

Chapter 3:

Integrals

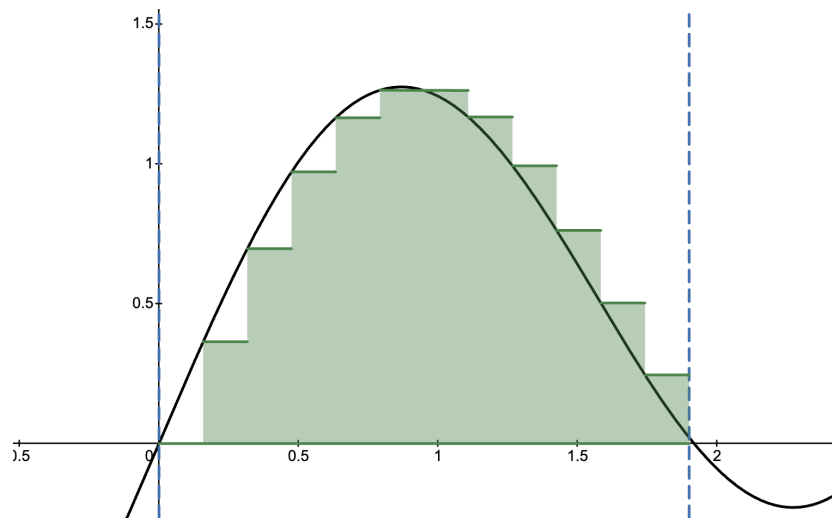
Chapter 3 Overview: Definite Integrals

In this chapter, we will study the Fundamental Theorem of Calculus, which establishes the link between the algebra and the geometry, with an emphasis on mechanics of how to find the definite integral. We will consider the differences implied between the context of the definite integral as an operation and as an area accumulator. We will learn some approximation techniques for definite integrals and see how they provide theoretical foundation for the integral. We will revisit graphical analysis in terms of the definite integral and view another typical AP context for it. Finally, we will consider what happens when trying to integrate at or near an asymptote.

As noted in the overview of the last chapter, antiderivatives are known as indefinite integrals because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an analytic-geometrical context of area. Though it is not necessary to know the theory behind this in order to calculate the integral, the theory is a major subject of integral calculus, so we will explore it briefly in here.

The Limit Definition of the Definite Integral

We know, from geometry, how to find the exact area of various polygons, but we never considered figures where one side is not made of a line segment. Here we want to consider an area bounded by some curve $y = f(x)$ on the top, the x -axis on the bottom, some arbitrary $x = a$ on the left, and $x = b$ on the right.



As we can see above, the area approximated by rectangles whose height is the y -value of the equation and whose width we will call Δx . The more rectangles we make, the better the approximation. For a good animation of this concept, consider the following video:

[Riemann sum approximation animation](#)

The area of each rectangle would be $f(x) \cdot \Delta x$, and the total area of n rectangles would be

$$A = \sum_{i=1}^n f(x_i) \cdot \Delta x.$$

This equation is known as the **Riemann summation**. Although this equation looks complicated, it represents a rather simple idea. We are adding up the areas of many thin rectangles to approximate the total area under the curve $y = f(x)$ between two points. As we increase the number of rectangles n , each rectangle becomes narrower, and thus our approximation becomes more accurate.

But, how can we find the exact area? With the Riemann sum, we are only coming up with better and better approximations right now. If we could make an infinite number of rectangles (which would be infinitely thin), we could potentially find the exact area under this curve. Luckily, we just so happen to have the mathematical tools to do this: we can take the limit as n approaches infinity.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \cdot \Delta x$$

This is where the definite integral comes in. The definite integral provides a precise way to calculate the exact area under a curve by taking the limit of the Riemann sum as the number of rectangles approaches infinity. In other words, instead of merely approximating the area with a finite number of rectangles, the definite integral captures what happens when the width

of each rectangle becomes infinitesimally small. We write this limit in a compact form as:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \cdot \Delta x,$$

where a is the "lower bound" and b is the "upper bound." Mathematicians sometimes nuance this statement as

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(a + k\Delta x) \cdot \Delta x, \text{ where } \Delta x = \frac{b-a}{n}$$

3.1: The Fundamental Theorem of Calculus

Up to this point, we've seen how integration can be used to find the exact area under a curve by taking the limit of Riemann sums. But what's truly remarkable is how integration connects so deeply with differentiation. This connection is captured in one of the most important results in all of calculus: the Fundamental Theorem of Calculus (FTC).

The Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a, b]$, then:

1. $\frac{d}{dx} \int_c^x f(t) dt = f(x)$ or $\frac{d}{dx} \int_c^u f(t) dt = f(u) \cdot \frac{du}{dx}$
2. If $F'(x) = f(x)$, then $\int_a^b f(x) dx = F(b) - F(a)$.

The first part of the Fundamental Theorem of Calculus simply says what we already know—that an integral is an anti-derivative. The second part of the Fundamental Theorem says that the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

The idea of the integral meaning the area may not make sense initially, mainly because we are used to geometry, where an area is always measured in square units. But, that is only because the length and width are always measured in the same kind of units, so multiplying length and width area always measured in the same kind of units, so multiplying length and width must yield square units. We are expanding our vision beyond that narrow view of things here. Consider a graph where the x -axis is time and the y -axis is velocity in feet per second. The area under the curve would be measured as seconds multiplied by feet per seconds, which is simply feet. So, the area under the curve is equal to the distance traveled in feet. In other words, the integral of velocity is distance.

OBJECTIVES

Evaluate Definite Integrals.

Find Average Value of a Continuous Function Over a Given Interval.

Differentiate Integral Expressions with the Variable in the Boundary.

Let us first consider part 2 of the Fundamental Theorem of Calculus, since it has a very practical application. This part of the Fundamental Theorem of Calculus gives us a clear method for evaluating definite integrals.

Ex 3.1.1: Evaluate $\int_2^8 (4x + 3) dx$

Sol 3.1.1: First, let's start by treating this as a regular antiderivative

$$\int (4x + 3) dx = 2x^2 + 3x$$

Note that our $+C$ will not be needed, as we will be taking a definite integral. Now, let's apply the Fundamental Theorem of Calculus.

$$\begin{aligned}\int_2^8 f(x) dx &= F(8) - F(2) \\ &= \left(2x^2 + 3x\right)\bigg|_2^8 \\ &= 2(8)^2 + 3(8) - \left(2(2)^2 + 3(2)\right) \\ &= \boxed{138}\end{aligned}$$

The vertical bar that you see is called the evaluation bar, and it's used to indicate that we are evaluating the antiderivative at the upper and lower limits of integration.

