AP Calculus BC

Saint Ignatius College Preparatory

Textbook Companion

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"Ad Majorem Dei Gloriam"

For the Greater Glory of God.

Created with care by the Mathematics Department
Saint Ignatius College Preparatory, San Francisco, CA

How To Use This Textbook?

This textbook has been heavily edited in regards to the original, KQuattrin.com version. However, the structure that this textbook follows has not changed. There are 11 chapters within this textbook, and each chapter has varying numbers of sections. Within each section, one can expect to find the following:

- Explanation of material
- Example problems and solutions
- Practice problems
- Practice exam

Answers to practice problems can be found at the end of every chapter.

Information within these boxes is important information.

Information within these boxes is the objectives for the section.

Information within these boxes is interesting sidenotes and extra relevant information.

Information within these boxes is definitions for important terms.

Information within these boxes is example problems.

Information within these boxes is solutions to example problems.

`These symbols` depict buttons on your TI-84.

Chapter 1:

Review of Derivatives

Chapter 1 Overview: Review of Derivatives

The purpose of this chapter is to review the "how" of differentiation. We will review all the derivative rules learned last year in Precalculus. In the next two chapters, we will review the "why." As a quick reference, here are those rules:

The Power Rule: $\dfrac{d}{dx}\left[u^n\right] = nu^{n-1}\dfrac{du}{dx}$

The Product Rule: $\dfrac{d}{dx}\left[u \cdot v\right] = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$

The Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{u(x)}{v(x)}\right] = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$

The Chain Rule: $\dfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}[\sin(u)] = (\cos(u))\frac{du}{dx} \qquad \frac{d}{dx}[\csc(u)] = (-\csc(u)\cot(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\cos(u)] = (-\sin(u))\frac{du}{dx} \qquad \frac{d}{dx}[\sec(u)] = (\sec(u)\tan(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2(u)\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot(u)] = \left(-\csc^2(u)\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right] = \left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1}(u)\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc^{-1}(u)\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1}(u)\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec^{-1}(u)\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1}(u)\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot^{-1}(u)\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

Here is a quick review from last year:

Identities: While all will eventually be used somewhere in Calculus, the ones that occur most often early are the Reciprocals and Quotients, the Pythagoreans, and the Double Angle Identities.

$$\tan(x) = \frac{\sin(x)}{\cos(x)}; \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}; \qquad \sec(x) = \frac{1}{\cos(x)} = \frac{1}{\sin(x)}$$

$$\sin^2(x) + \cos^2(x) = 1; \qquad \tan^2(x) + 1 = \sec^2(x); \qquad \cot^2(x) + 1 = \csc^2(x)$$

$$\sin(2x) = 2\sin(x)\cos(x); \qquad \cos(2x) = \cos^2(x) - \sin^2(x)$$

Inverses: Because of the quadrants, taking an inverse yields two answers, only one of which your calculator can show. How the second answer is found depends on the kind of inverse:

$$\cos^{-1}(x) = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ -\textit{calculator} \ \pm 2\pi n \end{array} \right\} \qquad \sin^{-1}(x) = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ \pi - \textit{calculator} \ \pm 2\pi n \end{array} \right\}$$

$$\tan^{-1}(x) = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ \pi + \textit{calculator} \ \pm 2\pi n \end{array} \right\} = \textit{calculator} \ \pm \pi n$$

Logarithm Rules: Here are some logarithm rules which you should recall:

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_a x^n = n \log_a x$$

1.1: The Power and Exponential Rules with the Chain Rule

In Precalculus we developed the idea of the derivative geometrically. That is, the derivative initially arose from our need to find the slope of the tangent line. In Chapter 2 and 3, that meaning, its link to limits, and other conceptualizations of the derivative will be explored. In this chapter, we are primarily interested in how to find the derivative and what it is used for.

Derivative → Definition: $f(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

→ Means: The function that yields the slope of the tangent line.

Numerical Derivative → Definition: $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

→ Means: The numerical value of the slope of the tangent line at $x = a$

Symbols for the Derivative

$\dfrac{dy}{dx}$ = "d - y - d - x" $\quad$ $f'(x)$ = "f prime of x" $\quad$ y' = "y prime"

$\dfrac{d}{dx}$ = "d - d - x" $\quad$ D_x = "d sub x"

OBJECTIVES

Use the Power Rule and Exponential Rules to Find Derivatives.
Find the Derivative of Composite Functions.

Key Idea from Precalculus: The derivative yields the slope of the tangent line. (But there is more to it than that).

The first and most basic derivative rule is the Power Rule. Among the last rules we learned in Precalculus were the Exponential Rules. They look similar to one another, therefore it would be a good idea to view them together.

The Power Rule:

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

The Exponential Rules:

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$\frac{d}{dx}\left[a^x\right] = a^x \cdot \ln a$$

The difference between these is where the variable is. The Power Rule applies when the variable is in the *base*, while the Exponential Rules apply when the variable is in the *exponent*. The difference between the two Exponential rules is what the base is. $e = 2.718281828459\ldots$, while a is any positive number other than 1.

Ex 1.1.1: Find a) $\frac{d}{dx}\left[x^5\right]$ and b) $\frac{d}{dx}\left[5^x\right]$.

Sol 1.1.1: The first is a case of the Power Rule while the second is a case of the second Exponential Rule. Therefore,

a) $\frac{d}{dx}\left[x^5\right] = \boxed{5x^4}$ b) $\frac{d}{dx}\left[5^x\right] = \boxed{5^x \ln 5}$

There are a few other basic rules that we need to remember.

$$\frac{d}{dx}[\text{constant}] = 0$$

$$\frac{d}{dx}\left[cx^n\right] = (cn)x^{n-1}$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

These rules allow us to easily differentiate a polynomial term by term.

Ex 1.1.2: $y = 3x^2 + 5x + 1$; find $\frac{dy}{dx}$.

Sol 1.1.2:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[3x^2+5x+1\right] \\ &= (3\cdot 2)x^{2-1}+(5\cdot 1)x^{1-1}+0 \\ &= \boxed{6x+5}\end{aligned}$$

Ex 1.1.3: $f(x)=x^2+4x-3+e^x$; find $f'(x)$.

Sol 1.1.3:

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left[x^2+4x-3+e^x\right] \\ &= (1\cdot 2)x^{2-1}+(4\cdot 1)x^{1-1}-0+e^x \\ &= \boxed{2x+4+e^x}\end{aligned}$$

Ex 1.1.4: $y=\sqrt{x^3}+\dfrac{4}{\sqrt{x}}-\sqrt[4]{x^3}+e^4$; find $\dfrac{dy}{dx}$.

Sol 1.1.4:

$$\begin{aligned}y &= \sqrt{x^3}+\frac{4}{\sqrt{x}}-\sqrt[4]{x^3}+e^4 \\ &= x^{\frac{3}{2}}+4x^{-\frac{1}{2}}-x^{\frac{3}{4}}+e^4 \\ \frac{dy}{dx} &= \frac{d}{dx}\left[x^{\frac{3}{2}}+4x^{-\frac{1}{2}}-x^{\frac{3}{4}}+e^4\right] \\ &= \left(1\cdot\frac{3}{2}\right)x^{\frac{3}{2}-1}+\left(4\cdot-\frac{1}{2}\right)x^{-\frac{1}{2}-1}-\left(1\cdot\frac{3}{4}\right)x^{\frac{3}{4}-1}+0 \\ &= \boxed{\frac{3}{2}x^{\frac{1}{2}}-2x^{-\frac{3}{2}}-\frac{3}{4}x^{-\frac{1}{4}}}\end{aligned}$$

Note in Ex 1.1.4 that e^4 is a constant. Therefore, its derivative is 0.

As we have seen, when the variable was in the exponent, we use the Exponential Rules. When the variable was in the base, we used the Power Rule. But what if the variable is in both places, such as $\frac{d}{dx}\left[(2x-1)^{x^2}\right]$? It is definitely an exponential problem, but the base is not a constant as the rules above have. The Change of Base Rule allows us to clarify the problem:

$$\frac{d}{dx}\left[(2x-1)^{x^2}\right] = \frac{d}{dx}\left[e^{x^2\ln(2x-1)}\right]$$

but we will need the Product Rule for this derivative. Therefore, we will save this for later.

Ex 1.1.5: If $y = (x^2+1)(x^3-4x)$, find $\frac{dy}{dx}$.

Sol 1.1.5:

$$\begin{aligned} y &= (x^2+1)(x^3-4x) \\ &= x^5 - 4x^3 + x^3 - 4x \\ &= x^5 - 3x^3 - 4x \\ \frac{dy}{dx} &= \frac{d}{dx}\left[x^5 - 3x^3 - 4x\right] \\ &= \boxed{5x^4 - 9x^2 - 4} \end{aligned}$$

Ex 1.1.6: If $y = \frac{x^2-4x+6}{\sqrt[3]{x}}$, find $\frac{dy}{dx}$.

Sol 1.1.6:

$$\begin{aligned} y &= \frac{x^2-4x+6}{\sqrt[3]{x}} \\ &= \frac{x^2-4x+6}{x^{\frac{1}{3}}} \\ &= x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 6x^{-\frac{1}{3}} \end{aligned}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 6x^{-\frac{1}{3}}\right]$$

$$= \boxed{\frac{5}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{-\frac{1}{3}} - 2x^{-\frac{4}{3}}}$$

The Chain Rule

Composite Function → Definition: A function made of two other functions, one within the other.

→ For example, $y = \sqrt{16x - x^3}$, $y = \sin\left(x^3\right)$. $y = \cos^3(x)$, and $y = \left(x^2 + 2x - 5\right)^3$. The general symbol is $f(g(x))$.

Ex 1.1.7: Given $f(x) = \cos^{-1}(x)$, $g(x) = x^2 - 1$, and $h(x) = \sqrt{1 + x^2}$, find a) $f\left(g\left(\sqrt{2}\right)\right)$, b) $h(g(1))$, and c) $f(h(g(1)))$.

Sol 1.1.7:

(a) $g\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^2 - 1 = 1,$ so $f\left(g\left(\sqrt{2}\right)\right) = f(1) = \cos^{-1}(1) = \boxed{0}$.

(b) $g(1) = 0,$ so $h(g(1)) = h(0) = \sqrt{1 + 0^2} = \boxed{1}$

(c) $g(1) = 0$ and $h(g(1)) = h(0) = \sqrt{1 + 0^2} = 1,$ so $f(h(g(1))) = \cos^{-1}(1) = \boxed{0}$

So. How do we take the derivative of a composite function? There are two (or more) functions that must be differentiated, but, since one is inside the other, the derivatives cannot be taken at the same time. Just as a radical cannot be distributed over addition, a derivative cannot be distributed concentrically. The composite function is like a matryoshka (Russian doll) that has a doll inside a doll. The derivative is akin to opening them. They cannot both be opened at the same time and, when one is opened, there is an unopened one within. The result is two open dolls adjacent to each other.

The Chain Rule: $\dfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

If you think of the inside function (the $g(x)$) as equaling u, we could write the Chain Rule like this:

$$\frac{d}{dx}[f(u)] = \frac{df}{du} \cdot \frac{du}{dx}$$

This is the way that most derivatives are written with the Chain Rule.

The Chain Rule is one of the cornerstones of Calculus. It can be embedded within each of the other rules, as seen in the introduction to this chapter. So the Power Rule and Exponential Rules in the last section really should have been stated as:

The Power Rule:

$$\frac{d}{dx}\left[u^n\right] = nu^{n-1} \cdot \frac{du}{dx}$$

The Exponential Rules:

$$\frac{d}{dx}\left[e^u\right] = e^u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln a\right) \cdot \frac{du}{dx}$$

(where u is a function of x)

Ex 1.1.8: $\dfrac{d}{dx}\left[(4x^2 - 2x - 1)^{10}\right]$

Sol 1.1.8:

$$u = 4x^2 - 2x - 1 \text{ and } f(u) = u^{10}$$

$$\begin{aligned} \frac{d}{dx}\left[f(u)\right] &= f'(u) \cdot \frac{du}{dx} \\ &= 10u^9 \cdot (8x - 2) \\ &= \boxed{10\left(4x^2 - 2x - 1\right)^9 (8x - 2)} \end{aligned}$$

Ex 1.1.9: $\dfrac{d}{dx}\left[e^{4x^2}\right]$

Sol 1.1.9:

$$u = 4x^2$$

$$\begin{aligned}\frac{d}{dx}\left[e^u\right] &= e^u \cdot \frac{du}{dx} \\ &= e^{4x^2} \cdot 8x \\ &= \boxed{8xe^{4x^2}}\end{aligned}$$

Ex 1.1.10: If $y = \sqrt{16 - x^3}$, find $\dfrac{dy}{dx}$.

Sol 1.1.10:

$$u = 16 - x^3 \text{ and } f(u) = \sqrt{u}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[f(u)\right] = f'(u) \cdot \frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}} \cdot -3x^2 \\ &= \boxed{\frac{-3x^2}{2\sqrt{16 - x^3}}}\end{aligned}$$

Ex 1.1.11: $\dfrac{d}{dx}\left[\sqrt{\left(x^2+1\right)^5 + 7}\right]$

Sol 1.1.11:

$$u = x^2 + 1,\ g(u) = u^5 + 7,\ \text{and } f(g(u)) = \sqrt{g(u)}$$

$$
\begin{aligned}
\frac{d}{dx}\left[f(g(u))\right] &= f'(g(u)) \cdot g'(u) \cdot \frac{du}{dx} \\
&= \frac{1}{2\sqrt{g\left(u\right)}} \cdot 5u^4 \cdot 2x \\
&= \frac{5\left(x^2+1\right)\left(2x\right)}{2\sqrt{\left(x^2+1\right)^5+7}} = \boxed{\frac{5x\left(x^2+1\right)}{\sqrt{\left(x^2+1\right)^5+7}}}
\end{aligned}
$$

1.1 Free Response Homework

Find the derivatives of the given functions. Simplify where possible.

1. $f(x) = x^2 + 3x - 4$

2. $f(t) = \frac{1}{4}\left(t^4 + 8\right)$

3. $y = x^{-\frac{2}{3}}$

4. $y = 5e^x + 3$

5. $v(r) = \frac{4}{3}\pi r^3$

6. $g(x) = x^2 + \frac{1}{x^2}$

7. $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$

8. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$

9. $z = \frac{A}{y^{10}} + Be^y$

10. $y = e^{x+1} + 1$

Complete the following.

11. $\frac{d}{dx}\left[x^7 - 4\sqrt[8]{x^7} + 7^x - \frac{1}{\sqrt[7]{x^4}} + \frac{1}{5x}\right]$

12. $\frac{d}{dx}\left[x^6 - 3\sqrt[6]{x^7} + 5^x - \frac{1}{\sqrt[3]{x^5}} + \frac{1}{8x}\right]$

13. $\frac{d}{dx}\left[x^4 - 14\sqrt[7]{x^9} + 8^x - \frac{1}{\sqrt[3]{x^7}} + \frac{1}{8x}\right]$

14. $\frac{d}{dx}\left[(x-1)\sqrt{x}\right]$

15. $\frac{d}{dz}\left[\left(z^2 - 4\right)\sqrt{z^3}\right]$

16. $\frac{d}{dx}\left[\left(x^2 - 4x + 3\right)\sqrt{x^5}\right]$

17. $\frac{d}{dt}\left[\left(4t^2 + 1\right)\left(3t^3 + 7\right)\right]$

18. $\frac{d}{dx}\left[\left(x^3 + 4x - \pi\right)^{-7}\right]$

19. $\frac{d}{dx}\left[\sqrt{3x^2 - 4x + 9}\right]$

20. $\frac{d}{dx}\left[\sqrt[7]{x^3 - 2x}\right]$

21. $\frac{d}{dy}\left[\frac{4y^3 - 2y^2 - 5y}{\sqrt{y}}\right]$

22. $\frac{d}{dv}\left[\frac{v^2 - 4v + 7}{2\sqrt{v}}\right]$

23. $\frac{d}{dw}\left[\frac{7w^2 - 4w + 1}{5w^3}\right]$

24. $\frac{d}{dw}\left[\frac{5w^2 - 3w - 4}{7w^2}\right]$

25. $f(x) = \sqrt[4]{1 + 2x + x^3}$, find $f'(x)$

26. $f(x) = \sqrt[5]{\left(\frac{1}{x} + 2x + e^x\right)^3}$, find $f'(x)$

27. $f(x) = \left(x^3 + 2x\right)^{37}$, find $f'(x)$

28. $f(x) = 3x^5 - 5x^3 + 3$, find $f'(x)$

29. $g(2) = 3,\ g'(2) = -4,\ f(x) = e^{g(x)}$, find $f'(2)$

30. $y = e^{\sqrt{x}}$, find $\frac{dy}{dx}$

31. $f(x)=\sqrt{4-\frac{4}{9}x^2}$, find $f'\left(\sqrt{5}\right)$

32. $f(x)=e^{\sqrt{9-x^2}}$, find $f'(x)$

33. $v(t)=\sqrt{\left(\frac{E(t)}{3}+3t\right)^{\frac{3}{7}}-4}$, find $v'(t)$

34. $v(t)=\sqrt[3]{\left(\frac{C(t)}{7}+4t^2\right)^{\frac{5}{7}}-1}$, find $v'(t)$

1.1 Multiple Choice Homework

1. If $f(x)=x^{\frac{3}{2}}$, then $f'(4)=$

a) -6 b) -3 c) 3 d) 6 e) 8

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2. The derivative of $\sqrt{x}-\frac{1}{x\sqrt[3]{x}}$

a) $\frac{1}{2}x^{-\frac{1}{2}}-x^{-\frac{4}{3}}$ b) $\frac{1}{2}x^{-\frac{1}{2}}+\frac{4}{3}x^{-\frac{7}{3}}$ c) $\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{1}{3}}$

d) $-\frac{1}{2}x^{-\frac{1}{2}}+\frac{4}{3}x^{-\frac{7}{3}}$ e) $-\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{1}{3}}$

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3. Given $f(x)=\frac{1}{2x}+\frac{1}{x^2}$, find $f'(x)$

a) $-\frac{1}{2x^2}-\frac{2}{x^3}$ b) $-\frac{2}{x^2}-\frac{2}{x^3}$ c) $\frac{2}{x^2}-\frac{2}{x^3}$

d) $-\frac{1}{2x^2}+\frac{2}{x^3}$ e) $\frac{1}{2x^2}-\frac{2}{x^3}$

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4. If $f(x)=e^{5x^2}+x^4$, then $f'(1)=$

a) e^5+1 b) $5e^4+4$ c) $5e^5+1$ d) $10e+4$ e) $10e^5+4$

---

5. If h is the function defined by $h(x)=e^{5x}+x+3$, then $h'(0)$ is

a) 2 b) 4 c) 5 d) 6 e) 8

---

6. If $y = \left(x^4 + 4\right)^2$, then $\dfrac{dy}{dx} =$

a) $2\left(x^4 + 4\right)$ b) $\left(4x^3\right)^2$ c) $2\left(4x^3 + 4\right)$

d) $4x^3\left(x^4 + 4\right)$ e) $8x^3\left(x^4 + 4\right)$

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7. If $h(x) = [f(x)]^2\, g(x)$ and $g(x) = 3$, then $h'(x) =$

a) $2f'(x)g'(x)$ b) $6f'(x)f(x)$ c) $g'(x)\,[f(x)]^2 + 2f(x)f'(x)g(x)$

d) $2f'(x)g(x) + g'(x)\,[f(x)]^2$ e) 0

---

8. Which of the following statements must be true?

I. $\dfrac{d}{dx}\left[\sqrt{e^x + 3}\right] = \dfrac{e^x}{2\sqrt{e^x + 3}}$

II. $\dfrac{d}{dx}\left[5^{3x^2}\right] = 6x\ln(5)\left(5^{3x^2}\right)$

III. $\dfrac{d}{dx}\left[6x^3 - \pi + \sqrt[3]{x^8} - \dfrac{2}{x^3}\right] = 18x^2 + \dfrac{8}{3}\sqrt[3]{x^5} + \dfrac{6}{x^4}$

a) I only b) II only c) I and III only

d) I and III only e) I, II, and III

1.2: Trig, Trig Inverse, and Log Rules

Trigonometric → Definition: A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure.

→ Means: An equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

Logarithmic → Definition: The inverse of an exponential function.

→ Means: An equation with log or ln in it.

Trig Derivative Rules

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

Log Derivative Rules

$$\frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx} \qquad \frac{d}{dx}[\log_a u] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

Note that all these rules are expressed in terms of the Chain Rule.

OBJECTIVES

Find Derivatives Involving Trig, Trig Inverse, and Logarithmic Functions.

Ex 1.2.1: $\frac{d}{dx}\left[\sin^3(x)\right]$

Sol 1.2.1:

$$\frac{d}{dx}\left[\sin^3(x)\right] = \boxed{3\sin^2(x)\cos(x)}$$

Ex 1.2.2: $\dfrac{d}{dx}\left[\sin\left(x^3\right)\right]$

Sol 1.2.2:

$$\frac{d}{dx}\left[\sin\left(x^3\right)\right] = \boxed{3x^2\cos\left(x^3\right)}$$

Ex 1.2.3: $\dfrac{d}{dx}\left[\ln\left(4x^3\right)\right]$

Sol 1.2.3:

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(4x^3\right)\right] &= \frac{1}{4x^3}\cdot 12x^2\\ &= \boxed{\frac{3}{x}}\end{aligned}$$

We could have also simplified algebraically before taking the derivative:

$$\begin{aligned}\ln\left(4x^3\right) &= \ln 4 + \ln x^3\\ &= \ln 4 + 3\ln x\\ \frac{d}{dx}\left[\ln 4 + 3\ln x\right] &= 0 + 3\cdot\frac{1}{x}\\ &= \boxed{\frac{3}{x}}\end{aligned}$$

Of course, composites can involve more than two functions. The Chain Rule has as many derivatives in the chain as there are functions.

Ex 1.2.4: $\dfrac{d}{dx}\left[\sec^5\left(3x^4\right)\right]$

Sol 1.2.4:

$$\begin{aligned}\frac{d}{dx}\left[\sec^5\left(3x^4\right)\right] &= 5\sec^4\left(3x^4\right)\cdot\sec\left(3x^4\right)\tan\left(3x^4\right)\cdot\left(12x^3\right)\\ &= \boxed{60x^3\sec^5\left(3x^4\right)\tan\left(3x^4\right)}\end{aligned}$$

Ex 1.2.5: $\dfrac{d}{dx}\ln\left(\cos\left(\sqrt{x}\right)\right)$

Sol 1.2.5:

$$\begin{aligned}\frac{d}{dx}\ln\left(\cos\left(\sqrt{x}\right)\right) &= \frac{1}{\cos\left(\sqrt{x}\right)}\cdot\left(-\sin\left(\sqrt{x}\right)\right)\cdot\frac{1}{2\left(\sqrt{x}\right)}\\ &= -\tan\left(\sqrt{x}\right)\cdot\frac{1}{2\left(\sqrt{x}\right)}\\ &= \boxed{\frac{-\tan\left(\sqrt{x}\right)}{2\sqrt{x}}}\end{aligned}$$

General inverses are not all that interesting. We are more interested in particular *transcendental* inverse functions, like the natural log. Another particular kind of inverse function that bears more study is the trig inverse function. Interestingly, as with the log functions, the derivatives of these transcendental functions become algebraic functions.

Inverse Trig Derivative Rules

$$\frac{d}{dx}\left[\sin^{-1}u\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc^{-1}u\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1}u\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec^{-1}u\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1}u\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot^{-1}u\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

Ex 1.2.6: $\frac{d}{dx}\left[\tan^{-1}\left(3x^4\right)\right]$

Sol 1.2.6:

$$\frac{d}{dx}\left[\tan^{-1}\left(3x^4\right)\right] = \frac{1}{(3x^4)^2+1}\cdot\left(12x^3\right)$$

$$= \boxed{\frac{12x^3}{9x^8+1}}$$

Ex 1.2.7: $\frac{d}{dx}\left[\sec^{-1}\left(x^2\right)\right]$

Sol 1.2.7:

$$\frac{d}{dx}\left[\sec^{-1}\left(x^2\right)\right] = \frac{1}{|x^2|\sqrt{(x^2)^2-1}}\cdot 2x$$

$$= \frac{2x}{(x^2)\sqrt{(x^2)^2-1}}$$

$$= \boxed{\frac{2}{x\sqrt{x^4-1}}}$$

General Inverse Derivative

$$\frac{d}{dx}\left[f^{-1}(x)\right] - \frac{1}{f'\left[f^{-1}(x)\right]}$$

Ex 1.2.8: If $f(x) = x^2+2x+3,\ g(x) = f^{-1}(x),$ and $g(1) = 2;$ find $g'(1)$.

Sol 1.2.8:

$$f'(x) = 2x+2 \ \therefore\ f'\left(g(x)\right) = 2\left(g(x)\right)+2$$

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left[f^{-1}(x)\right]} = \frac{1}{f'\left(g(x)\right)}$$

$$g'(1) = \frac{1}{f'\left(g(1)\right)} = 2\left(g(1)\right) + 2 = \boxed{6}$$

1.2 Free Response Homework Set A

Find the derivatives of the given functions. Simplify where possible.

1. $y = \sin(4x)$
2. $y = 4\sec\left(x^5\right)$
3. $f(t) = \sqrt[3]{1 + \tan t}$
4. $f(\theta) = \ln(\cos(\theta))$
5.* $y = a^3 + \cos^3(x)$
6.* $y = \cos\left(a^3 + x^3\right)$
7. $f(x) = \cos(\ln x)$
8. $f(x) = \sqrt[5]{\ln x}$
9. $f(x) = \log_{10}(2 + \sin(x))$
10. $f(x) = \log_2(1 - 3x)$
11. $y = \sin^{-1}(e^x)$
12. $y = \tan^{-1}(\sqrt{x})$

*Note that a is a constant.

Complete the following.

13. $\frac{d}{dx}\left[\sin^{-1}\left(e^{3x}\right)\right]$
14. $\frac{d}{dx}\left[\cot^{-1}\left(e^{2x}\right)\right]$
15. $\frac{d}{dx}\left[\tan^{-1}\left(x^2\right)\right]$
16. $\frac{d}{dx}\left[\cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1}(x)\right]$
17. $\frac{d}{dx}\left[3e^{x^2+2x}\right]$
18. $\frac{d}{dx}\left[3\cos\left(x^2 + 2x\right)\right]$
19. $\frac{d}{dx}\left[\sqrt[3]{16 + x^3}\right]$
20. $\frac{d}{dx}\left[\sec^{-1}\left(2x^2\right)\right]$
21. $\frac{d}{dx}\left[5e^{\tan(7x)}\right]$
22. $\frac{d}{dx}\left[\sqrt{\cos\left(1 - x^2\right)}\right]$
23. $\frac{d}{dx}\left[\ln^3\left(x^2 + 1\right)\right]$
24. $\frac{d}{dx}\left[\ln\left(\sin\left(x^3\right)\right)\right]$
25. $\frac{d}{dx}\left[\ln(\sec(x))\right]$
26. $\frac{d}{dx}\left[\cos\left(x^2\right)\right]$
27. $f(x) = \ln\left(x^2 + 3\right)$, find $f'(x)$
28. $g(x) = \ln\left(x^2 - 4x + 4\right)$, find $g'(x)$
29. $h(x) = \sqrt{x^2 + 5}$, find $h'(x)$
30. $F(x) = \sqrt[3]{3x^2 - 6x + 1}$, find $F'(x)$
31. $y = \sin^{-1}(\cos(x))$, find y'
32. $y = \sin\left(\cos^{-1}(x)\right)$, find y'
33. $y = \tan^2(3\theta)$, find y'
34. $y = \cot^7(\sin(\theta))$, find y'

35. $y = \sin^{-1}\left(x\sqrt{2}\right)$, find y'

36. $y = \sin^{-1}(2x+1)$, find y'

1.2 Free Response Homework Set B

Find the derivatives of the given functions. Simplify where possible.

1. $y = \cos^{-1}\left(e^{3z}\right)$

2. $y = \tan^{-1}\left(\sqrt{x^2-1}\right)$

3. $y = \sec^{-1}(4x) + \csc^{-1}(4x)$

4. $f(x) = \ln\left(\tan^{-1}(5x)\right)$

5. $g(w) = \sin^{-1}(5w) + \cos^{-1}(5w)$

6. $f(t) = \sec^{-1}\left(\sqrt{9+t^2}\right)$

Complete the following.

7. $\dfrac{d}{d\theta}\left[e^{\csc(\theta)} + \ln\left(\cot\left(\theta^2\right)\right) - \sec(\theta)\right]$

8. $\dfrac{d}{dx}\left[\ln\left(\sec^3\left(x^3 + 5\ln x + 7\right)\right)\right]$

9. $\dfrac{d}{dx}\left[\ln\left(\tan^3\left(x^2 + 5e^x + 7\right)\right)\right]$

10. $\dfrac{d}{dx}\left[\dfrac{\cos\left(\ln\left(5x^2\right)\right)}{\sin\left(\ln\left(5x^2\right)\right)}\right]$

11. $\dfrac{d}{dx}\left[\ln\left(\sqrt{x^2+4x-5}\right)\right]$

12. $\dfrac{d}{dt}\left[\sin^5\left(\ln(7t+3)\right)\right]$

13. $\dfrac{d}{dx}\left[\csc\left(\ln\left(7x^2+x\right)\right)\right]$

14. $\dfrac{d}{dx}\left[\ln\left(\sqrt{e^{4t^2+6}}\right)\right]$

15. $\dfrac{d}{dx}\left[\sqrt{9x - 27x^2 + \dfrac{5}{x^3}}\right]$

16. $\dfrac{d}{dx}\left[\sec(5x) + \cot\left(e^x\right) - 10\ln x\right]$

17. $z = \ln(\cos(t)) + \sec\left(e^t\right) + 7\pi^2$, find $\dfrac{dz}{dt}$

18. $z = \ln(\tan(t)) + \sin\left(e^t\right) + 7\pi^2$, find $\dfrac{dz}{dt}$

19. $z = \ln(\cot(\theta)) + \sec(\ln\theta) + 7\pi^2$, find $\dfrac{dz}{d\theta}$

20. $z = \ln(\cos(\theta)) + \sin(\ln\theta) + 7\pi^2$, find $\dfrac{dz}{d\theta}$

21. If $g(3) = \dfrac{\pi}{2}$, $g'(3) = \dfrac{\pi}{4}$, and $f(x) = x^3 g(x) + g\left(-3\cos\left(\dfrac{\pi}{3}x\right)\right) - e^{\sin(g(x))}$, find $f'(3)$

1.2 Multiple Choice Homework

1. If $y = \sin^{-1}\left(e^{3\theta}\right)$, then $\dfrac{dy}{d\theta} =$

a) $\dfrac{1}{\sqrt{1-e^{3\theta}}}$ b) $\dfrac{1}{\sqrt{1-e^{6\theta}}}$ c) $\dfrac{1}{\sqrt{1-e^{9\theta^2}}}$

d) $-3e^{3\theta}\cos^{-1}\left(e^{3\theta}\right)$ e) $\dfrac{3e^{3\theta}}{\sqrt{1-e^{6\theta}}}$

---

2. If $f(x) = \tan^{-1}(\cos x)$, then $f'(x) =$

a) $-\csc(x)\sec^{-2}(\cos(x))$ b) $-\sin(x)\sec^{-2}(\cos(x))$ c) $-\cos(x)\csc^{-2}(\cos(x))$

d) $\dfrac{-\cos(x)}{1-\sin^2(x)}$ e) $\dfrac{-\sin(x)}{\cos^2(x)+1}$

---

3. If $h(x) = \ln\left(x^2\right)\tan^{-1}(x)$, then $h'(1) =$

a) $\dfrac{\pi}{4}$ b) $\dfrac{\pi}{4}+1$ c) $\dfrac{\pi}{2}$ d) $\dfrac{\pi}{2}+1$ e) $\dfrac{\pi}{2}+2$

---

4. If $f(t) = t\sqrt{1-t^2}+\cos^{-1}(t)$, then $f'(t) =$

a) $\dfrac{t-2}{2\sqrt{t^2-1}}$ b) $\dfrac{-2t^2}{\sqrt{1-t^2}}$ c) $\dfrac{-2t^2+2}{\sqrt{1-t^2}}$ d) $\dfrac{-1-t^2}{\sqrt{1-t^2}}$ e) $\dfrac{1-t^2}{\sqrt{1-t^2}}$

---

5. If h is the function defined by $h(x) = e^{5x}+x+3$, then $h'(0) =$

a) 2 b) 4 c) 5 d) 6 e) 8

---

6. Given that $f(x) = 8\sin^2(5x)$, find $f'\left(\dfrac{\pi}{30}\right)$

a) $20\sqrt{3}$ b) $20\sqrt{2}$ c) 20 d) 100 e) 0

---

7. If $g(x) = \cos^2(2x)$, then $g'(x)$ is

a) $2\cos(2x)\sin(2x)$ b) $-4\cos(2x)\sin(2x)$ c) $2\cos(2x)$

d) $-2\cos(2x)$ e) $4\cos(2x)$

---

8. If $f(x) = \sin^2(3-x)$, then $f'(0) =$

a) $-2\cos(3)$ b) $-2\sin(3)\cos(3)$ c) $6\cos(3)$

d) $2\sin(3)\cos(3)$ e) $6\sin(3)\cos(3)$

---

9. If $f(x) = \cos^2(3-x)$, then $f'(0) =$

a) $-2\cos(3)$ b) $-2\sin(3)\cos(3)$ c) $6\cos(3)$

d) $2\sin(3)\cos(3)$ e) $6\sin(3)\cos(3)$

---

10. The function $f(x) = \tan(3^x)$ has one zero in the interval $[0, 1.4]$. The derivative at this point is

a) 0.411 b) 1.042 c) 3.451 d) 3.763 e) undefined

1.3: Trig, Trig Inverse, and Log Rules

Remember:

$$\text{The Product Rule: } f'(x) = U \cdot \frac{dV}{dx} + V \cdot \frac{dU}{dx}$$

$$\text{The Quotient Rule: } f'(x) = \frac{V \cdot \frac{dU}{dx} - U \cdot \frac{dV}{dx}}{V^2}$$

OBJECTIVES

Find the Derivative of a Product or Quotient of Two Functions.

The Product Rule

Ex 1.3.1: $\frac{d}{dx}\left[x^2 \sin(x)\right]$

Sol 1.3.1:

$$\begin{aligned}\frac{d}{dx}\left[x^2 \sin(x)\right] &= x^2 \cdot \cos(x) + \sin(x) \cdot (2x) \\ &= \boxed{x^2 \cos(x) + 2(x)\sin(x)}\end{aligned}$$

Ex 1.3.2: $\frac{d}{dx}\left[5^x \cos(x)\right]$

Sol 1.3.2:

$$\frac{d}{dx}\left[5^x \cos(x)\right] = 5^x \cdot (-\sin(x)) + \cos(x) \cdot (5^x \ln 5)$$

$$= \boxed{5^x \left(\ln(5)\cos(x) + \sin(x)\right)}$$

The product rule is pretty straightforward. The tricky part is simplifying the algebra.

Ex 1.3.3: If $f(x) = x^2 e^{-\frac{x}{2}}$, find $f'(x)$

Sol 1.3.3:

$$U = x^2, \ \frac{dU}{dx} = 2x$$

$$V = e^{-\frac{x}{2}}, \ \frac{dV}{dx} = e^{-\frac{x}{2}} \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-\frac{x}{2}}$$

$$f'(x) = x^2 \cdot \left(-\frac{1}{2}e^{-\frac{x}{2}}\right) + e^{-\frac{x}{2}} \cdot 2x$$

$$= \boxed{xe^{-\frac{x}{2}}\left(-\frac{1}{2}x + 2\right)}$$

Ex 1.3.4: $\frac{d}{dx}\left[x\sqrt{1-x^2}\right]$

Sol 1.3.4:

$$U = x, \ \frac{dU}{dx} = 1$$

$$V = \sqrt{1-x^2} = \left(1-x^2\right)^{\frac{1}{2}}, \ \frac{dV}{dx} = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[x\sqrt{1-x^2}\right] = x\cdot\left(-\frac{x}{\sqrt{1-x^2}}\right)+\sqrt{1-x^2}\cdot 1$$

$$= \frac{-x^2+(1-x^2)}{\sqrt{1-x^2}}$$

$$= \boxed{\frac{1-2x^2}{\sqrt{1-x^2}}}$$

Ex 1.3.5: $\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]$

Sol 1.3.5:

$U=(2x-3)^8,\ \frac{dU}{dx}=8(2x-3)^7\cdot 2=16(2x-3)^7$

$V=\left(3x^2-1\right)^7,\ \frac{dV}{dx}=7\left(3x^2-1\right)^6\cdot 6x=42x\left(3x^2-1\right)^6$

$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]=(2x-3)^8\cdot 42x\left(3x^2-1\right)^6+\left(3x^2-1\right)^7\cdot 16(2x-3)^7$

This, then, is factorable.

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right] = 42x(2x-3)^8\left(3x^2-1\right)^6+16\left(3x^2-1\right)^7 16(2x-3)^7$$

$$= 2(2x-3)^7\left(3x^2-1\right)^6\left(21x(2x-3)+8\left(3x^2-1\right)\right)$$

$$= 2(2x-3)^7\left(3x^2-1\right)^6\left(42x^2-63x+24x^2-8\right)$$

$$= \boxed{2(2x-3)^7\left(3x^2-1\right)^6\left(66x^2-63x-8\right)}$$

Remember that in Section 1.1 we said that we would need the Product Rule to deal with the derivative of a function where the variable is in both the base and the exponent. We can now address that situation.

Ex 1.3.6: $\frac{d}{dx}\left[(\cos(x))^{x^2}\right]$

Sol 1.3.6:

$$\frac{d}{dx}\left[(\cos(x))^{x^2}\right] = \frac{d}{dx}\left[e^{x^2\ln(\cos(x))}\right]$$

$$= e^{x^2\ln(\cos(x))}\cdot\left(x^2\cdot\frac{1}{\cos(x)}\cdot -(\sin(x)) + \ln(\cos(x))\cdot 2x\right)$$

$$= \boxed{(\cos(x))^{x^2}\left(2x\ln(\cos(x)) - x^2\tan(x)\right)}$$

The Quotient Rule

Ex 1.3.7: $\frac{d}{dx}\left[\frac{6x}{x^2+4}\right]$

Sol 1.3.7:

$$U = 6x,\ \frac{dU}{dx} = 6$$

$$V = x^2+4,\ \frac{dV}{dx} = 2x$$

$$\frac{d}{dx}\left[\frac{6x}{x^2+4}\right] = \frac{(x^2+4)\cdot 6 - 6x\cdot 2x}{(x^2+4)^2}$$

$$= \frac{6x^2+24-12x^2}{(x^2+4)}$$

$$= \boxed{\frac{24-6x^2}{(x^2+4)}}$$

Ex 1.3.8: $\dfrac{d}{dx}\left[\dfrac{x^2+2x-3}{x-4}\right]$

Sol 1.3.8:

$$U = x^2+2x-3,\ \frac{dU}{dx} = 2x+2$$

$$V = x-4,\ \frac{dV}{dx} = 1$$

$$\begin{aligned}\frac{d}{dx}\left[\frac{x^2+2x-3}{x-4}\right] &= \frac{(x-4)\cdot(2x+2)-(x^2+2x-3)\cdot 1}{(x-4)^2}\\ &= \frac{2x^2-6x-8-x^2-2x+3}{(x-4)^2}\\ &= \boxed{\frac{x^2-8x-5}{(x-4)^2}}\end{aligned}$$

Ex 1.3.9: $\dfrac{d}{dx}\left[\dfrac{x^2-4x+3}{2x^2-5x-3}\right]$

Sol 1.3.9: Notice that this problem becomes much easier if we simplify before applying the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\left[\frac{x^2-4x+3}{2x^2-5x-3}\right] &= \frac{d}{dx}\left[\frac{(x-1)(x-3)}{(2x+1)(x-3)}\right]\\ &= \frac{d}{dx}\left[\frac{x-1}{2x+1}\right]\end{aligned}$$

$$U = x-1,\ \frac{dU}{dx} = 1$$

$$V = 2x+1,\ \frac{dV}{dx} = 2$$

$$\frac{d}{dx}\left[\frac{x-1}{2x+1}\right] = \frac{(2x+1)\cdot 1-(x-1)\cdot 2}{(2x+1)^2}$$

$$= \boxed{\frac{3}{(2x+1)^2}}$$

Ex 1.3.10: $\frac{d}{dx}\left[\frac{\cot(3x)}{x^2+1}\right]$

Sol 1.3.10:

$$U=\cot(3x),\ \frac{dU}{dx} = -\csc^2(3x)\cdot 3 = -3\csc^2(3x)$$

$$V = x^2+1,\ \frac{dV}{dx} = 2x$$

$$\frac{d}{dx}\left[\frac{\cot(3x)}{x^2+1}\right] = \frac{(x^2+1)\cdot\left(-3\csc^2(3x)\right)-\cot(3x)\cdot 2x}{\left(x^2+1\right)^2}$$

$$= \frac{-3x^2\csc^2(3x)-3\csc^2(3x)-2x\cot(3x)}{\left(x^2+1\right)^2}$$

$$= \boxed{-\frac{\csc^2(3x)\left(3x^2+3\right)+2x\cot(3x)}{\left(x^2+1\right)^2}}$$

As with the Product Rule, the difficulty with the Quotient Rule arises from the algebra needed to simplify our answer.

Ex 1.3.11: If $y=\frac{4x}{\sqrt{x^2+4}}$, find $\frac{dy}{dx}$

Sol 1.3.11:

$$U = 4x,\ \frac{dU}{dx} = 4$$

$$V=\sqrt{x^2+4} = \left(x^2+4\right)^{\frac{1}{2}},\ \frac{dV}{dx} = \frac{1}{2}\left(x^2+4\right)^{-\frac{1}{2}}\cdot 2x = \frac{2x}{2\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+4}\cdot 4 - 4x\cdot\dfrac{2x}{2\sqrt{x^2+4}}}{x^2+4}$$

$$= \frac{\dfrac{4\left(x^2+4\right)}{\sqrt{x^2+4}} - \dfrac{4x^2}{\sqrt{x^2+4}}}{x^2+4}$$

$$= \frac{4x^2+16-4x^2}{(x^2+4)^{\frac{3}{2}}}$$

$$= \boxed{\frac{16}{(x^2+4)^{\frac{3}{2}}}}$$

Ex 1.3.12: Find the equation of the tangent line to $f(x) = \dfrac{x}{\sqrt{x^2+9}}$ at $x = -\sqrt{7}$.

Sol 1.3.12: As we recall, for the equation of a line, we need a point and a slope.

$$\text{The point: } f\left(-\sqrt{7}\right) = \frac{-\sqrt{7}}{\sqrt{\left(-\sqrt{7}\right)^2+9}}$$

$$= -\frac{\sqrt{7}}{4} \to \left(-\sqrt{7}, -\frac{\sqrt{7}}{4}\right)$$

The slope is the derivative at the given x-value:

$$U = x,\ \frac{dU}{dx} = 1$$

$$V = \sqrt{x^2+9} = \left(x^2+9\right)^{\frac{1}{2}},\ \frac{dV}{dx} = \frac{1}{2}\left(x^2+9\right)^{-\frac{1}{2}}\cdot 2x = \frac{2x}{2\sqrt{x^2+9}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+9}\cdot 1 - x\cdot\dfrac{2x}{2\sqrt{x^2+9}}}{x^2+9}$$

Rather than simplify the algebra, we can find the slope by substituting $x = -\sqrt{7}$:

$$\left.\frac{dy}{dx}\right|_{x=-\sqrt{7}} = \frac{\sqrt{\left(-\sqrt{7}\right)^2+9}\cdot 1 - \left(-\sqrt{7}\right)\cdot\dfrac{2\left(-\sqrt{7}\right)}{2\sqrt{\left(-\sqrt{7}\right)^2+9}}}{\left(-\sqrt{7}\right)^2+9}$$

$$= \frac{4 - \frac{7}{4}}{16}$$

$$= \frac{9}{64}$$

The tangent line equation is therefore:

$$\boxed{y + \frac{\sqrt{7}}{4} = \frac{9}{64}\left(x + \sqrt{7}\right)}$$

1.3 Free Response Homework Set A

Find the derivatives of the given functions.

1. $y = t^3 \cos(t)$

2. $y = (2x-5)^4 \left(8x^2-5\right)^{-3}$

3. $y = \dfrac{\tan(x)-1}{\sec(x)}$

4. $y = \dfrac{\sin(x)}{x^2}$

5. $y = xe^{-x^2}$

6. $y = \dfrac{r}{\sqrt{r^2+1}}$

7. $y = e^{x\cos(x)}$

8. $y = e^{-5x}\cos(3x)$

9. $y = x\sin\left(\dfrac{1}{x}\right)$

10. $y = \ln\left(e^{-x} + xe^{-x}\right)$

11. $y = \dfrac{\sec^{-1}(x)}{x}$

12. $y = \dfrac{\sec(x)}{x^3}$

13. $y = \left(1+x^2\right)\tan^{-1}(x)$

14. $y = \ln\left(x^2+4\right) - x\tan^{-1}\left(\dfrac{x}{2}\right)$

15. $f(x) = x\sqrt{\ln x}$

16. $g(x) = (1+4x)^5\left(3+x-x^2\right)^8$

17. $f(x) = x\cos^{-1}(x) - \sqrt{1-x^2}$

18. $g(x) = \cos^{-1}(x) + x\sqrt{1-x^2}$

Complete the following.

19. $\dfrac{d}{dx}\left[\dfrac{3x^2+4x-3}{x^2-9}\right]$

20. $\dfrac{d}{dx}\left[\dfrac{x^3-2x^2-5x+6}{x+2}\right]$

21. $\dfrac{d}{dx}\left[\dfrac{x^5-12x^3-19x}{3x^3}\right]$

22. $\dfrac{d}{dx}\left[\dfrac{3x+3}{x^3+1}\right]$

23. $\dfrac{d}{dx}\left[\dfrac{x-4}{x^2-9x+20}\right]$

24. $\dfrac{d}{dx}\left[\dfrac{\tan(x)+5}{\sin(x)}\right]$

25. $\dfrac{d}{dx}\left[\dfrac{\sin(x)}{1-\cos(x)}\right]$

26. $\dfrac{d}{dx}\left[\dfrac{x^2}{\cos(x)}\right]$

27. $y = \dfrac{x^2-3}{x^2-4}$, find $\dfrac{dy}{dx}$

28. $f(x) = \dfrac{x^2+2x-8}{x^2-x-3}$, find $f'(x)$

29. $y = \dfrac{x^2+2x-3}{x-4}$, find y'

30. $f(x) = \dfrac{x}{\ln x}$, find $f'(x)$

31. $h(t) = \left(\dfrac{1+t^2}{1-t^2}\right)^{17}$, find $h'(t)$

32. $y = \dfrac{\tan(x)}{\cos(x)-3}$, find $\dfrac{dy}{dx}$

33. $f(x) = \left(x\sin(2x) + \tan^4\left(x^7\right)\right)^5$, find $f'(x)$

34. $f(x) = e^x - x^2 \arctan x$, find $f'(x)$

35. $f(x) = \dfrac{\tan(x)}{\tan(x)+1}$, find $f'\left(\dfrac{\pi}{4}\right)$

36. $y = x^2\sqrt{5-x^2}$, find $y'(1)$

1.3 Free Response Homework Set B

Complete the following.

