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1 Orderings and Functions

1.1 Basic Notation

Definition 1.1.1.

(1) The *natural numbers* are defined as $\mathbf{N} = \{1, 2, 3, \ldots\}$,

(2) The *positive integers* are defined as $\mathbf{N}_0 = \mathbf{Z}^+ = \{0, 1, 2, 3, \ldots\}$,

(3) The *integers* are defined as $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$,

(4) The *rational numbers* are defined as $\mathbf{Q} = \{\frac{a}{b} \mid a, b \in \mathbf{Z}, b \neq 0\}$,

(5) The *real numbers* are "defined" (we will get more into this later) as the set $(-\infty, \infty)$,

(6) The *complex numbers* are defined as $\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R}, i^2 = -1\}$.

Example 1.1.1. Note that $\sqrt{2}, \pi, e \notin \mathbf{Q}$, as they cannot be expressed as fractions.

Definition 1.1.2. Let A and B be sets. The *cartesian product* is defined as $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Definition 1.1.3. A *relation* from A to B is a subset $R \subseteq A \times B$. Typically, when one says "a relation on A" that means a relation from A to A; i.e., $R \subseteq A \times A$.

Definition 1.1.4. Let A be a set and R a relation on A. Then R is:

(1) *reflexive* if $(a, a) \in R$ for all $a \in A$,

(2) *transitive* if $(a, b), (b, c) \in R$ implies $(a, c) \in R$,

(3) *symmetric* if $(a, b) \in R$ implies $(b, a) \in R$, and

(4) *antisymmetric* if $(a, b), (b, a) \in R$ implies $a = b$.

1.2 Orderings

Definition 1.2.1. Let A be a set. An *ordering* of A is a relation R on A that is reflexive, transitive, and antisymmetric. If this is the case, we write $(a, b) \in R$ as $a \leqslant_R b$. If A is an ordered set we write it as the ordered pair $(A, \leqslant_R)$ (or just A if the ordering is obvious by context).

Example 1.2.1.

(1) Let $m, n \in \mathbf{Z}$. The *algebraic ordering* $\leqslant_a$ is defined as follows: $m \leqslant_a n$ if and only if there exists an element $k \in \mathbf{N}_0$ with $m + k = n$.

(2) The set of natural numbers $\mathbf{N}$ equipped with the relation of divisibility form an ordering. Let $m, n \in \mathbf{N}$. Then $m \leqslant_d n$ if and only if $m \mid n$.

(3) Let S be any set. The subsets of S (i.e., elements of its power set) equipped with the relation of inclusion form an ordering. Let $A, B \in \mathcal{P}(S)$. Then $A \leqslant_{\mathcal{P}(S)} B$ if and only if $A \subseteq B$.

(4) The set of rational numbers $\mathbf{Q}$ form an algebraic ordering as follows: if $\frac{a}{b}, \frac{c}{d} \in \mathbf{Q}$, then $\frac{a}{b} \leqslant_a \frac{c}{d}$ if and only if $ad \leqslant_a bc$ (in $\mathbf{Z}$).

Definition 1.2.2. An ordered set $(A, \leqslant_R)$ is *total* (or *linear*) if for all $a, b \in A$ we have that $a \leqslant_R b$ or $b \leqslant_R a$.

Example 1.2.2. The ordered sets $(\mathbf{Z}, \leqslant_a)$ and $(\mathbf{Q}, \leqslant_a)$ are total orderings, whereas $(\mathbf{N}, \leqslant_d)$ and $(\mathcal{P}(S), \leqslant_{\mathcal{P}(S)})$ are not total orderings.

Definition 1.2.3. Let $(X, \leqslant)$ be an ordered set. Let $A \subseteq X$.

(1) A is called *bounded above* if there exists an element $u \in X$ with $a \leqslant u$ for all $a \in A$. Such a u (not necessarily unique) is called an *upperbound* for A.

(2) A is called *bounded below* if there exists an element $v \in X$ with $v \leqslant a$ for all $a \in A$. Such a v (not necessarily unique) is called a *lowerbound* for A.

(3) If A admits an upperbound u with $u \in A$, then u is called *the greatest element of A*.

(4) If A admits a lowerbound v with $v \in A$, then v is called *the least element of A*.

(5) Let A be bounded above. The *set of upperbounds of A* is defined as $\mathcal{U}_A = \{u \in X \mid u \text{ is an upperbound of } A\}$. If l is the least element of $\mathcal{U}_A$, we write $l = \sup(A)$ and call it *the supremum of A*.

(6) Let A be bounded below. The *set of lowerbounds of A* is defined as $\mathscr{L}_A = \{v \in X \mid v \text{ is a lowerbound of } A\}$. If g is the greatest element of $\mathscr{L}_A$, we write $g = \inf(A)$ and call it *the infimum of A*.

(7) A *maximal element of A* is an element $m \in A$ such that if $a \geqslant m$, then $a = m$ (not necessarily unique).

(8) A *minimal element of A* is an element $n \in A$ such that if $a \leqslant n$, then $a = n$ (not necessarily unique).

(9) If $(A, \leqslant)$ is a total ordering, then A is called a *chain*.

Proposition 1.2.1. *Let $(X, \leqslant)$ be an ordered set and $A \subseteq X$.*

(1) If A admits a greatest element, then it is unique,

(2) *If A admits a least element, then it is unique,*

(3) *If A admits a least upper bound, then it is unique,*

(4) *If A admits a greatest lower bound, then it is unique.*

Proof. Suppose u, u' are greatest elements of A, then $u, u' \in A$. Hence $u \leqslant u'$ and $u' \leqslant u$. By antisymmetry, $u = u'$, meaning the greatest element is unique. The proof for least elements being unique is identical, which establishes (1) and (2).

Note that $\mathcal{U}_A \subseteq X$. By definition the least element of $\mathcal{U}_A$ is defined to be the supremum of A, and since least elements are unique the supremum of A must be unique. Similarly, $\mathscr{L}_A \subseteq X$. By definition the greatest element of $\mathscr{L}_A$ is defined to be the infimum of A, and since greatest elements are unique the infimum of A must be unique. This establishes (3) and (4). □

Lemma 1.2.2 (Zorn's Lemma)**.** *Let X be an ordered set with the property that every chain has an upperbound. Then X contains a maximal element.*

Example 1.2.3. Considered the ordered set $(\mathbf{N}, \leqslant_d)$ and the subset $A = \{4, 7, 12, 28, 35\}$.

- A is bounded above with $4 \times 7 \times 12 \times 28 \times 35$ as an upperbound.
- The supremum of A is $\operatorname{lcm}(4, 7, 12, 28, 35)$.
- There does not exist a greatest element.
- $12, 28$, and 35 are maximal elements (no other element in A divides them).

Definition 1.2.4. Let $(X, \leqslant)$ be an ordered set and $A \subseteq X$. If A is bounded above and below, then we say A is *bounded*.

Definition 1.2.5. Let $(X, \leqslant)$ be an ordered set. Then $(X, \leqslant)$ is *complete* if, for every bounded set $A \subseteq X$, $\sup(A)$ and $\inf(A)$ exist.

1.3 Functions

Definition 1.3.1. Let X and Y be sets. A *function* from X to Y is a relation $f \subseteq X \times Y$ such that for all $x \in X$, there exists a unique $y_x \in Y$ with $(x, y_x) \in f$.

(1) The set X is the *domain* of f.

(2) The set Y is the *codomain* of f.

(3) The *image* of f is defined as $f(X) = \{f(x) \mid x \in X\} \subseteq Y$ (also sometimes denoted $\operatorname{im}(f)$).

(4) The *preimage* of f is defined as $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\} \subseteq X$.

(5) The *graph* of f is defined as $\operatorname{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

If f is a function, we denote it by $f : X \to Y$ or $X \xrightarrow{f} Y$.

Example 1.3.1. Let X be a set.

(1) The *identity map* $\mathrm{id}_X : X \to X$ is defined by $\mathrm{id}_X(x) = x$.

(2) If $X \subseteq Y$, the *inclusion map* $\iota : X \to Y$ is defined by $\iota(x) = x$.

(3) If $A \subseteq X$ is a set, the *characteristic function* (or *step function*) $\mathbf{1}_A : X \to \mathbf{R}$ is defined by

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Definition 1.3.2. Given $f, g : X \to \mathbf{R}$ and $\alpha \in \mathbf{R}$, the *pointwise operations* on f and g are:

- $(f \pm g)(x) = f(x) \pm g(x)$,
- $(\alpha f)(x) = \alpha f(x)$,
- $(fg)(x) = f(x)g(x)$,
- $(f/g)(x) = f(x)/g(x)$.

Definition 1.3.3. Let $f : X \to Y$ and $g : Y \to Z$ be maps between sets. The *composition* of f and g is denoted $g \circ f : X \to Z$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Definition 1.3.4. Let $f : X \to Y$ be a map between sets.

(1) f is *left-invertible* if there exists a map $g : Y \to X$ with $g \circ f = \mathrm{id}_X$.

(2) f is *right-invertible* if there exists a map $h : Y \to X$ with $f \circ h = \mathrm{id}_Y$.

(3) f is *invertible* if there exists a map $k : Y \to X$ with $k \circ f = \mathrm{id}_X$ and $f \circ k = \mathrm{id}_Y$.

Example 1.3.2. The *shift function* is a map $s : \mathbf{N} \to \mathbf{N}$ defined by $s(n) = n + 1$. Note that this function is left-invertible: define $g : \mathbf{N} \to \mathbf{N}$ by

$$g(n) = \begin{cases} n - 1, & n \geqslant 2 \\ n_0, & n = 1, \end{cases}$$

where n_0 is an arbitrary natural number, then $g \circ s = \mathrm{id}_{\mathbf{N}}$.

Suppose that s has a right inverse, that is, there exists a function $h : \mathbf{N} \to \mathbf{N}$ such that $s \circ h = \mathrm{id}_{\mathbf{N}}$. Observe that:

$$(s \circ h)(1) = s(h(1)) = h(1) + 1 = 1.$$

It must be the case that $h(1) = 0$, which is a contradiction. Hence s is not right-invertible.

Example 1.3.3. The function g defined above is right invertible, but not left invertible.

Proposition 1.3.1. *Let $f : X \to Y$ be a map between sets. The following are equivalent:*

(1) f is invertible,

(2) f is right-invertible and left-invertible.

Proof. Clearly (1) implies (2). Assume f to be left and right-invertible. Then there exists maps $h, g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$. Observe that:

$$\begin{aligned} h &= \mathrm{id}_X \circ h \\ &= (g \circ f) \circ h \\ &= g \circ (f \circ h) \\ &= g \circ \mathrm{id}_Y \\ &= g, \end{aligned}$$

establishing the proposition. □

Definition 1.3.5. Let $f : X \to Y$ be a map between sets.

(1) f is *injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$,

(2) f is *surjective* if $\mathrm{im}\,(f) = Y$, and

(3) f is *bijective* if it is injective and surjective.

Proposition 1.3.2. *Let $f : X \to Y$ be a map between sets.*

1. f is injective if and only if f is left-invertible.

2. f is surjective if and only if f is right-invertible.

3. f is bijective if and only if f is invertible.

Proof. (1) Do the forward direction yourself! Now assume $f : X \to Y$ is injective. Define $g : Y \to X$ by

$$g(y) = \begin{cases} x_0, & y \notin \mathrm{im}\,(f) \\ x_y, & y \in \mathrm{im}\,(f), \end{cases}$$

where x_y is the unique element in x mapping to y; i.e., $f(x_y) = y$. By our construction, $(g \circ f)(x) = x$ for all $x \in X$.

(2) Do the forward direction yourself! Now assume $f : X \to Y$ is onto. Note that the preimage of f is nonempty, so we can define $h : Y \to X$ by $h(y) = x_y$, where $x_y \in f^{-1}(X)$. By our construction $(f \circ h)(y) = f(x_y) = y$ for all $y \in Y$.

(3) Do this yourself its easy! □

Corollary 1.3.3. *Let A, B be sets. There exists an injection $A \hookrightarrow B$ if and only if there exists a surjection $B \twoheadrightarrow A$.*

Proof. If $f : A \to B$ is injective, then f is left invertible, that is, there exists a function $g : B \to A$ with $g \circ f = \mathrm{id}_A$. But this means g is right invertible, so g is onto. The other direction follows identically. □

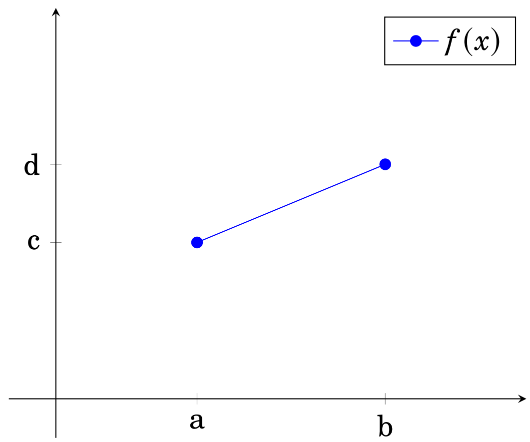
1.4 Cardinality

Definition 1.4.1. Let A, B be sets. Then $\mathrm{card}(A) = \mathrm{card}(B)$ if there exists a bijection $A \hookrightarrow\!\!\!\!\rightarrow B$.

Example 1.4.1.

(1) Define $f : \mathbf{N}_0 \to \mathbf{N}$ by $f(n) = n + 1$. This is a bijection, hence $\mathrm{card}(\mathbf{N}_0) = \mathrm{card}(\mathbf{N})$.

(2) Let $[a, b]$ and $[c, d]$ be intervals with $a < b$ and $c < d$. Define $f : [a, b] \to [c, d]$ by $f(x) = (\frac{d-c}{b-a})(x - a) + c$.



This is a bijection, hence $\mathrm{card}([a, b]) = \mathrm{card}([c, d])$. The result is the same had the intervals been open.

(3) Recall that $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbf{R}$ is a bijection. Consider the maps $(0, 1) \xrightarrow{g} (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow{\tan} \mathbf{R}$. Since g and $\tan$ are bijective, $\tan \circ g$ is bijective, hence $\mathrm{card}((0, 1)) = \mathrm{card}(\mathbf{R})$.

Definition 1.4.2. A set A is called *finite* if there exists an $N \in \mathbf{N}$ such that $\mathrm{card}(A) = \mathrm{card}(\{1, \ldots, N\})$. If not, then A is called *infinite*.

Proposition 1.4.1. *Given $m, n \in \mathbf{N}$, $m \neq n$, then* $\mathrm{card}(\{1, \ldots, m\}) \neq \mathrm{card}(\{1, \ldots, n\})$.