Ex 3.1.2: Evaluate $\int_1^4 \frac{1}{\sqrt{x}} dx$

Sol 3.1.2:

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}} dx &= 2\sqrt{x}\bigg|_1^4 \\ &= 2\sqrt{(4)} - 2\sqrt{(1)} \\ &= \boxed{2}\end{aligned}$$

Ex 3.1.3: Evaluate $\int_0^{\frac{\pi}{2}} \sin(x) dx$

Sol 3.1.3:

$$\int_0^{\frac{\pi}{2}} \sin(x) dx = -\cos(x)\bigg|_0^{\frac{\pi}{2}}$$

$$= -\cos\left(\frac{\pi}{2}\right) + \cos(0)$$

$$= \boxed{1}$$

Ex 3.1.4: Evaluate $\int_1^2 \frac{4+u^2}{u^3} du$

Sol 3.1.4:

$$\begin{aligned} \int_1^2 \frac{4+u^2}{u^3} du &= \int_1^2 (4u^{-3} + u^{-1}) du \\ &= \left(-2u^{-2} + \ln|u|\right) \Big|_1^2 \\ &= -2(2)^{-2} + \ln 2 - (-2(1)^{-2} + \ln 1) \\ &= \boxed{\frac{3}{2} + \ln 2} \end{aligned}$$

Ex 3.1.5: Evaluate $\int_{-5}^5 \frac{1}{x^3} dx$

Sol 3.1.5: When initially looking at the problem, one may simply proceed with finding the definite integral.

$$\begin{aligned} \int_{-5}^5 \frac{1}{x^3} dx &= -\frac{1}{x^2} \Big|_{-5}^5 \\ &= -\frac{1}{(-5)^2} + \frac{1}{(5)^2} \\ &= 0 \end{aligned}$$

But, this is a trap! We have to be careful here, because the function $\frac{1}{x^3}$ is *not defined* over the interval $[-5, 5]$, since $\frac{1}{0^3}$ is undefined. Therefore, the Fundamental Theorem of Calculus does not apply.

Just as we had many properties for the indefinite integral, we have many properties for the definite integral. There are three main properties that are utilized often on the AP exam.

Properties of Definite Integrals

1. $\int_a^b f(x) dx = -\int_b^a f(x) dx$
2. $\int_a^a f(x) dx = 0$
3. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, where $a < c < b$

Ex 3.1.6: If $\int_{-5}^2 f(x) dx = -17$ and $\int_5^2 f(x) dx = -4$, find $\int_{-5}^5 f(x) dx$.

Sol 3.1.6:

$$\begin{aligned}
 \int_{-5}^5 f(x) dx &= \int_{-5}^2 f(x) dx + \int_2^5 f(x) dx \\
 &= \int_{-5}^2 f(x) dx - \int_5^2 f(x) dx \\
 &= -17 - (-4) \\
 &= \boxed{-13}
 \end{aligned}$$

Part I of the Fundamental Theorem of Calculus is very important for the **theory** of calculus, but is limited (hehe) in the context of this course to L'Hospital problems which will explore in a later chapter. Here is how the formula may be applied.

Ex 3.1.7: Use the Fundamental Theorem of Calculus to find $f'(t)$ if $f(t) = \int_2^{3t^2} (4x + 3) dx$

Sol 3.1.7:

$$f'(t) = \frac{d}{dt} \int_2^{3t^2} (4x + 3) dx$$

$$= (4(3t^2) + 3)(6t)$$

$$= \boxed{72t^3 + 18t}$$

3.1 Free Response Homework

Use part II of the Fundamental Theorem of Calculus to evaluate the integral, or explain why the integral cannot be evaluated.

1. $\int_{-1}^3 x^5 dx$

2. $\int_2^7 (5x - 1) dx$

3. $\int_{-5}^5 \frac{2}{x^3} dx$

4. $\int_{-3}^{-1} \frac{x^7 - 4x^3 - 3}{x} dx$

5. $\int_1^2 \frac{3}{t^4} dt$

6. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc(y) \cot(y) dy$

7. $\int_0^{\frac{\pi}{4}} \sec^2(y) dy$

8. $\int_1^9 \frac{3}{2z} dz$

9. $\int_1^8 \frac{x^2 - 4}{\sqrt[3]{x}} dx$

10. $\int_{\pi}^{\frac{5\pi}{4}} \sin(y) dy$

11. $\int_1^4 \frac{x^4 - 4x^2 - 5}{x^2} dx$

12. $\int_3^5 (x^2 + 5x + 6) dx$

13. $\int_{\pi}^{\frac{\pi}{4}} \cos(y) dy$

14. $\int_1^4 \frac{x^3 - 2x^2 - 4x}{x^2} dx$

15. $\int_1^2 \frac{x^2 - 4x + 7}{x} dx$

16. $\int_1^{16} \frac{2x^2 - 1}{\sqrt[4]{x}} dx$

Use the following values for problems 17 - 27 to evaluate the given integrals.

$\int_{-2}^5 f(x) dx = -2$	$\int_1^5 f(x) dx = 3$
$\int_{-2}^1 g(x) dx = 4$	$\int_5^1 g(x) dx = 9$
$\int_1^5 h(x) dx = 7$	$\int_5^{-2} h(x) dx = -6$

17. $\int_{-2}^1 f(x) dx$

18. $\int_{-2}^5 g(x) dx$

19. $\int_{-2}^1 h(x) dx$

20. $\int_1^5 (f(x) - g(x)) dx$

$$21. \int_{-2}^5 (g(x) + h(x)) dx$$

$$22. \int_{-2}^1 (h(x) - f(x)) dx$$

$$23. \int_{-2}^5 (h(x) + f(x)) dx$$

$$24. \int_1^5 (2f(x) + 3h(x)) dx$$

$$25. \int_{-2}^1 (2f(x) - 3g(x)) dx$$

$$26. \int_{-2}^5 \left(\frac{1}{2}g(x) + 4h(x) \right) dx$$

$$27. \int_1^5 \left(\frac{1}{3}h(x) + 2f(x) \right) dx$$

$$28. \int_5^5 (f(x) + g(x) + h(x)) dx$$

Use part I of the Fundamental Theorem of Calculus to find the derivative of the function.

