Introduction to *Mathematics 1*

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WS 2023-24

TECHNISCHE HOCHSCHULE DEGGENDORF

Outline

Topic

Questions?

Schedule

- Weekly lectures and exercise sessions:
  - Usually lecture in block 1 (09:45-11:15) and exercises in block 2 (11:30-13:00)
  - In the exercise sessions we'll discuss *problem sheets*
    - problem sheets are *the most important* part of the course
    - more useful if you attempt to complete the problem sheet *before* the session
    - make sure you **ask questions** whenever you have difficulties
  - You may also **ask for clarifications** about the lecture during the exercise sessions
    - but feel free to **ask questions any time during the lecture** (unless I specifically ask for no interruptions, e.g., because we're running late)

Lectures

- Lectures will be in the traditional "blackboard"-style (but with a tablet replacing the blackboard for the benefit of the online participants).
  - Hybrid model: lectures will be streamed via Zoom, recorded, and the recordings will be made available via iLearn.
  - We won't follow a specific textbook, but references will be occasionally provided.
  - You are *strongly encouraged* to take your own notes during the lectures (either when following live or via a recording).

Ground Rules for Lectures

- To reiterate: lectures will be streamed via Zoom, recorded, and the recordings will be made available via iLearn.
- Therefore, **you do not need to attend physically**.
- If you do attend physically, then **do not talk or otherwise disturb the rest of the audience, unless you want to ask a question**.
  - If the lecture gets too boring, or you must discuss something with your friends, then **please leave the lecture room**.
  - No phones, smart or otherwise! If you must use your phone, **please leave the lecture room**. There is no problem with that, I will probably be doing that myself from time to time.
- Discussing the exercises with the others during exercise sessions is encouraged, but be respectful of the other groups.

Exam

- There will be a written exam during 26.01.–14.02.2024.
  - The exam lasts 90 minutes and contains exercises similar to those on the problem sheets and possibly bookwork. More details towards the end of the semester.
- You need to register for the exam in the period 07.11.–21.11.2023.
  - If you do not register for the exam, you will not be allowed to take it, even if you show up (and even if you insist).
  - Conversely, if you do register for the exam and you do not show up for it, it counts as failed ("automatic fail", in German "die Verwaltungsfünf", notation 5N).
    - If you register for the exam, but discover that you cannot make it, then you must deregister at least seven days before the exam.
    - Deregistration nearer to the exam than six days is only possible on exceptional grounds, e.g., medical.

Retakes and Deadlines (1)

- Attempting an exam (registering and then not de-registering) sets a clock ticking.
  - If the attempt is successful, the ticking stops.
  - If the attempt fails, then you must make a second attempt within six months. That means, you must register for the exam in the upcoming exam session.
    - Corollary: At the THD, by default every exam is offered in every exam session. Exceptions exist, but are rare (and almost exclusively in subjects evaluated by projects).
    - If you do not register for the retake, you get an automatic fail.
  - If the second attempt is also unsuccessful, then a final attempt must be made within one year (either in the immediately upcoming session, or in the one following it).
    - If the final attempt is also unsuccessful, then you will be exmatriculated. You will not be able to enrol in this program again, and you will not be able to enrol in similar programs in any other German university!

Exercise

- What happens if you do not register for the final attempt within one year of a second failed attempt?
- Assume you have attempted an exam in the SS 2024. What is the earliest that you can be exmatriculated because of that subject? Trick question: what is the latest that you can be exmatriculated because of that subject?

Retakes and Deadlines (2)

- The reason the previous question was a trick one is the possibility of a *sabbatical semester* (German: das Urlaubssemester). During a sabbatical, all exam clocks are paused.
- The *regular study duration* is seven semesters. You must attempt every exam in the course of study at least once within these seven semesters.

Exercise

- What is the latest you can take *Operating Systems*?
- Trick question: what is the latest you can take *Mathematics 1*?

Retakes and Deadlines (3)

- The reason the previous question was a trick one is that *Mathematics 1* belongs to the group of *foundational subjects*.
  - The other two are *Programming 1* and *Introduction to Artificial Intelligence*
  - The exams in these three subjects must be attempted within the first year.

