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Chapter 1

Vector Spaces, Algebras, and Normed Spaces

For the entirety of this chapter assume F to be $\mathbb{R}$ or $\mathbb{C}$.

§ 1.1. Vector Spaces

Definition 1.1.1. A *vector space* (or *linear space*) over F is a nonempty set V equipped with two operations:

$$\begin{aligned} V \times V \xrightarrow{+} V \ &\text{ defined by } \ (v, w) \mapsto v + w \\ F \times V \longrightarrow V \ &\text{ defined by } \ (\alpha, v) \mapsto \alpha v \end{aligned}$$

satisfying:

(1) $(V, +)$ is an abelian group:

  (i) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$;

  (ii) there exists 0_V such that $v + 0_V = 0_V + v = v$ for all $v \in V$;

  (iii) for all $v \in V$, there exists $w \in V$ satisfying $v + w = w + v = 0_V$;

  (iv) $v + w = w + v$ for all $v, w \in V$;

(2) $\alpha(v + w) = \alpha v + \alpha w$ for all $\alpha \in F$, $v, w \in V$;

(3) $\alpha(\beta v) = (\alpha\beta)v$ for all $\alpha, \beta \in F$, $v \in V$;

(4) $1_F v = v$ for all $v \in V$.

It can be shown that the vector 0_V is unique, the additive inverse in (iii) is unique (which we denote as $-v$), that $0v = 0_V$, and $(-1)v = -v$.

Exercise 1.1.1. Show (iv) follows from the other axioms.

Exercise 1.1.2. Show $nv = \underbrace{v + v + \ldots + v}_{n \text{ times}}$ for $n \in \mathbb{Z}_{\geqslant 1}$.

It can be shown that a subspace is a vector space in its own right.

Example 1.1.1. Let $\{W_i\}_{i\in I}$ be a family of vector spaces. Then $\bigcap_{i\in I} W_i$ is also a vector space.

Example 1.1.2. Planes and lines through the origin are subspaces of $\mathbb{R}^3$.

Definition 1.1.2. Let V be a vector space and $S \subseteq V$ a subset.

(1) A *linear combination* from S is a finite sum $\sum_{j=1}^{n} \alpha_j v_j$ with $\alpha_j \in F$, $v_j \in S$.

(2) The *linear span* of S is:

$$\operatorname{span}(S) := \left\{ \sum_{j=1}^{n} \alpha_j v_j \;\middle|\; n \in \mathbb{N}, \alpha_j \in F, v_j \in S \right\}.$$

Exercise 1.1.3. Show that $\operatorname{span}(S) \subseteq V$ is a subspace and:

$$\operatorname{span}(S) = \bigcap \{W \mid S \subseteq W, W \text{ is a subspace}\},$$

that is, $\operatorname{span}(S)$ is the smallest subspace of V containing S.

Definition 1.1.3. Let V be a vector space and $S \subseteq V$ a subset.

(1) S is *spanning* for V if $\operatorname{span}(S) = V$.

(2) S is *independent* if, given $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in F$, $v_1, \ldots, v_n \in S$, then $\sum_{j=1}^{n} \alpha_j v_j = 0$ implies $\alpha_j = 0$ for all j.

Our goal is to show that every vector space admits a basis. As such, recall the following definitions from a standard course in Real Analysis.

Definition 1.1.4. An *ordering* on a set X is a relation $R \subseteq X \times X$ on X that is reflexive, transitive, and antisymmetric. We write xRy as $x \leqslant_R y$. The pair $(X, \leqslant_R)$ is called an *ordered set.* An ordering $\leqslant$ on X is called *total* (or *linear*) if for all $x, y \in X$, $x \leqslant y$ or $y \leqslant x$.

Note that if $(X, \leqslant)$ is an ordered set and $Y \subseteq X$ is a subset, then $(Y, \leqslant)$ is an ordered set as well.

Definition 1.1.5. Let $(X, \leqslant)$ be an ordered set and $Y \subseteq X$. An *upper bound* for Y is an element $u \in X$ with $u \geqslant y$ for all $y \in Y$. An element $m \in X$ is called *maximal* if $x \in X$, $x \geqslant m$ implies $x = m$.

Lemma 1.1.1 (Zorn's Lemma)**.** *Let $(X, \leqslant_X)$ be an ordered set. Suppose every subset $Y \subseteq X$ for which $(Y, \leqslant_X)$ is totally ordered has an upper bound in X. Then X admits a maximal element.*

The proof of Zorn's Lemma is outside the interest of this text.

