Math 310

Homework 3

Due: 9/27/2024

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Exercise 1. *Find* $\sup(A)$ *and* $\inf(A)$ *where:*

(1) $A_1 = \left\{1 - \frac{(-1)^n}{n} \mid n \in \mathbf{N}\right\}$.

(2) $A_2 = \left\{\frac{1}{n} - \frac{1}{m} \mid m, n \in \mathbf{N}\right\}$.

(3) $A_3 = \left\{\frac{m}{n} \mid m, n \in \mathbf{N},\ m + n \leqslant 10\right\}$.

Proof. (1) Claim: $\inf(A_1) = \frac{1}{2}$. Note that $\frac{1}{2}$ is a lowerbound because $\frac{1}{2} \leqslant a$ for all $a \in A_1$. Let t be a lowerbound of A_1. If $t \leqslant \frac{1}{2}$, then we are done. If $t > \frac{1}{2}$, then $t - \frac{1}{2} > 0$. By the Archimedean Property, there exists an element $n \in \mathbf{N}$ with $t - \frac{1}{2} > \frac{1}{n}$ This gives $t > \frac{1}{2} + \frac{1}{n}$, which is a contradiction because $\frac{1}{2} + \frac{1}{n} \in A_1$ for all positive natural numbers. Thus $\inf(A_1) = \frac{1}{2}$. Note that $2 \geqslant 1 + \left|-\frac{(-1)^n}{n}\right| = 1 + \frac{(-1)^n}{n}$ for all $n \in \mathbf{N}$. Hence 2 is an upper bound. Furthermore, since $2 \in A_1$, it must be the case that $\sup(A_1) = 2$.

(2) Claim: $\sup(A_2) = 1$. Note that $1 \geqslant \frac{1}{n} + \frac{1}{m}$ for all $n, m \in \mathbf{N}$. Hence 1 is an upperbound. Fix $n = 1$. Let $\epsilon > 0$. Then $\epsilon > \frac{1}{m}$ for some $m \in \mathbf{N}$. Hence $1 - \epsilon < 1 - \frac{1}{m}$, establishing $\sup(A_2) = 1$. Claim: $\inf(A_2) = -1$. Note that $-1 \leqslant \frac{1}{n} - \frac{1}{m}$ for all $n, m \in \mathbf{N}$. Hence -1 is a lower bound. Fix $m = 1$. Let $\epsilon > 0$. Then $\epsilon > \frac{1}{n}$ for some $n \in \mathbf{N}$. Hence $-1 + \epsilon > \frac{1}{n} - 1$, meaning $\inf(A_2) = -1$.

(3) Note that $\frac{1}{9} \leqslant \frac{m}{n} \leqslant \frac{9}{1}$ for all $m, n \in \mathbf{N}$, $m + n \leqslant 10$. So $\frac{1}{9}$ is a lower bound of A_3 and $\frac{9}{1}$ is an upper bound of A_3. Since $\frac{1}{9}, \frac{9}{1} \in A_3$, it must be the case that $\inf(A_3) = \frac{1}{9}$ and $\sup(A_3) = \frac{9}{1}$. □

Exercise 2. *Suppose* $u = \sup(A)$ *such that* $u \notin A$. *Show that there is a strictly increasing sequence*

$$t_1 < t_2 < t_3 < \ldots$$

with $t_n \in A$ *and* $t_n + \frac{1}{n} > u$ *for all* $n \geqslant 1$.

Proof. Note that for all $\epsilon > 0$, there exists an a_ϵ with $u - \epsilon < a_\epsilon$. Define:

$$\begin{aligned} &t_1 > u - 1 \\ &t_2 > \max\left\{t_1, u - \frac{1}{2}\right\} \\ &t_3 > \max\left\{t_2, u - \frac{1}{3}\right\} \\ &\vdots \\ &t_n > \max\left\{t_{n-1}, u - \frac{1}{n}\right\}. \end{aligned}$$

If $\max\left\{t_{n-1}, u - \frac{1}{n}\right\} = t_{n-1}$, then clearly $t_n > t_{n-1}$. If $\max\left\{t_{n-1}, u - \frac{1}{n}\right\} = u - \frac{1}{n}$, then $t_n > u - \frac{1}{n} > t_{n-1}$. This gives $t_n + \frac{1}{n} > u$ for all $n \geqslant 1$, and furthermore, we obtain a strictly increasing sequence:

$$t_1 < t_2 < t_3 < \ldots$$

□

Exercise 3. *If m is a lower bound for $A \subseteq \mathbf{R}$, show that the following are equivalent:*

(1) $m = \inf(A)$.

