# Contents



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# 1

# Orderings and Functions

## 1.1 Basic Notation

**Definition 1.1.1.**

(1) The *natural numbers* are defined as $\mathbf{N} = \{1, 2, 3, \ldots\}$,

(2) The *positive integers* are defined as $\mathbf{N}_0 = \mathbf{Z}^+ = \{0, 1, 2, 3, \ldots\}$,

(3) The *integers* are defined as $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$,

(4) The *rational numbers* are defined as $\mathbf{Q} = \{\frac{a}{b} \mid a, b \in \mathbf{Z}, b \neq 0\}$,

(5) The *real numbers* are "defined" (we will get more into this later) as the set $(-\infty, \infty)$,

(6) The *complex numbers* are defined as $\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R}, i^2 = -1\}$.

**Example 1.1.1.** Note that $\sqrt{2}, \pi, e \notin \mathbf{Q}$, as they cannot be expressed as fractions.

**Definition 1.1.2.** Let $A$ and $B$ be sets. The *cartesian product* is defined as $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

**Definition 1.1.3.** A *relation* from $A$ to $B$ is a subset $R \subseteq A \times B$. Typically, when one says "a relation on $A$" that means a relation from $A$ to $A$; i.e., $R \subseteq A \times A$.

**Definition 1.1.4.** Let $A$ be a set and $R$ a relation on $A$. Then $R$ is:

(1) *reflexive* if $(a, a) \in R$ for all $a \in A$,

(2) *transitive* if $(a, b), (b, c) \in R$ implies $(a, c) \in R$,

(3) *symmetric* if $(a, b) \in R$ implies $(b, a) \in R$, and

(4) *antisymmetric* if $(a, b), (b, a) \in R$ implies $a = b$.

## 1.2 Orderings

**Definition 1.2.1.** Let $A$ be a set. An *ordering* of $A$ is a relation $R$ on $A$ that is reflexive, transitive, and antisymmetric. If this is the case, we write $(a, b) \in R$ as $a \leqslant_R b$. If $A$ is an ordered set we write it as the ordered pair $(A, \leqslant_R)$ (or just $A$ if the ordering is obvious by context).

**Example 1.2.1.**

(1) Let $m, n \in \mathbf{Z}$. The *algebraic ordering* $\leqslant_a$ is defined as follows: $m \leqslant_a n$ if and only if there exists an element $k \in \mathbf{N}_0$ with $m + k = n$.

(2) The set of natural numbers $\mathbf{N}$ equipped with the relation of divisibility form an ordering. Let $m, n \in \mathbf{N}$. Then $m \leqslant_d n$ if and only if $m \mid n$.

(3) Let $S$ be any set. The subsets of $S$ (i.e., elements of its power set) equipped with the relation of inclusion form an ordering. Let $A, B \in \mathcal{P}(S)$. Then $A \leqslant_{\mathcal{P}(S)} B$ if and only if $A \subseteq B$.

(4) The set of rational numbers $\mathbf{Q}$ form an algebraic ordering as follows: if $\frac{a}{b}, \frac{c}{d} \in \mathbf{Q}$, then $\frac{a}{b} \leqslant_a \frac{c}{d}$ if and only if $ad \leqslant_a bc$ (in $\mathbf{Z}$).

**Definition 1.2.2.** An ordered set $(A, \leqslant_R)$ is *total* (or *linear*) if for all $a, b \in A$ we have that $a \leqslant_R b$ or $b \leqslant_R a$.

**Example 1.2.2.** The ordered sets $(\mathbf{Z}, \leqslant_a)$ and $(\mathbf{Q}, \leqslant_a)$ are total orderings, whereas $(\mathbf{N}, \leqslant_d)$ and $(\mathcal{P}(S), \leqslant_{\mathcal{P}(S)})$ are not total orderings.

**Definition 1.2.3.** Let $(X, \leqslant)$ be an ordered set. Let $A \subseteq X$.

(1) $A$ is called *bounded above* if there exists an element $u \in X$ with $a \leqslant u$ for all $a \in A$. Such a $u$ (not necessarily unique) is called an *upperbound* for $A$.

(2) $A$ is called *bounded below* if there exists an element $v \in X$ with $v \leqslant a$ for all $a \in A$. Such a $v$ (not necessarily unique) is called a *lowerbound* for $A$.

(3) If $A$ admits an upperbound $u$ with $u \in A$, then $u$ is called *the greatest element of $A$*.

(4) If $A$ admits a lowerbound $v$ with $v \in A$, then $v$ is called *the least element of $A$*.

(5) Let $A$ be bounded above. The *set of upperbounds of $A$* is defined as $\mathcal{U}_A = \{u \in X \mid u \text{ is an upperbound of } A\}$. If $l$ is the least element of $\mathcal{U}_A$, we write $l = \sup(A)$ and call it *the supremum of $A$*.

(6) Let $A$ be bounded below. The *set of lowerbounds of $A$* is defined as $\mathscr{L}_A = \{v \in X \mid v \text{ is a lowerbound of } A\}$. If $g$ is the greatest element of $\mathscr{L}_A$, we write $g = \inf(A)$ and call it *the infimum of $A$*.

(7) A *maximal element of $A$* is an element $m \in A$ such that if $a \geqslant m$, then $a = m$ (not necessarily unique).

(8) A *minimal element of $A$* is an element $n \in A$ such that if $a \leqslant n$, then $a = n$ (not necessarily unique).

(9) If $(A, \leqslant)$ is a total ordering, then $A$ is called a *chain*.

**Proposition 1.2.1.** *Let $(X, \leqslant)$ be an ordered set and $A \subseteq X$.*

*(1) If $A$ admits a greatest element, then it is unique,*

*(2) If $A$ admits a least element, then it is unique,*

*(3) If $A$ admits a least upper bound, then it is unique,*

*(4) If $A$ admits a greatest lower bound, then it is unique.*

*Proof.* Suppose $u, u'$ are greatest elements of $A$, then $u, u' \in A$. Hence $u \leqslant u'$ and $u' \leqslant u$. By antisymmetry, $u = u'$, meaning the greatest element is unique. The proof for least elements being unique is identical, which establishes (1) and (2).

Note that $\mathcal{U}_A \subseteq X$. By definition the least element of $\mathcal{U}_A$ is defined to be the supremum of $A$, and since least elements are unique the supremum of $A$ must be unique. Similarly, $\mathscr{L}_A \subseteq X$. By definition the greatest element of $\mathscr{L}_A$ is defined to be the infimum of $A$, and since greatest elements are unique the infimum of $A$ must be unique. This establishes (3) and (4). $\square$

**Lemma 1.2.2** (Zorn's Lemma)**.** *Let $X$ be an ordered set with the property that every chain has an upperbound. Then $X$ contains a maximal element.*

**Example 1.2.3.** Considered the ordered set $(\mathbf{N}, \leqslant_d)$ and the subset $A = \{4, 7, 12, 28, 35\}$.

- $A$ is bounded above with $4 \times 7 \times 12 \times 28 \times 35$ as an upperbound.
- The supremum of $A$ is $\operatorname{lcm}(4, 7, 12, 28, 35)$.
- There does not exist a greatest element.
- $12, 28$, and $35$ are maximal elements (no other element in $A$ divides them).

**Definition 1.2.4.** Let $(X, \leqslant)$ be an ordered set and $A \subseteq X$. If $A$ is bounded above and below, then we say $A$ is <u>*bounded*</u>.

**Definition 1.2.5.** Let $(X, \leqslant)$ be an ordered set. Then $(X, \leqslant)$ is <u>*complete*</u> if, for every bounded set $A \subseteq X$, $\sup(A)$ and $\inf(A)$ exist.

## 1.3 Functions

**Definition 1.3.1.** Let $X$ and $Y$ be sets. A <u>*function*</u> from $X$ to $Y$ is a relation $f \subseteq X \times Y$ such that for all $x \in X$, there exists a unique $y_x \in Y$ with $(x, y_x) \in f$.

(1) The set $X$ is the <u>*domain*</u> of $f$.

(2) The set $Y$ is the <u>*codomain*</u> of $f$.

(3) The <u>*image*</u> of $f$ is defined as $f(X) = \{f(x) \mid x \in X\} \subseteq Y$ (also sometimes denoted $\operatorname{im}(f)$).

(4) The <u>*preimage*</u> of $f$ is defined as $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\} \subseteq X$.

(5) The <u>*graph*</u> of $f$ is defined as $\operatorname{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

If $f$ is a function, we denote it by $f : X \to Y$ or $X \xrightarrow{f} Y$.

**Example 1.3.1.** Let $X$ be a set.

(1) The *identity map* $\mathrm{id}_X : X \to X$ is defined by $\mathrm{id}_X(x) = x$.

(2) If $X \subseteq Y$, the *inclusion map* $\iota : X \to Y$ is defined by $\iota(x) = x$.

(3) If $A \subseteq X$ is a set, the *characteristic function* (or *step function*) $\mathbf{1}_A : X \to \mathbf{R}$ is defined by

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

**Definition 1.3.2.** Given $f, g : X \to \mathbf{R}$ and $\alpha \in \mathbf{R}$, the *pointwise operations* on $f$ and $g$ are:

- $(f \pm g)(x) = f(x) \pm g(x)$,
- $(\alpha f)(x) = \alpha f(x)$,
- $(fg)(x) = f(x)g(x)$,
- $(f/g)(x) = f(x)/g(x)$.

**Definition 1.3.3.** Let $f : X \to Y$ and $g : Y \to Z$ be maps between sets. The *composition* of $f$ and $g$ is denoted $g \circ f : X \to Z$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

**Definition 1.3.4.** Let $f : X \to Y$ be a map between sets.

(1) $f$ is *left-invertible* if there exists a map $g : Y \to X$ with $g \circ f = \mathrm{id}_X$.

(2) $f$ is *right-invertible* if there exists a map $h : Y \to X$ with $f \circ h = \mathrm{id}_Y$.

(3) $f$ is *invertible* if there exists a map $k : Y \to X$ with $k \circ f = \mathrm{id}_X$ and $f \circ k = \mathrm{id}_Y$.

**Example 1.3.2.** The *shift function* is a map $s : \mathbf{N} \to \mathbf{N}$ defined by $s(n) = n + 1$. Note that this function is left-invertible: define $g : \mathbf{N} \to \mathbf{N}$ by

$$g(n) = \begin{cases} n - 1, & n \geqslant 2 \\ n_0, & n = 1, \end{cases}$$

where $n_0$ is an arbitrary natural number, then $g \circ s = \mathrm{id}_{\mathbf{N}}$.

Suppose that $s$ has a right inverse, that is, there exists a function $h : \mathbf{N} \to \mathbf{N}$ such that $s \circ h = \mathrm{id}_{\mathbf{N}}$. Observe that:

$$(s \circ h)(1) = s(h(1)) = h(1) + 1 = 1.$$

It must be the case that $h(1) = 0$, which is a contradiction. Hence $s$ is not right-invertible.

**Example 1.3.3.** The function $g$ defined above is right invertible, but not left invertible.

**Proposition 1.3.1.** *Let $f : X \to Y$ be a map between sets. The following are equivalent:*

*(1) $f$ is invertible,*

*(2) $f$ is right-invertible and left-invertible.*

*Proof.* Clearly (1) implies (2). Assume $f$ to be left and right-invertible. Then there exists maps $h, g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$. Observe that:

$$\begin{aligned} h &= \mathrm{id}_X \circ h \\ &= (g \circ f) \circ h \\ &= g \circ (f \circ h) \\ &= g \circ \mathrm{id}_Y \\ &= g, \end{aligned}$$

establishing the proposition. □

**Definition 1.3.5.** Let $f : X \to Y$ be a map between sets.

(1) $f$ is *injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$,

(2) $f$ is *surjective* if $\mathrm{im}\,(f) = Y$, and

(3) $f$ is *bijective* if it is injective and surjective.

**Proposition 1.3.2.** *Let $f : X \to Y$ be a map between sets.*

1. *$f$ is injective if and only if $f$ is left-invertible.*
2. *$f$ is surjective if and only if $f$ is right-invertible.*
3. *$f$ is bijective if and only if $f$ is invertible.*

*Proof.* (1) Do the forward direction yourself! Now assume $f : X \to Y$ is injective. Define $g : Y \to X$ by

$$g(y) = \begin{cases} x_0, & y \notin \mathrm{im}\,(f) \\ x_y, & y \in \mathrm{im}\,(f), \end{cases}$$

where $x_y$ is the unique element in $x$ mapping to $y$; i.e., $f(x_y) = y$. By our construction, $(g \circ f)(x) = x$ for all $x \in X$.

(2) Do the forward direction yourself! Now assume $f : X \to Y$ is onto. Note that the preimage of $f$ is nonempty, so we can define $h : Y \to X$ by $h(y) = x_y$, where $x_y \in f^{-1}(X)$. By our construction $(f \circ h)(y) = f(x_y) = y$ for all $y \in Y$.

(3) Do this yourself its easy! □

**Corollary 1.3.3.** *Let $A, B$ be sets. There exists an injection $A \hookrightarrow B$ if and only if there exists a surjection $B \twoheadrightarrow A$.*

*Proof.* If $f : A \to B$ is injective, then $f$ is left invertible, that is, there exists a function $g : B \to A$ with $g \circ f = \mathrm{id}_A$. But this means $g$ is right invertible, so $g$ is onto. The other direction follows identically. □

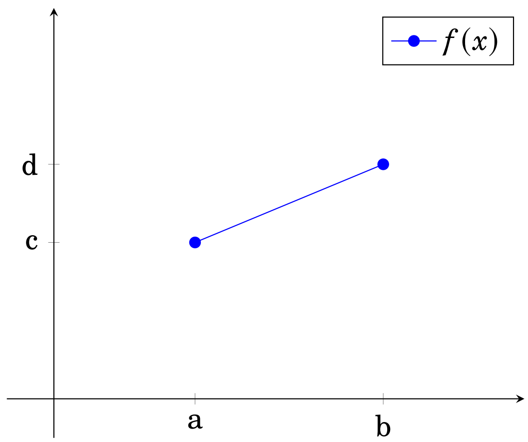
## 1.4 Cardinality

**Definition 1.4.1.** Let $A, B$ be sets. Then $\text{card}(A) = \text{card}(B)$ if there exists a bijection $A \hookrightarrow\!\!\!\!\rightarrow B$.

**Example 1.4.1.**

(1) Define $f : \mathbf{N}_0 \to \mathbf{N}$ by $f(n) = n + 1$. This is a bijection, hence $\text{card}(\mathbf{N}_0) = \text{card}(\mathbf{N})$.

(2) Let $[a, b]$ and $[c, d]$ be intervals with $a < b$ and $c < d$. Define $f : [a, b] \to [c, d]$ by $f(x) = (\frac{d-c}{b-a})(x - a) + c$.




This is a bijection, hence $\text{card}([a, b]) = \text{card}([c, d])$. The result is the same had the intervals been open.

(3) Recall that $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbf{R}$ is a bijection. Consider the maps $(0, 1) \xrightarrow{g} (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow{\tan} \mathbf{R}$. Since $g$ and tan are bijective, $\tan \circ g$ is bijective, hence $\text{card}((0, 1)) = \text{card}(\mathbf{R})$.

**Definition 1.4.2.** A set $A$ is called *finite* if there exists an $N \in \mathbf{N}$ such that $\text{card}(A) = \text{card}(\{1, \ldots, N\})$. If not, then $A$ is called *infinite*.

**Proposition 1.4.1.** *Given* $m, n \in \mathbf{N}$, $m \neq n$, *then* $\text{card}(\{1, \ldots, m\}) \neq \text{card}(\{1, \ldots, n\})$.