$$29. g(y) = \int_2^y t^2 \sin(t) dt$$

$$30. g(x) = \int_0^x \sqrt{1+2t} dt$$

$$31. F(x) = \int_x^2 \cos(t^2) dt$$

$$32. h(x) = \int_2^{\frac{1}{x}} \arctan(t) dt$$

$$33. y = \int_3^{\sqrt{x}} \frac{\cos(t)}{t} dt$$

$$34. f(x) = \int_e^{x^2} \ln(t^2 + 1) dt$$

$$35. f(x) = \int_{10}^{x^2} t \ln t dt$$

$$36. f(x) = \int_{e^x}^5 (t^3 + t + 1) dt$$

$$37. \text{ If } F(x) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u} du, \text{ find } F'(t).$$

$$38. \text{ If } h(x) = \int_{\pi}^{\sqrt{x}} e^{5t} dt, \text{ find } h'(x).$$

$$39. \text{ If } h(m) = \int_5^{\cos(m)} t^2 \cos^{-1}(t) dt, \text{ find } h'(m).$$

$$40. \text{ If } h(y) = \int_5^{\ln y} \frac{e^t}{t^4} dt, \text{ find } h'(y).$$

3.1 Multiple Choice Homework

$$1. \text{ If } \int_{-5}^2 f(x) dx = -17 \text{ and } \int_5^2 f(x) dx = -4, \text{ then } \int_{-5}^5 f(x) dx =$$

a) -21

b) -13

c) 0

d) 13

e) 21

$$2. \text{ Let } f \text{ and } g \text{ be continuous functions such that } \int_0^6 f(x) dx = 9, \int_3^6 f(x) dx = 5, \text{ and } \int_3^0 g(x) dx = -7. \text{ What is the value of } \int_0^3 \left(\frac{1}{2}f(x) - 3g(x) \right) dx.$$

- a) -23 b) -19 c) $-\frac{17}{2}$ d) 19 e) 23
-

3. Given that $\int_2^3 P(t) dt = 7$ and $\int_2^7 P(t) dt = -2$, what is $\int_7^3 P(t) dt$?

- a) -9 b) -5 c) 5 d) 9 e) not enough information
-

4. Based on the information below, find $\int_1^{-2} (g(x) + f(x)) dx$

$\int_{-2}^5 f(x) dx = -2$	$\int_1^5 f(x) dx = 3$
$\int_{-2}^1 g(x) dx = 4$	$\int_5^1 g(x) dx = 9$

- a) -9 b) -1 c) 0 d) 1 e) 9
-

5. Based on the information below, find $\int_5^{-2} (g(x) - f(x)) dx$

$\int_{-2}^5 f(x) dx = -2$	$\int_1^5 f(x) dx = 3$
$\int_{-2}^1 g(x) dx = 4$	$\int_5^1 g(x) dx = 9$

- a) -3 b) 3 c) 6 d) -6 e) 14
-

6. Using the table values from questions 5 and 6, which of the following cannot be determined?

- a) $\int_5^1 (g(x) + f(x)) dx$ b) $\int_1^{-2} (g(x) - f(x)) dx$
c) $\int_{-2}^5 3g(x)(-4(f(x))) dx$ d) $\int_1^5 (3g(x) + 4f(x)) dx$
-

3.2: Definite Integrals and the Substitution Rule

Now, it's time to revisit u -substitution within the context of definite integrals. Although the process is largely similar, there are some nuances that we must consider.

OBJECTIVES

Evaluate Definite Integrals Using the Fundamental Theorem of Calculus.
Evaluate Definite Integrals Applying the Substitution Rule, When Appropriate.
Use Proper Notation When Evaluating These Integrals

Ex 3.2.1: Evaluate $\int_0^2 t^2 \sqrt{t^3 + 1} dx$.

Sol 3.2.1: Now, this problem may look just like a regular u -substitution problem that we did in the previous chapter. However, when we switch our integration variable to u , we also need to make sure to switch our definite integration boundaries to match. Let's see what that means in the solution below.

$$\int_0^2 t^2 \sqrt{t^3 + 1} dt = \frac{1}{3} \int_0^2 3t^2 \sqrt{t^3 + 1} dt$$
$$\hookrightarrow u = t^3 + 1 \quad \Bigg| \quad \hookrightarrow du = 3t^2 dt$$

Here is where we have to be careful. We cannot simply rewrite our definite integral in terms of u , since our upper and lower boundaries are in terms of t ! Therefore, we need to also find our new boundaries for the u variables. Luckily, we can do this simply by plugging in our t values into our equation for u .

$$u(0) = (0)^3 + 1 = 1 \quad \Bigg| \quad u(2) = (2)^3 + 1 = 9$$

Now, we can continue integrating! (Note the boundary change in red.)

$$\begin{aligned} \frac{1}{3} \int_0^2 3t^2 \sqrt{t^3 + 1} dt &= \frac{1}{3} \int_1^9 \sqrt{u} du \\ &= \frac{1}{3} \left(\frac{2}{3} u^{\frac{3}{2}} \right) \Bigg|_1^9 \\ &= \frac{2}{9} \left(9^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) \\ &= \boxed{\frac{52}{9}} \end{aligned}$$

Ex 3.2.2: Evaluate $\int_0^{\sqrt{\pi}} x \cos(x^2) dx$.

Sol 3.2.2:

$$\begin{aligned} \hookrightarrow u = x^2 & \quad \Bigg| \quad \hookrightarrow du = 2x dx \\ u(0) = (0)^2 = 0 & \quad \Bigg| \quad u(\sqrt{\pi}) = (\sqrt{\pi})^2 = \pi \\ & = \frac{1}{2} \int_0^{\pi} \cos(u) du \\ & = \frac{1}{2} \sin(u) \Big|_0^{\pi} \\ & = \boxed{0} \end{aligned}$$

Ex 3.2.3: Evaluate $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx$.

Sol 3.2.3:

$$\begin{aligned} \hookrightarrow u = \frac{1}{x} & \quad \Bigg| \quad \hookrightarrow du = -\frac{1}{x^2} \\ u(1) = \frac{1}{(1)} = 1 & \quad \Bigg| \quad u(2) = \frac{1}{(2)} = \frac{1}{2} \\ \int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx & = - \int_1^{\frac{1}{2}} e^u du \\ & = -e^u \Big|_1^{\frac{1}{2}} \\ & = \boxed{-e^{\frac{1}{2}} + e} \end{aligned}$$

Ex 3.2.4: Evaluate $\int_1^{\sqrt{13}} \frac{x}{x^2 + 3} dx$.