Proof. Without loss of generality, let $m > n$. Suppose towards contradiction we have a bijection $\{1, \ldots, m\} \xrightarrow{f} \{1, \ldots, n\}$. By the pigeon-hole principle, it must be the case that —given any $i, j \in \{1, \ldots, m\}$ with $i \neq j$, we have that $f(i) = f(j)$. This is a contradiction (f is not injective), hence $\mathrm{card}(\{1, \ldots, m\}) \neq \mathrm{card}(\{1, \ldots, n\})$. □

Proposition 1.4.2. $\mathbf{N}$ *is infinite.*

Proof. Suppose towards contradiction we have a bijection $f : \mathbf{N} \to \{1, 2, \ldots, n\}$, where $n \in \mathbf{N}$. Consider the maps $\{1, 2, \ldots, n, n+1\} \xhookrightarrow{\iota} \mathbf{N} \xrightarrow{f} \{1, 2, \ldots, n\}$, it must be the case that the composition $f \circ \iota$ is injective. However, we established in Proposition 1.4.1 that this is false. Having reached a contradiction, it must be the case that $\mathbf{N}$ is infinite. □

Exercise 1.4.1. *If A is infinite, there exists an injection $\mathbf{N} \hookrightarrow A$.*

Proof. Let $\pi : \mathbf{N} \to A$ be a map. Pick $a_1 \in A$ and define $\pi(1) = a_1$. Since A is infinite, $A \setminus \{a_1\}$ is also infinite. Pick $a_2 \in A \setminus \{a_1\}$ and define $\pi(2) = a_2$. Inductively, we have an injection $\mathbf{N} \hookrightarrow A$. $\square$

Example 1.4.2. Define $k : \mathbf{Z} \to \mathbf{N}$ by $k(n) = (-1)^{n-1} \left\lfloor \frac{n}{2} \right\rfloor$. This is a bijection, hence $\mathrm{card}(\mathbf{Z}) = \mathrm{card}(\mathbf{N})$.

Definition 1.4.3. Let X and Y be sets.

(1) The <u>*power set*</u> of X is $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.

(2) The <u>*set of functions*</u> from X to Y is $Y^X = \{f \mid f : X \to Y\}$.

Lemma 1.4.3. *Let X be a set. There exists a bijection $\mathcal{P}(X) \hookrightarrow\!\!\!\!\rightarrow 2^X$.*

Proof. Let $A \subseteq X$. Define $\varphi : \mathcal{P}(X) \to 2^X$ by $A \mapsto \mathbf{1}_A$, where

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

is the *characteristic function* defined in Example 1.3.1. Let $\varphi(A) = \varphi(B)$. This is equivalent to $\mathbf{1}_A = \mathbf{1}_B$. Note that functions are equal if and only if $\mathbf{1}_A(x) = \mathbf{1}_B(x)$ for all $x \in X$. Hence $x \in A$ implies $\mathbf{1}_A(x) = 1 = \mathbf{1}_B(x)$, giving $x \in B$. The reverse inclusion is identical, hence $A = B$. Let $f \in 2^X$. Let $A = \{x \in X \mid f(x) = 1\}$. Then $\varphi(A) = \mathbf{1}_A = f$. Thus $\mathcal{P}(X) \hookrightarrow\!\!\!\!\rightarrow 2^X$. $\square$

Exercise 1.4.2. *Show that* $\mathrm{card}(\mathcal{P}(\{1, \ldots, n\})) = 2^n$.

Proof. Note that $\mathrm{card}(\mathcal{P}(\{1, \ldots, n\})) = \mathrm{card}(2^{\{1,\ldots,n\}})$. Let $f \in 2^{\{1,\ldots,n\}}$. For each $i \in \{1, \ldots, n\}$, there is a choice of two outputs for $f(i)$. Hence by the fundamental principle of counting $\mathrm{card}(\mathcal{P}(\{1, \ldots, N\})) = \mathrm{card}(2^{\{1,\ldots,n\}}) = 2^n$. $\square$

Theorem 1.4.4 (Cantor's Diagonal Argument)**.** $\mathrm{card}(\mathbf{N}) < \mathrm{card}((0,1))$.

Proof. Recall that every $\sigma \in (0,1)$ has a decimal expansion $\sigma = 0.\sigma_1\sigma_2\ldots = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$, where $\sigma_j \in \{0, 1, 2, \ldots, 9\}$ which does not terminate in 9's. By way of contradiction, suppose there exists a surjection $r : \mathbf{N} \to (0,1)$ defined by $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$, where $\sigma_j(n) \in \{0, 1, 2, \ldots, 9\}$ is the j^{th} digit in the decimal expansion.

Consider the map $\tau : \mathbf{N} \to \{0, 1, \ldots, 9\}$ defined by:

$$\tau(n) = \begin{cases} 3, & \sigma_n(n) = 2 \\ 2, & \sigma_n(n) = 3, \end{cases}$$

and let $t = 0.\tau(1)\tau(2)\tau(3)\ldots$ Observe the following:

$$\begin{aligned}
r(1) &= 0.\sigma_1(1)\sigma_2(1)\sigma_3(1)\sigma_4(1)\ldots \\
r(2) &= 0.\sigma_1(2)\sigma_2(2)\sigma_3(2)\sigma_4(2)\ldots \\
r(3) &= 0.\sigma_1(3)\sigma_2(3)\sigma_3(3)\sigma_4(3)\ldots \\
r(4) &= 0.\sigma_1(4)\sigma_2(4)\sigma_3(4)\sigma_4(4)\ldots \\
&\vdots \\
r(n) &= 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\sigma_4(n)\ \ldots\ \sigma_n(n).
\end{aligned}$$

Since r is surjective, there is an $m \in \mathbf{N}$ with $r(m) = t$. It follows that:

$$\begin{aligned}
r(m) &= 0.\sigma_1(m)\sigma_2(m)\sigma_3(m)\ldots\sigma_m(m)\ldots \\
&= 0.\tau(1)\tau(2)\tau(3)\ldots\tau(m)\ldots
\end{aligned}$$

which implies that $\sigma_m(m) = \tau(m)$. But recall how we defined $\tau(n)$ —if $\sigma_m(m) = 2$, then $\tau(2) = 3$ and if $\sigma_m(m) \neq 2$, then $\tau(2) = 2$. This is a contradiction, hence there does not exist a surjection $\mathbf{N} \xrightarrow{r} (0,1)$. □

Corollary 1.4.5. $\mathrm{card}(\mathbf{N}) \neq \mathrm{card}(\mathbf{R})$

Proof. It follows from Example 1.4.1 that $\mathrm{card}(\mathbf{N}) < \mathrm{card}((0,1)) = \mathrm{card}(\mathbf{R})$. □

Definition 1.4.4. Let A and B be sets.

(1) We write $\mathrm{card}(A) \leqslant \mathrm{card}(B)$ if there exists an injection $A \hookrightarrow B$.

(2) We write $\mathrm{card}(A) < \mathrm{card}(B)$ if $\mathrm{card}(A) \leqslant \mathrm{card}(B)$ and $\mathrm{card}(A) \neq \mathrm{card}(B)$

Example 1.4.3.

(1) If $A \subseteq B$, then the inclusion map $\iota : A \to B$ gives $\mathrm{card}(A) \leqslant \mathrm{card}(B)$.

(2) If $m > n$, then $\mathrm{card}\{1, \ldots, n\} < \mathrm{card}\{1, \ldots, m\}$

Proposition 1.4.6. *Let A be a set. Then* $\mathrm{card}(A) < \mathrm{card}(\mathcal{P}(A))$.

Proof. Define $f : A \to \mathcal{P}(A)$ by $a \mapsto \{a\}$. This is clearly an injective map. Now suppose towards contradiction that there exists a surjection $g : A \to \mathcal{P}(A)$ defined by $a \mapsto g(a)$. Then $g(a) \subseteq A$ (by the definition of a power set).

Let $S = \{a \in A \mid a \notin g(a)\}$. Then $S \subseteq A$. Since g is onto, there exists an element $x \in A$ with $g(x) = S$. Case 1: $x \in S$. This implies that $x \notin g(x)$. But $g(x) = S$, so $x \notin S$, a contradiction. Case 2: $x \notin S$. This implies that $x \notin g(x)$. But by definition this means $x \in S$, a contradiction. Since we have exhausted all the necessary cases, it must be that there does not exist a surjection from $A \to \mathcal{P}(A)$. Hence $\mathrm{card}(A) < \mathrm{card}(\mathcal{P}(A))$. □

Lemma 1.4.7. *Let A and B be sets. The following are equivalent:*

(1) $\mathrm{card}(A) \leqslant \mathrm{card}(B)$*;*

(2) there exists an injection $A \hookrightarrow B$*;*

(3) there exists a surjection $B \twoheadrightarrow A$*.*

Example 1.4.4.

(1) Define $\mathbf{N} \times \mathbf{Z} \to \mathbf{Q}$ by $(n, m) \mapsto \frac{m}{n}$. This is surjective, so $\mathrm{card}(\mathbf{Q}) \leqslant \mathrm{card}(\mathbf{N} \times \mathbf{Z})$.

(2) Define $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$ by $(m, n) \mapsto 2^m \cdot 3^n$. Then g is injective by the fundamental theorem of arithmetic. So $\mathrm{card}(\mathbf{N} \times \mathbf{N}) \leqslant \mathrm{card}(\mathbf{N})$.

(3) Recall from Example 1.4.2 that $k : \mathbf{N} \to \mathbf{Z}$ defined by $k(n) = (-1)^{n-1} \left\lfloor \frac{n}{2} \right\rfloor$ is a bijection. Define $K : \mathbf{Z} \times \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ by $(m, n) \mapsto (k^{-1}(m), n)$. This is a bijection, so $\mathrm{card}(\mathbf{Z} \times \mathbf{N}) = \mathrm{card}(\mathbf{N} \times \mathbf{N})$.

(4) From the previous examples, we've established that:

$$\mathrm{card}(\mathbf{N}) \leqslant \mathrm{card}(\mathbf{Q}) \leqslant \mathrm{card}(\mathbf{Z} \times \mathbf{N}) = \mathrm{card}(\mathbf{N} \times \mathbf{N}) \leqslant \mathrm{card}(\mathbf{N})$$

Theorem 1.4.8. *Let* $\mathfrak{N}$ *denote the class of cardinals. The pair* $(\mathfrak{N}, \leqslant)$ *forms a total ordering —where* $\leqslant$ *is defined by* $\mathrm{card}(A) \leqslant \mathrm{card}(B)$ *if and only if* $A \hookrightarrow B$*. In particular, if* A, B, C *are sets with* $\mathrm{card}(A), \mathrm{card}(B), \mathrm{card}(C) \in \mathrm{obj}(\mathfrak{N})$*, then we have the following:*

(1) $\mathrm{card}(A) \leqslant \mathrm{card}(A)$ *(reflexive).*

(2) If $\mathrm{card}(A) \leqslant \mathrm{card}(B) \leqslant \mathrm{card}(C)$*, then* $\mathrm{card}(A) \leqslant \mathrm{card}(C)$ *(transitive).*

(3) If $\mathrm{card}(A) \leqslant \mathrm{card}(B)$ *and* $\mathrm{card}(B) \leqslant \mathrm{card}(A)$*, then* $\mathrm{card}(A) = \mathrm{card}(B)$ *(antisymmetric).*

(4) Either $\mathrm{card}(A) \leqslant \mathrm{card}(B)$ *or* $\mathrm{card}(B) \leqslant \mathrm{card}(A)$ *(total).*

Proof. (1) and (2) follow by simply applying definitions. Note that any set bijects into itself, hence $A \hookrightarrow\!\!\!\!\twoheadrightarrow A$ implies $A \hookrightarrow A$, establishing $\mathrm{card}(A) \leqslant \mathrm{card}(A)$. Similarly, if there are bijections $A \hookrightarrow\!\!\!\!\twoheadrightarrow B \hookrightarrow\!\!\!\!\twoheadrightarrow C$, then clearly there is a bijection $A \hookrightarrow\!\!\!\!\twoheadrightarrow C$. Hence $\mathrm{card}(A) = \mathrm{card}(C)$.

(3) (Cantor-Shröder-Bernstein Theorem) We have injections $A \overset{f}{\hookrightarrow}$ and $B \overset{g}{\hookrightarrow} A$. Let:

$$\begin{aligned} A_0 &= \mathrm{im}\,(g)^{\complement} \\ A_1 &= (g \circ f)(A_0) \\ A_2 &= (g \circ f)(A_1) \\ &\vdots \\ A_n &= (g \circ f)(A_{n-1}). \end{aligned}$$

Note that $A_1 \cap A_0 = \emptyset$ because $A_1 \subseteq \mathrm{im}\,(g)$ and $A_0 = \mathrm{im}\,(g)^{\complement}$. We similarly have that $A_2 \cap A_0 = \emptyset$. Claim: $A_1 \cap A_2$. finish this

(4) Let $A \to B$ be a map. Let $\mathcal{F} = \{(D, f) \mid D \subseteq A, f : D \hookrightarrow B,\ f \text{ is injective}\}$. Note that $\mathcal{F} \neq \emptyset$ because $(\emptyset, k) \in \mathcal{F}$ for some map k. Define an ordering on $\mathcal{F}$ as follows: $(D, f) \leqslant_{\mathcal{F}} (E, g)$ if and only if $D \subseteq E$ and $g|_D = f$. Then $\mathcal{F}$ admits an upperbound of A. By Zorn's Lemma, there exists a

maximal element $(M, h) \in \mathcal{F}$. Suppose towards contradiction there are elements $a \in A$, $a \notin M$ and $b \in B$, $b \notin h(M)$. Consider the map:

$$h' : M \cup \{a\} \to B \text{ defined by } \begin{cases} h'(M) = h(M) \\ h'(a) = b \end{cases}.$$

This set is clearly injective, and furthermore we have that $(M, h) \leqslant (M \cup \{a\}, h')$. This is a contradiction, hence $M = A$ or $h(M) = B$. If $M = A$, then the injection $A \overset{h}{\hookrightarrow} B$ implies $\operatorname{card}(A) \leqslant \operatorname{card}(B)$. If $h(M) = B$, then the map $B \hookrightarrow M \overset{\iota}{\hookrightarrow} A$ implies $\operatorname{card}(B) \leqslant \operatorname{card}(A)$. □

Corollary 1.4.9. $\operatorname{card}(\mathbf{Q}) = \operatorname{card}(\mathbf{N})$.

Proof. This follows directly from Example 1.4.4 and Theorem 1.4.8 □

Definition 1.4.5. A set A is <u>*countable*</u> if $\operatorname{card}(A) \leqslant \operatorname{card}(\mathbf{N})$. Equivalently, there exists an injection $A \hookrightarrow \mathbf{N}$ and a surjection $\mathbf{N} \twoheadrightarrow A$. If A is countable and infinite, A is called <u>*denumerable*</u> (or more commonly referred to as <u>*countably infinity*</u>).

Definition 1.4.6. We say $\operatorname{card}(\mathbf{N}) = \operatorname{card}(\mathbf{Z}) = \operatorname{card}(\mathbf{Q}) := \aleph_0$, called <u>*aleph naught*</u>. We also define $\operatorname{card}(\mathbf{R}) = \mathfrak{c}$, called the <u>*continuum*</u>.

Example 1.4.5. By Theorem 1.4.4, $\aleph_0 < \mathfrak{c}$.

Corollary 1.4.10. *There does not exist an infinite set A with* $\operatorname{card}(A) < \aleph_0$. *In particular, if A is infinite and countable, then* $\operatorname{card}(A) = \aleph_0$.

Proof. By Exercise 1.4.1, $\operatorname{card}(\mathbf{N}) \leqslant \operatorname{card}(A)$, and by definition (since A is countable), $\operatorname{card}(A) \leqslant \operatorname{card}(\mathbf{N})$. So by Theorem 1.4.8, $\operatorname{card}(A) = \operatorname{card}(\mathbf{N}) = \aleph_0$. □

Example 1.4.6. $\operatorname{card}(\mathcal{P}(\mathbf{N})) > \operatorname{card}(\mathbf{N}) = \aleph_0$.

