Problem Sheet: Natural Numbers and Induction

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1. What is wrong with the following proof that people are of the same height? (Earl 2017)

   > Let $P(n)$ be the statement that n persons must be of the same height. Clearly $P(1)$ is true, as a person is the same height as themselves. Assume that $P(n)$ is true. If we have $n+1$ people, then we can ask one person to leave. Since $P(n)$ is true, we know all people must be equally tall. If we invite back the missing person and someone else leaves, then again the remaining n people are of equal height. Hence, all of the $n+1$ persons were of the same height, which establishes $P(n+1)$. By induction, everybody is of the same height.

2. Use induction to prove that the following identities hold for all $n \in \mathbb{N}_{\geq 1}$

   (a) $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

   (b) $1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

   (c) $1 + 3 + \ldots + (2n-1) = n^2$

   (d) $\frac{1}{3} + \frac{1}{15} + \ldots + \frac{1}{4n^2-1} = \frac{n}{2n+1}$

   (e) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

3. Prove **Bernoulli's** inequality: For all $n \in \mathbb{N}$ and $x \in \mathbb{R}_{>-1}$

$$(1+x)^n \geq 1 + nx$$

4. Prove that for all $m, n \in \mathbb{N}$: $n + m = m + n$.

5. Solve the equation

$$\binom{28}{x^2-5} = \binom{28}{3x+5}$$

6. Show that for all $n, k \in \mathbb{N}$ with $n \geq k+1 \geq 2$ we have

$$\binom{n+2}{k+2} = \binom{n}{k} + 2\binom{n}{k+1} + \binom{n}{k+2}$$

7. Consider the following function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

```
f(O, n)          =  O
f(S(m), O)       =  O
f(S(m), S(n))    =  S(f(m, n))
```

(a) What operation does f implement?

(b) Show that for all $n \in \mathbb{N}$:

$$f(m, n) = f(n, m)$$

8. The *Fibonacci numbers* are defined by fib : $\mathbb{N} \to \mathbb{N}$

```
fib(0)   = 0
fib(1)   = 1
fib(n+2) = fib(n+1) + fib(n)
```

We can implement this function directly in most programming languages. For example, in Python:

```python
def fib(n):
  if n == 0:
    return 0
  elif n == 1:
    return 1
  else:
    return fib(n-1) + fib(n-2)
```

(a) How many times do we have to evaluate `fib(2)` in order to obtain `fib(5)`?

(b) In order to improve the efficiency of this program, we introduce the function acfib : $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by

```
acfib(0, x, y) = x
acfib(n+1, x, y) = acfib(n, y, x + y)
```

Show that for all $n \in \mathbb{N}$, `acfib(n, 0, 1) = fib(n)`.

Hint: Show first that

`acfib(n+1, x, y) + acfib(n, x, y) = acfib(n+1, y, x+y)`