Problem Sheet: Sets and Logic

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In the following A, B, C are subsets of a given set S.

1. Show that
   (a) $A \subseteq A$
   (b) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
   (c) $A \cap B \ \subseteq \ A \ \subseteq \ A \cup B$
2. Which of the following statements are true for every A, B, C? Prove those that are true and provide counter-examples for the false ones.
   (a) If $A \in B$ and $B \in C$, then $A \in C$
   (b) If $A \subseteq B$ and $B \in C$, then $A \in C$
   (c) If $A \cap B \subseteq C^c$ and $A \cup C \subseteq B$, then $A \cap C = \emptyset$
   (d) If $A \neq B$ and $B \neq C$, then $A \neq C$
   (e) If $A \subseteq (B \cup C)^c$ and $B \subseteq (A \cup C)^c$, then $B = \emptyset$
3. Prove that for all $a, b, c, d \in S$

$$\{\{a\},\ \{a, b\}\} = \{\{c\}, \{c, d\}\} \iff a = c \ \wedge\ b = d$$

4. Express the operations $\cup$, $\cap$, $\setminus$ in terms of
   (a) Δ and $\cap$
   (b) Δ and $\cup$
   (c) $\setminus$ and Δ
5. Show that
   (a) $(A \cup B) \times C = (A \times C) \cup (B \times C)$
   (b) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
   (c) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$ (when does equality hold?)
   (d) If $A, B \neq \emptyset$, and $(A \times B) \cup (B \times A) = (C \times D)$, then $A = B = C = D$.

6. Prove the De Morgan laws using truth tables:

$$\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$$
$$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$$

7. In the following, S is non-empty set and P and R are statements indexed over elements of S and $S \times S$, respectively. Prove the following statements:

   (a) $\neg(\exists y \in S \forall x \in S \;\; R(x,y) \Leftrightarrow \neg R(x,x))$ (*Russell's paradox*)

   (b) $\exists x \in S \;\; (P(x) \Rightarrow \forall y \in S \;\; P(y))$ (*The drinker's paradox*)

8. The following questions relate to the two representations of natural numbers:

   (a) What does $m \cap n$ represent in the two representations?

   (b) Calculate $\mathbb{P}\,3$ for each of the representations.

   (c) How many elements does $\mathbb{P}\,n$ have in each of the representations?

   Define another representation, that is different from each of the two given in the notes, and answer the three questions for it as well.

9. Find dom(R), ran(R), R°, $R \circ R$, $R \circ R^\circ$, $R^\circ \circ R$ for the following relations

   (a) $R = \{(x,y) \mid x, y \in \mathbb{N},\; x \text{ divides } y\}$

   (b) $R = \{(x,y) \mid x, y \in \mathbb{R},\; x + y \leq 0\}$

   (c) $R = \{(x,y) \mid x, y \in \mathbb{R},\; 2x - 3y \geq 0\}$

   Which of these relations is an equivalence? Which are partial orders?

10. Which of the following statements are true? Prove those that are true and provide counter-examples for the false ones.

    - If R is an equivalence, then R is a partial order.
    - If R is a partial order, then it is an equivalence.
    - If R is an equivalence, then R is a total order.
    - If R is an equivalence, then R is a strict partial order.
    - If R is a strict partial order, then $R \cup id_A$ is a partial order.
    - If R_1 and R_2 are partial orders, then $R_1 \cup R_2$ is a partial order.

References

- *Problems in Set Theory, Mathematical Logic, and the Theory of Algorithms*, Lavrov and Maksimova, Springer 2003