Math 395

Homework 5

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Exercise 1. *Let* $A = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$. *Calculate* $m_A(x)$ *and* $c_A(x)$.

Proof. We have that $c_A(x) = \det(A - xI_4) = (x-5)^2(x-4)^2$. Note however that A is in Jordan canonical form. Since the largest Jordan block associated to $\lambda_1 = 5$ is size 2 and the largest Jordan block associated to $\lambda_2 = 4$ is size 1, it must be the case that $m_A(x) = (x-5)^2(x-4)$. □

Exercise 2. *Let* $T \in \mathrm{Hom}_F(V, V)$ *with* $\dim_F(V) = n$. *Suppose* T *has eigenvectors* $v_1, \ldots, v_k$ *with corresponding eigenvalues* $\lambda_1, \ldots, \lambda_k$. *If* $v \in V$ *can be written as a linear combination of the eigenvectors, say* $v = a_1 v_1 + \ldots + a_k v_k$, *prove that:*

$$T^m(v) = a_1 \lambda_1^m v_1 + \ldots + a_k \lambda_k^m v_k.$$

Proof. Observe that:

$$\begin{aligned} T^m(v) &= T^m(a_1 v_1 + a_2 v_2 + \ldots + a_k v_k) \\ &= a_1 T^m(v_1) + a_2 T^m(v_2) + \ldots + a_k T^m(v_k) \\ &= a_1 \lambda_1^m v_1 + a_2 \lambda_2^m v_2 + \ldots + a_k \lambda_k^m v_k. \end{aligned}$$

□

Exercise 3. *Let* $A \in \mathrm{Mat}_n(F)$.

(a) *Assume* A *has eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Prove that* $\det(A) = \lambda_1 \ldots \lambda_n$ *and* $\mathrm{tr}(A) = \lambda_1 + \ldots + \lambda_n$.

(b) *Suppose that* A *does not have* n *distinct eigenvalues, but* $c_A(x)$ *splits into linear factors over* F. *Can you characterize the determinant and trace of* A *in terms of the eigenvalues?*

Proof. Since A has n distinct eigenvalues, it is similar to a diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_n)$. Hence:

$$\begin{aligned}
\det(A) &= \det(P \,\text{diag}(\lambda_1, \ldots, \lambda_n) P^{-1}) \\
&= \det(P) \det(\text{diag}(\lambda_1, \ldots, \lambda_n)) \det(P^{-1}) \\
&= \det(P) \det(P^{-1}) \det(\text{diag}(\lambda_1, \ldots, \lambda_n)) \\
&= \det(PP^{-1}) \det(\text{diag}(\lambda_1, \ldots, \lambda_n)) \\
&= \det(\text{diag}(\lambda_1, \ldots, \lambda_n)) \\
&= \lambda_1, \ldots, \lambda_n
\end{aligned}$$

$$\begin{aligned}
\text{tr}(A) &= \text{tr}(P \,\text{diag}(\lambda_1, \ldots, \lambda_n) P^{-1}) \\
&= \text{tr}(P^{-1} P \,\text{diag}(\lambda_1, \ldots, \lambda_n)) \\
&= \text{tr}(\text{diag}(\lambda_1, \ldots, \lambda_n)) \\
&= \lambda_1 + \ldots + \lambda_n
\end{aligned}$$

for some $P \in \text{GL}_n(F)$. If A were to not have n distinct eigenvalues, then A is similar to a Jordan matrix J. We'd still be able to characterize the determinant and trace of A in terms of its eigenvalues because J is an upper-triangular matrix with eigenvalues along the diagonal. $\square$

Exercise 4. *Let* $A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{pmatrix}$.

(a) *Find the characteristic polynomial of A.*

(b) *Compute E_λ^j for all eigenvalues λ of A and all j.*

(c) *Give the Jordan canonical form of A and the Jordan basis $\mathcal{B}$ of F^4.*

Proof. We have that $c_A(x) = \det(A - xI_4) = (x-2)^2(x-4)^2$. Note that:

$$\begin{aligned}
(A - 2I_4) &= \begin{pmatrix} 0 & -4 & 2 & 2 \\ -2 & -2 & 1 & 3 \\ -2 & -2 & 1 & 3 \\ -2 & -6 & 3 & 5 \end{pmatrix}, \quad E_2^1 = \text{span}_F \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\} \\
(A - 2I_4)^2 &= \begin{pmatrix} 0 & -8 & 4 & 4 \\ -4 & -8 & 4 & 8 \\ -4 & -8 & 4 & 8 \\ -4 & -16 & 8 & 12 \end{pmatrix}, \quad E_2^2 = \text{span}_F \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\} = E_2^1.
\end{aligned} \tag{1}$$

Hence $E_2^\infty = E_2^1$. So we have two Jordan blocks of size 1 corresponding to eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$. Furthermore:

$$(A - 4I_4) = \begin{pmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{pmatrix}, \ E_4^1 = \mathrm{span}_F \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$(A - 4I_4)^2 = \begin{pmatrix} -4 & -8 & -4 & -4 \\ 4 & 4 & 0 & -4 \\ 4 & 0 & 4 & -4 \\ 4 & 8 & -4 & -4 \end{pmatrix}, \ E_4^2 = \mathrm{span}_F \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$(A - 4I_4)^3 = \begin{pmatrix} -8 & -16 & 8 & 8 \\ -8 & -8 & 0 & 8 \\ -8 & 0 & -8 & 8 \\ -8 & -16 & 8 & 8 \end{pmatrix}, \ E_4^3 = \mathrm{span}_F \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\} = E_4^2.$$

Hence $E_4^\infty = E_4^2$. So we will have one Jordan block of size 2. Solving the matrix equation:

$$\begin{pmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

gives $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$ as a generalized eigenvector. Thus $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}$ is the Jordan basis for A. We have:

$$P = [\mathrm{id}_{F^4}]_{\mathcal{E}_4}^{\mathcal{B}} = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & -2 & 1 & 2 \\ 0 & -2 & 1 & 1 \end{pmatrix}.$$

Thus:

$$[A]_{\mathcal{B}} = PAP^{-1} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \qquad \square$$