(2) For all $t > m$, there exists $a_t \in A$ with $a_t < t$.

(3) For all $\epsilon > 0$ there exists $a_\epsilon \in A$ with $m + \epsilon > a_\epsilon$.

Proof. Let $m = \inf(A)$. Assuming $t > m$, suppose towards contradiction there does not exist an $a \in A$ with $a < t$. Then it must be the case that $m < t \leqslant a$ for all $t > m$. This is a contradiction, because m is the greatest lower bound.

Now assume for all $t > m$, there exists $a_t \in A$ with $a_t < t$. Given $\epsilon > 0$, pick $t = m + \epsilon$. Then by (2) there exists an a_t with $m + \epsilon > a_t$.

Now assume for all $\epsilon > 0$ there exists $a_\epsilon \in A$ with $m + \epsilon > a_\epsilon$. Given that m is a lower bound for A, assume there exists another lower bound for A with $l > m$. Pick $\epsilon = l - m$, then there exists an $a \in A$ with $m + (l - m) > a$. Simplifying yields $l > a$, which contradicts l being a lower bound. Hence $\inf(A) = m$. □

Exercise 4. *Let $A, B \subseteq \mathbf{R}$ be bounded subsets.*

(1) Show that

$$\sup(A + B) = \sup(A) + \sup(B),$$
$$\inf(A + B) = \inf(A) + \inf(B).$$

(2) If $t > 0$, show that

$$\sup(tA) = t\sup(A),$$
$$\inf(tA) = t\inf(A).$$

Proof. (1) Define $\sup(A) = u$ and $\sup(B) = v$. Then for all $\epsilon > 0$, there exists $a_\epsilon \in A$, $b_\epsilon \in B$ with $u - \epsilon < a_\epsilon$ and $v - \epsilon < b_\epsilon$. Pick $\epsilon = \frac{\epsilon}{2}$. Then adding both inequalities gives $(u + v) - \epsilon < a_\epsilon + b_\epsilon \in A + B$. Hence $\sup(A + B) = u + v = \sup(A) + \sup(B)$. Similarly, define $\inf(A) = m$ and $\inf(B) = n$. Then for all $\epsilon > 0$, there exists $a_\epsilon \in A$, $b_\epsilon \in B$ with $m + \epsilon > a_\epsilon$ and $n + \epsilon > b_\epsilon$. Pick $\epsilon = \frac{\epsilon}{2}$. Then adding both inequalities gives $(m + n) + \epsilon > a_\epsilon + b_\epsilon \in A + B$. Hence $\inf(A + B) = m + n = \inf(A) + \inf(B)$.

(2) Let $\sup(A) = u$. Then $a \leqslant u$ for all $a \in A$. We have that $u - \epsilon < a$ for some $a \in A$. Pick $\epsilon = \frac{\epsilon}{t}$. Then $tu - \epsilon < ta$ for some $ta \in tA$. Hence $\sup(tA) = tu = t\sup(A)$. Similarly, let $\inf(A) = m$. Then $m \leqslant a$ for all $a \in A$. We have that $m + \epsilon > a$ for some $a \in A$. Pick $\epsilon = \frac{\epsilon}{t}$. Then $tm + \epsilon > ta$ for some $ta \in tA$. Hence $\inf(tA) = tm = t\inf(A)$. □

Exercise 5. *Let $I = (0, 1)$ denote the open interval and consider the function*

$$F : I \times I \to \mathbf{R} \ \text{ defined by } \ F(x, y) = 2x + y.$$

Compute

$$\sup_{y \in I}\left(\inf_{x \in I} F(x, y)\right),$$

and

$$\inf_{y\in I}\left(\sup_{x\in I} F(x,y)\right).$$

Are they equal?