1. Find the first derivative for the following function: $x(t) = e^{t^2}\sin\left(t^2 - 5t^4\right)$

2. Find the first derivative for the following function: $x(t) = e^{5t}\tan\left(3t^4\right)$

3. Find the first derivative for the following function: $y = \dfrac{x^2+2x-15}{x-3}$

4. Find the first derivative for the following function: $x(t) = e^t\left(t^2 - 5t^4\right)$

5. $\dfrac{d}{dx}\left[\dfrac{e^x + 7x^2 + 5}{\sin\left(x^3\right)}\right]$

6. $\dfrac{d}{dx}\left[e^{\sin(x)}\ln\left(\cot\left(e^x\right)\right)\right]$

7. $\dfrac{d}{dx}\left[x^2\sin\left(x^2\right) + \dfrac{x+1}{\ln x}\right]$

8. $\dfrac{d}{dx}\left[x^2\cos\left(x^2\right) + \dfrac{e^x}{x}\right]$

9. $\dfrac{d}{dx}\left[x^5\ln(5x+4) + \dfrac{x}{\ln x}\right]$

10. $\dfrac{d}{dx}\left[\dfrac{\cos\left(x^2-3\right)}{e^{-5x}}\right]$

11. $\dfrac{d}{dx}\left[e^{x^2}\cos(x)\right]$

12. $\dfrac{d}{dx}\left[\dfrac{1+\tan(x)}{\ln(4x)}\right]$

13. $\dfrac{d}{dx}\left[\sin(t)\tan(t)\right]$

14. $\dfrac{d}{dx}\left[\dfrac{1+\ln x}{\csc(x)}\right]$

15. $\dfrac{d}{dx}\left[e^{5x^4}\ln\left(\sin(x)\right)\right]$

16. $\dfrac{d}{dx}\left[5x\sin(x) + e^{2x} - \ln\left(3x^2+1\right) + \dfrac{x}{x^2+1}\right]$

17. $\dfrac{d}{dx}\tan\left(e^x\right)\left(x^4 - 5x^3 + x\right)$

18. $\dfrac{d}{dx}\left[\dfrac{5x+2}{\ln(3x+7)}\right]$

19. $\dfrac{d}{dx}\left[\dfrac{x^5-12x^3-19x}{3x^3}\right]$

20. $\dfrac{d}{dx}\left[8\cos(4x+2)\sin(4x+2)\right]$

21. $g(z)=\left(\dfrac{e^{5z}}{1+\ln z}\right)^{118}$, find $g'(z)$

22. $g(t)=\left(\dfrac{t^2-4}{1-t^2}\right)^{15}$, find $g'(t)$

23. $y=\tan^{-1}\left(\dfrac{2e^x}{1-e^{2x}}\right)$, find y'

24. $f(x)=x^2\arccos(x)$, find $f'(x)$

25. $y=\ln\left(u^2+1\right)-u\cot^{-1}(u)$, find $\dfrac{dy}{du}$

26. $y=\cos^{-1}\left(\dfrac{x-1}{x+1}\right)$

27. $f(t)=c\sin^{-1}\left(\dfrac{t}{c}\right)-\sqrt{c^2-t^2}$, find $f'(t)$

28. $y=4\sin^{-1}\left(\dfrac{1}{2}x\right)+x\sqrt{4-x^2}$

29. If $h(1)=5$ and $h'(1)=3$, find $f'(1)$ if $f(x)=(h(x))^4+x\ln(h(x))$

1.3 Multiple Choice Homework

1. If $y=x^2\cos(2x)$, then $\dfrac{dy}{dx}=$

   a) $-2x\sin(2x)$ b) $-4x\sin(2x)$ c) $2x(\cos(2x)-\sin(2x))$

   d) $2x(\cos(2x)-x\sin(2x))$ e) $2x(\cos(2x)+\sin(2x))$

---

2. If $x(t)=2t\cos\left(t^2\right)$, find $x'(t)$.

   a) $x'(t)=\sin\left(t^2\right)+3$ b) $x'(t)=-\sin\left(t^2\right)+4$ c) $x'(t)=\sin\left(t^2\right)+2$

   d) $x'(t)=-4t^2\sin\left(t^2\right)$ e) $x'(t)=-4t^2\sin\left(t^2\right)+2\cos\left(t^2\right)$

---

3. If $f(x)=x\tan(x)$, then $f'\left(\dfrac{\pi}{4}\right)=$

   a) $1-\dfrac{\pi}{2}$ b) $1+\dfrac{\pi}{2}$ c) $1+\dfrac{\pi}{4}$ d) $1-\dfrac{\pi}{4}$ e) $\dfrac{\pi}{2}-1$

---

4. If f is a function that is differentiable throughout its domain and is defined by $f(x)=$

$\dfrac{1+e^x}{\sin(x^2)}$, then the value of $f'(0) =$

a) -1 b) 0 c) 1 d) e e) nonexistent

---

5. If $y = \dfrac{5x-4}{4x-5}$, then $\dfrac{dy}{dx} =$

a) $-\dfrac{9}{(4x-5)^2}$ b) $\dfrac{9}{(4x-5)^2}$ c) $\dfrac{40x-41}{(4x-5)^2}$ d) $\dfrac{40x+41}{(4x-5)^2}$ e) $\dfrac{5}{4}$

---

6. If $y = \dfrac{3-2x}{3x+2}$, then $\dfrac{dy}{dx} =$

a) $\dfrac{12x+2}{(3x+2)^2}$ b) $\dfrac{12x-2}{(3x+2)^2}$ c) $\dfrac{13}{(3x+2)^2}$ d) $-\dfrac{13}{(3x+2)^2}$ e) $-\dfrac{2}{3}$

---

7. If $y = \dfrac{3}{4+x^2}$, then $\dfrac{dy}{dx} =$

a) $-\dfrac{6x}{(4+x^2)^2}$ b) $\dfrac{3x}{(4+x^2)^2}$ c) $\dfrac{6x}{(4+x^2)^2}$ d) $-\dfrac{3x}{(4+x^2)^2}$ e) $\dfrac{3}{2x}$

---

8. Which of the following statements must be true?

I. $\dfrac{d}{dx}\left[x\tan(x)\right] = x\tan(x) + x\sec^2(x)$

II. $\dfrac{d}{dx}\left[\dfrac{3}{4+x^2}\right] = \dfrac{-6x}{(4+x^2)^2}$

III. $\dfrac{d}{dx}\left[\sqrt{1-x}\right] = \dfrac{1}{2\sqrt{1-x}}$

a) I only b) II only c) III only

d) I and II only e) I, II, and III

1.4: Higher Order Derivatives

What we've been calling the derivative is actually the first derivative. There can be successive uses of the derivative rules, and they have meanings other than the slope of the tangent line. In this section, we will explore the process of finding the higher-order derivatives.

Second Derivative $\rightarrow$ Definition: The derivative of the derivative.

Just as with the first derivative, there are several symbols for the second derivative.

Higher Order Derivative Symbols

$$\frac{d^2y}{dx^2} = \text{“d squared y, d - x squared”} \rightarrow \frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4} \cdots \frac{d^ny}{dx^n}$$

$$\frac{d^2}{dx^2} = \text{“d squared, d - x squared”} \rightarrow \frac{d^3}{dx^3},\ \frac{d^4}{dx^4} \cdots \frac{d^n}{dx^n}$$

$$f''(x) = \text{“f double prime of x”} \rightarrow f'''(x),\ f^{IV}(x) \ldots f^n(x)$$

$$y'' = \text{“y double prime”}$$

OBJECTIVES

Find Higher Order Derivatives.

Ex 1.4.1: $\dfrac{d^2}{dx^2}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right]$

Sol 1.4.1:

$$\begin{aligned}\frac{d^2}{dx^2}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right] &= \frac{d}{dx}\left[\frac{d}{dx}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right]\right] \\ &= \frac{d}{dx}\left[4x^3 - 21x^2 - 6x + 2\right] \\ &= \boxed{12x^2 - 42x - 6}\end{aligned}$$

Ex 1.4.2: Find $\dfrac{d^3y}{dx^3}$ if $y = \sin(3x)$.

Sol 1.4.2:

$$\begin{aligned} y &= \sin(3x) \\ \frac{dy}{dx} &= \cos(3x)\cdot 3 = 3\cos(3x) \\ \frac{d^2y}{dx^2} &= 3(-\sin(3x))\cdot 3 = -9\sin(3x) \\ \frac{d^3y}{dx^3} &= -9\cos(3x)\cdot 3 = \boxed{-27\cos(3x)} \end{aligned}$$

More complicated functions, in particular composite functions, have a more complicated process. When the Chain Rule is applied, the result often includes a product or a quotient. Therefore, the second derivative will require the Product or Quotient Rules, as well as possibly the Chain Rule.

Ex 1.4.3: $y = e^{3x^2}$, find $\dfrac{d^y}{dx^2}$.

Sol 1.4.3:

$$\begin{aligned} \frac{dy}{dx} &= e^{3x^2}\cdot 6x = 6xe^{3x^2} \\ \frac{d^2y}{dx^2} &= 6x\left(e^{3x^2}\cdot 6x\right) + e^{3x^2}\cdot 6 \\ &= 36x^2e^{3x^2} + 6e^{3x^2} \\ &= \boxed{6e^{3x^2}\left(6x^2+1\right)} \end{aligned}$$

Ex 1.4.4: $y = \sin^3(x)$, find y''.

Sol 1.4.4:

$$y' = 3\sin^2(x)\cdot\cos(x)$$

$$y'' = 3\sin^2(x)(-\sin(x)) + \cos(x)(6\sin(x)\cdot\cos(x))$$

$$= \boxed{3\sin(x)\left(2\cos^2(x) - \sin^2(x)\right)}$$

Ex 1.4.5: $f(x) = \ln\left(x^2 + 3x - 1\right)$, find $f''(x)$.

Sol 1.4.5:

$$f'(x) = \frac{1}{x^2 + 3x - 1}(2x + 3) = \frac{2x + 3}{x^2 + 3x - 1}$$

$$f''(x) = \frac{\left(x^2 + 3x - 1\right)(2) - (2x + 3)(2x + 3)}{\left(x^2 + 3x - 1\right)^2}$$

$$= \frac{\left(2x^2 + 6x - 2\right) - \left(4x^2 + 12x + 9\right)}{\left(x^2 + 3x - 1\right)^2}$$

$$= \boxed{-\frac{2x^2 - 6x - 11}{\left(x^2 + 3x - 1\right)^2}}$$

Ex 1.4.6: $g(x) = \sqrt{4x^2 + 1}$, find $g''(x)$

Sol 1.4.6:

$$g'(x) = \frac{1}{2}\left(4x^2 + 1\right)^{-\frac{1}{2}}(8x) = \frac{4x}{\left(4x^2 + 1\right)^{\frac{1}{2}}}$$

$$
\begin{aligned}
g''(x) &= \frac{\left(4x^2+1\right)^{\frac{1}{2}}(4) - (4x)\left[\frac{1}{2}\left(4x^2+1\right)^{-\frac{1}{2}}(8x)\right]}{\left[\left(4x^2+1\right)^{\frac{1}{2}}\right]^2} \\
&= \frac{\left(4x^2+1\right)^{\frac{1}{2}}(4) - \dfrac{16x^2}{\left(4x^2+1\right)^{\frac{1}{2}}}}{\left(4x^2+1\right)} \\
&= \frac{\left(4x^2+1\right)(4) - 16x^2}{\left(4x^2+1\right)^{\frac{3}{2}}} \\
&= \boxed{\frac{4}{\left(4x^2+1\right)^{\frac{3}{2}}}}
\end{aligned}
$$

1.4 Free Response Homework

Find the second derivatives of the given functions. Simplify where possible.

1. $f(x) = x^5 + 6x^2 - 7x$
2. $h(x) = 5x^4 + 9x^3 - 4x^2 + x - 8$
3. $y = \left(x^3 + 1\right)^{\frac{2}{3}}$
4. $H(t) = \tan(3t)$
5. $g(t) = t^3 e^{5t}$
6. $y = e^{3x^2}$
7. $y = \sin^4(x)$
8. $f(t) = t\cos(t)$
9. $y = -\dfrac{4x}{x^2 + 4}$
10. $y = \dfrac{x^2 - 1}{x^2 - 4}$
11. $f(x) = x\sqrt{8 - x^2}$
12. $y = \dfrac{1}{2}x + \sin(x)$
13. $g(t) = te^{-t}$
14. $y = e^{-x^2}$
15. $y = \dfrac{x}{x^2 - 9}$
16. $B(x) = 2x - x^{\frac{2}{3}}$
17. $y = x^3 + x^2 - 7x + 15$
19. $y = 3x^4 - 20x^3 + 42x^2 - 36x + 16$

Complete the following:

21. $y = \cos\left(x^2\right)$, find y''
22. $y = \tan^2(x)$, find y''
23. $y = \sec(3x)$, find $\dfrac{d^2y}{dx^2}$
24. $y = xe^{2x}$, find $\dfrac{d^2y}{dx^2}$
25. $f(x) = \ln\left(x^2 + 3\right)$, find $f''(x)$
25. $g(x) = \ln\left(x^2 - 4x + 4\right)$, find $g''(x)$
27. $h(x) = \sqrt{x^2 + 5}$, find $h''(x)$
28. $F(x) = \sqrt{3x^2 - 2x + 1}$, find $F''(x)$
29. $y = \dfrac{x^2 - 3}{x^2 - 10}$, find $\dfrac{d^2y}{dx^2}$
30. $y = \dfrac{3x + 3}{x^3 + 1}$, find $\dfrac{d^2y}{dx^2}$

1.4 Multiple Choice Homework

1. If f and g are twice differentiable and if $h(x) = g\left(f(x)\right)$, then $h''(x) =$

   a) $g''\left(f(x)\right)$
   b) $g''\left(f(x)\right) f''(x)$
   c) $g''\left(f(x)\right)\left[f'(x)\right]^2$

d) $g'(f(x))\left[f'(x)\right]^2 + f'(x)\left(f''(x)\right)$ e) $g'(f(x))\, f''(x) + \left[f'(x)\right]^2 g''(f(x))$

---

2. Find $\dfrac{d^2y}{dx^2}$ if $y = \dfrac{x+2}{x-3}$

   a) $-\dfrac{2}{(x-3)^2}$ b) 0 c) $\dfrac{10}{(x-3)^3}$ d) $\dfrac{2}{(x-3)^2}$ e) None of these

---

3. If $y = \ln(\cos(x))$ and $0 \le x \le \pi$, then $\dfrac{d^2y}{dx^2}$ is

   a) $-\tan(x)$ b) $-\sec^2(x)$ c) $\tan(x)$ d) $\sec^2(x)$ e) $\sec(x)\tan(x)$

---

4. If $y = \ln\left(x^2+4\right)$, then $\dfrac{d^2y}{dx^2}$ is

   a) $\dfrac{1}{x^2+4}$ b) $\dfrac{2x}{x^2+4}$ c) $\dfrac{-2x^2+8}{x^2+4}$ d) $\dfrac{2x}{\left(x^2+4\right)^2}$ e) $\dfrac{-2x^2+8}{\left(x^2+4\right)^2}$

---

5. If $y = e^{x^2}$, then $\dfrac{d^2y}{dx^2} =$

   a) e^{x^2} b) $2e^{x^2}\left(2x^2+1\right)$ c) $2xe^{x^2}$ d) $4x^2e^{x^2}$ e) $2e^{x^2}\left(2x^2-1\right)$

---

6. If $h(t) = \ln\left(t^2+1\right)$, then $h''(-1) =$

   a) $\ln 2$ b) 0 c) -1 d) -2 e) DNE

---

7. If $y = \sin\left(e^x\right)$, then $\dfrac{d^2y}{dx^2} =$

   a) $\cos\left(e^x\right)$ b) $e^x\cos\left(e^x\right)$ c) $e^x\sin\left(e^x\right) + e^x\cos\left(e^x\right)$

   d) $-e^x\sin\left(e^x\right) + e^x\cos\left(e^x\right)$ e) $-e^x\left(e^x\sin\left(e^x\right) - \cos\left(e^x\right)\right)$

1.5: Implicit Differentiation and Second Derivative Applications

Implicit differentiation is a technique that is considered here because of its direct impact on related rates in section 8 of this chapter. Implicit differentiation is an application of the Chain Rule where the y-function is not easily defined explicitly.

One of the most useful aspects of the Chain Rule is that we can take derivatives of more complicated equations that would be difficult to take the derivative of otherwise. One of the key elements to remember is that we already know the derivative of y with respect to x — that is, $\dfrac{dy}{dx}$. This can be a powerful tool as it allows us to take the derivative of relations as well as functions while bypassing a lot of tedious algebra. When y cannot easily be isolated, we can treat y like we treat $g(x)$. In other words:

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \cdot \left[g'(x)\right] \text{ is the same as } \frac{d}{dx}\left[f(y)\right] = f'(y) \cdot \left(\frac{dy}{dx}\right)$$

OBJECTIVES

Take Derivatives of Relations Implicitly.
Use Implicit Differentiation to Find Higher Order Derivatives.
Use the Second Derivative Test to Determine Whether a Point is at a Maximum, Minimum, or Neither.

Ex 1.5.1: Find $\dfrac{dy}{dx}$ if $x^2 + y^2 = 25$

Sol 1.5.1:

$$\frac{d}{dx}\left[x^2 + y^2 = 25\right]$$

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = \frac{d}{dx}[25]$$

$$\rightarrow 2x + 2y\frac{dy}{dx} = 0$$

We can now isolate $\dfrac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \boxed{-\frac{x}{y}}$$

With this function, notice that y could have been isolated and $\frac{dy}{dx}$ could've been found **explicitly**.

$$x^2 + y^2 = 25$$

$$y^2 = 25 - x^2$$

$$y = \sqrt{25 - x^2}$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{25 - x^2}}$$

Notice that this is the same answer as we found with implicit differentiation. You could substitute y for $\sqrt{25 - x^2}$ in the denominator and come up with the same derivative, $\frac{dy}{dx} = -\frac{x}{y}$.

Ex 1.5.2: Find the derivative of $x^2 - 3y^2 + 4x - 12y - 2 = 0$ implicitly.

Sol 1.5.2:

$$\frac{d}{dx}\left[x^2 - 3y^2 + 4x - 12y - 2 = 0\right]$$

$$2x - 6y\frac{dy}{dx} + 4 - 12\frac{dy}{dx} = 0$$

$$(-6y - 12)\frac{dy}{dx} = -2x - 4$$

$$\frac{dy}{dx} = \boxed{\frac{-2x - 4}{-6y - 12}}$$

When considering functions, implicit differentiation may not seem to be a particularly powerful tool, because it is often simple to isolate y. But consider a non-function, like this circle, ellipse, or hyperbola, where y is not so easily isolated.

Ex 1.5.3: Find $\frac{dy}{dx}$ for the hyperbola $x^2 - 3xy + 3y^2 = 2$

Sol 1.5.3: It would be very difficult to solve for y here, so implicit differentiation is really our only option.

$$\frac{d}{dx}\left[x^2 - 3xy + 3y^2 = 2\right]$$

Note that $-3xy$ is a product. It will require the Product Rule.

$$2x - 3x\frac{dy}{dx} - 3y + 6y\frac{dy}{dx} = 0$$

$$(-3x + 6y)\frac{dy}{dx} = -2x + 3y$$

$$\frac{dy}{dx} = \boxed{\frac{-2x + 3y}{-3x + 6y}}$$

Ex 1.5.4: Find the equation of the line tangent to $x^3 - y^2 + 6y = -3$ at $y = 1$

a) $3x^2 - 2y = 6$

b) $3x - y = -7$

c) $3x + y = -5$

d) $x + 3y = 1$

e) $x - 3y = -5$

Sol 1.5.4: First, let's find the point of tangency.

$$x^3 - y^2 + 6y = -3 \to x^3 - (1)^2 + 6(1) = -3$$

$$x^3 = -8 \therefore x = -2$$

$$\frac{d}{dx}\left[x^3 - y^2 + 6y = -3\right]$$

$$3x^2 - 2y\frac{dy}{dx} + 6\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3x^2}{-2y + 6}$$

Now, let's plug in our point that we found, $(-2, 1)$

$$\frac{dy}{dx} = \frac{-3(-2)^2}{-2(1) + 6}$$

$$\frac{dy}{dx} = -3$$

$$y - 1 = -3(x + 2)$$

$$y - 1 = -3x - 6$$

$$\rightarrow \boxed{\text{c) } 3x + y = -5}$$

Of course, if we want to find a second derivative, we can use implicit differentiation a second time.

Ex 1.5.5: Given $\dfrac{dy}{dx} = \dfrac{x+2}{3y+6}$, find $\dfrac{d^2y}{dx^2}$

Sol 1.5.5:

$$\frac{d}{dx}\left[\frac{dy}{dx} = \frac{x+2}{3y+6}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(3y+6) - (x+2)\left(3\dfrac{dy}{dx}\right)}{(3y+6)^2}$$

Since we already know $\dfrac{dy}{dx}$, we can substitute

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{(3y+6) - (x+2)(3)\left(\dfrac{x+2}{3y+6}\right)}{(3y+6)^2} \\
&= \frac{(3y+6) - (x+2)(3)\left(\dfrac{x+2}{3y+6}\right)}{(3y+6)^2} \cdot \frac{3y+6}{3y+6} \\
&= \boxed{\frac{(3y+6)^2 - 3(x+2)^2}{(3y+6)^3}}
\end{aligned}$$

Ex 1.5.6: Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for $\sin(y) = 2\cos(3x)$.

Sol 1.5.6:

$$\frac{d}{dx}\left[\sin(y) = 2\cos(3x)\right]$$

$$\cos(y)\frac{dy}{dx} = -6\sin(3x)$$

$$\frac{dy}{dx} = \boxed{-\frac{6\sin(3x)}{\cos(y)}}$$

$$\frac{d^2y}{dx^2} = \frac{-18\cos(y)\cos(3x) - 6\sin(3x)\sin(y)\dfrac{dy}{dx}}{\cos^2(y)}$$

$$= \frac{-18\cos(y)\cos(3x) - 6\sin(3x)\sin(y)\left(-\dfrac{6\sin(3x)}{\cos(y)}\right)}{\cos^2(y)}$$

$$= \boxed{\frac{-18\cos(y)\cos(3x) + 36\sin^2(3x)\sin(y)}{\cos^3(y)}}$$

AP-Style Implicit Differentiation Problems

Common Sub-topics:

- Demonstrating implicit differentiation
- Finding the equation of a tangent line
- Finding points where the tangent line is horizontal and/or vertical
- Finding points on a curve with a particular slope
- Finding the second derivative and apply the Second Derivative Test
- Finding the particular solution (this will have to wait for the next chapter)

Remember:

The Second Derivative Test

For a function f:

1) If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at c

2) If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at c

This is necessary because one cannot create a sign pattern without an **explicitly** stated function, so the First Derivative Test will not work on problems which require implicit differentiation to find the derivative.

OBJECTIVES

Take Derivatives of Relations Implicitly.
Use Implicit Differentiation to Find Higher Order Derivatives.
Use Separation of Variables to Find the Particular Solution to a Differential Equation.

Ex 1.5.7: Consider the curve given by $x^2 + 4xy + y^2 = -12$.

(a) Show that $\dfrac{dy}{dx} = -\dfrac{x+2y}{2x+y}$

(b) Find the point(s) where the equation of the tangent line(s) is/are horizontal.

(c) Find the value(s) of $\dfrac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

Sol 1.5.7:

(a) Because we have a $4xy$ term, we need to use the product rule

$$\frac{d}{dx}\left[x^2 + 4xy + y^2 = -12\right]$$

$$2x + 4x\frac{dy}{dx} + 4y(1) + 2y\frac{dy}{dx} = 0$$

$$4x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - 4y$$

$$(4x + 2y)\frac{dy}{dx} = -2x - 4y$$

$$\frac{dy}{dx} = \frac{-2x-4y}{4x+2y} = \boxed{-\frac{x+2y}{2x+y}}$$

(b) Horizontal lines have a slope of 0, so we need to find when $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 0$$

$$-\frac{x+2y}{2x+y} = 0$$

$$x + 2y = 0$$

$$x = -2y$$

To be on the curve, $x = -2y$ must satisfy the original equation.

$$(-2y)^2 + 4(-2y)y + y^2 = -12$$

$$4y^2 - 8y^2 + y^2 = -12$$

$$-3y^2 = -12$$

$$y^2 = 4 \therefore y = \pm 2$$

$$x = -2y \rightarrow \boxed{(-4, 2) \text{ and } (4, -2)}$$

(c) What is really asked here is to apply the Second Derivative Test, because we cannot create a sign pattern for non-functions. The y is not isolated in the equation.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx} = -\frac{x+2y}{2x+y}\right]$$

$$= \frac{(2x+y)\left(1+2\frac{dy}{dx}\right) - (x+2y)\left(2+\frac{dy}{dx}\right)}{(2x+y)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-4,2)} = \frac{(2(-4)+2)\,(1+2(0)) - (-4+2(2))\,(2+0)}{(2(-4)+2)^2}$$

$$= \frac{6}{(-6)^2} > 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{(4,-2)} = \frac{(2(4)-2)(1+2(0)) - (4+2(-2))(2+0)}{(2(4)+2)^2}$$

$$= \frac{6}{(-6)^2} < 0$$

$\boxed{(-4,2) \text{ will be a minimum}}$ because the second derivative is positive.

$\boxed{(4,-2) \text{ will be a maximum}}$ because the second derivative is negative.

Be Careful!! There is a lot of algebraic simplification that happens in these problems, and it is easy to make mistakes. Take your time with the simplifications so that you don't make careless mistakes.

1.5 Free Response Homework

Find $\frac{dy}{dx}$ for each of these equations, first by implicit differentiation, then by solving for y and differentiating. Show that $\frac{dy}{dx}$ is the same in both cases.

1. $x^2 + y^2 = 1$
2. $x^3 + 4y^2 = 16$
3. $\frac{1}{x} + \frac{1}{y} = 1$
4. $\sqrt{x} + \sqrt{y} = 4$

Find $\frac{dy}{dx}$ for each of these equations by implicit differentiation.

5. $x^2 + xy = 10$
6. $x^3 + 10x^2y + 7y^2 = 60$
7. $x^2 + xy - 4y - 1 = 0$
8. $xy + 2x + 3x^2 = 4$
9. $x^2 + 4xy - 5y^2 = 4$
10. $3x^2 + xy - 4y^2 = 5$
11. $x^2 = \frac{x-y}{x+y}$
12. $x^2 + y^2 = \frac{x}{y}$
13. $y^2 = \frac{x-y}{x+y}$
14. $y^2 = \frac{x^2-1}{x+2}$
15. $x^2y^2 + x\sin(y) = 4$
16. $4\cos(x)\sin(y) = 1$
17. $e^{x^2y} = x + y$
18. $\tan(x-y) = \frac{y}{1+x^2}$
19. Find the equation of the line tangent to $x^2 - y^2 - 6y - 3 = 0$ at $\left(\sqrt{3}, 0\right)$.
20. Find the equation of the line tangent to $9x^2 + 4y^2 + 36x - 8y - 32 = 0$ at $(0, 2)$.
21. Find the equation of the line tangent to $12x^2 - 4y^2 + 72x + 16y + 44 = 0$ at $(-1, -3)$.
22. Find the equation of the line tangent to $x^3 + \frac{y}{x} + y^2 = 7$ at $(1, 2)$.
23. Find the equation of the lines tangent and normal to $y - \frac{4}{\pi^2}x^2 = 2e^{y\sin(x)} + y^3 - 3$ through the point $\left(\frac{\pi}{2}, 0\right)$.

24. Find the equation of the lines tangent and normal to $x^2 + 3xy + y^2 = 11$ through the point $(1, 2)$.

25. Find $\dfrac{d^2y}{dx^2}$ if $xy + y^2 = 1$.

26. Find $\dfrac{d^2y}{dx^2}$ if $4x^2 + 9y^2 = 36$.

27. Find $\dfrac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$

28. Find $\dfrac{d^2y}{dx^2}$ if $x^3 + 4y^2 = 16$.

29. Consider the curve given by $3x^2 - 4xy + 5y^2 = 25$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x - 2y}{2x - 5y}$.

(b) Determine point(s) P on the curve for which the x-coordinate is equal to 2.

(c) Find the equation(s) of the line(s) tangent to $3x^2 - 4xy + 5y^2 = 25$ at the point(s) P found in part (b).

(d) Find the point(s) on $3x^2 - 4xy + 5y^2 = 25$ where the tangent line is horizontal.

30. Consider the curve given by $x^2 - xy + y^2 = 4$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$.

(b) Determine point(s) P on the curve for which the x-coordinate is equal to 2.

(c) Find the equation(s) of the line(s) tangent to $x^2 - xy + y^2 = 4$ at the point(s) P found in part (b).

(d) Find the point(s) on $x^2 - xy + y^2 = 4$ where the tangent line is vertical.

31. Consider the curve given by $2x^2 - xy + y^2 = 44$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{4x - y}{x - 2y}$.

(b) Determine point(s) P on the curve for which the x-coordinate is equal to 5.

(c) Find the equation(s) of the line(s) tangent to $2x^2 - xy + y^2 = 44$ at the point(s) P found in part (b).

(d) Find the point(s) on $x^2 - xy + y^2 = 4$ where the tangent line is vertical.

32. Consider the curve given by $x^2 + xy + y^2 = 12$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{-y-2x}{2y+x}$.

(b) Find the point(s) P on $x^2 + xy + y^2 = 12$ where the tangent line is horizontal.

(c) Find the value(s) of $\dfrac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

33. Consider the curve given by $xy + y^3 = 4x$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{4-y}{3y^2 = x}$.

(b) Show that there are no points on the curve where the tangent line is horizontal.

(c) Find the point(s) on $xy + y^3 = 4x$ where the tangent line is vertical.

34. Consider the curve given by $x^2 + xy + y^2 = 4$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{-2x-y}{x+2y}$.

(b) Find the point(s) on $x^2 + xy + y^2 = 4$ where the tangent line is horizontal.

(c) Find the y-coordinates of the point(s) where the tangent line is vertical.

1.5 Multiple Choice Homework

1. Use implicit differentiation to find the points on $x^3 - y^2 + x^2 = 0$ where the tangent line is vertical.

a) $(0, 0)$ only b) $(-1, 0)$ only c) $\left(1, \sqrt{2}\right)$ only

d) $(-1, 0)$ and $(0, 0)$ e) No such points exist

---

2. If $x^2 + xy = 10$, then when $x = 2$, $\dfrac{dy}{dx} =$

a) $-\dfrac{7}{2}$ b) -2 c) $\dfrac{2}{7}$ d) $\dfrac{3}{2}$ e) $\dfrac{7}{2}$

---

3. What is the slope of the line tangent to the curve $y^2 + x = -2xy - 5$ at the point $(2, 1)$.

a) $-\frac{4}{3}$ b) $-\frac{3}{4}$ c) $-\frac{1}{2}$ d) $-\frac{1}{4}$ e) 0

---

4. Given $3x^3 - 4xy - 4y^2 = 1$, determine the change in y with respect to x.

a) $\frac{6x - 4y}{4x + 4}$ b) $\frac{9x^2 - 4}{4x + 8y}$ c) $\frac{9x^2 - 4}{4 + 8y}$ d) $\frac{9x^2 - 4y}{4x + 8y}$ e) $\frac{9x^2 - 4y}{4 + 8y}$

---

5. Given $x + xy + 2y^2 = 6$, then $\left.\frac{dy}{dx}\right|_{(2,1)} =$

a) $\frac{2}{3}$ b) $\frac{1}{3}$ c) $-\frac{1}{3}$ d) $-\frac{1}{5}$ e) $-\frac{3}{4}$

---

6. Consider the closed curve in the xy-plane given by $2x^2 + 5x + y^2 + y = 8$. Which of the following is correct?

a) $\frac{dy}{dx} = -\frac{4x + 5}{8x + 2y + 1}$ b) $\frac{dy}{dx} = \frac{4x + 5}{2y + 1}$ c) $\frac{dy}{dx} = -\frac{4x + 5}{8x + 2y}$
d) $\frac{dy}{dx} = \frac{4x + 5}{8x + 2y}$ e) $\frac{dy}{dx} = \frac{4x + 5}{2y + 1}$

---

7. The slope of the line tangent to $xy - y^3 + 6 = 0$ at $(1, 2)$ is

a) 0 b) $-\frac{1}{12}$ c) $\frac{2}{11}$ d) $\frac{1}{6}$ e) $\frac{1}{4}$

---

8. Find the equation of the line tangent to the curve $\sec\left(x^2\right) + xy^3 = 2 - y$ at $x = 0$.

a) $y = -x$ b) $y - 1 = -x$ c) $y - 2 = -x$ d) $y - 1 = x$ e) $y - 2 = x$

---

9. If $\sin^{-1}(x) = \ln y$, then $\frac{dy}{dx} =$

a) $\dfrac{y}{\sqrt{1-x^2}}$ b) $\dfrac{xy}{\sqrt{1-x^2}}$ c) $\dfrac{y}{1+x^2}$ d) $e^{\sin^{-1}(x)}$ e) $\dfrac{e^{\sin^{-1}(x)}}{1+x^2}$

---

10. If $x^2y + yx^2 = 6$, then at $(1,3)$, $\dfrac{d^2y}{dx^2} =$

a) -18 b) -6 c) 6 d) 12 e) 18

---

11. If $y = x + \sin(xy)$, then $\dfrac{dy}{dx} =$

a) $1 + \cos(xy)$ b) $1 + y\cos(xy)$ c) $\dfrac{1}{1-\cos(xy)}$

d) $\dfrac{1}{1 - x\cos(xy)}$ e) $\dfrac{1 + y\cos(xy)}{1-\cos(xy)}$

---

12. If $\sin(xy) = x^2$, then $\dfrac{dy}{dx} =$

a) $2x\sec(xy)$ b) $\dfrac{\sec(xy)}{x^2}$ c) $2x\sec(xy) - y$

d) $\dfrac{2x\sec(xy)}{y}$ e) $\dfrac{2x\sec(xy) - y}{x}$

---

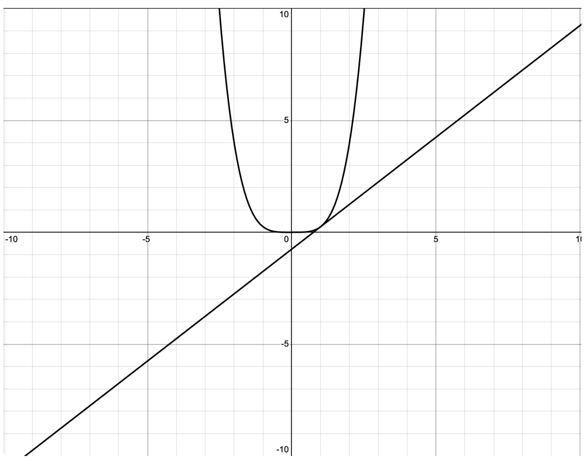
13. Given $y = \ln\left(x^2 + y^2\right)$, find $\dfrac{dy}{dx}$ at the point $(1,0)$.

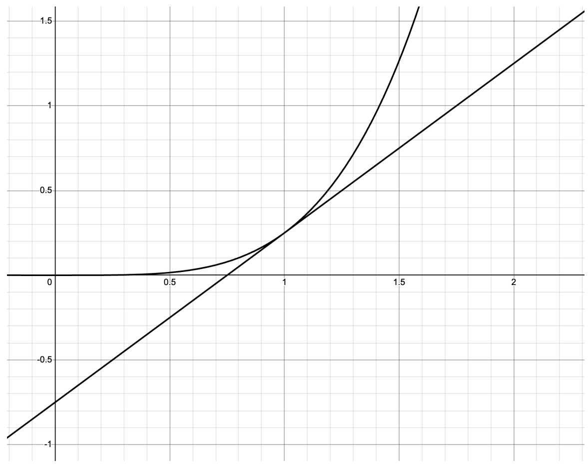
a) 0 b) 0.5 c) 1 d) 2 e) undefined

1.6: Local Linearity, Euler's Method, and Approximations

Before calculators, one of the most valuable uses of the derivative was to find approximate function values from a tangent line. Since the tangent line only shares one point on the function, y values on the line are very close to y values on the function. This idea is called local linearity—near the point of tangency, the function curve appears to be a line. This can be easily demonstrated with the graphing calculator by zooming in on the point of tangency.

Consider the graphs of $y = \frac{1}{4}x^4$ and its tangent line at $x = 1$, given by the equation $y = x - \frac{3}{4}$:





The closer you zoom in, the more the line and the curve become one. The y values on the line are good approximations of the y values on the curve. For a good animation of this concept, see the following:

tangent line approximation animation

Since it's easier to find the y value of a line arithmetically than for other functions — especially transcendental functions — the tangent line approximation is useful if you have no calculator.

OBJECTIVES

Use the Equation of a Tangent Line to Approximate Function Values.

Ex 1.6.1: Find the equations of the line tangent to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$.

Sol 1.6.1: The slope of the tangent line will be $f'(-1)$

$$f'(x) = 4x^3 - 3x^2 - 4x$$

$$f'(-1) = 4(-1)^3 - 3(-1)^2 - 4(-1) = -3$$

(Note that we could've gotten this more easily with the nDeriv function on our calculator.)

$$f(-1) = 1, \therefore \boxed{y - 1 = -3(x + 1)} \text{ or } \boxed{y = -3x - 2}$$

One of the many uses of the tangent line is based on the idea of local linearity. This means that in small areas, algebraic curves act like lines — namely their tangent lines. Therefore, one can get an approximate y value for points near the point of tangency by plugging x values into the equation of the tangent line.

Ex 1.6.2: Use the tangent line equation found in **Ex 1.6.1** to get an approximate value of $f(-0.9)$.

Sol 1.6.2: While we can find the exact value of $f(-0.9)$ with a calculator, we can get a quick approximation from the tangent line.

If $x = -0.9$ on the tangent line, then:

$$f(-0.9) \approx y(-0.9) = -3(-0.9) - 2 = \boxed{0.7}$$

This last example was somewhat trite in that we could've just plugged -0.9 into $f(x) = x^4 - x^3 - 2x^2 + 1$ and figured out the exact value even without a calculator. It would have been a pain, but it is doable. Consider the next example, though.

Ex 1.6.3: Find the tangent line equation to $f(x) = e^{2x}$ at $x = 0$ and use it to approximate the value of $e^{0.2}$.

Sol 1.6.3: Without a calculator, we could not find the exact value of $e^{0.2}$. In fact, even the calculator gives us an approximate value.

$$f'(x) = 2e^{2x} \text{ and } f'(0) = 2e^{2(0)} = 2$$

$$f(0) = e^0 = 1$$

So the tangent line equation is $\boxed{y - 1 = 2(x - 0)}$ or $y = \boxed{2x + 1}$

$$e^{0.2} \approx 2(0.2) + 1 = \boxed{1.2}$$

Note that the value you get from a calculator of $e^{0.2}$ is $1.221403\ldots$. Our approximation of 1.2 seems very reasonable.

Though not as useful as practically useful (in 2 dimensions) as the tangent line, another context for the derivative is finding the equation of the normal line.

Normal Line $\rightarrow$ Definition: The line perpendicular to a curve.

Ex 1.6.4: Find the equation of the line normal to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$.

Sol 1.6.4: In **Ex 1.6.1**, we saw that the slope of the tangent line was $f'(-1) = -3$. The normal line is perpendicular to the tangent line and, therefore, has the negative reciprocal slope of $\frac{1}{3}$. This gives us

$$\boxed{y - 1 = \frac{1}{3}(x + 1)} \text{ or } \boxed{y = \frac{1}{3}x - \frac{4}{3}}$$

for the equation of the normal line.

Euler's Method

OBJECTIVES

Use Euler's Method to Approximate a Numerical Solution to a Differential Equation at a Given Point.

In the previous section, we learned a little regarding approximations with tangent lines. Euler's Method is just a better approximation method. It uses more than one tangent line to get the job done.

The process is similar to approximating with tangent lines. We use $\frac{dy}{dx}$ to find a tangent line, then use that tangent line to find an approximate value for y. We then use that y value and another x value to create another "tangent line." Of course, it isn't actually a tangent line because our y value wasn't actually on the curve. We then repeat the process until we get to the value we want to approximate.

Steps to Euler's Method

1. Identify your starting point and step size.
2. Use $\dfrac{dy}{dx}$ to find the slope and make it a tangent line.
3. Find an approximate y value by plugging in $x + (\mathit{1\ step\ size})$ to the tangent line.
4. Use the approximate y value and the next x step over to make a new tangent line.
5. Repeat steps 3 and 4 until you reach your final x value – the one you actually want an approximation for.

Ex 1.6.5: Use Euler's Method with a step size of $\dfrac{1}{2}$ to estimate $f(3)$, where $f(x) = \ln(x)$.

Sol 1.6.5:

$$f(1) = \ln 1 = 0 \qquad \text{We start with 1 because we know } \ln 1 = 0.$$

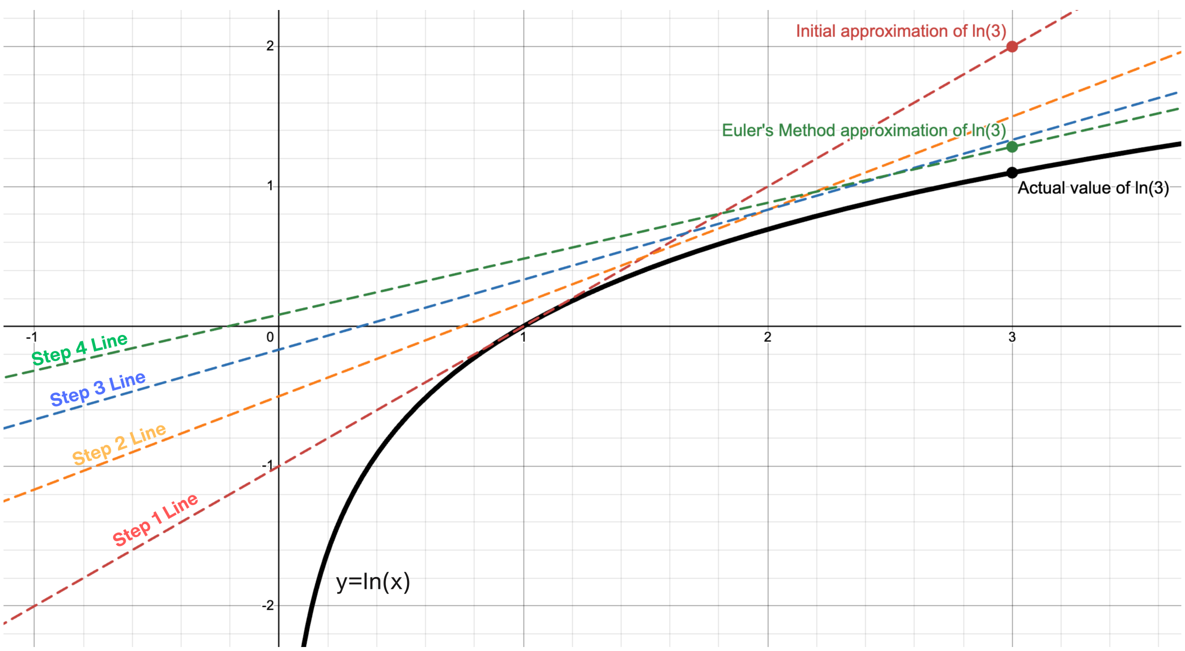
$$f'(x) = \frac{1}{x} \qquad \text{We start by taking the derivative.}$$

Note that in our chart below, we are getting our "New y" from our tangent line (step 3 above). Our "New x" comes from the "$x + (\mathit{1\ step\ size})$" step. Our new slope comes from plugging in the "New x" into $f'(x)$. For instance, for our first step below, the "New x" is equal to 1 plus the step size of $\dfrac{1}{2}$ given in the problem, our "New y" comes from plugging in $\dfrac{3}{2}$ for x and solving for y in $y - 0 = 1(x - 1)$, and our slope (for the next step) is a result of $f'\left(\dfrac{3}{2}\right) = \dfrac{1}{\frac{3}{2}} = \dfrac{2}{3}$.

Step	Point	$f'(x)$ (Slope)	Tangent Line Equation	New x	New y
1	$(1, 0)$	1	$y - 0 = 1(x - 1)$	$\frac{3}{2}$	$\frac{1}{2}$
2	$\left(\frac{3}{2}, \frac{1}{2}\right)$	$\frac{2}{3}$	$y - \frac{1}{2} = \frac{2}{3}\left(x - \frac{3}{2}\right)$	2	$\frac{5}{6}$
3	$\left(2, \frac{5}{6}\right)$	$\frac{1}{2}$	$y - \frac{5}{6} = \frac{1}{2}(x - 2)$	$\frac{5}{2}$	$\frac{13}{12}$
4	$\left(\frac{5}{2}, \frac{13}{12}\right)$	$\frac{2}{5}$	$y - \frac{13}{12} = \frac{2}{5}\left(x - \frac{5}{2}\right)$	3	$\frac{77}{60}$

So, $f(3) \approx y(3) = \boxed{\frac{77}{60}}$ or $\boxed{1.28\overline{3}}$. By way of comparison, $\ln 3 \approx 1.099$.

If we had just used the initial tangent line (the tangent line from step one) to get an approximation, we would've gotten $f(3) \approx 2$. Euler's method got us a much closer approximation.



We could also use this process to approximate a value for a curve when we only know its derivative and an initial value on the curve.

Ex 1.6.6: Use Euler's method with a step size of $\frac{1}{2}$ to estimate $f\left(\frac{5}{2}\right)$ for the function whose derivative is given by $\frac{dy}{dx} = 2x + y$ with an initial value of $f(1) = 4$.

Sol 1.6.6: For this problem, to find the slope for each step, we simply we need to plug in our point into the given differential equation.

Step	**Point**	$\frac{dy}{dx}$ **(Slope)**	**Tangent Line Equation**	**New** x	**New** y
1	$(1,4)$	6	$y - 4 = 6(x-1)$	$\frac{3}{2}$	7
2	$\left(\frac{3}{2},7\right)$	10	$y - 7 = 10\left(x - \frac{3}{2}\right)$	2	12
3	$(2,12)$	16	$y - 12 = 16(x-2)$	$\frac{5}{2}$	20

$f\left(\frac{5}{2}\right) \approx y\left(\frac{5}{2}\right) = \boxed{20}$. We cannot get an exact value for this function, because we have not learned techniques regarding how to solve this differential equation yet.

Ex 1.6.7: Is the approximation in **Ex 1.6.6** an overestimate or underestimate? Why?

