Quantum Fourier Transform

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1 Introduction

Briefly speaking, quantum Fourier transform (QFT) is discrete Fourier transform on quantum devices.

Discrete Fourier Transform

DFT transform a N dimension vector $x_0, x_1, ..., x_{N-1}$ to another N dimension vector $y_0, y_1, ..., y_{N-1}$ as following,

$$y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi ijk/N}. \tag{1}$$

Quantum Fourier Transform

On the other hand, QFT, operates the same transformation but with different form of notation. Consider an orthonormal basis $|0\rangle, |1\rangle, ..., |N-1\rangle$, QFT acting on this basis is defined to be

$$|j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle. \tag{2}$$

Assuming $N = 2^n$, noting that we can write $|k\rangle$ into the binary form $|k_1 \dots k_n\rangle$, specifically, $k = k_1 2^{n-1} + k_2 2^{n-2} + \dots + k_n 2^0$ and $\sum_{k=0}^{2^n-1} e^{k/2^n} |k\rangle = \sum_{k_1=0}^{1} \dots \sum_{k_n=0}^{1} e^{(\sum_{l=1}^{n} k_l 2^{-l})} |k_1 \dots k_n\rangle$. To implement QFT on quantum devices, it would be convenience to use the following representation,

$$\begin{aligned}
|j\rangle &\longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi ijk/2^n} |k\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \dots \sum_{k_n=0}^{1} e^{2\pi ij(\sum_{l=1}^{n} k_l 2^{-l})} |k_1 \dots k_n\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \dots \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi ijk_l/2^{-l}} |k_l\rangle \\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{k_l=0}^{1} e^{2\pi ijk_l/2^{-l}} |k_l\rangle \right] \\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi ij/2^{-l}} |1\rangle \right] \\
&= \frac{\big(|0\rangle + e^{2\pi i 0.j_n} |1\rangle\big)\big(|0\rangle + e^{2\pi i 0.j_{n-1}j_n} |1\rangle\big) \dots \big(|0\rangle + e^{2\pi i 0.j_1 j_2 \cdots j_n} |1\rangle\big)}{2^{n/2}}.
\end{aligned} \tag{3}$$

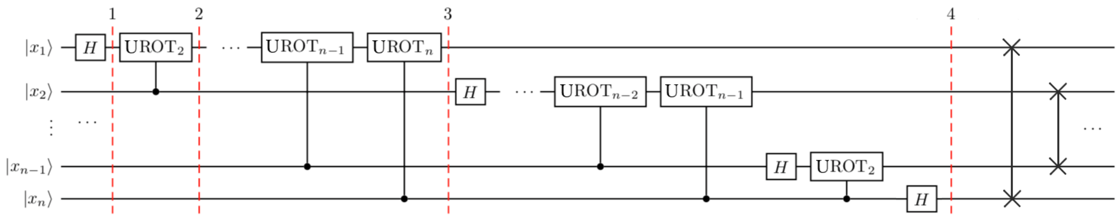

Figure 1: Quantum Fourier transform circuit.

2 Implementation

With the following representation

$$|j_1, \cdots, j_n\rangle \longrightarrow \frac{\big(|0\rangle + e^{2\pi i 0.j_n}|1\rangle\big)\big(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle\big)\ldots\big(|0\rangle + e^{2\pi i 0.j_1 j_2 \cdots j_n}|1\rangle\big)}{2^{n/2}}, \tag{4}$$

we are able to construct an efficient circuit for quantum Fourier transform. To implement QFT circuit, we need the following two gates, single-qubit Hadamard gate H and two-qubit controlled rotation $CROT_j$,

$$\begin{aligned} \hat{H}|j\rangle &= \frac{1}{\sqrt{2}}\big(|0\rangle + e^{2\pi i j/2}|1\rangle\big), \\ C\hat{ROT}_j|0j\rangle &= |0j\rangle, \\ C\hat{ROT}_j|1j\rangle &= e^{2\pi i j/2^j}|1j\rangle. \end{aligned} \tag{5}$$

The QFT circuit is shown on figure 1. Please refer to the Jupyter notebook for details of the implementation example of 3 qubits quantum Fourier transform.