Modified Linear Tangent Guidance

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1 Introduction

The aim of this project is to devise a trajectory tobe followed by a rocket to reach orbit using minimum fuel. To accomplish this, we make use of optimal control theory. Linear tangent guidance[1] is a guidance law derived using optimal control theory. According to it, the angle α made by the rocket with the vertical at time t is given by

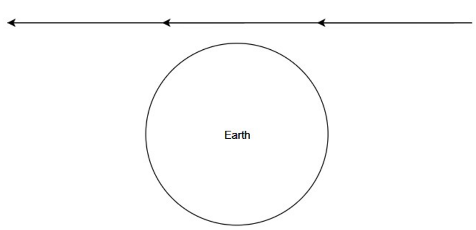
$$\tan(\alpha) = At + B$$

This law, however has one glaring assumption- It assumes that the earth is flat. This is reasonable as long as we only deal with orbits close to the earth, because the diameter of the earth is large. We, however tried to eliminate the need for this assumption and improve the accuracy of the guidance law by deriving it for a spherical earth.

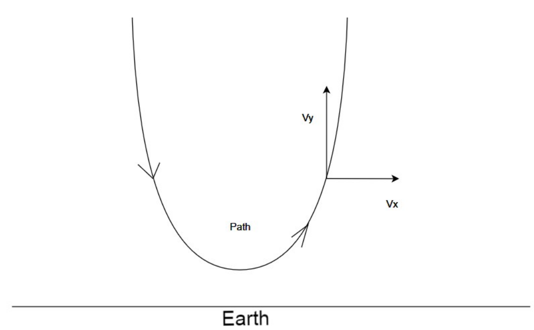
2 From curved to flat

In order to derive the Linear tangent guidance law for a spherical earth, we decided to assume the earth as flat and assume that there are some forces acting on the body of our concern. These forces, in reality arise from the change in the direction of the gravitational force vector.

Note that the path is not a straight line - there are pseudo forces acting on the body that cause it to change direction. We shall now proceed to calculate the magnitude of these forces.



(a) The path of a body moving under no external force.



(b) The path of the body, if we now change our coordinate system such that the Earth is flat.

Figure 1: Conversion from a (a) curved Earth reference frame to (b) flat Earth.

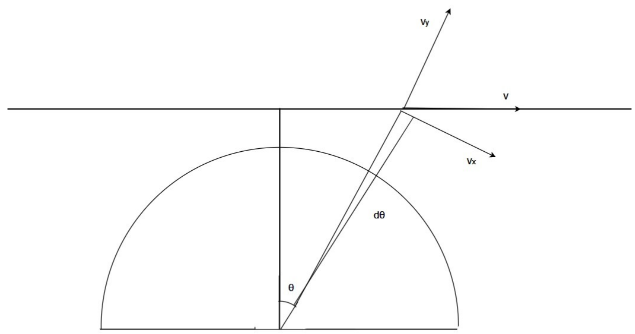


Figure 2: Components of velocities

We have from centripetal acceleration,

$$\begin{aligned} \mathrm{d}v_y &= \mathrm{d}(v\sin\theta) \\ &= \frac{v_x^2}{R}\mathrm{dt} \\ \Rightarrow \frac{\mathrm{d}(v_y)}{\mathrm{dt}} &= \frac{v_x^2}{R} \end{aligned} \tag{1}$$

Also,

$$\mathrm{d}v_x = \mathrm{d}(v\cos(\theta)) - v_x\frac{\mathrm{d}r}{R} \tag{2}$$

differentiating and putting $v\sin(\theta) = v_y$ and $v\cos(\theta) = v_x$, we get,

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -2\frac{v_x v_y}{R} \tag{3}$$

We can, therefore treat our rocket as launched from a flat earth, with the aforementioned two forces acting on it.

3 Major Loop Guidance

In this section we will try to find the optimal guidance law, Pontryagin's minimum principle[2] and some suitable approximations. The method used here is similar to the one used for deriving Linear Tangent Guidance law.

We start by defining our cost function

$$J = \varphi(t_0, x(t_0), t_1, x(t_1)) + \int_{t_0}^{t_1} L(x(t), v(t), t)\mathrm{d}t \tag{4}$$

We chose our $t_0 = 0$, *i.e.* the current point of calculation is the point where $t_0 = 0$.