Theorem 1.1.2. *Every vector space admits a basis. Moreover, every independent set is contained in a basis.*

Proof. Let $S \subseteq V$ be linearly independent. Define:

$$\mathfrak{T}(S) = \{T \subseteq V \mid S \subseteq T, T \text{ linearly independent}\}.$$

Let $\mathfrak{C} \subseteq \mathfrak{T}(S)$ be a totally ordered subset. Set $R = \bigcup_{T \in \mathfrak{C}} T$. Clearly $R \supseteq S$. Assume $\sum_{j=1}^{n} \alpha_j v_j = 0$, where $\alpha_j \in F$ and $v_j \in R$. Since $\mathfrak{C}$ is totally ordered, there exists $T_0 \in \mathfrak{C}$ with $v_j \in T_0$ for all $j = 1, \ldots, n$. Since T_0 is independent, $\alpha_j = 0$ for all $j = 1, \ldots, n$. Thus R is independent as well. Whence R is an upper bound for $\mathfrak{C}$. By Zorn's Lemma, $\mathfrak{T}(S)$ admits a maximal element, call it B.

Claim: B is a basis for V. Suppose towards contradiction it's not, then there exists $v_0 \in V \setminus \text{span}(B)$. Consider $B \cup \{v_0\}$ and let $\alpha_0 v_0 + \sum_{j=1}^{n} \alpha_j v_j = 0_V$. If $\alpha_0 \neq 0$, then $\sum_{j=1}^{n} \alpha_j v_j = -\alpha_0 v_0$, giving $v_0 \in \text{span}(B)$ which is a contradiction. If $\alpha_0 = 0$, then $\sum_{j=1}^{n} \alpha_j v_j = 0_V$. Since B is independent, $\alpha_j = 0$ for all $j = 1, \ldots, n$. Thus $B \cup \{v_0\}$ is independent, contradicting the maximality of B. Whence B is a basis for V. □

Theorem 1.1.3. *If* B_1 *and* B_2 *are bases for* V*, then* $\text{card}(B_1) = \text{card}(B_2)$.

Definition 1.1.6. If V is a vector space, its *dimension* is the cardinality of any of its bases.

Corollary 1.1.4. *If* B *is a basis for* V*, then every* $v \in V$ *can be written* $v = \sum_{j=1}^{n} \alpha_k \beta_k$, $\alpha_k \in F$, $b_k \in B$ *in a unique way.*

Theorem 1.1.5. *Let* V *be a linear space and* $B \subseteq V$ *a subset. The following are equivalent:*

(1) B *is a basis for* V*;*

(2) B *is a maximal element in* $\mathfrak{T} = \{T \subseteq V \mid T \text{ independent}\}$*;*

(3) B *is a minimal element in* $\mathfrak{S} = \{S \subseteq V \mid S \text{ spans } V\}$*;*

Definition 1.1.7. Let $\{V_i\}_{i \in I}$ be a family of vector spaces over a field F.

(1) The *product* of $\{V_i\}_{i \in I}$ is denoted:

$$\prod_{i \in I} V_i := \{(v_i)_{i \in I} \mid v_i \in V_i\}.$$

(2) The *co-product* (or *sum*) is denoted

$$\bigoplus_{i \in I} V_i := \left\{(v_i)_{i \in I} \mid v_i \in V_i, \text{supp}\big((v_i)_{i \in I}\big) < \infty\right\}.$$

Exercise 1.1.4.

(1) Show that $\prod_{i\in I} V_i$ equipped with pointwise operations:

$$(v_i)_{i\in I} + (w_i)_{i\in I} = (v_i + w_i)_{i\in I}$$
$$\alpha(v_i)_{i\in I} = (\alpha v_i)_{i\in I}$$

is a linear space.

(2) Show that $\bigoplus_{i\in I} V_i$ is a subspace of $\prod_{i\in I} V_i$.

Proposition 1.1.6. *Let V be a vector space over F and $W \subseteq V$. The (additive, abelian) quotient group V/W can be made into a vector space by defining multiplication by scalars as $\alpha(v+W) = \alpha v + W$ for all $\alpha \in F$, $v + W \in V/W$.*

Example 1.1.3.

(1) The set $F^n = \{(x_1, ..., x_n) \mid x_j \in F\}$ with component-wise operations is a vector space.

(2) The set $M_{n,m}(F) = \{(a_{ij}) \mid a_{ij} \in F\}$ with linear operations is a vector space.

