Math 395

Homework 2

Due: 9/12/2024

Name: Gianluca Crescenzo

Collaborators: Avinash Iyer, Noah Smith, Carly Venenciano, Clarissa Ly

For these problems F is assumed to be a field.

Exercise 1. Let $T \in \mathrm{Hom}_F(V, W)$. We get an induced map $T^\vee \in \mathrm{Hom}_F(W^\vee, V^\vee)$ such that $T^\vee(\varphi) = \varphi \circ T$ for some $\varphi \in W^\vee$. The following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, dashed, "T^\vee(\varphi)"'] & W \arrow[d, "\varphi"] \\ & F \end{tikzcd}$$

Prove that $T^\vee \in \mathrm{Hom}_F(W^\vee, V^\vee)$.

Proof. Let $\varphi_1, \varphi_2 \in W^\vee$ and $\alpha \in F$. Then

$$\begin{aligned} T^\vee(\varphi_1 + \alpha\varphi_2)(v) &= (\varphi_1 + \alpha\varphi_2)(T(v)) \\ &= \varphi_1(T(v)) + \alpha(\varphi_2(T(v))) \\ &= T^\vee(\varphi_1)(v) + \alpha T^\vee(\varphi_2)(v) \\ &= (T^\vee(\varphi_1) + \alpha T^\vee(\varphi_2))(v). \end{aligned}$$

Thus $T^\vee$ is a linear map. $\square$

Exercise 11. Let $T \in \mathrm{Hom}_F(P_7(F), P_7(F))$ be defined by $T(f(x)) = f'(x)$, where $f'(x)$ denotes the usual derivative of a polynomial $f(x) \in P_7(F)$. For each of the fields below, determine a basis for the image and kernel of T:

(a) $F = \mathbf{R}$.

(b) $F = \mathbf{F}_3$.

Proof. Let $F = \mathbf{R}$. Observe that:

$$\begin{gathered} T(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + 7a_7x^6 \\ T(a_0) = 0. \end{gathered}$$

So a basis for $\mathrm{im}\,(T)$ is $\{1, x, x^2, x^3, x^4, x^5, x^6\}$ and a basis for $\ker(T)$ is $\{1\}$.

Now let $F = \mathbf{F}_3$. Observe that:

$$\begin{gathered} T(\overline{a_0} + \overline{a_1}x + \overline{a_2}x^2 + \overline{a_3}x^3 + \overline{a_4}x^4 + \overline{a_5}x^5 + \overline{a_6}x^6 + \overline{a_7}x^7) = \overline{a_1} + \overline{2a_2}x + \overline{a_4}x^3 + \overline{2a_5}x^4 + \overline{a_7}x^6 \\ T(\overline{a_0} + \overline{a_1}x^3 + \overline{a_2}x^6) = \overline{0}. \end{gathered}$$

So a basis for $\mathrm{im}\,(T)$ is $\{1, x, x^3, x^4, x^6\}$ and a basis for $\ker(T)$ is $\{1, x^3, x^6\}$. $\square$

Exercise 12. Let $T \in \mathrm{Hom}_F(V, F)$. Prove that if $v \in V$ is not in $\ker(T)$, then

$$V = \ker(T) \oplus \{cv \mid c \in F\}.$$

Proof. Let $I = \{cv \mid c \in F\}$, we must first show that it is a subspace of V. Let $av, bv \in I$ and $c \in F$. Note that I is nonempty because $0v = 0 \in I$. Let $av, bv \in I$ and $c \in F$. Then $av + c(bv) = av + (cb)v = (a + cb)v \in I$.

Let $x + y \in \ker(T) + I$. Since $\ker(T)$ and I are subspaces, $x + y \in V$. Now let $w \in V$. If $w \in \ker(T)$, then we are done (because $w = w + 0v \in \ker(T) + I$). Suppose $w \notin \ker(T)$. Then $T(w) = \alpha$ for some $\alpha \in F, \alpha \neq 0$, which is equivalent to $\alpha^{-1}T(w) = 1$. Since $v \notin \ker(T)$, suppose $T(v) = \beta$ for some $\beta \in F, \beta \neq 0$. It follows that $\alpha^{-1}T(w) = \beta^{-1}T(v)$. Simplifying yields $T(w - \alpha\beta^{-1}v) = 0$, giving $w - \alpha\beta^{-1}v = k$ for some $k \in \ker(T)$. Thus $w = k + \alpha\beta^{-1}v \in \ker(T) + I$, establishing $V = \ker(T) + I$.

