

Quantum Fourier Transform

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1 Introduction

Briefly speaking, quantum Fourier transform (QFT) is discrete Fourier transform on quantum devices.

Discrete Fourier Transform

DFT transform a N dimension vector x_0, x_1, \dots, x_{N-1} to another N dimension vector y_0, y_1, \dots, y_{N-1} as following,

$$y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i j k / N}. \quad (1)$$

Quantum Fourier Transform

On the other hand, QFT, operates the same transformation but with different form of notation. Consider an orthonormal basis $|0\rangle, |1\rangle, \dots, |N-1\rangle$, QFT acting on this basis is defined to be

$$|j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i j k / N} |k\rangle. \quad (2)$$

Assuming $N = 2^n$, noting that we can write $|k\rangle$ into the binary form $|k_1 \dots k_n\rangle$, specifically, $k = k_1 2^{n-1} + k_2 2^{n-2} + \dots + k_n 2^0$ and $\sum_{k=0}^{2^n-1} e^{k/2^n} |k\rangle = \sum_{k_1=0}^1 \dots \sum_{k_n=0}^1 e^{(\sum_{l=1}^n k_l 2^{-l})} |k_1 \dots k_n\rangle$. To implement QFT on quantum devices, it would be convenience to use the following representation,

$$\begin{aligned} |j\rangle &\longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i j k / N} |k\rangle \\ &= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i j k / 2^n} |k\rangle \\ &= \frac{1}{2^{n/2}} \sum_{k_1=0}^1 \dots \sum_{k_n=0}^1 e^{2\pi i j (\sum_{l=1}^n k_l 2^{-l})} |k_1 \dots k_n\rangle \\ &= \frac{1}{2^{n/2}} \sum_{k_1=0}^1 \dots \sum_{k_n=0}^1 \bigotimes_{l=1}^n e^{2\pi i j k_l / 2^{-l}} |k_l\rangle \\ &= \frac{1}{2^{n/2}} \bigotimes_{l=1}^n \left[\sum_{k_l=0}^1 e^{2\pi i j k_l / 2^{-l}} |k_l\rangle \right] \\ &= \frac{1}{2^{n/2}} \bigotimes_{l=1}^n \left[|0\rangle + e^{2\pi i j / 2^{-l}} |1\rangle \right] \\ &= \frac{(|0\rangle + e^{2\pi i 0 \cdot j_n} |1\rangle)(|0\rangle + e^{2\pi i 0 \cdot j_{n-1} j_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0 \cdot j_1 j_2 \dots j_n} |1\rangle)}{2^{n/2}}. \end{aligned} \quad (3)$$

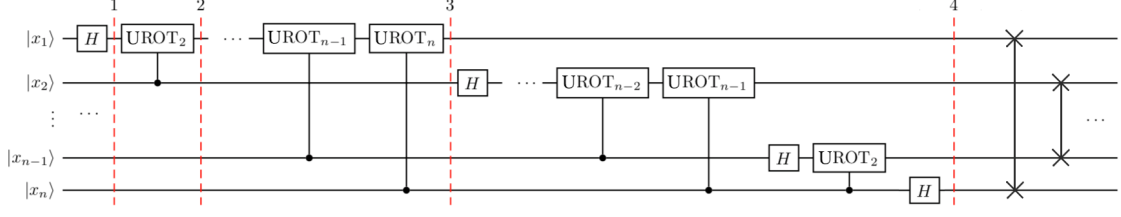


Figure 1: Quantum Fourier transform circuit.

2 Implementation

With the following representation

$$|j_1, \dots, j_n\rangle \longrightarrow \frac{(|0\rangle + e^{2\pi i 0 \cdot j_n} |1\rangle)(|0\rangle + e^{2\pi i 0 \cdot j_{n-1} j_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0 \cdot j_1 j_2 \dots j_n} |1\rangle)}{2^{n/2}}, \quad (4)$$

we are able to construct an efficient circuit for quantum Fourier transform. To implement QFT circuit, we need the following two gates, single-qubit Hadamard gate H and two-qubit controlled rotation $CROT_j$,

$$\begin{aligned} \hat{H}|j\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i j/2} |1\rangle), \\ CROT_j|0j\rangle &= |0j\rangle, \\ CROT_j|1j\rangle &= e^{2\pi i j/2^j} |1j\rangle. \end{aligned} \quad (5)$$

The QFT circuit is shown on figure 1. Please refer to the Jupyter notebook for details of the implementation example of 3 qubits quantum Fourier transform.

References

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