Linear Algebra and Differential Equations

Lecture Notes

August 27, 2025

At the core, linear algebra surrounds the solving of systems of equations. Take the following linear system for instance:

$$\begin{aligned} 2x_1 + 3x_2 &= 10 \\ 4x_1 + 11x_2 &= 16 \end{aligned}$$

Traditionally, one of two methods is employed. One can use elimination to eliminate a variable:

We first begin by multiplying the first equation by -2.

$$\begin{aligned} -2(2x_1 + 3x_2) &= -2(10) \\ 4x_1 + 11x_2 &= 16 \end{aligned}$$

$$\begin{aligned} -4x_1 + -6x_2 &= -20 \\ 4x_1 + 11x_2 &= 16 \end{aligned}$$

We then sum both equations to eliminate x_1.

$$-4x_1 + -6x_2 + (4x_1 + 11x_2) = -20 + 16$$

This leaves us with:

$$5x_2 = -4$$

$$x_2 = -\frac{4}{5}$$

We can then substitute x_2 back into equation 1 to find x_1.

$$2x_1 + 3\left(-\frac{4}{5}\right) = 10$$

$$2x_1 - \frac{12}{5} = 10$$

$$2x_1 = \frac{62}{5}$$

$$x_1 = \frac{31}{5}$$

Therefore, our solution set is:

$$\boxed{(x_1, x_2) = \left(\frac{31}{5}, -\frac{4}{5}\right)}$$

The other method traditional method for solving linear systems is solving one variable in terms of another:

We can first express x_1 in terms of x_2.

$$x_1 = \frac{10 - 3x_2}{2}$$

We can then substitute x_2 for x_1 in the second equation.

$$4\left(\frac{10 - 3x_2}{2}\right) + 11x_2 = 16$$

Now, we are able to solve for x_2.

$$\frac{40 - 12x_2}{2} + 11x_2 = 16$$

$$\frac{40 - 12x_2 + 22x_2}{2} = 16$$

$$40 + 10x_2 = 32$$

$$10x_2 = -8$$

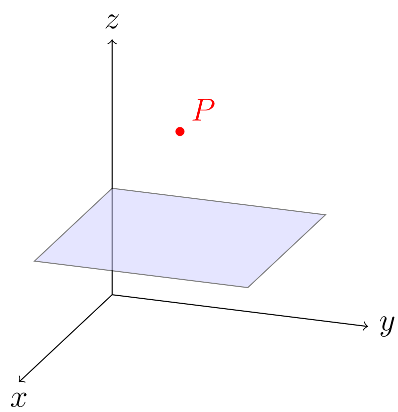
$$x_2 = -\frac{4}{5}$$

Again, as shown above, we can now substitute x_2 back into equation 1 to find x_1. Our solution once again turns out to be

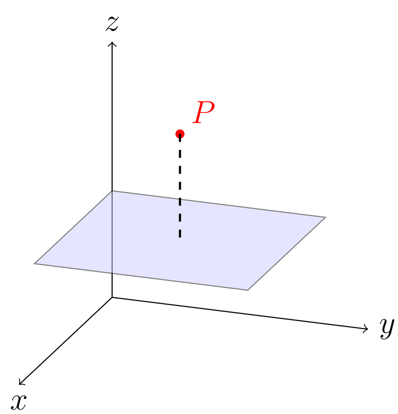
$$\boxed{(x_1, x_2) = \left(\frac{31}{5}, -\frac{4}{5}\right)}.$$

However, what if we have 10 equations and 10 variables? It's clear that although we could use the aforementioned methods, there would be many more steps and will take much longer. This is where linear algebra comes in. There are primarily two contexts where linear algebra is especially useful.

Context 1: Extension into Multiple Dimensions
Imagine you have a 3D space, and in that 3D space, there lies a plane and a point.



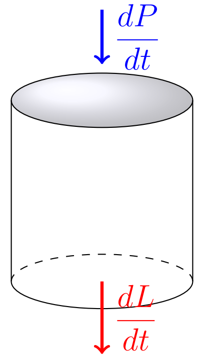
To find the distance from point P to the plane, one can use many methods, including but not limited to using the distance formula or using the cross product.



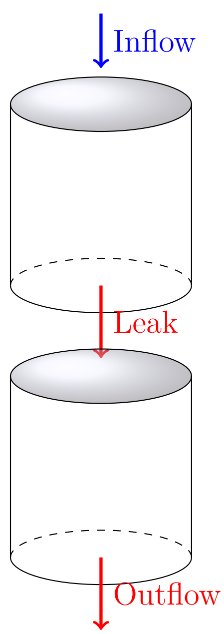
However, what if in 10-dimensions, one is asked to find the distance between a point and a 7-dimensional piece? Although the distance formula could still hold, some of the concepts that we know (such as the cross product) can only be applied in lower dimensions. Therefore, we would have to employ concepts from linear algebra to tackle the problem.

Context 2: Mixing

Imagine you have a cylindrical can. Water is being poured into this can at some rate $\frac{dP}{dt}$ and water is leaking out of this can at some rate $\frac{dL}{dt}$.



It's clear that the rate of change of total volume of water in the can is the differential equation $\frac{dV}{dt} = \frac{dP}{dt} - \frac{dL}{dt}$. However, what if I have two cans, with one leaking into the other?



This would result in a **system of differential equations**, which ties together the linear algebra and the differential equations portion of this class.