{5t}{t^2+1}\,dt$

8. $\displaystyle\int_{-1}^2 \frac{1}{2x+5}\,dx$

9. $\displaystyle\int_{\sqrt{3}}^2 ye^{y^2-3}\,dy$

10. $\displaystyle\int_0^1 \frac{v^2}{8-v^3}\,dv$

11. $\displaystyle\int_3^{e^2+2} \frac{1}{x-2}\,dx$

12. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin(\theta)}{\cos^2(\theta)}d\theta$

13. $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx$

14. $\displaystyle\int_0^{\pi} \sec^2\left(\frac{t}{4}\right)\,dt$

15. $\displaystyle\int_0^{\pi} \frac{\sin(x)}{2-\cos(x)}\,dx$

16. $\displaystyle\int_2^4 \frac{1}{x\ln x}\,dx$

17. $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\,dx$

18. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^5(x)\sin(x)\,dx$

19. $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$

20. $\displaystyle\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2(\ln y)}{y}\,dy$

21. $\displaystyle\int_0^{\pi} \frac{\cos(x)}{2+\sin(x)}\,dx$

22. $\displaystyle\int_0^{\pi} \frac{\sin(y)}{2+\cos(y)}\,dy$

23. $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} m\sec\left(m^2\right)\tan\left(m^2\right)\,dm$

24. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2(x)\tan^3(x)\,dx$

25. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos^9(x)\sin(x)\,dx$

26. $\displaystyle\int_0^{\pi} \cos^6\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)\,dx$

27. $\displaystyle\int_{\pi}^{2\pi} \cos\left(\frac{1}{2}\theta\right)\,d\theta$

28. $\displaystyle\int_2^{e^3+1} \frac{(\ln(x-1))^4}{x-1}\,dx$

29. $\displaystyle\int_0^{e^2-1} \frac{1}{x+1}\,dx$

30. $\displaystyle\int_5^{e^3+4} \frac{1}{x-4}\,dx$

Find the average value of each of the following functions over the given interval.

31. $F(x) = (x-3)^2$ on $x \in [3,\ 7]$

32. $H(x) = \sqrt{x}$ on $x \in [0,\ 3]$

33. $F(x) = \sec^2(x)$ on $x \in \left[0,\ \frac{\pi}{4}\right]$

34. $F(x) = \frac{1}{x}$ on $x \in [1\,3]$

35. $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [1,\ 4]$

36. $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [4,\ 9]$

37. $f(x) = \cos(x)\sin^4(x)$ on $x \in [0,\ \pi]$

38. $g(x) = xe^{-x^2}$ on $x \in [1,\ 5]$

39. $G(x) = \frac{x}{(1+x^2)^3}$ on $x \in [0,\ 2]$

40. $h(x) = \frac{x}{(1+x^2)^2}$ on $x \in [0,\ 4]$

41. If a cookie taken out of a 450° F oven cools in a 60° F room, then according to Newton's Law of Cooling, the temperature of the cookie t minutes after it has been taken out of the oven is given by

$$T(t) = 60 + 390e^{-0.205t}.$$

What is the average value of the cookie's temperature during its first 10 minutes out of the oven?

42. We know as the seasons change so do the length of the days. Suppose the length of the day varies sinusoidally with time by the equation

$$L(t) = 10 - 3\cos\left(\frac{\pi t}{182}\right),$$

where t is the number of days after the winter solstice (December 22, 2007). What was the average day length from January 1, 2008 to March 31, 2008?

43. During one summer in the Sunset, the temperature is modeled by the function

$$T(t) = 50 + 15\sin\left(\frac{\pi}{12}t\right),$$

where T is measured in F° and t is measured in hours after 7 a.m. What is the average temperature in the Sunset during the six-hour chemistry class that runs from 9 a.m. to 3 p.m.?

3.2 Multiple Choice Homework

1. $\displaystyle\int_1^4 \frac{1}{(1+\sqrt{x})^2\sqrt{x}}\,dx$

a) $\frac{6}{5}$ b) $\frac{1}{3}$ c) $\frac{2}{3}$ d) $\frac{4}{9}$ e) $\frac{3}{2}$

2. If $\int_1^4 h(x)\,dx = 6$, then $\int_1^4 h(5-x)\,dx =$

a) -6 b) -1 c) 0 d) 3 e) 6

3. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx =$

a) $\ln(\ln 2)$ b) $\dfrac{2}{e^2}$ c) $\ln 2$ d) $\dfrac{1-2e}{2e^2}$ e) DNE

4. Determine the average value of $y = e^{6x}$ on $x \in [0,\, 4]$.

a) $\dfrac{e^{24}-1}{4}$ b) $\dfrac{e^{24}-1}{6}$ c) $\dfrac{e^{24}}{24}$ d) $\dfrac{e^{24}}{6}$ e) $\dfrac{e^{24}-1}{24}$

5. Determine the average value of $g(x) = (2x+3)^2$ on $x \in [-3,\, -1]$.

a) $\dfrac{7}{3}$ b) -4 c) 5 d) $\dfrac{14}{3}$ e) 3

6. Determine the average value of $g(x) = e^{7x}$ on $x \in [0,\, 4]$.

a) $\dfrac{1}{14}e^{14}$ b) $\dfrac{1}{7}\left(e^{14}-1\right)$ c) $\dfrac{1}{14}\left(e^{14}-1\right)$ d) $\dfrac{1}{2}\left(e^{14}-1\right)$ e) $\dfrac{1}{7}e^{14}$