Sol 3.2.4:

$$\begin{aligned} \hookrightarrow u = x^2 + 3 & \quad \Bigg| \quad \hookrightarrow du = 2x \, dx \\ u(1) = (1)^2 + 3 = 4 & \quad \Bigg| \quad u\left(\sqrt{13}\right)^2 + 3 = 16 \\ \int_1^{\sqrt{13}} \frac{x}{x^2 + 3} \, dx &= \frac{1}{2} \int_1^{\sqrt{13}} \frac{2x}{x^2 + 3} \, dx \\ &= \frac{1}{2} \int_4^{16} \frac{1}{u} \, du \\ &= \frac{1}{2} \ln |u| \Bigg|_4^{16} \\ &= \boxed{\frac{1}{2} \ln(16) - \frac{1}{2} \ln 4} \end{aligned}$$

Now, technically, $\frac{1}{2} \ln(16) - \frac{1}{2} \ln 4$ is the correct answer. However, when you are taking the multiple choice portion of the AP Calc BC exam, the expectation is that trivial logarithms should be simplified, so this answer may not appear as a choice. So, we must utilize our log rules:

$$\begin{aligned} \frac{1}{2} \ln(16) - \frac{1}{2} \ln 4 &= \frac{1}{2} \ln\left(\frac{16}{4}\right) \\ &= \frac{1}{2} \ln 4 \\ &= \ln 4^{\frac{1}{2}} \\ &= \boxed{\ln 2} \end{aligned}$$

The Average Value

One simple application of the definite integral is the Average Value Theorem. Recall that the average of a finite set of numbers is the total of the numbers divided by how many numbers we are averaging. Or, in technical terms:

$$\text{avg} = \frac{\sum_{i=1}^n x_i}{n}$$

But, what does it mean to take the average value of a continuous function? Let's say you drive from home to school—what was your average velocity? What was the average temperature today? What was your average height for the first 15 years of your life? All these questions can be answered with the following formula:

The Average Value Formula

The average value of a function f on a closed interval $[a, b]$ is defined by

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

If we look at this formula in the context of the Fundamental Theorem of Calculus, it will start to make a little more sense.

$$\begin{aligned} \int_a^b f(x) dx &= F(b) - F(a) \\ \frac{1}{b-a} \int_a^b f(x) dx &= \frac{1}{b-a} (F(b) - F(a)) \\ &= \frac{F(b) - F(a)}{b-a} \end{aligned}$$

Notice that this is just the average slope for $F(x)$ on $x \in [a, b]$. Since the derivative $F'(x)$ gives the instantaneous rate of change of $F(x)$, the average slope of $F(x)$ is the same as average value of $F'(x)$. But, since the definition in the Fundamental Theorem of Calculus says that $F'(x) = f(x)$, this is actually just the average value of $f(x)$.

Ex 3.2.5: Find the average value of $f(x) = x^2 + 1$ on $[0, 5]$.

Sol 3.2.5: We first start with our formula:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

Then, we can substitute in the function and the interval.

$$f_{avg} = \frac{1}{5-0} \int_0^5 (x^2 + 1) dx$$

Using either MATH 9 on our TI-84 or integrating analytically gives us $f_{avg} = \boxed{\frac{28}{3}}$.

Ex 3.2.6: Find the average value of $h(\theta) = \sec(\theta) \tan(\theta)$ on $\left[0, \frac{\pi}{4}\right]$.

Sol 3.2.6:

$$\begin{aligned} h_{avg} &= \frac{1}{b-a} \int_a^b h(\theta) d\theta \\ &= \frac{1}{\frac{\pi}{4} - 0} \int_0^{\frac{\pi}{4}} \sec(\theta) \tan(\theta) d\theta \\ &= \boxed{\frac{4}{\pi} (\sqrt{2} - 1) \approx 0.524} \end{aligned}$$

3.2 Free Response Homework

1. $\int_0^1 x^2 (1 + 2x^3)^5 dx$

3. $\int_{-1}^1 x \sqrt{4 - x^2} dx$

5. $\int_0^3 \frac{10t + 15}{\sqrt[4]{t^2 + 3t + 1}} dt$

7. $\int_1^3 \frac{5t}{t^2 + 1} dt$

9. $\int_{\sqrt{3}}^2 ye^{y^2-3} dy$

11. $\int_3^{e^2+2} \frac{1}{x-2} dx$

13. $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} dx$

15. $\int_0^\pi \frac{\sin(x)}{2 - \cos(x)} dx$

17. $\int_0^{\ln 2} \frac{e^x}{1 + e^{2x}} dx$

19. $\int_0^{\frac{\pi}{8}} \sec^2(2x) dx$

21. $\int_0^\pi \frac{\cos(x)}{2 + \sin(x)} dx$

23. $\int_0^{\sqrt{\frac{\pi}{4}}} m \sec(m^2) \tan(m^2) dm$

25. $\int_{\frac{\pi}{2}}^\pi \cos^9(x) \sin(x) dx$

27. $\int_\pi^{2\pi} \cos\left(\frac{1}{2}\theta\right) d\theta$

29. $\int_0^{e^2-1} \frac{1}{x+1} dx$

2. $\int_0^1 x^3 (x^4 + 5)^3 dx$

4. $\int_1^2 \frac{x^2}{\sqrt[3]{9 - x^3}} dx$

6. $\int_1^2 \frac{x+1}{\sqrt{x^2 + 2x + 4}} dx$

8. $\int_{-1}^2 \frac{1}{2x+5} dx$

10. $\int_0^1 \frac{v^2}{8 - v^3} dv$

12. $\int_0^{\frac{\pi}{3}} \frac{\sin(\theta)}{\cos^2(\theta)} d\theta$

14. $\int_0^\pi \sec^2\left(\frac{t}{4}\right) dt$

16. $\int_2^4 \frac{1}{x \ln x} dx$

18. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^5(x) \sin(x) dx$

20. $\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2(\ln y)}{y} dy$

22. $\int_0^\pi \frac{\sin(y)}{2 + \cos(y)} dy$

24. $\int_0^{\frac{\pi}{4}} \sec^2(x) \tan^3(x) dx$

26. $\int_0^\pi \cos^6\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) dx$

28. $\int_2^{e^3+1} \frac{(\ln(x-1))^4}{x-1} dx$

30. $\int_5^{e^3+4} \frac{1}{x-4} dx$

Find the average value of each of the following functions over the given interval.