*Proof.* Without loss of generality, let $m > n$. Suppose towards contradiction we have a bijection $\{1, \ldots, m\} \xrightarrow{f} \{1, \ldots, n\}$. By the pigeon-hole principle, it must be the case that —given any $i, j \in \{1, \ldots, m\}$ with $i \neq j$, we have that $f(i) = f(j)$. This is a contradiction ($f$ is not injective), hence $\text{card}(\{1, \ldots, m\}) \neq \text{card}(\{1, \ldots, n\})$. □

**Proposition 1.4.2.** $\mathbf{N}$ *is infinite.*

*Proof.* Suppose towards contradiction we have a bijection $f : \mathbf{N} \to \{1, 2, \ldots, n\}$, where $n \in \mathbf{N}$. Consider the maps $\{1, 2, \ldots, n, n+1\} \xrightarrow{\iota} \mathbf{N} \xrightarrow{f} \{1, 2, \ldots, n\}$, it must be the case that the composition $f \circ \iota$ is injective. However, we established in Proposition 1.4.1 that this is false. Having reached a contradiction, it must be the case that $\mathbf{N}$ is infinite. □

**Exercise 1.4.1.** *If $A$ is infinite, there exists an injection $\mathbf{N} \hookrightarrow A$.*

*Proof.* Let $\pi : \mathbf{N} \to A$ be a map. Pick $a_1 \in A$ and define $\pi(1) = a_1$. Since $A$ is infinite, $A \setminus \{a_1\}$ is also infinite. Pick $a_2 \in A \setminus \{a_1\}$ and define $\pi(2) = a_2$. Inductively, we have an injection $\mathbf{N} \hookrightarrow A$. □

**Example 1.4.2.** Define $k : \mathbf{Z} \to \mathbf{N}$ by $k(n) = (-1)^{n-1} \left\lfloor \frac{n}{2} \right\rfloor$. This is a bijection, hence $\text{card}(\mathbf{Z}) = \text{card}(\mathbf{N})$.

**Definition 1.4.3.** Let $X$ and $Y$ be sets.

(1) The <u>*power set*</u> of $X$ is $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.

(2) The <u>*set of functions*</u> from $X$ to $Y$ is $Y^X = \{f \mid f : X \to Y\}$.

**Lemma 1.4.3.** *Let $X$ be a set. There exists a bijection $\mathcal{P}(X) \hookrightarrow\!\!\!\!\rightarrow 2^X$.*

*Proof.* Let $A \subseteq X$. Define $\varphi : \mathcal{P}(X) \to 2^X$ by $A \mapsto \mathbf{1}_A$, where

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

is the *characteristic function* defined in Example 1.3.1. Let $\varphi(A) = \varphi(B)$. This is equivalent to $\mathbf{1}_A = \mathbf{1}_B$. Note that functions are equal if and only if $\mathbf{1}_A(x) = \mathbf{1}_B(x)$ for all $x \in X$. Hence $x \in A$ implies $\mathbf{1}_A(x) = 1 = \mathbf{1}_B(x)$, giving $x \in B$. The reverse inclusion is identical, hence $A = B$. Let $f \in 2^X$. Let $A = \{x \in X \mid f(x) = 1\}$. Then $\varphi(A) = \mathbf{1}_A = f$. Thus $\mathcal{P}(X) \hookrightarrow\!\!\!\!\rightarrow 2^X$. □

**Exercise 1.4.2.** *Show that* $\text{card}(\mathcal{P}(\{1, \dots, n\})) = 2^n$.

*Proof.* Note that $\text{card}(\mathcal{P}(\{1, \dots, n\})) = \text{card}(2^{\{1,\dots,n\}})$. Let $f \in 2^{\{1,\dots,n\}}$. For each $i \in \{1, \dots, n\}$, there is a choice of two outputs for $f(i)$. Hence by the fundamental principle of counting $\text{card}(\mathcal{P}(\{1, \dots, N\})) = \text{card}(2^{\{1,\dots,n\}}) = 2^n$. □

**Theorem 1.4.4** (Cantor's Diagonal Argument)**.** $\text{card}(\mathbf{N}) < \text{card}((0, 1))$.

*Proof.* Recall that every $\sigma \in (0, 1)$ has a decimal expansion $\sigma = 0.\sigma_1\sigma_2 \dots = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$, where $\sigma_j \in \{0, 1, 2, \dots, 9\}$ which does not terminate in 9's. By way of contradiction, suppose there exists a surjection $r : \mathbf{N} \to (0, 1)$ defined by $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, where $\sigma_j(n) \in \{0, 1, 2, \dots, 9\}$ is the $j^{\text{th}}$ digit in the decimal expansion.

Consider the map $\tau : \mathbf{N} \to \{0, 1, \dots, 9\}$ defined by:

$$\tau(n) = \begin{cases} 3, & \sigma_n(n) = 2 \\ 2, & \sigma_n(n) = 3, \end{cases}$$

and let $t = 0.\tau(1)\tau(2)\tau(3)\ldots$ Observe the following:

$$\begin{aligned}
r(1) &= 0.\sigma_1(1)\sigma_2(1)\sigma_3(1)\sigma_4(1)\ldots \\
r(2) &= 0.\sigma_1(2)\sigma_2(2)\sigma_3(2)\sigma_4(2)\ldots \\
r(3) &= 0.\sigma_1(3)\sigma_2(3)\sigma_3(3)\sigma_4(3)\ldots \\
r(4) &= 0.\sigma_1(4)\sigma_2(4)\sigma_3(4)\sigma_4(4)\ldots \\
&\vdots \\
r(n) &= 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\sigma_4(n)\ \ldots\ \sigma_n(n).
\end{aligned}$$

Since $r$ is surjective, there is an $m \in \mathbf{N}$ with $r(m) = t$. It follows that:

$$\begin{aligned}
r(m) &= 0.\sigma_1(m)\sigma_2(m)\sigma_3(m)\ldots\sigma_m(m)\ldots \\
&= 0.\tau(1)\tau(2)\tau(3)\ldots\tau(m)\ldots
\end{aligned}$$

which implies that $\sigma_m(m) = \tau(m)$. But recall how we defined $\tau(n)$ —if $\sigma_m(m) = 2$, then $\tau(2) = 3$ and if $\sigma_m(m) \neq 2$, then $\tau(2) = 2$. This is a contradiction, hence there does not exist a surjection $\mathbf{N} \xrightarrow{r} (0,1)$. □

**Corollary 1.4.5.** $\text{card}(\mathbf{N}) \neq \text{card}(\mathbf{R})$

*Proof.* It follows from Example 1.4.1 that $\text{card}(\mathbf{N}) < \text{card}((0,1)) = \text{card}(\mathbf{R})$. □

**Definition 1.4.4.** Let $A$ and $B$ be sets.

(1) We write $\text{card}(A) \leqslant \text{card}(B)$ if there exists an injection $A \hookrightarrow B$.

(2) We write $\text{card}(A) < \text{card}(B)$ if $\text{card}(A) \leqslant \text{card}(B)$ and $\text{card}(A) \neq \text{card}(B)$

**Example 1.4.3.**

(1) If $A \subseteq B$, then the inclusion map $\iota : A \to B$ gives $\text{card}(A) \leqslant \text{card}(B)$.

(2) If $m > n$, then $\text{card}\{1, \ldots, n\} < \text{card}\{1, \ldots, m\}$

**Proposition 1.4.6.** *Let $A$ be a set. Then* $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

*Proof.* Define $f : A \to \mathcal{P}(A)$ by $a \mapsto \{a\}$. This is clearly an injective map. Now suppose towards contradiction that there exists a surjection $g : A \to \mathcal{P}(A)$ defined by $a \mapsto g(a)$. Then $g(a) \subseteq A$ (by the definition of a power set).

Let $S = \{a \in A \mid a \notin g(a)\}$. Then $S \subseteq A$. Since $g$ is onto, there exists an element $x \in A$ with $g(x) = S$. Case 1: $x \in S$. This implies that $x \notin g(x)$. But $g(x) = S$, so $x \notin S$, a contradiction. Case 2: $x \notin S$. This implies that $x \notin g(x)$. But by definition this means $x \in S$, a contradiction. Since we have exhausted all the necessary cases, it must be that there does not exist a surjection from $A \to \mathcal{P}(A)$. Hence $\text{card}(A) < \text{card}(\mathcal{P}(A))$. □

**Lemma 1.4.7.** *Let $A$ and $B$ be sets. The following are equivalent:*

*(1)* $\operatorname{card}(A) \leqslant \operatorname{card}(B)$*;*

*(2) there exists an injection* $A \hookrightarrow B$*;*

*(3) there exists a surjection* $B \twoheadrightarrow A$*.*

**Example 1.4.4.**

(1) Define $\mathbf{N} \times \mathbf{Z} \to \mathbf{Q}$ by $(n, m) \mapsto \frac{m}{n}$. This is surjective, so $\operatorname{card}(\mathbf{Q}) \leqslant \operatorname{card}(\mathbf{N} \times \mathbf{Z})$.

(2) Define $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$ by $(m, n) \mapsto 2^m \cdot 3^n$. Then $g$ is injective by the fundamental theorem of arithmetic. So $\operatorname{card}(\mathbf{N} \times \mathbf{N}) \leqslant \operatorname{card}(\mathbf{N})$.

(3) Recall from Example 1.4.2 that $k : \mathbf{N} \to \mathbf{Z}$ defined by $k(n) = (-1)^{n-1} \left\lfloor \frac{n}{2} \right\rfloor$ is a bijection. Define $K : \mathbf{Z} \times \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ by $(m, n) \mapsto (k^{-1}(m), n)$. This is a bijection, so $\operatorname{card}(\mathbf{Z} \times \mathbf{N}) = \operatorname{card}(\mathbf{N} \times \mathbf{N})$.

(4) From the previous examples, we've established that:

$$\operatorname{card}(\mathbf{N}) \leqslant \operatorname{card}(\mathbf{Q}) \leqslant \operatorname{card}(\mathbf{Z} \times \mathbf{N}) = \operatorname{card}(\mathbf{N} \times \mathbf{N}) \leqslant \operatorname{card}(\mathbf{N})$$

**Theorem 1.4.8.** *Let* $\mathfrak{N}$ *denote the class of cardinals. The pair* $(\mathfrak{N}, \leqslant)$ *forms a total ordering —where* $\leqslant$ *is defined by* $\operatorname{card}(A) \leqslant \operatorname{card}(B)$ *if and only if* $A \hookrightarrow B$*. In particular, if* $A, B, C$ *are sets with* $\operatorname{card}(A), \operatorname{card}(B), \operatorname{card}(C) \in \operatorname{obj}(\mathfrak{N})$*, then we have the following:*

*(1)* $\operatorname{card}(A) \leqslant \operatorname{card}(A)$ *(reflexive).*

*(2) If* $\operatorname{card}(A) \leqslant \operatorname{card}(B) \leqslant \operatorname{card}(C)$*, then* $\operatorname{card}(A) \leqslant \operatorname{card}(C)$ *(transitive).*

*(3) If* $\operatorname{card}(A) \leqslant \operatorname{card}(B)$ *and* $\operatorname{card}(B) \leqslant \operatorname{card}(A)$*, then* $\operatorname{card}(A) = \operatorname{card}(B)$ *(antisymmetric).*

*(4) Either* $\operatorname{card}(A) \leqslant \operatorname{card}(B)$ *or* $\operatorname{card}(B) \leqslant \operatorname{card}(A)$ *(total).*

*Proof.* (1) and (2) follow by simply applying definitions. Note that any set bijects into itself, hence $A \hookrightarrow\!\!\!\!\twoheadrightarrow A$ implies $A \hookrightarrow A$, establishing $\operatorname{card}(A) \leqslant \operatorname{card}(A)$. Similarly, if there are bijections $A \hookrightarrow\!\!\!\!\twoheadrightarrow B \hookrightarrow\!\!\!\!\twoheadrightarrow C$, then clearly there is a bijection $A \hookrightarrow\!\!\!\!\twoheadrightarrow C$. Hence $\operatorname{card}(A) = \operatorname{card}(C)$.

(3) (Cantor-Shröder-Bernstein Theorem) We have injections $A \overset{f}{\hookrightarrow}$ and $B \overset{g}{\hookrightarrow} A$. Let:

$$\begin{aligned} A_0 &= \operatorname{im}(g)^{\complement} \\ A_1 &= (g \circ f)(A_0) \\ A_2 &= (g \circ f)(A_1) \\ &\vdots \\ A_n &= (g \circ f)(A_{n-1}). \end{aligned}$$

Note that $A_1 \cap A_0 = \emptyset$ because $A_1 \subseteq \operatorname{im}(g)$ and $A_0 = \operatorname{im}(g)^{\complement}$. We similarly have that $A_2 \cap A_0 = \emptyset$. Claim: $A_1 \cap A_2$. finish this

(4) Let $A \to B$ be a map. Let $\mathcal{F} = \{(D, f) \mid D \subseteq A, f : D \hookrightarrow B,\ f \text{ is injective}\}$. Note that $\mathcal{F} \neq \emptyset$ because $(\emptyset, k) \in \mathcal{F}$ for some map $k$. Define an ordering on $\mathcal{F}$ as follows: $(D, f) \leqslant_{\mathcal{F}} (E, g)$ if and only if $D \subseteq E$ and $g|_D = f$. Then $\mathcal{F}$ admits an upperbound of $A$. By Zorn's Lemma, there exists a

maximal element $(M,h) \in \mathcal{F}$. Suppose towards contradiction there are elements $a \in A$, $a \notin M$ and $b \in B$, $b \notin h(M)$. Consider the map:

$$h' : M \cup \{a\} \to B \text{ defined by } \begin{cases} h'(M) = h(M) \\ h'(a) = b \end{cases} .$$

This set is clearly injective, and furthermore we have that $(M,h) \leqslant (M \cup \{a\}, h')$. This is a contradiction, hence $M = A$ or $h(M) = B$. If $M = A$, then the injection $A \overset{h}{\hookrightarrow} B$ implies $\text{card}(A) \leqslant \text{card}(B)$. If $h(M) = B$, then the map $B \hookrightarrow M \overset{\iota}{\hookrightarrow} A$ implies $\text{card}(B) \leqslant \text{card}(A)$. □

**Corollary 1.4.9.** $\text{card}(\mathbf{Q}) = \text{card}(\mathbf{N})$.

*Proof.* This follows directly from Example 1.4.4 and Theorem 1.4.8 □

**Definition 1.4.5.** A set $A$ is <u>*countable*</u> if $\text{card}(A) \leqslant \text{card}(\mathbf{N})$. Equivalently, there exists an injection $A \hookrightarrow \mathbf{N}$ and a surjection $\mathbf{N} \twoheadrightarrow A$. If $A$ is countable and infinite, $A$ is called <u>*denumerable*</u> (or more commonly referred to as <u>*countably infinity*</u>).

**Definition 1.4.6.** We say $\text{card}(\mathbf{N}) = \text{card}(\mathbf{Z}) = \text{card}(\mathbf{Q}) := \aleph_0$, called <u>*aleph naught*</u>. We also define $\text{card}(\mathbf{R}) = \mathfrak{c}$, called the <u>*continuum*</u>.

**Example 1.4.5.** By Theorem 1.4.4, $\aleph_0 < \mathfrak{c}$.

**Corollary 1.4.10.** *There does not exist an infinite set $A$ with* $\text{card}(A) < \aleph_0$. *In particular, if $A$ is infinite and countable, then* $\text{card}(A) = \aleph_0$.