Proposition 1.4.11. *The countable union of countable sets is countable. More precisely, if A_i is countable for all $i \in \mathbf{N}$, then $\bigcup_{i=1}^{\infty} A_i$ is countable.*

Proof. By definition, there exist surjections $\pi_i : \mathbf{N} \to A_i$. Define $\pi : \mathbf{N} \times \mathbf{N} \to \bigcup_{i=1}^{\infty} A_i$ by $\pi(i, j) = \pi_i(j)$. Claim: π is onto. Let $x \in \bigcup_{i=1}^{\infty} A_i$, then there exists an i_0 with $x \in A_{i_0}$. Since π_{i_0} is onto, there exists a $j_0 \in \mathbf{N}$ with $\pi_{i_0}(j_0) = x$. So $\pi(i_0, j_0) = x$, establishing that π is surjective as well. Therefore $\operatorname{card}(\bigcup_{i=1}^{\infty} A_i) \leqslant \operatorname{card}(\mathbf{N} \times \mathbf{N}) = \operatorname{card}(\mathbf{N})$. □

Lemma 1.4.12. $\operatorname{card}([0, 1]) \leqslant \operatorname{card}(2^{\mathbf{N}})$.

Proof. Recall that every $\sigma \in [0, 1]$ has a binary expansion $\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$, where $\sigma_k \in \{0, 1\}$. Consider the map $\varphi : 2^{\mathbf{N}} \to [0, 1]$ defined by $\varphi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{2^k}$. Letting $f(k) = \sigma_k$ gives φ is surjective. □

Lemma 1.4.13. $\operatorname{card}(\mathbf{R}) = \operatorname{card}([0, 1])$.

Proof. By inclusion $[0,1] \overset{\iota}{\hookrightarrow} \mathbf{R}$, which implies that $\mathrm{card}([0,1]) \leqslant \mathrm{card}(\mathbf{R})$. Recall that $\mathbf{R} \xhookrightarrow{\tan} (0,1) \overset{\iota}{\hookrightarrow} [0,1]$, which implies that $\mathrm{card}(\mathbf{R}) \leqslant \mathrm{card}([0,1])$. Then Theorem 1.4.8 gives the desired result. $\square$

Lemma 1.4.14. $\mathrm{card}(2^{\mathbf{N}}) \leqslant \mathrm{card}([0,1])$.

Proof. Consider the map $\lambda : 2^{\mathbf{N}} \to [0,1]$ defined by $\lambda(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$. Claim: λ is injective. Let $f, g \in 2^{\mathbf{N}}$ with $f \neq g$. Let k_0 be the *smallest point k where f and g are different*. So in particular:

$$\begin{aligned} f(1) &= g(1) \\ f(2) &= g(2) \\ &\vdots \\ f(k_0 - 1) &= g(k_0 - 1) \\ f(k_0) &\neq g(k_0). \end{aligned}$$

Let:

$$\begin{aligned} t_1 &= \sum_{k>k_0} \frac{f(k)}{3^k} && \text{sum past } k_0 \\ t_2 &= \sum_{k>k_0} \frac{g(k)}{3^k} && \text{sum past } k_0 \\ s_1 &= \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} && \text{sum before } k_0 \\ s_1 &= \sum_{k=1}^{k_0-1} \frac{g(k)}{3^k} && \text{sum before } k_0 \end{aligned}$$

We have that:

$$\begin{aligned} \lambda(f) &= s_1 + \frac{f(k_0)}{3^{k_0}} + t_1 \\ \lambda(g) &= s_2 + \frac{g(k_0)}{3^{k_0}} + t_2 \end{aligned}$$

Because f and g differ at k_0, without loss of generality let $f(k_0) = 0$ and $g(k_0) = 1$. Then

$\lambda(g) - \lambda(f) = \frac{1}{3^{k_0}} + t_2 - t_1$. Observe that:

$$\begin{aligned}
|t_2 - t_2| &= \left| \sum_{k>k_0} \frac{g(k) - f(k)}{3^k} \right| \\
&\leqslant \sum_{k>k_0} \frac{|g(k) - f(k)|}{3^k} && \text{By triangle inequality} \\
&\leqslant \sum_{k>k_0} \frac{1}{3^k} && \text{By comparison test} \\
&= \frac{1}{3^{k_0+1}} \sum_{k \geqslant 0} \frac{1}{3^k} \\
&= \frac{1}{3^{k_0+1}} \cdot \frac{1}{1 - \frac{1}{3}} \\
&= \frac{3}{2 \cdot 3^{k_0+1}} \\
&= \frac{1}{2 \cdot 3^{k_0}} \\
&< \frac{1}{3^{k_0}}.
\end{aligned}$$

Since $|t_2 - t_2| < \frac{1}{3^{k_0}}$, $\lambda(g) - \lambda(f) \neq 0$, establishing λ as an injection. Thus $\mathrm{card}(2^{\mathbf{N}}) \leqslant \mathrm{card}([0, 1])$. □

Theorem 1.4.15. $\mathrm{card}(2^{\mathbf{N}}) = \mathrm{card}(\mathcal{P}(\mathbf{N})) = \mathrm{card}(\mathbf{R})$.

Proof. This follows from Lemma 1.4.12, Lemma 1.4.13, and Lemma 1.4.14. □

2
Ordered Fields

2.1 Ordering of $\mathbb{Z}$

Definition 2.1.1. Define $\mathbf{Z}^+ = \{n \in \mathbf{Z} \mid n \geqslant_a 0\}$, where $\geqslant_a$ is the *algebraic ordering* from Example 1.2.1. We call $\mathbf{Z}^+$ the cone of positive integers, and they admit the following axioms:

(1) If $m, n \in \mathbf{Z}^+$, then $m + n \in \mathbf{Z}^+$ and $mn \in \mathbf{Z}^+$.

(2) For all $m \in \mathbf{Z}$, $m \in \mathbf{Z}^+$ or $-m \in \mathbf{Z}^+$.

(3) If $m \in \mathbf{Z}^+$ and $-m \in \mathbf{Z}^+$, then $m = 0$.

Proposition 2.1.1 (Properties of $\leqslant_a$)**.**

(1) $m \leqslant_a n$ if and only if $n - m \in \mathbf{Z}^+$.

(2) If $m \leqslant_a n$ and $p \leqslant_a q$, then $m + p \leqslant_a n + q$.

(3) If $m \leqslant_a n$ and $p \in \mathbf{Z}^+$, then $pm \leqslant_a pn$.

(4) If $m \leqslant_a n$ then $-n \leqslant_a -m$.

(5) $(\mathbf{Z}, \leqslant_a)$ forms a total ordering.

(6) If $m >_a 0$ and $mn >_a 0$, then $n >_a 0$.

(7) If $m >_a 0$ and $mn \geqslant_a mp$, then $n \geqslant_a p$.

Proof. (5) Let $m, n \in \mathbf{Z}$, since $\mathbf{Z}$ is closed under subtraction $m - n \in \mathbf{Z}$. So either $m - n \in \mathbf{Z}^+$ or $n - m \in \mathbf{Z}^+$. Then by (1) $n \leqslant_a m$ or $m \leqslant_a n$. Thus $(\mathbf{Z}, \leqslant_a)$ is a total ordering.

(6) We have $mn >_a 0$ with $m >_a 0$. If $n = 0$, we are done. So now assume $n \neq 0$. Then either $n \in \mathbf{Z}^+$ or $-n \in \mathbf{Z}^+$. If $-n \in \mathbf{Z}^+$, then $m(-n) = -(mn) \in \mathbf{Z}^+$. But we had assumed $mn >_a 0$; i.e., $mn \in \mathbf{Z}^+$, hence it must be the case that $mn = 0$, a contradiction. Therefore it must be that $n \in \mathbf{Z}^+$. □

2.2 Ordering of $\mathbb{Q}$

Proposition 2.2.1. *Define $Q := \mathbf{Z} \times \mathbf{N}$. Show that $\sim$ forms an equivalence relation, where $(a, b) \sim (c, d)$ if and only if $ad = bc$.*

Proof. I dont wanna do this □

Definition 2.2.1. The set of equivalence classes of Q is $\mathbf{Q} = Q/\sim = \{[(a, b)] \mid (a, b) \in Q\}$. We call this set the rational numbers, and denote the equivalence classes $[(a, b)]$ as $\frac{a}{b}$.

Proposition 2.2.2. *The operations*

$$+ : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q} \text{ defined by } [(a,b)] + [(c,d)] = [(ad+bc, bd)]$$
$$\cdot : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q} \text{ defined by } [(a,b)] \cdot [(c,d)] = [(ac, bd)]$$

are well-defined. Furthermore, $(\mathbf{Q}, +, \cdot)$ *forms a field.*

Proof. I dont wana □

Lemma 2.2.3. *There is an injective map* $\mathbf{Z} \xhookrightarrow{j} \mathbf{Q}$ *defined by* $j(n) = \frac{n}{1}$ *satisfying the properties*

$$j(n+m) = j(n) + j(m)$$
$$j(nm) = j(n)j(m).$$

Proof. Note that $j(n) = j(m)$ if and only if $\frac{n}{1} + \frac{m}{1}$. By definition this is equivalent to $n = m$, hence j is injective.

Observe that $j(n+m) = \frac{n+m}{1} = \frac{n}{1} + \frac{m}{1} = j(n) + j(m)$ and $j(nm) = \frac{nm}{1} = \frac{n}{1} \cdot \frac{m}{1} = j(n)j(m)$. □

Theorem 2.2.4. $(\mathbf{Q}, \leqslant_Q)$ *is a total ordering, where* $\leqslant_Q$ *is a well-defined ordering defined by* $\frac{a}{b} \leqslant_Q \frac{c}{d}$ *if and only if* $ad \leqslant_a bc$ *in* $(\mathbf{Z}, \leqslant_a)$*. Furthermore, the map* $j : \mathbf{Z} \hookrightarrow \mathbf{Q}$ *is order preserving, that is, if* $n \leqslant_a m$ *in* $(\mathbf{Z}, \leqslant_a)$*, then* $j(n) \leqslant_Q j(m)$ *in* $(\mathbf{Q}, \leqslant_Q)$.

Proof. i REALLY dont □

Definition 2.2.2. Define $\mathbf{Q}_+ := \{q \in \mathbf{Q} \mid q \geqslant_Q 0\}$ as the *cone of positive rationals*, and they admit the following axioms:

(1) If $q_1, q_2 \in \mathbf{Q}^+$, then $q_1 + q_2 \in \mathbf{Z}^+$ and $q_1 q_2 \in \mathbf{Z}^+$.

(2) For all $q \in \mathbf{Q}$, $q \in \mathbf{Q}^+$ or $-q \in \mathbf{Q}^+$.

(3) If $q \in \mathbf{Q}^+$ and $-q \in \mathbf{Q}^+$, then $q = 0$.

(4) $q_1 \leqslant_Q q_2$ if and only if $q_2 - q_1 \in \mathbf{Q}^+$.

Proposition 2.2.5. *Let* $r, s, t, u \in \mathbf{Q}$

(1) If $r \leqslant_Q s$ *and* $t \leqslant_Q u$*, then* $r + t \leqslant_Q s + u$.

(2) If $r \leqslant_Q s$ *and* $t \geqslant_Q 0$*, then* $tr \leqslant_Q ts$.

Proof. do this shi later □

2.3 Rings and Fields

Definition 2.3.1. A *ring* is a non-empty set R equipped with two binary operations:

$$R \times R \xrightarrow{a} R \text{ defined by } a(r,s) = r + s$$
$$R \times R \xrightarrow{m} R \text{ defined by } m(r,s) = rs,$$

such that they admit the following axioms:

(1) R is an *abelian group* under addition:

  (i) $r + (s + t) = (r + s) + t$ for all $r, s, t \in R$,

  (ii) there exists an element $0_R \in R$ with $r + 0_R = r = 0_R = r$ for all $r \in R$,

  (iii) For all $r \in R$ there exists an $s \in R$ such that $r + s = 0_R = s + r$ (such an s is unique, and is denoted $-r$),

  (iv) $r + s = s + r$ for all $r, s \in R$.

(2) $r(st) = (rs)t$ for all $r, s, t \in R$,

(3) $(r + s)t = rt + rs$ and $r(s + t) = rs + rt$ for all $r, s, t \in R$.

If R contains an element 1_R such that $1_R r = r = r 1_R$, then we say R is *unital*. If $rs = sr$ for all $r, s \in R$, then we say R is *commutative*. If R is a unital ring such that $1_R \neq 0_R$ *and* for all $r \in R$ there exists an $s \in R$ such that $rs = 1_R = sr$ (such an s is unique, and denoted r^{-1}), then we say R is a *division ring*.

Definition 2.3.2. A *field* is a commutative division ring.

Example 2.3.1.

(1) $\mathbf{Q}$ is a field.

(2) $\mathbf{Z}/p\mathbf{Z}$ is a field.

(3) $\mathbf{C}_{\mathbf{Q}} = \{r + si \mid r, s \in \mathbf{Q}, i^2 = -1\}$ with addition and multiplication defined by

$$(r + si) + (t + ui) := (r + t) + (s + u)i$$
$$(r + si)(t + ui) := (rt - su) + (ru + st)i$$

is a field. We call this set the *complex rationals*.

Definition 2.3.3. An *ordered field* is a field F equipped with a total ordering $\leqslant_F$ such that:

(1) If $x \leqslant_F y$ and $u \leqslant_F v$, then $x + u \leqslant_F y + v$.

(2) If $x \leqslant_F y$ and $z \geqslant_F 0$, then $xz \leqslant_F zy$.

We similarly define $F^+ = \{x \in F \mid x \geqslant_F 0\}$ as the *cone of positive elements*.

Proposition 2.3.1. *Let $(F, \leqslant_F)$ be an ordered field.*

(1) If $x, y \in F^+$, then $x + y \in F^+$ and $xy \in F^+$.

(2) If $x \in F$, then $-x \in F^+$ or $x \in F^+$.

(3) If $x, -x \in F^+$, then $x = 0$.

Proof. need to do □

Example 2.3.2.

(1) $\mathbf{Q}$ is an ordered field.

(2) Is $\mathbf{C}_{\mathbf{Q}}$ an ordered field?

Proposition 2.3.2. *Let $(F, \leqslant)$ be an ordered field with $1_F \neq 0_F$.*

(1) For all $a \in F$, $a^2 \in F$.

(2) $0, 1 \in F^+$.

(3) If $n \in \mathbf{N}$, then $n \cdot 1_F := \underbrace{1_F + 1_F + \ldots + 1_F}_{n \ \ times}$, implying $n \cdot 1_F \in F^+$.

(4) If $x \in F^+$ and $x \neq 0$, then $x^{-1} \in F^+$.

(5) If $xy \in F^+$ and $xy \neq 0$, then $x, y \in F^+$ or $-x, -y \in F^+$.

(6) If $0 < x \leqslant y$, then $y^{-1} \leqslant x^{-1}$.

(7) If $x \leqslant y$, then $-y \leqslant -x$.

(8) If $x \geqslant 1_F$, then $x^2 \geqslant x$.

(9) If $x \leqslant 1_F$, then $x^2 \leqslant x$.