So,

$$J = \varphi(t_1, x(t_1)) + \int_{0}^{t_1} L(x(t), v(t), t)\mathrm{d}t \tag{5}$$

To minimize the fuel usage, we should essentially minimise t_1, provided the engines aren't throttling, *i.e.*,

$$\varphi = t_1 \;\Rightarrow\; J = t_1 \tag{6}$$

We seek to minimize this cost function J subject to the following constraints (f), where α is the angle the ship needs to make with the horizontal.

$$\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v_x} &= a\cos\alpha - \frac{2v_xv_y}{R} \\
\dot{v_y} &= a\sin\alpha - g + \frac{2v_x^2}{R}
\end{aligned} \tag{7}$$

Note, we have already added the correction terms to v_x and v_y.

It should match the boundary constraints:

$$\begin{aligned}
y(0) &= h \\
v_x(0) &= v_{x0} \\
v_y(0) &= v_{y0} \\
y(t_1) &= H \\
v_x(t_1) &= v_T \\
v_y(t_1) &= 0
\end{aligned} \tag{8}$$

where, h is the current altitude, H is the orbit altitude and v_T is the final orbital speed at H.

We now compute the Hamiltonian,

$$\begin{aligned}
H &= \langle \lambda, f \rangle - L \\
&= \lambda_1 v_x + \lambda_2 v_y + \lambda_3\left(a\cos\alpha - \frac{2v_xv_y}{R}\right) + \lambda_4\left(a\sin\alpha - g + \frac{2v_x^2}{R}\right)
\end{aligned} \tag{9}$$

Typically, R would change as one rises through the orbit, but for our purposes, we eventually take it to be a constant (R doesn't change much).

Now computing the costate equations,

$$
\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x} \\
\Rightarrow \lambda_1 &= c_1 \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial y} \\
\Rightarrow \lambda_2 &= c_2 \\
\dot{\lambda}_3 &= -\frac{\partial H}{\partial v_x} \\
&= -\lambda_1 + \frac{2\lambda_3 v_y}{R} - \frac{2\lambda_4 v_x}{R} \\
\dot{\lambda}_4 &= -\frac{\partial H}{\partial v_y} \\
&= -\lambda_2 + \frac{2\lambda_3 v_x}{R}
\end{aligned}
\tag{10}
$$

Now, let $\frac{2v_x}{R} = \omega_x$, $\frac{2v_y}{R} = \omega_y$,

$$
\Rightarrow \dot{\lambda}_3 = -\lambda_1 + \frac{2\lambda_3 v_y}{R} - \frac{2\lambda_4 v_x}{R}
\tag{11}
$$

$$
\Rightarrow \dot{\lambda}_4 = -\lambda_2 + \frac{2\lambda_3 v_x}{R}
\tag{12}
$$

Now the factors ω_x and ω_y are very small, so we first neglect them and use perturbation theory to take in the first order term and get,

$$
\begin{aligned}
\Rightarrow \lambda_3 &= -c_1 t + c_3 \\
\Rightarrow \lambda_4 &= -c_2 t + c_4
\end{aligned}
\tag{13}
$$

Now, before applying perturbation to it, we first look into how the optimal guidance looks, for that,

$$
\begin{aligned}
\frac{\partial H}{\partial \alpha} = 0 \ &\Rightarrow \ -\lambda_3 a \sin\alpha + \lambda a \cos\alpha = 0 \\
&\Rightarrow \tan\alpha = \frac{\lambda_4}{\lambda_3} \\
&\Rightarrow \tan\alpha = \frac{c_2 t - c_4}{c_1 t - c_3}
\end{aligned}
\tag{14}
$$

Now taking only independent variables, we get,

$$
\tan\alpha = \frac{k_1 + k_2 t}{1 + k_3 t}
\tag{15}
$$

This is the bilinear tangent guidance law.

Now we compute the transversatility conditions. By not fixing the downrange distance we can say that $\lambda_1 = 0$ and the boundary conditions don't explicitly depend on t_1. Also, since the initial time and position are fixed and the final y, v_x, v_y are fixed,

$$\left(\frac{\partial \varphi}{\partial t_1} + H_1\right) \delta t_1 + \left(\frac{\partial \varphi}{\partial x(t_1)} - \lambda_1(t_1)\right) \delta(x(t_1)) = 0 \tag{16}$$

$$\begin{aligned} \Rightarrow H_{(1)} &= -\frac{\partial \varphi}{\partial t_1} = -1 \\ \lambda_1(t_1) &= \frac{\partial \varphi}{\partial x(t_1)} = 0 \end{aligned} \tag{17}$$

$$\Rightarrow c_1 = 0$$

i.e., our final guidance law becomes,

$$\tan\alpha = k_1 + k_2 t$$

i.e.,

$$\begin{aligned} \lambda_4 &= k_1 + k_2 t \\ \lambda_3 &= 1 \end{aligned} \tag{18}$$