*Proof.* By Exercise 1.4.1, $\text{card}(\mathbf{N}) \leqslant \text{card}(A)$, and by definition (since $A$ is countable), $\text{card}(A) \leqslant \text{card}(\mathbf{N})$. So by Theorem 1.4.8, $\text{card}(A) = \text{card}(\mathbf{N}) = \aleph_0$. □

**Example 1.4.6.** $\text{card}(\mathcal{P}(\mathbf{N})) > \text{card}(\mathbf{N}) = \aleph_0$.

**Proposition 1.4.11.** *The countable union of countable sets is countable. More precisely, if $A_i$ is countable for all $i \in \mathbf{N}$, then $\bigcup_{i=1}^{\infty} A_i$ is countable.*

*Proof.* By definition, there exist surjections $\pi_i : \mathbf{N} \to A_i$. Define $\pi : \mathbf{N} \times \mathbf{N} \to \bigcup_{i=1}^{\infty} A_i$ by $\pi(i,j) = \pi_i(j)$. Claim: $\pi$ is onto. Let $x \in \bigcup_{i=1}^{\infty} A_i$, then there exists an $i_0$ with $x \in A_{i_0}$. Since $\pi_{i_0}$ is onto, there exists a $j_0 \in \mathbf{N}$ with $\pi_{i_0}(j_0) = x$. So $\pi(i_0, j_0) = x$, establishing that $\pi$ is surjective as well. Therefore $\text{card}(\bigcup_{i=1}^{\infty} A_i) \leqslant \text{card}(\mathbf{N} \times \mathbf{N}) = \text{card}(\mathbf{N})$. □

**Lemma 1.4.12.** $\text{card}([0,1]) \leqslant \text{card}(2^{\mathbf{N}})$.

*Proof.* Recall that every $\sigma \in [0,1]$ has a binary expansion $\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$, where $\sigma_k \in \{0,1\}$. Consider the map $\varphi : 2^{\mathbf{N}} \to [0,1]$ defined by $\varphi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{2^k}$. Letting $f(k) = \sigma_k$ gives $\varphi$ is surjective. □

**Lemma 1.4.13.** $\text{card}(\mathbf{R}) = \text{card}([0,1])$.

*Proof.* By inclusion $[0,1] \overset{\iota}{\hookrightarrow} \mathbf{R}$, which implies that $\operatorname{card}([0,1]) \leqslant \operatorname{card}(\mathbf{R})$. Recall that $\mathbf{R} \xhookrightarrow{\tan} (0,1) \overset{\iota}{\hookrightarrow} [0,1]$, which implies that $\operatorname{card}(\mathbf{R}) \leqslant \operatorname{card}([0,1])$. Then Theorem 1.4.8 gives the desired result. $\square$

**Lemma 1.4.14.** $\operatorname{card}(2^{\mathbf{N}}) \leqslant \operatorname{card}([0,1])$.

*Proof.* Consider the map $\lambda : 2^{\mathbf{N}} \to [0,1]$ defined by $\lambda(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$. Claim: $\lambda$ is injective. Let $f, g \in 2^{\mathbf{N}}$ with $f \neq g$. Let $k_0$ be the *smallest point $k$ where $f$ and $g$ are different*. So in particular:

$$\begin{aligned} f(1) &= g(1) \\ f(2) &= g(2) \\ &\vdots \\ f(k_0-1) &= g(k_0-1) \\ f(k_0) &\neq g(k_0). \end{aligned}$$

Let:

$$\begin{aligned} t_1 &= \sum_{k>k_0} \frac{f(k)}{3^k} && \text{sum past } k_0 \\ t_2 &= \sum_{k>k_0} \frac{g(k)}{3^k} && \text{sum past } k_0 \\ s_1 &= \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} && \text{sum before } k_0 \\ s_1 &= \sum_{k=1}^{k_0-1} \frac{g(k)}{3^k} && \text{sum before } k_0 \end{aligned}$$

We have that:

$$\begin{aligned} \lambda(f) &= s_1 + \frac{f(k_0)}{3^{k_0}} + t_1 \\ \lambda(g) &= s_2 + \frac{g(k_0)}{3^{k_0}} + t_2 \end{aligned}$$

Because $f$ and $g$ differ at $k_0$, without loss of generality let $f(k_0) = 0$ and $g(k_0) = 1$. Then

$\lambda(g) - \lambda(f) = \frac{1}{3^{k_0}} + t_2 - t_1$. Observe that:

$$
\begin{aligned}
|t_2 - t_2| &= \left| \sum_{k > k_0} \frac{g(k) - f(k)}{3^k} \right| \\
&\leqslant \sum_{k > k_0} \frac{|g(k) - f(k)|}{3^k} && \text{By triangle inequality} \\
&\leqslant \sum_{k > k_0} \frac{1}{3^k} && \text{By comparison test} \\
&= \frac{1}{3^{k_0+1}} \sum_{k \geqslant 0} \frac{1}{3^k} \\
&= \frac{1}{3^{k_0+1}} \cdot \frac{1}{1 - \frac{1}{3}} \\
&= \frac{3}{2 \cdot 3^{k_0+1}} \\
&= \frac{1}{2 \cdot 3^{k_0}} \\
&< \frac{1}{3^{k_0}}.
\end{aligned}
$$

Since $|t_2 - t_2| < \frac{1}{3^{k_0}}$, $\lambda(g) - \lambda(f) \neq 0$, establishing $\lambda$ as an injection. Thus $\operatorname{card}(2^{\mathbf{N}}) \leqslant \operatorname{card}([0,1])$. □

**Theorem 1.4.15.** $\operatorname{card}(2^{\mathbf{N}}) = \operatorname{card}(\mathcal{P}(\mathbf{N})) = \operatorname{card}(\mathbf{R})$.

*Proof.* This follows from Lemma 1.4.12, Lemma 1.4.13, and Lemma 1.4.14. □

# 2
# Ordered Fields

## 2.1 Ordering of $\mathbb{Z}$

**Definition 2.1.1.** Define $\mathbf{Z}^+ = \{n \in \mathbf{Z} \mid n \geqslant_a 0\}$, where $\geqslant_a$ is the *algebraic ordering* from Example 1.2.1. We call $\mathbf{Z}^+$ the cone of positive integers, and they admit the following axioms:

(1) If $m, n \in \mathbf{Z}^+$, then $m + n \in \mathbf{Z}^+$ and $mn \in \mathbf{Z}^+$.

(2) For all $m \in \mathbf{Z}$, $m \in \mathbf{Z}^+$ or $-m \in \mathbf{Z}^+$.

(3) If $m \in \mathbf{Z}^+$ and $-m \in \mathbf{Z}^+$, then $m = 0$.

**Proposition 2.1.1** (Properties of $\leqslant_a$)**.**

*(1) $m \leqslant_a n$ if and only if $n - m \in \mathbf{Z}^+$.*

*(2) If $m \leqslant_a n$ and $p \leqslant_a q$, then $m + p \leqslant_a n + q$.*

*(3) If $m \leqslant_a n$ and $p \in \mathbf{Z}^+$, then $pm \leqslant_a pn$.*

*(4) If $m \leqslant_a n$ then $-n \leqslant_a -m$.*

*(5) $(\mathbf{Z}, \leqslant_a)$ forms a total ordering.*

*(6) If $m >_a 0$ and $mn >_a 0$, then $n >_a 0$.*

*(7) If $m >_a 0$ and $mn \geqslant_a mp$, then $n \geqslant_a p$.*

*Proof.* (5) Let $m, n \in \mathbf{Z}$, since $\mathbf{Z}$ is closed under subtraction $m - n \in \mathbf{Z}$. So either $m - n \in \mathbf{Z}^+$ or $n - m \in \mathbf{Z}^+$. Then by (1) $n \leqslant_a m$ or $m \leqslant_a n$. Thus $(\mathbf{Z}, \leqslant_a)$ is a total ordering.

(6) We have $mn >_a 0$ with $m >_a 0$. If $n = 0$, we are done. So now assume $n \neq 0$. Then either $n \in \mathbf{Z}^+$ or $-n \in \mathbf{Z}^+$. If $-n \in \mathbf{Z}^+$, then $m(-n) = -(mn) \in \mathbf{Z}^+$. But we had assumed $mn >_a 0$; i.e., $mn \in \mathbf{Z}^+$, hence it must be the case that $mn = 0$, a contradiction. Therefore it must be that $n \in \mathbf{Z}^+$. □

## 2.2 Ordering of $\mathbb{Q}$

**Proposition 2.2.1.** *Define $Q := \mathbf{Z} \times \mathbf{N}$. Show that $\sim$ forms an equivalence relation, where $(a, b) \sim (c, d)$ if and only if $ad = bc$.*

*Proof.* I dont wanna do this □

**Definition 2.2.1.** The set of equivalence classes of $Q$ is $\mathbf{Q} = Q/\sim = \{[(a, b)] \mid (a, b) \in Q\}$. We call this set the rational numbers, and denote the equivalence classes $[(a, b)]$ as $\frac{a}{b}$.

**Proposition 2.2.2.** *The operations*

$$+ : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q} \text{ defined by } [(a,b)] + [(c,d)] = [(ad+bc, bd)]$$
$$\cdot : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q} \text{ defined by } [(a,b)] \cdot [(c,d)] = [(ac, bd)]$$

*are well-defined. Furthermore,* $(\mathbf{Q}, +, \cdot)$ *forms a field.*

*Proof.* I dont wana □

**Lemma 2.2.3.** *There is an injective map* $\mathbf{Z} \overset{j}{\hookrightarrow} \mathbf{Q}$ *defined by* $j(n) = \frac{n}{1}$ *satisfying the properties*

$$j(n+m) = j(n) + j(m)$$
$$j(nm) = j(n)j(m).$$

*Proof.* Note that $j(n) = j(m)$ if and only if $\frac{n}{1} + \frac{m}{1}$. By definition this is equivalent to $n = m$, hence $j$ is injective.

Observe that $j(n+m) = \frac{n+m}{1} = \frac{n}{1} + \frac{m}{1} = j(n) + j(m)$ and $j(nm) = \frac{nm}{1} = \frac{n}{1} \cdot \frac{m}{1} = j(n)j(m)$. □

**Theorem 2.2.4.** $(\mathbf{Q}, \leqslant_Q)$ *is a total ordering, where* $\leqslant_Q$ *is a well-defined ordering defined by* $\frac{a}{b} \leqslant_Q \frac{c}{d}$ *if and only if* $ad \leqslant_a bc$ *in* $(\mathbf{Z}, \leqslant_a)$*. Furthermore, the map* $j : \mathbf{Z} \hookrightarrow \mathbf{Q}$ *is order preserving, that is, if* $n \leqslant_a m$ *in* $(\mathbf{Z}, \leqslant_a)$*, then* $j(n) \leqslant_Q j(m)$ *in* $(\mathbf{Q}, \leqslant_Q)$*.*

*Proof.* i REALLY dont □

**Definition 2.2.2.** Define $\mathbf{Q}_+ := \{q \in \mathbf{Q} \mid q \geqslant_Q 0\}$ as the <u>*cone of positive rationals*</u>, and they admit the following axioms:

(1) If $q_1, q_2 \in \mathbf{Q}^+$, then $q_1 + q_2 \in \mathbf{Z}^+$ and $q_1 q_2 \in \mathbf{Z}^+$.

(2) For all $q \in \mathbf{Q}$, $q \in \mathbf{Q}^+$ or $-q \in \mathbf{Q}^+$.

(3) If $q \in \mathbf{Q}^+$ and $-q \in \mathbf{Q}^+$, then $q = 0$.

(4) $q_1 \leqslant_Q q_2$ if and only if $q_2 - q_1 \in \mathbf{Q}^+$.

**Proposition 2.2.5.** *Let* $r, s, t, u \in \mathbf{Q}$

*(1) If* $r \leqslant_Q s$ *and* $t \leqslant_Q u$*, then* $r + t \leqslant_Q s + u$*.*

*(2) If* $r \leqslant_Q s$ *and* $t \geqslant_Q 0$*, then* $tr \leqslant_Q ts$*.*

*Proof.* do this shi later □

## 2.3 Rings and Fields

**Definition 2.3.1.** A *ring* is a non-empty set $R$ equipped with two binary operations:

$$R \times R \xrightarrow{a} R \text{ defined by } a(r,s) = r + s$$
$$R \times R \xrightarrow{m} R \text{ defined by } m(r,s) = rs,$$

such that they admit the following axioms:

(1) $R$ is an *abelian group* under addition:

   (i) $r + (s + t) = (r + s) + t$ for all $r, s, t \in R$,

   (ii) there exists an element $0_R \in R$ with $r + 0_R = r = 0_R = r$ for all $r \in R$,

   (iii) For all $r \in R$ there exists an $s \in R$ such that $r + s = 0_R = s + r$ (such an $s$ is unique, and is denoted $-r$),

   (iv) $r + s = s + r$ for all $r, s \in R$.

(2) $r(st) = (rs)t$ for all $r, s, t \in R$,

(3) $(r + s)t = rt + rs$ and $r(s + t) = rs + rt$ for all $r, s, t \in R$.

If $R$ contains an element $1_R$ such that $1_R r = r = r 1_R$, then we say $R$ is *unital*. If $rs = sr$ for all $r, s \in R$, then we say $R$ is *commutative*. If $R$ is a unital ring such that $1_R \neq 0_R$ *and* for all $r \in R$ there exists an $s \in R$ such that $rs = 1_R = sr$ (such an $s$ is unique, and denoted $r^{-1}$), then we say $R$ is a *division ring*.

**Definition 2.3.2.** A *field* is a commutative division ring.

**Example 2.3.1.**

(1) $\mathbf{Q}$ is a field.

(2) $\mathbf{Z}/p\mathbf{Z}$ is a field.

(3) $\mathbf{C}_{\mathbf{Q}} = \{r + si \mid r, s \in \mathbf{Q}, i^2 = -1\}$ with addition and multiplication defined by

$$(r + si) + (t + ui) := (r + t) + (s + u)i$$
$$(r + si)(t + ui) := (rt - su) + (ru + st)i$$

is a field. We call this set the *complex rationals*.

**Definition 2.3.3.** An *ordered field* is a field $F$ equipped with a total ordering $\leqslant_F$ such that:

(1) If $x \leqslant_F y$ and $u \leqslant_F v$, then $x + u \leqslant_F y + v$.

(2) If $x \leqslant_F y$ and $z \geqslant_F 0$, then $xz \leqslant_F zy$.

We similarly define $F^+ = \{x \in F \mid x \geqslant_F 0\}$ as the *cone of positive elements*.

**Proposition 2.3.1.** *Let $(F, \leqslant_F)$ be an ordered field.*

*(1) If $x, y \in F^+$, then $x + y \in F^+$ and $xy \in F^+$.*

*(2) If $x \in F$, then $-x \in F^+$ or $x \in F^+$.*

*(3) If $x, -x \in F^+$, then $x = 0$.*

*Proof.* need to do □

**Example 2.3.2.**

(1) $\mathbf{Q}$ is an ordered field.

(2) Is $\mathbf{C}_\mathbf{Q}$ an ordered field?

**Proposition 2.3.2.** *Let $(F, \leqslant)$ be an ordered field with $1_F \neq 0_F$.*

*(1) For all $a \in F$, $a^2 \in F$.*

*(2) $0, 1 \in F^+$.*

*(3) If $n \in \mathbf{N}$, then $n \cdot 1_F := \underbrace{1_F + 1_F + \ldots + 1_F}_{n \ \text{times}}$, implying $n \cdot 1_F \in F^+$.*

*(4) If $x \in F^+$ and $x \neq 0$, then $x^{-1} \in F^+$.*

*(5) If $xy \in F^+$ and $xy \neq 0$, then $x, y \in F^+$ or $-x, -y \in F^+$.*

*(6) If $0 < x \leqslant y$, then $y^{-1} \leqslant x^{-1}$.*

*(7) If $x \leqslant y$, then $-y \leqslant -x$.*

*(8) If $x \geqslant 1_F$, then $x^2 \geqslant x$.*

*(9) If $x \leqslant 1_F$, then $x^2 \leqslant x$.*

*Proof.* (1) If $a \in F^+$, then $a \cdot a = a^2 \in F^+$. If $-a \in F^+$, then $(-a) \cdot (-a) = a^2 \in F^+$.

