Math 395

Homework 8
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Exercise 1. Let V be an F-vector space. Recall that a bilinear form is said to be *skew-symmetric* if $\varphi(v, w) = -\varphi(w, v)$ and $\varphi(v, v) = 0$. Show that the second condition is redundant if $\mathrm{char}(\mathsf{F}) \neq 2$.

Proof. If we assume $\mathrm{char}(\mathsf{F}) \neq 2$, then $\varphi(v, v) = -\varphi(v, v)$ is equivalent to $2\varphi(v, v) = 0$. Therefore $\varphi(v, v) = 0$, meaning that the first condition implies the second condition (hence its redundancy). Had it been the case that $\mathrm{char}(\mathsf{F}) = 2$, then $2\varphi(v, v) = 0$ does not imply $\varphi(v, v) = 0$, as $2 \equiv 0 \pmod 2$. □

Exercise 2. Let $W \subseteq V$ be a subspace. Show that $W^\perp$ is a subspace.

Proof. Note that $W^\perp \neq \emptyset$ because $\varphi(w, 0) = 0_{\mathsf{F}}$. Let $v_1, v_2 \in W^\perp$ and $c \in \mathsf{F}$. Then:

$$\begin{aligned}
\varphi(w, v_1 + cv_2) &= \varphi(w, v_1) + c\varphi(w, v_2) \\
&= 0_{\mathsf{F}}.
\end{aligned}$$

Hence $W^\perp$ is a subspace of V. □

Exercise 3. Let $\mathsf{T} \in \mathrm{Hom}_{\mathsf{F}}(V, V)$ and let φ be a bilinear form on V. Prove that $\psi(v, w) = \varphi(\mathsf{T}(v), w)$ is a bilinear form on V.

Proof. Observe that:

$$\begin{aligned}
\psi(v_1 + cv_2, w) &= \varphi(\mathsf{T}(v_1 + cv_2), w) \\
&= \varphi(\mathsf{T}(v_1) + c\mathsf{T}(v_2), w) \\
&= \varphi(\mathsf{T}(v_1), w) + c\varphi(\mathsf{T}(v_2), w) \\
&= \psi(v_1, w) + c\psi(v_2, w)
\end{aligned}$$

$$\begin{aligned}
\psi(v, w_1 + cw_2) &= \varphi(\mathsf{T}(v), w_1 + cw_2) \\
&= \varphi(\mathsf{T}(v), w_1) + c\varphi(\mathsf{T}(v), w_2) \\
&= \psi(v, w_1) + c\psi(v, w_2).
\end{aligned}$$

Hence ψ is a bilinear form on V. □

Exercise 4. Let φ be a bilinear form on V and assume $\mathrm{char}(\mathsf{F}) \neq 2$. Prove that φ is skew-symmetric if and only if the diagonal function $V \to \mathsf{F}$ given by $v \mapsto \varphi(v, v)$ is additive.

Proof. Suppose φ is skew-symmetric. Let $v, w \in V$. Then:

$$\begin{aligned}\varphi(v+w, v+w) &= \varphi(v, v+w) + \varphi(w, v+w)\\ &= \varphi(v,v) + \varphi(v,w) + \varphi(w,v) + \varphi(w,w)\\ &= \varphi(v,v) + \varphi(w,w).\end{aligned}$$

Hence the diagonal function is additive. Now suppose the diagonal function is additive. Then:

$$\varphi(v+w, v+w) = \varphi(v,v) + \varphi(w,w).$$

Moreover, we have:

$$\varphi(v+w, v+w) = \varphi(v,v) + \varphi(v,w) + \varphi(w,v) + \varphi(w,w).$$

This gives:

$$\varphi(v,v) + \varphi(w,w) = \varphi(v,v) + \varphi(v,w) + \varphi(w,v) + \varphi(w,w).$$

Simplifying yields $\varphi(v,w) = -\varphi(w,v)$. Hence φ is skew-symmetric. $\square$