Sol 1.6.7: To determine this, we need to look at the concavity of the curve – this requires the second derivative.

$$\frac{dy}{dx} = 2x + y$$

$$\frac{d^2y}{dx^2} = 2 + \frac{dy}{dx} \qquad \text{Don't forget implicit differentiation!}$$

$$\frac{d^2y}{dx^2} = 2 + (2x + y)$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,4)} = 2 + 2(1) + 4 = 8 \qquad \text{Plug in our initial value}$$

Since the second derivative is positive, our curve is concave up at this point, which means our tangent line lies under the curve. This is indicative of an $\boxed{\text{underestimate}}$.

Generally:

> → Your approximation will be an **overestimate** if the curve is **concave down** (since your "tangent lines" will be above the curve).
>
> → Your approximation will be an **underestimate** if the curve is **concave up** (since your "tangent lines" will be below the curve).

Ex 1.6.8: Use Euler's method with four equal step sizes to approximate $f(2)$ for $\frac{dy}{dx} = 3y - x$, given $f(0) = 1$. Is this an overestimate or an underestimate?

Sol 1.6.8: First, let's figure out our step sizes. We know that we start with $x = 0$, and we need to end up at $x = 2$. Therefore, four equal step sizes means that each step must be $+\frac{1}{2}$.

Next, let's make our table:

Step	**Point**	$\frac{dy}{dx}$ **(Slope)**	**Tangent Line Equation**	**New** x	**New** y
1	$(0, 1)$	3	$y - 1 = 3(x - 0)$	$\frac{1}{2}$	$\frac{5}{2}$
2	$\left(\frac{1}{2}, \frac{5}{2}\right)$	7	$y - \frac{5}{2} = 7\left(x - \frac{1}{2}\right)$	1	6
3	$(1, 6)$	17	$y - 6 = 17(x - 1)$	$\frac{3}{2}$	$\frac{29}{2}$
4	$\left(\frac{3}{2}, \frac{29}{2}\right)$	42	$y - \frac{29}{2} = 42\left(x - \frac{3}{2}\right)$	2	$\frac{71}{2}$

We can see that $f(2) \approx y(2) = \boxed{\frac{71}{2}}$. Now, let's solve for the second derivative to find out if this is an overestimate or underestimate.

$$\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 1 = 9y - 3x - 1$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 9(1) - 3(0) - 1 = 8$$

Because the positive second derivative indicates that y is concave up at $(0, 1)$, our Euler's Method result is going to be an $\boxed{\text{underestimate}}$.

1.6 Free Response Homework

Complete the following:

1. Find the equation of the line tangent to $y = x^4 + 2e^x$ at the point $(0, 2)$.

2. Find the equation of the line tangent to $y = x + \cos(x)$ at the point $(0, 1)$.

3. Find the equation of the line tangent to $y = \sec(x) - 2\cos(x)$ at the point $\left(\frac{\pi}{3}, 0\right)$.

4. Find the equation of the line tangent to $y = x^2 e^{-x}$ at the point $\left(1, \frac{1}{e}\right)$.

5. Find the equation of the line tangent to $y = \frac{2}{\pi}x + \cos(4x)$ when $x = \frac{\pi}{2}$.

6. Find the equation of the line tangent to $y = \frac{x^2 - 3}{x^2 - 4}$ when $x = -1$.

7. Find the equation of the line tangent to $f(x) = x\sqrt[4]{7 + x^2}$ when $x = 3$.

8. Find the equation of the line tangent to $y = e^{x\sin(4x)} + 2$ when $x = 0$.

9. Find the equation of the line tangent to $f(x) = x^5 - 5x + 1$ when $x = -2$ and use it to get an approximate value of $f(-1.9)$.

10. Find the equation of the line tangent to $f(x) = x\sqrt[3]{1 - x^2}$ when $x = 3$ and use it to get an approximate value of $f(3.1)$.

11. Find all points on the graph of $y = 2\sin(x) + \sin^2(x)$ where the tangent line is horizontal.

12. Find all points on the graph of $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.

13. Find the equation of the lines tangent and normal to $y = -\frac{2x}{x^2 + 16}$ at $x = -1$.

14. Find the equation of the lines tangent and normal to $y = -\frac{3x}{x^2 + 1}$ at $x = 1$.

15. Find the equation of the lines tangent and normal to $y = \dfrac{x^2 - 4x + 3}{2x^2 - 5x - 3}$ at $x = 2$.

16. Find the equation of the lines tangent and normal to $y = x \sin\left(\dfrac{\pi}{2} \ln x\right)$ when $x = e$.

17. Find the equation of the lines tangent and normal to $y = x \sin\left(\dfrac{1}{x}\right)$ when $x = \dfrac{4}{\pi}$.

18. Use Euler's Method with 2 equal step sizes to find an approximation for $f(0)$, given that $f(-1) = 2$ and $\dfrac{dy}{dx} = 6x^2 - x^2 y$.

19. Use Euler's Method with 4 equal step sizes to find an approximation for $f(1.4)$, given that $f(1) = 0$ and $f(x) = \ln(2x - 1)$.

20. Use Euler's Method with 3 equal step sizes to find an approximation for $f(2.6)$, given that $f(2) = -2$ and $\dfrac{dy}{dx} = 2x + y$.

1.6 Multiple Choice Homework

1. Let f be the function given by $f(x) = 2e^{4x^2}$. For what value of x is the slope of the line tangent to the graph of f at $(x, f(x))$ equal to 3?

   a) 0.168 b) 0.274 c) 0.318 d) 0.342 e) 0.551

---

2. Which of the following is an equation of the line tangent to the graph of $f(x) = x^6 + x^5 + x^2$ at the point where $f'(x) = -1$?

   a) $-3x - 2$ b) $-3x + 4$ c) $-x + 0.905$ d) $-x + 0.271$ e) $-x - 0.271$

---

3. At what point on the graph of $y = \dfrac{1}{2}x^2$ is the tangent line parallel to the line $2x - 4y = 3$?

   a) $\left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$ b) $\left(\dfrac{1}{2}, -\dfrac{1}{8}\right)$ c) $\left(\dfrac{1}{2}, -\dfrac{1}{4}\right)$ d) $\left(1, -\dfrac{1}{2}\right)$ e) $(2, 2)$

---

4. A normal line to the graph of a function f at the point $(x, f(x))$ is defined to be the line perpendicular to the tangent line at that point. An equation of the normal line to the curve $y = \sqrt[3]{x^2 - 1}$ at the point where $x = 3$ is

a) $y + 12x = 38$ b) $y - 4x = 10$ c) $y + 2x = 4$ d) $y + 2x = 8$ e) $y - 2x = -4$

---

5. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = \dfrac{4x}{y}$ with the initial condition $f(0) = 1$. What is the best approximation for $f(1)$ using Euler's Method, starting at $x = 0$ with a step size of 0.5?

a) 1 b) 2 c) $\sqrt{5}$ d) 2.5 e) 3

---

6. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = x - y^2$ with the initial condition $f(0) = 1$. What is the best approximation for $f(2)$ using Euler's Method, starting at $x = 0$ with a step size of 1?

a) -1 b) 0 c) 1 d) 2 e) 3

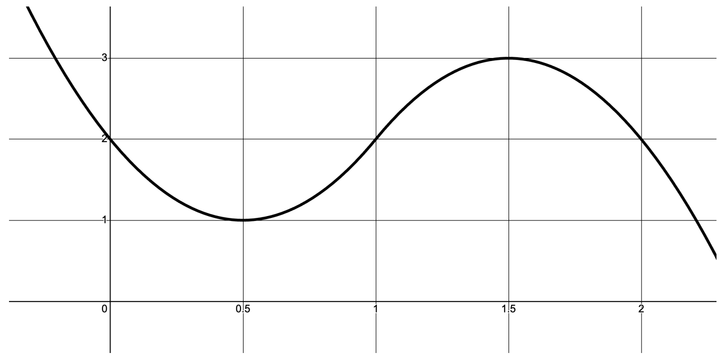
---

7. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = y - x$ with the initial condition $f(1) = 2$. What is the best approximation for $f(2)$ using Euler's Method, starting at $x = 1$ with a step size of 0.5?

a) 1 b) 2 c) 3 d) 4.5 e) 6

---

8. The graph of $y = f'(x)$ is given below. Use this information and the fact that $f(0) = 3$ to find an approximate value of $f(1)$ using Euler's method with 2 equal step sizes.



a) 2.5 b) 3.5 c) 4 d) 4.5 e) 5

---

9. The table below gives selected values for the derivative of a function g on the interval $-1 \leq x \leq 2$. If $g(-1) = -2$ and Euler's Method with a step size of 1.5 is used to approximate $g(2)$, what is the resulting approximation?

x	-1.0	-0.5	0	0.5	1.0	1.5	2.0
$f'(x)$	2	4	3	1	0	-3	-6

a) -6.5 b) -1.5 c) 1.5 d) 2.5 e) 3

---

10. The equation of the line **normal** to the graph of $y = \dfrac{3x+4}{4x-3}$ at $(1, 7)$ is

a) $25x + y = 32$ b) $25x - y = 18$ c) $7x - y = 0$

d) $x - 25y = -174$ e) $x + 25y = 176$

---

11. The equation of the line **normal** to the graph of $y = 3x\sqrt{x^2+6} - 3$ at $(0, -3)$ is

a) $3\sqrt{6}x + y = -3$ b) $3\sqrt{6}x - y = -3$ c) $x + 3\sqrt{6}y = -3$

d) $x - 3\sqrt{6}y = 9\sqrt{6}$ e) $x + 3\sqrt{6}y = -9\sqrt{6}$

1.7: Intro to AP: Basic Derivatives Numerically and Graphically

Traditionally, calculus was an algebraically heavy subject. One of the philosophical changes that the CollegeBoard made in the 1990s was to emphasize that calculus should be understood in a variety of modes. As they state in their enduring understanding:

> *"Students should be able to work with functions represented in a variety of ways: graphical, numerical, analytical or verbal. They should understand the connections among these representations."*

Later, they added that students should be able to verbalize their understanding and be able to communicate that understanding through proper writing. We will consider this later as we consider more context-oriented problems.

OBJECTIVES

Determine Derivative Values from Numerical or Graphical Data.

Ex 1.7.1: Assume $h(x) = f(x)g(x)$. Given the table of values below, find $h'(2)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

Sol 1.7.1:

$$h(x) = f(x)g(x)$$

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$

$$h'(2) = f(2)g'(2) + g(2)f'(2)$$

$$= 1 \cdot 7 + 8 \cdot 5$$

$$= \boxed{47}$$

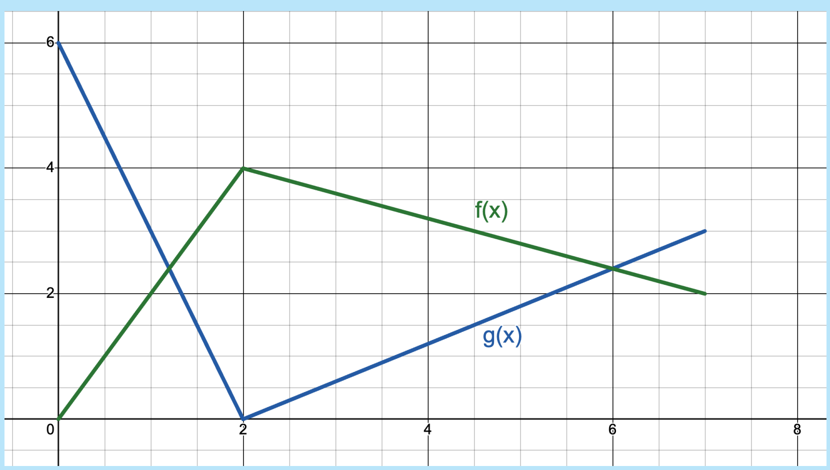
Ex 1.7.2: Using the table of values in **Ex 1.7.1:**, find $\dfrac{d}{dx}[f(g(x))]$ and $\dfrac{d}{dx}[g(f(x))]$ at $x = 1$.

Sol 1.7.2: These are two different but similar problems, so let's consider them individually.

$$\begin{aligned}\frac{d}{dx}\left[f(g(x))\right]\Big|_{x=1} &= f'(g(1))\cdot g'(1)\\ &= f'(2)\cdot 6\\ &= 5\cdot 6\\ &= \boxed{30}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left[g(f(x))\right]\Big|_{x=1} &= g'(f(1))\cdot f'(1)\\ &= g'(3)\cdot 4\\ &= 9\cdot 4\\ &= \boxed{36}\end{aligned}$$

Ex 1.7.3: Given the graph below, find (a) $w'(1)$ if $w = \dfrac{g(x)}{f(x)}$ and (b) $v'(1)$ if $v = g(f(x))$.



Sol 1.7.3:

(a)

$$w = \frac{g(x)}{f(x)} \therefore w' = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}$$

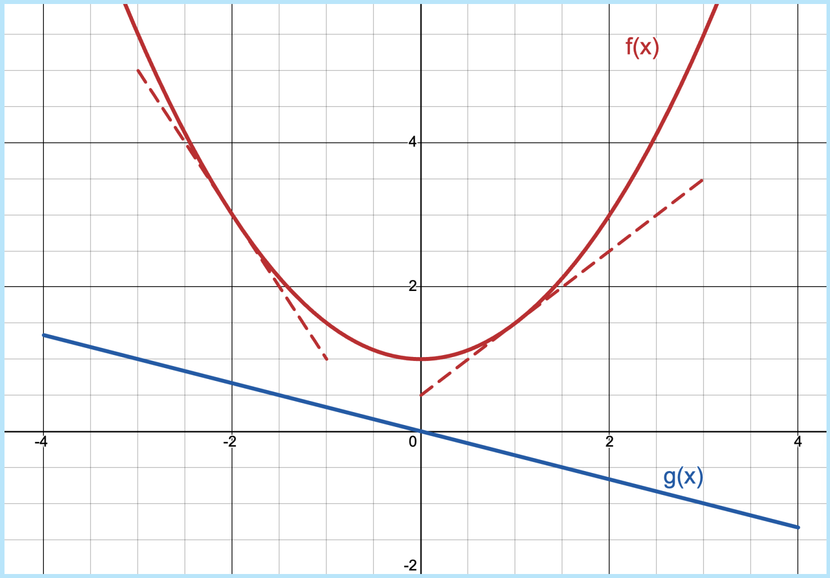
$$\begin{aligned} w'(1) &= \frac{f(1)g'(1) - g(1)f'(1)}{[f(1)]^2} \\ &= \frac{2 \cdot (-3) - 3 \cdot 2}{2^2} \\ &= \boxed{-3} \end{aligned}$$

(b)

$$\begin{aligned} v(x) &= g(f(x)) \\ v'(x) &= g'(f(x)) \cdot f'(x) \\ v'(1) &= g'(f(1)) \cdot f'(1) \\ &= g'(2) \cdot 2 \\ &= \boxed{DNE} \end{aligned}$$

(Note that $g'(2)$ does not exist. The slope cannot be determined at $x = 1$ because the slopes to the left and right of $x = 1$ are different. The is called a corner point, or a cusp point, and will be explored further in a later chapter.)

Ex 1.7.4: The figure below shows the graph of the functions f and g. The graphs of the lines tangent to the graph of f at $x = -2$ and $x = 1$ are also shown. If $B(x) = f(x) \cdot g(x)$, what is $B'(1)$?



a) $-\dfrac{5}{6}$ b) $-\dfrac{1}{2}$ c) $\dfrac{1}{6}$ d) $\dfrac{1}{3}$ e) $\dfrac{1}{2}$

Sol 1.7.4:

$$B(x) = f(x) \cdot g(x)$$

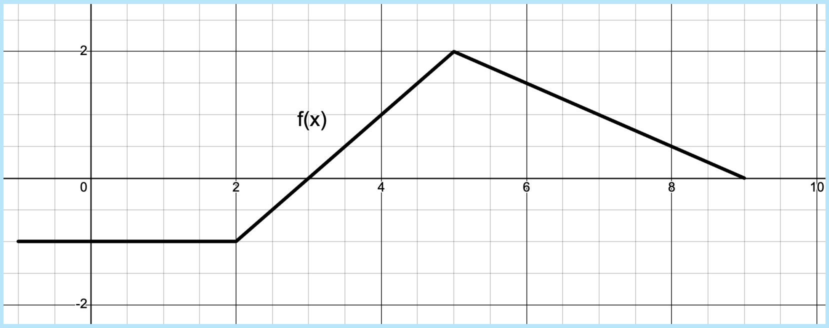
$$B'(x) = f(x)g'(x) + g(x)f'(x)$$

$$B'(1) = f(1)g'(1) + g(1)f'(1)$$

$$= \frac{3}{2} \cdot \left(-\frac{1}{3}\right) + \left(-\frac{2}{3}\right) \cdot 1$$

$$= \boxed{\text{a)} \ -\frac{5}{6}}$$

Ex 1.7.5: Let $f(x)$ be the function whose graph is given below and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given in the table below. Furthermore, let h be the function defined by $h(x) = \ln\left(x^2 + 4\right)$.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

(a) Find the equation of the line tangent to $f(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = h(f(x))$. Find $K'(3)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(6)$.

(d) Let J be the function defined by $J(x) = \dfrac{g(x)}{h\left(\frac{1}{2}x\right)}$. Find $J'(8)$.

Sol 1.7.5:

(a) $f(4) = 1$ and $f'(4) = 1$. Therefore, the tangent line equation is

$$\boxed{y - 1 = 1(x - 4)}$$

(b)

$$h(x) = \ln\left(x^2 + 4\right)$$

$$h'(x) = \frac{2x}{x^2 + 4}$$

$$K(x) = h(f(x)) \therefore K = h'(f(x)) \cdot f'(x)$$

$$K'(3) = h'(f(3)) \cdot f'(3)$$

$$= h'(0) \cdot 1$$

$$= 0 \cdot 1$$

$$= \boxed{0}$$

(c)

$$M(x) = g(x) \cdot f(x)$$

$$M'(x) = g(x) \cdot f'(x) + f(x) \cdot g'(x)$$

$$\begin{aligned} M'(6) &= g(6) \cdot f'(6) + f(6) \cdot g'(6) \\ &= 6 \cdot \frac{1}{2} + 12 \cdot \frac{3}{2} \\ &= \boxed{15} \end{aligned}$$

(d)

$$J(x) = \frac{g(x)}{h\left(\frac{1}{2}x\right)}$$

$$J'(x) = \frac{h\left(\frac{1}{2}x\right) \cdot g'(x) - g(x) \cdot h'\left(\frac{1}{2}x\right) \cdot \frac{1}{2}}{\left[h\left(\frac{1}{2}x\right)\right]^2}$$

$$\begin{aligned} J'(8) &= \frac{h(4) \cdot g'(8) - g(8) \cdot h'(4) \cdot \frac{1}{2}}{[h(4)]^2} \\ &= \boxed{\frac{8\ln 8 - \frac{4}{5}}{\ln^2 8}} \end{aligned}$$

1.7 Free Response Homework

1. Given the following table of values, find the indicated derivatives.

x	$f(x)$	$f'(x)$
2	1	7
8	5	-3

a) $g'(2)$, where $g(x) = [f(x)]^3$

b) $h'2$, where $h(x) = f(x^3)$

2. The following table shows some values of $g(x),\ g'(x),$ and $h(x)$, where $h(x) = g^{-1}(x)$.

x	$g(x)$	$h(x)$	$g'(x)$	$h'(x)$
1	2	3	$\frac{1}{2}$	$\frac{1}{3}$
3	1	2	-2	$\frac{1}{2}$

a) Find $g'(1)$

b) Find $h'(1)$

For problems 3 – 14, refer to the values in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	8	1
4	2	8	4	3
8	8	-12	2	4

3. If $h(x) = f(g(x))$, find $h'(8)$

4. If $h(x) = f(g(x))$, find $h'(2)$

5. If $k(x) = g(f(x))$, find $k'(2)$

6. If $m(x) = f(f(x))$, find $m'(4)$

7. If $P_1(x) = f(x)g(x)$, find $P_1'(2)$

8. If $P_1(x) = f(x)g(x)$, find $P_1'(8)$

9. If $P_2(x) = f(2x)g(x)$, find $P_2'(2)$

10. If $P_3(x) = f(x)g\left(\frac{1}{2}x\right)$, find $P_3'(4)$

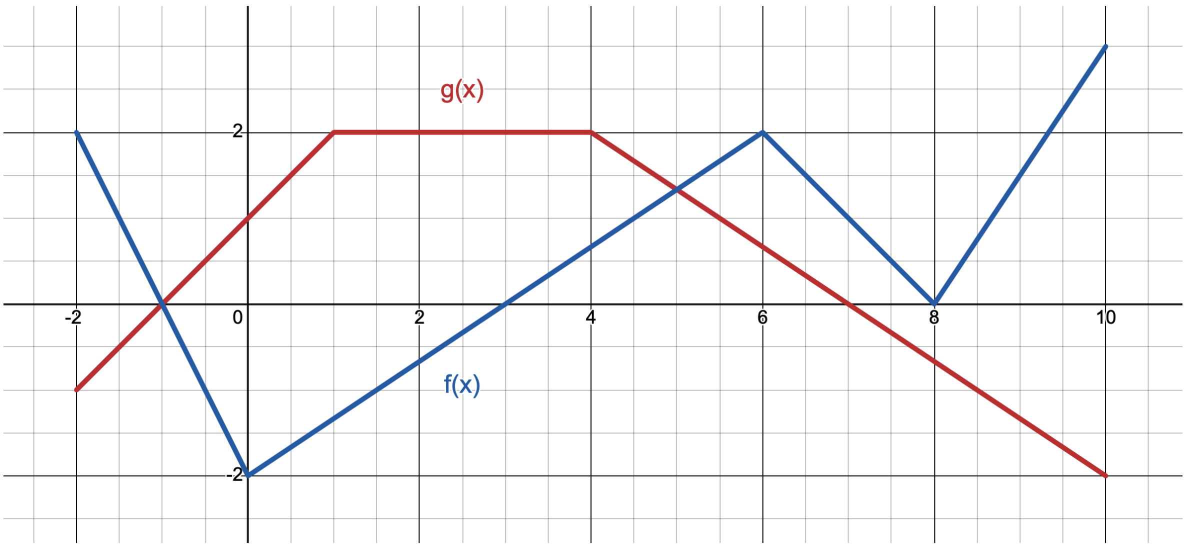
11. If $Q_1(x) = \dfrac{f(x)}{g(x)}$, find $Q_1'(2)$

12. If $Q_2(x) = \dfrac{g(x)}{f(x)}$, find $Q_2'(8)$

13. If $Q_3(x) = \dfrac{f(2x)}{g(x)}$, find $Q_3'(4)$

14. If $Q_4(x) = \dfrac{g\left(\frac{1}{2}x\right)}{f(2x)}$, find $Q_4'(4)$

For problems 15 – 26, the graphs of $f(x)$ and $g(x)$ are given below.



15. If $u = f(g(x))$, find $u'(2)$

16. $v = g(f(x))$, find $v'(4)$

17. If $w = g(g(x))$, find $w'(6)$

18. If $t = f(f(x))$, find $t'(8)$

19. If $P_1(x) = f(x)g(x)$, find $P_1'(2)$

20. If $P_1(x) = f(x)g(x)$, find $P_1'(8)$

21. If $P_2(x) = f(2x)g(x)$, find $P_2'(2)$

10. If $P_3(x) = f(x)g\left(\frac{1}{2}x\right)$, find $P_3'(2)$

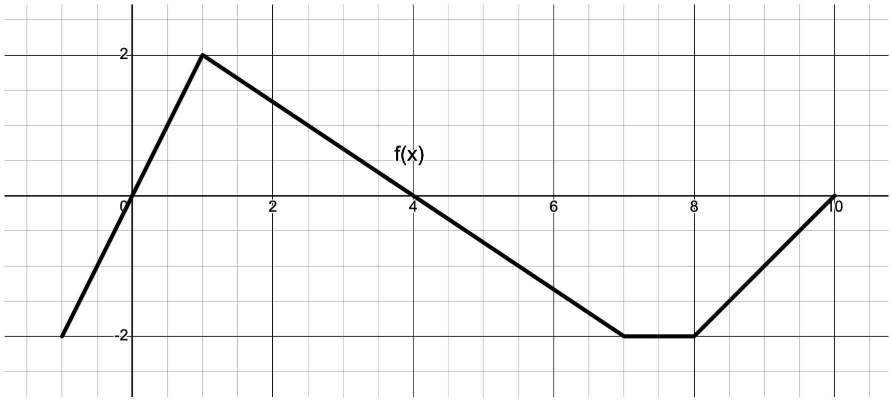
23. If $Q_1(x) = \dfrac{f(x)}{g(x)}$, find $Q_1'(2)$

24. If $Q_2(x) = \dfrac{g(x)}{f(x)}$, find $Q_2'(8)$

25. If $Q_3(x) = \dfrac{f(2x)}{g(x)}$, find $Q_3'(4)$

26. If $Q_4(x) = \dfrac{g\left(\frac{1}{2}x\right)}{f(2x)}$, find $Q_4'(4)$

27. Let $f(x)$ be the function whose graph is given below, and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given in the table below.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12

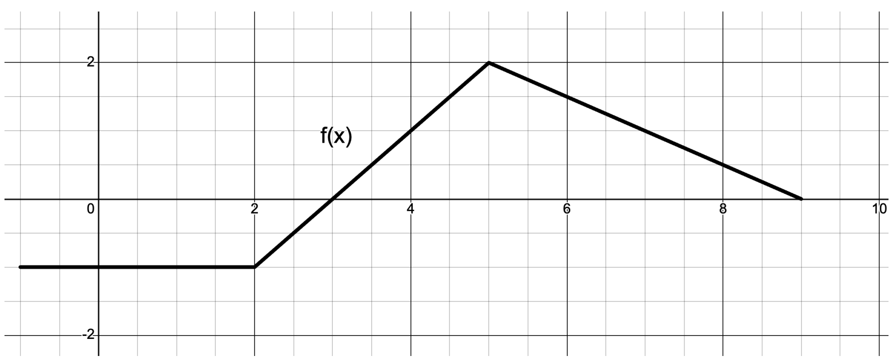
(a) Find the equation of the line tangent to $f(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = g(f(x))$. Find $K'(1)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(4)$.

(d) Let J be the function defined by $J(x) = \dfrac{f(2x)}{g(x)}$. Find $J'(2)$.

28. Let $f(x)$ be the function whose graph is given below, and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given in the table below.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

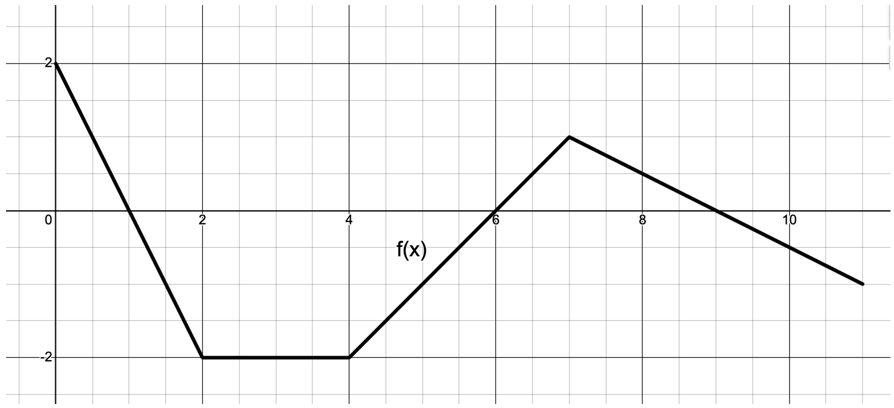
(a) Find the equation of the line tangent to $g(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = g(g(x))$. Find $K'(8)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(4)$.

(d) Let J be the function defined by $J(x) = \dfrac{g(2x)}{f(x)}$. Find $J'(1)$.

29. Let $f(x)$ be the function defined by $f(x) = 4x - x^3$, let $h(x)$ be the function whose graph is given below, and let $g(x)$ be a differentiable function with selected values of $g(x)$ and $g'(x)$ given in the table below.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

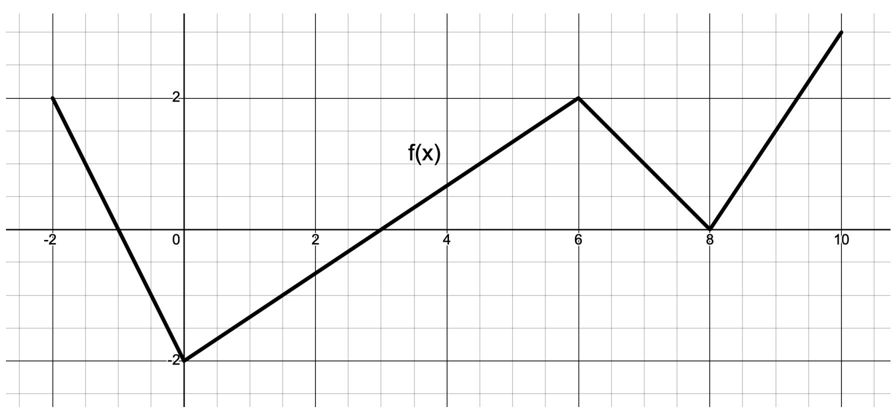
(a) Find the equation of the line tangent to $g(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = h(f(x))$. Find $K'(1)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(6)$.

(d) Let J be the function defined by $J(x) = \dfrac{g(x)}{f(x)}$. Find $J'(4)$.

30. Let h be the function defined by $h(x) = \sin(x) + e^{\cos(3x)}$, let f(x) be the function whose graph is given below, and let g(x) be a differentiable function with selected values of $g(x)$ and $g'(x)$ given in the table below.



x	$g(x)$	$g'(x)$
-4	3	2
-2	5	-1
0	7	0
2	5	-1
4	3	2

(a) Find the equation of the line tangent to $h(x)$ at $x = \dfrac{\pi}{2}$.

(b) Let K be the function defined by $K(x) = f(h(x))$. Find $K'\left(\dfrac{\pi}{2}\right)$.

(c) Let M be the function defined by $M(x) = f(x) \cdot g(x)$. Find $M'(0)$.

(d) Let J be the function defined by $J(x) = g(2x) \cdot f(x)$. Find $J'(2)$.

31. 2017 AP Calculus AB #6

1.7 Multiple Choice Homework

1. Let the function f be differentiable on the interval $[0, 2.5]$ and g be defined by $g(x) = f(f(x))$. Use the table to find $g'(1.5)$.

x	0	0.5	1	1.5	2	2.5
$f(x)$	0.5	1.5	2	2.5	1	0
$f'(x)$	0.1	0.3	0.6	1.1	2	2.2

a) 0 b) 1.24 c) 1.65 d) 2.08 e) 2.42

---

2. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that have values given in the table below, find $h'(2)$, where $h(x) = g(x) \cdot f(x)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	4	-2	8	1
4	10	8	4	3
8	6	-12	2	4

a) -12 b) -2 c) 0 d) 30 e) 64

---

3. Let $f(x)$ and $g(x)$ be differentiable functions. The table below gives the values of $f(x)$ and $g(x)$, and their derivatives, at several values of x.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	-6
2	1	8	-5	7
3	7	-2	7	9

If $h(x) = \dfrac{f(x)}{g(x)}$, what is the value of $h'(2)$?

a) -4 b) -63 c) 51 d) $-\dfrac{47}{64}$ e) $-\dfrac{33}{64}$

---

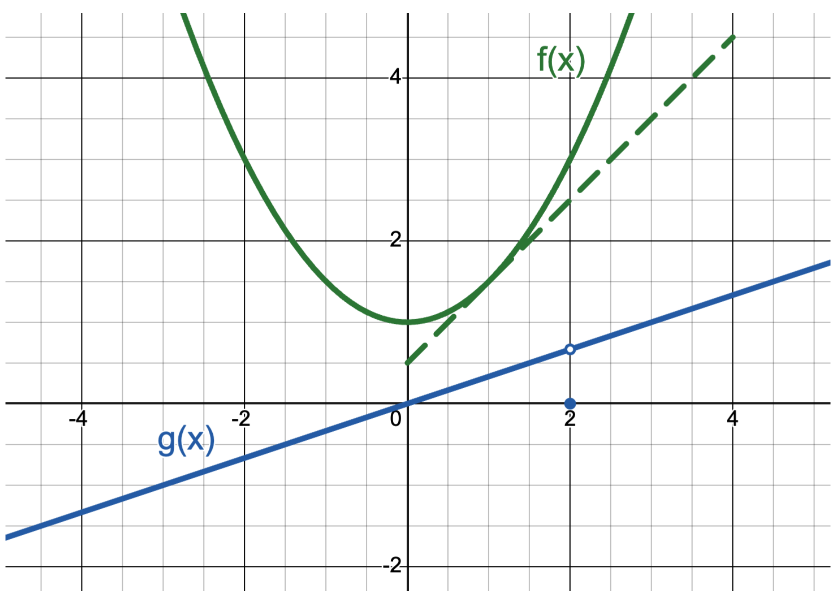
4. Let $h(x) = g(x) \cdot f\left(x^3\right)$. According to the table below, what is the value of $h'(2)$?

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	-3	0	9	10
2	4	6	-4	1
4	9	2	3	3
8	-1	1	2	5

a) -6 b) 2 c) 11 d) 24 e) 143

---

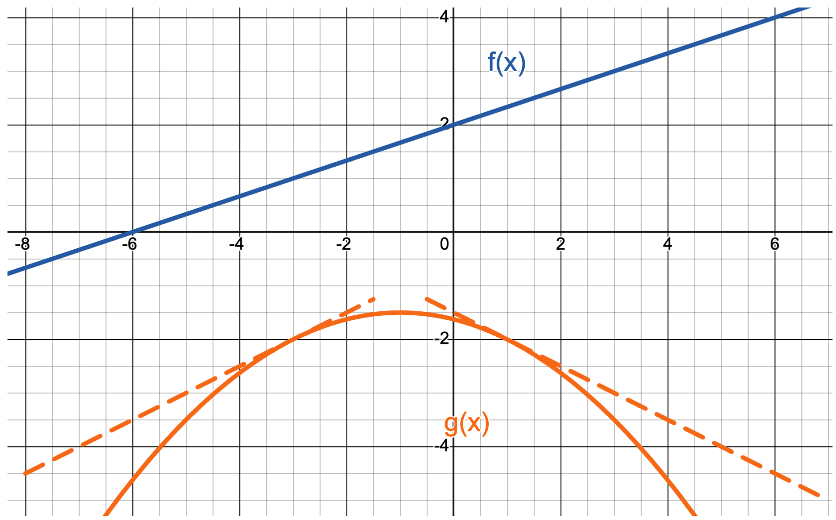
5. The figure below shows the graph of the functions f and g. The graph of the line tangent to the graph of f at $x = 1$ is also shown. If $B(x) = f(x) \cdot g(x)$, what is $B'(1)$?



a) $\dfrac{5}{6}$ b) $-\dfrac{1}{2}$ c) $-\dfrac{1}{6}$ d) $\dfrac{1}{3}$ e) $\dfrac{7}{6}$

---

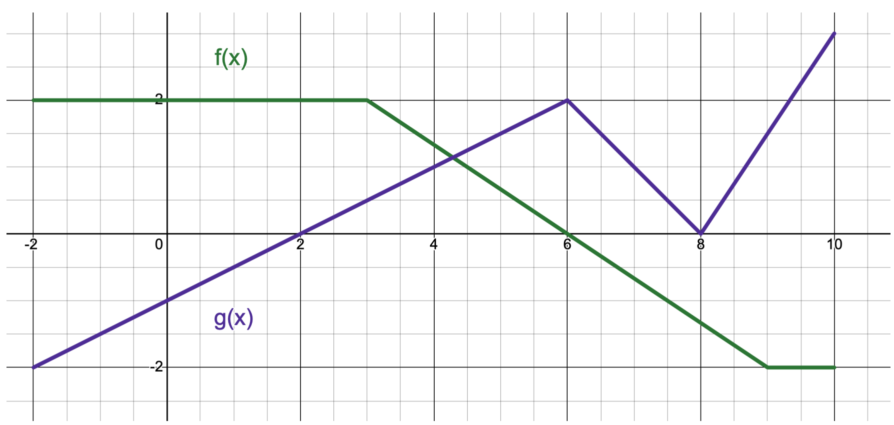
6. The figure below shows the graph of the functions f and g. The graphs of the lines tangent to the graph of g at $x = -3$ and $x = 1$ are also shown. If $B(x) = f(g(x))$, what is $B'(1)$?



a) $-\dfrac{1}{2}$ b) $-\dfrac{1}{6}$ c) $\dfrac{1}{6}$ d) $\dfrac{1}{3}$ e) $\dfrac{1}{2}$

---

7. Given the graphs of the two functions below and the fact that $B(x) = f(g(x))$, $B'(4) =$



a) 0 b) 1 c) $\dfrac{1}{2}$ d) $-\dfrac{1}{3}$ e) DNE

1.8: Related Rates

In this course, derivatives have primarily been interpreted as the slope of the tangent line. But, as with rectilinear motion, there are other contexts for the derivative. One overarching concept is that the derivative is a **rate of change**. The tendency is to think of rates as distance per time unit, like miles per hour or meters per second, but even slope is a rate of change—it is just that the rise and run are both measured as distances.

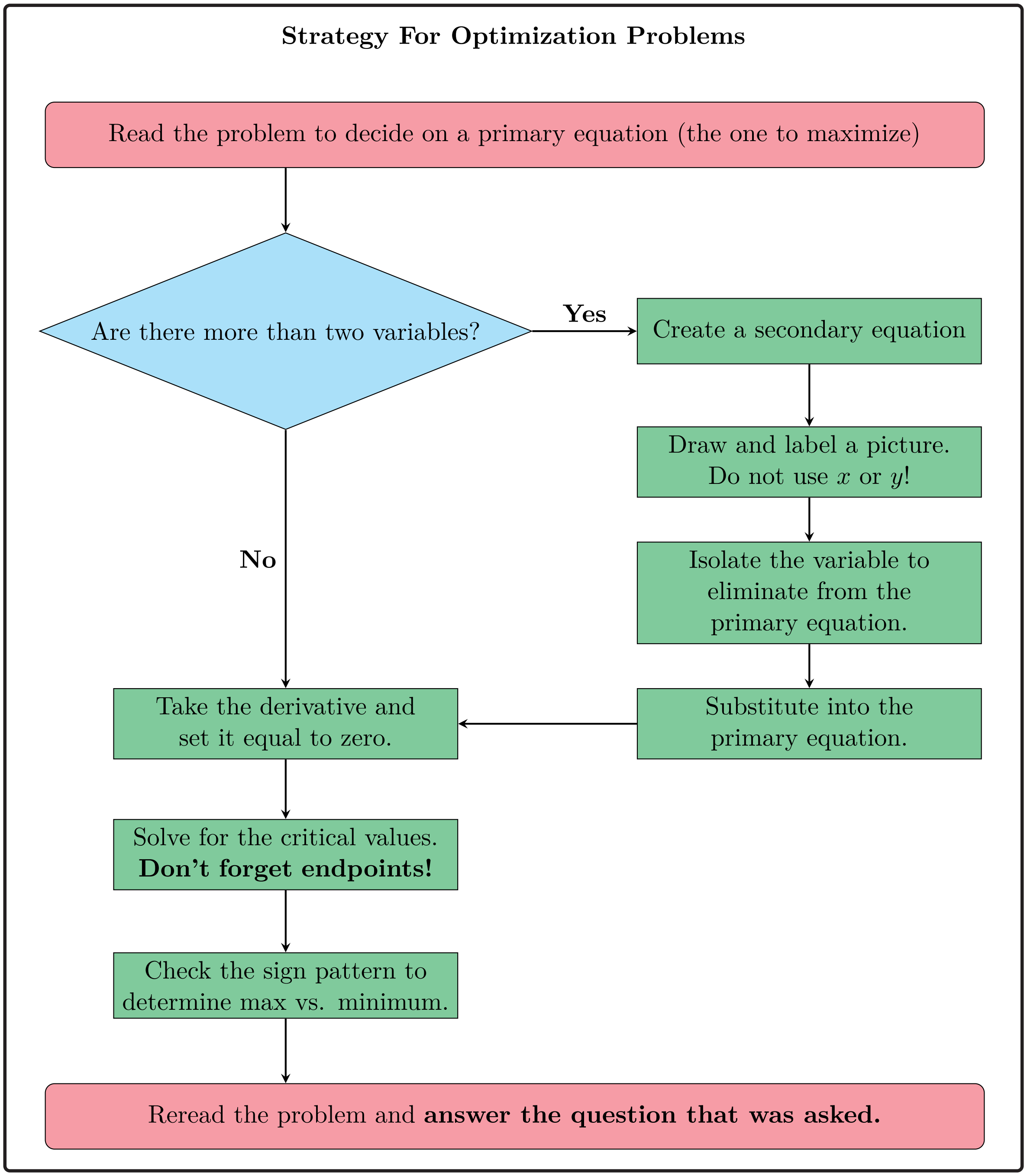
The idea behind related rates is two-fold. First, change is occurring in two or more measurements that are related to each other by the geometry (or algebra) of the situation. Second, an implicit Chain Rule situation exists in that the x and y-values are functions of time, which may or may not be a variable in the problem. Therefore, when taking the derivative of an x or y, an **implicit rate term** $\left(\dfrac{dx}{dx} \text{ or } \dfrac{dy}{dt}\right)$ often occurs.

> OBJECTIVES
>
> Solve Related Rates Problems.

At first glance, related rates problems might seem like optimization problems that we've seen last year. Consider the following example:

> **Ex 1.8.1:** The volume of a cylindrical cola can is 32π in^3. What is the minimum surface area for such a can?

The word "minimum" tells us that we have an optimization problem. Recall our workflow for tackling optimization problems:



So, let's tackle our example.

Sol 1.8.1: The problem asks to minimize surface area, which is determined by:

$$S = 2\pi r^2 + 2\pi rh$$

As there are more than two variables in this equation, either r or h needs to be eliminated in this formula before differentiating. The volume is $V = \pi r^2 h = 32\pi$, so $h = \dfrac{32}{r^2}$ and

$$S = 2\pi r^2 + 2\pi r\left(\frac{32}{r^2}\right)$$

$$= 2\pi r^2 + \frac{64\pi}{r}$$

$$S' = 4\pi r - \frac{64\pi}{r^2} = 0$$

$$\therefore 4\pi r = \frac{64\pi}{r^2}$$

$$r^3 = 16 \to r = 2.5198$$

Now, let's make a sign pattern to determine if this critical value is a minimum.

S'	$-$	0	$+$
r		2.520	

Because the derivative changes from negative to positive, we know that 2.5198 is a minimum. We can plug it back into our surface area equation to find our minimum surface area.

$$S(2.5198) = 2\pi(2.5198)^2 + \frac{64\pi}{2.5198}$$

$$S(2.5198) = 119.687 \text{ in}^2$$

Therefore, the minimum surface area of the cola can is $\boxed{119.687 \text{ in}^2}$.

A related rates problem is characterized by various measurements that are changing **in relation to each other**. The variables are still related to each other through a geometric or physical relationship. The key difference from other differentiation problems is that we differentiate implicitly with respect to time, rather than a variable like x. In other words,

Optimization Problems:	Apply $\dfrac{d}{dx}$
Related Rates Problems:	Apply $\dfrac{d}{dt}$

Now, let's take a look at this different (related rates) cola problem.

Ex 1.8.2: The volume of a cylindrical cola can is 32π in^3. The height of the can is changing at $\frac{1}{4}$ in/sec. If the radius changes at the same time so as to maintain the volume, how fast is the radius shrinking when the can is 4 inches tall?

Sol 1.8.2:

$$V = \pi r^2 h = 32\pi$$

$$h = \frac{32}{r^2}$$

$$\frac{d}{dt}\left[h = \frac{32}{r^2}\right]$$

$$\frac{dh}{dt} = -\frac{64}{r^2}\left(\frac{dr}{dt}\right)$$

Now that we have a method to find what we are looking for, $\frac{dr}{dt}$, let's substitute in the values that we are given in the problem.

$$h = 4 \to \frac{32}{r^2} = 4 \to r^2 = 8$$

$$\frac{1}{4} = -\frac{64}{8}\left(\frac{dr}{dt}\right)$$

$$\frac{dr}{dt} = \boxed{-\frac{1}{32}\text{ in/sec}}$$

As we can see, many of the steps that we take in the related rates cola problem is similar to those of the optimization cola problem.

Process For Related Rates Problems

1. Draw a visual for the problem.
2. Label the visual with what is given and what is being asked.
    a. Use variables for any quantities that are **changing**.
    b. Pay particular attention to the units.
3. Determine the equation(s) that relate the variables to each other.

a. Decide which equation will be differentiated.

b. If there is a product of two variables, eliminate the product by either multiplying the equation out or substituting a secondary equation.

4. **Differentiate in terms of time!** This is the key step.

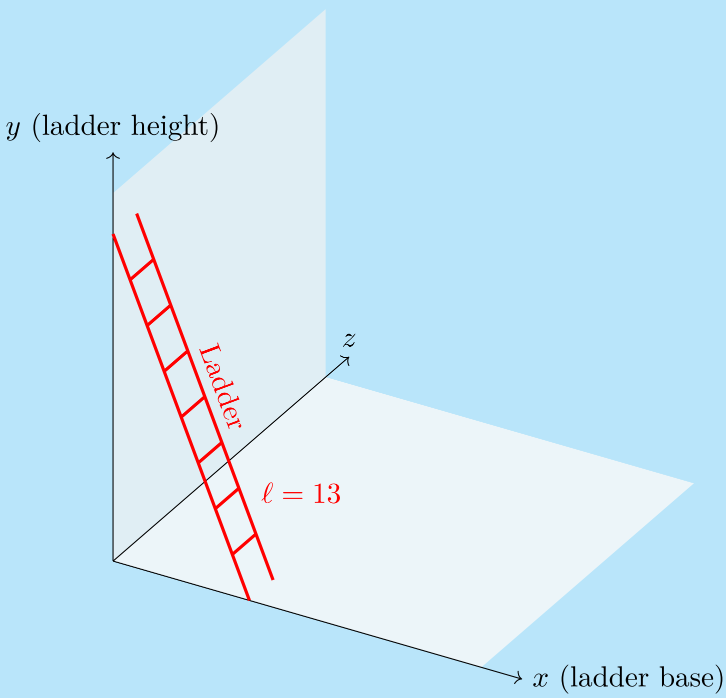
a. Do not forget implicit fractions.

5. Substitute the given information and solve for the missing variable.

6. Reread the problem and make sure to answer the question that was asked.

A classic related rates problem is the falling ladder.

Ex 1.8.3: A 13-foot ladder is leaning against a wall. The bottom of the ladder slides away from the wall at 4 ft/sec. How fast is the top of the ladder moving down the wall when the ladder is 5 feet from the wall?



Sol 1.8.3: As can be seen in the visual, the height of the top of the ladder and the distance the bottom of the ladder is from the wall is related through the Pythagorean theorem. Both are variables, because the ladder is moving.

$$x^2 + y^2 = 13^2$$

We are given the information that the bottom of the ladder slides away from the wall at 4 ft/sec. Therefore, we can say that our $\frac{dx}{dt} = 4$. To find $\frac{dy}{dt}$, we differentiate $x^2 + y^2 = 13^2$ to get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Although this may seem like a complex four equation variable, we already know x and $\frac{dx}{dt}$. We also can determine that $y = 12$ by the Pythagorean theorem. So,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(5)(4) + 2(12)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \boxed{-\frac{5}{3} \text{ ft/sec}}$$

It should make sense that $\frac{dy}{dt}$ is negative because the top of the ladder is sliding down.