(2) From part (1) we have that $0 = 0 \cdot 0 \in F^+$. Similarly, $1 = 1 \cdot 1 \in F^+$ and $(-1) \cdot (-1) \in F^+$

(3) Since $F^+$ is closed under addition, we can inductively show that $n \cdot 1 = 1 + 1 + \ldots + 1 \in F^+$.

(4) Suppose towards contradiction $x^{-1} \notin F^+$. Then $-(x^{-1}) \in F^+$, so $(-(x^{-1})) \cdot x = -1(x^{-1} \cdot x) = -1 \in F^+$. But $-1, 1 \in F^+$ implies $1 = 0$, a contradiction. Thus $x^{-1} \in F^+$.

(6) $y \geqslant x > 0$ implies $x, y \in F^+$. So $x^{-1}, y^{-1} \in F^+$. Then $y^{-1}xx^{-1} \leqslant y^{-1}yx^{-1}$, and simplifying yields $y^{-1} \leqslant x^{-1}$. finish the rest (i'm not going to) □

# 3

# The Real Numbers

## 3.1 The Completion of $\mathbb{Q}$

**Definition 3.1.1.** A <u>*Dedekind cut*</u> is a nonempty subset $D$ of $\mathbf{Q}$ with the following properties:

(1) $D \neq \mathbf{Q}$;

(2) If $b \in D$, then $a \in D$ for all $a \in \mathbf{Q}$ with $a < b$;

(3) D does not contain a largest element.

**Example 3.1.1.** The following examples are Dedekind cuts:

(1) $\{a \in \mathbf{Q} \mid a < 3\}$ (the set of all rational numbers less than 3).

(2) $\{a \in \mathbf{Q} \mid a < 0 \text{ or } a^2 < 2\}$ (the set of all rational numbers less than $\sqrt{2}$).

(3) $\{a \in \mathbf{Q} \mid a < 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \text{ for some } n \in \mathbf{Z}^+\}$ (the set of all rational numbers less than $e$).

**Definition 3.1.2.** Let $C$ and $D$ be Dedekind cuts.

will probably not finish this

## 3.2 Ordering of $\mathbb{R}$

**Axiom 1.** $\mathbf{R}$ *is an ordered field.*

**Proposition 3.2.1.** $\mathbf{Q}^+ \subseteq \mathbf{R}^+$.

*Proof.* If $x \in \mathbf{Q}^+$, then $x = \frac{p}{q}$ with $p \in \mathbf{Z}^+$ and $q \in \mathbf{N}$. Write $p = \underbrace{1 + 1 + \ldots + 1}_{p \text{ times}}$, then $p \in \mathbf{R}^+$. Similarly, write $q = \underbrace{1 + 1 + \ldots + 1}_{q \text{ times}}$. Then $q \in \mathbf{R}^+$, which implies that $q^{-1} \in \mathbf{R}^+$. Hence $\frac{p}{q} \in \mathbf{R}^+$, establishing $\mathbf{Q}^+ \subseteq \mathbf{R}^+$. □

**Proposition 3.2.2.** *The maps $Z \overset{j}{\hookrightarrow} \mathbf{Q} \overset{i}{\hookrightarrow} \mathbf{R}$ are order embeddings (defined in Lemma 2.2.3 and Theorem 2.2.4).*

*Proof.* Suppose $i(q_1) \leqslant_Q i(g_2)$. Then $q_1 \leqslant_{\mathbf{R}} q_2$, hence $q_2 - q_1 \in \mathbf{R}^+$. Now If $q_2 - q_2 \in \mathbf{Q}^+$, then $q_2 - q_1 \in \mathbf{R}^+$. Hence $q_1 \leqslant_{\mathbf{R}} q_2$. wtf is this saying? □

**Proposition 3.2.3.** *Let $a, b \in \mathbf{R}$. If $a \leqslant b$ (or $a < b$), then $a \leqslant \frac{1}{2}(a + b) \leqslant b$ (or $a < \frac{1}{2}(a + b) < b$).*

*Proof.* By the order axioms, $a \leqslant b$ implies $a + a \leqslant a + b$. So $2a \leqslant a + b$, which is equivalent to $a \leqslant \frac{1}{2}(a + b)$. Similarly, $a + b \leqslant b + b$, which similarly gives $\frac{1}{2}(a + b) \leqslant b$, establishing the proposition. □

**Corollary 3.2.4.** *Given $b > 0$, we have $0 < \frac{1}{2}b < b$.*

*Proof.* From Proposition 3.2.3, setting $a = 0$ yields the desired result. □

**Proposition 3.2.5.** *Suppose $a \in \mathbf{R}$. For all $\epsilon > 0$, if $0 \leqslant a \leqslant \epsilon$, then $a = 0$.*

*Proof.* If $a = 0$ we are done. If $a > 0$, by Corollary 3.2.4 $0 \leqslant \frac{1}{2}a \leqslant a$. Pick $\epsilon = \frac{1}{2}a$, then $a \leqslant \frac{1}{2}a$, a contradiction. Thus $a = 0$. □

**Definition 3.2.1.** Let $a_1, a_2, \ldots, a_n > 0$. The <u>*arithmetic mean*</u> is $\frac{1}{2}\left(\sum_{j=1}^{n} a_j\right)$. The <u>*geometric mean*</u> is $\left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}$.

**Proposition 3.2.6** (AM-GM Inequality)**.** *For all $a_1, a_2, \ldots, a_n \geqslant 0$, then $\left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}} \leqslant \frac{1}{2}\left(\sum_{j=1}^{n} a_j\right)$.*

*Proof.* We will only prove the $n = 2$ case. Consider the fact that $(a_1 - a_2)^2 \geqslant 0$, and expanding gives $a_1^2 - 2a_1a_2 + a_2^2$. So $2a_1a_2 \leqslant a_1^2 + a_2^2$. Adding $2a_1a_2$ to both sides yields $4a_1a_2 \leqslant a_1^2 + 2a_1a_2 + a_2^2$, which is equivalent to $4a_1a_2 \leqslant (a_1 + a_2)^2$. Then simplifying yields the desired result of $(a_1a_2)^{\frac{1}{2}} \leqslant \frac{1}{2}(a_1 + a_2)$. □

**Proposition 3.2.7** (Bernoulli's Inequality)**.** *If $x > -1$, then $(1 + x)^n \geqslant 1 + nx$ for all $n \in \mathbf{N}_0$.*

*Proof.* We proceed with induction with base case $n = 0$ and $n = 1$; these hold by inspection. Assume the inequality holds true for $n = k$. For $n = k + 1$:

$$\begin{aligned}(1 + x)^{k+1} &= (1 + x)^k(1 + x) \\ &\geqslant (1 + nx)(1 + x)^{1} \\ &= 1 + (n + 1)x + nx^2 \\ &\geqslant 1 + (n + 1)x.\end{aligned}$$

□

**Proposition 3.2.8** (Cauchy-Schwartz Inequality)**.** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{R}^n$. Then:*

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leqslant \left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}.$$

[1] Because order is preserved under multiplication by positive elements.

*Proof.* Consider the map $F : \mathbf{R}^n \to \mathbf{R}^n$ defined by $F(t) = \sum_{j=1}^{n}(a_j - b_j t)^2$. Note that $\sum_{j=1}^{n}(a_j - b_j t)^2 \geqslant 0$. Observe that:

$$\begin{aligned}\sum_{j=1}^{n}(a_j - b_j t)^2 &= \sum_{j=1}^{n}(a_j^2 - 2a_j b_j t + b_j^2 t^2) \\ &= \sum_{j=1}^{n} a_j^2 - \sum_{j=1}^{n} 2a_j b_j t + \sum_{j=1}^{n} b_j^2 t^2.\end{aligned}$$

This is a quadratic equation, and since $F(t) \geqslant 0$, the discriminant will be less than or equal to 0. Hence:

$$\Delta = \left(\sum_{j=1}^{n} 2a_j b_j\right)^2 - 4\left(\sum_{j=1}^{n} b_j^2\right)\left(\sum_{j=1}^{n} a_j^2\right) \leqslant 0.$$

Simplifying gives:

$$\left(\sum_{j=1}^{n} 2a_j b_j\right)^2 \leqslant 4\left(\sum_{j=1}^{n} b_j^2\right)\left(\sum_{j=1}^{n} a_j^2\right).$$

Pulling 2 out from the left-hand side, dividing both sides by 4, and then square-rooting gives the desired result. □

**Question.** When do we have equality?

*Answer.* When $\Delta = 0$, there exists a $t_0 \in \mathbf{R}$ with $F(t_0) = 0$. So $\sum_{j=1}^{n}(a_j - b_j t_0) = 0$ implies $a_j - b_j t_0 = 0$ for all $j$. Hence there is equality only when $a_j = b_j t_0$ for all $j$.

**Proposition 3.2.9** (Triangle Inequality)**.** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{R}^n$. Then:*

$$\left(\sum_{j=1}^{n}(a_j + b_j)^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}.$$

*Proof.* Observe that:

$$\begin{aligned}\sum_{j=1}^{n}(a_j + b_j)^2 &= \sum_{j=1}^{n}(a_j^2 + 2a_j b_j + b_j^2) \\ &= \sum_{j=1}^{n} a_j^2 + \sum_{j=1}^{n} 2a_j b_j + \sum_{j=1}^{n} b_j^2 \\ &\leqslant \sum_{j=1}^{n} a_j^2 + 2\left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}} + \sum_{j=1}^{n} b_j^2. \\ &= \left(\left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}}\right)^2.\end{aligned}$$

Squaring both sides gives the desired result. □

## 3.3 Metrics and Norms on $\mathbb{R}^n$

**Definition 3.3.1.** The *absolute value* is a function $|\cdot| : \mathbf{R} \to \mathbf{R}$ defined by:

$$|x| = \begin{cases} x, & x \in \mathbf{R}^+ \\ -x, & -x \in \mathbf{R}^+. \end{cases}$$

**Proposition 3.3.1.** *Let $a, b \in \mathbf{R}$ and $\delta > 0$.*

*(1)* $|ab| = |a||b|$.

*(2)* $|a|^2 = |a^2|$.

*(3)* $|-a| = |a|$.

*(4)* $|a| \in \mathbf{R}+$.

*(5)* $-|a| \leqslant a \leqslant |a|$.

*(6)* $|a| \leqslant \delta$ *if and only if* $-\delta \leqslant a \leqslant \delta$.

*(7)* $|a+b| \leqslant |a| + |b|$.

*(8)* $|a-b| \leqslant |a| + |b|$.

*(9)* $||a| - |b|| \leqslant |a-b|$.

*Proof.* do later □

**Lemma 3.3.2.** $\pm x \leqslant \delta$ *if and only if* $|x| \leqslant \delta$.

*Proof.* do lter □

**Lemma 3.3.3.** *$A \subseteq \mathbf{R}$ is bounded if and only if there exists an $r > 0$ such that $|a| < r$ for all $a \in A$.*

*Proof.* Suppose $A \subseteq \mathbf{R}$ is bounded. Then there exists an $l, u \in \mathbf{R}$ with $l \leqslant a \leqslant u$ for all $a \in A$. We have that:

$$-|l| \leqslant l \leqslant a \leqslant u \leqslant |u|.$$

Let $r = \max\{|l|, |u|\} \geqslant 0$. So $-r \leqslant |l| \leqslant a \leqslant |u| \leqslant r$. Thus $|a| \leqslant r$.

Conversely, suppose there exists an $r > 0$ with $|a| \leqslant r$ for all $a \in A$. Then $-r \leqslant a \leqslant r$ for all $a \in A$, hence $A$ is bounded. □

**Definition 3.3.2.** A function $f : D \to \mathbf{R}$ is *bounded* if $\operatorname{im}(f) \subseteq \mathbf{R}$ is a bounded subset. Equivalently, there exists a $c > 0$ such that $|f(x)| < c$ for all $x \in D$.

**Example 3.3.1.** Consider the function $f : [3,7] \to \mathbf{R}$ defined by $f(x) = \frac{x^2+2x+1}{x-1}$. Since $3 \leqslant x \leqslant 7$, observe that:

$$\begin{aligned} |x^2 + 2x + 1| &\leqslant |x^2| + |2x| + 1 \\ &= |x|^2 + 2|x| + 1 \quad \text{Evaluate at 7} \\ &= 64 \end{aligned}$$

Likewise, $3 \leqslant x \leqslant 7$ implies $|x-1| \geqslant 2$, hence $\frac{1}{|x-1|} \leqslant \frac{1}{2}$. Together, we have that:

$$\left|\frac{x^2+2x+1}{x-1}\right| \leqslant \frac{64}{2} = 32.$$

**Definition 3.3.3.** Let $s, t \in \mathbf{R}$. We define the *distance* between $s$ and $t$ as $d(s,t) = |s-t|$.

**Definition 3.3.4.** Let $X$ be a nonempty set equipped with a map $d : X \times X \to \mathbf{R}^+$. We say $(X,d)$ is a *semi-metric* if for all $x, y, z \in X$,

(1) $d(x,y) = d(y,x)$,

(2) $d(x,z) \leqslant d(x,y) + d(y,z)$, and

(3) $d(x,x) = 0$.

We say $(X,d)$ is a *metric space* if it satisfies the additional axiom:

(4) $d(x,y) = 0$ implies $x = y$.

**Proposition 3.3.4.**

*(1)* $(\mathbf{R}, d_1(s,t) = |s-t|)$ *is a metric space.*

*(2)* $\left(\mathbf{R}^n, d_1(\vec{x},\vec{y}) = \sum_{j=1}^n |y_j - x_j|\right)$ *is a metric space.*

*(3)* $\left(\mathbf{R}^n, d_\infty(\vec{x},\vec{y}) = \max_{j=1}^n \{|y_j - x_j\})\right)$ *is a metric space.*

*(4)* $\left(\mathbf{R}^n, d_2(\vec{x},\vec{y}) = \left(\sum_{j=1}^n |y_j - x_j|^2\right)^{\frac{1}{2}}\right)$ *is a metric space.*

*(5)* $\left(\mathbf{R}^n, d_p(\vec{x},\vec{y}) = \left(\sum_{j=1}^n |y_j - x_j|^p\right)^{\frac{1}{p}}\right)$ *for some* $p \in \mathbf{Q}$ *is a metric space.*

*Proof.* (1) We have $d(s,t) = |s-t| = |t-s| = d(t,s)$. Similarly, $d(s,r) = |s-r| = |s-t+t-r| \leqslant |s-t| + |t-r| = d(s,t) + d(t,r)$. Clearly $d(s,s) = |s-s| = 0$. Lastly, if $d(s,t) = 0$, then $|s-t| = 0$, which is equivalent to $s - t = 0$; i.e., $s = t$. Thus $(\mathbf{R}, d_1)$ is a metric space.