7. If the function $y = x^3$ has an average value of 9 on $x \in [0,\, k]$, then $k =$.

a) 3 b) $\sqrt{3}$ c) $\sqrt[3]{18}$ d) $\sqrt[4]{36}$ e) $\sqrt[3]{36}$

8. Find the average rate of change of $y = x^2 + 5x + 14$ on $x \in [-1\,2]$

a) 3 b) 6 c) 9 d) $\dfrac{65}{6}$ e) 18

9. If the average of the function $f(x) = |x - a|$ on $[-1\,1]$ is $\frac{5}{4}$, what is/are the values of a?

a) ± 1 b) $\pm\frac{1}{2}$ c) $\pm\frac{1}{4}$ d) 0 e) None of these

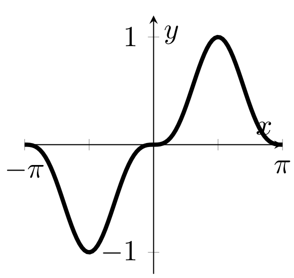
10. What is the average rate of change of the function $f(x) = x^4 - 5x$ on the closed interval $[0,\, 3]$?

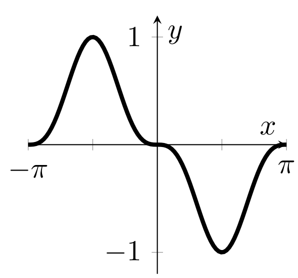
a) 8.5 b) 8.7 c) 22 d) 23 e) 66

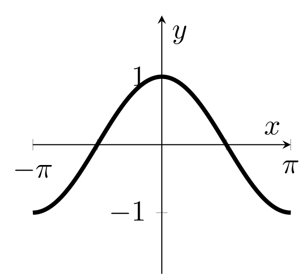
11. Determine the average value of $y = e^x \cos(x)$ on $x \in \left[0,\, \frac{\pi}{2}\right]$

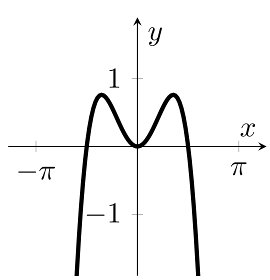
a) 0 b) 1.213 c) 1.905 d) 2.425 e) 3.810

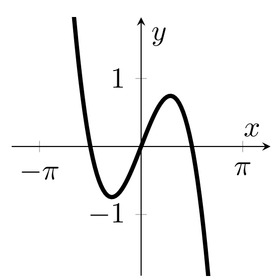
12. The graphs of five functions are shown below. Which function has a nonzero average value over the closed interval $x = [-\pi,\, \pi]$.

a) 

b) 

c) 

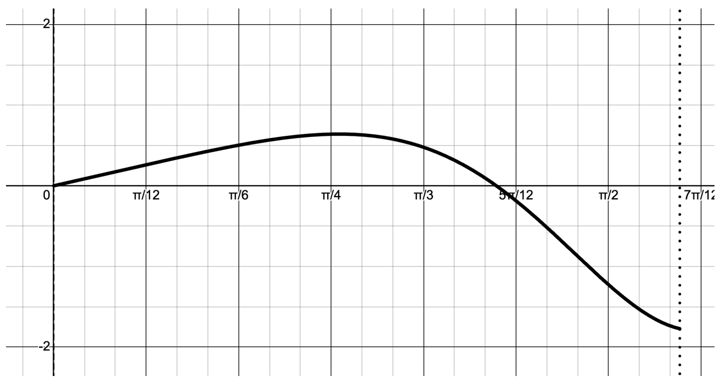
d) 

e) 

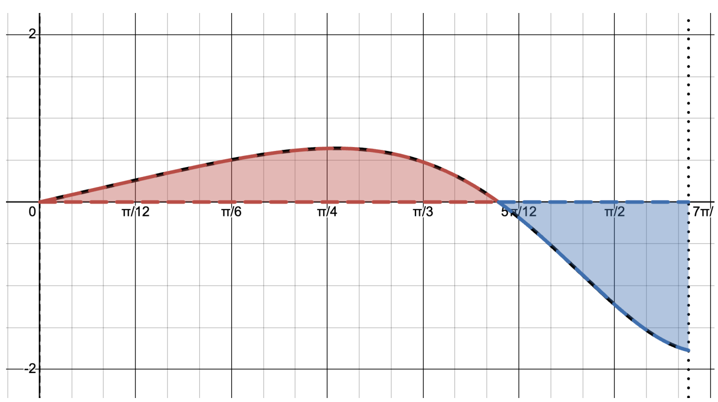
3.3: Context For Definite Integrals: Area, Displacement, and Net Change

Since we originally defined the definite integral in terms of "area under a curve," we need to consider what this idea of "area" really means in relation to the definite integral.

Recall **Ex 3.2.2**, where we had a function $y = x\cos\left(x^2\right)$ on $x \in [0\,\sqrt{\pi}]$. The graph looks like this:



In **Ex 3.2.2**, we found that $\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)\, dx = 0$. But, there's clearly area under the curve, so how can the integral equal both the area and 0? Well, as it turns out, because the integral was created from rectangles with width dx and height $f(x)$, a negative $f(x)$ will result in a rectangle with "negative area." Take a look at the following graph:



It's clear that by our definition of the definite integral, the "area" in red would cancel out the "area" in blue. So, how would we find the actual, positive area? That is what we are going to talk about in this section.

OBJECTIVES

Relate Definite Integrals to Area Under a Curve.
Understand the Difference Between Displacement and Distance.
Understand Displacement and Distance in Other Contexts.

Ex 3.3.1: What is the area under $y = x\cos\left(x^2\right)$ on $x \in [0, \sqrt{x}]$?

Sol 3.3.1: First, let's make sure we're clear on terminology. In this context, "area under" means "area between the graph and x-axis." Now, to find the area under this graph, we have two methods.
The first method is