Proof. Observe that:

$$\begin{aligned}
\sup_{y\in I}\left(\inf_{x\in I}(2x+y)\right) &= \sup_{y\in I}\left(2\inf_{x\in I} x + \inf_{x\in I} y\right) \\
&= \sup_{y\in I} y \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
\inf_{y\in I}\left(\sup_{x\in I}(2x+y)\right) &= \inf_{y\in I}\left(\sup_{x\in I} 2x + \sup_{x\in I} y\right) \\
&= \inf_{y\in I}(2+y) \\
&= \inf_{y\in I} 2 + \inf_{y\in I} y \\
&= 2.
\end{aligned}$$

□

Exercise 6. *Let D be a nonempty set and consider the set of all bounded functions:*

$$\ell_\infty(D) := \{f \mid f : D \to \mathbf{R} \textit{ is bounded}\}$$

with point-wise addition and scalar multiplication. Show that

$$d_u(f,g) := \sup_{x\in D}|f(x)-g(x)|$$

*defines a metric on $\ell_\infty(D)$. We call d_u the **uniform metric**.*

Proof. Observe that:

$$\begin{aligned}
d_u(f,g) &= \sup_{x\in D}(|f(x)-g(x)|) \\
&= \sup_{x\in D}(|g(x)-f(x)|) \\
&= d_u(g,f).
\end{aligned}$$

Thus (ℓ_∞, d_u) is symmetric. We also have that:

$$\begin{aligned}
d_u(f,h) &= \sup_{x\in D}(|f(x)-h(x)|) \\
&= \sup_{x\in D}(|f(x)-g(x)+g(x)-h(x)|) \\
&\leqslant \sup_{x\in D}(|f(x)-g(x)|+|g(x)-h(x)|) \\
&= \sup_{x\in D}(|f(x)-g(x)|) + \sup_{x\in D}(|g(x)-h(x)|) \\
&= d(f,g)+d(g,h).
\end{aligned}$$

Hence (ℓ_∞, d_u) satisfies the triangle-inequality. Furthermore:

$$\begin{aligned} d_u(f,f) &= \sup_{x\in D} (|f(x) - f(x)|) \\ &= \sup_{x\in D} 0 \\ &= 0. \end{aligned}$$

Lastly $d_u(f,g) = 0$ implies $\sup_{x\in D} (|f(x) - g(x)|) = 0$. By definition of the absolute value, $|f(x) - g(x)| \geqslant 0$, so it must be the case that $|f(x) - g(x)| = 0$. Hence $f(x) = g(x)$, establishing that (ℓ_∞, d_u) forms a metric space. □

Exercise 7. *Let $f, g : D \to \mathbf{R}$ be bounded functions. Show that*

(1) $\sup_{x\in D}(f+g)(x) \leqslant \sup_{x\in D} f(x) + \sup_{x\in D} g(x)$.

(2) $\inf_{x\in D}(f+g)(x) \geqslant \inf_{x\in D} f(x) + \inf_{x\in D} g(x)$.

(3) $|\sup_{x\in D} f(x) - \sup_{x\in D} g(x)| \leqslant \sup_{x\in D} |f(x) - g(x)|$.

Proof. (1) Note that $f(x) \leqslant \sup_{x\in D} f(x)$ and $g(x) \leqslant \sup_{x\in D} g(x)$. Hence:

$$(f+g)(x) = f(x) + g(x) \leqslant \sup_{x\in D} f(x) + \sup_{x\in D} g(x).$$

However, $\sup_{x\in D} f(x) + \sup_{x\in D} g(x)$ is merely an upper bound of $(f+g)(x)$. Hence $\sup_{x\in D}(f+g)(x) \leqslant \sup_{x\in D} f(x) + \sup_{x\in D} g(x)$.

(2) Note that $f(x) \geqslant \inf_{x\in D} f(x)$ and $g(x) \geqslant \inf_{x\in D} g(x)$. Hence:

$$(f+g)(x) = f(x) + g(x) \geqslant \inf_{x\in D} f(x) + \inf_{x\in D} g(x).$$

However, $\inf_{x\in D} f(x) + \inf_{x\in D} g(x)$ is merely a lower bound of $(f+g)(x)$. Hence $\inf_{x\in D}(f+g)(x) \geqslant \inf_{x\in D} f(x) + \inf_{x\in D} g(x)$.