31. $F(x) = (x - 3)^2$ on $x \in [3, 7]$

32. $H(x) = \sqrt{x}$ on $x \in [0, 3]$

33. $F(x) = \sec^2(x)$ on $x \in \left[0, \frac{\pi}{4}\right]$

34. $F(x) = \frac{1}{x}$ on $x \in [1, 3]$

35. $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [1, 4]$

36. $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [4, 9]$

37. $f(x) = \cos(x) \sin^4(x)$ on $x \in [0, \pi]$

38. $g(x) = xe^{-x^2}$ on $x \in [1, 5]$

39. $G(x) = \frac{x}{(1 + x^2)^3}$ on $x \in [0, 2]$

40. $h(x) = \frac{x}{(1 + x^2)^2}$ on $x \in [0, 4]$

41. If a cookie taken out of a 450° F oven cools in a 60° F room, then according to Newton's Law of Cooling, the temperature of the cookie t minutes after it has been taken out of the oven is given by

$$T(t) = 60 + 390e^{-0.205t}.$$

What is the average value of the cookie's temperature during its first 10 minutes out of the oven?

42. We know as the seasons change so do the length of the days. Suppose the length of the day varies sinusoidally with time by the equation

$$L(t) = 10 - 3 \cos\left(\frac{\pi t}{182}\right),$$

where t is the number of days after the winter solstice (December 22, 2007). What was the average day length from January 1, 2008 to March 31, 2008?

43. During one summer in the Sunset, the temperature is modeled by the function

$$T(t) = 50 + 15 \sin\left(\frac{\pi}{12}t\right),$$

where T is measured in F° and t is measured in hours after 7 a.m. What is the average temperature in the Sunset during the six-hour chemistry class that runs from 9 a.m. to 3 p.m.?

3.2 Multiple Choice Homework

1. $\int_1^4 \frac{1}{(1 + \sqrt{x})^2 \sqrt{x}} dx$

a) $\frac{6}{5}$

b) $\frac{1}{3}$

c) $\frac{2}{3}$

d) $\frac{4}{9}$

e) $\frac{3}{2}$

2. If $\int_1^4 h(x) dx = 6$, then $\int_1^4 h(5-x) dx =$

- a) -6 b) -1 c) 0 d) 3 e) 6
-

3. $\int_e^{e^2} \frac{1}{x \ln x} dx =$

- a) $\ln(\ln 2)$ b) $\frac{2}{e^2}$ c) $\ln 2$ d) $\frac{1-2e}{2e^2}$ e) DNE
-

4. Determine the average value of $y = e^{6x}$ on $x \in [0, 4]$.

- a) $\frac{e^{24}-1}{4}$ b) $\frac{e^{24}-1}{6}$ c) $\frac{e^{24}}{24}$ d) $\frac{e^{24}}{6}$ e) $\frac{e^{24}-1}{24}$
-

5. Determine the average value of $g(x) = (2x+3)^2$ on $x \in [-3, -1]$.

- a) $\frac{7}{3}$ b) -4 c) 5 d) $\frac{14}{3}$ e) 3
-

6. Determine the average value of $g(x) = e^{7x}$ on $x \in [0, 4]$.

- a) $\frac{1}{14}e^{14}$ b) $\frac{1}{7}(e^{14}-1)$ c) $\frac{1}{14}(e^{14}-1)$ d) $\frac{1}{2}(e^{14}-1)$ e) $\frac{1}{7}e^{14}$
-

7. If the function $y = x^3$ has an average value of 9 on $x \in [0, k]$, then $k =$.

- a) 3 b) $\sqrt{3}$ c) $\sqrt[3]{18}$ d) $\sqrt[4]{36}$ e) $\sqrt[3]{36}$
-

8. Find the average rate of change of $y = x^2 + 5x + 14$ on $x \in [-1, 2]$

- a) 3 b) 6 c) 9 d) $\frac{65}{6}$ e) 18

9. If the average of the function $f(x) = |x - a|$ on $[-1, 1]$ is $\frac{5}{4}$, what is/are the values of a ?

- a) ± 1 b) $\pm \frac{1}{2}$ c) $\pm \frac{1}{4}$ d) 0 e) None of these
-

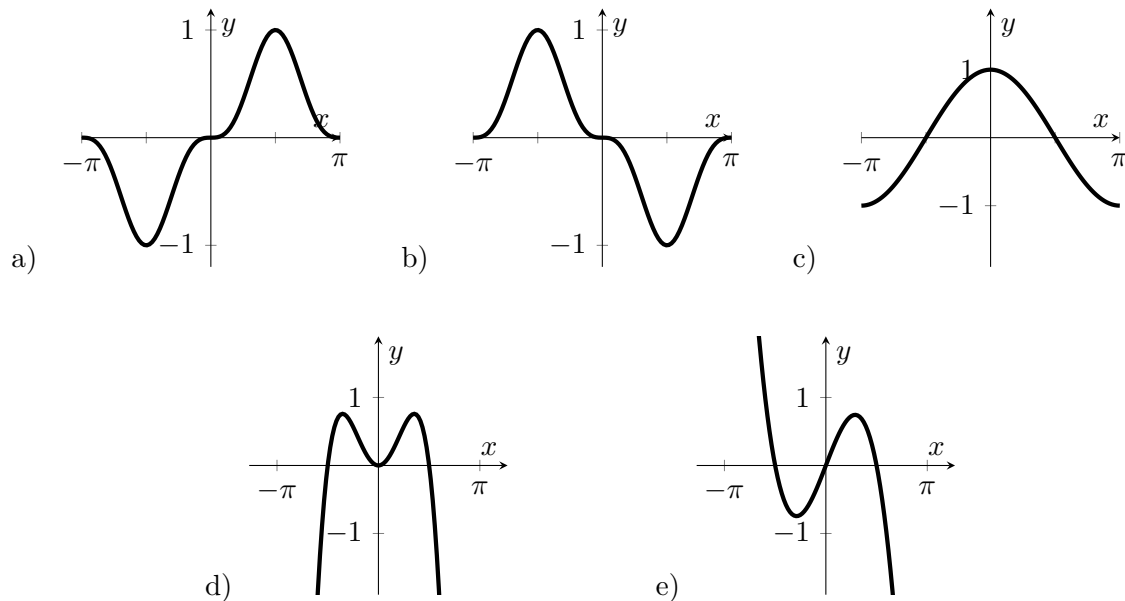
10. What is the average rate of change of the function $f(x) = x^4 - 5x$ on the closed interval $[0, 3]$?