Now, let's add small perturbations to it,

$$\begin{aligned} \lambda_4 &= k_1 + k_2 t + \delta_1 \\ \lambda_3 &= 1 + \delta_2 \end{aligned} \tag{19}$$

For ease of solving, we can take ω_x and ω_y to be constants equal to their average values.
Now, from 11,

$$\dot{\delta_1} = \omega_x \tag{20}$$

and from 12

$$\dot{\delta_2} = \omega_y - \omega_x k_1 \tag{21}$$

$$\Rightarrow \delta_1 = \omega_x t,\ \delta_2 = (\omega_y - \omega_x k_1)\, t \tag{22}$$

So, finally we have,

$$\begin{aligned} \tan\alpha &= \frac{k_1 + (k_2 + \omega_x)\, t}{1 + (\omega_y - \omega_x k_1)\, t} \\ \Rightarrow \tan\alpha &= \frac{A + Bt}{1 + (\omega_y - \omega_x A)\, t} \end{aligned} \tag{23}$$

where $A = k_1$ and $B = k_2 + \omega_x$.

Now we basically need to find the values of A, B for our given initial and final positions. This is the most involving part of the process.

4 Solving for A and B

Solving for A and B at each part of the trajectory is the most difficult part of the solution and is computationally expensive.

$$\begin{aligned}\int_0^T (a\cos\alpha - \frac{2v_x v_y}{R})\mathrm{d}t = \Delta v_x \\ \int_0^T (a\sin\alpha - g + \frac{v_x^2}{R})\mathrm{d}t = -v_{0_y} \\ \int_0^T (v_{0_y} + \int_0^t (a\sin\alpha - g + \frac{v_x^2}{R})\mathrm{d}t)\mathrm{d}t = \Delta h\end{aligned} \tag{24}$$

For the sake of simplicity, we will be treating v_x, v_y, R and g to be constant and equal to the average of their initial and final values.

By integrating by parts,

$$\begin{aligned}&v_{0_y}T + T\int_0^T (a\sin\alpha - g + \frac{v_x^2}{R})\mathrm{d}t - \int_0^T (t(a\sin\alpha - g + \frac{v_x^2}{R}))\mathrm{d}t = \Delta h \\ &v_{0_y}T - Tv_{0_y} - \int_0^T (t(a\sin\alpha - g + \frac{v_x^2}{R}))\mathrm{d}t = \Delta h \\ &\int_0^T t(a\sin\alpha - g + \frac{v^2}{R})\mathrm{d}t = -\Delta h\end{aligned} \tag{25}$$

Here, a is the acceleration at any point of time. Note that this is a piecewise continuous function of time.

We now have three integral equations to solve. We may solve them by assuming some A, B and T. A good guess would be the values from the previous iteration. We then substitute the values in the equation, representing the small deviations by deltas.

$$\begin{aligned}\delta\int_0^T (a\cos\alpha - \frac{2v_x v_y}{R})\mathrm{d}t = \delta\Delta v_x \\ \delta\int_0^T (a\sin\alpha - g + \frac{v_x^2}{R})\mathrm{d}t = -\delta v_{0_y} \\ \delta\int_0^T t(a\sin\alpha - g + \frac{v^2}{R})\mathrm{d}t = -\delta\Delta h\end{aligned} \tag{26}$$

https://www.overleaf.com/project/5fb1242f6183a1674d10441b On taking the partial derivatives to account for the error, we get

$$\begin{aligned}(a\cos\alpha_T - 2\frac{v_x v_y}{R})\delta T + (\int_0^T a(\frac{\partial\cos\alpha}{\partial A}))\delta A + (\int_0^T a(\frac{\partial\cos\alpha}{\partial B}))\delta B = \delta\Delta v_x \\ (a\sin\alpha_T - g + \frac{v_x^2}{R})\delta T + (\int_0^T a(\frac{\partial\sin\alpha}{\partial A}))\delta A + (\int_0^T a(\frac{\partial\sin\alpha}{\partial B}))\delta B = -\delta v_{0_y} \\ (a\sin\alpha_T - g + \frac{v_x^2}{R})T\delta T + (\int_0^T ta(\frac{\partial\sin\alpha}{\partial A}))\delta A + (\int_0^T ta(\frac{\partial\sin\alpha}{\partial B}))\delta B = -\delta\Delta h\end{aligned} \tag{27}$$

We now have three linear equations in δT, δA and $deltaB$, which can be solved at each point in the trajectory. We can then revise our guess of (T, A, B) to ($T + \delta T$, $A + \delta A$, $B + \delta B$) and run this process throughout the flight.