Proof. (1) If $a \in F^+$, then $a \cdot a = a^2 \in F^+$. If $-a \in F^+$, then $(-a) \cdot (-a) = a^2 \in F^+$.

(2) From part (1) we have that $0 = 0 \cdot 0 \in F^+$. Similarly, $1 = 1 \cdot 1 \in F^+$ and $(-1) \cdot (-1) \in F^+$

(3) Since F^+ is closed under addition, we can inductively show that $n \cdot 1 = 1 + 1 + \ldots + 1 \in F^+$.

(4) Suppose towards contradiction $x^{-1} \notin F^+$. Then $-(x^{-1}) \in F^+$, so $(-(x^{-1})) \cdot x = -1(x^{-1} \cdot x) = -1 \in F^+$. But $-1, 1 \in F^+$ implies $1 = 0$, a contradiction. Thus $x^{-1} \in F^+$.

(6) $y \geqslant x > 0$ implies $x, y \in F^+$. So $x^{-1}, y^{-1} \in F^+$. Then $y^{-1}xx^{-1} \leqslant y^{-1}yx^{-1}$, and simplifying yields $y^{-1} \leqslant x^{-1}$. finish the rest (i'm not going to) □

3
The Real Numbers

3.1 The Completion of $\mathbb{Q}$

Definition 3.1.1. A *Dedekind cut* is a nonempty subset D of $\mathbf{Q}$ with the following properties:

(1) $D \neq \mathbf{Q}$;

(2) If $b \in D$, then $a \in D$ for all $a \in \mathbf{Q}$ with $a < b$;

(3) D does not contain a largest element.

Example 3.1.1. The following examples are Dedekind cuts:

(1) $\{a \in \mathbf{Q} \mid a < 3\}$ (the set of all rational numbers less than 3).

(2) $\{a \in \mathbf{Q} \mid a < 0 \text{ or } a^2 < 2\}$ (the set of all rational numbers less than $\sqrt{2}$).

(3) $\{a \in \mathbf{Q} \mid a < 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \text{ for some } n \in \mathbf{Z}^+\}$ (the set of all rational numbers less than e).

Definition 3.1.2. Let C and D be Dedekind cuts.

will probably not finish this

3.2 Ordering of $\mathbb{R}$

Axiom 1. $\mathbf{R}$ *is an ordered field.*

Proposition 3.2.1. $\mathbf{Q}^+ \subseteq \mathbf{R}^+$.

Proof. If $x \in \mathbf{Q}^+$, then $x = \frac{p}{q}$ with $p \in \mathbf{Z}^+$ and $q \in \mathbf{N}$. Write $p = \underbrace{1 + 1 + \ldots + 1}_{p \text{ times}}$, then $p \in \mathbf{R}^+$. Similarly, write $q = \underbrace{1 + 1 + \ldots + 1}_{q \text{ times}}$. Then $q \in \mathbf{R}^+$, which implies that $q^{-1} \in \mathbf{R}^+$. Hence $\frac{p}{q} \in \mathbf{R}^+$, establishing $\mathbf{Q}^+ \subseteq \mathbf{R}^+$. □

Proposition 3.2.2. *The maps $Z \overset{j}{\hookrightarrow} \mathbf{Q} \overset{i}{\hookrightarrow} \mathbf{R}$ are order embeddings (defined in Lemma 2.2.3 and Theorem 2.2.4).*

Proof. Suppose $i(q_1) \leqslant_Q i(g_2)$. Then $q_1 \leqslant_{\mathbf{R}} q_2$, hence $q_2 - q_1 \in \mathbf{R}^+$. Now If $q_2 - q_2 \in \mathbf{Q}^+$, then $q_2 - q_1 \in \mathbf{R}^+$. Hence $q_1 \leqslant_{\mathbf{R}} q_2$. wtf is this saying? □

Proposition 3.2.3. *Let $a, b \in \mathbf{R}$. If $a \leqslant b$ (or $a < b$), then $a \leqslant \frac{1}{2}(a+b) \leqslant b$ (or $a < \frac{1}{2}(a+b) < b$).*

Proof. By the order axioms, $a \leqslant b$ implies $a + a \leqslant a + b$. So $2a \leqslant a + b$, which is equivalent to $a \leqslant \frac{1}{2}(a + b)$. Similarly, $a + b \leqslant b + b$, which similarly gives $\frac{1}{2}(a + b) \leqslant b$, establishing the proposition. □

Corollary 3.2.4. *Given $b > 0$, we have $0 < \frac{1}{2}b < b$.*

Proof. From Proposition 3.2.3, setting $a = 0$ yields the desired result. □

Proposition 3.2.5. *Suppose $a \in \mathbf{R}$. For all $\epsilon > 0$, if $0 \leqslant a \leqslant \epsilon$, then $a = 0$.*

Proof. If $a = 0$ we are done. If $a > 0$, by Corollary 3.2.4 $0 \leqslant \frac{1}{2}a \leqslant a$. Pick $\epsilon = \frac{1}{2}a$, then $a \leqslant \frac{1}{2}a$, a contradiction. Thus $a = 0$. □

Definition 3.2.1. Let $a_1, a_2, ..., a_n > 0$. The <u>*arithmetic mean*</u> is $\frac{1}{2}\left(\sum_{j=1}^{n} a_j\right)$. The <u>*geometric mean*</u> is $\left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}$.

Proposition 3.2.6 (AM-GM Inequality)**.** *For all $a_1, a_2, ..., a_n \geqslant 0$, then $\left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}} \leqslant \frac{1}{2}\left(\sum_{j=1}^{n} a_j\right)$.*

Proof. We will only prove the $n = 2$ case. Consider the fact that $(a_1 - a_2)^2 \geqslant 0$, and expanding gives $a_1^2 - 2a_1a_2 + a_2^2$. So $2a_1a_2 \leqslant a_1^2 + a_2^2$. Adding $2a_1a_2$ to both sides yields $4a_1a_2 \leqslant a_1^2 + 2a_1a_2 + a_2^2$, which is equivalent to $4a_1a_2 \leqslant (a_1 + a_2)^2$. Then simplifying yields the desired result of $(a_1a_2)^{\frac{1}{2}} \leqslant \frac{1}{2}(a_1 + a_2)$. □

Proposition 3.2.7 (Bernoulli's Inequality)**.** *If $x > -1$, then $(1 + x)^n \geqslant 1 + nx$ for all $n \in \mathbf{N}_0$.*

Proof. We proceed with induction with base case $n = 0$ and $n = 1$; these hold by inspection. Assume the inequality holds true for $n = k$. For $n = k + 1$:

$$\begin{aligned}(1 + x)^{k+1} &= (1 + x)^k(1 + x) \\ &\geqslant (1 + nx)(1 + x)^{1} \\ &= 1 + (n + 1)x + nx^2 \\ &\geqslant 1 + (n + 1)x.\end{aligned}$$

□

Proposition 3.2.8 (Cauchy-Schwartz Inequality)**.** *Let $a_1, ..., a_n, b_1, ..., b_n \in \mathbf{R}^n$. Then:*

$$\left|\sum_{j=1}^{n} a_jb_j\right| \leqslant \left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}.$$

[1]Because order is preserved under multiplication by positive elements.

Proof. Consider the map $F : \mathbf{R}^n \to \mathbf{R}^n$ defined by $F(t) = \sum_{j=1}^n (a_j - b_j t)^2$. Note that $\sum_{j=1}^n (a_j - b_j t)^2 \geqslant 0$. Observe that:

$$\begin{aligned}\sum_{j=1}^n (a_j - b_j t)^2 &= \sum_{j=1}^n (a_j^2 - 2a_j b_j t + b_j^2 t^2) \\ &= \sum_{j=1}^n a_j^2 - \sum_{j=1}^n 2a_j b_j t + \sum_{j=1}^n b_j^2 t^2.\end{aligned}$$

This is a quadratic equation, and since $F(t) \geqslant 0$, the discriminant will be less than or equal to 0. Hence:

$$\Delta = \left(\sum_{j=1}^n 2a_j b_j\right)^2 - 4\left(\sum_{j=1}^n b_j^2\right)\left(\sum_{j=1}^n a_j^2\right) \leqslant 0.$$

Simplifying gives:

$$\left(\sum_{j=1}^n 2a_j b_j\right)^2 \leqslant 4\left(\sum_{j=1}^n b_j^2\right)\left(\sum_{j=1}^n a_j^2\right).$$

Pulling 2 out from the left-hand side, dividing both sides by 4, and then square-rooting gives the desired result. □

Question. When do we have equality?

Answer. When $\Delta = 0$, there exists a $t_0 \in \mathbf{R}$ with $F(t_0) = 0$. So $\sum_{j=1}^n (a_j - b_j t_0) = 0$ implies $a_j - b_j t_0 = 0$ for all j. Hence there is equality only when $a_j = b_j t_0$ for all j.

Proposition 3.2.9 (Triangle Inequality)**.** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{R}^n$. Then:*

$$\left(\sum_{j=1}^n (a_j + b_j)^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{j=1}^n a_j^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^n b_j^2\right)^{\frac{1}{2}}.$$

Proof. Observe that:

$$\begin{aligned}\sum_{j=1}^n (a_j + b_j)^2 &= \sum_{j=1}^n (a_j^2 + 2a_j b_j + b_j^2) \\ &= \sum_{j=1}^n a_j^2 + \sum_{j=1}^n 2a_j b_j + \sum_{j=1}^n b_j^2 \\ &\leqslant \sum_{j=1}^n a_j^2 + 2\left(\sum_{j=1}^n a_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^n b_j^2\right)^{\frac{1}{2}} + \sum_{j=1}^n b_j^2. \\ &= \left(\left(\sum_{j=1}^n a_j^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^n b_j^2\right)^{\frac{1}{2}}\right)^2.\end{aligned}$$

Squaring both sides gives the desired result. □

3.3 Metrics and Norms on $\mathbb{R}^n$

Definition 3.3.1. The <u>*absolute value*</u> is a function $|\cdot| : \mathbf{R} \to \mathbf{R}$ defined by:

$$|x| = \begin{cases} x, & x \in \mathbf{R}^+ \\ -x, & -x \in \mathbf{R}^+. \end{cases}$$

Proposition 3.3.1. *Let $a, b \in \mathbf{R}$ and $\delta > 0$.*

(1) $|ab| = |a||b|$.

(2) $|a|^2 = |a^2|$.

(3) $|-a| = |a|$.

(4) $|a| \in \mathbf{R}+$.

(5) $-|a| \leqslant a \leqslant |a|$.

(6) $|a| \leqslant \delta$ *if and only if* $-\delta \leqslant a \leqslant \delta$.

(7) $|a+b| \leqslant |a| + |b|$.

(8) $|a-b| \leqslant |a| + |b|$.

(9) $||a| - |b|| \leqslant |a-b|$.

Proof. do later □

Lemma 3.3.2. $\pm x \leqslant \delta$ *if and only if* $|x| \leqslant \delta$.

Proof. do lter □

Lemma 3.3.3. $A \subseteq \mathbf{R}$ *is bounded if and only if there exists an* $r > 0$ *such that* $|a| < r$ *for all* $a \in A$.

Proof. Suppose $A \subseteq \mathbf{R}$ is bounded. Then there exists an $l, u \in \mathbf{R}$ with $l \leqslant a \leqslant u$ for all $a \in A$. We have that:

$$-|l| \leqslant l \leqslant a \leqslant u \leqslant |u|.$$

Let $r = \max\{|l|, |u|\} \geqslant 0$. So $-r \leqslant |l| \leqslant a \leqslant |u| \leqslant r$. Thus $|a| \leqslant r$.

Conversely, suppose there exists an $r > 0$ with $|a| \leqslant r$ for all $a \in A$. Then $-r \leqslant a \leqslant r$ for all $a \in A$, hence A is bounded. □

Definition 3.3.2. A function $f : D \to \mathbf{R}$ is <u>*bounded*</u> if $\operatorname{im}(f) \subseteq \mathbf{R}$ is a bounded subset. Equivalently, there exists a $c > 0$ such that $|f(x)| < c$ for all $x \in D$.

Example 3.3.1. Consider the function $f : [3,7] \to \mathbf{R}$ defined by $f(x) = \frac{x^2+2x+1}{x-1}$. Since $3 \leqslant x \leqslant 7$, observe that:

$$\begin{aligned} |x^2 + 2x + 1| &\leqslant |x^2| + |2x| + 1 \\ &= |x|^2 + 2|x| + 1 \quad \text{Evaluate at 7} \\ &= 64 \end{aligned}$$

Likewise, $3 \leqslant x \leqslant 7$ implies $|x-1| \geqslant 2$, hence $\frac{1}{|x-1|} \leqslant \frac{1}{2}$. Together, we have that:

$$\left|\frac{x^2+2x+1}{x-1}\right| \leqslant \frac{64}{2} = 32.$$

Definition 3.3.3. Let $s, t \in \mathbf{R}$. We define the <u>*distance*</u> between s and t as $d(s,t) = |s-t|$.

Definition 3.3.4. Let X be a nonempty set equipped with a map $d : X \times X \to \mathbf{R}^+$. We say (X, d) is a <u>*semi-metric*</u> if for all $x, y, z \in X$,

(1) $d(x,y) = d(y,x)$,

(2) $d(x,z) \leqslant d(x,y) + d(y,z)$, and

(3) $d(x,x) = 0$.

We say (X,d) is a <u>*metric space*</u> if it satisfies the additional axiom:

(4) $d(x,y) = 0$ implies $x = y$.

Proposition 3.3.4.

(1) $(\mathbf{R}, d_1(s,t) = |s-t|)$ *is a metric space.*

(2) $\left(\mathbf{R}^n, d_1(\vec{x},\vec{y}) = \sum_{j=1}^n |y_j - x_j|\right)$ *is a metric space.*

(3) $\left(\mathbf{R}^n, d_\infty(\vec{x},\vec{y}) = \max_{j=1}^n \left\{|y_j - x_j\}|\right)\right)$ *is a metric space.*

(4) $\left(\mathbf{R}^n, d_2(\vec{x},\vec{y}) = \left(\sum_{j=1}^n |y_j - x_j|^2\right)^{\frac{1}{2}}\right)$ *is a metric space.*

(5) $\left(\mathbf{R}^n, d_p(\vec{x},\vec{y}) = \left(\sum_{j=1}^n |y_j - x_j|^p\right)^{\frac{1}{p}}\right)$ *for some* $p \in \mathbf{Q}$ *is a metric space.*

Proof. (1) We have $d(s,t) = |s-t| = |t-s| = d(t,s)$. Similarly, $d(s,r) = |s-r| = |s-t+t-r| \leqslant |s-t| + |t-r| = d(s,t) + d(t,r)$. Clearly $d(s,s) = |s-s| = 0$. Lastly, if $d(s,t) = 0$, then $|s-t| = 0$, which is equivalent to $s - t = 0$; i.e., $s = t$. Thus $(\mathbf{R}, d_1)$ is a metric space.