(4) Axioms 2 and 3 of metric spaces are clearly satisfied. If $d_2(\vec{x},\vec{y}) = 0$ then $|y_j - x_j|^2 = 0$ for all $j$. Hence $y_j - x_j = 0$; i.e., $y_j = x_j$ for all $j$, establishing axiom 4. Observe that:

$$\begin{aligned}
d_2(\vec{x},\vec{z}) &= \left(\sum_{j=1}^{n} |z_j - x_j|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j=1}^{n} |z_j - y_j + y_j - x_j|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j=1}^{n} (z_j - y_j + y_j - x_j)^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{j=1}^{n} (z_j - y_j)^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} (y_j - x_j)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j=1}^{n} |z_j - y_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} |y_j - x_j|^2\right)^{\frac{1}{2}} \\
&= d_2(\vec{x},\vec{y}) + d_2(\vec{y},\vec{z}).
\end{aligned}$$

Thus $(\mathbf{R}^n, d_2)$ is a metric space. □

**Definition 3.3.5.** Let $(X, d)$ be a metric space.

(1) The <u>*open ball*</u> centered at $x_0$ with radius $\delta > 0$ is $U(x_0,\delta) = \{y \in X \mid d(y,x_0) < \delta\}$.

(2) The <u>*closed ball*</u> centered at $x_0$ with radius $\delta > 0$ is $B(x_0,\delta) = \{y \in X \mid d(y,x_0) \leqslant \delta\}$.

(3) A subset $A \subseteq X$ is called <u>*open*</u> if for all $a \in A$, there exists a $\delta > 0$ such that $U(a,\delta) \subseteq A$.

(4) A subset $C \subseteq X$ is called <u>*closed*</u> if $\mathrm{compl}(C) = X \setminus C$ is open.

**Example 3.3.2.** Consider $X = \mathbf{R}$ and $d(s,t) = |s - t|$. Observe that:

$$\begin{aligned}
U(t,\delta) &= \{s \in \mathbf{R} \mid d(s,t) < \delta\} \\
&= \{s \in \mathbf{R} \mid |s - t| < \delta\} \\
&= \{s \in \mathbf{R} \mid -\delta < s - t < \delta\} \\
&= \{s \in \mathbf{R} \mid -\delta + t < s < \delta + t\} \\
&= (t - \delta, t + \delta).
\end{aligned}$$

It follows similarly that $B(t,\delta) = [t - \delta, t + \delta]$.

**Proposition 3.3.5.** *If $I$ is an open interval, then $I$ is open.*

*Proof.* Let $I = (a,b)$. Let $x \in I$. Let $\delta_x = \min\{x - a, b - x\} > 0$. Now let $t \in V_\delta(x)$. Then $t \in (x - \delta, x + \delta)$. Case 1: $\min\{x - a, b - x\} = x - a$. Then $x - (x - a) < t < x + x - a$, idk how to do this □

# 4

# Supremum, Infimum, and Completeness

## 4.1 Supremum and Infimum

**Theorem 4.1.1.** *Let $\emptyset \neq A \subseteq \mathbf{R}$. Let $u$ be an upperbound for $A$. The following are equivalent:*

*(1)* $u = \sup(A)$.

*(2) If $t < u$, then there exists an $a_t \in A$ with $t < a_t$.*

*(3) For all $\epsilon > 0$, there exists an $a_\epsilon \in A$ such that $u - \epsilon < a_\epsilon$.*

*Proof.* $[(1) \implies (2)]$ Assume $u = \sup(A)$. Let $t < u$. Suppose towards contradiction there does not exist and $a \in A$ with $a > t$. Then $a \leqslant t$ for all $a \in A$. But this implies $t$ is an upperbound of $A$ less than $u$, which is a contradiction because $u$ is the least upper bound. $[(2) \implies (3)]$ Given $\epsilon > 0$, let $t = u - \epsilon$. Then applying (2) gives the desired result. $[(3) \implies (1)]$ We know $u$ is an upperbound of $A$, we aim to show that it is the least upperbound. Let $v$ be an upperbound for $A$ with $v < u$. Pick $\epsilon = u - v > 0$. By (3), there exists an $a_\epsilon \in A$ such that $u - (u - v) < a_\epsilon$. So $v < a_\epsilon$, which is a contradiction ( $v$ is an upperbound, how can it be smaller than an element of $A$?). □

**Example 4.1.1.** Claim: $\sup([0,1)) = 1$. If $s \in [0,1)$, by definition $s < 1$, so 1 is an upper bound for $[0,1)$. Given $t < 1$, set $\delta = 1 - t > 0$. Then $0 < \frac{\delta}{2} < \delta$ this is not trivial, have to show $\delta - \delta/2$ is positive. This gives:

$$t < t + \frac{\delta}{2} < t + \delta = 1.$$

Pick $a_t = t + \frac{\delta}{2}$. By (2) of Theorem 4.1.1, $a_t \in [0,1)$, hence $1 = \sup([0,1))$.

**Proposition 4.1.2.** *Let $A, B \subseteq \mathbf{R}$ and $a \leqslant b$ for all $a \in A$ and $b \in B$. Then* $\sup(A) \leqslant \inf(B)$.

*Proof.* Fix a point $b_0 \in B$. Then $a \leqslant b_0$ for all $a \in A$. Then $b_0$ is an upperbound for $A$. This gives $u := \sup(A) \leqslant b_0$. But since $b_0$ was arbitrary, we have $u \leqslant b$ for all $b \in B$. So $u$ is a lower bound for $B$, therefore $u \leqslant \inf(B)$. □

**Axiom 2** (Completeness of $\mathbf{R}$). *Given any nonempty subset $A \subseteq \mathbf{R}$ which is bounded above,* $\sup(A)$ *exists.*

**Lemma 4.1.3.** *For $A \subseteq \mathbf{R}$ which is bounded below,* $\sup(-A) = -\inf(A)$.

*Proof.* If $A$ is bounded below, then $-A$ is bounded above. Then $\sup(-A)$ exists, define it as $u$. So for all $a \in A$, $-a \leqslant u$. Hence $-u$ is a lower bound for $A$. Suppose $v$ is another lower bound for $A$. Then $v \leqslant a$ for all $a \in A$. So $-v \geqslant -a$ for all $a \in A$. Thus $-v$ is an upper bound of $-A$. Therefore, since $u$ is the least upper bound, $-v \geqslant u$; i.e., $-u \geqslant v$. Thus $-u = \inf(A)$. □

**Axiom 3** (Well-Ordering Princple)**.** *Every nonempty subset $A \subseteq \mathbf{N}$ contains a least element.*

**Proposition 4.1.4** (Arcimedean Property 1)**.** *If $x \in \mathbf{R}$, then there exists $n_x \in \mathbf{N}$ with $x < n_x$.*

*Proof.* Suppose not. That is, suppose $n \leqslant x$ for all $n \in \mathbf{N}$. Then $x$ is an upper bound for $\mathbf{N}$. Thus $\sup(A) := u$ exists. From part (3) of Theorem 4.1.1, take $\epsilon = 1$. Then there exists an $n \in \mathbf{N}$ such that $u - 1 < n$. So $u < n + 1 \in \mathbf{N}$, which is a contradiction. □

**Proposition 4.1.5** (Archimedean Property 2)**.** *If $t > 0$, there exists $n_t \in \mathbf{N}$ with $\frac{1}{n_t} < t$.*

*Proof.* From Arcimedean Property 1, pick $x = \frac{1}{t}$. □

**Corollary 4.1.6.** *Given $t > 0$, there exists $m \in \mathbf{N}$ with $\frac{1}{2^m} < t$.*

*Proof.* By Archimedean Property 2 there exists an $n \in \mathbf{N}$ with $\frac{1}{n} < t$. Claim: $\frac{1}{2^n} < \frac{1}{n}$. It suffices to show that $2^n > n$. Proposition 1.4.6 gives $\operatorname{card}(\{1, 2, \ldots, n\}) < \operatorname{card}(\mathcal{P}(\{1, 2, \ldots, n\}))$. Then Exercise 1.4.2 gives:

$$n = \operatorname{card}(\{1, 2, \ldots, n\}) < \operatorname{card}(\mathcal{P}(\{1, 2, \ldots, n\})) = 2^n.$$

Alternatively, Bernoulli's Inequality gives $(1 + 1)^n \geqslant 1 + n$. Hence $2^n > n$. □

**Example 4.1.2.**

(1) Claim: $\inf\left\{\frac{1}{n} \mid n \in N\right\} = 0$. Note that 0 is indeed a lower bound because $0 < \frac{1}{n}$ for all $n \in \mathbf{N}$. Suppose $t$ is another lower bound. If $t \leqslant 0$, then we are done. If $t > 0$, by the Archimedean Property there exists an $n_t \in \mathbf{N}$ such that $\frac{1}{n_t} < t$, which is a contradiction (because we asserted that $t$ is a lower bound, and $\frac{1}{n_t} \in \inf\{\frac{1}{n} \mid n \in N\}$). Thus $\inf\left\{\frac{1}{n} \mid n \in N\right\} = 0$.

(2) Claim: $\inf\left\{\frac{1}{2^m} \mid m \in N\right\} = 0$. This follows from the above example and previous corollary.

**Corollary 4.1.7.** *Let $x \in \mathbf{R}$, Then there exists $n_x \in \mathbf{Z}$ with $n_x - 1 \leqslant x < n_x$.*

*Proof.* Case 1: $x \geqslant 0$. Let $S_x = \{n \in \mathbf{N} \mid x < n\}$. By Arcimedean Property 1 $S_x \neq 0$. By the Well-Ordering Princple, there exists a least element in this set, call it $n_x$. Since $n_x \in S_x$, it must be the case that $x < n_x$. But since $n_x$ is the least element, $n_x - 1 \notin S_x$. Since $S_x$ is the set of all natural numbers with lower bound $x$, $n_x - 1$ is not bounded below by $x$. Whence $n_x - 1 \leqslant x$.

Case 2: $x < 0$. Define $S_{-x} = \{n \in \mathbf{N} \mid n < -x\}$. As a consequence of the Well-Ordering Princple, any subset of the integers which is bounded above admits a greatest element, define it to be $n_{-x} \in \mathbf{Z}$. Then $n_{-x} + 1 \notin S_{-x}$, hence $n_{-x} < -x \leqslant n_{-x} + 1$. This establishes $-n_{-x} - 1 \leqslant x < -n_{-x}$. □

**Definition 4.1.1.** Let $I$ be an open interval. A subset $D \subseteq \mathbf{R}$ is <u>dense</u> if $I \cap D \neq \emptyset$.

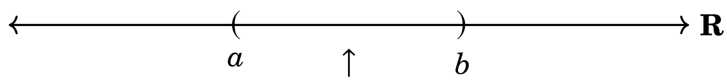
**Theorem 4.1.8.** $\mathbf{Q} \subseteq \mathbf{R}$ *is dense.*

*Proof.* Let $I$ be an open interval. Then there exists $a, b \in \mathbf{R}$ with $(a, b) \subseteq I$. We have that $b - a > 0$. By Archimedean Property 2 there exists $n \in \mathbf{N}$ with $\frac{1}{n} < b - a$. So $1 + na < nb$. By Corollary 4.1.7, there exists $m \in \mathbf{Z}$ with $m - 1 \leqslant na < m$. Equivalently, we have that $a < \frac{m}{n}$. We also have that $m \leqslant na + 1 < nb$, which yields $\frac{m}{n} < b$. Thus $\frac{m}{n} \in (a, b) \cap \mathbf{Q}$. □

**Corollary 4.1.9.** $\mathbf{R} \setminus \mathbf{Q} \subseteq \mathbf{R}$ *is dense.*

*Proof.* Let $a < b$. Consider $a' = a\sqrt{2}$ and $b' = b\sqrt{2}$. Then $a' < b'$. By Theorem 4.1.8, there exists a $q \in \mathbf{Q}$ with $a' < q < b'$. Thus $a < \frac{q}{\sqrt{2}} < b$. Since $\frac{q}{\sqrt{2}} \notin \mathbf{Q}$, the corollary is established.

Alternatively, observe the following picture:




If there is not an irrational number between $(a, b)$, then $(a, b) \subseteq \mathbf{Q}$, which is a contradiction. □

**Theorem 4.1.10.** *There exists a unique positive number $x$ with $x^2 = 2$.*

*Proof.* Consider the set $S = \{t \in \mathbf{R} \mid t > 0, t^2 < 2\}$. Note that $S \neq 0$ because $1 \in S$. If $t \geqslant 2$, then $t^2 \geqslant 2t > 4$, meaning it would not be an element of $S$. So $S$ is bounded above by 2. Hence there exists $u := \sup(S)$.

Scratchwork: Assume $u^2 < 2$. Find a sufficiently small $n$ so that $(u + \frac{1}{n})^2 \in S$; i.e., $(u + \frac{1}{n})^2 < 2$. Solving for $n$ yields:

$$u^2 + \frac{2u}{n} + \frac{1}{n^2} < 2$$

$$\Longleftrightarrow$$

$$\frac{2u}{n} + \frac{1}{n^2} < 2 - u^2$$

$$\Longleftrightarrow$$

$$\frac{1}{n}\left(2u + \frac{1}{n}\right) < 2 - u^2$$

$$\Longleftrightarrow$$

$$\frac{1}{n}(2u + 1) < 2 - u^2$$

$$\Longleftrightarrow$$

$$\frac{1}{n} < \frac{2 - u^2}{2u + 1} \in \mathbf{R}^+ \setminus \{0\}$$

If $u^2 < 2$, then $\frac{2-u^2}{2u+1} > 0$. By Archimedean Property 2, there exists an $n \in \mathbf{N}$ with $\frac{1}{n} < \frac{2-u^2}{2u+1}$. Simplifying yields $(u + \frac{1}{n})^2 < 2$, or equivalently $u + \frac{1}{n} \in S$, which is a contradiction. It must be the case that $u^2 \geqslant 2$; i.e., $u^2 - 2 \geqslant 0$. Now since $u = \sup(S)$, for all $m \in \mathbf{N}$, there exists $t_m \in S$ with $u - \frac{1}{m} < t_m$. We have that $(u - \frac{1}{m})^2 < t_m^2 < 2$. This simplifies to $u^2 - 2 < \frac{2u}{m} - \frac{1}{m^2} < \frac{2u}{m}$, or equivalently $\frac{u^2-2}{2u} < \frac{1}{m}$. But if $\frac{u^2-2}{2u} < \frac{1}{m}$ for all $m \in \mathbf{N}$, it must be that $\frac{u^2-2}{2u} = 0$, hence $u^2 = 2$.

Lastly we show that $u^2$ is unique. Suppose $u^2 = 2 = v^2$. Since $u, v \geqslant 0$, $(u^2 - v^2) = 0$. Then $(u-v)(u+v) = 0$. If $u + v = 0$, then $u = 0$ and $v = 0$, which is a contradiction. So $u - v = 0$ implies $u = v$. □

*Remark.* Picking 2 was completely arbitrary, we could have showed $x^2 = a$ for any $a \geqslant 0$.

*Remark.* Using the same argument, we have that for all $a > 0$, there exists a unique $b > 0$ with $b^2 = a$. So we have a map:

$$\mathbf{R}^+ \xrightarrow{\sqrt{}} \mathbf{R}^+,$$

where $\sqrt{x}$ is the unique positive number with $(\sqrt{x})^2 = x$.

*Remark.* We could have similarly defined $S$ as:

$$S' = \{t \in \mathbf{Q} \mid t > 0, t^2 < 2\},$$

and the proof would not have changed. However, $\sup(S') = \sqrt{2} \notin \mathbf{Q}$, meaning $\mathbf{Q}$ is *not* complete.

