Math 395

Homework 3

Due: 9/24/2024

Name: Gianluca Crescenzo

Collaborators: Avinash Iyer, Noah Smith, Carly Venenciano

For these problems V is a finite-dimensional F-vector space.

Exercise.

(1) Let $\mathcal{E}$ be a basis of U, $\mathcal{F}$ a basis of V and $\mathcal{G}$ a basis of W. Let $T_B \in \mathrm{Hom}_F(U, V)$ and $T_A \in \mathrm{Hom}_F(V, W)$. Show

$$[T_A \circ T_B]_{\mathcal{E}}^{\mathcal{G}} = [T_A]_{\mathcal{F}}^{\mathcal{G}} [T_B]_{\mathcal{E}}^{\mathcal{F}}.$$

(2) Let $[T_A]_{\mathcal{F}}^{\mathcal{G}} = A \in \mathrm{Mat}_{p,m}(F)$ and $[T_B]_{\mathcal{E}}^{\mathcal{F}} = B \in \mathrm{Mat}_{m,n}(F)$. Show that you can recover the definition of matrix multiplication by using part (1).

Proof. Since the following diagram commutes:

$$\begin{tikzcd}
U \arrow[r, "T_B"] \arrow[d, "T_{\mathcal{E}}"'] & V \arrow[r, "T_A"] \arrow[d, "T_{\mathcal{F}}"'] & W \arrow[d, "T_{\mathcal{G}}"'] \\
F^n \arrow[r, "{[T_B]_{\mathcal{E}}^{\mathcal{F}}}"'] & F^m \arrow[r, "{[T_A]_{\mathcal{F}}^{\mathcal{G}}}"'] & F^p,
\end{tikzcd}$$

we have that $[T_A \circ T_B]_{\mathcal{E}}^{\mathcal{G}} = [T_A]_{\mathcal{F}}^{\mathcal{G}} [T_B]_{\mathcal{E}}^{\mathcal{F}}$. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$, $\mathcal{F} = \{f_1, \ldots, f_m\}$, and $\mathcal{G} = \{g_1, \ldots, g_p\}$. The equations

$$T_B(e_j) = \sum_{k=1}^{m} b_{kj} f_k$$
$$T_B(f_k) = \sum_{i=1}^{p} a_{ik} g_i$$

give rise to the following:

$$\begin{aligned} T_A(T_B(e_j)) &= \sum_{i=1}^{p} \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right) g_i \\ &:= \sum_{i=1}^{p} c_{ij} g_i. \end{aligned}$$

Hence $[T_A \circ T_B]_{\mathcal{E}}^{\mathcal{G}} = AB = (c_{ij}) \in \mathrm{Mat}_{p,n}(F)$. □

Exercise 1. *Let $V = P_n(F)$. Let $\mathcal{B} = \{1, x, x^2, ..., x^n\}$ be a basis of V. Let $\lambda \in F$ and set $\mathcal{C} = \{1, (x-\lambda), (x-\lambda)^2, ..., (x-\lambda)^n\}$. Define a linear transformation $T \in \mathrm{Hom}_F(V, V)$ by defining $T(x^j) = (x-\lambda)^j$. Determine the matrix of this linear transformation. Use this to conclude that $\mathcal{C}$ is also a basis of V.*

Proof. Note that $T(x^j) = (x-\lambda)^j = \sum_{k=0}^{j} \binom{j}{k}(-\lambda)^{j-k}x^k$ for all $0 \leqslant j \leqslant n$. Hence:

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & -\lambda & \lambda^2 & \dots & (-\lambda)^n \\ 0 & 1 & -2\lambda & \dots & \binom{n}{1}(-\lambda)^{n-1} \\ 0 & 0 & 1 & \dots & \binom{n}{2}(-\lambda)^{n-2} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Note that this matrix is non-singular, hence it is an isomorphism. Thus there exists a T^{-1} defined by $T^{-1}((x-\lambda)^j) = x^j$, establishing that $\mathcal{C}$ forms a basis of V. □

Exercise 4. *Let $V = P_5(\mathbf{Q})$ and let $\mathcal{B} = \{1, x, ..., x^5\}$. Prove that the following are elements of $V^\vee$ and express them as linear combinations of the dual basis:*

(1) $\phi_1 : V \to \mathbf{Q}$ defined by $\phi(p(x)) = \int_0^1 t^2 p(t)dt$.