5 Terminal Guidance

Our major loop guidance works perfectly and gets us very close to our intended orbit, but there is a catch- since the solutions are only approximate when far from orbit, the guidance equations try to make too many corrections when very close to the orbit. The craft will have to execute some very difficult maneuvers to accomplish this.Moreover, at some point, the path cannot be corrected anymore. At this time, we say that the solutions have diverged.

To avoid such scenario from happening,t most rockets stop solving the guidance equations when very close to orbit (in our case, we do it when there are 20 secondss of burn time remaining) and use the last calculated value to maneuver itself. This final phase of burn is known as Terminal Guidance. This process involves a lot of losses and is responsible for many errors during orbital insertion.

Our rocket, however does something different from this. We get around 15m of accuracy, only being limited by the fact that we are unable to control our engine thrust precisely. The main problem is the fact that our rate of turn increases quite a lot when we are close to the orbit. So, we shall attempt to solve our problem by orienting the rocket in a fixed direction.

Note that we are very close to orbit when the terminal guidance begins(<1km). So we should, in theory be more accurate if we target our velocity vector somewhere, instead of 'blindly' following the previous iteration. On doing so, it turns out that 'targeting' our velocity vector in the direction of($\vec{v'}$)-current velocity vector($\vec{v}$) gives us improves accuracy.

$$\tan(\alpha) = \frac{-v_y}{v' - v_x} \tag{28}$$

Using this method, we were able to get an orbit accurate to about 40-50 m. That's a lot of improvement. However, in theory, we may even get errors of up to 200m.(It just works out for our case), because we aren't fixing a target altitude anymore. We are only fixing a target velocity. Therefore, this Terminal guidance law will not give us an exact orbit of our desire. It is, however more accurate than just following the previously calculated values.

To fix this theoretical issue, we need another variable that doesn't affect the rate of change of alpha directly.

More accuracy is achieved by using throttlable engines. If we can control the engine thrust, we basically regain control over our altitude. Using throttle has an advantage over using the major loop guidance. The throttle can change drastically, it won't have much effect on the ship, but turning fast while your engines are on, is an issue.

When at terminal guidance, we are very close to our intended orbit, so the pseudo force along x term goes to 0 as v_y is small.
Further we can assume the gravitational acceleration is exactly cancelled by the pseudo force along y.

And the last assumption is that the mass of the ship doesn't change much, since in only 20 secs, the percentage change in mass will not be that high.
We have

$$\frac{F\cos(\alpha)T}{m} = v' - v_x$$
$$\frac{F\sin(\alpha)T}{m} = -v_y \tag{29}$$
$$\frac{v_y}{2}T = \Delta H$$

The last equation is a direct consequence of the fact that the throttle is constant and mass isn't changing and the angle is fixed, so acceleration is constant.
And hence from the last two equations,

$$\begin{aligned} \frac{2F\sin(\alpha)\Delta H}{mv_y} &= -v_y \\ \Rightarrow F &= -\frac{mv_y^2}{2\sin(\alpha)\Delta H} \end{aligned} \tag{30}$$

So, our throttle will be just this divided by the total engine thrust (F_{max})

$$throttle = -\frac{mv_y^2}{2\sin(\alpha)\Delta H F_{max}} \tag{31}$$

Now, what if you also want the burn to end at a particular downrange distance? This isn't necessary for getting to orbit, but is for maneuvers like trans-lunar injection or trans mars injection. We basically fall short of another variable. That can be achieved by using a throttle with two independent variables, like:

$$throttle = \epsilon + \tau t \tag{32}$$

6 Conclusion

Our guidance equations work perfect taking the minimum amount of time to get to orbit.

We could have made the simulation more accurate using numerical derivatives instead of constant pseudo forces.

The terminal guidance part gives it the best possible accuracy. Typically, rockets get to around 5km accuracy using this process (the apogee is around 305km and perigee is around 295km for a target orbit of 300km). But, using the throttle as a variable has eliminated the need to stop solving the guidance equations.

Thus we get a phenomenal accuracy of less than 50m. This is quite an achievement!

References

[1] Fred Teren. Explicit guidance equations for multistage boost trajectories. 1966.

[2] Stefano Campagnola. Optimal space trajectories, 2010.