Below are some of the equations you should know for the AP exam:

Common Formulas For Optimization/Related Rates Problems

Pythagorean Theorem

$$x^2 + y^2 = r^2$$

Area Formulas

Circle: $A = \pi r^2$

Rectangle: $A = lw$

Triangle: $A = \frac{1}{2}bh$

Trapezoid: $A = \frac{1}{2}h\left(b_1 + b_2\right)$

Volume Formulas

Sphere: $V = \frac{4}{3}\pi$

Right Prism: $V = Bh$

Cylinder: $V = \pi r^2 h$

Cone: $V = \frac{1}{3}\pi r^2 h$

Right Pyramid: $V = \frac{1}{3}Bh$

*Washer: $V = \pi(R^2 - r^2)h$

Surface Area Formulas

Sphere: $S = 4\pi r^2$

Cylinder: $S = 2\pi r^2 + 2\pi rh$

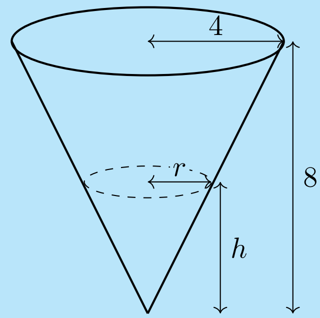
Cone: $S = \pi r^2 + \pi rl$

*Right Prism: $S = 2B + Ph$

N.B. The two equations with asterisks are less commonly used.

Another common related rates problem is where a tank of a particular shape is filling or draining.

Ex 1.8.4: A tank shaped like an inverted cone 8 feet in height and with a base diameter of 8 feet is filling at a rate of 10 ft^3/min. How fast is the height changing when the water is 6 feet deep?



Sol 1.8.4: The units on the rate of change tell us that this is a change in volume, or $\frac{dV}{dt}$. Therefore, we use the equation

$$V = \frac{1}{3}\pi r^2 h$$

But, there are too many variables in this equation to differentiate as it stands. Since the rate of the change of the height—$\frac{dh}{dt}$—is what we are looking for, we can eliminate r from the equation. By similar triangles, we have

$$\frac{r}{h} = \frac{4}{8}$$

$$r = \frac{1}{2}h$$

Substitution gives us a volume equation in terms of only height.

$$V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h$$

$$V = \frac{\pi}{12}h^3$$

Differentiate and plug in our given values to solve for $\frac{dh}{dt}$.

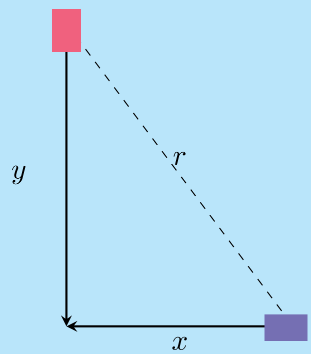
$$V = \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$10 = \frac{\pi}{4}(6)^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \boxed{\frac{10}{9\pi}\text{ ft/min}}$$

Ex 1.8.5: Two cars approach an intersection, one traveling south at 20 miles per hour and the other traveling west at 30 miles per hour. How fast is the direct distance between them decreasing when the westbound car is 0.6 miles and the southbound car is 0.8 miles away from the intersection?



Sol 1.8.5: As we can see in the picture, the distance between the two cars are related by the Pythagorean theorem.

$$x^2 + y^2 = r^2$$

We know several pieces of information. The southbound car is moving at 20 miles per hour; i.e. $\frac{dy}{dt} = -20$. By similar logic, we can deduce each of the following:

$$\frac{dy}{dt} = -20 \qquad \frac{dx}{dt} = -30$$

$$y = 0.8 \qquad x = 0.6$$

And, by the Pythagorean Theorem, $r = 1$. Now we take the derivative of the Pythagorean theorem to get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

This is essentially an equation in six variables. But, we know five of those variables, so let's substitute and solve.

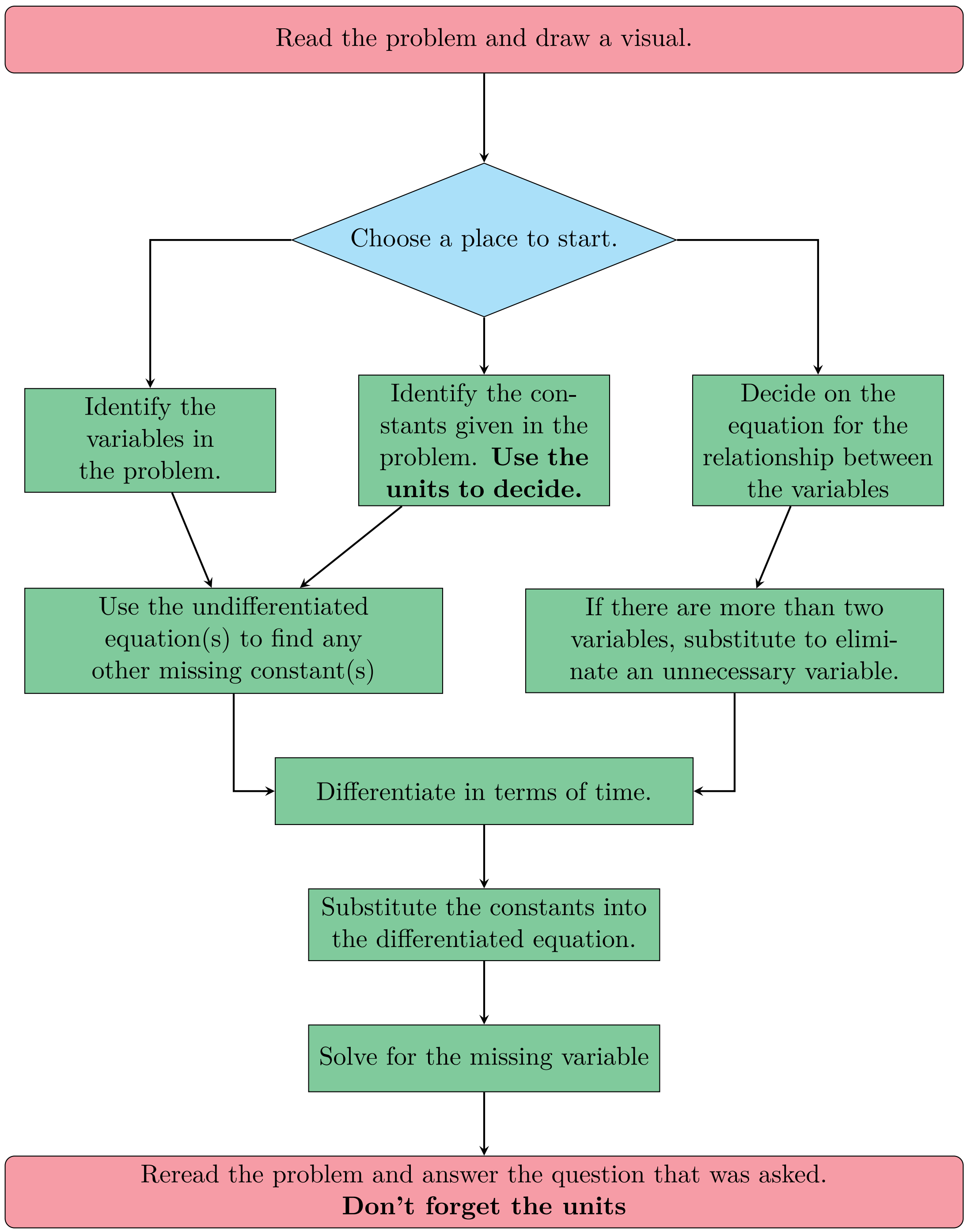
$$2(0.8)(-30) + (2)(0.6)(-20) = 2(1.0)\frac{dr}{dt}$$

$$\frac{dr}{dt} = \boxed{-36 \text{ mph}}$$

It should make sense that $\frac{dr}{dt}$ is negative since the two cars are approaching one another.

The units also make sense: since r is in miles and h is in hours, the final units should be miles per hour.

Strategy For Related Rates Problems



1.8 Free Response Homework

1. Two boats leave an island at the same time, one heading north and one heading east. The northbound boat is moving at 12 mph and the eastbound boat is moving at 5 mph. At $t = 0.2$ hours, the northbound boat is 1.4 miles away from the island and the eastbound boat is 1 mile away from the island.

(a) Draw a picture of the situation at any time t.

(b) What variables are present in the problem?

(c) What known quantities are given? What quantities can be determined directly from the given information? And what is the unknown for which to solve?

(d) What equation(s) relates the quantities? Which one will be differentiated?

(e) How fast is the distance between the two ships decreasing at $t = 0.2$ hours?

2. A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the intersection and watches an eastbound train traveling 60 m/sec.

(a) Draw a picture of the situation at any time t.

(b) What variables are present in the problem?

(c) What known quantities are given? What quantities can be determined directly from the given information? And what is the unknown for which to solve?

(d) What equation(s) relates the quantities? Which one will be differentiated?

(e) At how many meters per second is the train moving away from the observer 4 seconds after it passes the intersection?

3. A circular ink stain is spreading (i.e. the radius is changing) at half an inch per second.

(a) Draw a picture of the situation at any time t.

(b) What variables are present in the problem?

(c) What known quantities are given? What quantities can be determined directly from the given information? And what is the unknown for which to solve?

(d) What equation(s) relates the quantities? Which one will be differentiated?

(e) How fast is the area changing when the stain has a 1 inch diameter?

4. A screensaver has a rectangular logo that expands and contracts as it moves around the screen. The ratio of the sides stay constant, with the long side being 1.5 times the short side. At a particular moment, the long side is 3 cm, while the perimeter is changing by 0.25 cm/sec.

(a) Draw a picture of the situation at any time t.

(b) What variables are present in the problem?

(c) What known quantities are given? What quantities can be determined directly from the given information? And what is the unknown for which to solve?

(d) What equation(s) relates the quantities? Which one will be differentiated?

(e) How fast is the area of the screensaver changing at the given moment?

5. Sand is being dumped onto a pile at 30π ft/min. The pile forms a cone with the height always equal to the base diameter.

(a) Draw a picture of the situation at any time t.

(b) What variables are present in the problem?

(c) What known quantities are given? What quantities can be determined directly from the given information? And what is the unknown for which to solve?

(d) What equation(s) relates the quantities? Which one will be differentiated?

(e) How fast is the height changing when the pile is 5 feet high?

6. A cylindrical oil tank of height $30'$ and radius $10'$ is leaking at a rate of 300 ft^3/min.

(a) Draw a picture of the situation at any time t.

(b) What variables are present in the problem?

(c) What known quantities are given? What quantities can be determined directly from the given information? And what is the unknown for which to solve?

(d) What equation(s) relates the quantities? Which one will be differentiated?

(e) How fast is the oil level dropping?

7. Water is leaking out of an inverted conical tank at a rate of 5000 cm^3/min. The tank is 8 m tall and has a diameter of 4 m.

(a) Draw a picture of the situation at any time t.

(b) What variables are present in the problem?

(c) What known quantities are given? What quantities can be determined directly from the given information? And what is the unknown for which to solve?

(d) What equation(s) relates the quantities? Which one will be differentiated?

(e) Find the rate at which the height is decreasing when the water level is at 3 m.

(f) Find the rate of change of the radius at the same instant as part (e).

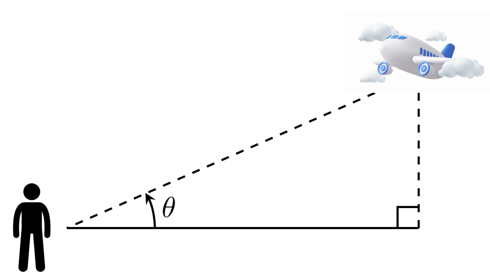
8. Sand is dumped onto a pile at 30π ft^3/min. The pile forms a cone with the height always equal to the base diameter. How fast is the base area changing when the pile is 10 feet high?

9. A spherical balloon is being inflated so that its volume is increasing at a rate of 6 ft^3/min. How fast is the radius changing when $r = 10$ ft?

10. The edge of a cube is expanding at a constant rate of 6 inches per second. What is the rate of the change of the volume, in inches cubed per second, when the total surface area of the cube is 54 in^2.

11. A 25-foot tall ladder is leaning against a wall. The bottom of the ladder is pushed toward the wall at 5 ft/sec. How fast is the top of the ladder moving up the wall when it is 7 feet up?

12. You are standing outside. A plane flies overhead, approaching you at constant altitude and a constant speed of 600 miles per hour. When the plane flies over a house 13 miles away from where you are standing, the angle of elevation is 0.647 radians. How quickly is the direct (diagonal) distance between you and the plane changing at that moment?



13. The altitude of a triangle is increasing at a rate of 2 cm/sec at the same time that the area of the triangle is increasing at a rate of 5 cm^2/sec. At what rate is the base increasing when the altitude is 12 cm and the area is 144 cm^2?

14. Two cars start moving away from the same point. One travels south at 60 mph, and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

15. According to Boyle's Law, gas pressure varies directly with temperature and inversely with volume, as described by the following equation:

$$P = \frac{kT}{V}$$

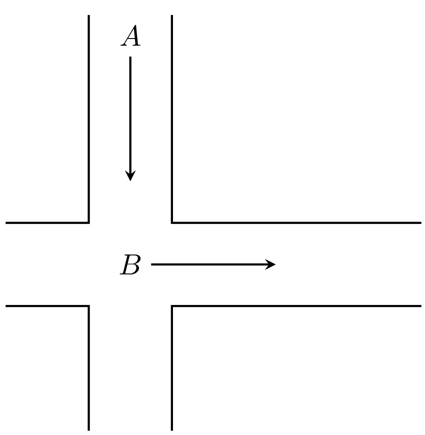
Suppose that the temperature is held constant while the pressure increases at at 20 kPa/min. What is the rate of change of the volume when the volume is 600 in^3 and the pressure is 150 kPa?

16. The Adiabatic Law for expansion of air can be represented with the equation

$$P \cdot V^{1.4} = \frac{4}{81},$$

where P represents pressure and V represents volume. If, at a specific instant, $P = 108 \text{ lb/in}^2$ and is increasing at 27 lb/in^2 per second, what is the rate of change of the volume?

17. Person A is 220 feet north of an intersection and walking toward it at 10 ft/sec. Person B starts at the intersection and walks east at 5 ft/sec.



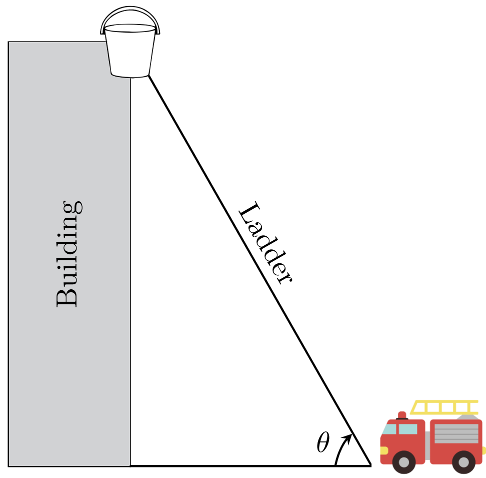
(a) At $t = 10$ seconds, how far is each person from the intersection?

(b) At $t = 10$ seconds, how far apart are the two people?

(c) How fast is the distance between the two people changing at $t = 10$ seconds?

(d) At time $t = 10$ seconds, if person A looks at person B, how fast is the angle between person A's line of sight to person B and the eastward direction changing?

18. A fire truck is parked 7 feet away from the base of a building and its ladder is extended to the top of the building. The ladder retracts at a rate of 0.5 feet per second, while the angle of the ladder changes such that the bucket at the end of the ladder comes down vertically.



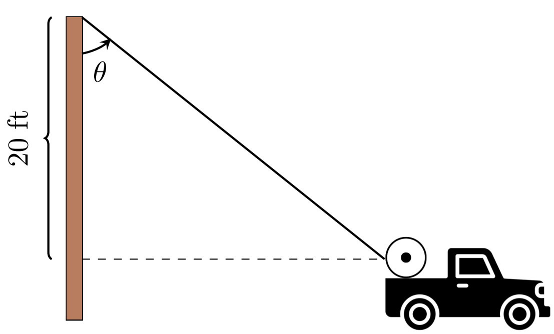
(a) How far is the ladder extended when the bucket is 10 feet above the ground?

(b) Find the rate at which the bucket is dropping vertically when the bucket is 10 feet above the ground.

(c) What is the relationship between the angle θ and the height of the bucket? Find θ, in radians, when the bucket is 10 feet above the ground.

(d) Find the rate, in radians per second, at which the angle the ladder forms with the ground is changing when the bucket is 10 feet above the ground.

19. A telephone crew is replacing a phone line from one telephone pole to the next. The line is on a spool on the back of a truck, and one end is attached to the top of a 25-foot pole. The vertical distance from the top of the pole to the level of the spool is 20 feet. The truck moves down the street at 20 ft/sec.



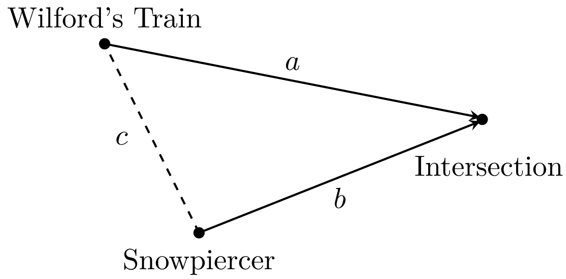
(a) Find the length of line that has been rolled out when $t = 15$ seconds.

(b) Find the rate at which the telephone line is coming off the spool when the truck is 50 feet from the pole.

(c) What is the relationship between the angle and the truck's distance from the pole? Find θ, in radians, when the truck is 40 feet from the pole.

(d) Find the rate, in radians per second, at which the angle the line forms with horizontal is changing when the truck is 40 feet from the pole.

20. At the end of the first season of the series *Snowpiercer*, a second train (Big Alice) controlled by the industrialist Mr. Wilford came down another track to intercept and stop Snowpiercer. If the two train tracks meet at a 30° then the Law of Cosines would apply such that $c^2 = a^2 + b^2 - 1.969ab$ as in the figure below. Snowpiercer is described as being two-and-a-half stories tall and $1,001$ cars long, with an average speed of 100 km/hour. Wilford's train Big Alice is much shorter, so it can average 120 km/hour.

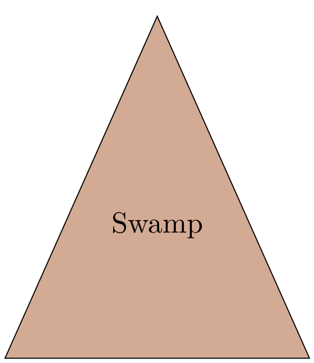


(a) If Snowpiercer is 38 km from the junction, how soon will it reach the junction?

(b) If Wilford's train is 45 km from the junction, which train will reach the junction first?

(c) How fast is the distance c between the two trains changing when Snowpiercer is 38 km from the junction and Big Alice is 45 km from the junction.

21. The triangle-shaped swamp on Oak Island has been proven to have been manmade sometime around 1250 AD, possibly by the Knights Templar. It has been drained several times, revealing a stone-paved wharf and paths, an ancient clay mine, and the remains of a galleon which had been destroyed by fire and sunk in the swamp to conceal it. The triangle is roughly isosceles, with legs measuring 730 feet and a base of 640 feet. The apex (top angle) of the triangle measures 0.79 radians.



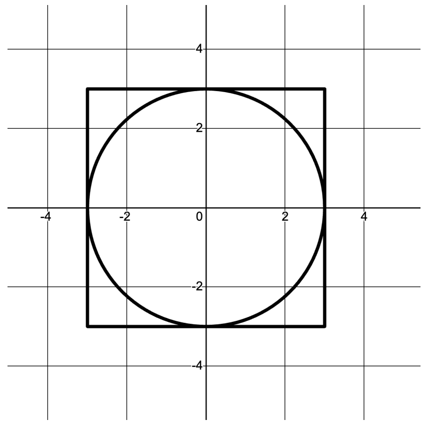
(a) Based on the Law of Sines, the area of a triangle can be determined by the equation $Area = \dfrac{1}{2}ab\sin(C)$, where a and b are the lengths of the legs and $\angle C$ is the measure of the angle in between the two legs. Find the surface area of the swamp before it was drained. Indicate the units.

(b) The length of third side of the triangle can be found using the Law of Cosine, $c^2 = a^2 + b^2 - 2ab\cos(C)$, where c is the third side and a, b, and $\angle C$ are the same as those of part (a). How long is the third side when the legs are each 370 feet?

(c) At $t = 24$ hours, the legs are 370 feet and their rate of change is -15.2 ft/hr. How fast is the third side of the triangle changing?

(d) Find the rate of change of the surface area. [Hint: Use the Law of Sines.] Indicate the units.

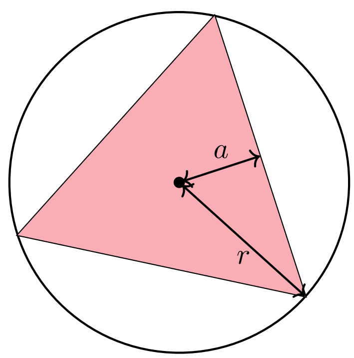
22. A circle is inscribed in a square as shown. The circumference of the circle is increasing at a constant rate of 4 inches per second. As the circle expands, the square expands to maintain the condition of tangency.



(a) Find the rate of change of the perimeter of the square.

(b) At the instant when the area of the circle is 16π square inches, find the rate at which the area *between the square and the circle* is increasing.

23. An equilateral triangle is inscribed in a circle. The circle's circumference is expanding at 6π in/sec and the triangle maintains the contact of its corners with the circle.



Given that the area of an equilateral triangle is equal to half the apothem a times the perimeter p, find out how fast the area inside the circle but outside the triangle is expanding when the area of the circle is 64π in. [Hint: Find p and a in terms of r.]

1.8 Multiple Choice Homework

1. The width of a square is increasing at a constant rate of 0.5 cm/sec. In terms of the perimeter P, what is the rate of change of the area of the square in centimeters squared

per second?

a) $-\frac{1}{2}P$ b) $-P$ c) $\frac{1}{4}P$ d) $\frac{1}{2}$ e) P

---

2. When $x = 18$, the rate at which $y = \sqrt{0.5x}$ is increasing is k times the rate at which x is increasing. What is the value of $\frac{1}{k}$?

a) $\frac{1}{12}$ b) $\frac{1}{6}$ c) 1 d) 6 e) 12

---

3. The side of a cube is expanding at a constant rate of 6 inches per second. What is the rate of change of the volume, in cubic inches per second, when the total surface area of the cube is 54 in^2/sec?

a) 324 b) 108 c) 18 d) 162 e) 54

---

4. If the volume of a cube is increasing at 20 cubic inches per second when each edge is 10 inches long, how fast is the surface area increasing?

a) $\frac{4}{3}$ b) 2 c) 4 d) 6 e) 8

---

5. When the height of a cylinder is 12 cm and the radius is 4 cm, the circumference is increasing at a rate of $\frac{\pi}{4}$

a) 4π b) 12π c) 20π d) 80π e) 100π

---

6. At what approximate rate (in cubic meters per minute) is the volume of a sphere changing at the instant when the surface area is 3 square meters and the radius is increasing at the rate of $\frac{1}{5}$ meters per minute?

a) 1.228 b) 1.905 c) 0.649 d) 0.600 e) 0.620

7. The radius of a sphere is decreasing at a rate of 2 centimeters per second. At the instant when the radius of the sphere is 3 cm, what is the rate of change, in square centimeters per second, of the surface area of the sphere? [Hint: The surface area S of a sphere with radius r is $4\pi r^2$.]

a) -108π b) -72π c) -48π d) -24π e) -16π

---

8. Water is flowing into a spherical tank with a 6-foot radius at the constant rate of 30π ft^3/hour. When the water is h feet deep, the volume of the water in the tank is given by the equation

$$V = \frac{\pi h^2}{3}(18 - h).$$

What is the rate at which the depth of the water in the tank is increasing the moment when the water is 2 feet deep?

a) 0.5 ft/hour b) 1.0 ft/hour c) 1.5 ft/hour d) 2.0 ft/hour e) 2.5 ft/hour

---

9. The radius of a circle is in increasing at a constant rate of 0.2 meters per second. What is the rate of increase in the area of the circle at the instant when the circumference of the circle is 20π meters?

a) 0.04π m^2/sec b) 0.4π m^2/sec c) 4π m^2/sec d) 40π m^2/sec e) 100π m^2/sec

---

10. If the rate of change of a number x with respect to time t, is x, what is the rate of change of the reciprocal of the number when $x = -\frac{1}{4}$?

a) -16 b) -4 c) $-\frac{1}{48}$ d) $\frac{1}{48}$ e) 4

---

11. Gravel is being dumped from a conveyor belt at a rate of 35 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 15 ft high?

a) 0.27 ft/min b) 1.24 ft/min c) 0.14 ft/min d) 0.20 ft/min e) 0.60 ft/min

---

12. Two cars start moving from the same point. One travels south at 28 mi/h and the other travels west at 70 mi/h. At what rate is the distance between the cars increasing 5 hours later?

a) 75.42 mi/h b) 75.49 mi/h c) 76.40 mi/h d) 75.39 mi/h e) 75.38 mi/h

---

13. A Golden Rectangle is one where the ratio (called ϕ) of the length of the short side w to the long side l is equal to the ratio of the long side to the sum of the two sides. In other words, $l = 1.618w$. If a Golden Rectangle changes such that w is growing at 2 in/min, how fast is the area changing when w is 5 inches?

a) 1.618 in^2/min b) 16.18 in^2/min c) 3.236 in^2/min d) 32.36 in^2/min e) 37.78 in^2/min

---

1.9: Logarithmic Differentiation

With implicit differentiation and the Chain Rule, we learned some powerful tools for differentiating functions and relations. The Product and Quotient Rules also allowed us to take derivatives of certain functions that would otherwise be impossible to differentiate. Sometimes, however, with very complex functions, it becomes easier to manipulate an equation so that it is easier to take the derivative. This is where logarithmic differentiation comes in.

OBJECTIVES

Determine When It Is Appropriate to Use Logarithmic Differentiation.
Use Logarithmic Differentiation to Take The Derivatives of Complicated Functions.

Before we begin, it would be helpful to look at a few exponent and logarithm rules that we should recall from algebra and precalculus.

$$a^x a^y = a^{x+y} \qquad \log_a x + \log_a y = \log_a xy$$

$$\frac{a^x}{a^y} = a^{x-y} \qquad \log_a x - \log_a y = \log_a \frac{x}{y}$$

$$(a^x)^y = a^{xy} \qquad \log_a x^n = n \log_a x$$

Since logarithms are exponents expressed in a different form, all of the above rules are derived from those of exponents.and you can see the corresponding exponential rule. Because of our algebraic rules, we can do whatever we want to both sides of an equation. In algebra, we usually used this to solve for a variable. In calculus, we can use this principle to make many derivative problems significantly easier.

Ex 1.9.1: Find the derivative of $y = \left(x^2 + 7x - 3\right)(\sin(x))$.

Sol 1.9.1: Traditionally, we would use the Product Rule to take the derivative of this function.

$$\frac{d}{dx}\left[y = \left(x^2 + 7x - 3\right)(\sin(x))\right]$$

$$\frac{dy}{dx} = \boxed{\left(x^2 + 7x - 3\right)\cos(x) + (2x + 7)(\sin(x))}$$

Obviously, this is a straightforward problem that can be easily done using the product rule. If, however, I took the natural log of both sides of the equation, I can achieve the same results, and never use the product rule.

$$\ln y = \ln\left[\left(x^2+7x-3\right)(\sin(x))\right]$$

Let's simplify using our log rules.

$$\ln y = \ln\left(x^2+7x-3\right) + \ln(\sin(x))$$

$$\frac{d}{dx}\left[\ln(y) = \ln\left(x^2+7x-3\right) + \ln(\sin(x))\right]$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2x+7}{x^2+7x-3} + \frac{\cos(x)}{\sin(x)}$$

$$\frac{dy}{dx} = \left(\frac{2x+7}{x^2+7x-3} + \frac{\cos(x)}{\sin(x)}\right)(y)$$

Now just substitute y back in and simplify.

$$\frac{dy}{dx} = \left(\frac{2x+7}{x^2+7x-3} + \frac{\cos(x)}{\sin(x)}\right)\left(\left(x^2+7x-3\right)(\sin(x))\right)$$

$$\frac{dy}{dx} = \boxed{\left(x^2+7x-3\right)\cos(x) + (2x+7)(\sin(x))}$$

Clearly, we got the same answer that we got from the product rule, but with significantly more effort.

Logarithmic differentiation is a tool we can use, but we have to use it judiciously, as we don't want to make problems more difficult than they have to be. Where logarithmic differentiation has the potential to be really useful is with functions that are excessively painful to work with (or impossible to take the derivative of any other way) because of multiple operations. Consider the following example:

Ex 1.9.2: Find $\frac{dy}{dx}$ for $y = \frac{(x^2+5)\sin(3x^3)}{\tan(5x+2)}$.

Sol 1.9.2: We could take the derivative by applying the Chain Rule, Quotient Rule, and Product Rule, but that would be a time-consuming and tedious process. It's much easier to take the natural log of both sides, simplify and then take the derivative.

$$\ln y = \ln\left[\frac{(x^2+5)\sin(3x^3)}{\tan(5x+2)}\right]$$

$$\ln y = \ln\left(x^2+5\right) + \ln\left(\sin\left(3x^3\right)\right) - \ln(\tan(5x+4))$$

$$\frac{d}{dx}\left[\ln y = \ln\left(x^2+5\right) + \ln\left(\sin\left(3x^3\right)\right) - \ln(\tan(5x+4))\right]$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2x}{x^2+5} + \frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)} - \frac{5\sec^2(5x+2)}{\tan(5x+2)}$$

$$\frac{dy}{dx} = \left(\frac{2x}{x^2+5} + \frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)} - \frac{5\sec^2(5x+2)}{\tan(5x+2)}\right)(y)$$

$$\frac{dy}{dx} = \boxed{\left(\frac{2x}{x^2+5} + \frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)} - \frac{5\sec^2(5x+2)}{\tan(5x+2)}\right)\left(\frac{(x^2+5)\sin\left(3x^3\right)}{\tan(5x+2)}\right)}$$

Now that may seem long and messy, but try it any other way, and you might end up taking a lot more time, with a lot more algebra and a lot more potential spots to make mistakes.

Ex 1.9.3: Find $f'(\pi)$ for $f(z) = z^{\cos(z)}$.

Sol 1.9.3:

$$\ln(f(z)) = \ln\left(z^{\cos(z)}\right)$$

$$\frac{d}{dz} = \left[\ln(f(z)) = \ln\left(z^{\cos(z)}\right)\right]$$

$$\frac{f'(z)}{f(z)} = \frac{\cos(z)}{z} - (\ln z)(\sin(z))$$

$$f'(z) = \left(\frac{\cos(z)}{z} - (\ln z)(\sin(z))\right)(f(z))$$

$$f'(z) = \left(\frac{\cos(z)}{z} - (\ln z)(\sin(z))\right)\left(z^{\cos(z)}\right)$$

$$f'(\pi) = \left(\frac{\cos(\pi)}{\pi} - (\ln \pi)(\sin(\pi))\right)\left(\pi^{\cos(\pi)}\right) = \boxed{-\frac{1}{\pi^2}}$$

We could have also done this problem using the change of base property that we learned in precalculus, and we would get the same answer in roughly the same number of steps.

Again, there are often more than one way to do a specific problem, and part of what we do as mathematicians is decide on the simplest **correct** method to solving a problem. The issue many people have when learning more difficult mathematical concepts is that they try to oversimplify a problem and end up getting it wrong as a result.

1.9 Free Response Homework

Find the derivatives of the following functions. Only use logarithmic differentiation **when appropriate**.

1. $y = (2x+1)^4 \left(x^3 - 3\right)^5$

2. $z = \left(y^3 - 3\right) e^{(2y+1)}$

3. $y = \dfrac{\sin^2(x)\tan^4(x)}{\left(x^2+5\right)^2}$

4. $g(t) = t \ln t$

5. $y = \ln^x x$

6. $p(v) = v^{e^v}$

Complete the following

7. Find $\dfrac{dt}{dt}$ if $t^u = u^t$.

8. Find $\dfrac{dy}{dx}$ for the function $y = \left(e^{17x^4}\right)\left(\sin^7(x)\right)(5x-17)^{12}(\cot(5x))$.

9. Use logarithmic differentation to find $\dfrac{dq}{dt}$ if $q = \dfrac{e^{t^4-15}\sin^5(3t)}{(\ln t)^{10}}$.

10. Use logarithmic differentation to find $\dfrac{dy}{dx}$ if $y = e^{150x-19}\ln^{100}(\sin(x))\sqrt{x^2-1}$.

11. Use logarithmic differentiation to prove the Product Rule.

12. Use logarithmic differentiation to prove the Quotient Rule.

Full Name: AP Calculus BC

Date: Chapter 1 Practice Test

Multiple Choice Section **20 Minutes; No Calculator**

Show All Work

1. Which of the following equations is true?

a) $\dfrac{d}{dx}\left[\cot^{-1}(4x)\right] = -\dfrac{4}{16x^2+1}$

b) $\dfrac{d}{dx}\left[e^{\cos(x)}\right] = -e^{-\sin(x)}\cos(x)$

c) $\dfrac{d}{dx}\left[\ln^3\left(1-x^2\right)\right] = \dfrac{3\ln^2\left(1-x^2\right)}{1-x^2}$

d) $\dfrac{d}{dx}\left[e^{2x}\cos^{-1}(x)\right] = \dfrac{2e^{2x}}{\sqrt{1-xe^2}}$

---

2. If $f(x) = \cot^{-1}(\sin(x))$, then $\dfrac{d}{dx}[f(x)] =$

a) $-\csc^2(\sin(x))\cos^2(x)$ b) $\sec(x)$ c) $\dfrac{\cos(x)}{1+\sin^2(x)}$ d) $-\dfrac{\cos(x)}{1+\sin^2(x)}$ e) -1

---

3. Let $y = f(x)$ be a solution to the differential equation $\dfrac{dy}{dx} = x - y^2$ with the initial condition $f(0) = 1$. What is the best approximation for $f(2)$ if Euler's Method is used, starting at $x = 0$ with a step size of 1.0

a) -1 b) 0 c) 1 d) 2 e) 3

---

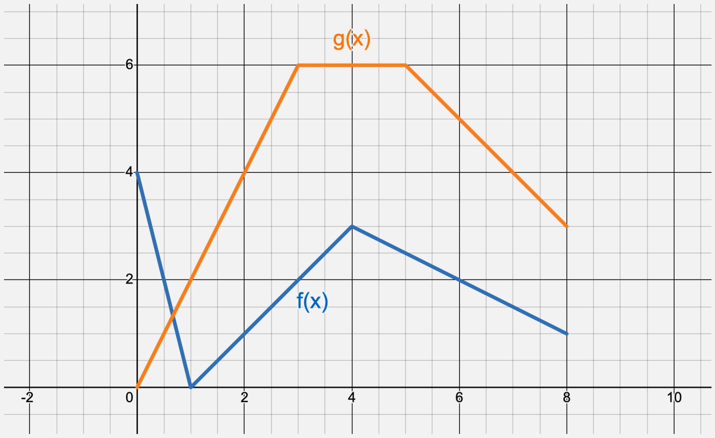
4. Selected values of f, g, and their derivatives are indicated in the table below. Let $h(x) = g\left(f\left(\sqrt{x}\right)\right)$. What is the value of $h'(4)$?

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	4	3	4	-1
4	2	-1	7	8
16	1	2	2	1

a) 6 b) $\frac{1}{4}$ c) 1 d) 2 e) 24

---

5. The figure below shows the graph of the functions f and g. If $B(x) = \dfrac{g(x)}{f(x)}$. what is $B'(6)$?



a) $-\frac{9}{2}$ b) $\frac{1}{8}$ c) -2 d) 0 e) DNE

---

6. If $f(x) = 3\sin(x) + 4\cos^2(x)$, then $f''\left(\frac{\pi}{2}\right)$

a) 3 b) 0 c) 5 d) -8 e) -3

---

7. Which of the following is an equation of the line tangent to the graph of $f(x) = x^6 + x^5 + x^2$ at the point where $f'(x) = -1$?

a) $-3x - 2$ b) $-3x + 4$ c) $-x + 0.905$ d) $-x + 0.271$ e) $-x - 0.271$

---

8. A biologist is tracking the growth of a circular colony of bacteria in a Petri dish. She observes that the colony is expanding at rate of 15 mm^2/hour. Find the rate at which the radius is increasing when the diameter is 5 mm.

a) 3 b) 3π c) $\dfrac{3}{\pi}$ d) $\dfrac{3}{2\pi}$ e) $\dfrac{3\pi}{2}$

---

9. What is the slope of the line tangent to the curve $y^2 + x = -2xy - 5$ at the point $(2, 1)$?

a) $-\dfrac{4}{3}$ b) $-\dfrac{3}{4}$ c) $-\dfrac{1}{2}$ d) $-\dfrac{1}{4}$ e) 0

Free Response Section **45 Minutes; No Calculator**

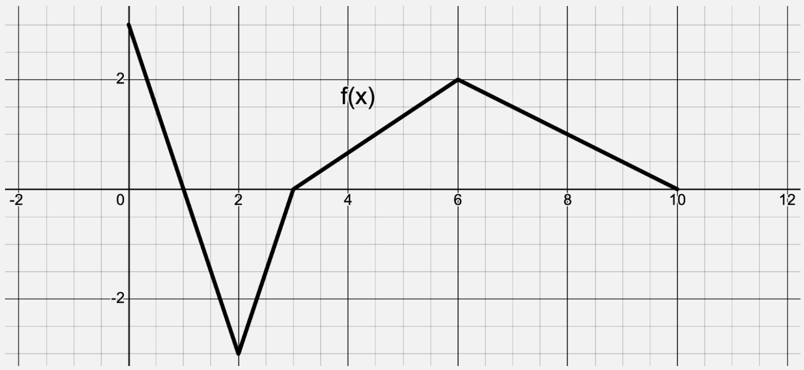
Show All Work

1. Compute the following derivatives.

   (a) $\dfrac{d}{dx}\left[\sin^{-1}\left(e^{2x}\right)\right]$

   (b) $\dfrac{d}{dx}\left[\ln\left(\cot\left(\sqrt{x}\right)\right)\right]$

2. Let $f(x)$ be the function defined by $f(x) = \sin\left(\frac{\pi}{2}\right)x$, let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given in the table below, and let $h(x)$ be a differentiable function whose graph is given below.



x	$g(x)$	$g'(x)$
0	-2	12
2	0	-3
4	5	5
6	3	8
8	-4	11

(a) Find the equation of the line tangent to $h(x)$ at $x = 8$.

(b) Let K be the function defined by $K(x) = g(f(x))$. Find $K'(6)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(4)$.

(d) Let J be the function defined by $J(x) = \dfrac{h(x)}{g\left(\frac{1}{2}x\right)}$. Find $J'(8)$.

3. If $F(x) = \ln\left(3x^2 - 2x + 1\right)$, find $F''(x)$.

4. Find the equations of the lines tangent and normal to $g(x) = e^{4x}\cos(x)$ at $x = 0$ and use it to approximate $g(-0.2)$

5. Consider the curve given by $2y - x + xy = 8$

(a) Show that $\dfrac{dy}{dx} = \dfrac{1-y}{2+x}$.

(b) Find the coordinates of the point(s) where the tangent line is horizontal or prove why there is no such point(s).

(c) Find the value of $\dfrac{d^2y}{dx^2}$ at $(1,3)$. Does the curve have a relative maximum, a relative minimum, or neither at $(0,1)$? Justify your answer.

6. Two people on bikes are at the same place. One of the bikers starts riding directly north at a rate of 8 m/sec. Five seconds after the first biker started riding north, the second starts to ride directly east at a rate of 5 m/sec At what rate is the distance between the two riders increasing 20 seconds after the *second* person started riding?

Chapter 1 Answer Key

1.1 Free Response Answers

1. $f'(x) = \boxed{2x + 3}$

2. $f'(t) = \boxed{t^3}$

3. $y' = \boxed{-\frac{2}{3}x^{-\frac{5}{3}}}$

4. $y' = \boxed{5e^x}$

5. $v'(r) = \boxed{4\pi r^2}$

6. $g'(x) = \boxed{2x - 2x^{-3}}$

7. $y' = \boxed{\frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}}$

8. $u' = \boxed{\frac{2}{3}t^{-\frac{1}{3}} + 3t^{\frac{1}{2}}}$

9. $z' = \boxed{-\frac{10A}{y^{11}} + Be^y}$

10. $y' = \boxed{e^{x+1}}$

11. $\boxed{7x^6 - \frac{7}{2}x^{-\frac{1}{8}} + 7^x \ln 7 + \frac{4}{7}x^{-\frac{11}{7}} - \frac{1}{5}x^{-2}}$

12. $\boxed{6x^5 - \frac{7}{2}x^{\frac{1}{6}} + 5^x \ln 5 + \frac{5}{3}x^{-\frac{8}{3}} - \frac{1}{8}x^{-2}}$

13. $\boxed{4x^3 - 18x^{\frac{2}{7}} + 8^x \ln 8 + \frac{7}{3}x^{-\frac{10}{3}} - \frac{1}{8}x^{-2}}$

14. $\boxed{\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}}$

15. $\boxed{\frac{7}{3}z^{\frac{4}{3}} - \frac{4}{3}z^{-\frac{2}{3}}}$

16. $\boxed{\frac{11}{5}x^{\frac{6}{5}} - \frac{24}{5}x^{\frac{1}{5}} + \frac{3}{5}x^{-\frac{4}{5}}}$

17. $\boxed{60t^4 + 9t^2 + 56t}$

18. $\boxed{\frac{-7\left(3x^2+4\right)}{\left(x^3+4x-\pi\right)^8}}$

19. $\boxed{\frac{3x-2}{\left(3x^2-4x+9\right)^2}}$

20. $\boxed{\frac{3x^2-2}{7\left(x^3-2x\right)^{\frac{6}{7}}}}$

21. $\boxed{10y^{\frac{3}{2}} - 3y^{\frac{1}{2}} - \frac{5}{2}y^{-\frac{1}{2}}}$

22. $\boxed{\frac{3}{4}v^{\frac{1}{2}} - v^{-\frac{1}{2}} - \frac{7}{4}v^{-\frac{3}{2}}}$

23. $\boxed{-\frac{7}{5}w^{-2} + \frac{8}{5}w^{-3} - \frac{3}{5}w^{-4}}$

24. $\boxed{\frac{3}{7}w^{-2} + \frac{8}{7}w^{-3}}$

25. $f'(x) = \boxed{\frac{2+3x^2}{4\left(1+2x+x^3\right)^{\frac{3}{4}}}}$

26. $f'(x) = \boxed{\frac{3\left(-x^{-2}+2+e^x\right)}{5\left(x^{-1}+2x+e^x\right)^{\frac{2}{5}}}}$

27. $f'(x) = \boxed{37\left(3x^2+2\right)\left(x^3+2x\right)^{36}}$

28. $f'(x) = \boxed{15x^4 - 15x^2}$

29. $f'(2) = \boxed{-4e^3}$

30. $\dfrac{dy}{dx} = \boxed{\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}}$

31. $f'\left(\sqrt{5}\right) = \boxed{-\dfrac{\sqrt{5}}{3}}$

32. $f'(x) = \boxed{-\dfrac{xe^{\sqrt{9-x^2}}}{\sqrt{9-x^2}}}$

33. $\boxed{v'(t) = \dfrac{1}{2}\left[\left(\dfrac{E(t)}{3} + 3t\right)^{\frac{3}{7}} - 4\right]^{-\frac{1}{2}} \left[\dfrac{3}{7}\left(\dfrac{E(t)}{3} + 3t\right)^{-\frac{4}{7}}\right] \left(\dfrac{1}{3}E'(t) + 3\right)}$

34. $\boxed{v'(t) = \dfrac{1}{3}\left[\left(\dfrac{C(t)}{7} + 4t^2\right) - 1\right]^{-\frac{2}{3}} \left[\dfrac{5}{7}\left(\dfrac{C(t)}{7} + 4t^2\right)^{-\frac{2}{7}}\right] \left(\dfrac{1}{7}C'(t) + 8t\right)}$

1.1 Multiple Choice Answers

1	2	3	4	5	6	7	8
C	**B**	**A**	**E**	**D**	**E**	**B**	**E**

1.2 Free Response Set A Answers

1. $y' = \boxed{4\cos(4x)}$

2. $y' = \boxed{20x^4 \sec\left(x^5\right) \tan\left(x^5\right)}$

3. $f'(t) = \boxed{\dfrac{\sec^2(t)}{3\left(\tan(t) + 1\right)^{\frac{2}{3}}}}$

4. $f'(\theta) = \boxed{-\tan(\theta)}$

5. $y' = \boxed{-3\cos^2(x)\sin(x)}$

6. $y' = \boxed{-3x^2 \sin\left(a^3 + x^3\right)}$

7. $f'(x) = \boxed{-\dfrac{\sin(\ln x)}{x}}$

8. $f'(x) = \boxed{\dfrac{1}{5x(\ln x)^{\frac{4}{5}}}}$

9. $f'(x) = \boxed{\dfrac{\cos(x)}{(\ln(10))(\sin(x)+2)}}$

10. $f'(x) = \boxed{-\dfrac{3}{(\ln 2)(1-3x)}}$

11. $y' = \boxed{\dfrac{e^x}{\sqrt{1-e^{2x}}}}$

12. $y' = \boxed{\dfrac{1}{2\sqrt{x}(x+1)}}$

13. $\boxed{\dfrac{3e^{3x}}{\sqrt{1-e^{6x}}}}$

14. $\boxed{-\dfrac{2e^{2x}}{e^{4x}+1}}$

15. $\boxed{\dfrac{2x}{x^4+1}}$

16. $\boxed{0}$

17. $\boxed{3(2x+2)e^{x^2+2x}}$

18. $\boxed{-3(2x+2)\sin\left(x^2+2x\right)}$

19. $\boxed{\dfrac{x^2}{(16+x^3)^{\frac{2}{3}}}}$

20. $\boxed{\dfrac{1}{x^3\sqrt{1-\frac{1}{4}x^{-4}}}}$

21. $\boxed{35e^{\tan(7x)}\sec^2(7x)}$

22. $\boxed{-\dfrac{x\sin\left(x^2-1\right)}{\sqrt{\cos\left(x^2-1\right)}}}$

23. $\boxed{\dfrac{6x\ln^2\left(x^2+1\right)}{x^2+1}}$

24. $\boxed{3x^2\cot\left(x^3\right)}$

25. $\boxed{\tan(x)}$

26. $\boxed{-2x\sin\left(x^2\right)}$

27. $f'(x) = \boxed{\dfrac{2x}{x^2+3}}$

28. $g'(x) = \boxed{\dfrac{2x-4}{x^2-4x+4}}$

29. $h'(x) = \boxed{\dfrac{x}{\sqrt{x^2+5}}}$

30. $F'(x) = \boxed{\dfrac{6x-6}{3\left(3x^2-6x+1\right)^{\frac{2}{3}}}}$

31. $y' = \boxed{-\dfrac{\sin(x)}{\sqrt{1-\cos^2(x)}}}$

32. $y' = \boxed{-\dfrac{x}{\sqrt{1-x^2}}}$

33. $y' = \boxed{6\sec^2(3\theta)\tan(3\theta)}$

34. $y' = \boxed{-7\cot^6(\sin(x))\csc^2(\sin(x))\cos(x)}$

35. $y' = \boxed{\dfrac{\sqrt{2}}{\sqrt{1-2x^2}}}$

36. $y' = \boxed{\dfrac{2}{\sqrt{1-(2x+1)^2}}}$

For **problem 31**, a common trap is to use the Pythagorean Identity to simplify $\sqrt{1-\cos^2(x)}$ to $\sin(x)$, and then simplify $-\dfrac{\sin(x)}{\sin(x)}$ to -1. However, one must note that the result of a square root must be a positive value. Therefore, the simplification really becomes $-\dfrac{\sin(x)}{|\sin(x)|}$. This structure is also known as the negative **signum** function for sine, also denoted as $-\operatorname{sgn}(\sin(x))$. For more information about the signum function, take a look at this article.