- a) 8.5 b) 8.7 c) 22 d) 23 e) 66
-

11. Determine the average value of $y = e^x \cos(x)$ on $x \in \left[0, \frac{\pi}{2}\right]$

- a) 0 b) 1.213 c) 1.905 d) 2.425 e) 3.810
-

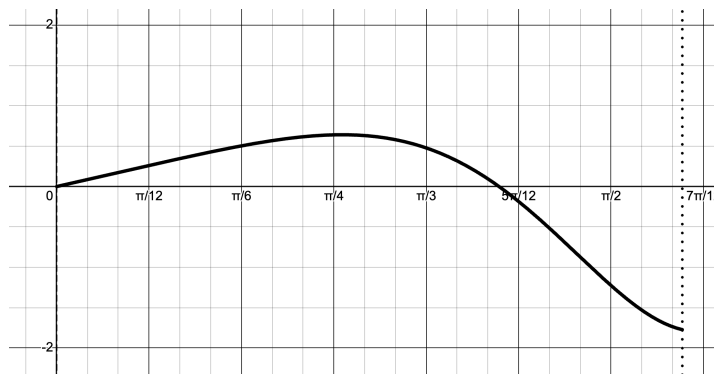
12. The graphs of five functions are shown below. Which function has a nonzero average value over the closed interval $x = [-\pi, \pi]$.



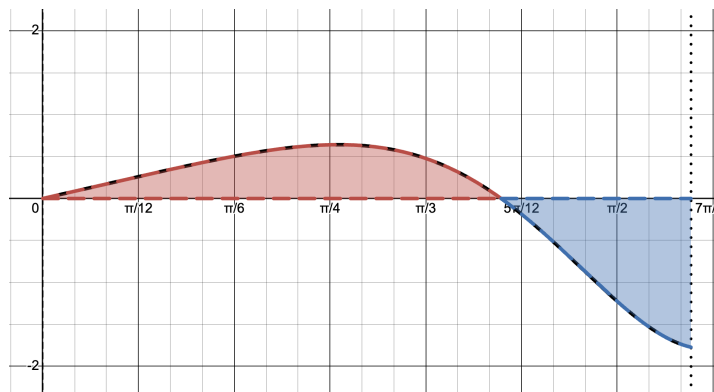
3.3: Context For Definite Integrals: Area, Displacement, and Net Change

Since we originally defined the definite integral in terms of “area under a curve,” we need to consider what this idea of “area” really means in relation to the definite integral.

Recall **Ex 3.2.2**, where we had a function $y = x \cos(x^2)$ on $x \in [0, \sqrt{\pi}]$. The graph looks like this:



In **Ex 3.2.2**, we found that $\int_0^{\sqrt{\pi}} x \cos(x^2) dx = 0$. But, there's clearly area under the curve, so how can the integral equal both the area and 0? Well, as it turns out, because the integral was created from rectangles with width dx and height $f(x)$, a negative $f(x)$ will result in a rectangle with “negative area.” Take a look at the following graph:



It's clear that by our definition of the definite integral, the “area” in red would cancel out the “area” in blue. So, how would we find the actual, positive area? That is what we are going to talk about in this section.

OBJECTIVES

Relate Definite Integrals to Area Under a Curve.

Understand the Difference Between Displacement and Distance.

Understand Displacement and Distance in Other Contexts.

Ex 3.3.1: What is the area under $y = x \cos(x^2)$ on $x \in [0, \sqrt{\pi}]$?

Sol 3.3.1: First, let's make sure we're clear on terminology. In this context, "area under" means "area between the graph and x -axis." Now, to find the area under this graph, we have two methods.

The first method is to split the graph into two parts: one part above the x -axis and one part below the x -axis. We can integrate the part above the x -axis as usual, and for the part below the x -axis, we simply negate the result of the definite integral. The sum of these two parts should equal the area under the curve. We can begin by finding the point where the graph passes the x -axis.

$$x \cos(x^2) = 0$$

$$x = 1.253$$

Now, by the rule $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, where $a < c < b$, we can split our integral into two parts. Remember that we are negating the second term to compensate for the area being below the x -axis.

$$\int_0^{\sqrt{\pi}} x \cos(x^2) dx \rightarrow \int_0^{1.253} x \cos(x^2) dx + \left(- \int_{1.253}^{\sqrt{\pi}} x \cos(x^2) dx \right)$$

Finally, using **MATH** **9**, we arrive at the answer **1**.

However, there is a much cleaner way to do this problem. As it turns out, if we simply take the integral of the absolute value of the integrand (the function that is being integrated), we arrive at the same answer. This is because the absolute value automatically accounts for both the parts of the graph that are above and below the x -axis, turning any negative "signed area" into positive area. Therefore, we can write

$$\int_0^{\sqrt{\pi}} |x \cos(x^2)| dx,$$

which turns out to also evaluate to **1**.

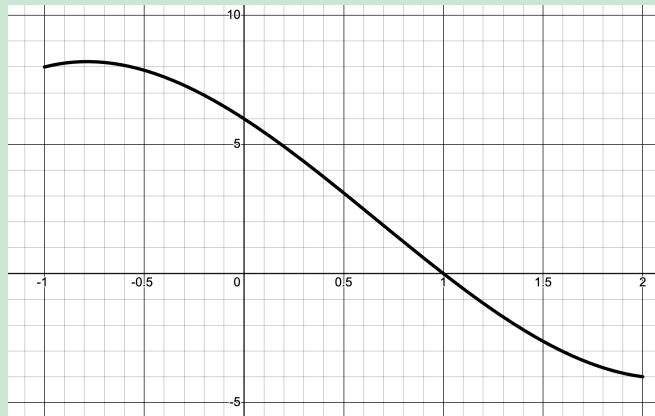
Now that we've seen an example, let's generalize some steps to finding the total area described by a definite integral.

Steps to Finding Total Area

1. Draw the function.
2. Find the zeroes between $x = a$ and $x = b$.
3. Set up separate integrals representing the area above and below the x -axis.
4. Change the sign on those integrals which represent the negative values (i.e., those where the curve is below the x -axis).
5. Solve the integral expression.

Ex 3.3.2: Find the area under $y = x^3 - 2x^2 - 5x + 6$ on $x \in [-1, 2]$.

Sol 3.3.2: Let's take a quick look at the graph:



It's clear that the graph crosses the x -axis at $x = 1$. Therefore, to get the total area, let's set up our two-term integration and solve.

