Quantum Phase Estimation

Tzu Hsuan Chang

December 14, 2020

1 Introduction

Quantum phase estimation (QPE) is used to find the eigenvalues of a unitary matrix. Suppose we want to find the eigenvalues $e^{2\pi i\theta_i}$ corresponding to the eigenvector $|u_i\rangle$ of an unitary operator $\hat{U}$ such that $\hat{U}|u_i\rangle = e^{2\pi i\theta_i}|u_i\rangle$. The QPE have the following operation,

$$|0\rangle |u_i\rangle \longrightarrow \left|\tilde{\theta}_i\right\rangle |u_i\rangle, \tag{1}$$

where $\tilde{\theta}_i$ is an estimate for θ_i.

As shown on figure 1, the QPE circuit write the phase of $\hat{U}$ to n ancillary qubits $|0\rangle^{\otimes n}$ in the Fourier basis and using inverse QFT to transform them back to the computational basis. The following is the mathematical details.

Mathematical details

As shown in figure 1, assuming ψ is the eigenvector of the unitary operator $\hat{U}$ with eigenvalue $e^{2\pi i\theta}$. Initially, we have

$$|\psi_0\rangle = |0\rangle^{\otimes n}\psi. \tag{2}$$

After applying n-bit Hadamard gates on the ancillary qubits,

$$|\psi_1\rangle = \frac{1}{2^{n/2}}(|0\rangle + |1\rangle)^{\otimes n}\psi. \tag{3}$$

Now we apply controlled unitary operator $C-U$. Since $|\psi\rangle$ is the eigenvector fo $\hat{U}$, we have,

$$\hat{U}^{2^j}|\psi\rangle = \hat{U}^{2^j-1}\, e^{2\pi i\theta}|\psi\rangle = e^{2\pi i 2^j\theta}|\psi\rangle. \tag{4}$$

By apply all the n controlled operations,

$$\begin{aligned} |\psi_2\rangle &= \frac{1}{2^{\frac{n}{2}}}\big(|0\rangle + e^{2\pi i\theta 2^{n-1}}|1\rangle\big)\otimes\ldots\otimes\big(|0\rangle + e^{2\pi i\theta 2^1}|1\rangle\big)\otimes\big(|0\rangle + e^{2\pi i\theta 2^0}|1\rangle\big)\otimes|\psi\rangle \\ &= \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle\otimes|\psi\rangle, \end{aligned} \tag{5}$$

where k is the integer representation of n ancillary qubits. Finally, we are going to apply inverse QFT to get the phase estimate of θ. Note that QFT is the following operation,

$$QFT|j\rangle = \frac{1}{2^{\frac{n}{2}}}\big(|0\rangle + e^{2\pi i\frac{j}{2}}|1\rangle\big)\otimes\big(|0\rangle + e^{2\pi i\frac{j}{2^2}}|1\rangle\big)\otimes\ldots\otimes\big(|0\rangle + e^{2\pi i\frac{j}{2^n}}|1\rangle\big). \tag{6}$$

With j replaced by $2^n\theta$, we have the following states after applying inverse QFT to the ancillary qubits,

$$|\psi_3\rangle = QFT_n^\dagger|\psi_2\rangle = QFT_n^\dagger\frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle\otimes|\psi\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{-\frac{2\pi ik}{2^n}(j-2^n\theta)}|j\rangle\otimes|\psi\rangle. \tag{7}$$

We can find that there is a peak near $j = 2^n\theta$. Therefore, by measuring ancillary qubits in the computational basis, we would get the phase with high probability,

$$|\psi_4\rangle = |2^n\theta\rangle\otimes|\psi\rangle. \tag{8}$$

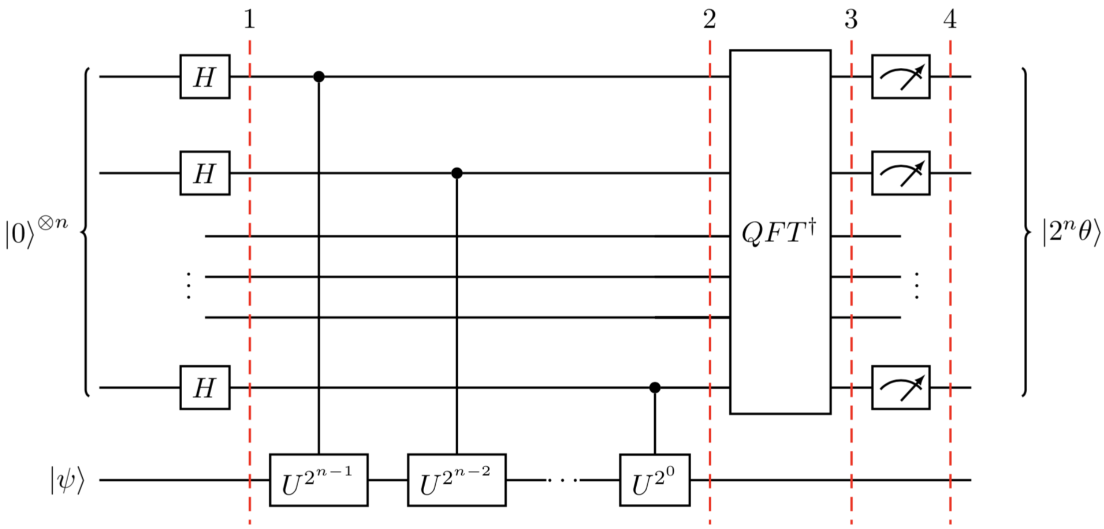

Figure 1: Quantum phase estimation circuit.[1]

References

[1] A. Asfaw, L. Bello, Y. Ben-Haim, S. Bravyi, N. Bronn, L. Capelluto, A. C. Vazquez, J. Ceroni, R. Chen, A. Frisch, J. Gambetta, S. Garion, L. Gil, S. D. L. P. Gonzalez, F. Harkins, T. Imamichi, D. McKay, A. Mezzacapo, Z. Minev, R. Movassagh, G. Nannicni, P. Nation, A. Phan, M. Pistoia, A. Rattew, J. Schaefer, J. Shabani, J. Smolin, K. Temme, M. Tod, S. Wood, and J. Wootton. Learn quantum computation using qiskit, 2020.