1.2 Free Response Set B Answers

1. $y' = \boxed{-\dfrac{3e^{3z}}{\sqrt{1-e^{6z}}}}$

2. $y' = \boxed{\dfrac{2x}{(x^2-1)^2+1}}$

3. $y' = \boxed{\dfrac{1}{|x|\sqrt{16x^2-1}} - \dfrac{4}{\sqrt{1-16x^2}}}$

4. $f'(x) = \boxed{\dfrac{5}{(25x^2+1)\tan^{-1}(5x)}}$

5. $g'(w) = \boxed{0}$

6. $f'(t) = \boxed{\dfrac{t}{(t^2+9)\sqrt{t^2+8}}}$

7. $\boxed{-e^{\csc(\theta)}\cot(\theta)\csc(\theta) - \dfrac{2\theta\csc^2(\theta^2)}{\cot(\theta^2)} - \sec(\theta)\tan(\theta)}$

8. $\boxed{3\left(3x^2+\dfrac{5}{x}\right)\tan\left(5\ln x + x^3 + 7\right)}$

9. $\boxed{\dfrac{3(2x+5e^x)\sec^2(x^2+5e^x+7)}{\tan(x^2+5e^x+7)}}$

10. $\boxed{-\dfrac{2\csc^2(\ln(5x^2))}{x}}$

11. $\boxed{\dfrac{2x+4}{2(x^2+4x-5)}}$

12. $\boxed{\dfrac{35\sin^4(\ln(7t+3))\cos(\ln(7t+3))}{7t+3}}$

13. $\boxed{-\dfrac{(14x+1)\cot(\ln(7x^2+x))\csc(\ln(7x^2+x))}{7x^2+x}}$

14. $\boxed{4t}$

15. $\boxed{\dfrac{9-54x-15x^{-4}}{2\sqrt{9x-27x^2+5x^{-1}}}}$

16. $\boxed{5\sec(5x)\tan(5x) - e^x\csc^2(e^x) - \dfrac{10}{x}}$

17. $\dfrac{dz}{dt} = \boxed{-\tan(t) + e^t\sec\left(e^t\right)\tan\left(e^t\right)}$

18. $\dfrac{dz}{dt} = \boxed{\dfrac{\sec^2(t)}{\tan(t)} + e^t\cos\left(e^t\right)}$

19. $\dfrac{dz}{d\theta} = \boxed{-\dfrac{\csc^2(\theta)}{\cot(\theta)} + \dfrac{\sec(\ln(\theta))\tan(\ln(\theta))}{\theta}}$

20. $\dfrac{dz}{d\theta} = \boxed{-\tan(\theta) + \dfrac{\cos(\ln(\theta))}{\theta}}$

21. $f'(3) = \boxed{\dfrac{81\pi}{4}}$

1.2 Multiple Choice Answers

1	2	3	4	5	6	7	8	9	10
E	**E**	**C**	**B**	**D**	**A**	**B**	**B**	**D**	**C**

1.3 Free Response Set A Answers

1. $y' = \boxed{-t^3\sin(t) + 3t^2\cos(t)}$

2. $y' = \boxed{-48x(2x-5)^4\left(8x^2-5\right)^{-4} + 8(2x-5)^3\left(8x^2-5\right)^{-3}}$

3. $y' = \boxed{\dfrac{\sec^3(x) - (\tan(x)-1)(\sec(x)\tan(x))}{\sec^2(x)}}$

4. $y' = \boxed{\dfrac{x^2\cos(x) - 2x\sin(x)}{x^4}}$

5. $y' = \boxed{-2x^2e^{-x^2} + e^{-x^2}}$

6. $y' = \boxed{\dfrac{\left(r^2+1\right)^{\frac{1}{2}} - r^2\left(r^2+1\right)^{-\frac{1}{2}}}{r^2+1}}$

7. $y' = \boxed{e^{x\cos(x)}\left(-x\sin(x) + \cos(x)\right)}$

8. $y' = \boxed{-3e^{-5x}\sin(3x) - 5e^{-5x}\cos(3x)}$

9. $y' = \boxed{-\dfrac{\cos\left(x^{-1}\right)}{x} + \sin\left(x^{-1}\right)}$

10. $y' = \boxed{-\dfrac{xe^{-x}}{e^{-x} + xe^{-x}}}$

11. $y' = \boxed{\dfrac{\frac{x}{|x|\sqrt{x^2-1}} - \sec^{-1}(x)}{x^2}}$

12. $y' = \boxed{\dfrac{x^3\sec(x)\tan(x) - 3x^2\sec(x)}{x^6}}$

13. $y' = \boxed{1 + 2x\tan^{-1}(x)}$

14. $y' = \boxed{\dfrac{2x}{x^2+4} - \dfrac{x}{2\left(1+\frac{x^2}{4}\right)} - \tan^{-1}\left(\dfrac{x}{2}\right)}$

15. $f'(x) = \boxed{\dfrac{1}{2\sqrt{\ln(x)}} + \sqrt{\ln(x)}}$

16. $g'(x) = \boxed{8(-2x+1)\left(3+x-x^2\right)^7 + 20(4x+1)^4\left(3+x-x^2\right)^8}$

17. $f'(x) = \boxed{\cos^{-1}(x)}$

18. $g'(x) = \boxed{-\sqrt{1-x^2} + \dfrac{x^2}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}}}$

19. $\boxed{\dfrac{\left(x^2-9\right)(6x+4) - \left(3x^2+4x-3\right)(2x)}{\left(x^2-9\right)^2}}$

20. $\boxed{\dfrac{(x+2)\left(3x^2-4x-5\right) - \left(x^3-2x^2-5x+6\right)}{(x+2)^2}}$

21. $\boxed{\dfrac{\left(3x^3\right)\left(5x^4-36x^2-19\right) - \left(x^5-12x^3-19x\right)\left(9x^2\right)}{9x^6}}$

22. $\boxed{\dfrac{\left(x^3+1\right)(3) - (3x)\left(3x^2\right)}{\left(x^3+1\right)^2}}$

23. $\boxed{\dfrac{\left(x^2-9x+20\right) - (x-4)(2x-9)}{\left(x^2-9x+20\right)^2}}$

24. $\boxed{\dfrac{(\sin(x))(\sec^2(x)) - (\tan(x)+5)(\cos(x))}{\sin^2(x)}}$

25. $\boxed{\dfrac{(1-\cos(x))(\cos(x)) - \sin^2(x)}{(1-\cos(x))^2}}$

26. $\boxed{\dfrac{2x\cos(x) + x^2\sin(x)}{\cos^2(x)}}$

27. $\dfrac{dy}{dx} = \boxed{\dfrac{\left(x^2-4\right)(2x) - \left(x^2-3\right)(2x)}{\left(x^2-4\right)^2}}$

28. $f'(x) = \boxed{\dfrac{\left(x^2-x-3\right)(2x+2) - \left(x^2+2x-8\right)(2x-1)}{\left(x^2-x-3\right)^2}}$

29. $y' = \boxed{\dfrac{(x-4)(2x+2) - \left(x^2+2x-3\right)(x-4)}{(x-4)^2}}$

30. $f'(x) = \boxed{\dfrac{\ln(x)-1}{\ln^2(x)}}$

31. $h'(t) = \boxed{17\left(\dfrac{1+x^2}{1-x^2}\right)^{16}\left(\dfrac{\left(1-x^2\right)(2x) - \left(1+x^2\right)(-2x)}{\left(1-x^2\right)^2}\right)}$

32. $\dfrac{dy}{dx} = \boxed{\dfrac{(\cos(x)-3)\left(\sec^2(x)\right) - (\tan(x))(-\sin(x))}{(\cos(x)-3)^2}}$

33. $f'(x) = \boxed{5\left(x\sin(2x) + \tan^4\left(x^7\right)\right)^4\left(2x\cos(2x) + \sin(2x) + 28\tan^3\left(x^7\right)\sec^2\left(x^7\right)\right)}$

34. $f'(x) = \boxed{e^x - \dfrac{x^2}{x^2+1} - 2x\arctan(x)}$

35. $f'\left(\dfrac{\pi}{4}\right) = \boxed{\dfrac{1}{2}}$

36. $y'(1) = \boxed{\dfrac{7}{2}}$

1.3 Free Response Set B Answers

1. $x'(t) = \boxed{e^{t^2}\cos\left(t^2-5t^4\right)\left(2t-20t^3\right)+2te^{t^2}\sin\left(t^2-5t^4\right)}$

2. $x'(t) = \boxed{12t^3e^{5t}\sec^2\left(3t^4\right)+5e^{5t}\tan\left(3t^4\right)}$

3. $y' = \boxed{\dfrac{(x-3)(2x+2)-(x^2+2x-15)}{(x-3)^2}}$

4. $x'(t) = \boxed{e^t\left(2t-20t^3\right)+\left(t^2-5t^4\right)e^t}$

5. $\boxed{\dfrac{\left(\sin\left(x^3\right)\right)\left(e^x+14x\right)-\left(e^x+7x^2+5\right)\left(3x^2\cos\left(x^3\right)\right)}{\sin^2\left(x^3\right)}}$

6. $\boxed{-\dfrac{e^{\sin(x)}e^x\csc^2\left(e^x\right)}{\cot\left(e^x\right)}+e^{\sin(x)}\cos(x)\ln\left(\cot\left(e^x\right)\right)}$

7. $\boxed{2x^3\cos\left(x^2\right)+2x\sin\left(x^2\right)+\dfrac{\ln x-\frac{x+1}{x}}{\ln^2 x}}$

8. $\boxed{-2x^3\sin\left(x^2\right)+2x\cos\left(x^2\right)+\dfrac{xe^x-e^x}{x^2}}$

9. $\boxed{\dfrac{5x^5}{5x+4}+5x^4\ln(5x+4)+\dfrac{\ln x-1}{\ln^2 x}}$

10. $\boxed{\dfrac{-2xe^{-5x}\sin\left(x^2-3\right)+5e^{-5x}\cos\left(x^2-3\right)}{e^{-10x}}}$

11. $\boxed{-e^{x^2}\sin(x)+2x\cos(x)}$

12. $\boxed{\dfrac{\ln(4x)\sec^2(x)-\frac{1}{x}(1+\tan(x))}{\ln^2(4x)}}$

13. $\boxed{\sin(x)\sec^2(x)+\tan(x)\cos(x)}$

14. $\boxed{\dfrac{\frac{1}{x}\csc(x)+(1+\ln x)\csc(x)\cot(x)}{\csc^2(x)}}$

15. $\boxed{e^{5x^4}\cot(x)+20x^3e^{5x^4}\ln(\sin(x))}$

16. $\boxed{5x\cos(x)+5\sin(x)+2e^{2x}-\dfrac{6x}{3x^2+1}+\dfrac{\left(x^2+1\right)-2x^2}{\left(x^2+1\right)^2}}$

17. $\boxed{\tan\left(e^x\right)\left(4x^3-15x^2+1\right)+e^x\left(x^4-5x^3+x\right)\sec^2\left(e^x\right)}$

18. $\boxed{\dfrac{5\ln(3x+7)-\frac{3(5x+2)}{3x+7}}{\ln^2(3x+7)}}$

19. $\boxed{\dfrac{\left(3x^3\right)\left(5x^4-36x^2-19\right)-\left(x^5-12x^3-19x\right)\left(9x^2\right)}{9x^6}}$

20. $\boxed{32\cos^2(4x+2)-32\sin^2(4x+2)}$

21. $g'(z) = \boxed{118\left(\dfrac{e^{5z}}{1+\ln z}\right)^{117}\left(\dfrac{(\ln z+1)\left(5e^{5z}\right)-e^{5z}\left(\frac{1}{z}\right)}{(1+\ln z)^2}\right)}$

22. $g'(t) = \boxed{15\left(\dfrac{t^2-4}{1-t^2}\right)^{14}\left(\dfrac{\left(1-t^2\right)(2t)-\left(t^2-4\right)(-2t)}{\left(1-t^2\right)^2}\right)}$

23. $y' = \boxed{\left(\dfrac{1}{\left(\frac{2e^x}{1-e^{2x}}\right)^2+1}\right)\left(\dfrac{\left(1-e^{2x}\right)\left(2e^x\right)-\left(2e^x\right)\left(-2e^{2x}\right)}{\left(1-e^{2x}\right)^2}\right)}$

24. $f'(x) = \boxed{x^2\left(-\dfrac{1}{\sqrt{1-x^2}}\right)+\arccos(x)(2x)}$

25. $\dfrac{dy}{du} = \boxed{\dfrac{2u}{u^2+1} - \dfrac{u}{u^2+1} + \cot^{-1}(u)}$

26. $\boxed{\dfrac{\frac{(x+1)-(x-1)}{(x+1)^2}}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}}$

27. $f'(t) = \boxed{\dfrac{1}{\sqrt{1-\left(\frac{t}{c}\right)^2}} + \dfrac{t}{\sqrt{c^2-t^2}}}$

28. $y' = \boxed{\dfrac{2}{\sqrt{1-\frac{x^2}{4}}} - \dfrac{x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2}}$

29. $f'(1) = \boxed{\dfrac{7503}{5} + \ln 5}$

1.3 Multiple Choice Answers

1	2	3	4	5	6	7	8
D	**E**	**B**	**E**	**A**	**D**	**A**	**B**

1.4 Free Response Homework

1. $f''(x) = \boxed{20x^3 + 12}$

2. $h''(x) = \boxed{60x^2 + 54x - 8}$

Chapter 2:

Intro To Anti-Derivatives

Chapter 2 Overview: Anti-Derivatives

As noted in the introduction, Calculus is essentially comprised of four operations:

- Limits
- Derivatives
- Indefinite Integrals (Or Anti-Derivatives)
- Definite Integrals

As mentioned above, there are two types of integrals — the definite integral and the indefinite integral. The definite integral was explored first as a way to determine the area bounded by a curve, rather than bounded by a polygon. The summation of infinite rectangles is

$$A = \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x,$$

and the representation

$$\int_a^b f(x)\, dx$$

is the exact amount, with $\displaystyle\int$ being an elongated and stylized s for "sum".

Newton and Leibnitz made the connection between the definite integral and the antiderivative, showing that the process of reversing the derivative results in an infinite summation. The antiderivative and indefinite integral are inverses of each other, just as squares and square roots or exponential and log functions. In this chapter, we will consider how to reverse the differentiation process. In a later chapter, we will dive deeper into the definite integral. Let's start by reviewing our derivative rules, as they will be necessary for us to take the antiderivative.

You must know the derivative rules in order to know the antiderivative rules!

The Power Rule: $\dfrac{d}{dx}\left[u^n\right] = nu^{n-1}\dfrac{du}{dx}$

The Product Rule: $\dfrac{d}{dx}\left[u \cdot v\right] = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$

The Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{u(x)}{v(x)}\right] = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$

The Chain Rule: $\dfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}[\sin(u)] = (\cos(u))\frac{du}{dx} \qquad \frac{d}{dx}[\csc(u)] = (-\csc(u)\cot(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\cos(u)] = (-\sin(u))\frac{du}{dx} \qquad \frac{d}{dx}[\sec(u)] = (\sec(u)\tan(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2(u)\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot(u)] = \left(-\csc^2(u)\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right] = \left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1}(u)\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc^{-1}(u)\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1}(u)\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec^{-1}(u)\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1}(u)\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot^{-1}(u)\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

2.1: The Anti-Power Rule

As we have seen, we can deduce things about a function if its derivative is known. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called antidifferentiation, or integration.

Symbol for the Integral	
$\int f(x)\,dx$	"the integral of f of x, d-x"

The dx is called the differential. For now, we will treat it as part of the integral symbol. It tells us the independent variable of the function [usually, but not always, x]. It does have a meaning on its own, but we will explore that later.

Looking at the integral as an antiderivative, we should be able to figure out the basic process. Remember:

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

and

$$\frac{d}{dx}[\text{constant}] = 0$$

It follows that if we are starting with the derivative and want to reverse the process, the power must increase by one and we should divide by this new power. Formally,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

The $+C$ is to account for any constant that might've been in the equation before the derivative was taken. Note that $n = -1$ does not work with this rule because it results in a division by zero. However, we know from our derivative rules that the derivative of $\ln x$ yields x^{-1}. Therefore, we can append our anti-power rule.

The Complete Anti-Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Since $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)]+[g(x)]$ and $\frac{d}{dx}[cx^n] = c\frac{d}{dx}[x^n]$, it follows that:

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c(f(x))\,dx = c\int f(x)\,dx$$

These allow us to integrate a polynomial by integrating each term separately.

OBJECTIVES

Find the Anti-Derivative of a Polynomial.
Integrate Functions Using Transcendental Operations
Use Integration to Solve Rectilinear Motion Problems

Ex 2.1.1: $\int \left(3x^2+4x+5\right)\,dx$

Sol 2.1.1:

$$\begin{aligned}\int \left(3x^2+4x+5\right)\,dx &= 3\frac{x^{2+1}}{2+1}+4\frac{x^{1+1}}{1+1}+5\frac{x^{0+1}}{0+1}+C\\ &= \frac{3x^3}{3}+\frac{4x^2}{2}+\frac{5x^1}{1}+C\\ &= \boxed{x^3+2x^2+5x+C}\end{aligned}$$

Ex 2.1.2: $\int \left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)\,dx$

Sol 2.1.2:

$$\begin{aligned}\int \left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)\,dx &= \frac{x^{4+1}}{4+1}+\frac{4x^{2+1}}{2+1}+\frac{5x^{0+1}}{0+1}+\ln|x|-\frac{x^{-5+1}}{-5+1}+C\\ &= \boxed{\frac{1}{5}x^5+\frac{4}{3}x^3+5x+\ln|x|+\frac{1}{4x^4}+C}\end{aligned}$$

Ex 2.1.3: $\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right)\, dx$

Sol 2.1.3:

$$\begin{aligned}\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right)\, dx &= \int \left(x^2 + x^{\frac{1}{3}} - \frac{4}{x}\right)\, dx \\ &= \frac{x^{2+1}}{2+1} + \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - 4\ln|x| + C \\ &= \boxed{\frac{1}{3}x^3 - \frac{3}{4}x^{\frac{4}{3}} + 4\ln|x| + C}\end{aligned}$$

Integrals of products and quotients can be done easily IF they can be turned into a polynomial.

Ex 2.1.4: $\int \left(x^2 + \sqrt[3]{x}\right)(2x+1)\, dx$

Sol 2.1.4:

$$\begin{aligned}\int \left(x^2 + \sqrt[3]{x}\right)(2x+1)\, dx &= \int \left(2x^3 + 2x^{\frac{4}{3}} + x^2 + x^{\frac{1}{3}}\right)\, dx \\ &= \frac{2x^4}{4} + \frac{2x^{\frac{7}{3}}}{\frac{7}{3}} + \frac{x^3}{3} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} \\ &= \boxed{\frac{1}{2}x^4 + \frac{6}{7}x^{\frac{7}{3}} + \frac{1}{3}x^3 + \frac{3}{4}x^{\frac{4}{3}} + C}\end{aligned}$$

The next example is called an initial value problem. It has an ordered pair (or initial value pair) that allows us to solve for C.

Ex 2.1.5: $f'(x) = 4x^3 - 6x + 3$. Find $f(x)$ if $f(0) = 13$.

Sol 2.1.5:

$$\begin{aligned}f(x) &= \int \left(4x^3 - 6x + 3\right)\, dx \\ &= x^4 - 3x^2 + 3x + C\end{aligned}$$

$$f(0) = 0^4 - 3(0)^2 + 3(0) + C$$

$$= 13 \therefore C = 13$$

$$\therefore \boxed{f(x) = x^4 - 3x^2 + 3x + 13}$$

Now, let's take a look at a type of problem called a *rectilinear motion* problem. In these problems, we study the motion of an object moving along a straight line—its position, velocity, and acceleration.

Ex 2.1.6: The acceleration of particle is described by $a(t) = 3t^2 + 8t + 1$. Find the distance equation for $x(t)$ if $v(0) = 3$ and $a(0) = 1$.

Sol 2.1.6:

$$v(t) = \int a(t)\,dt$$

$$= \int \left(3t^2 + 8t + 1\right)\,dt$$

$$= t^3 + 4t^2 + t + C_1$$

$$3 = (0)^3 + 4(0)^2 + (0) + C_1 \therefore 3 = C_1$$

$$v(t) = t^3 + 4t^2 + t + 3$$

$$x(t) = \int v(t)\,dt$$

$$= \int \left(t^3 + 4t^2 + t + 3\right)\,dt$$

$$= \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + C_2$$

$$1 = \frac{1}{4}(0)^4 + \frac{4}{3}(0)^3 + \frac{1}{2}(0)^2 + 3(0) + C_2 \therefore 1 = C_2$$

$$\boxed{x(t) = \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + 1}$$

Ex 2.1.7: The acceleration of a particle is described by $a(t) = 12t^2 - 6t + 4$. Find the distance equation for $x(t)$ if $v(1) = 0$ and $x(1) = 3$.

Sol 2.1.7:

$$
\begin{aligned}
v(t) &= \int a(t)\,dt \\
&= \int \left(12t^2 - 6t + 4\right) \\
&= 4t^3 - 3t^2 - 4t + C_1
\end{aligned}
$$

$$0 = 4(1)^3 - 3(1)^2 + 4(1) + C_1 \therefore -5 = C_1$$

$$v(t) = 4t^3 - 3t^2 - 4t - 5$$

$$
\begin{aligned}
x(t) &= \int v(t)\,dt \\
&= \int \left(4t^3 - 3t^2 - 4t - 5\right)\,dt \\
&= t^4 - t^3 - 2t^2 - 5t + C_2
\end{aligned}
$$

$$3 = (1)^4 - (1)^3 - 2(1)^2 - 5(1) + C_2 \therefore 6 = C_2$$

$$\boxed{x(t) = t^4 - t^3 - 2t^2 - 5t + 6}$$

The proof of all the transcendental integral rules can be left to a more formal Calculus course. But, since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

Transcendental Integral Rules

$$\int \cos(u)\,du = \sin(u) + C \qquad \int \csc(u)\cot(u)\,du = -\csc(u) + C$$

$$\int \sin(u)\,du = -\cos(u) + C \qquad \int \sec(u)\tan(u)\,du = \sec(u) + C$$

$$\int \sec^2(u)\,du = \tan(u) + C \qquad \int \csc^2(u)\,du = -\cot(u) + C$$

$$\int e^u\,du = e^u + C \qquad \int \frac{1}{u}\,du = \ln|u| + C$$

$$\int a^u\,du = \frac{a^u}{\ln|a|} + C \qquad \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$$

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C \qquad \int \frac{1}{u\sqrt{u^2-1}}\,du = sec^{-1} + C$$

Note that there are only three integrals that yield inverse trig functions, but there were six inverse trig derivatives. This is because the other three derivative rules are just the negatives of the first three.

Ex 2.1.8: $\int (\sin(x) + 3\cos(x))\ dx$

Sol 2.1.8:

$$\begin{aligned}\int (\sin(x) + 3\cos(x))\ dx &= \int \sin(x)\,dx + 3\int \cos(x)\,dx \\ &= \boxed{-\cos(x) + 3\sin(x) + C}\end{aligned}$$

Ex 2.1.9: $\int \left(e^x + 4 + 3\csc^2(x)\right)\ dx$

Sol 2.1.9:

$$\int \left(e^x + 4 + 3\csc^2(x)\right)\ dx = \int e^x\,dx + 4\int\ dx + 3\int \csc^2(x)\,dx$$

$$= \boxed{e^x + 4x - 3\cot(x) + C}$$

Now, let's take a look at some more complex integrals that yield inverse trig functions. These more general forms extend the earlier rules by introducing a constant a, and they are especially useful when working with substitutions or integrals that don't simplify neatly to the unit case.

Trig Inverse Integral Rules

$$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u^2 + a^2}\, du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$

Ex 2.1.10: Find $\displaystyle\int \frac{1}{x^2 + 4}\, dx$

Sol 2.1.10: All we need to do is apply our formula above.

$$\int \frac{1}{x^2 + 4}\, dx = \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

Ex 2.1.11: If $\dfrac{dy}{dx} = \sec(x)(\sec(x)\tan(x))$, find $y(x)$ if $y(0) = 0$.

Sol 2.1.11:

$$\begin{aligned} y &= \int (\sec(x)(\sec(x)\tan(x)))\, dx \\ &= \int \left(\sec^2(x)\right)\, dx + \int (\sec(x)\tan(x))\, dx \\ &= \tan(x) + \sec(x) + C \end{aligned}$$

$$0 = \tan(0) + \sec(0) + C$$

$$0 = 0 + 1 + C \therefore C = -1$$

$$\boxed{y = \tan(x) + \sec(x) - 1}$$

2.1 Free Response Homework

Perform the antidifferentiation.

1. $\displaystyle\int \left(6x^2 - 2x + 3\right)\, dx$

2. $\displaystyle\int \left(x^3 + 3x^2 - 2x + 4\right)\, dx$

3. $\displaystyle\int \frac{2}{\sqrt[3]{x}}\, dx$

4. $\displaystyle\int \left(8x^4 - 4x^3 + 9x^2 + 2x + 1\right)\, dx$

5. $\displaystyle\int x^3 \left(4x^2 + 5\right)\, dx$

6. $\displaystyle\int (4x - 1)(3x + 8)\, dx$

7. $\displaystyle\int \left(\sqrt{x} - \frac{6}{\sqrt{x}}\right)\, dx$

8. $\displaystyle\int \frac{x^2 + \sqrt{x} + 3}{x}\, dx$

9. $\displaystyle\int (x + 1)^3\, dx$

10. $\displaystyle\int (4x - 3)^2\, dx$

11. $\displaystyle\int \left(\sqrt{x} + 3\sqrt{3} - \frac{6}{\sqrt{x}}\right)\, dx$

12. $\displaystyle\int \frac{4x^3 + \sqrt{x} + 3}{x^2}\, dx$

13. $\displaystyle\int \left(x^2 + 5x + 6\right)\, dx$

14. $\displaystyle\int \frac{x^2 - 4x + 7}{x}\, dx$

15. $\displaystyle\int \frac{x^5 - 7x^3 + 2x - 9}{2x}\, dx$

16. $\displaystyle\int \frac{x^3 + 3x^2 + 3x + 1}{x + 1}\, dx$

17. $\displaystyle\int \left(y^2 + 5\right)^2\, dy$

18. $\displaystyle\int \left(4t^2 + 1\right)\left(3t^3 + 7\right)\, dt$

Complete the following problems.

19. $f'(x) = 3x^2 - 6x + 3$. Find $f(x)$ if $f(0) = 2$.

20. $f'(x) = x^3 + x^2 - x + 3$. Find $f(x)$ if $f(1) = 0$.

21. $f'(x) = (\sqrt{x} - 2)(3\sqrt{x} + 1)$. Find $f(x)$ if $f(4) = 1$.

22. The acceleration of a particle is described by $a(t) = 36t^2 - 12t + 8$. Find the distance equation for $x(t)$ if $v(1) = 1$ and $x(1) = 3$.

23. The acceleration of a particle is described by $a(t) = t^2 - 2t + 4$. Find the distance equation for $x(t)$ if $v(0) = 2$ and $x(0) = 4$.

2.1 Multiple Choice Homework

1. $\displaystyle\int \frac{1}{x^2}\,dx =$

   a) $\ln\left(x^2\right)+C$ b) $-\ln\left(x^2\right)+C$ c) $\dfrac{1}{x}+C$ d) $-\dfrac{1}{x}+C$ e) $-\dfrac{2}{x^3}+C$

---

2. $\displaystyle\int x\left(10+8x^4\right)\,dx =$

   a) $5x^2+\dfrac{4}{3}x^6+C$ b) $5x^2+\dfrac{8}{5}x^5+C$ c) $10x+\dfrac{4}{3}x^6+C$

   d) $5x^2+8x^6+C$ e) $5x^2+\dfrac{8}{7}x^6+C$

---

3. $\displaystyle\int x\sqrt{3x}\,dx$

   a) $\dfrac{2\sqrt{3}}{5}x^{\frac{5}{2}}+C$ b) $\dfrac{5\sqrt{3}}{2}x^{\frac{5}{2}}+C$ c) $\dfrac{\sqrt{3}}{2}x^{\frac{1}{2}}+C$ d) $2\sqrt{3x}+C$ e) $\dfrac{5\sqrt{3}}{2}x^{\frac{3}{2}}+C$

---

4. $\displaystyle\int (x-1)\sqrt{x}\,dx =$

   a) $\dfrac{3}{2}\sqrt{x}-\dfrac{1}{\sqrt{x}}+C$ b) $\dfrac{2}{3}x^{\frac{2}{3}}+\dfrac{1}{2}x^{\frac{1}{2}}+C$ c) $\dfrac{1}{2}x^2-x+C$

   d) $\dfrac{2}{5}x^{\frac{5}{2}}-\dfrac{2}{3}x^{\frac{3}{2}}+C$ e) $\dfrac{1}{2}x^2+2x^{\frac{2}{3}}+C$

---

5. A particle is moving upward along the y-axis until it reaches the origin and then it moves downward such that $v(t)=8-2t$ for $t\geq 0$. The position of the particle at time t is given by

   a) $y(t)=-t^2+8t-16$ b) $y(t)=-t^2+8t+16$ c) $y(t)=2t^2-8t-16$

   d) $8t-t^2$ e) $8t-2t^2$

---

6. If a particle's acceleration is given by $a(t)=12t+4$, and $v(1)=5$ and $y(0)=2$, then

$y(2) =$

a) 20 b) 10 c) 4 d) 16 e) 12

2.2: Integration by U-Substitution

Reversing the Power Rule was fairly easy. The other three core derivative rules—the Product Rule, the Quotient Rule, and the Chain Rule—are a little more complicated to undo. This is because they yield a more complicated function as a derivative, one which has several algebraic simplification steps. The integral of a rational function is particularly difficult to unravel because, as we have seen, rational derivatives can be obtained by differentiating a composite function with a log or a radical, or by differentiating another rational function. The same goes for reversing the Product Rule.

Key Idea: There is no single Product or Quotient Rule for integrals.

Instead, there are several techniques that apply in different situations, and it is not always obvious at the outset which one will be most effective. The choice depends on the algebraic manipulations that produced the product or quotient in the first place.

Products can be a result of:	Quotients can be the result of:
• The Chain Rule • Differentiating a product • Differentiating some trig functions	• Common denominators • Differentiating a quotient • Differentiating a log with a composite function • Differentiating some trig inverse functions

Composite functions are among the most pervasive functions in math. Therefore, we will start with undoing products and quotients that involve composites.

Remember:

$$\text{The Chain Rule: } \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

The derivative of a composite function often becomes a product of two functions: one part still composite and the other not. So, when we see a product in an integral, it may have originated from the Chain Rule. Unlike differentiation, however, integration in this case does not follow a fixed formula. Instead, it involves a process of substitution that sometimes works and sometimes does not. We make an informed guess and check whether it simplifies the integral. In later parts of Calculus, you will learn additional techniques to use when this approach is not successful.

Steps to Integration by U-Substitution

1. Make sure that you are integrating a product or quotient.

2. Identify the inner function of the composite and set u equal to it.

3. Differentiate u to find du in terms of dx.

4. Adjust the integral by multiplying and dividing by a constant if needed so a factor matches du. [See **Ex 2.2.2**]

5. Rewrite the integral entirely in terms of u and du.

6. Integrate using the power rule or other appropriate rules.

7. Substitute back the original x-expression for u.

This is one of those mathematical processes that makes little sense when first seen. But, after seeing several examples, the meaning should become clear. *Be patient!*

OBJECTIVES

Use U-Substitution to Integrate Composite Expressions.

Ex 2.2.1: $\int 3x^2\left(x^3+5\right)^{10}\,dx$

Sol 2.2.1:

$\left(x^3+5\right)$ is the inner function.

$$\hookrightarrow u = x^3+5$$

$$\hookrightarrow du = 3x^2\,dx$$

$$\begin{aligned}\int 3x^2\left(x^3+5\right)^{10}\,dx &= \int u^{10}\,du\\ &= \frac{u^{11}}{11}+C\\ &= \boxed{\frac{1}{11}\left(x^3+5\right)^{11}}\end{aligned}$$

Ex 2.2.2: $\int x\left(x^2+5\right)^3\,dx$

Sol 2.2.2:

$$\left(x^2+5\right) \text{ is the inner function.}$$

$$\hookrightarrow\ u = x^2+5$$

$$\hookrightarrow\ du = 2x\,dx$$

$$\begin{aligned}\int x\left(x^2+5\right)^3\,dx &= \frac{1}{2}\int (2x)\left(x^2+5\right)^3\,dx \\ &= \frac{1}{2}\int u^3\,du \\ &= \frac{1}{2}\cdot\frac{u^4}{4}+C \\ &= \boxed{\frac{1}{8}\left(x^2+5\right)^4+C}\end{aligned}$$

Notice how the factor of 2 from $du = 2x\,dx$ is accounted for by multiplying by $\frac{1}{2}$ when substituting. This ensures the integral is correctly expressed in terms of u.

Ex 2.2.3: $\displaystyle\int \left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\,dx$

Sol 2.2.3:

$$\sqrt[4]{x^4+2x^2-5} \text{ is the inner function.}$$

$$\hookrightarrow\ u = x^4+2x^2-5$$

$$\hookrightarrow\ du = \left(4x^3+4x\right)\,dx = 4\left(x^3+x\right)\,dx$$

$$\begin{aligned}\int \left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\,dx &= \frac{1}{4}\int 4\left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\,dx \\ &= \frac{1}{4}\int \sqrt[4]{u}\,du \\ &= \frac{1}{4}\cdot\frac{4u^{\frac{5}{4}}}{5}+C \\ &= \boxed{\frac{1}{5}\left(x^4+2x^2-5\right)^{\frac{5}{4}}+C}\end{aligned}$$

Ex 2.2.4: $\displaystyle\int \frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3}\,dx$

Sol 2.2.4:

x^3+2x^2-5x+2 is the inner function.

$\hookrightarrow\ u = x^3+2x^2-5x+2$

$\hookrightarrow\ du = \left(3x^2+4x-5\right)\,dx$

$$\begin{aligned}\int \frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3}\,dx &= \int \frac{1}{u^3}\,du \\ &= -\frac{1}{2}u^{-2}+C \\ &= \boxed{-\frac{1}{2}\left(x^3+2x^2-5x+2\right)^{-2}+C}\end{aligned}$$

Of course, u-substitution will apply to the transcendental functions as well.

Ex 2.2.5: $\displaystyle\int \sin(5x)\,dx$

Sol 2.2.5:

$\hookrightarrow\ u = 5x$

$\hookrightarrow\ du = 5\,dx$

$$\begin{aligned}\int \sin(5x)\,dx &= \frac{1}{5}\int 5\sin(5x)\,dx\\ &= \frac{1}{5}\int \sin(u)\,du\\ &= \frac{1}{5}\cdot(-\cos(u))+C\\ &= \boxed{-\frac{1}{5}\cos(5x)+C}\end{aligned}$$

Ex 2.2.6: $\int \sin^6(x)\cos(x)\,dx$

Sol 2.2.6:

$$\hookrightarrow u=\sin(x)$$
$$\hookrightarrow du=\cos(x)\,dx$$

$$\begin{aligned}\int \sin^6(x)\cos(x)\,dx &= \int u^6\,du\\ &= \frac{1}{7}u^7+C\\ &= \boxed{\frac{1}{7}\sin^7(x)+C}\end{aligned}$$

Ex 2.2.7: $\int x^5\sin\left(x^6\right)\,dx$

Sol 2.2.7:

$$\hookrightarrow u=x^6$$
$$\hookrightarrow du=6x^5\,dx$$

$$\int x^5 \sin\left(x^6\right)\, dx = \frac{1}{6}\int 6x^5 \sin\left(x^6\right)\, dx$$

$$= \frac{1}{6}\int \sin(u)\, du$$

$$= -\frac{1}{6}\cos(u) + C$$

$$= \boxed{-\frac{1}{6}\cos\left(x^6\right) + C}$$

Ex 2.2.8: $\int \cot^3(x)\csc^2(x)\, dx$

Sol 2.2.8

$$\hookrightarrow\ u = \cot(x)$$

$$\hookrightarrow\ du = -\csc^2(x)\, dx$$

$$\int \cot^3(x)\csc^2(x)\, dx = -\int \cot^3(x)\left(-\csc^2(x)\right)\, dx$$

$$= -\int u^3\, du$$

$$= -\frac{1}{4}u^4\, du + C$$

$$= \boxed{-\frac{1}{4}\cot^4(x) + C}$$

Ex 2.2.9: $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\, dx$

Ex 2.2.9:

$$\hookrightarrow\ u = \sqrt{x}$$

$$\hookrightarrow\ du = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx = 2\int (\cos(\sqrt{x}))\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\,dx$$

$$= 2\int \cos(u)\,du$$

$$= 2\sin(u) + C$$

$$= \boxed{2\sin(\sqrt{x}) + C}$$

Ex 2.2.10: $\int xe^{x^2+1}\,dx$

Sol 2.2.10:

$$\hookrightarrow\; u = x^2 + 1$$

$$\hookrightarrow\; du = 2x\,dx$$

$$\int xe^{x^2+1}\,dx = \frac{1}{2}\int (2x)e^{x^2+1}\,dx$$

$$= \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C$$

$$= \boxed{\frac{1}{2}e^{x^2+1} + C}$$

Ex 2.2.11: $\int \frac{x}{\sqrt{1-x^4}}\,dx$

Sol 2.2.11:

$$\hookrightarrow\; u = x^2$$

$$\hookrightarrow\; du = 2x\,dx$$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int (2x)\frac{1}{\sqrt{1-(x^2)^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\sin^{-1}(u) + C$$

$$= \boxed{\frac{1}{2}\sin^{-1}\left(x^2\right) + C}$$

Ex 2.2.12: $\int \left(xe^{x^2} + 4x^2 - 3\sin(5x)\right)\,dx$

Ex 2.2.12:

$$\int \left(xe^{x^2} + 4x^2 - 3\sin(5x)\right)\,dx = \int xe^{x^2}\,dx + \int 4x^2\,dx - \int 3\sin(5x)\,dx$$

$\hookrightarrow u_1 = x^2$ $\qquad$ $\hookrightarrow u_2 = 5x$

$\hookrightarrow du_1 = 2x\,dx$ $\qquad$ $\hookrightarrow du_2 = 5\,dx$

$$\int xe^{x^2}\,dx + \int 4x^2\,dx - \int 3\sin(5x)\,dx = \frac{1}{2}\int (2x)e^{x^2}\,dx + 4\int x^2\,dx - \frac{3}{5}\int 5\sin(5x)\,dx$$

$$= \frac{1}{2}\int e^{u_1}\,du_1 + 4\int x^2\,dx - \frac{3}{5}\int \sin\left(u_2\right)\,du_2$$

$$= \frac{1}{2}e^{u_1} + 4\frac{x^3}{3} - \frac{3}{5}(-\cos(u_2)) + C$$

$$= \boxed{\frac{1}{2}e^{x^2} + \frac{4}{3}x^3 + \frac{3}{5}\cos(5x) + C}$$

2.2 Free Response Homework Set A

Perform the antidifferentiation.

1. $\int (5x+3)^3 \, dx$

2. $\int \left(x^3\left(x^4+5\right)\right)^{24} dx$

3. $\int \left(1+x^3\right)^2 dx$

4. $\int (2-x)^{\frac{2}{3}} \, dx$

5. $\int x\sqrt{2x^2+3}\, dx$

6. $\int \frac{1}{(5x+2)^3}\, dx$

7. $\int \frac{x^3}{\sqrt{1+x^4}}\, dx$

8. $\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\, dx$

9. $\int \left(x^5-\sin(3x)+xe^{x^2}\right) dx$

10. $\int \left(x^2\sec^2\left(x^3\right)+\frac{\ln^3 x}{x}\right) dx$

11. $\int x^4\cos\left(x^5\right) dx$

12. $\int \sin(7x+1)\, dx$

13. $\int \sec^2(3x-1)\, dx$

14. $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$

15. $\int \tan^4(x)\sec^2(x)\, dx$

16. $\int \frac{\ln x}{x}\, dx$

17. $\int e^{6x}\, dx$

18. $\int \frac{\cos(2x)}{\sin^3(2x)}\, dx$

19. $\int \frac{x\ln\left(x^2+1\right)}{x^2+1}\, dx$

20. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

21. $\int \sqrt{\cot(x)}\csc^2(x)\, dx$

22. $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right) dx$

23. $\int \frac{x}{1+x^4}\, dx$

24. $\int \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}\, dx$

2.2 Free Response Homework Set B

Perform the antidifferentiation.

1. $\int (2x+5)\left(x^2+5x+6\right)^6 dx$

2. $\int 3t^2\left(t^3+1\right)^5 dt$

3. $\int \frac{10m+15}{\sqrt[4]{m^2+3m+1}}\, dm$

4. $\int \frac{3x^2}{(1+x^3)^5}\, dx$

5. $\int (4s+1)^5\, ds$

6. $\int \frac{5t}{t^2+1}\, dt$

7. $\int \frac{3m^2}{m^3+8}\, dm$

8. $\int (181x+1)^5\, dx$

9. $\int \frac{v^2}{5-v^3}\, dv$

10. $\int \left(x^7 - \cot(5x) + xe^{x^2}\right)\, dx$

11. $\int \frac{\cos(x)}{1+\sin(x)}\, dx$

12. $\int \left(x^2 \sec^2\left(4x^3\right) + 2xe^{x^2}\right)\, dx$

13. $\int \sec^2(2x)\, dx$

14. $\int \frac{\csc^2\left(e^{-x}\right)}{e^x}\, dx$

15. $\int \frac{\sec(\ln x)\tan(\ln x)}{3x}\, dx$

16. $\int \left(x^5 + \frac{7}{x^2} - e^{2x} + \sec^2(x)\right)\, dx$

17. $\int e^x \csc\left(e^x\right)\cot\left(e^x\right)\, dx$

18. $\int \left(e^x-2\right)\left(e^x-1\right)\, dx$

19. $\int x^2 \sin\left(x^3\right)\, dx$

20. $\int te^{5t^2+1}\, dt$

21. $\int \left(e^y+1\right)^2\, dy$

22. $\int x\sec^2\left(x^2\right)\sqrt{\tan\left(x^2\right)}\, dx$

23. $\int \sin(3t)\cos^5(3t)\, dt$

24. $\int x\cos\left(x^2\right)e^{\sin\left(x^2\right)}\, dx$

25. $\int \tan(\theta)\ln(\sec(\theta))\, d\theta$

26. $\int \left(e^{4y} + 2y^2 - 7\cos(3y)\right)\, dy$

27. $\int \frac{\sin(x+4)}{\cos^7(x+4)}\, dx$

28. $\int \left(\frac{2x}{x^2+5} - \sec^2(3x) + xe^{x^2} - \pi\right)\, dx$

29. $\int e^{2t}\sec^2\left(e^{2t}\right)\, dt$

30. $\int \frac{18\ln m}{m}\, dm$

31. $\int \sec^2(y)\tan^5(y)$. Verify your answer by taking the derivative.

32. $\int \left(\cos(\theta)e^{\sin(\theta)} + \frac{\theta}{\theta^2+1}\right)\, d\theta$. Verify your answer by taking the derivative.

33. $\int t\sec^2\left(4t^2\right)\sqrt{\tan\left(4t^2\right)}\, dt$. Verify your answer by taking the derivative.

34. $\int \frac{2y\cos\left(y^2\right)}{\sin^4\left(y^2\right)}\, dy$. Verify your answer by taking the derivative.