We now need to show that $\ker(T)$ and I are independent. Let $k + \alpha\beta^{-1}v = k + cv = 0$. Then $T(k + cv) = T(cv) = cT(v) = 0$. But since $v \notin \ker(T)$, we must have that $c = 0$; i.e., $cv = 0$. Then $k + cv = k = 0$. Since $\ker(T)$ and I are independent, $V = \ker(T) \oplus I$. $\square$

Exercise 19. Let W be a subspace of a finite dimensional vector space V. Let $T \in \mathrm{Hom}_F(V, V)$ so that $T(W) \subset W$. Show that T induces a linear transformation $\overline{T} \in \mathrm{Hom}_F(V/W, V/W)$. Prove that T is nonsingular (i.e., injective) on V if and only if T is restricted to W and $\overline{T}$ on V/W are both nonsingular.

Proof. Define $\overline{T} : V/W \to V/W$ by $v + W \mapsto T(v) + W$. We must first show that $\overline{T}$ is well-defined. Suppose $v_1 + W = v_2 + W$, then $v_1 = v_2 + w$ for some $w \in W$. Observe that:

$$\begin{aligned}
\overline{T}(v_1 + W) &= T(v_1) + W \\
&= T(v_2 + w) + W \\
&= T(v_2) + T(w) + W && \text{Since } T \in \mathrm{Hom}_F(V, V) \\
&= T(v_2) + W && \text{Since } T(W) \subset W \\
&= \overline{T}(v_2 + W).
\end{aligned}$$

Let $v_1 + W, v_2 + W \in V/W$, and $\alpha \in F$. Then:

$$\begin{aligned}
\overline{T}((v_1 + W) + \alpha(v_2 + W)) &= \overline{T}((v_1 + W) + (\alpha v_2 + W)) \\
&= \overline{T}((v_1 + \alpha v_2) + W) \\
&= T(v_1 + \alpha v_2) + W \\
&= T(v_1) + \alpha T(v_2) + W \\
&= (T(v_1) + W) + \alpha(T(v_2) + W) \\
&= \overline{T}(v_1 + W) + \alpha\overline{T}(v_2 + W),
\end{aligned}$$

hence $\overline{T} \in \mathrm{Hom}_F(V/W, V/W)$. Now consider the maps $V \xrightarrow{T} V \xrightarrow{\pi} V/W$, where $\pi : V \to V/W$ is the canonical projection map. It can be proved that $\pi \circ T = \overline{T} \circ \pi$ as follows: let $v \in V$ and observe that $\pi(T(v)) = T(v) + W$, which is equivalent to $\overline{T}(\pi(v)) = \overline{T}(v + W) = T(v) + W$. We've established that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\pi"'] & V \arrow[d, "\pi"] \\
V/W \arrow[r, "\overline{T}"'] & V/W
\end{tikzcd}.$$

Let T be injective. Since T is finite dimensional, it must be bijective. Clearly $T|_W : W \to V$ is injective by inclusion. Let $v + W \in \ker(\overline{T})$. Then $\overline{T}(v + W) = T(v) + W = 0 + W$. This gives that $T(v) \in W$, and from the fact that T is bijective, $T(W) = W$, giving $v \in W$. Hence $v + W = 0 + W$.

Conversely, suppose $T|_W$ and $\overline{T}$ are injective. Let $v \in \ker(T)$. Then $T(v) = 0$ is equivalent to $\pi(T(v)) = 0 + W$. Using the fact that the above diagram commutes, we can write $\overline{T}(\pi(v)) = 0 + W$. Since $\overline{T}$ is injective, its kernel is trivial, hence it must be the case that $\pi(v) = 0 + W$. Thus $v \in W$, hence $T(v) = T|_W(v) = 0$, establishing that $v = 0$. Thus T is injective. □