(4) Axioms 2 and 3 of metric spaces are clearly satisfied. If $d_2(\vec{x},\vec{y}) = 0$ then $|y_j - x_j|^2 = 0$ for all j. Hence $y_j - x_j = 0$; i.e., $y_j = x_j$ for all j, establishing axiom 4. Observe that:

$$\begin{aligned}
d_2(\vec{x},\vec{z}) &= \left(\sum_{j=1}^{n} |z_j - x_j|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j=1}^{n} |z_j - y_j + y_j - x_j|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j=1}^{n} (z_j - y_j + y_j - x_j)^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{j=1}^{n} (z_j - y_j)^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} (y_j - x_j)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j=1}^{n} |z_j - y_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} |y_j - x_j|^2\right)^{\frac{1}{2}} \\
&= d_2(\vec{x},\vec{y}) + d_2(\vec{y},\vec{z}).
\end{aligned}$$

Thus $(\mathbf{R}^n, d_2)$ is a metric space. □

Definition 3.3.5. Let (X,d) be a metric space.

(1) The <u>*open ball*</u> centered at x_0 with radius $\delta > 0$ is $U(x_0,\delta) = \{y \in X \mid d(y,x_0) < \delta\}$.

(2) The <u>*closed ball*</u> centered at x_0 with radius $\delta > 0$ is $B(x_0,\delta) = \{y \in X \mid d(y,x_0) \leqslant \delta\}$.

(3) A subset $A \subseteq X$ is called <u>*open*</u> if for all $a \in A$, there exists a $\delta > 0$ such that $U(a,\delta) \subseteq A$.

(4) A subset $C \subseteq X$ is called <u>*closed*</u> if $\operatorname{compl}(C) = X \setminus C$ is open.

Example 3.3.2. Consider $X = \mathbf{R}$ and $d(s,t) = |s-t|$. Observe that:

$$\begin{aligned}
U(t,\delta) &= \{s \in \mathbf{R} \mid d(s,t) < \delta\} \\
&= \{s \in \mathbf{R} \mid |s-t| < \delta\} \\
&= \{s \in \mathbf{R} \mid -\delta < s - t < \delta\} \\
&= \{s \in \mathbf{R} \mid -\delta + t < s < \delta + t\} \\
&= (t-\delta, t+\delta).
\end{aligned}$$

It follows similarly that $B(t,\delta) = [t-\delta, t+\delta]$.

Proposition 3.3.5. *If I is an open interval, then I is open.*

Proof. Let $I = (a,b)$. Let $x \in I$. Let $\delta_x = \min\{x-a, b-x\} > 0$. Now let $t \in V_\delta(x)$. Then $t \in (x-\delta, x+\delta)$. Case 1: $\min\{x-a, b-x\} = x - a$. Then $x - (x-a) < t < x + x - a$, idk how to do this □

4

Supremum, Infimum, and Completeness

4.1 Supremum and Infimum

Theorem 4.1.1. *Let $\emptyset \neq A \subseteq \mathbf{R}$. Let u be an upperbound for A. The following are equivalent:*

(1) $u = \sup(A)$.

(2) If $t < u$, then there exists an $a_t \in A$ with $t < a_t$.

(3) For all $\epsilon > 0$, there exists an $a_\epsilon \in A$ such that $u - \epsilon < a_\epsilon$.

Proof. $[(1) \implies (2)]$ Assume $u = \sup(A)$. Let $t < u$. Suppose towards contradiction there does not exist and $a \in A$ with $a > t$. Then $a \leqslant t$ for all $a \in A$. But this implies t is an upperbound of A less than u, which is a contradiction because u is the least upper bound. $[(2) \implies (3)]$ Given $\epsilon > 0$, let $t = u - \epsilon$. Then applying (2) gives the desired result. $[(3) \implies (1)]$ We know u is an upperbound of A, we aim to show that it is the least upperbound. Let v be an upperbound for A with $v < u$. Pick $\epsilon = u - v > 0$. By (3), there exists an $a_\epsilon \in A$ such that $u - (u - v) < a_\epsilon$. So $v < a_\epsilon$, which is a contradiction (v is an upperbound, how can it be smaller than an element of A?). □

Example 4.1.1. Claim: $\sup([0,1)) = 1$. If $s \in [0,1)$, by definition $s < 1$, so 1 is an upper bound for $[0,1)$. Given $t < 1$, set $\delta = 1 - t > 0$. Then $0 < \frac{\delta}{2} < \delta$ this is not trivial, have to show $\delta - \delta/2$ is positive. This gives:

$$t < t + \frac{\delta}{2} < t + \delta = 1.$$

Pick $a_t = t + \frac{\delta}{2}$. By (2) of Theorem 4.1.1, $a_t \in [0,1)$, hence $1 = \sup([0,1))$.

Proposition 4.1.2. *Let $A, B \subseteq \mathbf{R}$ and $a \leqslant b$ for all $a \in A$ and $b \in B$. Then* $\sup(A) \leqslant \inf(B)$.

Proof. Fix a point $b_0 \in B$. Then $a \leqslant b_0$ for all $a \in A$. Then b_0 is an upperbound for A. This gives $u := \sup(A) \leqslant b_0$. But since b_0 was arbitrary, we have $u \leqslant b$ for all $b \in B$. So u is a lower bound for B, therefore $u \leqslant \inf(B)$. □

Axiom 2 (Completeness of **R**). *Given any nonempty subset $A \subseteq \mathbf{R}$ which is bounded above,* $\sup(A)$ *exists.*

Lemma 4.1.3. *For $A \subseteq \mathbf{R}$ which is bounded below,* $\sup(-A) = -\inf(A)$.

Proof. If A is bounded below, then $-A$ is bounded above. Then $\sup(-A)$ exists, define it as u. So for all $a \in A$, $-a \leqslant u$. Hence $-u$ is a lower bound for A. Suppose v is another lower bound for A. Then $v \leqslant a$ for all $a \in A$. So $-v \geqslant -a$ for all $a \in A$. Thus $-v$ is an upper bound of $-A$. Therefore, since u is the least upper bound, $-v \geqslant u$; i.e., $-u \geqslant v$. Thus $-u = \inf(A)$. □

Axiom 3 (Well-Ordering Princple)**.** *Every nonempty subset $A \subseteq \mathbf{N}$ contains a least element.*

Proposition 4.1.4 (Arcimedean Property 1)**.** *If $x \in \mathbf{R}$, then there exists $n_x \in \mathbf{N}$ with $x < n_x$.*

Proof. Suppose not. That is, suppose $n \leqslant x$ for all $n \in \mathbf{N}$. Then x is an upper bound for $\mathbf{N}$. Thus $\sup(A) := u$ exists. From part (3) of Theorem 4.1.1, take $\epsilon = 1$. Then there exists an $n \in \mathbf{N}$ such that $u - 1 < n$. So $u < n + 1 \in \mathbf{N}$, which is a contradiction. □

Proposition 4.1.5 (Archimedean Property 2)**.** *If $t > 0$, there exists $n_t \in \mathbf{N}$ with $\frac{1}{n_t} < t$.*

Proof. From Arcimedean Property 1, pick $x = \frac{1}{t}$. □

Corollary 4.1.6. *Given $t > 0$, there exists $m \in \mathbf{N}$ with $\frac{1}{2^m} < t$.*

Proof. By Archimedean Property 2 there exists an $n \in \mathbf{N}$ with $\frac{1}{n} < t$. Claim: $\frac{1}{2^n} < \frac{1}{n}$. It suffices to show that $2^n > n$. Proposition 1.4.6 gives $\operatorname{card}(\{1, 2, \ldots, n\}) < \operatorname{card}(\mathcal{P}(\{1, 2, \ldots, n\}))$. Then Exercise 1.4.2 gives:

$$n = \operatorname{card}(\{1, 2, \ldots, n\}) < \operatorname{card}(\mathcal{P}(\{1, 2, \ldots, n\})) = 2^n.$$

Alternatively, Bernoulli's Inequality gives $(1 + 1)^n \geqslant 1 + n$. Hence $2^n > n$. □

Example 4.1.2.

(1) Claim: $\inf\left\{\frac{1}{n} \mid n \in N\right\} = 0$. Note that 0 is indeed a lower bound because $0 < \frac{1}{n}$ for all $n \in \mathbf{N}$. Suppose t is another lower bound. If $t \leqslant 0$, then we are done. If $t > 0$, by the Archimedean Property there exists an $n_t \in \mathbf{N}$ such that $\frac{1}{n_t} < t$, which is a contradiction (because we asserted that t is a lower bound, and $\frac{1}{n_t} \in \inf\{\frac{1}{n} \mid n \in N\}$). Thus $\inf\left\{\frac{1}{n} \mid n \in N\right\} = 0$.

(2) Claim: $\inf\left\{\frac{1}{2^m} \mid m \in N\right\} = 0$. This follows from the above example and previous corollary.

Corollary 4.1.7. *Let $x \in \mathbf{R}$, Then there exists $n_x \in \mathbf{Z}$ with $n_x - 1 \leqslant x < n_x$.*

Proof. Case 1: $x \geqslant 0$. Let $S_x = \{n \in \mathbf{N} \mid x < n\}$. By Arcimedean Property 1 $S_x \neq 0$. By the Well-Ordering Princple, there exists a least element in this set, call it n_x. Since $n_x \in S_x$, it must be the case that $x < n_x$. But since n_x is the least element, $n_x - 1 \notin S_x$. Since S_x is the set of all natural numbers with lower bound x, $n_x - 1$ is not bounded below by x. Whence $n_x - 1 \leqslant x$.

Case 2: $x < 0$. Define $S_{-x} = \{n \in \mathbf{N} \mid n < -x\}$. As a consequence of the Well-Ordering Princple, any subset of the integers which is bounded above admits a greatest element, define it to be $n_{-x} \in \mathbf{Z}$. Then $n_{-x} + 1 \notin S_{-x}$, hence $n_{-x} < -x \leqslant n_{-x} + 1$. This establishes $-n_{-x} - 1 \leqslant x < -n_{-x}$. □

Definition 4.1.1. Let I be an open interval. A subset $D \subseteq \mathbf{R}$ is dense if $I \cap D \neq \emptyset$.

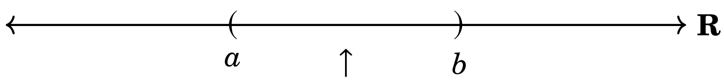
Theorem 4.1.8. $\mathbf{Q} \subseteq \mathbf{R}$ *is dense.*

Proof. Let I be an open interval. Then there exists $a, b \in \mathbf{R}$ with $(a, b) \subseteq I$. We have that $b - a > 0$. By Archimedean Property 2 there exists $n \in \mathbf{N}$ with $\frac{1}{n} < b - a$. So $1 + na < nb$. By Corollary 4.1.7, there exists $m \in \mathbf{Z}$ with $m - 1 \leqslant na < m$. Equivalently, we have that $a < \frac{m}{n}$. We also have that $m \leqslant na + 1 < nb$, which yields $\frac{m}{n} < b$. Thus $\frac{m}{n} \in (a, b) \cap \mathbf{Q}$. □

Corollary 4.1.9. $\mathbf{R} \setminus \mathbf{Q} \subseteq \mathbf{R}$ *is dense.*

Proof. Let $a < b$. Consider $a' = a\sqrt{2}$ and $b' = b\sqrt{2}$. Then $a' < b'$. By Theorem 4.1.8, there exists a $q \in \mathbf{Q}$ with $a' < q < b'$. Thus $a < \frac{q}{\sqrt{2}} < b$. Since $\frac{q}{\sqrt{2}} \notin \mathbf{Q}$, the corollary is established.

Alternatively, observe the following picture:



If there is not an irrational number between (a, b), then $(a, b) \subseteq \mathbf{Q}$, which is a contradiction. □

Theorem 4.1.10. *There exists a unique positive number x with $x^2 = 2$.*

Proof. Consider the set $S = \{t \in \mathbf{R} \mid t > 0, t^2 < 2\}$. Note that $S \neq 0$ because $1 \in S$. If $t \geqslant 2$, then $t^2 \geqslant 2t > 4$, meaning it would not be an element of S. So S is bounded above by 2. Hence there exists $u := \sup(S)$.

Scratchwork: Assume $u^2 < 2$. Find a sufficiently small n so that $(u + \frac{1}{n})^2 \in S$; i.e., $(u + \frac{1}{n})^2 < 2$. Solving for n yields:

$$u^2 + \frac{2u}{n} + \frac{1}{n^2} < 2$$

$$\Longleftrightarrow$$

$$\frac{2u}{n} + \frac{1}{n^2} < 2 - u^2$$

$$\Longleftrightarrow$$

$$\frac{1}{n}\left(2u + \frac{1}{n}\right) < 2 - u^2$$

$$\Longleftrightarrow$$

$$\frac{1}{n}(2u + 1) < 2 - u^2$$

$$\Longleftrightarrow$$

$$\frac{1}{n} < \frac{2 - u^2}{2u + 1} \in \mathbf{R}^+ \setminus \{0\}$$

If $u^2 < 2$, then $\frac{2-u^2}{2u+1} > 0$. By Archimedean Property 2, there exists an $n \in \mathbf{N}$ with $\frac{1}{n} < \frac{2-u^2}{2u+1}$. Simplifying yields $(u + \frac{1}{n})^2 < 2$, or equivalently $u + \frac{1}{n} \in S$, which is a contradiction. It must be the case that $u^2 \geqslant 2$; i.e., $u^2 - 2 \geqslant 0$. Now since $u = \sup(S)$, for all $m \in \mathbf{N}$, there exists $t_m \in S$ with $u - \frac{1}{m} < t_m$. We have that $(u - \frac{1}{m})^2 < t_m^2 < 2$. This simplifies to $u^2 - 2 < \frac{2u}{m} - \frac{1}{m^2} < \frac{2u}{m}$, or equivalently $\frac{u^2-2}{2u} < \frac{1}{m}$. But if $\frac{u^2-2}{2u} < \frac{1}{m}$ for all $m \in \mathbf{N}$, it must be that $\frac{u^2-2}{2u} = 0$, hence $u^2 = 2$.

Lastly we show that u^2 is unique. Suppose $u^2 = 2 = v^2$. Since $u, v \geqslant 0$, $(u^2 - v^2) = 0$. Then $(u-v)(u+v) = 0$. If $u + v = 0$, then $u = 0$ and $v = 0$, which is a contradiction. So $u - v = 0$ implies $u = v$. □

Remark. Picking 2 was completely arbitrary, we could have showed $x^2 = a$ for any $a \geqslant 0$.

Remark. Using the same argument, we have that for all $a > 0$, there exists a unique $b > 0$ with $b^2 = a$. So we have a map:

$$\mathbf{R}^+ \xrightarrow{\sqrt{}} \mathbf{R}^+,$$

where $\sqrt{x}$ is the unique positive number with $(\sqrt{x})^2 = x$.

Remark. We could have similarly defined S as:

$$S' = \{t \in \mathbf{Q} \mid t > 0, t^2 < 2\},$$

and the proof would not have changed. However, $\sup(S') = \sqrt{2} \notin \mathbf{Q}$, meaning $\mathbf{Q}$ is *not* complete.

4.2 Nested Intervals

Axiom 4. *Given any interval I, if $x, y \in I$ with $x < y$, then $[x, y] \in I$.*

Theorem 4.2.1. *Let $S \subseteq \mathbf{R}$ be any subset containing at least two points. If S satisfies Axiom 4, then S is an interval.*

Proof. We proceed with cases. Case 1: S is bounded. Write $a = \inf(S)$ and $b = \sup(S)$. Therefore $S \subseteq [a, b]$. If we show $(a, b) \subseteq S$, then it follows that $S = (a, b]$, or $[a, b)$, or (a, b) or $[a, b]$. We must use that S satisfies Axiom 4 and $a = \inf(S)$ and $b = \sup(S)$. Let $x \in (a, b)$. Since $x > a$, there exists and $s_1 \in S$ with $s_1 < x$. Since $x < b$, there exists an $s_2 \in S$ with $x < s_2$. Thus $s_1, s_2 \in S$ and $s_1 < s_2$. By Axiom 4 $[s_1, s_2] \subseteq S$. But $x \in [s_1, s_2]$ implies $x \in S$. Thus $(a, b) \subseteq S$.