(3) Let Ω be a nonempty set. Then $\mathcal{F}(\Omega, F) = \{f \mid f : \Omega \to F\}$ with pointwise operations is a vector space.

(4) The set $\ell_\infty(\Omega, F) = \{f \in \mathcal{F}(\Omega, F) \mid \|f\|_u < \infty\}$ with pointwise operations is a vector space.

Exercise 1.1.5. Show $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(5) Let $K \subseteq V$ be a convex subset of a vector space V, that is, for all $v, w \in K$ and $t \in [0,1]$, then $(1-t)v + tw \in K$. A function $f : K \to F$ is said to be *affine* if $x, y \in K$ and $t \in [0,1]$ implies $f((1-t)x + ty) = (1-t)f(x) + tf(y)$. The set $\text{Aff}(K, F) = \{f \in \mathcal{F}(\Omega, F) \mid f \text{ affine}\}$ with pointwise operations is a vector space.

Exercise 1.1.6. Show $\text{Aff}(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(6) The set $C([a,b], F) = \{f : [a,b] \to F \mid f \text{ continuous}\}$ with pointwise operations is a vector space.

Exercise 1.1.7. Explain why $C([a,b], F) \subseteq \ell_\infty([a,b], F)$ is a subspace.

(7) Consider the following sequence spaces:

- $s = \{(a_k)_k \mid a_k \in F\} = \mathcal{F}(\mathbb{N}, F)$;
- $\ell_\infty = \ell_\infty(\mathbb{N}, F) = \{(a_k)_k \mid \sup_{k\geqslant 1} |a_k| < \infty\}$;
- $c = \{(a_k)_k \mid (a_k)_k \text{ converges}\}$;
- $c_0 = \{(a_k)_k \mid (a_k)_k \to 0\}$;
- $c_{00} = \{(a_k)_k \mid \text{supp}(a_k)_k < \infty\}$;

- $\ell_1 = \left\{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k| \text{ converges}\right\}$.

These are all vector spaces with pointwise operations. In fact, $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subspaces.

Exercise 1.1.8. Show that $\ell_1 \subseteq c_0$ is a subspace.

(8) Consider the following continuous function spaces on $\mathbb{R}$:

- $C(\mathbb{R}) = \{f : \mathbb{R} \to F \mid f \text{ continuous}\}$;
- $C_b(\mathbb{R}) = C(\mathbb{R}) \cap \ell_\infty(\mathbb{R})$;
- $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \text{limit}_{x\to\pm\infty} f(x) = 0\}$;
- Recall that a function is *compactly supported* if for all $\epsilon > 0$, there exists $\alpha > 0$ such that $|x| \geqslant \alpha$ implies $f(x) = 0$. The set of compactly supported functions is denoted $C_c(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid f \text{ compactly supported}\}$.

These are all vector spaces with pointwise operations, and $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C_b(\mathbb{R}) \subseteq C(\mathbb{R})$ are all subspace inclusion.

Definition 1.1.8. If V and W are linear spaces over a common field F, a map $T : V \to W$ is called *linear* if $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha \in F$.

Example 1.1.4. Let $A \in M_{m,n}(F)$. Then $T_A : F^n \to F^m$ defined by $T_A(v) = Av$ is linear. Let $\{e_1, ..., e_n\}$ be a basis for F^n. If $T : F^n \to F^m$ is linear, set:

$$[T] = \Big(T(e_1) \,\Big|\, T(e_2) \,\Big|\, \dots \,\Big|\, T(e_n)\Big).$$

This gives $T(v) = [T]v$ for all $v \in F^n$. In fact, we also have $[T_A] = A$ and $T_{[T]} = T$.

Example 1.1.5. The *canonical projection* is linear:

$$\pi_j : \prod_{i\in I} V_i \to V_j \;\text{ defined by }\; \pi_j\big((v_i)_i\big) = v_i.$$

We also have that the *coordinate exclusions* are linear:

$$\iota_j : V_j \hookrightarrow \bigoplus_{i\in I} V_i \;\text{ defined by }\; \iota_j(v) = (v_i)_i\,, \;\text{ where }\; v_i = \begin{cases} 0_v, & i \neq j \\ v_j, & \text{otherwise.} \end{cases}$$

The *evaluation map* is linear as well. For $s \in S$, consider:

$$e_s : \mathcal{F}(S, F) \to F \;\text{ defined by }\; e_s(f) = f(s).$$