2.2 Multiple Choice Homework

1. $\displaystyle\int \frac{x}{x^2-4}\,dx =$

a) $-\dfrac{1}{4\left(x^2-4\right)^2}+C$ b) $\dfrac{1}{2\left(x^2-4\right)}+C$ c) $\dfrac{1}{2}\ln\left|x^2-4\right|+C$

d) $2\ln\left|x^2-4\right|+C$ e) $\dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right)+C$

---

2. $\displaystyle\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx =$

a) $\ln\left(\sqrt{x}\right)+C$ b) $x+C$ c) e^x+C d) $\dfrac{1}{2}e^{2\sqrt{x}}+C$ e) $e^{\sqrt{x}}+C$

---

3. When using the substitution $u=\sqrt{1+x}$, an antiderivative of $\displaystyle\int 60x\sqrt{1+x}\,dx$ is.

a) $20u^3-60u+C$ b) $15u^4-30u^2+C$ c) $30u^4-60u^2+C$

d) $24u^5-40u^3+C$ e) $12u^6-20u^4+C$

---

4. $\displaystyle\int \frac{3x^2}{\sqrt{x^3+3}}\,dx =$

a) $2\sqrt{x^3+3}+C$ b) $\dfrac{3}{2}\sqrt{x^3+3}+C$ c) $\sqrt{x^3+3}+C$

d) $\ln\left(\sqrt{x^3+3}\right)+C$ e) $\ln\left(x^3+3\right)+C$

---

5. $\displaystyle\int x\left(x^2-1\right)^4\,dx =$

a) $\dfrac{1}{10}x^2\left(x^2-1\right)^5+C$ b) $\dfrac{1}{10}\left(x^2-1\right)^5+C$ c) $\dfrac{1}{5}\left(x^3-x\right)^5+C$

d) $\dfrac{1}{5}\left(x^2-1\right)^5+C$ e) $\dfrac{1}{5}\left(x^2-x\right)^5+C$

---

6. $\displaystyle\int 4x^2\sqrt{3+x^3}\,dx =$

a) $\dfrac{16\left(3+x^3\right)^{\frac{3}{2}}}{9}+C$ b) $\dfrac{8\left(3+x^3\right)^{\frac{3}{2}}}{9}+C$ c) $\dfrac{8\left(3+x^3\right)^{\frac{3}{2}}}{3}+C$

d) $\dfrac{4}{3\left(3+x^3\right)^{\frac{1}{2}}}+C$ e) $\dfrac{8}{3\left(3+x^3\right)^{\frac{1}{2}}}+C$

---

7. $\displaystyle\int \left(x^3+2+\frac{1}{x^2+1}\right)dx =$

a) $\dfrac{x^4}{4}+2x+\tan^{-1}(x)+C$ b) $x^4+2+\tan^{-1}(x)+C$ c) $\dfrac{x^4}{4}+2x+\dfrac{3}{x^3+3}+C$

d) $\dfrac{x^4}{4}+2x+\tan^{-1}\left(2x^2\right)+C$ e) $4+2x+\tan^{-1}(x)+C$

---

8. $\displaystyle\int \cos(3-2x)\,dx =$

a) $\sin(3-2x)+C$ b) $-\sin(3-2x)+C$ c) $\dfrac{1}{2}\sin(3-2x)+C$

d) $-\sin(3-2x)+C$ e) $-\dfrac{1}{5}\sin(3-2x)+C$

---

9. $\displaystyle\int \frac{x-2}{x-1}\,dx =$

a) $-\ln|x-1|+C$ b) $x+\ln|x-1|+C$ c) $x-\ln|x-1|+C$

d) $x+\sqrt{x-1}+C$ e) $x-\sqrt{x-1}+C$

---

10. $\displaystyle\int \frac{e^{x^2}-2x}{e^{x^2}}\,dx =$

a) $x-e^{x^2}+C$ b) $x-e^{-x^2}+C$ c) $x+e^{-x^2}+C$ d) $-e^{x^2}+C$ e) $-e^{-x^2}+C$

---

11. $\displaystyle\int 6\sin(x)\cos^2(x)\,dx =$

a) $2\sin^3(x)+C$ b) $-2\sin^3(x)+C$ c) $2\cos^3(x)+C$

d) $-2\cos^3(x)+C$ e) $3\sin^2(x)\cos^2(x)+C$

---

12. $\displaystyle\int \frac{4x}{1+x^2}\,dx =$

a) $4\arctan(x)+C$ b) $\dfrac{4}{x}\arctan(x)+C$ c) $\dfrac{1}{2}\ln\left(1+x^2\right)+C$

d) $2\ln\left(1+x^2\right)+C$ e) $2x^2+4\ln|x|+C$

---

13. $\displaystyle\int \frac{x}{4+x^2}\,dx =$

a) $\tan^{-1}\left(\dfrac{x}{2}\right)+C$ b) $\ln\left(4+x^2\right)+C$ c) $\tan^{-1}(x)+C$

d) $\dfrac{1}{2}\ln\left(4+x^2\right)+C$ e) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+C$

---

14. $\displaystyle\int \left(2^x-4e^{2\ln x}\right)\,dx =$

a) $2^x\ln 2-\dfrac{4}{3}e^{2\ln x}+C$ b) $x2^{x-1}-\dfrac{4}{3}x^3+C$ c) $\dfrac{2^x}{\ln 2}-\dfrac{4}{3}e^{2\ln x}+C$

d) $x2^{x-1}-\dfrac{4}{3}e^{2\ln x}+C$ e) $\dfrac{2^x}{\ln 2}-\dfrac{4}{3}x^3+C$

---

15. The antiderivative of $2\tan(x)$ is:

a) $2\ln|\sec(x)|+C$ b) $2\sec^2(x)+C$ c) $\ln|\sec^2(x)|+C$

d) $2\ln|\cos(x)|+C$ e) $\ln|2\sec(x)|+C$

---

16. Which of the following statements are true?

I. $\displaystyle\int x^4\sin\left(x^5\right)\,dx = -\frac{1}{5}\cos\left(x^5\right)+C$

II. $\displaystyle\int \tan(x)\,dx = \sec^2(x) + C$

III. $\displaystyle\int \left(x^3 + x\right) \sqrt[4]{x^4 + 2x^2 - 5}\,dx = \frac{1}{5}\left(x^4 + 2x^2 - 5\right)^{\frac{5}{4}} + C$

a) I only b) II only c) III only d) I and II only

e) II and III only f) I and III only g) I, II, and III h) None of these

---

17. If $x'(t) = 2t\cos\left(t^2\right)$, find $x(t)$ when $x\left(\sqrt{\frac{\pi}{2}}\right) = 3$

a) $x(t) = -4t^2 \sin\left(t^2\right)$ b) $x(t) = -4t^2 \sin\left(t^2\right) + 2\cos\left(t^2\right)$ c) $x(t) = \sin\left(t^2\right) + 3$

d) $x(t) = -\sin\left(t^2\right) + 4$ e) $x(t) = \sin\left(t^2\right) + 2$

---

18. A particle moves along the y-axis so that at any time $t \geq 0$, its velocity is given $v(t) = \sin(2t)$. If the position of the particle at time $t = \frac{\pi}{2}$ is $y = 3$. What is the particle's position at time $t = 0$?

a) -4 b) 2 c) 3 d) 4 e) 6

2.3 Separable Differential Equations

Differential Equation → Definition: An equation that contains a derivative.

General Solution → Definition: All of the y-equations that would have the given equation as their derivative. Note the $+C$ which gives multiple equations.

Initial Condition → Definition: Constraint placed on a differential equation; sometimes called an initial value.

Particular Solution → Definition: Solution obtained from solving a differential equation when an initial condition allows you to solve for C.

Separable Differential Equation → Definition: A differential equation in which all terms with y's can be moved to the left side of an equals sign (=), and in which all terms with x's can be moved to the right side of an equals sign (=), by multiplication and division only.

Let's take a look at some examples of separable differential equations.

Ex 2.3.1: Separate the variables in the following differential equations:

a) $\dfrac{dy}{dx} = -\dfrac{x}{y}$

b) $\dfrac{dy}{dx} = x\sec(y)$

c) $y' = 2xy - 3y$

Sol 2.3.1:
a)

$$(y)\frac{dy}{dx} = -\frac{x}{y}(y)$$

$$(dx)\frac{y\,dy}{dx} = -x(dx)$$

$$\boxed{y\,dy = -x\,dx}$$

b)

$$\left(\frac{1}{\sec(y)}\right)\frac{dy}{dx} = x\sec(y)\left(\frac{1}{\sec(y)}\right)$$

$$(dx)\cos(y)\frac{dy}{dx} = x(dx)$$

$$\boxed{\cos(y)\,dy = x\,dx}$$

c)

$$\frac{dy}{dx} = (2x-3)y$$

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = (2x-3)y\left(\frac{1}{y}\right)$$

$$(dx)\left(\frac{1}{y}\right)\frac{dy}{dx} = (2x-3)(dx)$$

$$\boxed{\frac{1}{y}\,dy = (2x-3)\,dx}$$

OBJECTIVES

Given a Separable Differential Equation, Find the General Solution.
Given a Separable Differential Equation and an Initial Condition, Find a Particular Solution.

Steps to Solving Differential Equations

1. Separate the variables. Move all terms involving y (and dy) to one side and all terms involving x (and dx) to the other. Keep any constants on the right side of the equation.

2. Integrate both sides. Keep $+C$ only on the right side of the equation.

3. Solve for y, if possible. If the integration produces a natural log, isolate y. If not, solve for C. Note: $e^{\ln|y|} = y$, as e raised to any power is positive.

4. Apply initial conditions (if given). Substitute initial values to solve for C.

Ex 2.3.2: Find the general solution to the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

Sol 2.3.2:

$\frac{dy}{dx} = -\frac{x}{y}$	Start here.
$y\,dy = -x\,dx$	Separate all the y terms to the left side of the equation and all of the x terms to the right side.
$\int y\,dy = \int -x\,dx$	Integrate both sides.
$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$	You only need C on one side of the equation.
$y^2 = -x^2 + C$	Multiply both sides by 2. Note that $2C$ is still a constant, so we'll continue to denote it as C.
$x^2 + y^2 = C$	This is the family of circles with radius $\sqrt{C}$ centered at the origin.
$y = \pm\sqrt{C - x^2}$	Isolate y.

Since we usually solve our equation for y, our solution will be $\boxed{y = \pm\sqrt{C - x^2}}$.

Also, note that we can check our solution by taking its derivative.

$$x^2 + y^2 = C$$

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[C]$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \checkmark$$

Ex 2.3.3: Find the general solution to the differential equation $\frac{dm}{dt} = mt$

Sol 2.3.2:

$\frac{dm}{dt} = mt$	Start here.

$\frac{1}{m}\,dm = t\,dt$	Separate all the m terms to the left side of the equation and all of the t terms to the right side of the equation.
$\int \frac{1}{m}\,dm = \int t\,dt$	Integrate both sides.
$\ln\lvert m\rvert = \frac{1}{2}t^2 + C$	You only need C on one side of the equation.
$e^{\ln\lvert m\rvert} = e^{\frac{1}{2}t^2+C}$	e both sides of the equation to solve for y.
$m = e^{\frac{1}{2}t^2}e^C$	Pull out the constants from the equation.
$m = Ke^{\frac{1}{2}t^2}$	e^C is still a constant, which we will just denote as K.

Ex 2.3.4: Find the particular solution to $y' = 2xy - 3y$, given $y(3) = 2$.

Sol 2.3.4: To find the particular solution, recall that we first need to find the general solution.

$$\begin{aligned}
\frac{dy}{dx} &= (2x-3)y \\
\int \frac{1}{y}\,dy &= \int (2x-3)\,dx \\
\ln\lvert y\rvert &= x^2 - 3x + C \\
y &= e^{x^2-3x+C} \\
&= e^{x^2-3x}e^C \\
&= Ke^{x^2-3x}
\end{aligned}$$

Now, let's plug in our initial value to solve for K.

$$y(3) = 2 \therefore Ke^{3^2-3(3)} = 2 \therefore Ke^0 = 2 \therefore K = 2$$

So, our particular solution is $\boxed{y = 2e^{x^2-3x}}$.

Ex 2.3.5: Find the particular solution to $\frac{dy}{dx} = x^2 y$, given $y(0) = -2$.

Sol 2.3.5: Once again, let's find the general solution first.

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{1}{y} = \int x^2\, dx$$

$$\ln|y| = \frac{1}{3}x^3 + C$$

$$y = e^{\frac{1}{3}x^3 + C}$$

$$= e^{\frac{1}{3}}x^3 e^C$$

$$= Ke^{\frac{1}{3}}$$

Now, let's find K.

$$y(0) = -2 \therefore Ke^{\frac{1}{3}(0)} = -2 \therefore Ke^0 = -2 \therefore K = -2$$

So, our particular solution is $\boxed{y = -2e^{\frac{1}{3}x^3}}$.

Ex 2.3.6: Find the particular solution to $\frac{dy}{dx} = x^2 y^3$, given $y(0) = 1$.

Sol 2.3.6:

$$\frac{dy}{dx} = x^2 y^3$$

$$\int \frac{1}{y^3}\, dy = \int x^2\, dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + C$$

Usually, at this step, we would isolate y to find the general solution. However, notice that it's easier to solve for C here, because solving for y will likely result in nested fractions, which can be difficult to work with.

$$y(0) = 1 \therefore -\frac{1}{2(1)^2} = \frac{0^2}{2} + C \therefore C = -\frac{1}{2}$$

Now, we can continue solving for the particular solution.

$$-\frac{1}{2y^2} = \frac{x^2}{2} - \frac{1}{2}$$

$$\frac{1}{y^2} = -x^2 + 1$$

$$y^2 = \frac{1}{1 - x^2}$$

$$y = \frac{1}{\pm\sqrt{1 - x^2}}$$

Since $x = 0$ gave us $y = +1$, our particular solution must be:

$$\boxed{y = \frac{1}{\sqrt{1 - x^2}}}$$

Ex 2.3.7: Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = \dfrac{y+1}{x^2+9}$ and suppose the point $(0, -3)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Determine if the point $(0, -3)$ is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = \dfrac{y+1}{x^2+9}$ at $(0, -3)$.

Sol 2.3.7:
a)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{y+1}{x^2+9}\right]$$

$$= \frac{(x^2+9)\dfrac{dy}{dx} - (y+1)(2x)}{(x^2+9)^2}$$

$$= \frac{(x^2+9)\left(\dfrac{y+1}{x^2+9}\right) - (y+1)(2x)}{(x^2+9)}$$

$$= \frac{(y+1)-(y+1)(2x)}{(x^2+9)^2}$$

$$= \boxed{\frac{(y+1)(1-2x)}{(x^2+9)}}$$

b)

$$\left.\frac{d^2y}{dx^2}\right|_{(0,-3)} = \frac{(-3)+1}{(0)^2+9}$$

$$= -\frac{2}{9}$$

Because the derivative at the point $(0, -3)$ is not equal to zero, the point is $\boxed{\text{neither a maximum nor a minimum}}$.

c)

$$\frac{dy}{dx} = \frac{y+1}{x^2+9}$$

$$\int \frac{1}{y+1}\,dy = \int \frac{1}{x^2+9}\,dx$$

$$\ln|y+1| = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$y+1 = e^{\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C}$$

$$y = Ke^{\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)} - 1$$

$$y(0) = -3 \therefore -3 = Ke^{\frac{1}{3}\tan^{-1}\left(\frac{(0)}{3}\right)} - 1 \therefore K = -2$$

$$\boxed{y = -2e^{\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)} - 1}$$

2.3 Free Response Homework

Separate the variables for the following differential equations.

1. $\dfrac{dy}{dx} = \dfrac{y}{x}$

2. $\dfrac{dy}{dx} = xy^2$

3. $\left(x^2 + 1\right)\dfrac{dy}{dx} = xy$

4. $\dfrac{dy}{dt} = \dfrac{\sec^2(t)}{\sec(y)\tan(y)}$

5. $\dfrac{dv}{dt} = 2 + 2v + t + tv$

6. $\dfrac{dy}{dx} = \dfrac{x^2+1}{\sec(y)\tan(y)}$

7. $\dfrac{dy}{dt} = \dfrac{t}{y\sqrt{y^2+1}}$

8. $\dfrac{dy}{dx} = \dfrac{y^2+1}{xy}$

Find the particular solutions to these differential equations with the given initial conditions.

9. $\dfrac{dy}{dx} = \dfrac{3y^2}{1+x^2} \quad\Big|\quad y(1) = 5$

10. $\dfrac{dy}{dx} = \dfrac{x^2\sqrt{x^3-2}}{y^2} \quad\Big|\quad y(3) = 0$

11. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{4y^3} \quad\Big|\quad y(0) = 1$

12. $\dfrac{dy}{dx} = y^2\cos(x) \quad\Big|\quad y\left(\dfrac{\pi}{2}\right) = 1$

13. $\dfrac{dy}{dx} = 4xy^3 \quad\Big|\quad y(0) = -2$

14. $\dfrac{d\theta}{dr} = \dfrac{1+\sqrt{r}}{\sqrt{\theta}} \quad\Big|\quad \theta(4) = 9$

15. $\dfrac{dy}{dx} = 8xy \quad\Big|\quad y(0) = 5$

16. $\dfrac{dy}{dx} = 3y(2x+1) \quad\Big|\quad y(0) = 1$

17. $\dfrac{dy}{dx} = 6x^2y^2 \quad\Big|\quad y(1) = 3$

18. $\dfrac{dy}{dx} = y^2 + 1 \quad\Big|\quad y(1) = 0$

19. $\dfrac{dy}{dx} = \dfrac{2x^3}{3y^2} \quad\Big|\quad y\left(\sqrt{2}\right) = 0$

20. $\dfrac{dy}{dx} = \dfrac{\cos(x)}{y} \quad\Big|\quad y\left(\dfrac{\pi}{2}\right) = 3$

21. $\dfrac{dy}{dx} = xy^2 \quad\Big|\quad y(0) = 5$

22. $\dfrac{dy}{dx} = \dfrac{\sec(y)}{x} \quad\Big|\quad y(1) = \dfrac{\pi}{2}$

23. $\dfrac{dy}{dx} = y^2 + 1 \quad\Big|\quad y(1) = 0$

24. $\dfrac{dy}{dx} = \dfrac{2x}{y} \quad\Big|\quad y(0) = 1$

25. $\dfrac{du}{dt} = \dfrac{2t + \sec^2(t)}{2u} \quad\Big|\quad u(0) = -5$

26. $\dfrac{dy}{dx}\left(x^2+1\right)(2-y) \quad\Big|\quad y(1) = 3$

27. $\dfrac{dy}{dx} = \dfrac{y^2+1}{xy} \quad\Big|\quad y(0) = -1$

28. $\dfrac{dy}{dx} = \dfrac{xy^2+x}{y} \quad\Big|\quad y(0) = -1$

29. $\dfrac{dy}{dx} = \dfrac{\sin(x)}{\sin(y)} \quad \Big| \quad y(0) = \dfrac{\pi}{2}$

30. $\dfrac{dy}{dx} = yx - y\sin(x) \quad \Big| \quad y(0) = 5e$

31. Solve the equation $e^{-y}\dfrac{dy}{dx} + \cos(x) = 0$.

32. Find an equation of the curve that satisfies $\dfrac{dy}{dx} = 4x^3y$ and whose y-intercept is 7.

33. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = y^2(6 - 2x)$, and suppose the point $\left(3, -\dfrac{1}{3}\right)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Use the solution to a) to determine if the point $\left(3, -\dfrac{1}{3}\right)$ is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = y^2(6 - 2x)$ at $\left(3 - \dfrac{1}{3}\right)$.

34. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = xy + y$, and suppose the point $(-1, 2)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Use the solution to a) to determine if the point $(-1,\ 2)$ is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = xy + y$ at $(-1,\ 2)$.

35. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = \dfrac{3x^2}{y+2}$, and suppose the point $(-1, 2)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

(b) Use the solution to a) to determine if the point $(0,\ 1)$ is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\dfrac{dy}{dx} = \dfrac{3x^2}{y+2}$ at $(0,\ 1)$.

36. Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = (x-1)(y+2)$, and suppose the point $(1,0)$ is on the graph of $y = f(x)$.

(a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$.

(b) Use the solution to a) to determine if the point (1, 3) is at a maximum, a minimum, or neither.

(c) Find the particular solution to $\frac{dy}{dx} = (x-1)(y+2)$ at (1, 3).

2.3 Multiple Choice Homework

1. If $\frac{dy}{dx} = \sin(x)\cos^3(x)$ and if $y = 1$ when $x = \pi$, what is the value of y when $x = 0$?

a) 0 b) 1 c) 1.5 d) 2 e) 2.5

---

2. If $\frac{dy}{dx} = \cos(x)\sin^2(x)$ and if $y = 0$ when $x = \pi$, what is the value of y when $x = 0$?

a) -1 b) $-\frac{1}{3}$ c) 0 d) $\frac{1}{3}$ e) 1

---

3. The solution to the differential equation $\frac{dy}{dx} = 8xy$ with initial condition $y(0) = 5$ is

a) $\ln\left(4x^2+5\right)$ b) $e^{4x^2}+5$ c) $e^{4x^2}+4$ d) $5\ln\left(4x^2\right)$ e) $5e^{4x^2}$

---

4. Identify the first mistake (if any) in this process:

Problem:	$\frac{dy}{dx} = xy + x$
Step 1:	$\frac{1}{y+1}\,dy = x\,dx$
Step 2:	$\ln\lvert y+1\rvert = x^2 + C$
Step 3:	$y + 1 = e^{x^2} + C$
Step 4:	$y = e^{x^2} + C$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

---

5. Identify the first mistake (if any) in this process:

Problem:	$\dfrac{dy}{dx} = 6x^2y^2$
Step 1:	$\dfrac{1}{y^2}\,dy = 6x^2\,dx$
Step 2:	$\ln\lvert y^2\rvert = 2x^3 + C$
Step 3:	$y^2 = e^{2x^3+C}$
Step 4:	$y = \pm\sqrt{Ke^{2x^3}}$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

2.4 Integration by Back-Substitution

Sometimes when applying the Chain Rule, the other factor is not the du, or there are extra $x's$ that must be replaced with some form of u. The method of choosing u to equal the inside of the composite function remains the same, but there is more substitution necessary.

Steps to Integration by Back-Substitution

1. Find u and du, just as with u-substitution.

2. Handle extra x's. Identify any remaining "extra" x terms, and express x in terms of u.

3. Replace all x-expressions in the integral with u and du.

4. Integrate appropriately, and replace u with the original x-expression to get the final answer

All of this is best understood with some examples.

Ex 2.4.1: $\displaystyle\int x^3\left(x^2+4\right)^{\frac{3}{2}}\,dx$

Sol 2.4.1:

$$\hookrightarrow u = x^2+4 \therefore x^2 = u-4$$

$$\hookrightarrow du = 2x\,dx$$

$$\begin{aligned}\int x^3\left(x^2+4\right)^{\frac{3}{2}}\,dx &= \frac{1}{2}\int (2x)x^2\left(x^2+4\right)^{\frac{3}{2}}\,dx\\ &= \frac{1}{2}\int (u-4)u^{\frac{3}{2}}\,du\\ &= \frac{1}{2}\int\left(u^{\frac{5}{2}}-4u^{\frac{3}{2}}\right)du\\ &= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}}-\frac{8}{5}u^{\frac{5}{2}}\right)+C\\ &= \boxed{\frac{1}{7}\left(x^2+4\right)^{\frac{7}{2}}-\frac{4}{5}\left(x^2+4\right)^{\frac{5}{2}}+C}\end{aligned}$$

Notice how when we attempted u-substitution, an x^2 remained in the equation. That is the reason why we expressed x^2 in terms of u.

Ex 2.4.2: $\displaystyle\int (x+1)\sqrt{x-1}\,dx$

Sol 2.4.2:

$$\hookrightarrow u = x - 1 \therefore x = u + 1$$

$$\hookrightarrow du = dx$$

$$\begin{aligned}
\int (x+1)\sqrt{x-1}\,dx &= \int ((u+1)+1)\sqrt{u}\,du \\
&= \int (u+2)\sqrt{u}\,du \\
&= \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right)\,du \\
&= \frac{2}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + C \\
&= \boxed{\frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + C}
\end{aligned}$$

Ex 2.4.3: $\displaystyle\int (x+2)(x-3)^4\,dx$

Ex 2.4.3:

$$\hookrightarrow u = x - 3 \therefore x = u + 3$$

$$\hookrightarrow du = dx$$

$$\begin{aligned}\int (x+2)(x-3)^4\,dx &= ((u+3)+2)u^4\,du \\ &= \int (u+5)u^4\,du \\ &= \int u^5 + 5u^4\,du \\ &= \frac{1}{6}u^6 + u^5 + C \\ &= \boxed{\frac{1}{6}(x-3)^6 + (x-3)^5 + C}\end{aligned}$$

Ex 2.4.4: $\int \frac{x^2+4}{x+2}\,dx$

Sol 2.4.4: There are two ways to approach this problem. One could use polynomial long division to simplify before integrating:

$$\int \frac{x^2+4}{x+2}\,dx = \int \left(x - 2 + \frac{8}{x+2}\right)\,dx$$

Then

$$\hookrightarrow u = x+2$$

$$\hookrightarrow du = dx$$

$$\begin{aligned}\int \left(x - 2 + \frac{8}{x+2}\right)\,dx &= \int (x-2)\,dx + 8\int \frac{1}{u}\,du \\ &= \frac{1}{2}x^2 - 2x + 8\ln|u| + C \\ &= \boxed{\frac{1}{2}x^2 - 2x + 8\ln|x+2| + C}\end{aligned}$$

An alternative method to solving this problem would be to make the denominator u and use back-substitution:

$$\hookrightarrow u = x+2 \therefore x = u = 2$$

$$\hookrightarrow du = dx$$

$$
\begin{aligned}
\int \frac{x^2+4}{x+2}\,dx &= \int \frac{(u-2)^2+4}{u}\,du \\
&= \int \frac{u^2-4u+4+4}{u}\,du \\
&= \int \frac{u^2-4u+8}{u}\,du \\
&= \int \left(u-4+\frac{8}{u}\right)\,du \\
&= \frac{1}{2}u^2-4u+8\ln|u|+C \\
&= \boxed{\frac{1}{2}(x+2)^2-4(x+2)+8\ln|x+2|+C}
\end{aligned}
$$

From visual inspection, these answers may appear different. However, with FOILing and adding like terms, it can be shown that these are in fact the same answers.

As a rule of thumb, doing algebraic simplification before calculus will generally make the problem shorter and simpler. In this case, polynomial long division made the problem easier than back-substitution.

2.4 Free Response Homework

Perform the antidifferentiation.

1. $\displaystyle\int x\sqrt{4-x}\,dx$

2. $\displaystyle\int x^5\sqrt{x^3+4}\,dx$

3. $\displaystyle\int \frac{x+5}{2x+3}\,dx$

4. $\displaystyle\int x^3\left(x^2+1\right)^{12}$

5. $\displaystyle\int \frac{(3+\ln x)^2(2-\ln x)}{x}\,dx$

6. $\displaystyle\int \sqrt{4-\sqrt{x}}\,dx$

7. $\displaystyle\int x^5\left(x^2+4\right)^2\,dx$

8. $\displaystyle\int \sqrt{x+3}(x+1)^2\,dx$

9. $\displaystyle\int (t-1)(2t+4)^5\,dt$

10. $\displaystyle\int (z-3)(3z-1)^3\,dz$

11. $\displaystyle\int \frac{y^5}{\sqrt{y^3+5}}\,dy$

12. $\displaystyle\int \frac{w^5}{w^2+4}\,dw$

13. $\displaystyle\int \frac{x^5}{(x^2-1)^{\frac{5}{2}}}\,dx$

14. $\displaystyle\int \frac{x^7}{(x^4+4)^{\frac{3}{2}}}\,dx$

15. $\displaystyle\int (x+2)\sqrt[3]{x-1}\,dx$

16. $\displaystyle\int \sqrt{4-x}(2x+5)\,dx$

2.4 Multiple Choice Homework

1. $\displaystyle\int x^3\sqrt{1+x^2}\,dx =$

a) $\displaystyle\frac{x^4}{2}\cdot\frac{(1+x^2)^{\frac{3}{2}}}{3}+C$

b) $\displaystyle\frac{1}{2}\left(1+x^2\right)^{\frac{1}{2}}+\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

c) $\displaystyle-\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+\frac{1}{5}\left(1+x^2\right)^{\frac{5}{2}}+C$

d) $\displaystyle\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}-\frac{1}{5}\left(1+x^2\right)^{\frac{5}{2}}+C$

e) $\displaystyle\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

---

2. $\displaystyle\int x^5\sqrt{1+x^2}\,dx =$

a) $\displaystyle\frac{x^6\,(1+x^2)^{\frac{3}{2}}}{18}+C$

b) $\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+\frac{2}{7}\left(1+x^2\right)^{\frac{7}{2}}+C$

c) $\frac{1}{7}\left(1+x^2\right)^{\frac{7}{2}}-\frac{2}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

d) $\frac{2}{7}\left(1+x^2\right)^{\frac{7}{2}}-\frac{4}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

e) $\frac{1}{7}\left(1+x^2\right)^{\frac{7}{2}}+\frac{2}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

---

3. $\int \frac{x^3}{9-x^2}\,dx =$

a) $\frac{x^3}{3}\cdot\frac{(9-x^2)^{\frac{3}{2}}}{9}+C$

b) $\frac{9}{2}\ln|9-x^2|+\frac{1}{2}\left(9-x^2\right)+C$

c) $\frac{9}{2}\left(9-x^2\right)^2-\frac{1}{2}\left(9-x^2\right)+C$

d) $\frac{9}{2}\left(9-x^2\right)+\frac{1}{2}\left(9-x^2\right)^2+C$

e) $\frac{x^4}{36}-\frac{x^2}{2}+C$

---

4. $\int \frac{x^5}{x^2+5}\,dx =$

a) $\frac{1}{4}\left(x^2+5\right)^2-5\left(x^2+5\right)+\frac{25}{2}\ln|x^2+5|+C$

b) $\frac{1}{4}\left(x^2+5\right)^2-5\left(x^2+5\right)+\frac{25}{2}\tan^{-1}\left(x^2+5\right)+C$

c) $\frac{1}{4}\left(x^2+5\right)^2+5\left(x^2+5\right)+\frac{25}{2}\ln|x^2+5|+C$

d) $\frac{1}{4}\left(x^2+5\right)^2+5\left(x^2+5\right)+\frac{25}{2}\tan^{-1}\left(x^2+5\right)+C$

e) None of the above

---

5. $\int e^{2x}\sqrt{e^x+1}\,dx =$

a) $e^{2x}\left(e^x+1\right)^{\frac{3}{2}}+C$

b) $\frac{2}{5}\left(e^x+1\right)^{\frac{5}{2}}-3\left(e^x+1\right)^{\frac{3}{2}}+C$

c) $\dfrac{2}{5}\left(e^x+1\right)^{\frac{5}{2}}-\dfrac{2}{3}\left(e^x+1\right)^{\frac{3}{2}}+C$

d) $\dfrac{2}{5}\left(e^x+1\right)^{\frac{5}{2}}+3\left(e^x+1\right)^{\frac{3}{2}}+C$

e) $\dfrac{2}{5}e^{\frac{5x}{2}}-5e^{\frac{3x}{2}}+C$

---

6. $\dfrac{x^3}{\sqrt{4-x^2}}\,dx=$

a) $\dfrac{1}{3}\left(4-x^2\right)^{\frac{3}{2}}-4\left(4-x^2\right)^{\frac{1}{2}}+C$

b) $\dfrac{2}{3}\left(4-x^2\right)^{\frac{3}{2}}-2\left(4-x^2\right)^{\frac{1}{2}}+C$

c) $\dfrac{1}{3}\left(4-x^2\right)^{\frac{3}{2}}-4\sin^{-1}\left(\dfrac{x}{2}\right)+C$

d) $\dfrac{2}{3}\left(4-x^2\right)^{\frac{3}{2}}-2\sin^{-1}\left(\dfrac{x}{2}\right)+C$

e) $\dfrac{1}{3}\left(4-x^2\right)^{\frac{3}{2}}-\sin^{-1}\left(\dfrac{x}{2}\right)+C$

---

7. $\dfrac{4-x}{\sqrt{4-x^2}}\,dx=$

a) $4\sin^{-1}\left(\dfrac{x}{2}\right)+\left(4-x^2\right)^{\frac{1}{2}}+C$

b) $2\sin^{-1}\left(\dfrac{x}{2}\right)+\left(4-x^2\right)^{\frac{1}{2}}+C$

c) $4\left(4-x^2\right)^{\frac{1}{2}}+C$

d) $\sin^{-1}\left(\dfrac{x}{2}\right)-\left(4-x^2\right)^{\frac{1}{2}}+C$

e) $\dfrac{2}{3}\left(4-x^2\right)^{\frac{3}{2}}+C$

2.5 Powers of Trig Functions: Sine and Cosine

Another instance of back-substitution involves the trig functions and the Pythagorean Identities. As we saw in the previous section, since

$$\frac{d}{dx}[\cos(x)] = -\sin(x) \text{ and } \frac{d}{dx}[\sin(x)] = \cos(x),$$

one of these functions can serve as the du while the other serves as u. But, what about when higher exponents are involved? In general, what about integrals in the form

$$\int \sin^m(x)\cos^n(x)\,dx?$$

Remember:

$$\boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$$

OBJECTIVES

Use Integration by Substitution to Integrate Integrands Involving Sine and Cosine.

There are two cases of integration of this kind of integrand, depending on the powers m and n.

Case 1

The simpler (and more common on the AP Test) case is when either m, n, or both m and n are odd numbers. One of whichever function has the odd power will be the du and the rest of those functions can convert to the other trig function by means of the Pythagorean Identities.

Ex 2.5.1: $\int \sin^4(x)\cos^3(x)\,dx$

Sol 2.5.1: Since $\cos(x)$ has the odd power, we will make that our du.

$\hookrightarrow u = \sin(x)$

$\hookrightarrow du = \cos(x)\,dx$

$$
\begin{aligned}
\int \sin^4(x)\cos^3(x)\,dx &= \int \sin^4(x)\cos^2(x)\cos(x)\,dx \\
&= \int \sin^4(x)\left(1-\sin^2(x)\right)\cos(x)\,dx \\
&= \int u^4\left(1-u^2\right)\,du \\
&= \int \left(u^4-u^6\right)\,du \\
&= \frac{1}{5}u^5-\frac{1}{7}u^7+C \\
&= \boxed{\frac{1}{5}\sin^5(x)-\frac{1}{7}\sin^7(x)+C}
\end{aligned}
$$

Ex 2.5.2: $\int \sin^5(x)\cos^2(x)\,dx$

Sol 2.5.2: Since $\sin(x)$ has the odd power, we will make that our du.

$\hookrightarrow$ $u=\cos(x)$

$\hookrightarrow$ $du=-\sin(x)\,dx$

$$
\begin{aligned}
\int \sin^5(x)\cos^2(x) &= -\int \sin^4(x)(-\sin(x))\cos^2(x)\,dx \\
&= -\int \left(\sin^2(x)\right)^2 (-\sin(x))\cos^2(x)\,dx \\
&= -\int \left(1-\cos^2(x)\right)^2 (-\sin(x))\cos^2(x)\,dx \\
&= -\int \left(1-u^2\right)^2 u^2\,du \\
&= -\int \left(1-2u^2+u^4\right) u^2\,du \\
&= -\int \left(u^2-2u^4+u^6\right)\,du \\
&= -\left(\frac{1}{3}u^3-\frac{2}{5}u^5+\frac{1}{7}u^7\right)+C \\
&= \boxed{-\frac{1}{3}\cos^3(x)+\frac{2}{5}\cos^5(x)-\frac{1}{7}\cos^7(x)+C}
\end{aligned}
$$

If both powers are odd, either function can serve as the u. However, it is generally easier to choose $u=\sin(x)$ as there is no negative sign to deal with.

Ex 2.5.3: $\displaystyle\int \tan(x)\,dx$

Sol 2.5.3: At first, this does not appear to be a sine or cosine integral, but a basic substitution reveals that it is.

$$
\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx
$$

$$
\hookrightarrow u=\cos(x)
$$

$$
\hookrightarrow du=-\sin(x)\,dx
$$

$$\begin{aligned}
\int \frac{\sin(x)}{\cos(x)}\,dx &= -\int (-\sin(x))\frac{1}{\cos(x)}\,dx \\
&= -\int \frac{1}{u}\,du \\
&= -\ln|u| + C \\
&= -\ln|\cos(x)| + C \\
&= \boxed{\ln|\sec(x)| + C}
\end{aligned}$$

It may not be immediately apparent why $-\ln|\cos(x)|$ can be rewritten as $\ln|\sec(x)|$. The reason lies within the log rule $\log\left(a^b\right) = b\log a$:

$$\begin{aligned}
-\ln|\cos(x)| &= -\ln\left|\left(\frac{1}{\cos(x)}\right)^{-1}\right| \\
&= -1 \cdot -\ln\left|\frac{1}{\cos(x)}\right| \\
&= \ln\left|\frac{1}{\cos(x)}\right| \\
&= \ln|\sec(x)|
\end{aligned}$$

This gives us two more integral rules:

$$\int \tan(u)\,du = \ln|\sec(u)| + C \qquad \int \cot(u)\,du = \ln|\sin(u)| + C$$

Case 2

The more difficult situation is when both powers are even. In this case, variations on the half angle argument rules come into play.

Remember:

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

Ex 2.5.4: $\displaystyle\int \cos^2(x)\,dx$

Sol 2.5.4:

$$\int \cos^2(x) = \int \frac{1}{2}(1 + \cos(x))\,dx$$

$\hookrightarrow u = 2x$

$\hookrightarrow du = 2\,dx$

$$\begin{aligned}
\int \frac{1}{2}(1 + \cos(x))\,dx &= \frac{1}{2} \cdot \frac{1}{2} \int (1 + \cos(x))2\,dx \\
&= \frac{1}{4} \cdot (1 + \cos(u))\,du \\
&= \frac{1}{4}u + \frac{1}{4}\sin(u) + C \\
&= \boxed{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C}
\end{aligned}$$

This example alludes to two more integral equations that are helpful to know:

$$\int \cos^2(u)\,du = \frac{1}{2}u + \frac{1}{4}\sin(2u) + C \qquad \int \sin^2(u)\,du = \frac{1}{2}u - \frac{1}{4}\sin(2u) + C$$

Ex 2.5.5: $\displaystyle\int \sin^4(x)\cos^2(x)\,dx$

Sol 2.5.5:

$$\begin{aligned}
\int \sin^4(x)\cos^2(x)\,dx &= \int \left(\frac{1}{2}(1 - \cos(2x))\right)^2 \left(\frac{1}{2}(1 + \cos(2x))\right)\,dx \\
&= \frac{1}{8}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right)(1 + \cos(2x))\,dx
\end{aligned}$$

$$= \frac{1}{8} \int \Big(1 - \cos(2x) - \cos^2(2x) + \cos^3(2x)\Big)\ dx$$

$$\hookrightarrow\ u = 2x$$

$$\hookrightarrow\ du = 2\,dx$$

$$= \frac{1}{2} \cdot \frac{1}{8} \int \Big(1 - \cos(2x) - \cos^2(2x) + \cos^3(2x)\Big)\, 2\,dx$$

$$= \frac{1}{16} \int \Big(1 - \cos(u) - \cos^2(u) + \cos^3(u)\Big)\ du$$

$$= \frac{1}{16} \int\ du - \frac{1}{16} \int \cos(u)\,du - \frac{1}{16} \int \cos^2(u)\,du + \frac{1}{16} \int \cos^3(u)\,du$$

$$= \frac{1}{16}u - \frac{1}{16}\sin(u) - \frac{1}{16}\left(\frac{1}{2}u - \frac{1}{4}\sin(2u)\right)$$

$$+ \frac{1}{16} \int \cos^2(u)\cos(u)\,du + C$$

$$\hookrightarrow\ v = \sin(u)$$

$$\hookrightarrow\ dv = \cos(u)\,dx$$

$$= \frac{1}{16}u - \frac{1}{16}\sin(u) - \frac{1}{16}\left(\frac{1}{2}u - \frac{1}{4}\sin(2u)\right)$$

$$+ \frac{1}{16} \int \Big(1 - \sin^2(u)\Big)\cos(u)\,du + C$$

$$= \frac{1}{16}u - \frac{1}{16}\sin(u) - \frac{1}{32}u + \frac{1}{64}\sin(2u) + \frac{1}{16} \int \Big(1 - v^2\Big)\ dv + C$$

$$= \frac{1}{16}u - \frac{1}{16}\sin(u) - \frac{1}{32}u + \frac{1}{64}\sin(2u) + \frac{1}{16}\left(v - \frac{1}{3}v^3\right) + C$$

$$= \frac{1}{16}(2x) - \frac{1}{16}\sin(2x) - \frac{1}{32}(2x) + \frac{1}{64}\sin(2(2x))$$

$$+ \frac{1}{16}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right) + C$$

$$= \frac{1}{8}x - \frac{1}{16}\sin(2x) - \frac{1}{16}x + \frac{1}{64}\sin(4x) + \frac{1}{16}\sin(2x)$$

$$- \frac{1}{48}\sin^3(2x) + C$$

$$= \boxed{\frac{1}{16}x + \frac{1}{64}\sin(4x) - \frac{1}{48}\sin^3(2x) + C}$$

2.5 Free Response Homework

Perform the antidifferentiation.

1. $\int \sin^3(x)\cos^2(x)\,dx$
2. $\int \sin^4(x)\cos^5(x)\,dx$
3. $\int \sin^2(x)\cos^7(x)\,dx$
4. $\int \sin^5(x)\cos^6(x)\,dx$
5. $\int \sin(x)\cos^5(x)\,dx$
6. $\int \sin^5(x)\cos^5(x)\,dx$
7. $\int \sin^2(x)\cos^2(x)\,dx$
8. $\int \sin^2(x)\cos^4(x)\,dx$

2.5 Multiple Choice Homework

1. For $\int \sin^3(x)\cos^5(x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

2. For $\int \sin^3(5x)\cos^2(5x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

3. For $\int \sin^4(4x)\cos^5(4x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

4. For $\int \sin^2(x)\cos^4(x)\,dx$, the correct u-substitution is

   a) $u = \sin(x)$ b) $u = \cos(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

5. $\int \cos^2(2x)\,dx =$

a) $\sin(4x) + C$ b) $\frac{1}{2}x + \frac{1}{8}\sin(4x) + C$ c) $\frac{1}{2}x - \frac{1}{8}\sin(4x) + C$

d) $x + \frac{1}{4}\sin(4x) + C$ e) $x + \frac{1}{8}\cos(4x) + C$

---

6. $\int \cos^2\left(\frac{1}{2}x\right)\, dx =$

a) $\sin(4x) + C$ b) $\frac{1}{2}x + \frac{1}{4}\sin(x) + C$ c) $\frac{1}{2}x - \frac{1}{4}\sin(x) + C$

d) $\frac{1}{4}x + \frac{1}{2}\sin(x) + C$ e) $\frac{1}{4}x + \frac{1}{2}\cos(x) + C$

---

7. Identify the first mistake (if any) in this process:

Problem: $\int \sin^3(2x)\cos^4(2x) =$

Step 1: $= -\frac{1}{2}\int \sin^2(2x)\cos^4(2x)(-\sin(2x))2\, dx$

Step 2: $= -\frac{1}{2}\int \left(1 - u^2\right) u^4\, du$

Step 3: $= -\frac{1}{2}\int \left(u^4 - u^6\right)\, du$

Step 4: $= -\frac{1}{2}\left(\frac{1}{5}u^5 - \frac{1}{7}u^7 + C\right)$

Step 5: $= -\frac{1}{10}\sin^5(4x) + \frac{1}{14}\sin^7(4x) + C$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

2.6 Powers of Trig Functions: Secant, Tangent, and Beyond

As with sine and cosine, secant and tangent work together in a Pythagorean Identity, as well as cosecant and cotangent. So, we will be considering integrals of the form

$$\int \sec^m(x)\tan^n(x)\,dx \quad \text{and} \quad \int \csc^m(x)\cot^n(x)\,dx$$

to be cases of integration by u-substitution.

Remember:

$$\tan^2(\theta)+1=\sec^2(\theta) \qquad \cot^2(\theta)+1=\csc^2(\theta)$$

$$\frac{d}{dx}[\tan(u)]=\left(\sec^2(u)\right)\frac{du}{dx} \qquad \frac{d}{dx}[\sec(u)]=(\sec(u)\tan(u))\frac{du}{dx}$$

$$\frac{d}{dx}[\cot(u)]=\left(-\csc^2(u)\right)\frac{du}{dx} \qquad \frac{d}{dx}[\csc(u)]=(-\csc(u)\cot(u))\frac{du}{dx}$$

OBJECTIVES

Use Integration by Substitution to Integrate Integrands Involving Secant, Tangent, Cosecant, and Cotangent.

There are three cases of integration of this kind of integrand, depending on the powers m and n.

Case 1

The first case is when the secant's (or cosecant's) power is even. In this case, our u would be $\tan(x)$ or $\cot(x)$ and our du would be $\sec^2(x)\,dx$ or $-\csc^2(x)\,dx$.

Ex 2.6.1: $\int \sec^4(x)\tan^2(x)\,dx$

Sol 2.6.1:

$\hookrightarrow u=\tan(x)$

$\hookrightarrow du=\sec^2(x)$

$$
\begin{aligned}
\int \sec^4(x)\tan^2(x) &= \int \sec^2(x)\tan^2(x)\sec^2(x)\,dx \\
&= \int \Big(1+\tan^2(x)\Big)\tan^2(x)\sec^2(x)\,dx \\
&= \int \Big(1+u^2\Big)u^2\,du \\
&= \int \Big(u^2+u^4\Big)\,du \\
&= \frac{1}{3}u^3+\frac{1}{5}u^5+C \\
&= \boxed{\frac{1}{3}\tan^3(x)+\frac{1}{5}\tan^5(x)+C}
\end{aligned}
$$

Case 2

The second case is when the tangent's (or cotangent's) power is odd. In this case, our u would be $\sec(x)$ or $\csc(x)$ and our du would be $\sec(x)\tan(x)\,dx$ or $-\csc(x)\cot(x)\,dx$.

Ex 2.6.2: $\int \csc^3(x)\cot^3(x)\,dx$

Sol 2.6.2:

$\hookrightarrow$ $u=\csc(x)$

$\hookrightarrow$ $du=-\csc(x)\cot(x)\,dx$

$$
\begin{aligned}
\int \csc^3(x)\cot^3(x)\,dx &= -\int \csc^2(x)\cot^2(x)(-\csc(x)\cot(x))\,dx \\
&= -\int u^2\Big(u^2-1\Big)\,du \\
&= -\int \Big(u^4-u^2\Big)\,du \\
&= -\frac{1}{5}u^5+\frac{1}{3}u^3+C \\
&= \boxed{-\frac{1}{5}\csc^5(x)+\frac{1}{3}\csc^3(x)+C}
\end{aligned}
$$

If both cases are present (that is, the tangent's/cotangent's power is odd AND the secant's/cosecant's power is even), then any of the functions can serve as u.

Case 3

The third and final case is when the tangent's/cotangent's power is even AND the secant's/cosecant's power is odd. This case is no longer doable by integration by substitution, and requires a technique known as *integration by parts*. We will need to wait until Chapter 8 to approach this case.

Quick Summary of 2.5-2.6:

I. $\int \sin^m(x)\cos^n(x)\,dx$

  a) The odd power determines du. The other function is u.

  b) If both powers are even, use the half-angle formulas and simplify.

II. $\int \sec^m(x)\tan^n(x)\,dx$ or $\int \csc^n(x)\cot^m(x)\,dx$

  a) If both powers are even, $u = \tan(x)$ or $\cot(x)$ and $du = \sec^2(x)\,dx$ or $-\csc^2(x)\,dx$.

  b) If both powers are odd, $u = \sec(x)$ or $\csc(x)$ and $du = \sec(x)\tan(x)\,dx$ or $-\csc(x)\cot(x)\,dx$.

  c) If n is even and m is odd, either (a) or (b) will work.

  d) If n is odd and m is even, neither (a) nor (b) will work.

III. For any other mix of trig functions, convert all to sine and cosine and use I. above.

2.6 Free Response Homework

Perform the antidifferentiation.