Case 2: S is bounded above do this.

Case 3: S is bounded below need to do. □

Definition 4.2.1. A sequence of intervals $(I_n)_{n \geqslant 1}$ is said to be <u>nested</u> if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$.

Proposition 4.2.2. $\bigcap_{n \geqslant 1} \left[0, \frac{1}{n}\right) = \{0\}$.

Proof. Note that $0 \in \left[0, \frac{1}{n}\right)$ for all $n \geqslant 1$. So $0 \in \bigcap_{n \geqslant 1} \left[0, \frac{1}{n}\right)$. Let $a \in \bigcap_{n \geqslant 1} \left[0, \frac{1}{n}\right)$. Then $0 \leqslant a < \frac{1}{n}$ for all $n \geqslant 1$. Hence $a = 0$. □

Proposition 4.2.3. $\bigcap_{n \geqslant 1} [n, \infty) = \emptyset$.

Proof. Suppose towards contradiction there exists a $t \in \bigcap_{n \geqslant 1} [n, \infty) = \emptyset$. Then $t \in [n, \infty)$ for all $n \geqslant 1$. So $t \geqslant n$ for all $n \geqslant 1$. Hence $\mathbf{N}$ is bounded above, which is a contradiction. □

Theorem 4.2.4 (Nested Intervals)**.** *Let $(I_n)_{n \geqslant 1}$ be a sequence of closed and bounded nested intervals. Then $\bigcap_{n \geqslant 1} I_n \neq \emptyset$. Furthermore, if $\inf\{\text{length}(I_n) \mid n \geqslant 1\} = 0$, then $\bigcap_{n \geqslant 1} I_n = \{\xi\}$.*

Proof. Let $I_n = [a_n, b_n]$. Note that:

$$a_1 \leqslant a_2 \leqslant a_3 \leqslant \dots$$
$$b_1 \geqslant b_2 \geqslant b_3 \geqslant \dots$$

We have that $a_1 \leqslant a_n \leqslant b_n \leqslant b_1$ for all $n \geqslant 1$. So the set $\{a_n \mid n \geqslant 1\}$ is bounded above, and similarly $\{b_n \mid n \geqslant 1\}$ is bounded below. Let

$$\xi = \sup_{n\geqslant 1} \{a_n\}$$
$$\eta = \inf_{n\geqslant 1} \{b_n\}.$$

Claim: $\xi \leqslant b_n$ for all $n \geqslant 1$. Assume towards contradiction $\xi > b_m$ for some $m \geqslant 1$. Since $\xi = \sup_{n\geqslant 1}\{a_n\}$, there exists an a_k with $b_m < a_k \leqslant \xi$. If $k \geqslant m$, then $b_m < a_k \leqslant b_k \leqslant b_m$, which is a contradiction. If $k < m$, then $a_k \leqslant a_m \leqslant b_m < a_k$, which is a contradiction.

Claim: $a_n \leqslant \xi$ for all $n \geqslant 1$. Then $\xi \leqslant \eta$ since $\sup_{n\geqslant 1}\{a_n\} = \xi$. We have $[\xi, \eta] \subseteq [a_n, b_n]$ for all $n \in \mathbf{N}$. Let $x \in [\xi, \eta]$. Then:

$$a_n \leqslant \xi \leqslant x \leqslant \eta \leqslant b_n,$$

hence $x \in [a_n, b_n]$; i.e., $[\xi, \eta] \subseteq [a_n, b_n]$ for all $n \geqslant 1$. Thus $[[\xi, \eta] \subseteq \bigcap_{n\geqslant 1}[a_n, b_n]]$. Conversely, let $t \in [a_n, b_n]$ for all $n \geqslant 1$. Then $a_n \leqslant t \leqslant b_n$. This implies t is both an upper bound for $\{a_n\}_{n\geqslant 1}$ and a lower bound for $\{a_b\}_{n\geqslant 1}$. Hence $\xi \leqslant t \leqslant eta$, implying $t \in [\xi, \eta]$. This establishes $[\xi, \eta] = \bigcap_{n\geqslant 1}[a_n, b_n]$.

Now suppose $\inf\{\text{length}(I_n) \mid n \geqslant 1\} = 0$. Then:

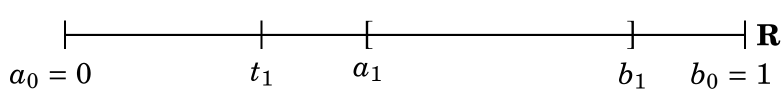
$$\begin{aligned} 0 &= \inf_{n\geqslant 1}(b_n - a_n) \\ &= \inf_{n\geqslant 1} b_n - \inf_{n\geqslant 1} a_n \\ &= \eta - \xi. \end{aligned}$$

Hence $\xi = \eta$, which establishes the theorem.

Alternatively, had we assumed $\xi \neq \eta$, then $\eta - \xi > 0$. So there exists an m such that $b_m - a_m < \eta - \xi$, which is a contradiction since $[\xi, \eta] \subseteq [a_m, b_m]$. □

Corollary 4.2.5. $[0, 1]$ *is uncountable.*

Proof. By way of contradiction, suppose $[0, 1] = \{t_1, t_2, t_3, \dots\}$. Consider the following picture:



Find $[a_1, b_1] \subseteq [0, 1]$ with $t_1 \notin [a_1, b_1]$. Find $[a_2, b_2] \subseteq [a_1, b_1]$ with $t_2 \notin [a_2, b_2]$. Inductively, find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $t_n \notin [a_n, b_n]$. Thus $[a_n, b_n]$ is nested. Now let $\xi \in \bigcap_{n\geqslant 1}[a_n, b_n]$. Then $\xi \in [0, 1]$. But $\xi \neq t_n$ for all n, which is a contradiction. □

5 Sequences

5.1 Basic Definitions and Examples

Definition 5.1.1. A <u>*sequence*</u> in a metric space X is a map $x : \mathbf{N} \to X$. We often write $x = (x_n)_n = (x_1, x_2, \dots)$, where $x_n = x(n)$. If $X = \mathbf{R}$, we call x a <u>*real sequence*</u>.

Example 5.1.1.

(1) Sequences defined explicitly:

(i) Constant sequences: $x_n = t$, $(x_n)_n = (t, t, t, \dots)$

(ii) Sequences defined by a function: $d_n = \left(1 + \frac{1}{n}\right)^n$.

(iii) Geometric sequences: fix $b \in \mathbf{R}$, then $(b^n)_n = (1, b, b^2, \dots)$.

(2) Sequences defined recursively:

(i) Let $a_1 = 1$, $a_{n+1} = 2a_n + 1$. Then $(a_n)_n = (1, 3, 7, 15, \dots)$.

(ii) Let $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$. Then $(f_n)_n = (1, 1, 2, 3, 5, 8, \dots)$. This is the *Fibonacci sequence*.

(iii) Let X be a metric space and $f : X \to X$ be an endomorphism. Fix $x_0 \in X$. Then define:

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) \\ &\vdots \\ x_n &= f(x_{n-1}). \end{aligned}$$

(3) New sequences from old:

(i) Let $(a_n)_n$ and $(b_n)_n$ be sequences. Define:

$$\begin{aligned} (a_n)_n + (b_n)_n &= (a_n + b_n)_n \\ t(a_n)_n &= (ta_n)_n \\ (a_n)_n \cdot (a_n)_n &= (a_n b_n)_n \\ \frac{(a_n)_n}{(b_n)_n} &= \left(\frac{a_n}{b_n}\right)_n, \ (b_n)_n \neq 0 \ \text{ for all } n \ . \end{aligned}$$

(ii) Given $(x_n)_n$ and $k \in \mathbf{N}$, consider $(x_{n+k})_n = (x_k, x_{k+1}, \dots)$. This is called a *shift* or the k^{th} *tail* of $(x_n)_n$.

(iii) If $(a_n)_n$ is a sequence, $a_n \neq 0$ for all n, consider:

$$r_n = \frac{a_{n+1}}{a_n}.$$

So $(r_n)_n = \left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \ldots\right)$. These are called *sequences of ratios*.

(iv) Given a real sequence $(x_k)_k$, consider the sequence $(s_n)_n$ where:

$$s_n = \sum_{k=1}^{n} x_k = s_{n-1} + x_k.$$

We call these n^{th} *partial sums*. An example of these are geometric sequences and telescoping sequences.

Example 5.1.2. Let F be a field. The set $F^{\mathbf{N}} = \{x \mid x : \mathbf{N} \to F\}$ is the set of all F-sequences. This forms an F-vector space under componentwise addition and scalar multiplication.

Definition 5.1.2. Let $(x_n)_n$ be a sequence.

(1) x_n is increasing if $x_1 \leqslant x_2 \leqslant x_3 \leqslant \ldots$

(2) x_n is decreasing if $x_1 \geqslant x_2 \geqslant x_3 \geqslant \ldots$

(3) x_n is strictly increasing if $x_1 < x_2 < x_3 < \ldots$

(4) x_n is strictly decreasing if $x_1 > x_2 > x_3 > \ldots$

Definition 5.1.3. A sequence is said to eventually have a certain property if it does not have the said property across all its ordered instances, but will after some instances have passed.

Definition 5.1.4. A sequence $(x_n)_n$ is monotone if it is either increasing, decreasing, strictly increasing, or strictly decreasing.

5.2 Convergence

Definition 5.2.1. Let $(x_n)_n$ be a sequence in a metric space X.

(1) $(x_n)_n$ converges to $x \in X$ if:

$$(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geqslant N_\epsilon \implies d(x_n, x) < \epsilon)).$$

We denote this as $(x_n)_n \to x$ or $\lim_{n\to\infty} x_n = x$.

(2) $(x_n)_n$ does not exist if:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(n \geqslant N \wedge d(x_n, n) \geqslant \epsilon_0).$$

We abbreviate this as D.N.E.

(3) $(x_n)_n$ <u>*diverges properly*</u> to $+\infty$ if:

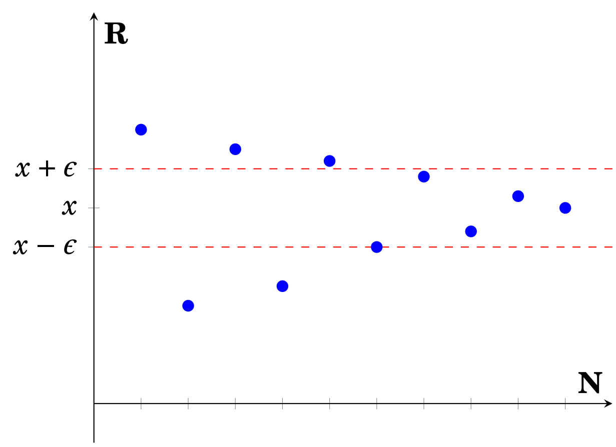
$$(\forall M > 0)(\exists N_M \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geqslant N_M \implies x_n \geqslant M).$$

We write $(x_n)_n \to +\infty$.

(4) $(x_n)_n$ <u>*diverges properly*</u> to $-\infty$ if:

$$(\forall M < 0)(\exists N_M \in \mathbf{N}) \ni (\forall n \geqslant N_M)(x_n \leqslant M).$$

Example 5.2.1. Let $(x_n)_n$ be a real sequence. Then $d(x_n, x) < \epsilon \iff |x_n - x| < \epsilon \iff x_n \in V_\epsilon(x)$. We can visually represent a sequence as follows:



If a sequence is convergent it will eventually be contained between the two dashed lines.

Example 5.2.2.

(1) Prove $\left(\frac{1}{n}\right)_n \to 0$.

Solution. Let $\epsilon > 0$. Find $N_\epsilon \in \mathbf{N}$ so that $\frac{1}{N_\epsilon} < \epsilon$. If $n \geqslant N_\epsilon$, then $\frac{1}{n} \leqslant \frac{1}{N_\epsilon} < \epsilon$. Hence $\frac{1}{n} = \left|\frac{1}{n} - 0\right| < \epsilon$.

(2) Prove $\left(\frac{5n-1}{3-n}\right)_{n=4}^{\infty} \to -5$.

Solution. Note that:

$$|x_n - x| = \left|\frac{5n-1}{3-n} + 5\right| = \frac{14}{|3-n|} = \frac{14}{n-3}.$$

Let $\epsilon > 0$. Find $N_\epsilon \in \mathbf{N}$ such that $N_\epsilon > \frac{14}{\epsilon} = 3$. If $n \geqslant N_\epsilon$, then $n > \frac{14}{\epsilon} + 3$ gives:

$$n - 3 > \frac{14}{\epsilon} \implies \frac{14}{n-3} < \epsilon \implies |x_n - x| < \epsilon.$$

Proposition 5.2.1. *Let (X, d) be a metric space. Then $(x_n)_n \to x$ if and only if $(d(x_n, x))_n \to 0$.*

Proof. Suppose $(x_n)_n \to x$. Given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $d(x_n, x) < \epsilon$. This is equivalent to $|d(x_n, x) - 0| < \epsilon$. The converse follows identically. $\square$

Theorem 5.2.2. *Let $(\epsilon_n)_n \to 0$ and $(x_n)_n$ be real sequences and $x \in \mathbf{R}$. If for some $c > 0$ and $N \in \mathbf{N}$ we have:*

$$|x_n - x| \leqslant c|\epsilon_n| \quad \text{for all } n \in \mathbf{N} \text{ such that } n \geqslant N,$$

then $(x_n)_n \to x$.

Proof. Let $\epsilon > 0$ be given. Since $(\epsilon_n)_n \to 0$ it follows there exists a natural number K such that if $n \geqslant K$ then

$$|a_n| = |a_n - 0| < \frac{\epsilon}{c}.$$

If both $n \geqslant K$ and $n \geqslant N$, then

$$|x_n - x| \leqslant c|\epsilon_n| < \epsilon.$$

Thus $(x_n)_n \to x$. □

Example 5.2.3.

(1) Prove $\left(\frac{\sin(n^2-1)}{n^2+3}\right)_n \to 0.$

Solution. Note that:

$$\left|\frac{\sin(n^2-1)}{n^2+3} - 0\right| = \frac{|\sin(n^2-1)|}{n^2+3} \leqslant \frac{1}{n^2+3} \leqslant \frac{1}{n^2} \leqslant \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $\left(\frac{\sin(n^2-1)}{n^2+3}\right)_n \to 0.$

(2) Prove $\left(\frac{1}{2^n}\right)_n \to 0.$

Solution. Note that:

$$\left|\frac{1}{2^n} - 0\right| \leqslant \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $\left(\frac{1}{2^n}\right)_n \to 0.$

(3) Prove $\left(\frac{1}{n} - \frac{1}{n+1}\right)_n \to 0.$

Solution. Note that:

$$\left|\frac{1}{n} - \frac{1}{n+1} - 0\right| \leqslant \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $\left(\frac{1}{n} - \frac{1}{n+1}\right)_n \to 0.$

Proposition 5.2.3. *Let $k \geqslant 1$ be fixed. Given a sequence $(x_n)_n$ in a metric space (X, d), $(x_n)_n \to x$ if and only if $(x_{k+n})_n \to x$.*

Proof. ($\Rightarrow$) Suppose $(x_n)_n \to x$. Let $\epsilon > 0$. We know there exists $N_\epsilon \in \mathbf{N}$ with $n \geqslant N_\epsilon$ implying $d(x_n, x) < \epsilon$. But if $n \geqslant N_\epsilon$, then $n + k \geqslant N_\epsilon$. Hence $d(x_{n+k}, x) < \epsilon$.