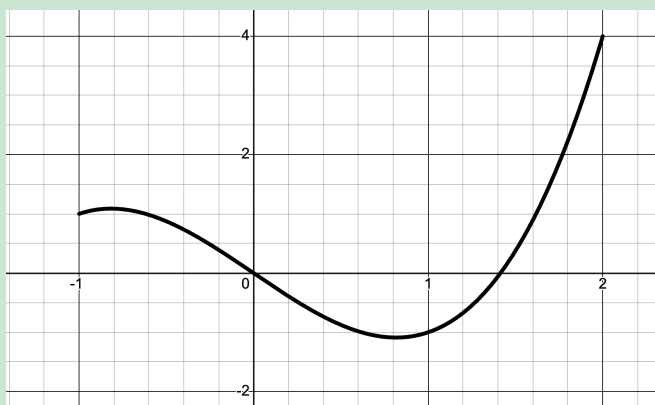
$$\begin{aligned} & \int_{-1}^1 (x^3 - 2x^2 + 5x + 6) \, dx + \left(- \int_1^2 (x^3 - 2x^2 + 5x + 6) \, dx \right) \\ &= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_{-1}^1 - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_1^2 \\ &= \frac{32}{3} + \frac{29}{12} \end{aligned}$$

$$= \boxed{\frac{157}{12}}$$

Now, two important things about this specific example. Instead of using the evaluation bar to denote evaluating the antiderivative, we instead can just use standard square brackets. These two notations can be used interchangeably. The second point is that the reason why we didn't use the quicker, absolute value method was because that method cannot be efficiently done by hand. On the AP exam, one can expect this problem to be on the no-calculator portion, and without a calculator, the absolute value method cannot be done.

Ex 3.3.3: Find the area under $y = x^3 - 2x$ on $x \in [-1, 2]$.

Sol 3.3.3: Once again, we begin with a sketch of the function:



Note that there are three regions here, so we will need to separate our integral into three terms. Our zeroes between $[-1, 2]$ are 0 and $\sqrt{2}$.

$$\begin{aligned} & \int_{-1}^0 (x^3 - 2x) \, dx + \left(- \int_0^{\sqrt{2}} (x^3 - 2x) \, dx \right) + \int_{\sqrt{2}}^2 (x^3 - 2x) \, dx \\ &= \left[\frac{x^4}{4} - x^2 \right]_{-1}^0 - \left[\frac{x^4}{4} - x^2 \right]_0^{\sqrt{2}} + \left[\frac{x^4}{4} - x^2 \right]_{\sqrt{2}}^2 \\ &= \frac{3}{4} - (-1) + 1 \\ &= \boxed{\frac{11}{4}} \end{aligned}$$

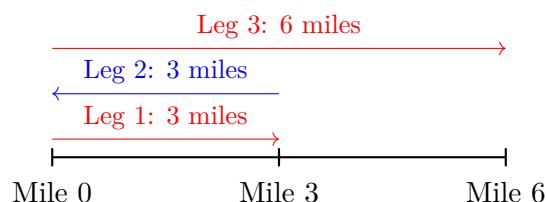
Distance and Displacement

Imagine you were leaving your house to go to school, and that school is 6 miles away. You leave your house and halfway to school you realize you have forgotten your AP Calc BC homework. You head back home, grab your assignment (that you just finished in the morning, of course), and then head to school.

There are two different questions that can be asked here. How far are you from where you started? And how far have you actually traveled? Although these questions seem extremely similar, their distinction is very important.

The first question is rather easy to answer. School is 6 miles from home, so we are 6 miles from where we started.

The second question is a little more difficult to answer. We first travel 3 miles to the halfway point, but end up traveling another 3 miles back to home to grab the homework. Finally, you travel the 6 miles to school. We can see this in the visual below.



Therefore, the total mileage that you've covered is 12 miles. These two questions are the root behind the concepts of **displacement** and **distance**.

Displacement → Definition: How far apart the starting position and ending position are. Note that this value can be positive or negative.

Total Distance → Definition: How far you travel in total. This value can only be positive.

Given v is the velocity function and x is the position function,

$$\text{Displacement} \rightarrow \int_a^b v \, dt$$

$$\text{Total Distance} \rightarrow \int_a^b |v| \, dt$$

$$\text{Position at } t = a \rightarrow x(a) + \int_a^b v \, dt$$

Ex 3.3.4: A particle moves along a line so that its velocity at any time t is $v(t) = t^2 + t - 6$ (measured in meters per second).

- (a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.
- (b) Find the distance traveled during the time period $1 \leq t \leq 4$.

Sol 3.3.4:

a)

$$\begin{aligned}\int_a^b v \, dt &= \int_1^4 (t^2 + t - 6) \, dt \\&= \left[\frac{t^3}{3} + \frac{t^2}{2} - 6t \right]_1^4 \\&= \boxed{-\frac{21}{2} \text{ meters}}\end{aligned}$$

b) $v(t) = 0$ when $t = 2$.

$$\begin{aligned}\int_a^b |v| \, dt &= -\int_1^2 (t^2 + t - 6) \, dt + \int_2^4 (t^2 + t - 6) \, dt \\&= -\left[\frac{t^3}{3} + \frac{t^2}{2} - 6t \right]_1^2 + \left[\frac{t^3}{3} + \frac{t^2}{2} - 6t \right]_2^4 \\&= \boxed{\frac{89}{6} \text{ meters}}\end{aligned}$$

Make sure to remember your units in your answer! The AP exam will dock points for unitless answers.

3.3 Free Response Homework

Find the area between the curve of the given equation and the x -axis on the given interval.

1a. $y = x^3$ on $x \in [0, 2]$

1b. $y = x^3$ on $x \in [-1, 2]$

2a. $y = 4x - x^3$ on $x \in [0, 2]$

2b. $y = 4x - x^3$ on $x \in [-1, 2]$

3a. $y = \sin(x)$ on $x \in [0, \pi]$

3b. $y = \sin(x)$ on $x \in [-\pi, \pi]$

4a. $y = 2x^2 - x^3$ on $x \in [0, 2]$

4b. $2x^2 - x^3$ on $x \in [-1, 2]$

5. $y = x^3 - 2x^2 - 3x$ on $x \in [-2, 2]$

6. $y = x^3 - 4x^2 + 4x$ on $x \in [-1, 2]$

7. $y = x^3 - 2x^2 - x + 2$ on $x \in [-3, 3]$

8. $y = \frac{\pi}{2} \cos(x) \sin(\pi + \pi \sin(x))$ on $x \in \left[-\frac{\pi}{2}, \pi\right]$

9. $y = -\frac{x}{x^2 + 4}$ on $x \in [-2, 2]$

10. $y = \frac{4 - x^2}{x^2 + 4}$ on $x \in [-3, 3]$

11. $y = \frac{\sin(\sqrt{x})}{\sqrt{x}}$ on $x \in [0.01, \pi^2]$

12. $y = x\sqrt{18 - 2x^2}$ on $x \in [-2, 1]$

13. $y = 3\sin(x)\sqrt{1 - \cos(x)}$ on $x \in \left[-\frac{\pi}{2}, \frac{\pi}{3}\right]$

14. $y = x^2 e^{x^3}$ on $x \in [0, 1.5]$

Answer the following questions regarding distance and displacement.