1. $\int \sec^2(x)\tan^5(x)\,dx$
2. $\int \sec^6(x)\tan^4(x)\,dx$
3. $\int \sec^5(x)\tan^7(x)\,dx$
4. $\int \sec^2(x)\tan^6(x)\,dx$
5. $\int \sec^6(x)\tan^3(x)\,dx$
6. $\int \csc^2(x)\cot^5(x)\,dx$
7. $\int \csc^4(x)\cot(x)\,dx$
8. $\int \csc^7(x)\cot^5(x)\,dx$

2.6. Multiple Choice Homework

1. For $\int \csc^3(x)\cot^5(x)\,dx$, the correct u-substitution is

   a) $u = \csc(x)$ b) $u = \cot(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

2. For $\int \csc^4(x)\cot^4(x)\,dx$, the correct u-substitution is

   a) $u = \csc(x)$ b) $u = \cot(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

3. For $\int \sec^4(x)\tan^5(x)\,dx$, the correct u-substitution is

   a) $u = \sec(x)$ b) $u = \tan(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

4. For $\int \sec^5(x)\tan^4(x)\,dx$, the correct u-substitution is

   a) $u = \sec(x)$ b) $u = \tan(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

5. Which of the following statements are true?

   I. $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

II. $\displaystyle\int \tan(x)\,dx = \sec^2(x) + C$

III. $\displaystyle\int x^2 \cot\left(x^3\right)\,dx = \frac{1}{3}\ln\left|\sin\left(x^3\right)\right| + C$

a) I only b) II only c) III only d) I and II only e) I and III only

---

6. Which of the following statements are **false**?

I. $\displaystyle\int x^5 \sec\left(x^6\right) = \frac{1}{6}\ln\left|\sec\left(x^6\right) + \tan\left(x^6\right)\right| + C$

II. $\displaystyle\int \cot(x)\,dx = -\csc^2(x) + C$

III. $\displaystyle\int \csc(x)\,dx = \ln|\csc(x) - \cot(x)| + C$

a) I only b) II only c) III only d) I and II only e) I and III only

---

7. Identify the first mistake (if any) in this process:

Problem:	$\displaystyle\int \sec^4(2x)\tan^3(2x) =$
Step 1:	$\displaystyle= \frac{1}{2}\int \sec^2(2x)\tan^3(2x)\sec^2(2x)2\,dx$
Step 2:	$\displaystyle= \frac{1}{2}\int \left(1-u^2\right)u^3\,du$
Step 3:	$\displaystyle= \frac{1}{2}\int \left(u^3-u^5\right)\,du$
Step 4:	$\displaystyle= \frac{1}{2}\left(\frac{1}{4}u^4 - \frac{1}{6}u^6 + C\right)$
Step 5:	$\displaystyle= -\frac{1}{8}\tan^4(2x) - \frac{1}{12}\tan^6(2x) + C$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

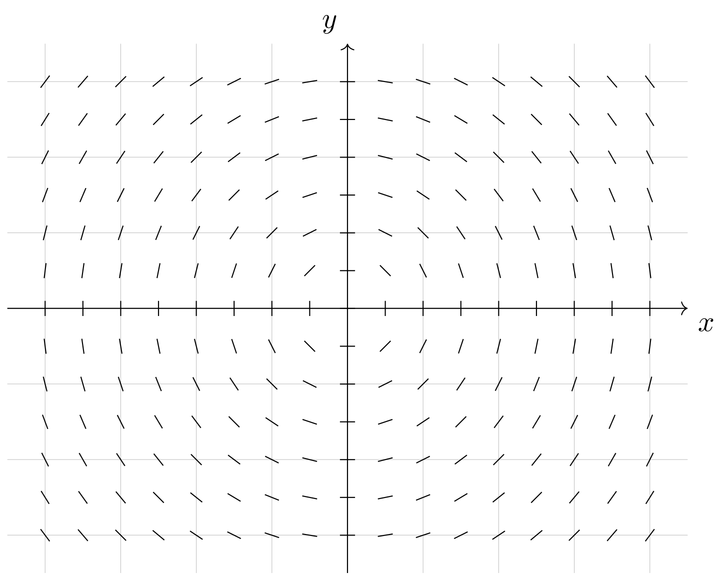
2.7 Intro to AP: Slope Fields

This section is very much an AP-driven section. Recall that one of the big emphasis of CollegeBoard regarding calculus was that it should be understood and explained in many different modes. The topic of differential equations fits nicely into this paradigm in that the visual is a graphical representation and the connection between the equation and the slopes is a numerical process.

Slope Field $\rightarrow$ Definition: Given any function f, a slope field is drawn by taking evenly spaced points on the Cartesian coordinate system (usually points of integer coordinates) and, at each point, drawing a small line with the slope of f.

IMPORTANT: For this section, assume the grids of all graphs are 1 unit in length and height

Here is an example of a slope field of the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$:



OBJECTIVES

Given a Differential Equation, Sketch its Slope Field.
Given a Slope Field, Sketch a Particular Solution Curve.
Given a Slope Field, Determine the Family of Functions to Which the Solution Curves Belong.
Given a Slope Field, Determine the Differential Equation that it Represents.

The are four ways that the AP Exam usually approaches slope fields:

1. Draw a slope field (free response).

2. Sketch the solution to a slope field (free response).

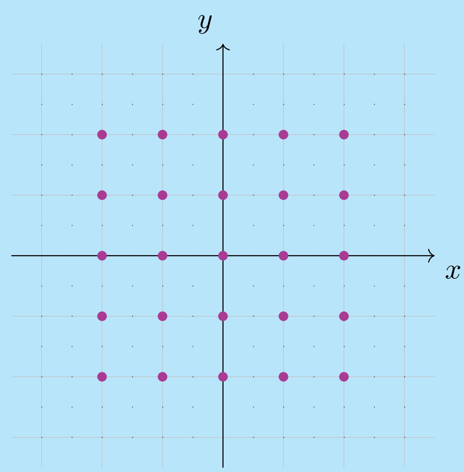
3. Identify the differential equation for a slope field (multiple choice).

4. Identify the solution equation to a slope field (multiple choice).

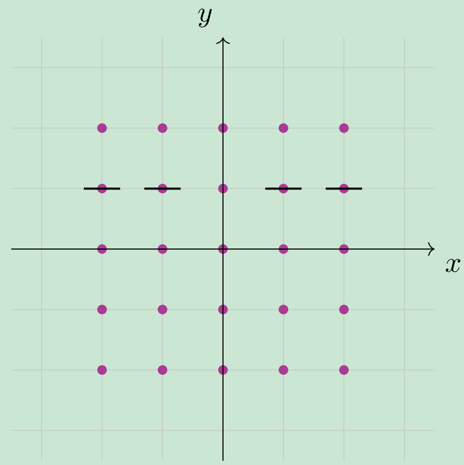
Two of these types of questions (items one and three) are numerically based and two (items two and four) are graphically oriented.

Slope Fields Numerically (FRQs)

Ex 2.7.1: Sketch the slope field for $\frac{dy}{dx} = \frac{1-y}{x}$ at the points indicated.



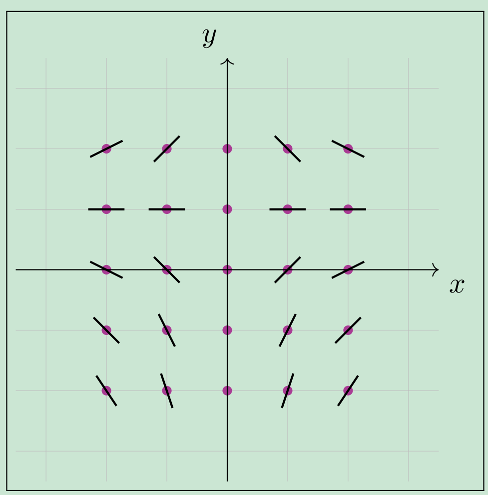
Sol 2.7.1: Note that wherever $y = 1$, $\frac{dy}{dx} = 0$. So, the segments at $y = 1$ will be horizontal. Also, where $x = 0$, the derivative does not exist (due to division by zero). So, there will be no segments on the x-axis.



To find the slant of the rest of the line segments, we can plug the numerical values of the points into $\dfrac{dy}{dx} = \dfrac{1-y}{x}$.

$(-2,2) \to \dfrac{dy}{dx} = \dfrac{1}{2}$	$(-1,2) \to \dfrac{dy}{dx} = 1$	$(1,2) \to \dfrac{dy}{dx} = -1$	$(2,2) \to \dfrac{dy}{dx} = -\dfrac{1}{2}$
$(-2,0) \to \dfrac{dy}{dx} = -\dfrac{1}{2}$	$(-1,0) \to \dfrac{dy}{dx} = -1$	$(1,0) \to \dfrac{dy}{dx} = 1$	$(2,0) \to \dfrac{dy}{dx} = \dfrac{1}{2}$
$(-2,-1) \to \dfrac{dy}{dx} = -1$	$(-1,-1) \to \dfrac{dy}{dx} = -2$	$(1,-1) \to \dfrac{dy}{dx} = 2$	$(2,-1) \to \dfrac{dy}{dx} = 1$
$(-2,-2) \to \dfrac{dy}{dx} = -\dfrac{3}{2}$	$(-1,-2) \to \dfrac{dy}{dx} = -3$	$(1,-2) \to \dfrac{dy}{dx} = 3$	$(2,-2) \to \dfrac{dy}{dx} = \dfrac{3}{2}$

These values give us our final slope field:



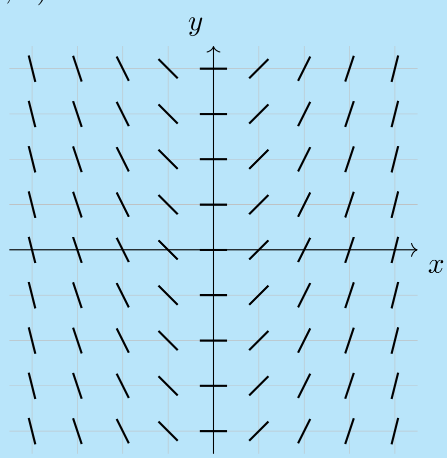
We can use the steps we took in this example to generalize some steps to sketching slope fields:

Steps to Sketching a Slope Field:

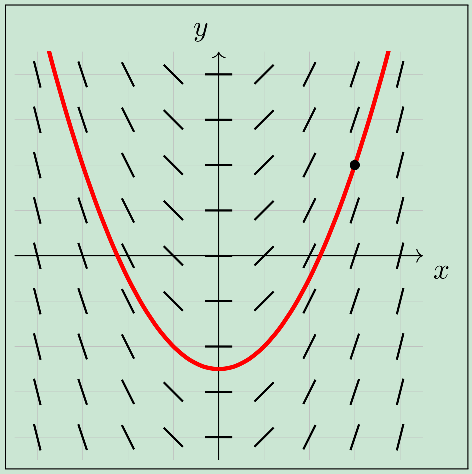
1. Determine the grid of points for which you need to sketch (generally, the points are given).
2. Pick your first point. note its x and y coordinate. Plug these numbers into the differential equation; the output represents the slope at that point.
3. Find the point on the graph. Make a little line/dash at that point whose slope represents the slope that you found in Step 2.
4. Repeat this process for all the points needed.

Slope Fields Graphically (FRQ)

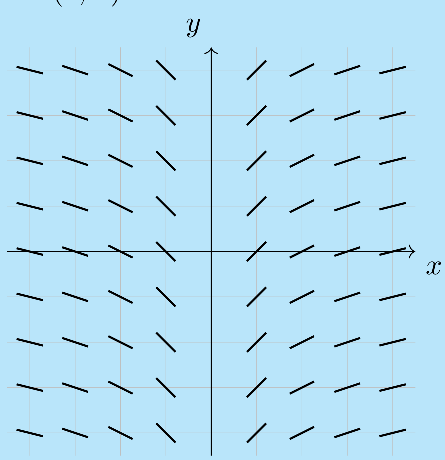
Ex 2.7.2: Given the slope field for $\dfrac{dy}{dx} = x$ below, sketch the particular solution given the initial condition of (3, 2).



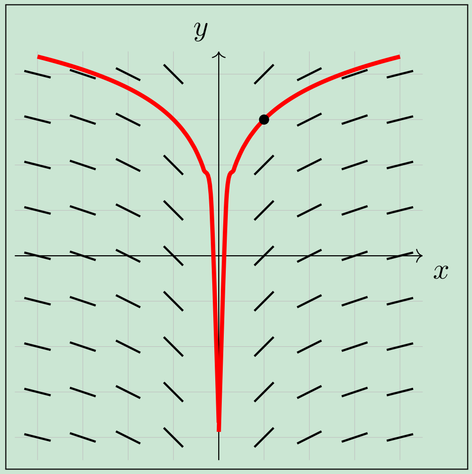
Sol 2.7.2: To find the particular solution, we simply need to start at (3, 2) and follow the slope segments:



Ex 2.7.3: Given the slope field for $x\dfrac{dy}{dx} = 1$ below, sketch the particular solution given the initial condition of (1, 3).



Sol 2.7.3: Once again, we simply need to start at (1, 3) and follow the slope segments:



Note that there appears to be a vertical asymptote at $y = 0$.

Slope Fields Numerically (MCQ)

Let's summarize what we know about slopes of lines in terms of numbers.

1. Horizontal lines have $\dfrac{dy}{dx} = 0$.

2. Vertical lines have an undefined derivative.

3. Lines with positive slopes go up from left to right.

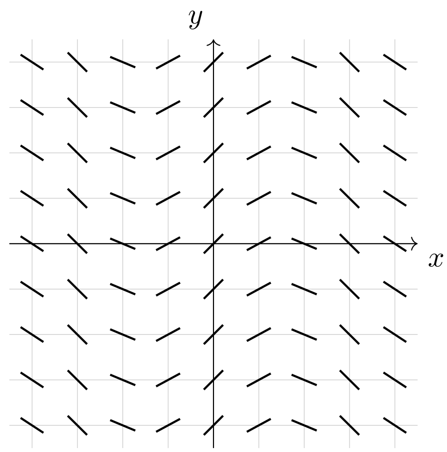
4. Lines with negative slopes go down from left to right.

Two other facts are obvious from viewing a slope field and its associated differential equation:

5. If all dashes in each **column** are parallel to each other, then $\dfrac{dy}{dx}$ has **no** y.

6. If all dashes in each **row** are parallel to each other, then $\dfrac{dy}{dx}$ has **no** x.

To make points five and six clearer, take a look at the following slope field:

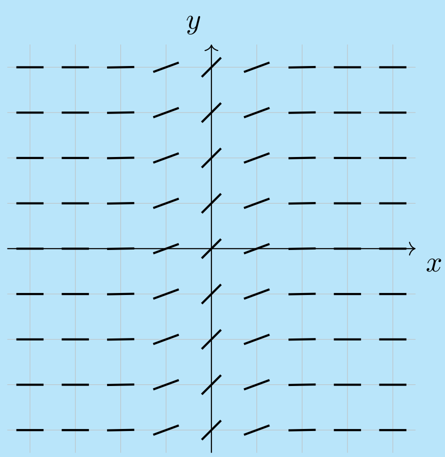


The differential equation that represents this slope field would not have a y in the equation, because the segments in each column are parallel with each other.

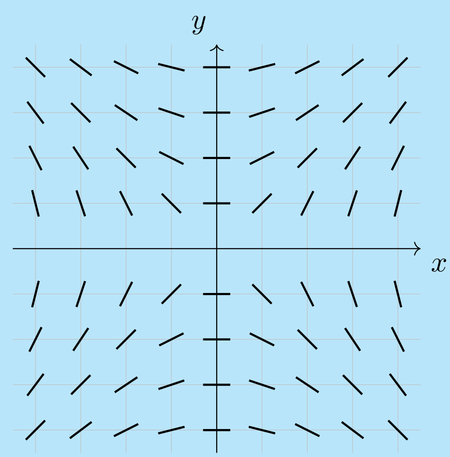
For those curious, the equation for this slope field is $\dfrac{dy}{dx} = \cos(x)$, which in fact does not contain a y!

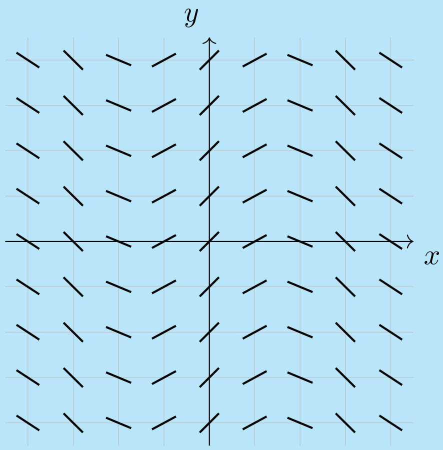
Ex 2.7.4: Which of the following slope fields matches $\dfrac{dy}{dx} = 3y - 4x$?

a)

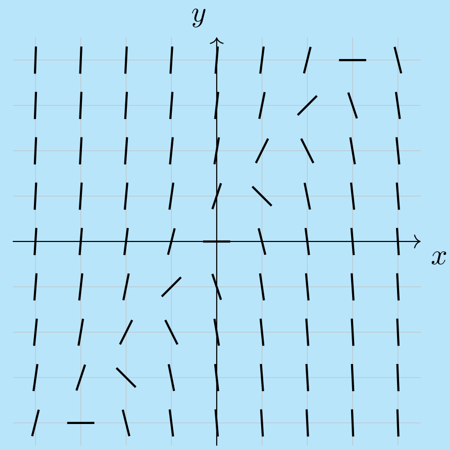


b)





c)



d)

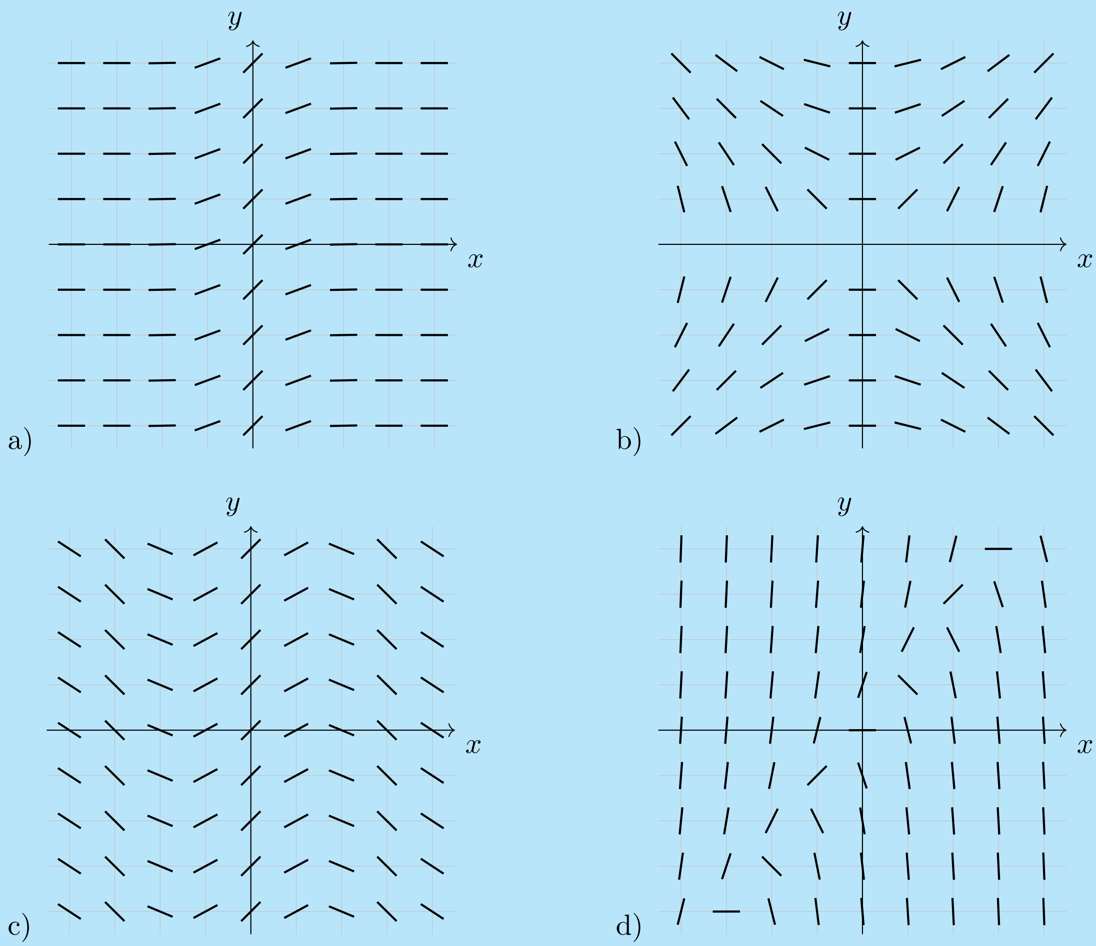
Sol 2.7.4: These types of problems are best done by a combination of process of elimination and substituting numbers. Options (a) and (c) all have slopes parallel to one another in each column. Therefore, they cannot be the slope field for the given differential equation, since the given equation has both x and y.

Now, since we cannot eliminate any other obvious choices, we can move to substitution. Option (b) appears to have horizontal slopes when $x = 0$, so let's plug in a point on the y axis into our differential equation to see if it is equal to zero.

$$\left.\frac{dy}{dx}\right|_{(0,\,3)} = 3(3) - 4(0) = 9 \neq 0$$

Clearly, option (b) is not our desired slope field. Therefore, the correct answer is option $\boxed{\text{(d)}}$.

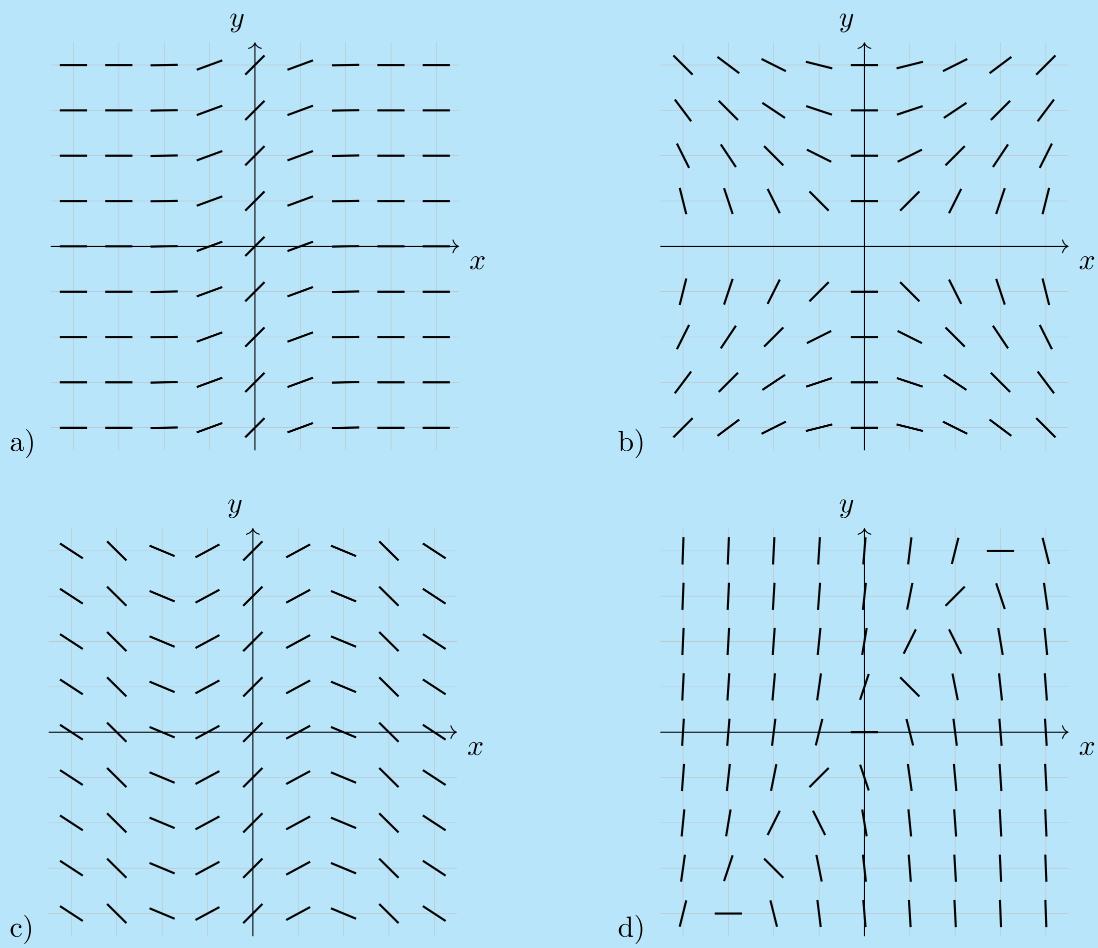
Ex 2.7.5: Which of the following slope fields matches $\dfrac{dy}{dx} = e^{-x^2}$?



Sol 2.7.5: Once again, let's begin with elimination. Since our differential equation only has one variable, we know that our slope field must consist of columns of parallel dashes. This eliminates options (b) and (d).

Now, lets look at the equation. We know that raising e to the power of anything will produce a positive number, so all our slopes must go up from left to right. Therefore, it's obvious that our answer is option $\boxed{\text{(a)}}$.

Ex 2.7.6: Which of the following slope fields matches $\dfrac{dy}{dx} = \dfrac{x}{y}$?



Sol 2.7.6: Using the techniques in **Ex 2.7.4** and **Ex 2.7.5**, we can directly eliminate options (a) and (c).

Now, yet again, we look at the equation. Specifically, we look at what happens when $x = 0$:

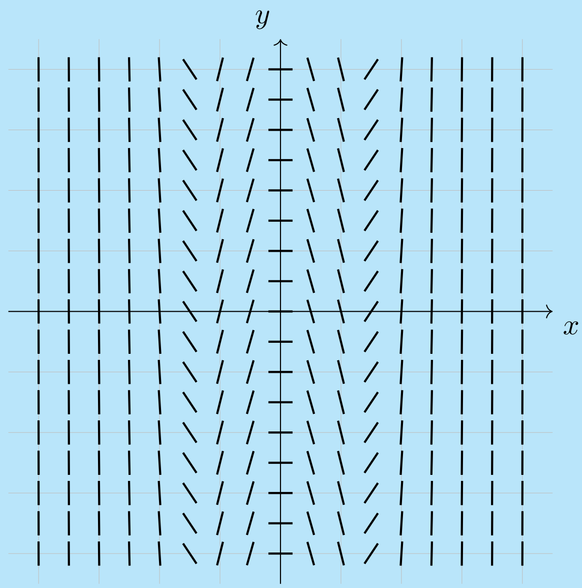
$$\left.\frac{dy}{dx}\right|_{(0,\,c)} = \frac{0}{c} = 0$$

So, our slope field must have a slope of zero when $x = 0$. This means that option $\boxed{\text{(b)}}$ is the correct slope field for this differential equation.

Slope Fields Graphically (FRQ)

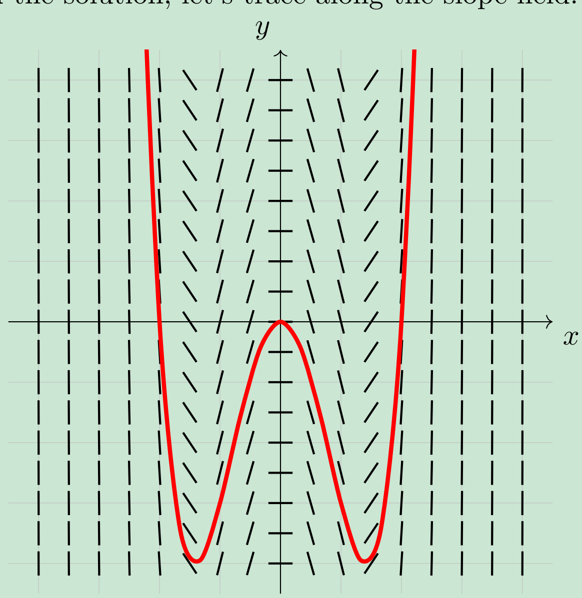
For these types of problems, one would typically sketch a solution and decide from among the options based on what was learned about families of functions in precalculus. For a refresher on families of functions, please see Chapter 0.

Ex 2.7.7: Which of the following equations might be the solution to the slope field shown in the figure below?



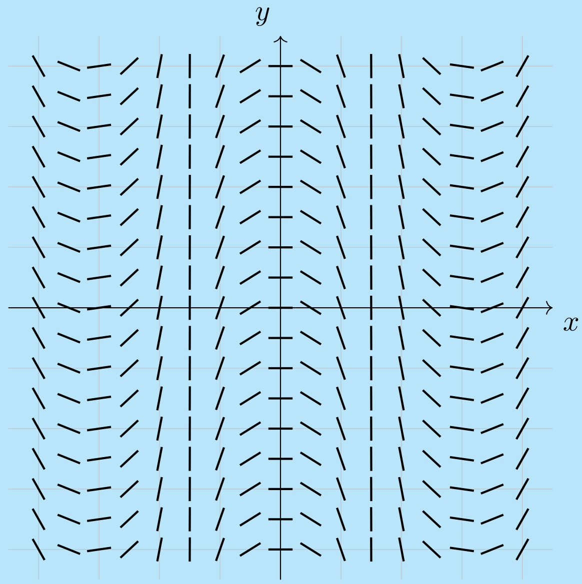
a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos(x)$ d) $y = -\sec(x)$

Sol 2.7.7: To find the solution, let's trace along the slope field.



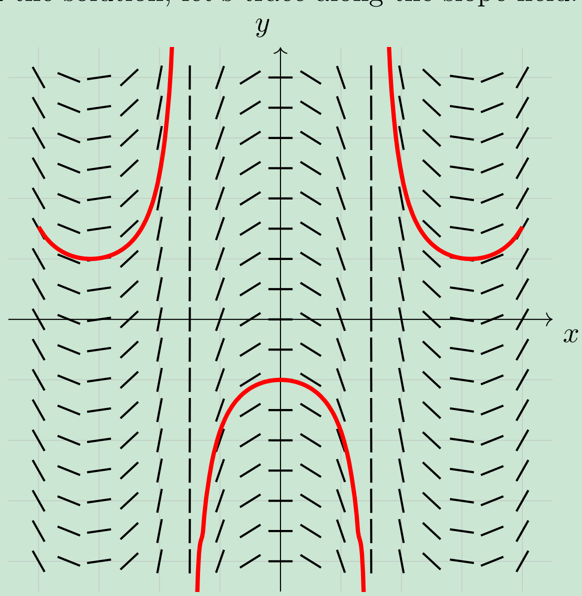
This is a quartic function, so our answer is option $\boxed{\text{a) } y = x^4 - 4x^2}$.

Ex 2.7.8: Which of the following equations might be the solution to the slope field shown in the figure below?



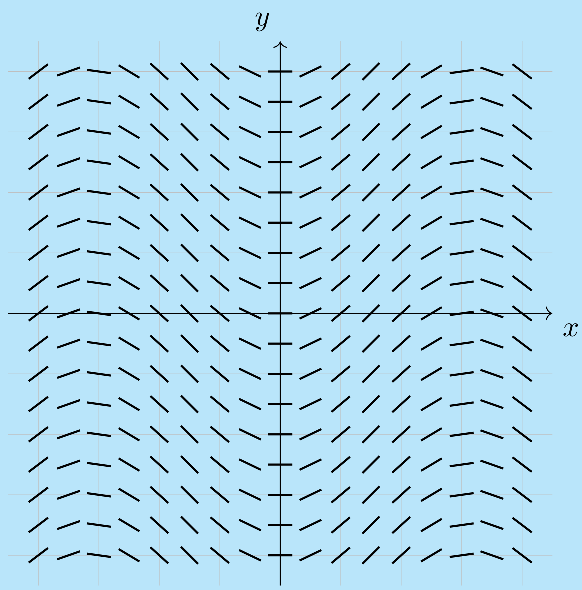
a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos(x)$ d) $y = -\sec(x)$

Sol 2.7.8: To find the solution, let's trace along the slope field.



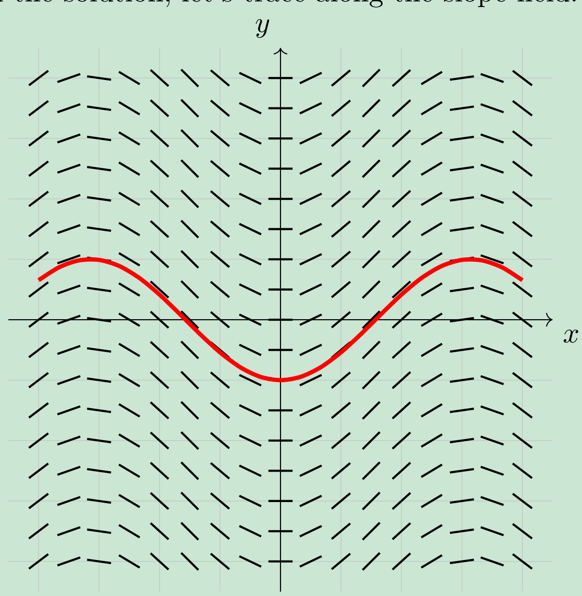
This is very clearly a secant function, so our answer is option $\boxed{\text{d) } y = -\sec(x)}$.

Ex 2.7.9: Which of the following equations might be the solution to the slope field shown in the figure below?



a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos(x)$ d) $y = -\sec(x)$

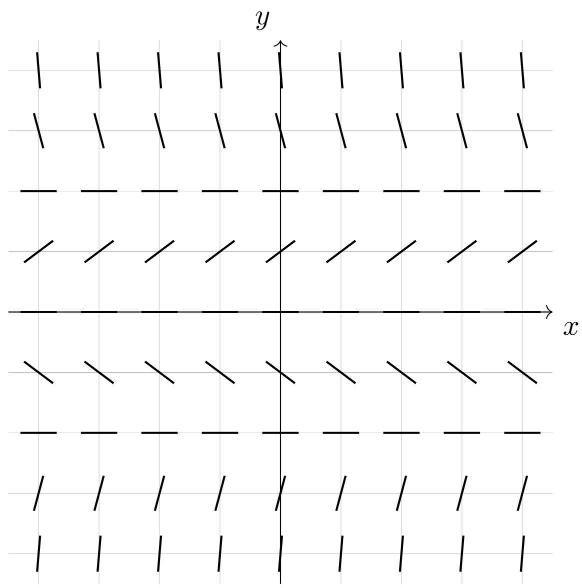
Sol 2.7.9: To find the solution, let's trace along the slope field.



This appears to be some sort of trig function. By noting that y appears to be -1 when $x = 0$, we can deduce that this is a negative cosine function. Therefore, our answer is option $\boxed{\text{c) } y = -\cos(x)}$.

2.7 Free Response Homework

1. A slope field for the differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$ is shown.



Sketch the graphs of the solutions that satisfy the given initial conditions:

(a) $y(0) = 1$

(b) $y(0) = -1$

(c) $y(0) = -3$

(d) $y(0) = 3$

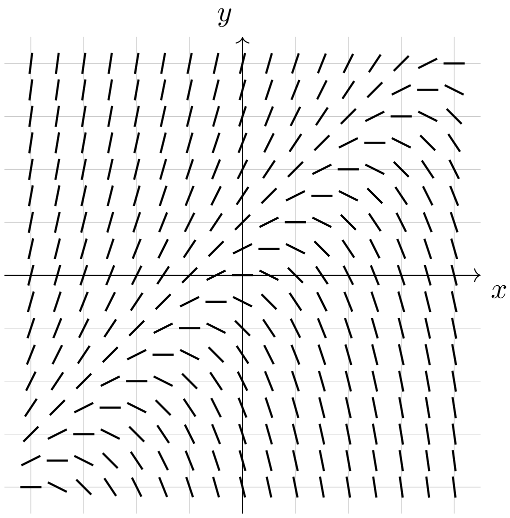
Match each differential equation with its slope field (labeled I-IV). Give reasons for your answer.

2. $\dfrac{dy}{dx} = y - 1$

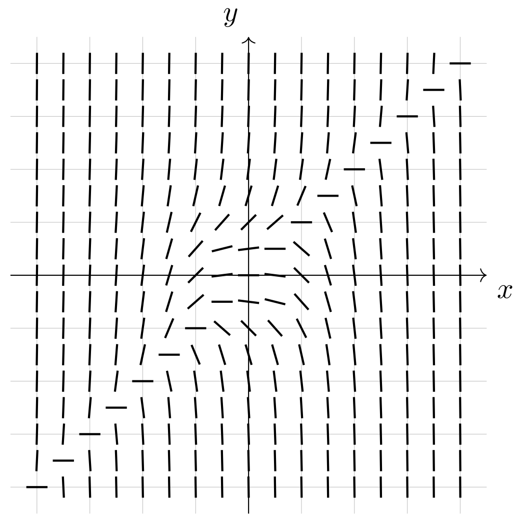
3. $\dfrac{dy}{dx} = y - x$

4. $\dfrac{dy}{dx} = y^2 - x^2$

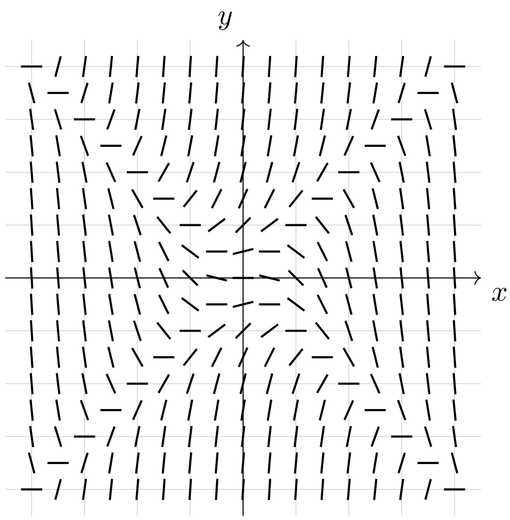
5. $\dfrac{dy}{dx} = y^3 - x^3$



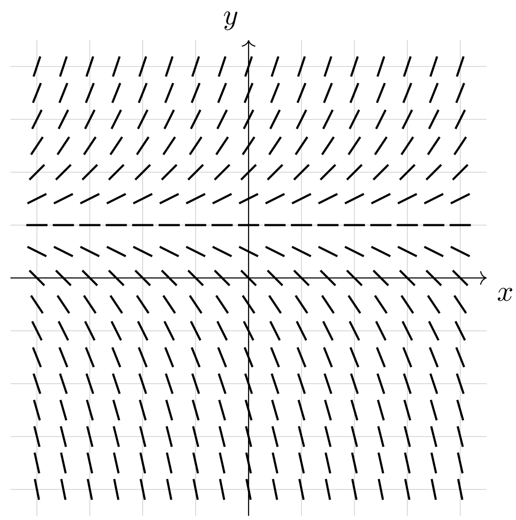
I.



II.



III.



IV.

6. Use the slope field labeled I. above to sketch the graph of the solutions that satisfy the given initial conditions.

(a) $y(0) = 1$

(b) $y(0) = 0$

(c) $y(0) = -1$

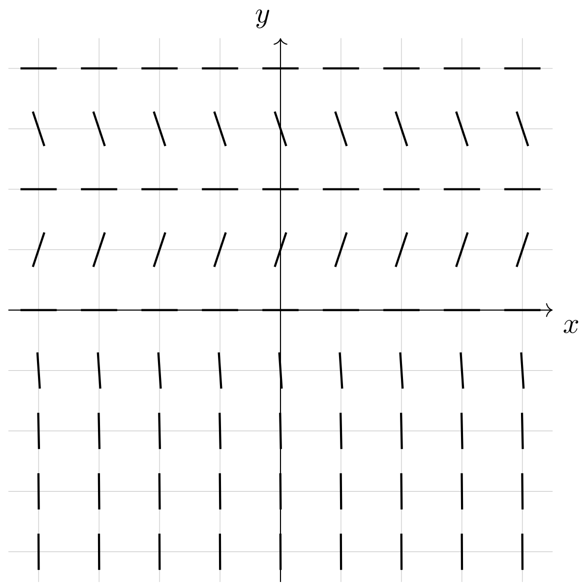
7. Sketch a slope field for the differential equation below. Then, use it to sketch three different solution curves.

$$y' = 1 + y$$

8. Sketch a slope field for the differential equation below. Then, use it to sketch a solution curve that passes through the point $(1, 0)$.

$$y' = y - 2x$$

9. A slope field for the differential equation $y' = y(y-2)(y-4)$ is shown.



For parts (a) through (d), sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0) = -0.3$

(b) $y(0) = 1$

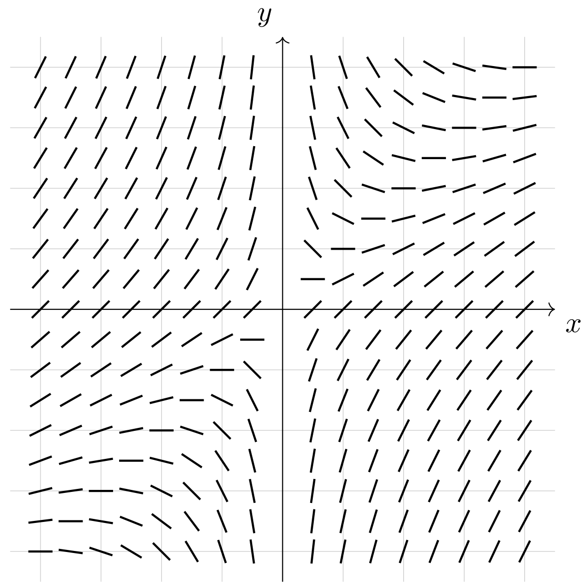
(c) $y(0) = 3$

(d) $y(0) = 4.3$

(e) If the initial condition is $y(0) = c$, for what values of c is $\lim_{t\to\infty} y(t)$ finite?

2.7 Multiple Choice Homework

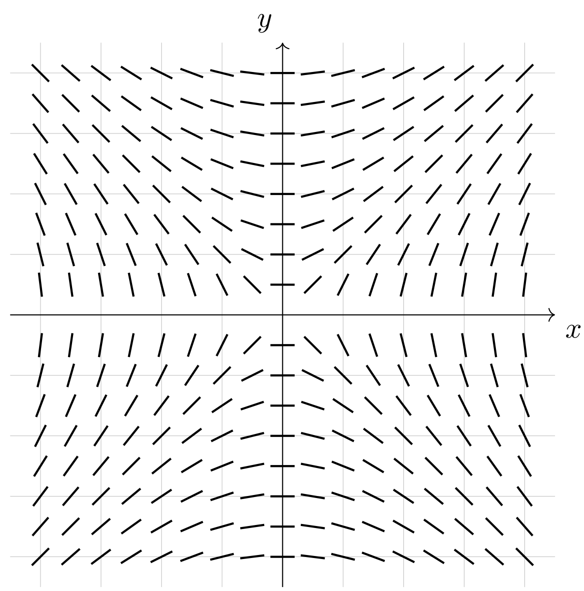
1. Which of the following differential equations corresponds to the slope field shown in the figure below?



a) $\dfrac{dy}{dx} = -\dfrac{y^2}{x}$ b) $\dfrac{dy}{dx} = 1 - \dfrac{y}{x}$ c) $\dfrac{dy}{dx} = y^3$ d) $\dfrac{dy}{dx} = x - \dfrac{1}{2}x^3$ e) $\dfrac{dy}{dx} = x + y$

---

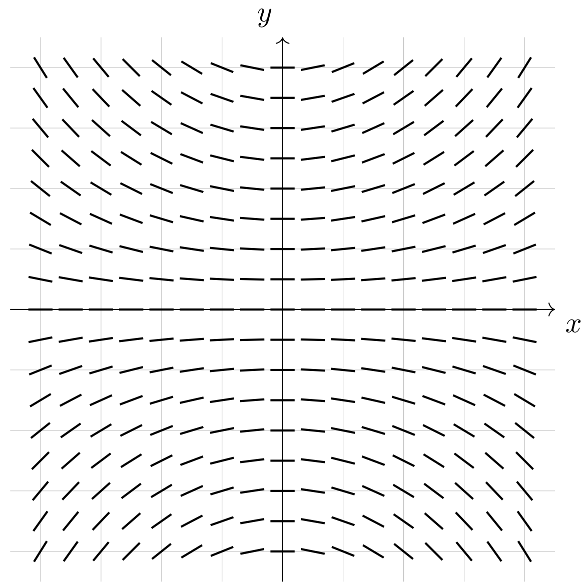
2. Which of the following differential equations corresponds to the slope field shown below?



a) $\dfrac{dy}{dx} = -\dfrac{y}{x}$ b) $\dfrac{dy}{dx} = 5xy$ c) $\dfrac{dy}{dx} = \dfrac{xy}{10}$ d) $\dfrac{dy}{dx} = \dfrac{y}{x}$ e) $\dfrac{dy}{dx} = \dfrac{x}{y}$

---

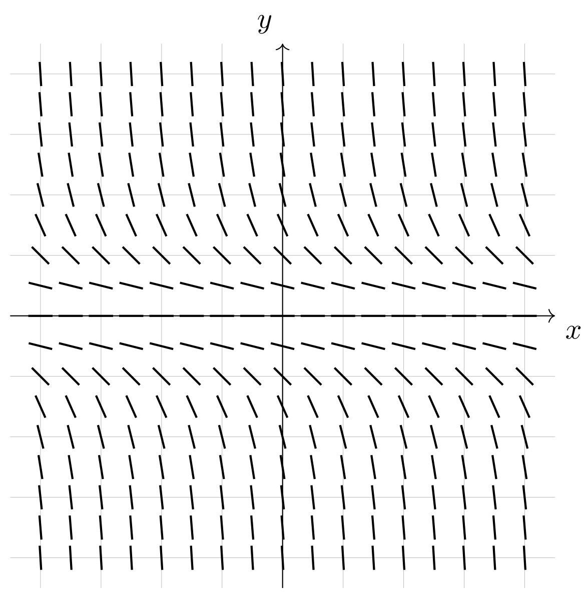
3. Which of the following differential equations corresponds to the slope field shown in the figure below?



a) $\dfrac{dy}{dx} = -\dfrac{y}{x}$ b) $\dfrac{dy}{dx} = 5xy$ c) $\dfrac{dy}{dx} = \dfrac{xy}{10}$ d) $\dfrac{dy}{dx} = \dfrac{y}{x}$ e) $\dfrac{dy}{dx} = \dfrac{x}{y}$

---

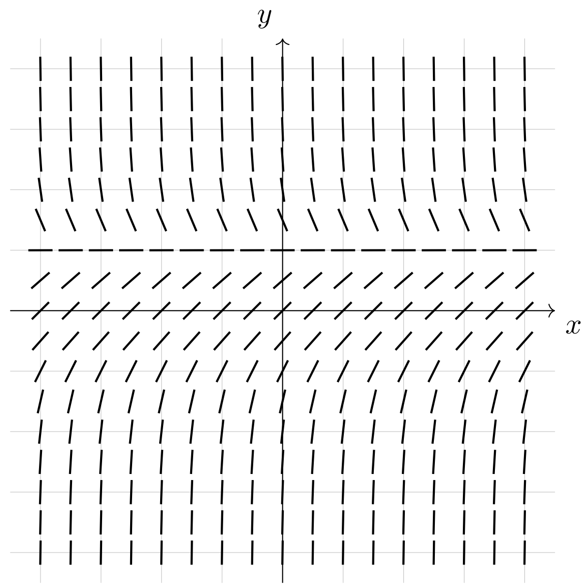
4. Which of the following equations might be the solution to the slope field shown in the figure below?



a) $y = 12x - x^3$ b) $-\cos(x)$ c) $\sec(x)$ d) $x = -y^2$ e) $x = -y^3$

---

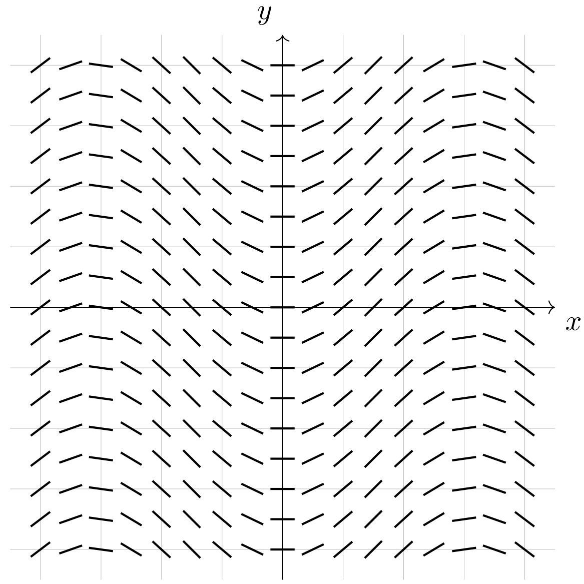
5. Which of the following differential equations corresponds to the slope field shown in the figure below?



a) $\dfrac{dy}{dx} = 1 - y^3$ b) $\dfrac{dy}{dx} = y^2 - 1$ c) $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$ d) $\dfrac{dy}{dx} = x^2 y$ e) $\dfrac{dy}{dx} = x + y$

---

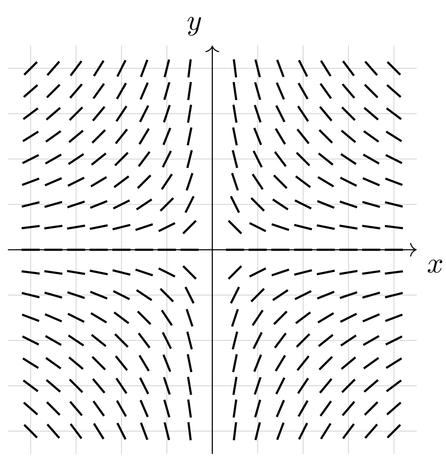
6. Which of the following differential equations corresponds to the slope field shown in the figure below?