($\Leftarrow$) Conversely, assume that $(x_{n+k}) \to 0$. Let $\epsilon > 0$. We know there exists $N_\epsilon \in \mathbf{N}$ such that $n \geqslant N_\epsilon$ implies $d(x_{n+k}, x) < \epsilon$. Consider $M = N_\epsilon + k$. Then if $n \geqslant M$, we have $n \geqslant N_\epsilon + k$, or equivalently $n - k \geqslant N_\epsilon$. Hence $d(x_{(n-k)+k}, x) = d(x_n, x) < \epsilon$. □

Proposition 5.2.4. *If $(x_n)_n$ is a real sequence with $\left(\left|\frac{x_{n+1}}{x_n}\right|\right) \to L < 1$, then $(x_n)_n \to 0$.*

Proof. Since $L < 1$, let ρ be an number satisfying $L < \rho < 1$. Pick $\epsilon = \rho - L$ Since $\left(\left|\frac{x_{n+1}}{x_n}\right|\right) \to L$, there exists $N_\epsilon \in \mathbf{N}$ such that $n \geqslant N_\epsilon$ implies $\left|\frac{x_{n+1}}{x_n}\right| \in V_\epsilon(L)$, or equivalently $L - \epsilon < \frac{|x_{n+1}|}{|x_n|} < L + \epsilon$. Then $\frac{|x_{n+1}|}{|x_n|} < \rho$, which gives $|x_{n+1}| < \rho|x_n|$. Observe that:

$$\begin{aligned}
&|x_{N+1}| < \rho|x_N| \\
&|x_{N+2}| < \rho|x_{N+1}| = \rho^2|x_N| \\
&|x_{N+3}| < \rho|x_{N+2}| = \rho^3|x_N| \\
&\qquad\vdots \\
\text{Inductively,}\quad &|x_{N+n}| = \rho^n|x_N|.
\end{aligned}$$

Since $(\rho^n)_n \to 0$ (and taking $c = |x_N|$), we have that $(x_{N+n})_n \to 0$. Thus $(x_n)_n \to 0$. □

Remark. Consider $(n)_n \to +\infty$. Then $\left(\frac{n+1}{n}\right)_n \to 1$. Now consider $\left(\frac{1}{n}\right)_n \to 0$. Then $\left(\frac{n}{n+1}\right) \to 1$. We gain no information if $L = 1$.

Example 5.2.4.

(1) Prove $((-1)^n)_n$ does not exist.

Solution. Suppose $((-1)^n)_n \to x$. We want to find some $\epsilon_0 > 0$ such that for all $N \in \mathbf{N}$, we can find an $n \in \mathbf{N}$ satisfying:

$$n \geqslant N \text{ and } |x_n - x| = |(-1)^n - x| \geqslant \epsilon_0.$$

Pick $\epsilon_0 = \max\{|x-1|, |x+1|\}$. Let $N \in \mathbf{N}$. Set $n = 2N$. This gives:

$$\begin{aligned}
(-1)^{2N} &= 1 \\
(-1)^{2N+1} &= -1
\end{aligned}$$

So we have $n \geqslant N$ and:

$$|(-1)^{2N} - x| = |1 - x| \geqslant \epsilon_0 \quad \text{or} \quad |(-1)^{2N+1} - x| = |1 + x| \geqslant \epsilon_0.$$

(2) Prove $(\sin(n))_n$ does not exist.

Solution.

Proposition 5.2.5. *Let (X, d) be a metric space. A sequence $(x_n)_n$ can have at most one limit.*

Proof. Suppose $(x_n)_n \to L_1$ and $(x_n)_n \to L_2$. Set $\epsilon = \frac{|L_1 - L_2|}{2}$. Then $V_\epsilon(L_1) \cap V_\epsilon(L_2) = \emptyset$. Since $(x_n)_n \to L_1$, there exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $x_n \in V_\epsilon(L_1)$. Likewise, since $(x_n)_n \to L_2$, there exists $N_2 \in \mathbf{N}$ such that $n \geqslant N_2$ implies $x_n \in V_\epsilon(L_2)$. Pick $N = \max\{N_1, N_2\}$. Then $x_N \in V_\epsilon(L_1) \cap V_\epsilon(L_2)$, which is a contradiction. □

Lemma 5.2.6. *If $(x_n)_n \to x$, then $(|x_n|)_n \to |x|$.*

Proof. Since $(x_n)_n \to x$, then there exists $N \in \mathbf{N}$ such that $n \geqslant \mathbf{N}$ implies $|x_n - x| < \epsilon$. The triangle inequality gives:

$$||x_n| - |x|| \leqslant |x_n - x| < \epsilon,$$

hence $(|x_n|)_n \to |x|$. Note that the converse does not hold in general, as:

$$(|(-1)^n|)_n \to 1 \text{ while } ((-1)^n)_n \text{ does not exist.}$$ □

Lemma 5.2.7. *Let $(t_n)_n$ be a sequence in (X, d). $(t_n)_n \to 0$ if and only if $(|t_n|)_n \to 0$.*

Proof. ($\Rightarrow$) The forward direction follows from Lemma 5.2.6. ($\Leftarrow$) Suppose $(|t_n|)_n \to 0$. We have that:

$$||t_n| - 0| \leqslant$$

□

Lemma 5.2.8. *If $(x_n)_n \to x \in \mathbf{R}$ with $x_n \geqslant 0$, then $(\sqrt{x_n})_n \to \sqrt{x}$.*

Proof. Case 1: $x = 0$. Let $\epsilon > 0$ be given. Since $(x_n)_n \to 0$, there exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $0 \leqslant x_n = |x_n - 0| < \epsilon^2$. Hence $0 \leqslant \sqrt{x_n} < \epsilon$. Since $\epsilon > 0$, was arbitrary, $(\sqrt{x_n})_n \to 0$.

Case 2: $x > 0$. Then $\sqrt{x} > 0$, and:

$$|\sqrt{x_n} - \sqrt{x}| = \left|(\sqrt{x_n} - \sqrt{x})\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right| = \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right| \leqslant \left(\frac{1}{\sqrt{x}}\right)|x_n - x|.$$

Hence the convergence $(\sqrt{x_n})_n$ is a consequence of $(x_n)_n \to x$. □

Example 5.2.5.

(1) Prove $(\sqrt{n})_n \to +\infty$.

Solution. Let $M > 0$ be given. Find N_M so that $N_M = \lceil M^2 \rceil$. Hence $N_M \geqslant M^2$. Then $n \geqslant M$ implies $n \geqslant M^2$, or equivalently $\sqrt{n} \geqslant M$.

(2) Prove $(n - \sqrt{n})_n \to +\infty$.

Solution. Write $(n - \sqrt{n})_n = (n)_n \overset{\to 1}{\cancel{(1 - \frac{1}{\sqrt{n}})_n}} = (n)_n$. Since $(n)_n$ trivially converges to $+\infty$, we have $(n - \sqrt{n})_n \to +\infty$.

(3) Prove:

$$(b^n)_{n=0}^{\infty} \to \begin{cases} 0, & |b| < 1 \\ 1, & b = 1 \\ +\infty, & b > 1 \\ \text{D.N.E.}, & b \leqslant -1 \end{cases}$$

Solution. Cases $b = 0$ and $b = 1$ are trivial. We showed case $b = -1$ in Example 5.2.4.

Case 1: $0 < b < 1$. Then $b < 1$ implies $\frac{1}{b} > 1$. We have $\frac{1}{b} = 1 + a$ for some $a > 0$, now observe that:

$$\left(\frac{1}{b}\right)^n = (1+a)^n \geqslant 1 + na.$$

This gives:

$$|b^n - 0| \leqslant \frac{1}{1+na} \leqslant \frac{1}{na} = \left(\frac{1}{a}\right)\frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $(b^n)_n \to 0$.

Case 2: $-1 < b < 0$. Since $(|b^n|)_n = (|b|^n)_n$, case 1 gives $(b^n)_n \to 0$ when $-1 < b < 0$.

Case 3: $b > 1$. Then $b = 1 + a$ for some $a > 0$. We have:

$$b^n = (1+a)^n \geqslant 1 + na \geqslant na.$$

Let $M > 0$ be given. Pick $N_M = \frac{\lceil M \rceil}{a}$. Then $N_M \geqslant \frac{M}{a}$. If $n \geqslant N_M$, then $n \geqslant \frac{M}{a}$, which simplifies to $na \geqslant M$. Hence $b^n \geqslant na \geqslant M$ gives $(b^n)_n \to +\infty$.

Case 4: $b < 1$. We prove that $(b^n)_n$ does not exist by contradiction. Suppose $(b_n)_n \to L$ for some $L \in \mathbf{R}$. Then $(|b_n|)_n \to |L|$. But this is a contradiction via the $b > 1$ case. Now if $(b^n)_n \to +\infty$, there exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $b^n \geqslant 1$. But for n odd, $b^n < 0$, which is a contradiction. Assuming $(b^n)_n \to -\infty$ leads to a similar contradiction, establishing the proof.

Example 5.2.6.

(1) Prove if $c > 0$, $(c^{\frac{1}{n}})_n \to 1$.

Solution. If $c = 1$, then clearly $(1^{\frac{1}{n}})_n \to 1$. Suppose $c > 1$, then $c^{\frac{1}{n}} > 1$. Write $c^{\frac{1}{n}} = 1 + a_n$, where $a_n > 0$ for all $n \in \mathbf{N}$. We have:

$$c = (c^{\frac{1}{n}})^n = (1+a_n)^n \geqslant 1 + na_n \geqslant na_n.$$

So $0 < na_n \leqslant c$, giving $a_n \leqslant \frac{c}{n}$. We have:

$$|c^{\frac{1}{n}} - 1| = a_n \leqslant \frac{c}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, $(c^{\frac{1}{n}})_n \to 1$. Now suppose $0 < c < 1$, then $c^{\frac{1}{n}} < 1$. Write $c^{\frac{1}{n}} = 1 + (-a_n)$ with $-1 < -a_n < 0$ for all n. Then:

$$c = (c^{\frac{1}{n}})^n = (1 + (-a_n))^n \geqslant 1 + n(-a_n) \geqslant n(-a_n).$$

So $n(-a_n) \leqslant c$, giving $-a_n \leqslant \frac{c}{n}$. We have:

$$|c^{\frac{1}{n}} - 1| = -a_n \leqslant \frac{c}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, $(c^{\frac{1}{n}})_n \to 1$.

(2) Prove $(n^{\frac{1}{n}})_n \to 1$.

Proof. Note that $n^{\frac{1}{n}} > 1$ for all $n > 1$. Write $n^{\frac{1}{n}} = 1 + a_n$. Then:

$$n = (1 + a_n)^n = \sum_{k=0}^{n} \binom{n}{k} a_n^k \geqslant \binom{n}{0} + \binom{n}{2} a_n^2 = 1 + \frac{n(n-1)}{2} a_n^2.$$

We have:

$$n - 1 \geqslant \frac{n(n-1)}{2} a_n^2,$$

which simplifies to:

$$\frac{2}{n} \geqslant a_n^2.$$

Hence $a_n \leqslant \sqrt{2}\,\frac{1}{n}$, thus by our lemma $(a_n)_n^\infty \to 0$. Therefore:

$$|n^{\frac{1}{n}} - 1| = d_n,$$

establishing that $(n^{\frac{1}{n}})_n \to 1$. □

Proposition 5.2.9. *A convergent sequence is bounded.*

Proof. Suppose $(x_n)_n \to x$. Since $(x_n)_n$ is convergent, we know for all $\epsilon > 0$ that $|x_n - x| < \epsilon$. Pick $\epsilon = 1$. Eventually the entire sequence will be contained in $V_1(x)$. More formally, there exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $x_n \in V_1(x)$. Define:

$$c = \max\left\{|x_1|, |x_2|, \ldots, |x_N|, |x - 1|, |x + 1|\right\}.$$

If $n \leqslant N$, then $|x_n| \leqslant c$. If $n \geqslant N_1$, then $x - 1 < x_n < x + 1$; i.e., $|x_n| \leqslant c$. □

Theorem 5.2.10. *Let x_n, y_n, z_n be convergent sequences with $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$ and $t \in \mathbf{R}$. Moreover, let $z_n \neq 0$ for all n and $z \neq 0$. We have:*

(1) $(x_n \pm y_n)_n \to x \pm y$.

(2) $(tx_n)_n \longrightarrow tx$.

(3) $(x_n y_n)_n \longrightarrow xy$.

(4) $\left(\frac{1}{z_n}\right)_n \longrightarrow \frac{1}{z}$.

(5) $\left(\frac{x_n}{z_n}\right)_n \longrightarrow \frac{x}{z}$.

Proof. (3) We have:

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \\ &= |(x_n - x) y_n + x(y_n - y)| \\ &\leqslant |(x_n - x) y_n| + |x(y_n - y)| \\ &= |x_n - x||y_n| + |x||y_n - y|. \end{aligned}$$

Since y_n is convergent, it is bounded. So there exists a $c > 0$ with $|y_n| \leqslant c$ for all $n \geqslant 1$. Hence:

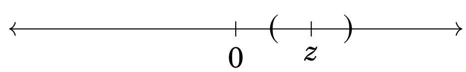
$$|x_n - x||y_n| + |x||y_n - y| \leqslant \overset{\to 0}{\cancel{|x_n - x|}}\, c + |x|\, \overset{\to 0}{\cancel{|y_n - y|}}.$$

Thus $(|x_n y_n - xy|)_n \longrightarrow 0$, which implies $(x_n y_n)_n \longrightarrow xy$.