15. The velocity function (in meters per second) for a particle moving along a line is $v(t) = 3t - 5$.

(a) Find the displacement of the particle during the time period $0 \leq t \leq 3$.

(b) Find the distance traveled during the time period $0 \leq t \leq 3$.

16. The velocity function (in meters per second) for a particle moving along a line is $v(t) = t^2 - 2t - 8$.

(a) Find the displacement of the particle during the time period $1 \leq t \leq 6$.

(b) Find the distance traveled during the time period $1 \leq t \leq 6$.

For problems #17-20, show the setup to determine the area and solve the integral. Do not use the absolute value method.

17. Find the area under the curve $f(x) = e^{-x^2} - x$ on $x \in [-1, 2]$.

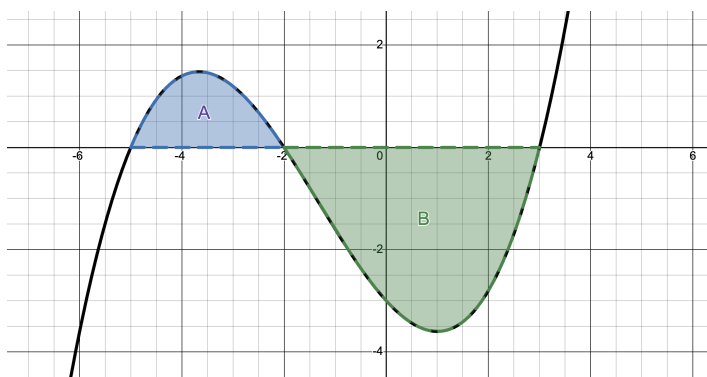
18. Find the area under the curve $f(x) = e^{-x^2} - 2x$ on $x \in [-1, 2]$.

19. Find the area under the curve $f(x) = \frac{x}{x^2 + 1} + \cos(x)$ on $x \in [0, \pi]$.

20. Find the area under the curve $g(x) = -1 - x \sin(x)$ on $x \in [0, 2\pi]$.

3.3 Multiple Choice Homework

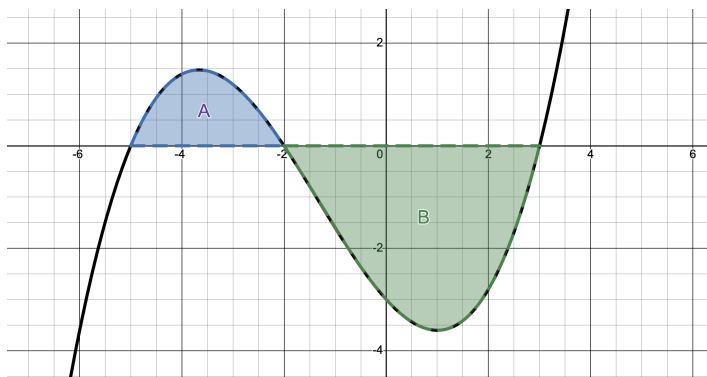
1. The graph of $y = f(x)$ is shown below. A and B are positive numbers that represent the areas between the curve and the x -axis.



In terms of A and B , $\int_{-5}^3 f(x) dx + \int_{-2}^3 f(x) dx =$

- a) A b) $A - B$ c) $2A - B$ d) $A + B$ e) $A - 2B$

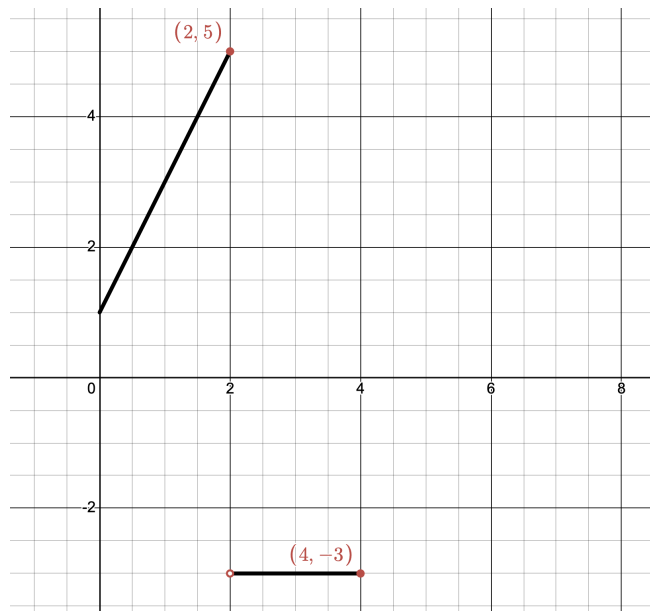
2. The graph of $y = f(x)$ is shown below. A and B are positive numbers that represent the areas between the curve and the x -axis.



In terms of A and B , $2 \int_{-5}^3 f(x) dx + 3 \int_{-2}^3 f(x) dx =$

- a) A b) $A - B$ c) $2A - B$ d) $A + B$ e) $A - 2B$
-

3. The graph of $f(x)$ on $0 \leq x \leq 4$ is shown.



- a) -1 b) 0 c) 2 d) 6 e) 12
-

4. A particle moves along the x -axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t + 1) - 2t + 1$. What is the total **distance**, in meters, traveled by the particle during the time interval $0 \leq t \leq 3$?

- a) -3.455 b) 0.704 c) 1.540 d) 2.667 e) 4.291
-

5. A particle travels along a straight line with a velocity of $v(t) = 3e^{-t^2} \sin(2t)$ meters per second. What is the total **distance**, in meters, traveled by the particle during the time interval $0 \leq t \leq 2$?

- a) 0.835 b) 1.625 c) 1.661 d) 2.261 e) 5.350
-

6. A particle moves along the x -axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t + 1) - 2t + 1$. What is the total **displacement**, in meters, of the particle during the time interval $0 \leq t \leq 3$ seconds?

a) -3.455 b) 0.704 c) 1.540 d) 2.667 e) 4.291

7. A particle travels along a straight line with a velocity of $v(t) = 3e^{-t^2} \sin(2t)$ meters per second. What is the total **distance**, in meters, traveled by the particle during the time interval $0 \leq t \leq 2$?

a) 0.835 b) 1.625 c) 1.661 d) 2.261 e) 5.350
