

# Math 310

## Homework 4

Due: 10/9/2024

Name: Gianluca Crescenzo

**Exercise 1.** Prove the following limits:

(1)  $\left(\frac{2n}{n+1}\right)_n \rightarrow 2.$

(2)  $\left(\frac{\sqrt{n}}{n+1}\right)_n \rightarrow 0.$

(3)  $\left(\frac{(-1)^n}{\sqrt{n+7}}\right)_n \rightarrow 0.$

(4)  $(n^k b^n)_n \rightarrow 0$ , where  $0 \leq b < 1$  and  $k \in \mathbf{N}$ .

(5)  $\left(\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \rightarrow 3.$

**Exercise 2.** Show that the sequence  $(\cos(n))_n$  does not converge.

**Exercise 3.** If  $(x_n)_n$  is a real sequence converging to  $x$ , show that

$$(|x_n|)_n \rightarrow |x|.$$

Is the converse true?

**Exercise 4.** If  $(x_n)_n$  is a real sequence converging to  $x > 0$ , show that there is an  $N \in \mathbf{N}$  and  $c > 0$  such that

$$x_n \geq c$$

for all  $n \geq N$ .

**Exercise 5.** If  $(x_n)_n$  is a real sequence of positive terms converging to  $x$ , show that  $x \geq 0$  and

$$(\sqrt{x_n})_n \rightarrow \sqrt{x}.$$

*Proof.* Observe that:

$$|\sqrt{x_n} - \sqrt{x}| \leq |\sqrt{x_n} - \sqrt{x}| |\sqrt{x_n} + \sqrt{x}| = |x_n - x| < \epsilon.$$

Hence  $(\sqrt{x_n})_n \rightarrow \sqrt{x}$ . If  $x < 0$ , then  $\sqrt{x} \notin \mathbf{R}$ , contradicting the definition of a real sequence. □

**Exercise 6.** If  $(x_n)_n$  and  $(y_n)_n$  are sequences with  $(x_n)_n \rightarrow 0$  and  $(y_n)_n$  bounded, show that

$$(x_n y_n)_n \rightarrow 0.$$

*Proof.* Since  $(y_n)_n$  is bounded,  $|y_n| \leq c$  for some  $c > 0$ . We have:

$$|x_n y_n| \leq c |x_n|.$$

Taking  $\epsilon_n = |x_n|$  and using "Lemma" gives  $(x_n y_n)_n \rightarrow 0$ . □

**Exercise 7.** If  $(x_n)_n$  is a sequence of positive terms such that

$$\left( \frac{x_{n+1}}{x_n} \right)_n \rightarrow L > 1,$$

show that  $(x_n)_n$  is not bounded hence not convergent. If  $L = 1$ , can we make any conclusion?