## 4.2 Nested Intervals

**Axiom 4.** *Given any interval $I$, if $x, y \in I$ with $x < y$, then $[x, y] \in I$.*

**Theorem 4.2.1.** *Let $S \subseteq \mathbf{R}$ be any subset containing at least two points. If $S$ satisfies Axiom 4, then $S$ is an interval.*

*Proof.* We proceed with cases. Case 1: $S$ is bounded. Write $a = \inf(S)$ and $b = \sup(S)$. Therefore $S \subseteq [a, b]$. If we show $(a, b) \subseteq S$, then it follows that $S = (a, b]$, or $[a, b)$, or $(a, b)$ or $[a, b]$. We must use that $S$ satisfies Axiom 4 and $a = \inf(S)$ and $b = \sup(S)$. Let $x \in (a, b)$. Since $x > a$, there exists and $s_1 \in S$ with $s_1 < x$. Since $x < b$, there exists an $s_2 \in S$ with $x < s_2$. Thus $s_1, s_2 \in S$ and $s_1 < s_2$. By Axiom 4 $[s_1, s_2] \subseteq S$. But $x \in [s_1, s_2]$ implies $x \in S$. Thus $(a, b) \subseteq S$.

Case 2: $S$ is bounded above do this.

Case 3: $S$ is bounded below need to do. □

**Definition 4.2.1.** A sequence of intervals $(I_n)_{n\geqslant 1}$ is said to be <u>*nested*</u> if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$.

**Proposition 4.2.2.** $\bigcap_{n\geqslant 1} \left[0, \frac{1}{n}\right) = \{0\}$.

*Proof.* Note that $0 \in \left[0, \frac{1}{n}\right)$ for all $n \geqslant 1$. So $0 \in \bigcap_{n\geqslant 1} \left[0, \frac{1}{n}\right)$. Let $a \in \bigcap_{n\geqslant 1} \left[0, \frac{1}{n}\right)$. Then $0 \leqslant a < \frac{1}{n}$ for all $n \geqslant 1$. Hence $a = 0$. □

**Proposition 4.2.3.** $\bigcap_{n\geqslant 1} [n, \infty) = \emptyset$.

*Proof.* Suppose towards contradiction there exists a $t \in \bigcap_{n\geqslant 1} [n, \infty) = \emptyset$. Then $t \in [n, \infty)$ for all $n \geqslant 1$. So $t \geqslant n$ for all $n \geqslant 1$. Hence $\mathbf{N}$ is bounded above, which is a contradiction. □

**Theorem 4.2.4** (Nested Intervals)**.** *Let $(I_n)_{n\geqslant 1}$ be a sequence of closed and bounded nested intervals. Then $\bigcap_{n\geqslant 1} I_n \neq \emptyset$. Furthermore, if $\inf\{\text{length}(I_n) \mid n \geqslant 1\} = 0$, then $\bigcap_{n\geqslant 1} I_n = \{\xi\}$.*

*Proof.* Let $I_n = [a_n, b_n]$. Note that:

$$a_1 \leqslant a_2 \leqslant a_3 \leqslant \dots$$
$$b_1 \geqslant b_2 \geqslant b_3 \geqslant \dots$$

We have that $a_1 \leqslant a_n \leqslant b_n \leqslant b_1$ for all $n \geqslant 1$. So the set $\{a_n \mid n \geqslant 1\}$ is bounded above, and similarly $\{b_n \mid n \geqslant 1\}$ is bounded below. Let

$$\xi = \sup_{n \geqslant 1} \{a_n\}$$
$$\eta = \inf_{n \geqslant 1} \{b_n\}.$$

Claim: $\xi \leqslant b_n$ for all $n \geqslant 1$. Assume towards contradiction $\xi > b_m$ for some $m \geqslant 1$. Since $\xi = \sup_{n \geqslant 1} \{a_n\}$, there exists an $a_k$ with $b_m < a_k \leqslant \xi$. If $k \geqslant m$, then $b_m < a_k \leqslant b_k \leqslant b_m$, which is a contradiction. If $k < m$, then $a_k \leqslant a_m \leqslant b_m < a_k$, which is a contradiction.

Claim: $a_n \leqslant \xi$ for all $n \geqslant 1$. Then $\xi \leqslant \eta$ since $\sup_{n \geqslant 1} \{a_n\} = \xi$. We have $[\xi, \eta] \subseteq [a_n, b_n]$ for all $n \in \mathbf{N}$. Let $x \in [\xi, \eta]$. Then:

$$a_n \leqslant \xi \leqslant x \leqslant \eta \leqslant b_n,$$

hence $x \in [a_n, b_n]$; i.e., $[\xi, \eta] \subseteq [a_n, b_n]$ for all $n \geqslant 1$. Thus $[[\xi, \eta] \subseteq \bigcap_{n \geqslant 1} [a_n, b_n]]$. Conversely, let $t \in [a_n, b_n]$ for all $n \geqslant 1$. Then $a_n \leqslant t \leqslant b_n$. This implies $t$ is both an upper bound for $\{a_n\}_{n \geqslant 1}$ and a lower bound for $\{a_b\}_{n \geqslant 1}$. Hence $\xi \leqslant t \leqslant eta$, implying $t \in [\xi, \eta]$. This establishes $[\xi, \eta] = \bigcap_{n \geqslant 1} [a_n, b_n]$.

Now suppose $\inf \{\text{length}(I_n) \mid n \geqslant 1\} = 0$. Then:

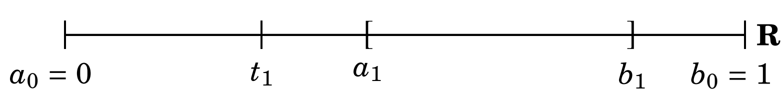
$$\begin{aligned} 0 &= \inf_{n \geqslant 1} (b_n - a_n) \\ &= \inf_{n \geqslant 1} b_n - \inf_{n \geqslant 1} a_n \\ &= \eta - \xi. \end{aligned}$$

Hence $\xi = \eta$, which establishes the theorem.

Alternatively, had we assumed $\xi \neq \eta$, then $\eta - \xi > 0$. So there exists an $m$ such that $b_m - a_m < \eta - \xi$, which is a contradiction since $[\xi, \eta] \subseteq [a_m, b_m]$. □

**Corollary 4.2.5.** $[0, 1]$ *is uncountable.*

*Proof.* By way of contradiction, suppose $[0, 1] = \{t_1, t_2, t_3, \dots\}$. Consider the following picture:




Find $[a_1, b_1] \subseteq [0, 1]$ with $t_1 \notin [a_1, b_1]$. Find $[a_2, b_2] \subseteq [a_1, b_1]$ with $t_2 \notin [a_2, b_2]$. Inductively, find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $t_n \notin [a_n, b_n]$. Thus $[a_n, b_n]$ is nested. Now let $\xi \in \bigcap_{n \geqslant 1} [a_n, b_n]$. Then $\xi \in [0, 1]$. But $\xi \neq t_n$ for all $n$, which is a contradiction. □

# 5
# Sequences

## 5.1 Basic Definitions and Examples

**Definition 5.1.1.** A <u>sequence</u> in a metric space $X$ is a map $x : \mathbf{N} \to X$. We often write $x = (x_n)_n = (x_1, x_2, \ldots)$, where $x_n = x(n)$. If $X = \mathbf{R}$, we call $x$ a <u>real sequence</u>.

**Example 5.1.1.**

(1) Sequences defined explicitly:

 (i) Constant sequences: $x_n = t$, $(x_n)_n = (t, t, t, \ldots)$
 (ii) Sequences defined by a function: $d_n = \left(1 + \frac{1}{n}\right)^n$.
 (iii) Geometric sequences: fix $b \in \mathbf{R}$, then $(b^n)_n = (1, b, b^2, \ldots)$.

(2) Sequences defined recursively:

 (i) Let $a_1 = 1$, $a_{n+1} = 2a_n + 1$. Then $(a_n)_n = (1, 3, 7, 15, \ldots)$.
 (ii) Let $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$. Then $(f_n)_n = (1, 1, 2, 3, 5, 8, \ldots)$. This is the *Fibonacci sequence*.
 (iii) Let $X$ be a metric space and $f : X \to X$ be an endomorphism. Fix $x_0 \in X$. Then define:

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) \\ &\vdots \\ x_n &= f(x_{n-1}). \end{aligned}$$

(3) New sequences from old:

 (i) Let $(a_n)_n$ and $(b_n)_n$ be sequences. Define:

$$\begin{aligned} (a_n)_n + (b_n)_n &= (a_n + b_n)_n \\ t(a_n)_n &= (ta_n)_n \\ (a_n)_n \cdot (a_n)_n &= (a_n b_n)_n \\ \frac{(a_n)_n}{(b_n)_n} &= \left(\frac{a_n}{b_n}\right)_n, \ (b_n)_n \neq 0 \ \text{ for all } n \quad . \end{aligned}$$

 (ii) Given $(x_n)_n$ and $k \in \mathbf{N}$, consider $(x_{n+k})_n = (x_k, x_{k+1}, \ldots)$. This is called a *shift* or the $k^{th}$ *tail* of $(x_n)_n$.

(iii) If $(a_n)_n$ is a sequence, $a_n \neq 0$ for all $n$, consider:

$$r_n = \frac{a_{n+1}}{a_n}.$$

So $(r_n)_n = \left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \dots\right)$. These are called *sequences of ratios*.

(iv) Given a real sequence $(x_k)_k$, consider the sequence $(s_n)_n$ where:

$$s_n = \sum_{k=1}^{n} x_k = s_{n-1} + x_k.$$

We call these $n^{th}$ *partial sums*. An example of these are geometric sequences and telescoping sequences.

**Example 5.1.2.** Let $F$ be a field. The set $F^{\mathbf{N}} = \{x \mid x : \mathbf{N} \to F\}$ is the set of all $F$-sequences. This forms an $F$-vector space under componentwise addition and scalar multiplication.

**Definition 5.1.2.** Let $(x_n)_n$ be a sequence.

(1) $x_n$ is increasing if $x_1 \leqslant x_2 \leqslant x_3 \leqslant \dots$

(2) $x_n$ is decreasing if $x_1 \geqslant x_2 \geqslant x_3 \geqslant \dots$

(3) $x_n$ is strictly increasing if $x_1 < x_2 < x_3 < \dots$

(4) $x_n$ is strictly decreasing if $x_1 > x_2 > x_3 > \dots$

**Definition 5.1.3.** A sequence is said to eventually have a certain property if it does not have the said property across all its ordered instances, but will after some instances have passed.

**Definition 5.1.4.** A sequence $(x_n)_n$ is monotone if it is either increasing, decreasing, strictly increasing, or strictly decreasing.

## 5.2 Convergence

**Definition 5.2.1.** Let $(x_n)_n$ be a sequence in a metric space $X$.

(1) $(x_n)_n$ converges to $x \in X$ if:

$$(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geqslant N_\epsilon \implies d(x_n, x) < \epsilon)).$$

We denote this as $(x_n)_n \to x$ or $\lim_{n\to\infty} x_n = x$.

(2) $(x_n)_n$ does not exist if:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(n \geqslant N \wedge d(x_n, n) \geqslant \epsilon_0).$$

We abbreviate this as D.N.E.

(3) $(x_n)_n$ *diverges properly* to $+\infty$ if:

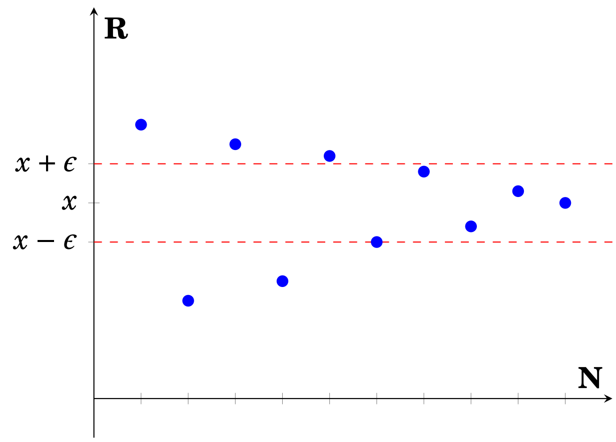
$$(\forall M > 0)(\exists N_M \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geqslant N_M \implies x_n \geqslant M).$$

We write $(x_n)_n \longrightarrow +\infty$.

(4) $(x_n)_n$ *diverges properly* to $-\infty$ if:

$$(\forall M < 0)(\exists N_M \in \mathbf{N}) \ni (\forall n \geqslant N_M)(x_n \leqslant M).$$

**Example 5.2.1.** Let $(x_n)_n$ be a real sequence. Then $d(x_n, x) < \epsilon \iff |x_n - x| < \epsilon \iff x_n \in V_\epsilon(x)$. We can visually represent a sequence as follows:




If a sequence is convergent it will eventually be contained between the two dashed lines.

**Example 5.2.2.**

(1) Prove $\left(\frac{1}{n}\right)_n \longrightarrow 0$.

*Solution.* Let $\epsilon > 0$. Find $N_\epsilon \in \mathbf{N}$ so that $\frac{1}{N_\epsilon} < \epsilon$. If $n \geqslant N_\epsilon$, then $\frac{1}{n} \leqslant \frac{1}{N_\epsilon} < \epsilon$. Hence $\frac{1}{n} = \left|\frac{1}{n} - 0\right| < \epsilon$.

(2) Prove $\left(\frac{5n-1}{3-n}\right)_{n=4}^{\infty} \longrightarrow -5$.

*Solution.* Note that:

$$|x_n - x| = \left|\frac{5n-1}{3-n} + 5\right| = \frac{14}{|3-n|} = \frac{14}{n-3}.$$

Let $\epsilon > 0$. Find $N_\epsilon \in \mathbf{N}$ such that $N_\epsilon > \frac{14}{\epsilon} = 3$. If $n \geqslant N_\epsilon$, then $n > \frac{14}{\epsilon} + 3$ gives:

$$n - 3 > \frac{14}{\epsilon} \implies \frac{14}{n-3} < \epsilon \implies |x_n - x| < \epsilon.$$

**Proposition 5.2.1.** *Let $(X, d)$ be a metric space. Then $(x_n)_n \longrightarrow x$ if and only if $(d(x_n, x))_n \longrightarrow 0$.*

*Proof.* Suppose $(x_n)_n \longrightarrow x$. Given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $d(x_n, x) < \epsilon$. This is equivalent to $|d(x_n, x) - 0| < \epsilon$. The converse follows identically. $\square$

**Theorem 5.2.2.** *Let $(\epsilon_n)_n \to 0$ and $(x_n)_n$ be real sequences and $x \in \mathbf{R}$. If for some $c > 0$ and $N \in \mathbf{N}$ we have:*

$$|x_n - x| \leqslant c|\epsilon_n| \quad \text{for all } n \in \mathbf{N} \text{ such that } n \geqslant N,$$

*then $(x_n)_n \to x$.*

*Proof.* Let $\epsilon > 0$ be given. Since $(\epsilon_n)_n \to 0$ it follows there exists a natural number $K$ such that if $n \geqslant K$ then

$$|a_n| = |a_n - 0| < \frac{\epsilon}{c}.$$

If both $n \geqslant K$ and $n \geqslant N$, then

$$|x_n - x| \leqslant c|\epsilon_n| < \epsilon.$$

Thus $(x_n)_n \to x$. □

**Example 5.2.3.**

(1) Prove $\left(\frac{\sin(n^2-1)}{n^2+3}\right)_n \to 0$.

*Solution.* Note that:

$$\left|\frac{\sin(n^2-1)}{n^2+3} - 0\right| = \frac{|\sin(n^2-1)|}{n^2+3} \leqslant \frac{1}{n^2+3} \leqslant \frac{1}{n^2} \leqslant \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $\left(\frac{\sin(n^2-1)}{n^2+3}\right)_n \to 0$.

(2) Prove $\left(\frac{1}{2^n}\right)_n \to 0$.

*Solution.* Note that:

$$\left|\frac{1}{2^n} - 0\right| \leqslant \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $\left(\frac{1}{2^n}\right)_n \to 0$.

(3) Prove $\left(\frac{1}{n} - \frac{1}{n+1}\right)_n \to 0$.