(3) Without loss of generality, let $\sup_{x\in D} f(x) - \sup_{x\in D} g(x) > 0$. We have:

$$\left|\sup_{x\in D} f(x) - \sup_{x\in D} g(x)\right| = \sup_{x\in D} f(x) - \sup_{x\in D} g(x) = \sup_{x\in D} (f(x) - g(x)).$$

Since $f(x) - g(x) \leqslant |f(x) - g(x)|$, we have:

$$\sup_{x\in D} (f(x) - g(x)) \leqslant \sup_{x\in D} |f(x) - g(x)|.$$

Hence

$$\left|\sup_{x\in D} f(x) - \sup_{x\in D} g(x)\right| \leqslant \sup_{x\in D} |f(x) - g(x)|.$$

□

Exercise 8. *Find $\bigcap_{n=1}^{\infty} I_n$ where*

(1) $I_n = \left[0, \frac{1}{n}\right]$,

(2) $I_n = \left(0, \frac{1}{n}\right)$,

(3) $I_n = [n, \infty)$.

Proof. (1) Note that $[0, \frac{1}{n}]$ is closed and bounded for all $n \geqslant 1$. Note that:

$$\inf\{\text{length}\,([0, 1/n]) \mid n \geqslant 1\} = \inf_{n \geqslant 1}\left(\frac{1}{n} - 0\right) = 0.$$

By the Nested Interval Theorem:

$$\bigcap_{n=1}^{\infty}\left[0, \frac{1}{n}\right] = \sup_{n \geqslant 1} 0 = \inf_{n \geqslant 1}\frac{1}{n} = 0.$$

(2) Claim: $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right) = \emptyset$. Suppose towards contradiction there exists $t \in \bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$. Then $t \in \left(0, \frac{1}{n}\right)$ for all $n \geqslant 1$. So $t < \frac{1}{n}$ implies $\frac{1}{t} > n$ for all $n \geqslant 1$, meaning $\mathbf{N}$ is bounded above. This is a contradiction, hence $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right) = \emptyset$.

(3) Claim: $\bigcap_{n=1}^{\infty}[n, \infty) = \emptyset$. Suppose towards contradiction there exists $t \in \bigcap_{n=1}^{\infty}[n, \infty)$. Then $t \in [n, \infty)$ for all $n \geqslant 1$. So $t \geqslant n$ for all $n \geqslant 1$. Hence $\mathbf{N}$ is bounded above, which is a contradiction. Thus $\bigcap_{n=1}^{\infty}[n, \infty) = \emptyset$. □

Exercise 9. *If $x > 0$, show that there is an $n \in \mathbf{N}$ with $\frac{1}{2^n} < x$.*

Proof. By the Archimedean Property 2, there exists $n \in \mathbf{N}$ such that $0 < \frac{1}{n} < x$. Claim: $\frac{1}{2^n} < \frac{1}{n}$. It suffices to show that $2^n > n$. Bernoulli's inequality gives $(1+1)^n \geqslant 1 + n$, hence $2^n > n$. □

Exercise 10. *The* ***Dyadic Rationals*** *are defined as*

$$\mathbf{D} := \left\{\frac{m}{2^n} \mid m, n \in \mathbf{Z}\right\}.$$

Show that $\mathbf{D} \subseteq \mathbf{R}$ is dense.

Proof. Let $I = (a, b)$. Then $b - a > 0$. By Archimedean Property 2 there exists $n \in \mathbf{N}$ such that $b - a > \frac{1}{n}$. Exercise 9 gives that $b - a > \frac{1}{2^n}$ for some $n \in \mathbf{Z}$. This simplifies to $2^n b > 1 + 2^n a$. Since $2^n a \in \mathbf{R}$, there exists $m \in \mathbf{Z}$ with $m - 1 \leqslant 2^n a < m$, implying that $a < \frac{m}{2^n}$. Furthermore, we also have that $m \leqslant 1 + 2^n a < m + 1$, and substituting for $2^n b$ gives $m < 2^n b$. So $\frac{m}{2^n} < b$, which means $\frac{m}{2^n} \in (a, b)$. Thus $I \cap D \neq \emptyset$. □