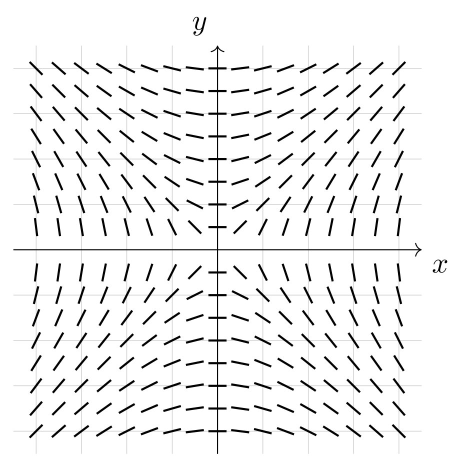
a) $y = 4x - x^3$ b) $y = -\cos(x)$ c) $y = \sec(x)$ d) $x = -y^2$ e) $x = -y^3$

---

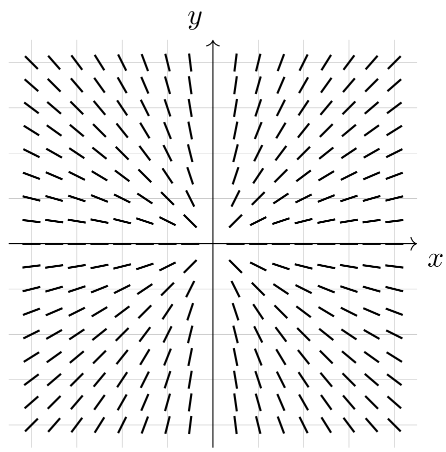
7. Which of the slope fields shown below corresponds to $\dfrac{dy}{dx} = -\dfrac{y}{x}$?



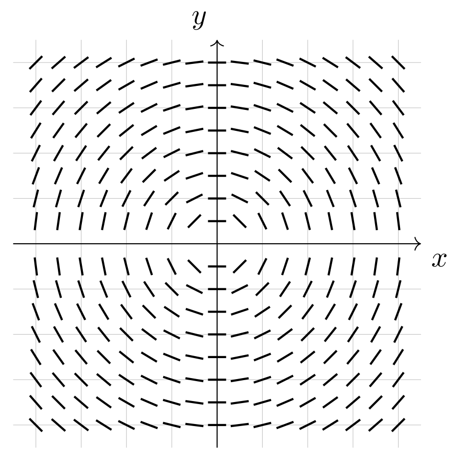
a)



b)

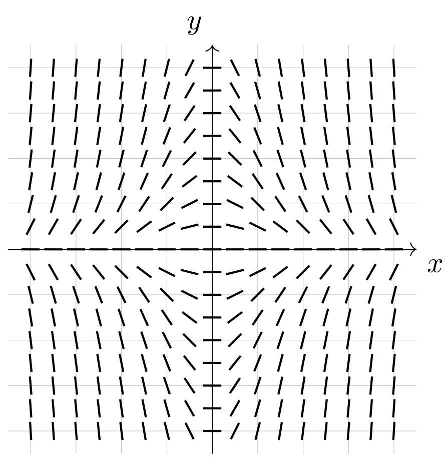


c)

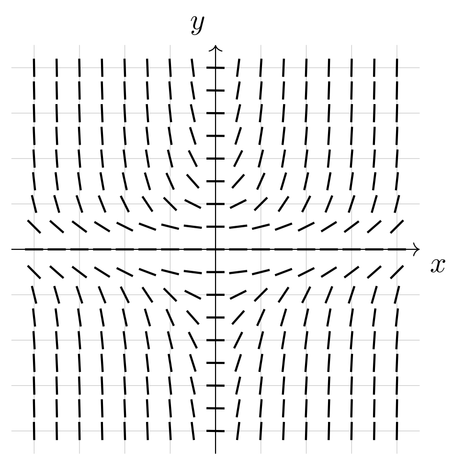


d)

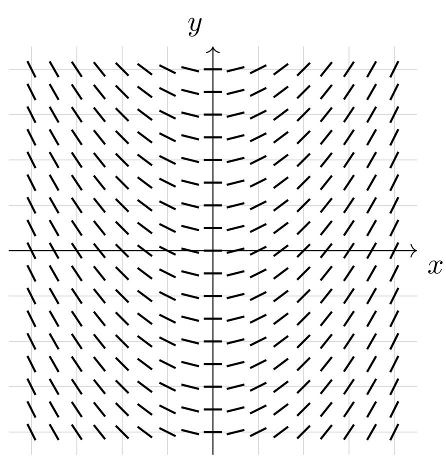
8. Which of the slope fields shown below corresponds to $\dfrac{dy}{dx} = xy$?



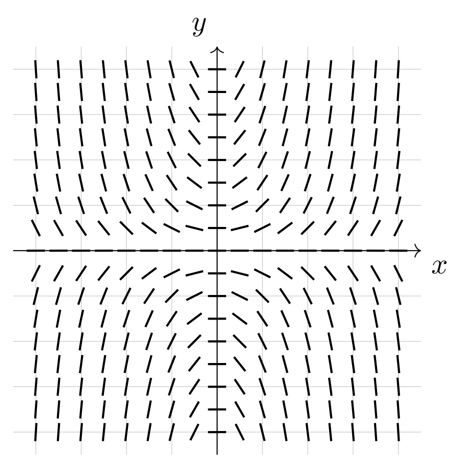
a)



b)

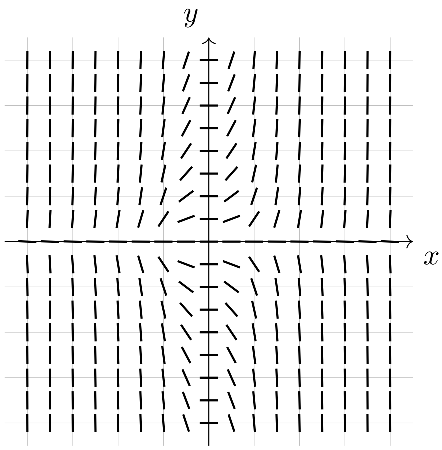


c)

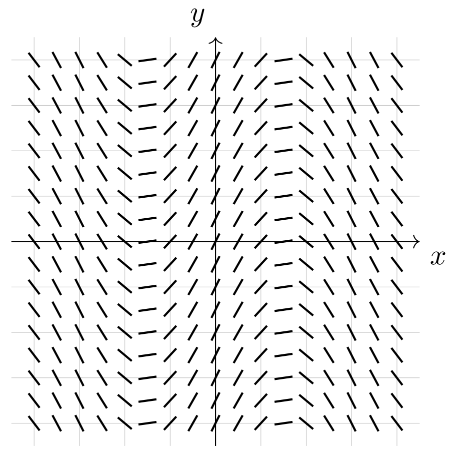


d)

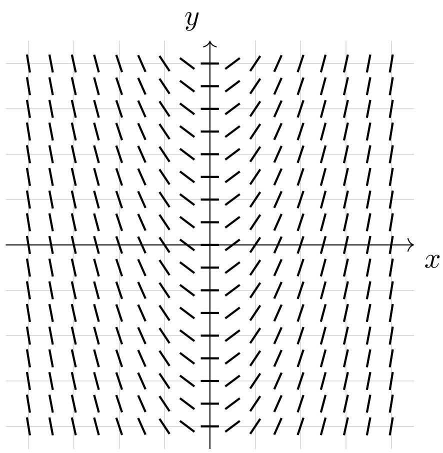
9. Which of the slope fields shown below corresponds to $|y| = e^{x^3}$?



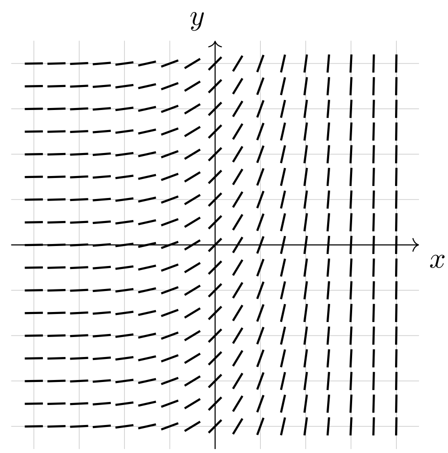
a)



b)



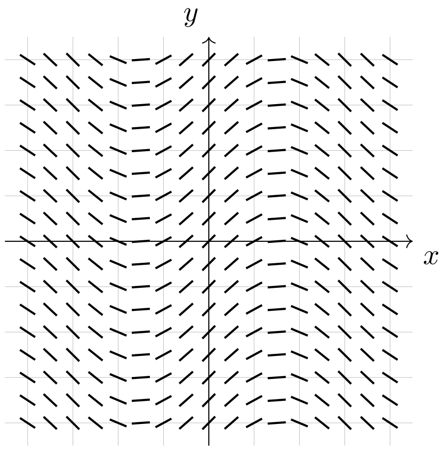
c)



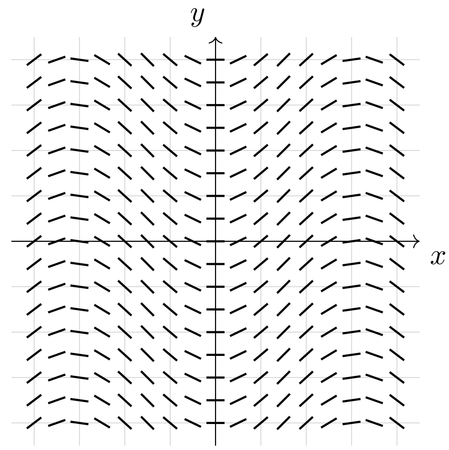
d)

---

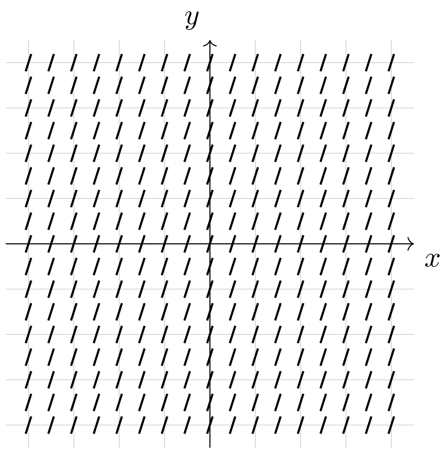
10. Which of the slope fields shown below corresponds to $y = \sec(x)$?



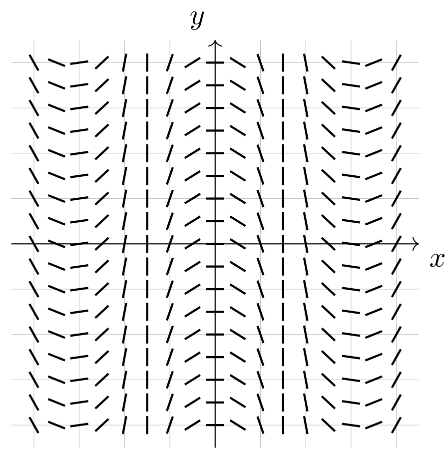
a)



b)



c)



d)

Full Name: AP Calculus BC

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Multiple Choice Section **20 Minutes; No Calculator**

Show All Work

1. Which of the following is **true**?

a) $\int \left(5^x + 2e^{-x}\right)\, dx = \frac{5^x}{\ln 5} + 2e^{-x} + C$

b) $\int \cot^3(x)\csc^2(x)\, dx = \frac{1}{4}\cot^4(x) + C$

c) $\int (x-1)\sqrt{x}\, dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{1}{2}} + C$

d) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = -2\cos\left(\sqrt{x}\right) + C$

---

2. $\int \frac{x}{x^2-4}\, dx =$

a) $-\frac{1}{4\left(x^2-4\right)^2} + C$

b) $-\frac{1}{2\left(x^2-4\right)} + C$

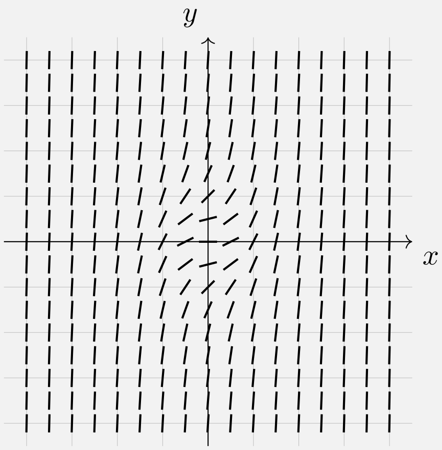
c) $\frac{1}{2}\ln\left|x^2-4\right| + C$

d) $2\ln\left|x^2-4\right| + C$

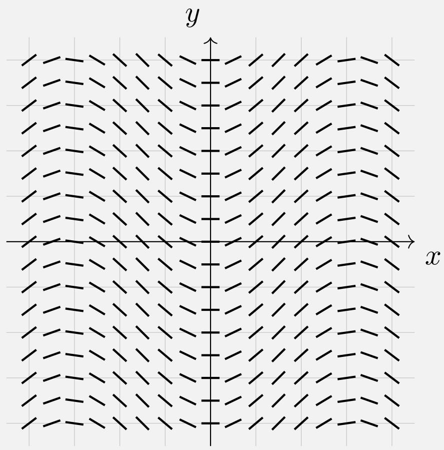
e) $\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

---

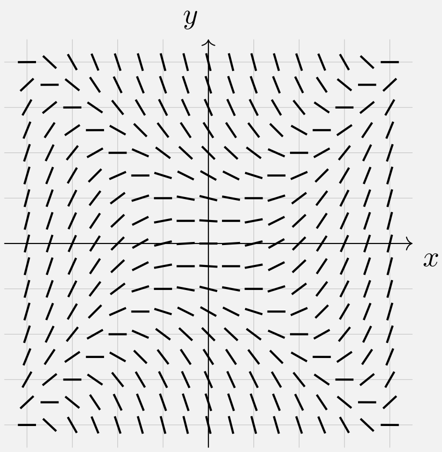
3. Which of the following slope fields presents $\dfrac{dy}{dx} = y^2 + 2x^2$?



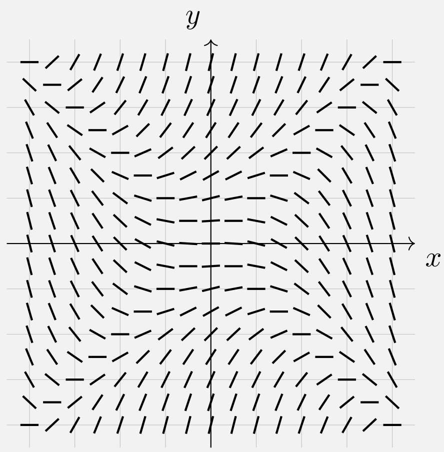
a)



b)

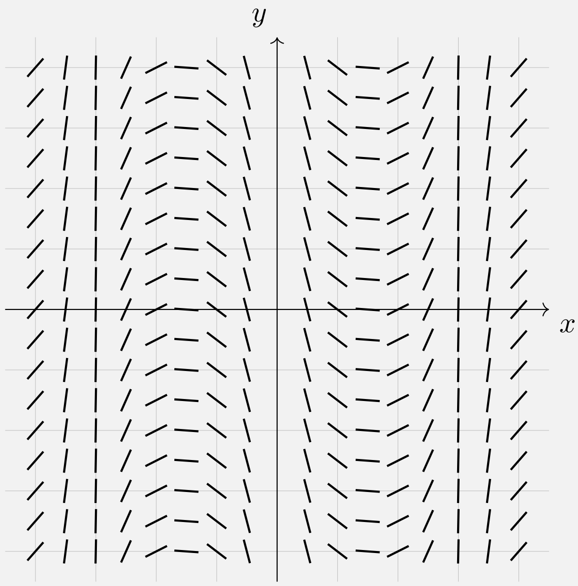


c)



d)

4. Which of the following equations corresponds to the solution to the slope field shown in the figure below?



a) $y = x^4 - 8x^2$ b) $y = 8x^2 - 4x^4$ c) $y = \csc(x)$ d) $y = -\sec(x)$

---

5. Which of the following is a solution to the differential equation $\dfrac{dy}{dx} = \dfrac{x-1}{y}$ with the initial condition $y(0) = -2$?

a) $y = -2e^{x^2-2x}$ b) $y = -2 + e^{x^2-2}$ c) $y = \sqrt{x^2 - 2x - 4}$

d) $y = -\sqrt{x^2 - 2x + 4}$ e) $y = -\sqrt{x^2 - 2x - 4}$

---

6. $\displaystyle\int \frac{4y^3 - 2y^2 - 5y}{\sqrt{y}}\, dy =$

a) $\left(y^4 - \dfrac{2}{3}y^3 - \dfrac{5}{2}y^2\right) 2y^{\frac{1}{2}}$ b) $4y^{\frac{5}{2}} - 2y^{\frac{3}{2}} - 5y^{\frac{1}{2}} + C$

c) $\dfrac{8}{7}y^{\frac{7}{2}} - \dfrac{4}{5}y^{\frac{5}{2}} - \dfrac{10}{3}y^{\frac{3}{2}} + C$ d) $10y^{\frac{3}{2}} - 3y^{\frac{1}{2}} - \dfrac{5}{2}y^{-\frac{1}{2}} + C$

---

7. Identify the first mistake (if any) in this process:

Problem:	$\dfrac{dy}{dx} = x - xy$
Step 1:	$\dfrac{1}{1-y}\,dy = x\,dx$
Step 2:	$-\ln\|1-y\| = \dfrac{1}{2}x^2 + C$
Step 3:	$\|1-y\| = e^{-\frac{1}{2}x^2+C}$
Step 4:	$y = 1 + Ke^{\frac{1}{2}x^2}$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

---

8. An object moves with velocity $v(t) = \sec^2(2t)$. It is known that the particle's position at time 0 is 2. What is the particle's position function?

a) $s(t) = \tan(2t) + 2$ b) $s(t) = \dfrac{1}{2}\tan(2t) + 2$ c) $s(t) = \sec^2(2t)\tan^2(2t) + 2$

d) $s(t) = \ln|\sec(2t)| + 2$ e) $s(t) = \dfrac{1}{2}\ln|\sec(2t)| + 2$

Free Response Section **35 Minutes; No Calculator**

Show All Work

1. Compute the following antiderivatives.

   (a) $\int \left(2x^5 + 2^x - \frac{17}{\sqrt[5]{x^2}} - \frac{1}{5x^2}\right) dx$

   (b) $\int e^{5x} \csc\left(e^{5x}\right) dx$

(c) $\displaystyle\int \left(\cos(3x) + \cos^2(5x) + \sin(7x)\sqrt{\cos(7x)}\right)\, dx$

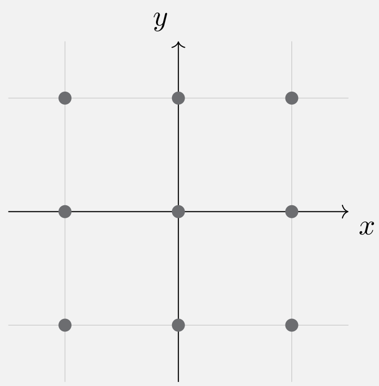
---

2. The acceleration of a particle is described by $a(t) = 98e^{-7t}$. Find the distance equation for $x(t)$ if $v(0) = 0$ and $x(0) = -2$.

---

3. Let's define a differential equation $\dfrac{dy}{dx} = \dfrac{y}{x^2+4}$.

(a) On the axis system provided, sketch the slope field for $\dfrac{dy}{dx}$ at all points plotted on the graph. You may assume that all gridlines have length 1.



(b) Find the particular solution $w = f(t)$ that passes through $y(0) = -2$.

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Multiple Choice Section **20 Minutes; No Calculator**

1. Which of the following statements are **true**?

   I. $\int \left(x^3+x\right) \sqrt[4]{x^4+2x^2-5}\, dx = \frac{1}{5}\left(x^4-2x^2-5\right)^{\frac{5}{4}} + C$

   II. $\int x^5 \sin\left(x^6\right)\, dx = -\frac{1}{6}\cos\left(x^6\right) + C$

   III. $\int \csc(x)\, dx = \ln|\csc(x)+\cot(x)| + C$

   a) I only b) II only c) III only d) I and II only e) II and III only

---

2. $\int \frac{x-2}{x-1}\, dx =$

   a) $-\ln|x-1|+C$ b) $x+\ln|x-1|+C$ c) $x-\ln|x-1|+C$
   d) $x-\sqrt{x-1}+C$ e) $x+\sqrt{x-1}+C$

---

3. If $\frac{dy}{dx} = \sin(x)\cos^3(x)$ and if $y=1$ when $x=\pi$, what is the value of y when $x=0$?

   a) -3 b) -2 c) 1 d) 2 e) 3

---

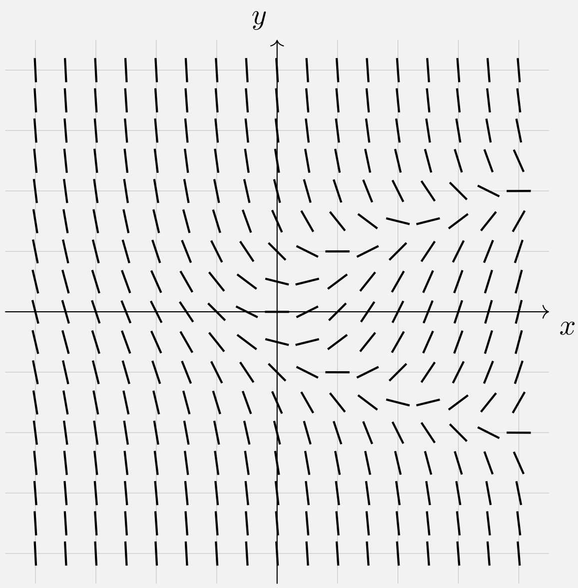
4. $\int x\sqrt{1-x^2}\,dx =$

a) $\dfrac{\left(1-x^2\right)^{\frac{3}{2}}}{3}+C$ b) $-\left(1-x^2\right)^{\frac{3}{2}}+C$ c) $\dfrac{x^2\left(1-x^2\right)^{\frac{3}{2}}}{3}+C$

d) $-\dfrac{x^2\left(1-x^2\right)^{\frac{3}{2}}}{3}+C$ e) $-\dfrac{\left(1-x^2\right)^{\frac{3}{2}}}{3}+C$

---

5. Which of the following differential equations corresponds to the slope field shown in the figure below?



a) $\dfrac{dy}{dx}=x-y^2$ b) $\dfrac{dy}{dx}=1-\dfrac{y}{x}$ c) $\dfrac{dy}{dx}=-y^3$ d) $\dfrac{dy}{dx}=x-\dfrac{1}{2}x^3$ e) $\dfrac{dy}{dx}=x+y$

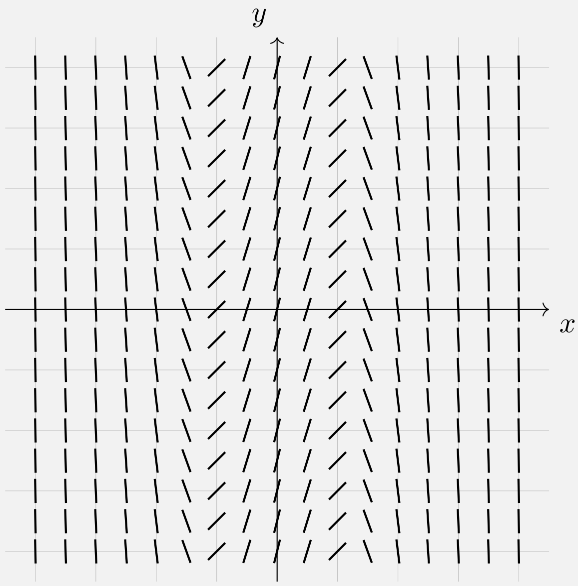
---

6. For $\int \sec^2(x)\tan^2(x)\,dx$, the correct u-substitution is

a) $u=\sec(x)$ b) $u=\tan(x)$ c) Either (a) or (b) d) Neither (a) nor (b)

---

7. Which of the following equations might be the solution to the slope field shown in the figure below?



a) $y = 4x - x^3$ b) $y = x^3 - 4x$ c) $y = 4x^4$ d) $y = x^3 - 15x^5$ e) $y = \sec(x)$

---

8. Identify the first mistake (if any) in this process:

Problem: $\dfrac{dy}{dx} = xy + x$

Step 1: $\dfrac{1}{y+1}\,dy = x\,dx$

Step 2: $\ln|y+1| = x^2 + C$

Step 3: $|y+1| = e^{x^2} + C$

Step 4: $y = e^{x^2} + C$

a) Step 1 b) Step 2 c) Step 3 d) Step 4 e) No mistake

Free Response Section **35 Minutes; No Calculator**

Show All Work

1. Compute the following antiderivatives.

(a) $\displaystyle\int \frac{t^3 - 4t - 3}{5t^{\frac{2}{3}}}\, dt$

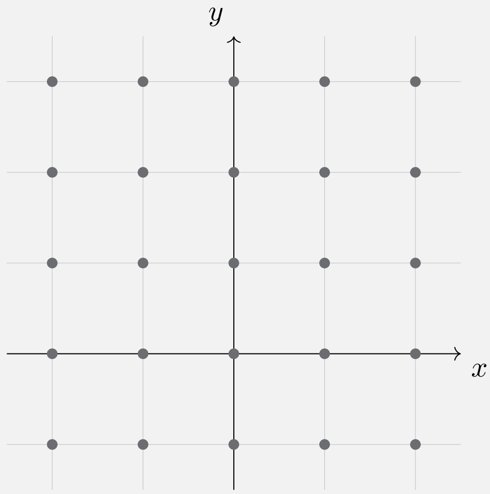
(b) $\displaystyle\int \frac{x^2}{(x^3 - 1)^{\frac{3}{2}}}\, dx$

(c) $\int x\sqrt{-3x^2+17}\,dx$

2. The acceleration of a particle is described by $a(t) = 48t^2 - 18t + 6$. Find the distance equation for $x(t)$ if $v(1) = 1$ and $x(1) = 3$.

3. Let's define a differential equation $\dfrac{dy}{dx} = \dfrac{y-2}{x+1}$.

(a) On the axis system provided, sketch the slope field for $\dfrac{dy}{dx}$ at all points plotted on the graph. You may assume that all gridlines have length 1.



(b) If the solution curve passes through the point (0, 0), sketch the solution curve on the same set of axes as your slope field.

(c) Find the equation for the solution curve of $\dfrac{dy}{dx} = \dfrac{y-2}{x+1}$ given that $y(0) = 5$.

Chapter 2 Answer Key

Chapter 3:

Integrals

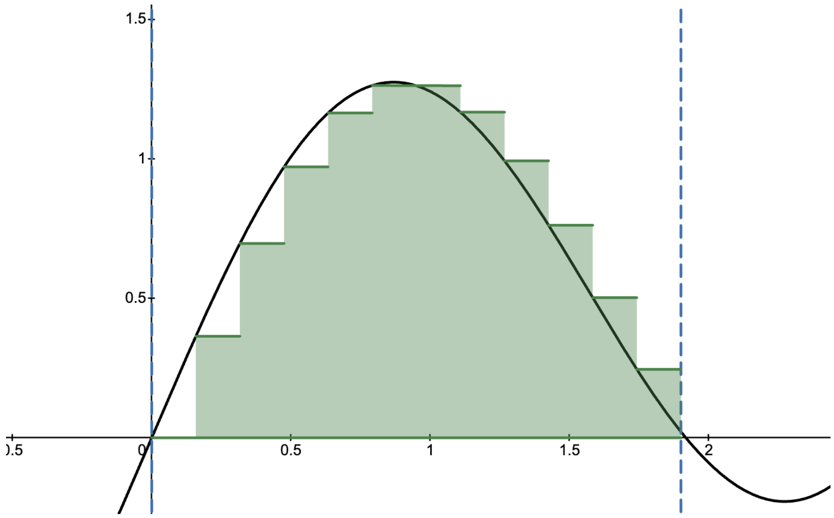
Chapter 3 Overview: Definite Integrals

In this chapter, we will study the Fundamental Theorem of Calculus, which establishes the link between the algebra and the geometry, with an emphasis on mechanics of how to find the definite integral. We will consider the differences implied between the context of the definite integral as an operation and as an area accumulator. We will learn some approximation techniques for definite integrals and see how they provide theoretical foundation for the integral. We will revisit graphical analysis in terms of the definite integral and view another typical AP context for it. Finally, we will consider what happens when trying to integrate at or near an asymptote.

As noted in the overview of the last chapter, antiderivatives are known as indefinite integrals because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an analytic-geometrical context of area. Though it is not necessary to know the theory behind this in order to calculate the integral, the theory is a major subject of integral calculus, so we will explore it briefly in here.

The Limit Definition of the Definite Integral

We know, from geometry, how to find the exact area of various polygons, but we never considered figures where one side is not made of a line segment. Here we want to consider an area bounded by some curve $y = f(x)$ on the top, the x-axis on the bottom, some arbitrary $x = a$ on the left, and $x = b$ on the right.



As we can see above, the area approximated by rectangles whose height is the y-value of the equation and whose width we will call Δx. The more rectangles we make, the better the approximation. For a good animation of this concept, consider the following video:

Riemann sum approximation animation

The area of each rectangle would be $f(x) \cdot \Delta x$, and the total area of n rectangles would be

$$A = \sum_{i=1}^{n} f(x_i) \cdot \Delta x.$$

This equation is known as the **Riemann summation**. Although this equation looks complicated, it represents a rather simple idea. We are adding up the areas of many thin rectangles to approximate the total area under the curve $y = f(x)$ between two points. As we increase the number of rectangles n, each rectangle becomes narrower, and thus our approximation becomes more accurate.

But, how can we find the exact area? With the Riemann sum, we are only coming up with better and better approximations right now. If we could make an infinite number of rectangles (which would be infinitely thin), we could potentially find the exact area under this curve. Luckily, we just so happen to have the mathematical tools to do this: we can take the limit as n approaches infinity.

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

This is where the definite integral comes in. The definite integral provides a precise way to calculate the exact area under a curve by taking the limit of the Riemann sum as the number of rectangles approaches infinity. In other words, instead of merely approximating the area with a finite number of rectangles, the definite integral captures what happens when the width of each rectangle becomes infinitesimally small. We write this limit in a compact form as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x,$$

where a is the "lower bound" and b is the "upper bound." Mathematicians sometimes nuance this statement as

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x, \text{ where } \Delta x = \frac{b-a}{n}}$$

3.1: The Fundamental Theorem of Calculus

Up to this point, we've seen how integration can be used to find the exact area under a curve by taking the limit of Riemann sums. But what's truly remarkable is how integration connects so deeply with differentiation. This connection is captured in one of the most important results in all of calculus: the Fundamental Theorem of Calculus (FTC).

The Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a, b]$, then:

1. $\dfrac{d}{dx}\displaystyle\int_c^x f(t)\,dt = f(x)$ or $\dfrac{d}{dx}\displaystyle\int_c^u f(t)\,dt = f(u)\cdot\dfrac{du}{dx}$

2. If $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

The first part of the Fundamental Theorem of Calculus simply says what we already know– that an integral is an anti-derivative. The second part of the Fundamental Theorem says that the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

The idea of the integral meaning the area may not make sense initially, mainly because we are used to geometry, where an area is always measured in square units. But, that is only because the length and width are always measured in the same kind of units, so multiplying length and width area always measured in the same kind of units, so multiplying length and width must yield square units. We are expanding our vision beyond that narrow view of things here. Consider a graph where the x-axis is time and the y-axis is velocity in feet per second. The area under the curve would be measured as seconds multiplied by feet per seconds, which is simply feet. So, the area under the curve is equal to the distance traveled in feet. In other words, the integral of velocity is distance.

OBJECTIVES

Evaluate Definite Integrals.
Find Average Value of a Continuous Function Over a Given Interval.
Differentiate Integral Expressions with the Variable in the Boundary.

Let us first consider part 2 of the Fundamental Theorem of Calculus, since it has a very practical application. This part of the Fundamental Theorem of Calculus gives us a clear method for evaluating definite integrals.

Ex 3.1.1: Evaluate $\displaystyle\int_2^8 (4x+3)\,dx$

Sol 3.1.1: First, let's start by treating this as a regular antiderivative

$$\int (4x+3)\,dx = 2x^2 + 3x$$

Note that our $+C$ will not be needed, as we will be taking a definite integral. Now, let's apply the Fundamental Theorem of Calculus.

$$\begin{aligned}\int_2^8 f(x)\,dx &= F(8) - F(2) \\ &= 2x^2 + 3x\Big|_2^8 \\ &= 2(8)^2 + 3(8) - \big(2(2)^2 + 3(2)\big) \\ &= \boxed{138}\end{aligned}$$

The vertical bar that you see is called the evaluation bar, and it's used to indicate that we are evaluating the antiderivative at the upper and lower limits of integration.

Ex 3.1.2: Evaluate $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

Sol 3.1.2:

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}}\,dx &= 2\sqrt{x}\Big|_1^4 \\ &= 2\sqrt{(4)} - 2\sqrt{(1)} \\ &= \boxed{2}\end{aligned}$$

Ex 3.1.3: Evaluate $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$

Sol 3.1.3:

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = -\cos(x)\Big|_0^{\frac{\pi}{2}}$$

$$= -\cos\left(\frac{\pi}{2}\right) + \cos(0)$$

$$= \boxed{1}$$

Ex 3.1.4: Evaluate $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\, du$

Sol 3.1.4:

$$\int_1^2 \frac{4+u^2}{u^3}\, du = \int_1^2 \left(4u^{-3} + u^{-1}\right) du$$

$$= \left(-2u^{-2} + \ln|u|\right)\Big|_1^2$$

$$= -2(2)^{-2} + \ln 2 - (-2(1)^{-2} + \ln 1)$$

$$= \boxed{\frac{3}{2} + \ln 2}$$

Ex 3.1.5: Evaluate $\displaystyle\int_{-5}^5 \frac{1}{x^3}\, dx$

Sol 3.1.5: When initially looking at the problem, one may simply proceed with finding the definite integral.

$$\int_{-5}^5 \frac{1}{x^3}\, dx = -\frac{1}{x^2}\Big|_{-5}^5$$

$$= -\frac{1}{(-5)^2} + \frac{1}{(5)^2}$$

$$= 0$$

But, this is a trap! We have to be careful here, because the function $\dfrac{1}{x^3}$ is *not defined* over the the interval $[-5, 5]$, since $\dfrac{1}{0^3}$ is undefined. Therefore, the Fundamental Theorem of Calculus $\boxed{\text{does not apply}}$.

Just as we had many properties for the indefinite integral, we have many properties for the definite integral. There are three main properties that are utilized often on the AP exam.

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2. $\int_a^a f(x)\,dx = 0$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

Ex 3.1.6: If $\int_{-5}^{2} f(x)\,dx = -17$ and $\int_{5}^{2} f(x)\,dx = -4$, find $\int_{-5}^{5} f(x)\,dx$.

Sol 3.1.6:

$$\begin{aligned}\int_{-5}^{5} f(x)\,dx &= \int_{-5}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx \\ &= \int_{-5}^{2} f(x)\,dx - \int_{5}^{2} f(x)\,dx \\ &= -17-(-4) \\ &= \boxed{-13}\end{aligned}$$

Part I of the Fundamental Theorem of Calculus is very important for the **theory** of calculus, but is limited (hehe) in the context of this course to L'Hospital problems which will will explore in a later chapter. Here is how the formula may be applied.

Ex 3.1.7: Use the Fundamental Theorem of Calculus to find $f'(t)$ if $f(t) = \int_2^{3t^2} (4x+3)\,dx$

Sol 3.1.7:

$$f'(t) = \frac{d}{dt}\int_2^{3t^2} (4x+3)\,dx$$

$$= \left(4\left(3t^2\right)+3\right)(6t)$$

$$= \boxed{72t^3+18t}$$

3.1 Free Response Homework

Use part II of the Fundamental Theorem of Calculus to evaluate the integral, or explain why the integral cannot be evaluated.

1. $\displaystyle\int_{-1}^{3} x^5\,dx$

2. $\displaystyle\int_{2}^{7} (5x-1)\,dx$

3. $\displaystyle\int_{-5}^{5} \frac{2}{x^3}\,dx$

4. $\displaystyle\int_{-3}^{-1} \frac{x^7-4x^3-3}{x}\,dx$

5. $\displaystyle\int_{1}^{2} \frac{3}{t^4}\,dt$

6. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc(y)\cot(y)\,dy$

7. $\displaystyle\int_{0}^{\frac{\pi}{4}} \sec^2(y)\,dy$

8. $\displaystyle\int_{1}^{9} \frac{3}{2z}\,dz$

9. $\displaystyle\int_{1}^{8} \frac{x^2-4}{\sqrt[3]{x}}\,dx$

10. $\displaystyle\int_{\pi}^{\frac{5\pi}{4}} \sin(y)\,dy$

11. $\displaystyle\int_{1}^{4} \frac{x^4-4x^2-5}{x^2}\,dx$

12. $\displaystyle\int_{3}^{5} \left(x^2+5x+6\right)\,dx$

13. $\displaystyle\int_{\pi}^{\frac{\pi}{4}} \cos(y)\,dy$

14. $\displaystyle\int_{1}^{4} \frac{x^3-2x^2-4x}{x^2}\,dx$

15. $\displaystyle\int_{1}^{2} \frac{x^2-4x+7}{x}\,dx$

16. $\displaystyle\int_{1}^{16} \frac{2x^2-1}{\sqrt[4]{x}}$

Use the following values for problems 17 - 27 to evaluate the given integrals.

$\displaystyle\int_{-2}^{5} f(x)\,dx = -2$	$\displaystyle\int_{1}^{5} f(x)\,dx = 3$
$\displaystyle\int_{-2}^{1} g(x)\,dx = 4$	$\displaystyle\int_{5}^{1} g(x)\,dx = 9$
$\displaystyle\int_{1}^{5} h(x)\,dx = 7$	$\displaystyle\int_{5}^{-2} h(x)\,dx = -6$

17. $\displaystyle\int_{-2}^{1} f(x)\,dx$

18. $\displaystyle\int_{-2}^{5} g(x)\,dx$

19. $\displaystyle\int_{-2}^{1} h(x)\,dx$

20. $\displaystyle\int_{1}^{5} (f(x)-g(x))\,dx$

21. $\displaystyle\int_{-2}^{5} (g(x) + h(x))\,dx$

22. $\displaystyle\int_{-2}^{1} (h(x) - f(x))\,dx$

23. $\displaystyle\int_{-2}^{5} (h(x) + f(x))\,dx$

24. $\displaystyle\int_{1}^{5} (2f(x) + 3h(x))\,dx$

25. $\displaystyle\int_{-2}^{1} (2f(x) - 3g(x))\,dx$

26. $\displaystyle\int_{-2}^{5} \left(\frac{1}{2}g(x) + 4h(x)\right)\,dx$

27. $\displaystyle\int_{1}^{5} \left(\frac{1}{3}h(x) + 2f(x)\right)\,dx$

28. $\displaystyle\int_{5}^{5} (f(x) + g(x) + h(x))\,dx$

Use part I of the Fundamental Theorem of Calculus to find the derivative of the function.

29. $\displaystyle g(y) = \int_{2}^{y} t^2 \sin(t)\,dt$

30. $\displaystyle g(x) = \int_{0}^{x} \sqrt{1 + 2t}\,dt$

31. $\displaystyle F(x) = \int_{x}^{2} \cos\left(t^2\right)\,dt$

32. $\displaystyle h(x) = \int_{2}^{\frac{1}{x}} \arctan(t)\,dt$

33. $\displaystyle y = \int_{3}^{\sqrt{x}} \frac{\cos(t)}{t}\,dt$

34. $\displaystyle f(x) = \int_{e}^{x^2} \ln\left(t^2 + 1\right)\,dt$

35. $\displaystyle f(x) = \int_{10}^{x^2} t \ln t\,dt$

36. $\displaystyle f(x) = \int_{e^x}^{5} \left(t^3 + t + 1\right)\,dt$

37. If $\displaystyle F(x) = \int_{1}^{t^2} \frac{\sqrt{1 + u^4}}{u}\,du$, find $F'(t)$.

38. If $\displaystyle h(x) = \int_{\pi}^{\sqrt{x}} e^{5t}\,dt$, find $h'(x)$.

39. If $\displaystyle h(m) = \int_{5}^{\cos(m)} t^2 \cos^{-1}(t)\,dt$, find $h'(m)$.

40. If $\displaystyle h(y) = \int_{5}^{\ln y} \frac{e^t}{t^4}\,dt$, find $h'(y)$.

3.1 Multiple Choice Homework

1. If $\displaystyle\int_{-5}^{2} f(x)\,dx = -17$ and $\displaystyle\int_{5}^{2} f(x)\,dx = -4$, then $\displaystyle\int_{-5}^{5} f(x)\,dx =$

   a) -21 b) -13 c) 0 d) 13 e) 21

---

2. Let f and g be continuous functions such that $\displaystyle\int_{0}^{6} f(x)\,dx = 9$, $\displaystyle\int_{3}^{6} f(x)\,dx = 5$, and $\displaystyle\int_{3}^{0} g(x)\,dx = -7$. What is the value of $\displaystyle\int_{0}^{3} \left(\frac{1}{2}f(x) - 3g(x)\right)\,dx$.

a) -23 b) -19 c) $-\frac{17}{2}$ d) 19 e) 23

---

3. Given that $\int_2^3 P(t)\,dt = 7$ and $\int_2^7 P(t)\,dt = -2$, what is $\int_7^3 P(t)\,dt$?

a) -9 b) -5 c) 5 d) 9 e) not enough information

---

4. Based on the information below, find $\int_1^{-2} (g(x) + f(x))\,dx$

$\int_{-2}^5 f(x)\,dx = -2$	$\int_1^5 f(x)\,dx = 3$
$\int_{-2}^1 g(x)\,dx = 4$	$\int_5^1 g(x)\,dx = 9$

a) -9 b) -1 c) 0 d) 1 e) 9

---

5. Based on the information below, find $\int_5^{-2} (g(x) - f(x))\,dx$

$\int_{-2}^5 f(x)\,dx = -2$	$\int_1^5 f(x)\,dx = 3$
$\int_{-2}^1 g(x)\,dx = 4$	$\int_5^1 g(x)\,dx = 9$

a) -3 b) 3 c) 6 d) -6 e) 14

---

6. Using the table values from questions 5 and 6, which of the following cannot be determined?

a) $\int_5^1 (g(x) + f(x))\,dx$ b) $\int_1^{-2} (g(x) - f(x))\,dx$

c) $\int_{-2}^5 3g(x)(-4(f(x)))\,dx$ d) $\int_1^5 (3g(x) + 4f(x))\,dx$

3.2: Definite Integrals and the Substitution Rule

Now, it's time to revisit u-substitution within the context of definite integrals. Although the process is largely similar, there are some nuances that we must consider.

OBJECTIVES

Evaluate Definite Integrals Using the Fundamental Theorem of Calculus.
Evaluate Definite Integrals Applying the Substitution Rule, When Appropriate.
Use Proper Notation When Evaluating These Integrals

Ex 3.2.1: Evaluate $\int_0^2 t^2\sqrt{t^3+1}\,dx$.

Sol 3.2.1: Now, this problem may look just like a regular u-substitution problem that we did in the previous chapter. However, when we switch our integration variable to u, we also need to make sure to switch our definite integration boundaries to match. Let's see what that means in the solution below.

$$\int_0^2 t^2\sqrt{t^3+1}\,dt = \frac{1}{3}\int_0^2 3t^2\sqrt{t^3+1}\,dt$$

$$\hookrightarrow\ u = t^3+1 \quad\Big|\quad \hookrightarrow\ du = 3t^2\,dt$$

Here is where we have to be careful. We cannot simply rewrite our definite integral in terms of u, since our upper and lower boundaries are in terms of x! Therefore, we need to also find our new boundaries for the u variables. Luckily, we can do this simply by plugging in our t values into our equation for u.

$$u(0) = (0)^3+1 = 1 \quad\Big|\quad u(2) = (2)^3+1 = 9$$

Now, we can continue integrating! (Note the boundary change in red.)

$$\begin{aligned}\frac{1}{3}\int_{0}^{2} 3t^2\sqrt{t^3+1}\,dt &= \frac{1}{3}\int_{1}^{9}\sqrt{u}\,du\\ &= \frac{1}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_1^9\\ &= \frac{2}{9}\left(9^{\frac{3}{2}}-1^{\frac{3}{2}}\right)\\ &= \boxed{\frac{52}{9}}\end{aligned}$$

Ex 3.2.2: Evaluate $\int_0^{\pi} x\cos\left(x^2\right)\,dx$.

Sol 3.2.2:

$$\hookrightarrow u = x^2 \quad \Big| \quad \hookrightarrow du = 2x\,dx$$

$$u(0) = (0)^2 = 0 \quad \Big| \quad u\left(\sqrt{\pi}\right) = \left(\sqrt{\pi}\right)^2 = \pi$$

$$= \frac{1}{2}\int_0^{\pi} \cos(u)\,du$$

$$= \frac{1}{2}\sin(u)\Big|_0^{\pi}$$

$$= \boxed{0}$$

Ex 3.2.3: Evaluate $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$