Exercise

- Assuming no attempts are made, what is the latest a student can be exmatriculated?
- Assuming that a student has passed the three foundational subjects, what is the latest he can be exmatriculated?

Prior Credits (1)

- Some students might have already studied one or more semesters at another university. It is possible to have the credits accumulated there transferred to the THD. This is known here as "recognition" (German: die Anerkennung)
- Previous studies can only be recognized if the appropriate forms are submitted **within the first semester**.
  - The forms must be submitted in accordance to the procedure described at https://th-deg.de/en/students/documents#accordion-650c8ae01d4b5-1
  - The deadline for this is the last day of lectures, **however...**
    - the Exam Board (German: die Prüfungskommission) will not have time to consider your request unless you submit the documents **six weeks before the last day of lectures**.
  - The only decision that matters is that of the Exam Board! Even if the lecturer agrees with the credit transfer, that does not guarantee a favourable decision of the Exam Board (and conversely).

Exercise

- Suppose you want to have *Mathematics 1* recognized and submit your application some time in November. The decision of the Exam Board will not be communicated before January. Do you register for the *Mathematics 1* exam?
- Suppose you submit your application in December. The decision of the Exam Board will not be communicated before March 2024. Complete the following table:

Registered	Show Up	Board Decision	Final Result
No	No	Favourable	
No	Yes	Favourable	
Yes	No	Favourable	
Yes	Yes, pass	Favourable	
Yes	Yes, fail	Favourable	
Yes	Yes, pass	Unfavourable	

Prior Credits (2)

- The previous question was underspecified. If the Exam Board has confirmed the results of the examination, then it cannot overturn these in favour of prior work.
- In general, the exam grades are considered **before** the remaining applications for credit transfer.
- Conclusion: submit the application as early as possible. If you are sure of its success, then do not register for the exam. Recall that a failed attempt sets the clocks ticking!

Study Abroad

- It is not uncommon for students to study for a semester at a foreign university (German: das Auslandsstudium). The subjects you study there should ideally have counterparts in your THD study programme, so that you can transfer the credits accumulated there.
- The procedure is similar to the one for prior credit recognition, but not as urgent, so we won't go into that here.
- When the time comes, check with the course assistant first!

Resources (1)

- During the course of your studies, you will have many administrative questions.
  - The answers are in the vast majority of cases to be found on the THD website, particularly under
    - https://th-deg.de/de/studierende/antraege-und-organisatorisches (German), or
    - https://th-deg.de/en/students/documents (English)
  - Use the search function!
- In some cases, the answer will come from the course assistant: enrolment keys, schedule, optional subjects, etc.
- In others, from the contact person in the Center of Studies: extension requests, prior credits, study abroad, etc.
- Very rarely, from the lecturers: can't think of anything.
  - Subject related questions must be asked in the context of the lectures or exercise sessions, so that all students can benefit from the answers (if there are any).

Contact

- If you need to contact me for some reason, then do so via email **from your THD account**.
  - Emails sent from other accounts will go unanswered.
  - Do **NOT** use the iLearn messaging function to contact me!
- Office hours by appointment.

Resources (2)

- Springer provides full-text, non-drm, no questions asked access to many of their technical textbooks (you might have to access the site via our library page https://opac.fh-deggendorf.de and/or from within the THD network (or via VPN).
  - In particular:
    - Mathematik für Hochschule und duales Studium, Guido Walz, 2020
    - Mathematik, Arens et al., 2022

    Our course will cover most of Walz, at a level somewhere between Walz and Arens.
- Other publishers (e.g., Hanser) provide similar services.
- Some books have unlimited viewing, but limited downloading (e.g., Norvig and Russell *AIMA 4th Ed*). You can still download some parts of these books. Make sure not to hog these resources, close the windows when you are done so that others can access them.
- The majority of these textbooks are in German: an excellent opportunity for you to practice it!