*Solution.* Note that:

$$\left|\frac{1}{n} - \frac{1}{n+1} - 0\right| \leqslant \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $\left(\frac{1}{n} - \frac{1}{n+1}\right)_n \to 0$.

**Proposition 5.2.3.** *Let $k \geqslant 1$ be fixed. Given a sequence $(x_n)_n$ in a metric space $(X, d)$, $(x_n)_n \to x$ if and only if $(x_{k+n})_n \to x$.*

*Proof.* ($\Rightarrow$) Suppose $(x_n)_n \to x$. Let $\epsilon > 0$. We know there exists $N_\epsilon \in \mathbf{N}$ with $n \geqslant N_\epsilon$ implying $d(x_n, x) < \epsilon$. But if $n \geqslant N_\epsilon$, then $n + k \geqslant N_\epsilon$. Hence $d(x_{n+k}, x) < \epsilon$.

($\Leftarrow$) Conversely, assume that $(x_{n+k}) \to 0$. Let $\epsilon > 0$. We know there exists $N_\epsilon \in \mathbf{N}$ such that $n \geqslant N_\epsilon$ implies $d(x_{n+k}, x) < \epsilon$. Consider $M = N_\epsilon + k$. Then if $n \geqslant M$, we have $n \geqslant N_\epsilon + k$, or equivalently $n - k \geqslant N_\epsilon$. Hence $d(x_{(n-k)+k}, x) = d(x_n, x) < \epsilon$. □

**Proposition 5.2.4.** *If $(x_n)_n$ is a real sequence with $\left(\left|\frac{x_{n+1}}{x_n}\right|\right) \to L < 1$, then $(x_n)_n \to 0$.*

*Proof.* Since $L < 1$, let $\rho$ be an number satisfying $L < \rho < 1$. Pick $\epsilon = \rho - L$ Since $\left(\left|\frac{x_{n+1}}{x_n}\right|\right) \to L$, there exists $N_\epsilon \in \mathbf{N}$ such that $n \geqslant N_\epsilon$ implies $\left|\frac{x_{n+1}}{x_n}\right| \in V_\epsilon(L)$, or equivalently $L - \epsilon < \frac{|x_{n+1}|}{|x_n|} < L + \epsilon$. Then $\frac{|x_{n+1}|}{|x_n|} < \rho$, which gives $|x_{n+1}| < \rho|x_n|$. Observe that:

$$\begin{aligned}
&|x_{N+1}| < \rho|x_N| \\
&|x_{N+2}| < \rho|x_{N+1}| = \rho^2|x_N| \\
&|x_{N+3}| < \rho|x_{N+2}| = \rho^3|x_N| \\
&\quad\vdots \\
\text{Inductively,}\quad &|x_{N+n}| = \rho^n|x_N|.
\end{aligned}$$

Since $(\rho^n)_n \to 0$ (and taking $c = |x_N|$), we have that $(x_{N+n})_n \to 0$. Thus $(x_n)_n \to 0$. □

*Remark.* Consider $(n)_n \to +\infty$. Then $\left(\frac{n+1}{n}\right)_n \to 1$. Now consider $\left(\frac{1}{n}\right)_n \to 0$. Then $\left(\frac{n}{n+1}\right) \to 1$. We gain no information if $L = 1$.

**Example 5.2.4.**

(1) Prove $((-1)^n)_n$ does not exist.

*Solution.* Suppose $((-1)^n)_n \to x$. We want to find some $\epsilon_0 > 0$ such that for all $N \in \mathbf{N}$, we can find an $n \in \mathbf{N}$ satisfying:

$$n \geqslant N \ \text{ and } \ |x_n - x| = |(-1)^n - x| \geqslant \epsilon_0.$$

Pick $\epsilon_0 = \max\{|x-1|, |x+1|\}$. Let $N \in \mathbf{N}$. Set $n = 2N$. This gives:

$$\begin{aligned}
(-1)^{2N} &= 1 \\
(-1)^{2N+1} &= -1
\end{aligned}$$

So we have $n \geqslant N$ and:

$$|(-1)^{2N} - x| = |1 - x| \geqslant \epsilon_0 \quad \text{or} \quad |(-1)^{2N+1} - x| = |1 + x| \geqslant \epsilon_0.$$

(2) Prove $(\sin(n))_n$ does not exist.

*Solution.*

**Proposition 5.2.5.** *Let $(X, d)$ be a metric space. A sequence $(x_n)_n$ can have at most one limit.*

*Proof.* Suppose $(x_n)_n \to L_1$ and $(x_n)_n \to L_2$. Set $\epsilon = \frac{|L_1-L_2|}{2}$. Then $V_\epsilon(L_1) \cap V_\epsilon(L_2) = \emptyset$. Since $(x_n)_n \to L_1$, there exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $x_n \in V_\epsilon(L_1)$. Likewise, since $(x_n)_n \to L_2$, there exists $N_2 \in \mathbf{N}$ such that $n \geqslant N_2$ implies $x_n \in V_\epsilon(L_2)$. Pick $N = \max\{N_1, N_2\}$. Then $x_N \in V_\epsilon(L_1) \cap V_\epsilon(L_2)$, which is a contradiction. □

**Lemma 5.2.6.** *If $(x_n)_n \to x$, then $(|x_n|)_n \to |x|$.*

*Proof.* Since $(x_n)_n \to x$, then there exists $N \in \mathbf{N}$ such that $n \geqslant \mathbf{N}$ implies $|x_n - x| < \epsilon$. The triangle inequality gives:

$$||x_n| - |x|| \leqslant |x_n - x| < \epsilon,$$

hence $(|x_n|)_n \to |x|$. Note that the converse does not hold in general, as:

$$(|(-1)^n|)_n \to 1 \text{ while } ((-1)^n)_n \text{ does not exist.}$$ □

**Lemma 5.2.7.** *Let $(t_n)_n$ be a sequence in $(X, d)$. $(t_n)_n \to 0$ if and only if $(|t_n|)_n \to 0$.*

*Proof.* (⇒) The forward direction follows from Lemma 5.2.6. (⇐) Suppose $(|t_n|)_n \to 0$. We have that:

$$||t_n| - 0| \leqslant$$

□

**Lemma 5.2.8.** *If $(x_n)_n \to x \in \mathbf{R}$ with $x_n \geqslant 0$, then $(\sqrt{x_n})_n \to \sqrt{x}$.*

*Proof.* Case 1: $x = 0$. Let $\epsilon > 0$ be given. Since $(x_n)_n \to 0$, there exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $0 \leqslant x_n = |x_n - 0| < \epsilon^2$. Hence $0 \leqslant \sqrt{x_n} < \epsilon$. Since $\epsilon > 0$, was arbitrary, $(\sqrt{x_n})_n \to 0$.

Case 2: $x > 0$. Then $\sqrt{x} > 0$, and:

$$|\sqrt{x_n} - \sqrt{x}| = \left|(\sqrt{x_n} - \sqrt{x})\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right| = \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right| \leqslant \left(\frac{1}{\sqrt{x}}\right)|x_n - x|.$$

Hence the convergence $(\sqrt{x_n})_n$ is a consequence of $(x_n)_n \to x$. □

**Example 5.2.5.**

(1) Prove $(\sqrt{n})_n \to +\infty$.

*Solution.* Let $M > 0$ be given. Find $N_M$ so that $N_M = \lceil M^2 \rceil$. Hence $N_M \geqslant M^2$. Then $n \geqslant M$ implies $n \geqslant M^2$, or equivalently $\sqrt{n} \geqslant M$.

(2) Prove $(n - \sqrt{n})_n \to +\infty$.

*Solution.* Write $(n - \sqrt{n})_n = (n)_n \overset{\to 1}{\cancel{(1 - \frac{1}{\sqrt{n}})}}_n = (n)_n$. Since $(n)_n$ trivially converges to $+\infty$, we have $(n - \sqrt{n})_n \to +\infty$.

(3) Prove:

$$(b^n)_{n=0}^{\infty} \to \begin{cases} 0, & |b| < 1 \\ 1, & b = 1 \\ +\infty, & b > 1 \\ \text{D.N.E.}, & b \leqslant -1 \end{cases}$$

*Solution.* Cases $b = 0$ and $b = 1$ are trivial. We showed case $b = -1$ in Example 5.2.4.

Case 1: $0 < b < 1$. Then $b < 1$ implies $\frac{1}{b} > 1$. We have $\frac{1}{b} = 1 + a$ for some $a > 0$, now observe that:

$$\left(\frac{1}{b}\right)^n = (1+a)^n \geqslant 1 + na.$$

This gives:

$$|b^n - 0| \leqslant \frac{1}{1+na} \leqslant \frac{1}{na} = \left(\frac{1}{a}\right)\frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $(b^n)_n \to 0$.

Case 2: $-1 < b < 0$. Since $(|b^n|)_n = (|b|^n)_n$, case 1 gives $(b^n)_n \to 0$ when $-1 < b < 0$.

Case 3: $b > 1$. Then $b = 1 + a$ for some $a > 0$. We have:

$$b^n = (1+a)^n \geqslant 1 + na \geqslant na.$$

Let $M > 0$ be given. Pick $N_M = \frac{\lceil M \rceil}{a}$. Then $N_M \geqslant \frac{M}{a}$. If $n \geqslant N_M$, then $n \geqslant \frac{M}{a}$, which simplifies to $na \geqslant M$. Hence $b^n \geqslant na \geqslant M$ gives $(b^n)_n \to +\infty$.

Case 4: $b < 1$. We prove that $(b^n)_n$ does not exist by contradiction. Suppose $(b_n)_n \to L$ for some $L \in \mathbf{R}$. Then $(|b_n|)_n \to |L|$. But this is a contradiction via the $b > 1$ case. Now if $(b^n)_n \to +\infty$, there exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $b^n \geqslant 1$. But for $n$ odd, $b^n < 0$, which is a contradiction. Assuming $(b^n)_n \to -\infty$ leads to a similar contradiction, establishing the proof.

**Example 5.2.6.**

(1) Prove if $c > 0$, $(c^{\frac{1}{n}})_n \to 1$.

*Solution.* If $c = 1$, then clearly $(1^{\frac{1}{n}})_n \to 1$. Suppose $c > 1$, then $c^{\frac{1}{n}} > 1$. Write $c^{\frac{1}{n}} = 1 + a_n$, where $a_n > 0$ for all $n \in \mathbf{N}$. We have:

$$c = (c^{\frac{1}{n}})^n = (1 + a_n)^n \geqslant 1 + na_n \geqslant na_n.$$

So $0 < na_n \leqslant c$, giving $a_n \leqslant \frac{c}{n}$. We have:

$$|c^{\frac{1}{n}} - 1| = a_n \leqslant \frac{c}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, $(c^{\frac{1}{n}})_n \to 1$. Now suppose $0 < c < 1$, then $c^{\frac{1}{n}} < 1$. Write $c^{\frac{1}{n}} = 1 + (-a_n)$ with $-1 < -a_n < 0$ for all $n$. Then:

$$c = (c^{\frac{1}{n}})^n = (1 + (-a_n))^n \geqslant 1 + n(-a_n) \geqslant n(-a_n).$$

So $n(-a_n) \leqslant c$, giving $-a_n \leqslant \frac{c}{n}$. We have:

$$|c^{\frac{1}{n}} - 1| = -a_n \leqslant \frac{c}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, $(c^{\frac{1}{n}})_n \to 1$.

(2) Prove $(n^{\frac{1}{n}})_n \to 1$.

*Proof.* Note that $n^{\frac{1}{n}} > 1$ for all $n > 1$. Write $n^{\frac{1}{n}} = 1 + a_n$. Then:

$$n = (1 + a_n)^n = \sum_{k=0}^{n} \binom{n}{k} a_n^k \geqslant \binom{n}{0} + \binom{n}{2} a_n^2 = 1 + \frac{n(n-1)}{2} a_n^2.$$

We have:

$$n - 1 \geqslant \frac{n(n-1)}{2} a_n^2,$$

which simplifies to:

$$\frac{2}{n} \geqslant a_n^2.$$

Hence $a_n \leqslant \sqrt{2}\,\frac{1}{n}$, thus by our lemma $(a_n)_n^\infty \to 0$. Therefore:

$$|n^{\frac{1}{n}} - 1| = d_n,$$

establishing that $(n^{\frac{1}{n}})_n \to 1$. □

**Proposition 5.2.9.** *A convergent sequence is bounded.*

*Proof.* Suppose $(x_n)_n \to x$. Since $(x_n)_n$ is convergent, we know for all $\epsilon > 0$ that $|x_n - x| < \epsilon$. Pick $\epsilon = 1$. Eventually the entire sequence will be contained in $V_1(x)$. More formally, there exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $x_n \in V_1(x)$. Define:

$$c = \max \{|x_1|, |x_2|, \ldots, |x_N|, |x - 1|, |x + 1|\}.$$

If $n \leqslant N$, then $|x_n| \leqslant c$. If $n \geqslant N_1$, then $x - 1 < x_n < x + 1$; i.e., $|x_n| \leqslant c$. □

**Theorem 5.2.10.** *Let $x_n$, $y_n$, $z_n$ be convergent sequences with $(x_n)_n \to x$, $(y_n)_n \to y$, and $(z_n)_n \to z$ and $t \in \mathbf{R}$. Moreover, let $z_n \neq 0$ for all $n$ and $z \neq 0$. We have:*

*(1)* $(x_n \pm y_n)_n \to x \pm y$.

*(2)* $(tx_n)_n \longrightarrow tx$.

*(3)* $(x_n y_n)_n \longrightarrow xy$.

*(4)* $\left(\frac{1}{z_n}\right)_n \longrightarrow \frac{1}{z}$.

*(5)* $\left(\frac{x_n}{z_n}\right)_n \longrightarrow \frac{x}{z}$.

*Proof.* (3) We have:

$$\begin{aligned}|x_n y_n - xy| &= |x_n y_n - xy_n + xy_n - xy| \\ &= |(x_n - x)y_n + x(y_n - y)| \\ &\leqslant |(x_n - x)y_n| + |x(y_n - y)| \\ &= |x_n - x||y_n| + |x||y_n - y|.\end{aligned}$$

Since $y_n$ is convergent, it is bounded. So there exists a $c > 0$ with $|y_n| \leqslant c$ for all $n \geqslant 1$. Hence:

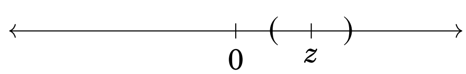
$$|x_n - x||y_n| + |x||y_n - y| \leqslant \overset{\to 0}{\cancel{|x_n - x|}}\, c + |x|\, \overset{\to 0}{\cancel{|y_n - y|}}.$$

Thus $(|x_n y_n - xy|)_n \longrightarrow 0$, which implies $(x_n y_n)_n \longrightarrow xy$.

(4) We have:

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \frac{|z - z_n|}{|z||z_n|}.$$

Since $z \neq 0$, it won't be "near" zero. We have the following picture:



Let $\delta = \frac{|z|}{2} > 0$. There exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $z_n \in V_\delta(z)$. We have:

$$\begin{aligned}& z - \delta < z_n < z + \delta \\ \Longrightarrow\ & z - \frac{|z|}{2} < z_n \\ \Longrightarrow\ & \frac{|z|}{2} < |z_n|.\end{aligned}$$

Since $|z_n| \geqslant \frac{|z|}{2}$, we have $\frac{1}{|z_n|} < \frac{2}{|z|}$. So for $n \geqslant N$,

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \frac{|z - z_n|}{|z||z_n|} \leqslant \frac{2}{|z|^2}|z - z_n|.$$

Thus $\left(\frac{1}{z_n}\right)_n \longrightarrow \frac{1}{z}$. □

**Theorem 5.2.11.** *Suppose $(x_n) \to x$ and $(y_n)_n \to y$ with $x_n \leqslant y_n$ for all $n$. Then $x \leqslant y$.*

*Proof.* We have that $(y_n - x_n)_n \longrightarrow y - x$, and $y_n - x_n \geqslant 0$ for all $n$. Thus $y - x \geqslant 0$. □

**Corollary 5.2.12.** *If* $(x_n)_n \to x$ *and* $a \leqslant x_n \leqslant b$, *then* $a \leqslant x \leqslant b$.

*Proof.* Taking $(y_n)_n = (a, a, a, \dots)$ and $(y_n)_n = (b, b, b, \dots)$ gives the desired result. □

**Theorem 5.2.13** (Squeeze Theorem). *Let* $(x_n)_n$, $(y_n)_n$, *and* $(z_n)_n$ *be sequences with* $(x_n)_n \leqslant (y_n)_n \leqslant (z_n)_n$ *for all* $n \geqslant 1$. *If* $\lim x_n = \lim z_n = L$, *then* $(y_n)_n \to L$.