(4) We have:

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \frac{|z - z_n|}{|z||z_n|}.$$

Since $z \neq 0$, it won't be "near" zero. We have the following picture:



Let $\delta = \frac{|z|}{2} > 0$. There exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $z_n \in V_\delta(z)$. We have:

$$\begin{aligned} & z - \delta < z_n < z + \delta \\ \Longrightarrow\ & z - \frac{|z|}{2} < z_n \\ \Longrightarrow\ & \frac{|z|}{2} < |z_n|. \end{aligned}$$

Since $|z_n| \geqslant \frac{|z|}{2}$, we have $\frac{1}{|z_n|} < \frac{2}{|z|}$. So for $n \geqslant N$,

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \frac{|z - z_n|}{|z||z_n|} \leqslant \frac{2}{|z|^2}|z - z_n|.$$

Thus $\left(\frac{1}{z_n}\right)_n \longrightarrow \frac{1}{z}$. □

Theorem 5.2.11. *Suppose $(x_n) \longrightarrow x$ and $(y_n)_n \longrightarrow y$ with $x_n \leqslant y_n$ for all n. Then $x \leqslant y$.*

Proof. We have that $(y_n - x_n)_n \longrightarrow y - x$, and $y_n - x_n \geqslant 0$ for all n. Thus $y - x \geqslant 0$. □

Corollary 5.2.12. *If $(x_n)_n \to x$ and $a \leqslant x_n \leqslant b$, then $a \leqslant x \leqslant b$.*

Proof. Taking $(y_n)_n = (a, a, a, \dots)$ and $(y_n)_n = (b, b, b, \dots)$ gives the desired result. □

Theorem 5.2.13 (Squeeze Theorem). *Let $(x_n)_n$, $(y_n)_n$, and $(z_n)_n$ be sequences with $(x_n)_n \leqslant (y_n)_n \leqslant (z_n)_n$ for all $n \geqslant 1$. If $\lim x_n = \lim z_n = L$, then $(y_n)_n \to L$.*

Proof. Let $\epsilon > 0$. There exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $x_n \in V_\epsilon(L)$. Likewise, there exists $N_2 \in \mathbf{N}$ such that $n \geqslant N_2$ implies $z_n \in V_\epsilon(L)$. So for $n \geqslant \max\{N_1, N_2\} := N$, both $x_n, z_n \in V_\epsilon(L)$. We have:

$$L - \epsilon < x_n \leqslant y_n < z_n \leqslant L + \epsilon.$$

Thus $y_n \in V_\epsilon(L)$ for $n \geqslant N$. □

Theorem 5.2.14 (Monotone Convergence Theorem). *Let $(x_n)_n$ be a monotone sequence. $(x_n)_n$ is convergent if and only if $(x_n)_n$ is bounded. Moreover,*

(a) If $(x_n)_n$ is increasing and bounded above, $\lim x_n = \sup\{x_n \mid n \in \mathbf{N}\}$.

(b) If $(x_n)_n$ is decreasing and bounded below, $\lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}$.

Proof. ($\Rightarrow$) We showed this direction in Proposition 5.2.9. ($\Leftarrow$) (a) Suppose $(x_n)_n$ is bounded above and increasing. Let $u = \sup\{x_n \mid n \in \mathbf{N}\}$. Given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $u - \epsilon < x_N$. But for $n \geqslant N$, $u - \epsilon < x_N \leqslant x_n \leqslant u < u + \epsilon$. Hence $x_n \in V_\epsilon(u)$, establishing that $(x_n)_n \to u$.

(b) Consider $y_n = -x_n$, we get y_n is increasing and bounded above. By (a), we get:

$$\begin{aligned} \lim y_n = \sup\{y_n \mid n \in \mathbf{N}\} &\implies -\lim x_n = \sup\{-x_n \mid n \in \mathbf{N}\} \\ &\implies -\lim x_n = -\inf\{x_n \mid n \in \mathbf{N}\} \\ &\implies \lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}. \end{aligned}$$

□

Example 5.2.7.

(1) Consider the recursively defined sequence $x_1 = 8$, $x_{n+1} = \frac{1}{2}x_n + 2$. We will show by induction that it is bounded below by 4. Clearly $x_1 = 8 \geqslant 4$. Now assume $x_k \geqslant 4$. Then:

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_n + 2 \\ &\geqslant \frac{1}{2}(4) + 2 \\ &= 4. \end{aligned}$$

Therefore $(x_n)_n$ is bounded below by 4. Now observe that:

$$\begin{aligned} x_{n+1} \leqslant x_n &\iff \frac{1}{2}x_n + 2 \leqslant x_n \\ &\iff 4 \leqslant x_n. \end{aligned}$$

Hence $(x_n)_n$ is decreasing. By the Monotone Convergence Theorem, $(x_n)_n \to L$. Now observe that:

$$\begin{aligned}(x_{n+1})_n = \left(\frac{1}{2}x_n + 2\right)_n &\iff L = \frac{1}{2}L + 2 \\ &\iff L = 4.\end{aligned}$$

(2) Let $x_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9}$. We will show that this sequence converges. Clearly $x_n \leqslant x_{n+1}$, so it is increasing. We will use the fact that $k^2 \geqslant k(k-1)$ as follows:

$$\begin{aligned}x_n &= \sum_{k=1}^{n} \frac{1}{k^2} \\ &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} \\ &\leqslant 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ &= 1 + \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] \\ &= 1 + 1 - \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &\leqslant 2.\end{aligned}$$

So $(x_n)_n$ is increasing and bounded above, hence it has a limit.

(3) Given $a > 0$, we will find a sequence $(x_n)_n$ which converges to $\sqrt{a}$. Consider the recursively defined sequence $x_1 = 1$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. Claim: $x_n^2 \geqslant a$ for all $n \geqslant 2$. Note that:

$$\begin{aligned}2x_{n+1} = x_n + \frac{a}{x_n} &\implies 2x_{n+1}x_n = x_n^2 + a \\ &\implies 0 = x_n^2 - 2x_{n+1}x_n = a.\end{aligned}$$

This polynomial has a real root, hence $\Delta \geqslant 0$. We get:

$$\Delta = 4x_{n+1}^2 - 4a \geqslant 0 \implies x_{n+1}^2 \geqslant a$$

We will now show that $(x_n)_n$ is eventually decreasing. Observe that:

$$\begin{aligned}x_n \geqslant x_{n+1} &\iff x_n \geqslant \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ &\iff 2x_n \geqslant x_n + \frac{a}{x_n} \\ &\iff x_n \geqslant \frac{a}{x_n} \\ &\iff x_n^2 \geqslant a.\end{aligned}$$

By the Monotone Convergence Theorem, $(x_n)_n \to L$. We have:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \implies L = \frac{1}{2}\left(L = \frac{a}{L}\right)$$
$$\implies L^2 = a$$
$$\implies L = \sqrt{a}.$$

Example 5.2.8 (Euler's Number)**.** I will do this later.

Proposition 5.2.15. *If $(x_n)_n$ is increasing and unbounded, then $(x_n)_n$ diverges properly to $+\infty$.*

Proof. Let M be arbitrarily big. Since $(x_n)_n$ is unbounded, there exists $N \in \mathbf{N}$ with $x_n > M$. Hence if $n \geqslant M$, $x_n \geqslant x_N > M$ because $(x_n)_n$ is increasing. □

Example 5.2.9. We will show that $h_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges properly to $+\infty$. do this later

5.3 Subsequences

Definition 5.3.1. A *natural sequence* is a strictly increasing sequence of natural numbers: $(n_k)_{k=1}^{\infty}$ with $n_k \in \mathbf{N}$, $n_1 < n_2 < \dots$

Example 5.3.1.

(1) $(2k+1)_k = (3, 5, 7, \dots)$

(2) $(k^2)_k = (1, 4, 9, \dots)$

Exercise 5.3.1. *Given a natural sequence $(n_k)_k$, prove $n_k \geqslant k$.*

Definition 5.3.2. Let $(x_n)_n$ be a sequence. A *subsequence* of $(x_n)_n$ is a sequence $(x_{n_k})_{k=1}^{\infty}$ where $(n_k)_k$ is a natural sequence. Formally, a subsequence is a composition of maps:

$$\mathbf{N} \underset{k \mapsto n_k}{\to} \mathbf{N} \underset{n_k \mapsto x_{n_k}}{\to} X.$$

Example 5.3.2.

(1) Consider $(x_n)_n \to \frac{1}{n}$. Let $n_k = 2_k$. Then $(x_{n_k})_k = \left(\frac{1}{2k}\right)_k = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right)$.

(2) Conider $(x_n)_n = (-1)^n$ Then $(x_{2k})_k = (1, 1, 1, \dots)$ and $(x_{2k+1})_k = (-1, -1, -1, \dots)$

Proposition 5.3.1. *Suppose $(x_n)_n \to x$. For any subsequence $(x_{n_k})_k$, we have $(x_{n_k})_k \to x$.*

Proof. Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $|x_n - x| < \epsilon$. Take $K = N$. Then $k \geqslant K$ implies $k \geqslant N$. But by Exercise **??**, $n_k \geqslant k \geqslant N$. Hence $|x_{n_k} - x| < \epsilon$. □

Example 5.3.3. We give an alternate proof of $(b^n)_n \to 0$ for $0 < b < 1$. Clearly $b^{n+1} < b^n$ if and only if $b < 1$. So b^n is decreasing and bounded below by 0. By the Monotone Convergence Theorem, $(b^n)_n \to L$ for some L. But we also have that $(b^{2k})_k \to L$. So we have:

$$\begin{aligned}(b^{2k})_k = (b^k)_k^2 &\iff L = L^2 \\ &\iff L(1-L) = 0.\end{aligned}$$

Since $L \neq 1$, it must be that $L = 0$.

Proposition 5.3.2. *Let $(x_n)_n$ be a sequence. Then $(x_n)_n \nrightarrow x$ if and only if there exists an $\epsilon_0 > 0$ and subsequence $(x_{n_k})_k$ such that $d(x_{n_k}, x) > \epsilon_0$.*

Proof. ($\Leftarrow$) If $(x_n)_n \to x$, then any subsequence $(x_{n_k})_k$ converges to x.

($\Rightarrow$) Since $(x_n)_n \nrightarrow x$, we have:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N})(\exists n \geqslant N) \ni (x_n \notin V_{\epsilon_0}(x)).$$

With this ϵ_0, we will construct our subsequence x_{n_k}. Note that:

$$\begin{aligned}N = 1 &\implies (\exists n_1 \geqslant 1) \ni (x_{n_1} \notin V_{\epsilon_0}(x)) \\ N = n_1 + 1 &\implies (\exists n_2 \geqslant n_1) \ni (x_{n_2} \notin V_{\epsilon_0}(x)) \\ N = n_2 + 1 &\implies (\exists n_3 \geqslant n_2) \ni (x_{n_3} \notin V_{\epsilon_0}(x)) \\ &\vdots \\ \text{Inductively, } N = n_k + 1 &\implies (\exists n_{k+1} \geqslant n_k) \ni (x_{k+1} \notin V_{\epsilon_0}(x))\end{aligned}$$

Thus $(x_{n_k})_k$ is a subsequence with $x_{n_k} \notin V_{\epsilon_0}(x)$, so $|x_{n_k} - x| \geqslant \epsilon_0$ for all $k = 1, 2, 3, \ldots$ □

Definition 5.3.3. If $(x_n)_n$ is a sequence of real numbers, a <u>*peak*</u> of the sequence is a term x_m satisfying $x_m \geqslant x_n$ for all $n \geqslant m$.

Proposition 5.3.3. *Let $(x_n)_n$ be a real sequence. There is a subsequence that is monotone.*

Proof. Case 1: There are infinitely many peaks. Let $x_{n_1}, x_{n_2}, x_{n_3} \ldots$ be an enumeration of peaks. Then $(x_{n_k})_k$ is decreasing by definition.

Case 2: There are finitely many peaks. Let $x_{m_1}, x_{m_2}, \ldots, x_{m_r}$ be the peaks of our sequence where $m_1 < m_2 < \ldots < m_r$. Let $n_1 = m_r + 1$. Since x_{n_1} is not a peak, there exists $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Since x_{n_2} is not a peak, there exists $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$. Inductively, we obtain a sequence $(x_{n_k})_k = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ with $x_{n_k} < x_{n_{k+1}}$. □

Theorem 5.3.4 (Bolzano-Weierstass Theorem)**.** *If $(x_n)_n$ is a real sequence that is bounded, it admits a convergent subsequence.*

Proof. By Proposition 5.3.3, there exists a subsequence (x_{n_k}) which is monotone and bounded. By the Monotone Convergence Theorem, $(x_{n_k})_k$ converges. □

5.4 Limit Inferior and Limit Superior

Definition 5.4.1. Let $X = (x_n)_n$ be a fixed bounded sequence who's limit may not exist. Then

$$\overline{X} = \{t \in \mathbf{R} \mid t = \lim_{k\to\infty} x_{n_k},\ x_{n_k} \text{ some subsequence}\}$$

is the set containing all *subsequential limits* (or *limit points*) of X.

Example 5.4.1. Let $X = ((-1)^n)_n$. Then $\overline{X} = \{-1, 1\}$.

Example 5.4.2. Fix a bounded sequence $(x_n)_n$. Let

$$u_1 = \sup_{n\geqslant 1}(x_n),$$
$$l_1 = \inf_{n\geqslant 1}(x_n).$$

If a subsequence $(x_{n_k})_k \to x$, we know $x \in [l_1, u_1]$ because $l_1 \leqslant x \leqslant u_1$. Hence $l_1 \leqslant x_{n_k} \leqslant u_1$. Now let

$$u_2 = \sup_{n\geqslant 2}(x_n),$$
$$l_2 = \inf_{n\geqslant 1}(x_n).$$

We have $u_2 \leqslant u_1$ (we know u_1 is an upper bound for all $n \geqslant 2$, hence u_2 must be the least upper bound) and $l_1 \leqslant l_2$. Similarly, if $(x_{n_k})_k \to x$ for some subsequence, then $x \in [l_2, u_2]$ because $l_2 \leqslant x_{n_k} \leqslant u_2$ for k large enough. Inductively:

$$u_m = \sup_{n\geqslant m} x_n,$$
$$l_m = \inf_{n\geqslant m} x_n.$$

We get:

$$l_1 \leqslant l_2 \leqslant \ldots \leqslant l_m \leqslant u_m \leqslant \ldots \leqslant u_2 \leqslant u_1.$$

This holds for all $m \geqslant 1$. Let $I_m = [l_m, u_m]$. Then $(I_m)_m$ is a sequence of closed and bounded nested intervals. So

$$\bigcap_{m\geqslant 1} I_m = [l, u]$$

where

$$l = \sup_{m\geqslant 1} l_m = \sup_{m\geqslant 1}\left(\inf_{n\geqslant m} x_n\right),$$
$$u = \inf_{m\geqslant 1} u_m = \inf_{m\geqslant 1}\left(\sup_{n\geqslant m} x_n\right).$$

Note that:

$$\sup_{m\geqslant 1} l_m = \lim_{m\to\infty} l_m$$
$$\inf_{m\geqslant 1} u_m = \lim_{m\to\infty} u_m.$$

This follows from the Monotone Convergence Theorem, as $(l_m)_m$ is an increasing sequence bounded above and $(u_m)_m$ is a decreasing sequence bounded below.

Definition 5.4.2. Let $(x_n)_n$ be a bounded sequence.

(1) $l = \lim_{m\to\infty} l_m = \lim_{m\to\infty}\left(\inf_{n\geqslant m} x_n\right) := \liminf x_n$.

(2) $u = \lim_{m\to\infty} u_m = \lim_{m\to\infty}\left(\sup_{n\geqslant m} x_n\right) := \limsup x_n$.

Proposition 5.4.1. *Let $X = (x_n)_n$ be a bounded sequence with $l = \liminf x_n$ and $u = \limsup x_n$. If $x \in X$, then $x \in [l, u]$. We have:*

$$l_{n_k} = \inf_{n\geqslant n_k} x_n \leqslant x_{n_k}.$$

Taking the limit as $k \to \infty$ yields $l \leqslant x$. Similarly, we have:

$$u_{n_k} = \sup_{n\geqslant n_k} x_n \geqslant x_{n_k}.$$

Taking the limit as $k \to \infty$ yields $x \leqslant u$. Thus $x \in [l, u]$.