Math 397

Homework 2

Name: Gianluca Crescenzo

Exercise 5. Let $\lambda = (\lambda_k)_k$ belong to ℓ_∞. Show that the map:

$$D_\lambda : \ell_2 \to \ell_2 \text{ defined by } D_\lambda((\xi_k)_k) = (\lambda_k \xi_k)_k$$

is well-defined, linear, and bounded with $\|D_\lambda\|_{\text{op}} = \|\lambda\|_\infty$.

Proof. Let $(x_k)_k, (y_k)_k \in \ell_2$ with $(x_k)_k = (y_k)_k$. Then $x_k = y_k$ for all $k \in \mathbf{N}$. We have:

$$\begin{aligned} D_\lambda((x_k)_k) &= (\lambda_k x_k)_k \\ &= (\lambda_k y_k)_k \\ &= D_\lambda((y_k)_k). \end{aligned}$$

Thus D_λ is well-defined. Let $\alpha \in F$. Observe that:

$$\begin{aligned} D_\lambda\Big((x_k)_k + \alpha(y_k)_k\Big) &= D_\lambda\Big((x_k + \alpha y_k)_k\Big) \\ &= \Big(\lambda_k(x_k + \alpha y_k)\Big)_k \\ &= (\lambda_k x_k + \alpha \lambda_k y_k)_k \\ &= D_\lambda((x_k)_k) + \alpha D_\lambda((y_k)_k). \end{aligned}$$

Whence D_λ is linear. It only remains to show that D_λ is bounded. We have:

$$\begin{aligned} \|D_\lambda\|_{\text{op}} &= \sup_{(x_k)_k \in B_{\ell_2}} \|(\lambda_k x_k)_k\|_{\ell_2} \\ &= \sup_{(x_k)_k \in B_{\ell_2}} \left(\sum_{k=1}^\infty |\lambda_k x_k|^2 \right)^{\frac{1}{2}} \\ &= \sup_{(x_k)_k \in B_{\ell_2}} \left(\sum_{k=1}^\infty |\lambda_k|^2 |x_k|^2 \right)^{\frac{1}{2}} \\ &\leqslant \sup_{(x_k)_k \in B_{\ell_2}} \left(\sum_{k=1}^\infty \left(\sup_{i=1}^\infty |\lambda_i| \right)^2 |x_k|^2 \right)^{\frac{1}{2}} \\ &= \sup_{(x_k)_k \in B_{\ell_2}} \left(\sup_{i=1}^\infty |\lambda_i| \cdot \left(\sum_{k=1}^\infty |x_k|^2 \right)^{\frac{1}{2}} \right) \\ &= \sup_{i=1}^\infty |\lambda_i| \cdot \sup_{(x_k)_k \in B_{\ell_2}} \left(\sum_{k=1}^\infty |x_k|^2 \right)^{\frac{1}{2}} \\ &= \sup_{i=1}^\infty |\lambda_i| \cdot \sup_{(x_k)_k \in B_{\ell_2}} \|(x_k)_k\|_{\ell_2} \\ &\leqslant \sup_{i=1}^\infty |\lambda_i| \cdot 1 \\ &= \|(\lambda_k)_k\|_\infty < \infty. \end{aligned}$$

$\square$

Exercise 6. Consider the vector space $C([0,2\pi])$ equipped with the pairing:

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt.$$

(1) Show that this pairing defines an inner product on $C([0,2\pi])$.

(2) For $n \in \mathbf{Z}$ set $e_n(t) := \cos(nt) + i\sin(nt)$. Show that the family $\{e_n\}_{n\in\mathbf{Z}}$ is orthonormal.

Proof. (1) We must first show that $\langle\cdot,\cdot\rangle$ defined as above is sesquilinear. We have:

$$\begin{aligned}\langle f, g_1 + \alpha g_2\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{(g_1+\alpha g_2)(t)}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g_1(t)}dt + \frac{\overline{\alpha}}{2\pi}\int_0^{2\pi} f(t)\overline{g_2(t)}dt \\ &= \langle f, g_1\rangle + \overline{\alpha}\langle f, g_2\rangle\end{aligned}$$

$$\begin{aligned}\langle f_1 + \alpha f_2, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (f_1+\alpha f_2)(t)\overline{g(t)}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f_1(t)\overline{g(t)}dt + \frac{\alpha}{2\pi}\int_0^{2\pi} f_2(t)\overline{g(t)}dt \\ &= \langle f_1, g\rangle + \alpha\langle f_2, g\rangle.\end{aligned}$$