(2) $\phi_2 : V \to \mathbf{Q}$ defined by $\phi(p(x)) = p'(5)$ where $p'(x)$ denotes the derivative of $p(x)$.

Proof. Let $p_1, p_2 \in P_5(\mathbf{Q})$. Then:

$$\begin{aligned} \phi_1((p_1 + cp_2)(x)) &= \int_0^1 t^2(p_1 + cp_2)(x)dt \\ &= \int_0^1 t^2(p_1(x) + cp_2(x))dt \\ &= \int_0^1 t^2 p_1(x)dt + c\int_0^1 t^2 p_2(x)dt \\ &= \phi_1(p_1(x)) + c\phi_1(p_2(x)). \end{aligned}$$

$$\begin{aligned} \phi_2((p_1 + cp_2)(x)) &= (p_1 + cp_2)'(5) \\ &= p_1'(5) + cp_2'(5) \\ &= \phi_2(p_1(x)) + c\phi_2(p_2(x)). \end{aligned}$$

Thus $\phi_1, \phi_2 \in V^\vee$. Note that elements of the dual basis $\mathcal{B}^\vee = \{1^\vee, x^\vee, ..., x^{5^\vee}\}$ are defined as follows:

$$x^{i^\vee}(x^j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise,} \end{cases}$$

and furthermore ${x^i}^\vee(p) = a_i$ for some $p \in P_5(\mathbf{Q})$ and $i \in \{0, 1, ..., 5\}$. Thus we can express any $\phi \in V^\vee$ in terms of its dual basis as follows:

$$\begin{aligned}
\phi(p) &= \phi\left(\sum_{i=0}^{5} a_i x^i\right) \\
&= \sum_{i=0}^{5} a_i \phi(x^i) \\
&= \sum_{i=0}^{5} {x^i}^\vee(p)\phi(x^i).
\end{aligned}$$

Hence:

$$\begin{aligned}
\phi_1(p) &= \sum_{i=0}^{5}\left[{x^i}^\vee(p)\left(\int_0^1 t^{2+i}dt\right)\right] = \sum_{i=0}^{5} {x^i}^\vee(p)\cdot\frac{1}{3+i} \\
\phi_2(p) &= \sum_{i=0}^{5} {x^i}^\vee(p)\cdot i x^{i-1}
\end{aligned}$$

□

Exercise 5. *Let V be a vector space over F and let $T \in \mathrm{Hom}_F(V, V)$. A nonzero $v \in V$ satisfying $T(v) = \lambda v$ for some $\lambda \in F$ is called an eigenvector of T with eigenvalue λ.*

(a) Prove that for any fixed $\lambda \in F$ the collection of eigenvectors of T with eigenvalue λ together with 0_V forms a subspace of V.

(b) Prove that if V has a basis $\mathcal{B}$ consisting of eigenvectors for T then $[T]_{\mathcal{B}}^{\mathcal{B}}$ is a diagonal matrix with the eigenvalues of T as diagonal entries.

Proof. Let $E = \{v_i \mid T(v_i) = \lambda v_i,\ v_i \in V\} \cup \{0_V\}$. Clearly E is nonempty. Let $v_1, v_2 \in E$. Then $T(v_1 + cv_2) = T(v_1) + cT(v_2) = \lambda v_1 + c\lambda v_2 = \lambda(v_1 + cv_2)$. Thus E is a subspace of V.

Now let $\mathcal{B} = \{v_1, ..., v_n\}$ where each v_i is an eigenvector associated with a unique eigenvalue λ_i. Observe that:

$$\begin{aligned}
T(v_1) &= \lambda_1 v_1 = \lambda_1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + ... + 0 \cdot v_n \\
T(v_2) &= \lambda_2 v_2 = 0 \cdot v_1 + \lambda_2 \cdot v_2 + 0 \cdot v_3 + ... + 0 \cdot v_n \\
T(v_3) &= \lambda_3 v_3 = 0 \cdot v_1 + 0 \cdot v_2 + \lambda_3 \cdot v_3 + ... + 0 \cdot v_n \\
&\vdots \\
T(v_n) &= \lambda_n v_n = 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + ... + \lambda_n v_n.
\end{aligned}$$

Hence:

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_n \end{pmatrix}.$$

□