Topic

Why do we have to study this? (1)

1. *Mathematics is useful*.
   - data analysis and machine learning make *heavy use* of linear algebra and continuous optimization
   - a good grasp of limits and sequences is vital for estimating the resources (run time and memory) required by an algorithm

Why do we have to study this? (2)

...continued:

- logic and set theory play an important role in the formal specification of programs
- A relevant quote from a course description (Discrete Mathematics, Oxford 2022-23):
  In order to be able to formulate what a computer system is supposed to do, or to prove that it does meet its specification, or to reason about its efficiency, one needs the precision of mathematical notation and techniques. For instance, to specify computational problems precisely one needs to abstract the detail and then use mathematical objects such as sets, functions, relations, orders, and sequences. To prove that a proposed solution does work as specified, one needs to apply the principles of mathematical logic, and to use proof techniques such as induction.
- ...

Why do we have to study this? (2)

2. *Mathematics trains the thought processes required for programming*
   - there is a precise analogy between mathematical proofs and programs
   - mathematical concepts are usually *simpler* and *cleaner* than a programmer's data structures.
   - the main problem of programming is *managing complexity*, and the main tool for tackling this problem is *abstraction*. Mathematics is (also a) training ground for abstraction.

Why do we have to study this? (3)

- A popular *misunderstanding*:

  We only need to know how to use mathematics, not why it works. After all, one can drive a car without having any idea of how the engine works.

  The *opposite* is the case. Not understanding the concepts involved and why they work is not at all similar to not knowing how the engine works: it is in fact like *driving blind*.

Driving Blind

- Virtually any computational exercise, such as finding limits of sequences, derivatives and integrals of functions, solutions of (systems of) equations, optimization, etc. can be solved by a program such as Mathematica or the freeley available sympy.
- However, we still need to understand the results!

Example

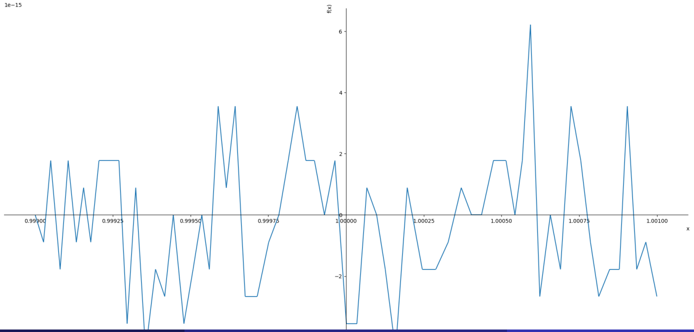
Using sympy, let us plot the polynomial

$$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$$

in a small region around 1.

```
>>> plot(x**6 - 6*x**5 + 15*x**4 - 20*x**3 + 15*x**2 - 6*x + 1, (x, 0.999, 1.001), axis_c
```

We obtain the following result:



Example (2)

The plot we obtained makes no sense!

- The polynomial we have plotted is of degree 6, and therefore can have at most six roots, but the graph we have obtained shows a function with at least 30 zeros, which therefore corresponds to a polynomial of degree at least thirty.
- Even worse: our polynomial is the expansion of the expression $(x-1)^6$, so there is exactly one real root, namely 1:

```
>>> expand((x-1)**6)
x**6 - 6*x**5 + 15*x**4 - 20*x**3 + 15*x**2 - 6*x + 1
>>> plot(expand((x-1)**6), (x, 0.999, 1.001), axis_center=(1, 0))
```

Yet if we blindly trusted sympy's plot, we could be chasing for zeros in the vicinity of 1 for a long time.

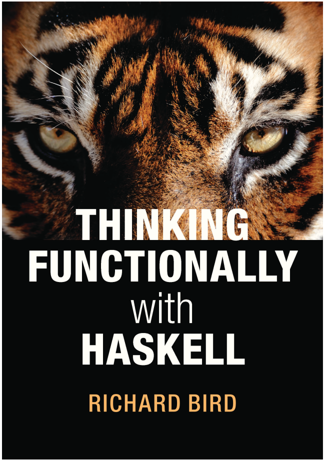
Topic

Working with Expressions

- An essential skill, hopefully acquired in school, is the ability to manipulate algebraic expressions (German: *Term Umformungen*).
  - Example: $(a+b)^2 = a^2 + 2ab + b^2$
- The expressions we will work with usually denote numbers, functions, or sets.
- Trignometric identities are a good way of reminding ourselves not just what the trignometric functions are, but also how to work with fractions, polynomials, and expressions in general.