The map $\langle\cdot,\cdot\rangle$ is Hermitian because:

$$\begin{aligned}\overline{\langle g, f\rangle} &= \overline{\frac{1}{2\pi}\int_0^{2\pi} g(t)\overline{f(t)}dt} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \overline{g(t)}f(t)dt \\ &= \langle f, g\rangle.\end{aligned}$$

It remains to show that $\langle\cdot,\cdot\rangle$ is positive-definite. Observe that:

$$\begin{aligned}\langle f, f\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{f(t)}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt \\ &> 0.\end{aligned}$$

Thus $\Big(C([0,2\pi]), \langle\cdot,\cdot\rangle\Big)$ is an inner product space.

(2) Observe that:

$$
\begin{aligned}
\|e_n\|_2 &= \langle e_n, e_n \rangle^{\frac{1}{2}} \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} (\cos(nt) + i\sin(nt))(\cos(nt) - i\sin(nt)) dt \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} (\cos^2(nt) + \sin^2(nt)) dt \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} 1 dt \right)^{\frac{1}{2}} \\
&= 1.
\end{aligned}
$$

Thus $\{e_n\}_{n\in\mathbf{Z}}$ is a family of unit vectors. We also have:

$$
\begin{aligned}
\langle e_j, e_k \rangle &= \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\cos(jt) + i\sin(jt)\right) \left(\cos(kt) + i\sin(kt)\right) dt \right) \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\cos(jt)\cos(kt) + i\sin(jt)\cos(kt) - i\sin(kt)\cos(jt) + \sin(jt)\sin(kt)\right) dt \right) \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\cos((j-k)t) + i\sin((j-k)t)\right) dt \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{j-k} \left(\sin((j-k)t) - i\cos((j-k)t) \Big|_0^{2\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{j-k} (-i + i) \\
&= 0.
\end{aligned}
$$

Thus the family $\{e_n\}_{n\in\mathbf{Z}}$ is orthonormal. $\square$

Exercise 7. Let V be any normed space, $p \in [1, \infty]$, and suppose $T : \ell_p^n \to V$ is linear. Show that T is bounded.

Proof. Let $\mathfrak{B} = \{e_1, ..., e_n\}$ be a basis for ℓ_p^n with $\|e_i\|_p = 1$ for all i. Observe that:

$$\begin{aligned}
\|T\|_{\text{op}} &= \sup_{v \in B_{\ell_p^n}} \|T(v)\|_p \\
&= \sup_{e_i \in B_{\ell_p^n}} \left\| \sum_{i=1}^{n} \alpha_i T(e_i) \right\|_p \\
&\leqslant \sum_{i=1}^{n} |\alpha_i| \sup_{e_i \in B_{\ell_p^n}} \|T(e_i)\|_p \\
&= \sum_{i=1}^{n} |\alpha_i| \max_{e_i \in B_{\ell_p^n}} \|T(e_i)\|_p \\
&= \sum_{i=1}^{n} |\alpha_i| \, \|T(e_M)\|_p \\
&< \infty.
\end{aligned}$$

Thus T is bounded. □

Exercise 9. Let V be an infinite-dimensional normed space. Show that there is a linear functional $\varphi : V \to F$ that is unbounded.

Proof. Let $\mathfrak{B}$ be a basis for V. Since the cardinality of $\mathfrak{B}$ is infinite, we have that $\mathbf{N} \hookrightarrow \mathfrak{B}$. Let $\{v_1, v_2, ...\} \subseteq \mathfrak{B}$. Define $\varphi : \mathfrak{B} \to F$ by:

$$\varphi(v) = \begin{cases} n, & v \in \{v_1, v_2, ...\} \\ 0, & v \notin \{v_1, v_2, ...\} \end{cases}$$

Since this is a map of basis elements, there exists a unique linear map $\varphi_F : V \to F$ with $\varphi_F(v) = \varphi(v)$ for all $v \in \mathfrak{B}$. Whence:

$$\begin{aligned}
\|\varphi_F\|_{\text{op}} &= \sup_{v \in B_V} \|\varphi_F(v)\| \\
&\geqslant \sup_{n=1}^{\infty} \|\varphi_F(v_n)\| \\
&= \sup_{n=1}^{\infty} n \\
&= \infty.
\end{aligned}$$

Thus $\varphi_F : V \to F$ is unbounded. □