Example

Trigonometric identities used as an example of algebraic reasoning in Richard Bird's *Thinking Functionally with Haskell*, Cambridge University Press, 2014.



Exercise A

Express $\sin 3\alpha$ in terms of $\sin\alpha$.

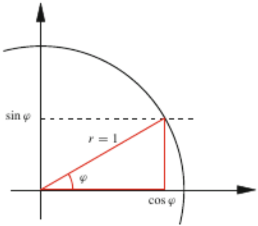
Answers

Answer to Exercise A

$$
\begin{array}{ll}
& \sin 3\alpha \\
= & \quad \{\text{arithmetic}\} \\
& \sin(2\alpha+\alpha) \\
= & \quad \{\text{since } \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\} \\
& \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha \\
= & \quad \{\text{since } \sin 2\alpha = 2\sin\alpha\cos\alpha\} \\
& 2\sin\alpha\cos^2\alpha + \cos 2\alpha\sin\alpha \\
= & \quad \{\text{since } \cos 2\alpha = \cos^2\alpha - \sin^2\alpha\} \\
& 2\sin\alpha\cos^2\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha \\
= & \quad \{\text{since } \sin^2\alpha + \cos^2\alpha = 1\} \\
& \sin\alpha(3 - 4\sin^2\alpha)
\end{array}
$$

The above proof format was, I believe, invented by Wim Feijen. It will be used throughout the book.

The Trigonometric Functions



- In the unit circle: the length of the opposite side (German: die Gegenkathete) is equal to the sinus, that of the adjacent side (German: die Ankathete) is equal to the cosinus.
- In an arbitrary right triangle (das rechwinglige Dreieck), the sinus of an angle is the ratio between the length of the opposite and that of the hypothenuse.
  - Question: Is the ratio the same for all right triangles with the same angle? Why?
- From Pythagora's theorem we then have:

$$\sin^2\alpha + \cos^2\alpha = 1$$

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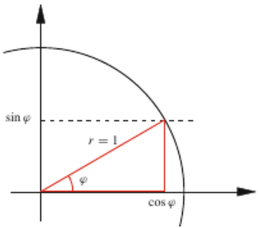

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Degrees vs Radians

- In mathematics, we usually measure angles in *radian* (German: der Radiant). The measure of an angle is the length of the arc (German: der Bogen) of the unit circle subtended by the angle.
- Therefore:
  - $360°$ corresponds to the angle that subtends the whole unit circle, the length of which is 2π
  - $180°$ corresponds to π
  - 1 radian corresponds to the angle subtending an arc of length 1
  - $\frac{\pi}{6}$ corresponds to $30°$

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First Identities

- The geometric interpretation of the trigonometric functions justifies the following identities:
  - $\sin 45° = \frac{\sqrt{2}}{2} = \cos 45°$
  - $\sin(90° - \alpha) = \cos \alpha$

  An important special angle is $30°$:
  - $\sin 30° = \frac{1}{2}$

  With this, you should be able to fill in the table

α (degrees)	α (in radians)	$\sin \alpha$	$\cos \alpha$	$\tan \alpha$	$\cot \alpha$
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$30°$					
$45°$					
$60°$					
$90°$					

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The Sum Rule

The following identity is very important and has its own name: *the sum rule*

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

You probably have seen a proof of it in school, but it's been a while, so we'll give a proof of it here.

Derived Rules

Given the sum rule, we can easily derive the following rules

- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Sum Rule for cos

We only prove the second of the identities on the previous slide, the rest are left as exercises.

$$
\begin{aligned}
& \cos(\alpha + \beta) \\
= \quad & \sin(90° - (\alpha + \beta)) \\
= \quad & \sin((90° - \alpha) - \beta) \\
= \quad & \sin(90° - \alpha)\cos\beta - \cos(90° - \alpha)\sin\beta \\
= \quad & \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}
$$