*Proof.* Let $\epsilon > 0$. There exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $x_n \in V_\epsilon(L)$. Likewise, there exists $N_2 \in \mathbf{N}$ such that $n \geqslant N_2$ implies $z_n \in V_\epsilon(L)$. So for $n \geqslant \max\{N_1, N_2\} := N$, both $x_n, z_n \in V_\epsilon(L)$. We have:

$$L - \epsilon < x_n \leqslant y_n < z_n \leqslant L + \epsilon.$$

Thus $y_n \in V_\epsilon(L)$ for $n \geqslant N$. □

**Theorem 5.2.14** (Monotone Convergence Theorem). *Let* $(x_n)_n$ *be a monotone sequence.* $(x_n)_n$ *is convergent if and only if* $(x_n)_n$ *is bounded. Moreover,*

*(a) If* $(x_n)_n$ *is increasing and bounded above,* $\lim x_n = \sup\{x_n \mid n \in \mathbf{N}\}$.

*(b) If* $(x_n)_n$ *is decreasing and bounded below,* $\lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}$.

*Proof.* $(\Rightarrow)$ We showed this direction in Proposition 5.2.9. $(\Leftarrow)$ (a) Suppose $(x_n)_n$ is bounded above and increasing. Let $u = \sup\{x_n \mid n \in \mathbf{N}\}$. Given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $u - \epsilon < x_N$. But for $n \geqslant N$, $u - \epsilon < x_N \leqslant x_n \leqslant u < u + \epsilon$. Hence $x_n \in V_\epsilon(u)$, establishing that $(x_n)_n \to u$.

(b) Consider $y_n = -x_n$, we get $y_n$ is increasing and bounded above. By $(a)$, we get:

$$\begin{aligned} \lim y_n = \sup\{y_n \mid n \in \mathbf{N}\} &\implies -\lim x_n = \sup\{-x_n \mid n \in \mathbf{N}\} \\ &\implies -\lim x_n = -\inf\{x_n \mid n \in \mathbf{N}\} \\ &\implies \lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}. \end{aligned}$$

□

**Example 5.2.7.**

(1) Consider the recursively defined sequence $x_1 = 8$, $x_{n+1} = \frac{1}{2}x_n + 2$. We will show by induction that it is bounded below by 4. Clearly $x_1 = 8 \geqslant 4$. Now assume $x_k \geqslant 4$. Then:

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_n + 2 \\ &\geqslant \frac{1}{2}(4) + 2 \\ &= 4. \end{aligned}$$

Therefore $(x_n)_n$ is bounded below by 4. Now observe that:

$$\begin{aligned} x_{n+1} \leqslant x_n &\iff \frac{1}{2}x_n + 2 \leqslant x_n \\ &\iff 4 \leqslant x_n. \end{aligned}$$

Hence $(x_n)_n$ is decreasing. By the Monotone Convergence Theorem, $(x_n)_n \to L$. Now observe that:

$$\begin{aligned}(x_{n+1})_n = \left(\frac{1}{2}x_n + 2\right)_n &\iff L = \frac{1}{2}L + 2 \\ &\iff L = 4.\end{aligned}$$

(2) Let $x_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9}$. We will show that this sequence converges. Clearly $x_n \leqslant x_{n+1}$, so it is increasing. We will use the fact that $k^2 \geqslant k(k-1)$ as follows:

$$\begin{aligned}x_n &= \sum_{k=1}^{n} \frac{1}{k^2} \\ &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} \\ &\leqslant 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ &= 1 + \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] \\ &= 1 + 1 - \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &\leqslant 2.\end{aligned}$$

So $(x_n)_n$ is increasing and bounded above, hence it has a limit.

(3) Given $a > 0$, we will find a sequence $(x_n)_n$ which converges to $\sqrt{a}$. Consider the recursively defined sequence $x_1 = 1$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. Claim: $x_n^2 \geqslant a$ for all $n \geqslant 2$. Note that:

$$\begin{aligned}2x_{n+1} = x_n + \frac{a}{x_n} &\implies 2x_{n+1}x_n = x_n^2 + a \\ &\implies 0 = x_n^2 - 2x_{n+1}x_n = a.\end{aligned}$$

This polynomial has a real root, hence $\Delta \geqslant 0$. We get:

$$\Delta = 4x_{n+1}^2 - 4a \geqslant 0 \implies x_{n+1}^2 \geqslant a$$

We will now show that $(x_n)_n$ is eventually decreasing. Observe that:

$$\begin{aligned}x_n \geqslant x_{n+1} &\iff x_n \geqslant \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ &\iff 2x_n \geqslant x_n + \frac{a}{x_n} \\ &\iff x_n \geqslant \frac{a}{x_n} \\ &\iff x_n^2 \geqslant a.\end{aligned}$$

By the Monotone Convergence Theorem, $(x_n)_n \to L$. We have:

$$\begin{aligned} x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) &\implies L = \frac{1}{2}\left(L = \frac{a}{L}\right) \\ &\implies L^2 = a \\ &\implies L = \sqrt{a}. \end{aligned}$$

**Example 5.2.8** (Euler's Number)**.** I will do this later.

**Proposition 5.2.15.** *If $(x_n)_n$ is increasing and unbounded, then $(x_n)_n$ diverges properly to $+\infty$.*

*Proof.* Let $M$ be arbitrarily big. Since $(x_n)_n$ is unbounded, there exists $N \in \mathbf{N}$ with $x_n > M$. Hence if $n \geqslant M$, $x_n \geqslant x_N > M$ because $(x_n)_n$ is increasing. □

**Example 5.2.9.** We will show that $h_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges properly to $+\infty$. do this later

## 5.3 Subsequences

**Definition 5.3.1.** A *natural sequence* is a strictly increasing sequence of natural numbers: $(n_k)_{k=1}^{\infty}$ with $n_k \in \mathbf{N}$, $n_1 < n_2 < \dots$

**Example 5.3.1.**

(1) $(2k+1)_k = (3, 5, 7, \dots)$

(2) $(k^2)_k = (1, 4, 9, \dots)$

**Exercise 5.3.1.** *Given a natural sequence $(n_k)_k$, prove $n_k \geqslant k$.*

**Definition 5.3.2.** Let $(x_n)_n$ be a sequence. A *subsequence* of $(x_n)_n$ is a sequence $(x_{n_k})_{k=1}^{\infty}$ where $(n_k)_k$ is a natural sequence. Formally, a subsequence is a composition of maps:

$$\mathbf{N} \underset{k \mapsto n_k}{\to} \mathbf{N} \underset{n_k \mapsto x_{n_k}}{\to} X.$$

**Example 5.3.2.**

(1) Consider $(x_n)_n \to \frac{1}{n}$. Let $n_k = 2_k$. Then $(x_{n_k})_k = \left(\frac{1}{2k}\right)_k = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right)$.

(2) Conider $(x_n)_n = (-1)^n$ Then $(x_{2k})_k = (1, 1, 1, \dots)$ and $(x_{2k+1})_k = (-1, -1, -1, \dots)$

**Proposition 5.3.1.** *Suppose $(x_n)_n \to x$. For any subsequence $(x_{n_k})_k$, we have $(x_{n_k})_k \to x$.*

*Proof.* Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $|x_n - x| < \epsilon$. Take $K = N$. Then $k \geqslant K$ implies $k \geqslant N$. But by Exercise **??**, $n_k \geqslant k \geqslant N$. Hence $|x_{n_k} - x| < \epsilon$. □

**Example 5.3.3.** We give an alternate proof of $(b^n)_n \to 0$ for $0 < b < 1$. Clearly $b^{n+1} < b^n$ if and only if $b < 1$. So $b^n$ is decreasing and bounded below by 0. By the Monotone Convergence Theorem, $(b^n)_n \to L$ for some $L$. But we also have that $(b^{2k})_k \to L$. So we have:

$$\begin{aligned}(b^{2k})_k = (b^k)_k^2 &\iff L = L^2 \\ &\iff L(1-L) = 0.\end{aligned}$$

Since $L \neq 1$, it must be that $L = 0$.

**Proposition 5.3.2.** *Let $(x_n)_n$ be a sequence. Then $(x_n)_n \nrightarrow x$ if and only if there exists an $\epsilon_0 > 0$ and subsequence $(x_{n_k})_k$ such that $d(x_{n_k}, x) > \epsilon_0$.*

*Proof.* ($\Leftarrow$) If $(x_n)_n \to x$, then any subsequence $(x_{n_k})_k$ converges to $x$.

($\Rightarrow$) Since $(x_n)_n \nrightarrow x$, we have:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N})(\exists n \geqslant N) \ni (x_n \notin V_{\epsilon_0}(x)).$$

With this $\epsilon_0$, we will construct our subsequence $x_{n_k}$. Note that:

$$\begin{aligned} N = 1 &\implies (\exists n_1 \geqslant 1) \ni (x_{n_1} \notin V_{\epsilon_0}(x)) \\ N = n_1 + 1 &\implies (\exists n_2 \geqslant n_1) \ni (x_{n_2} \notin V_{\epsilon_0}(x)) \\ N = n_2 + 1 &\implies (\exists n_3 \geqslant n_2) \ni (x_{n_3} \notin V_{\epsilon_0}(x)) \\ &\vdots \\ \text{Inductively, } N = n_k + 1 &\implies (\exists n_{k+1} \geqslant n_k) \ni (x_{k+1} \notin V_{\epsilon_0}(x)) \end{aligned}$$

Thus $(x_{n_k})_k$ is a subsequence with $x_{n_k} \notin V_{\epsilon_0}(x)$, so $|x_{n_k} - x| \geqslant \epsilon_0$ for all $k = 1, 2, 3, \ldots$ □

**Definition 5.3.3.** If $(x_n)_n$ is a sequence of real numbers, a <u>*peak*</u> of the sequence is a term $x_m$ satisfying $x_m \geqslant x_n$ for all $n \geqslant m$.

**Proposition 5.3.3.** *Let $(x_n)_n$ be a real sequence. There is a subsequence that is monotone.*

*Proof.* Case 1: There are infinitely many peaks. Let $x_{n_1}, x_{n_2}, x_{n_3} \ldots$ be an enumeration of peaks. Then $(x_{n_k})_k$ is decreasing by definition.

Case 2: There are finitely many peaks. Let $x_{m_1}, x_{m_2}, \ldots, x_{m_r}$ be the peaks of our sequence where $m_1 < m_2 < \ldots < m_r$. Let $n_1 = m_r + 1$. Since $x_{n_1}$ is not a peak, there exists $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Since $x_{n_2}$ is not a peak, there exists $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$. Inductively, we obtain a sequence $(x_{n_k})_k = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ with $x_{n_k} < x_{n_{k+1}}$. □

**Theorem 5.3.4** (Bolzano-Weierstass Theorem)**.** *If $(x_n)_n$ is a real sequence that is bounded, it admits a convergent subsequence.*

*Proof.* By Proposition 5.3.3, there exists a subsequence $(x_{n_k})$ which is monotone and bounded. By the Monotone Convergence Theorem, $(x_{n_k})_k$ converges. □

## 5.4 Limit Inferior and Limit Superior

**Definition 5.4.1.** Let $X = (x_n)_n$ be a fixed bounded sequence who's limit may not exist. Then

$$\overline{X} = \{t \in \mathbf{R} \mid t = \lim_{k\to\infty} x_{n_k},\ x_{n_k} \text{ some subsequence}\}$$

is the set containing all *subsequential limits* (or *limit points*) of $X$.

**Example 5.4.1.** Let $X = ((-1)^n)_n$. Then $\overline{X} = \{-1, 1\}$.

**Example 5.4.2.** Fix a bounded sequence $(x_n)_n$. Let

$$u_1 = \sup_{n\geqslant 1}(x_n),$$
$$l_1 = \inf_{n\geqslant 1}(x_n).$$

If a subsequence $(x_{n_k})_k \to x$, we know $x \in [l_1, u_1]$ because $l_1 \leqslant x \leqslant u_1$. Hence $l_1 \leqslant x_{n_k} \leqslant u_1$. Now let

$$u_2 = \sup_{n\geqslant 2}(x_n),$$
$$l_2 = \inf_{n\geqslant 1}(x_n).$$

We have $u_2 \leqslant u_1$ (we know $u_1$ is an upper bound for all $n \geqslant 2$, hence $u_2$ must be the least upper bound) and $l_1 \leqslant l_2$. Similarly, if $(x_{n_k})_k \to x$ for some subsequence, then $x \in [l_2, u_2]$ because $l_2 \leqslant x_{n_k} \leqslant u_2$ for $k$ large enough. Inductively:

$$u_m = \sup_{n\geqslant m} x_n,$$
$$l_m = \inf_{n\geqslant m} x_n.$$

We get:

$$l_1 \leqslant l_2 \leqslant \ldots \leqslant l_m \leqslant u_m \leqslant \ldots \leqslant u_2 \leqslant u_1.$$

This holds for all $m \geqslant 1$. Let $I_m = [l_m, u_m]$. Then $(I_m)_m$ is a sequence of closed and bounded nested intervals. So

$$\bigcap_{m\geqslant 1} I_m = [l, u]$$

where

$$l = \sup_{m\geqslant 1} l_m = \sup_{m\geqslant 1}\left(\inf_{n\geqslant m} x_n\right),$$
$$u = \inf_{m\geqslant 1} u_m = \inf_{m\geqslant 1}\left(\sup_{n\geqslant m} x_n\right).$$

Note that:

$$\sup_{m\geqslant 1} l_m = \lim_{m\to\infty} l_m$$
$$\inf_{m\geqslant 1} u_m = \lim_{m\to\infty} u_m.$$

This follows from the Monotone Convergence Theorem, as $(l_m)_m$ is an increasing sequence bounded above and $(u_m)_m$ is a decreasing sequence bounded below.

**Definition 5.4.2.** Let $(x_n)_n$ be a bounded sequence.

(1) $l = \lim_{m\to\infty} l_m = \lim_{m\to\infty} \left(\inf_{n\geqslant m} x_n\right) := \liminf x_n$.

(2) $u = \lim_{m\to\infty} u_m = \lim_{m\to\infty} \left(\sup_{n\geqslant m} x_n\right) := \limsup x_n$.

**Proposition 5.4.1.** *Let $X = (x_n)_n$ be a bounded sequence with $l = \liminf x_n$ and $u = \limsup x_n$. If $x \in X$, then $x \in [l, u]$. We have:*

$$l_{n_k} = \inf_{n\geqslant n_k} x_n \leqslant x_{n_k}.$$

*Taking the limit as $k \to \infty$ yields $l \leqslant x$. Similarly, we have:*

$$u_{n_k} = \sup_{n\geqslant n_k} x_n \geqslant x_{n_k}.$$

*Taking the limit as $k \to \infty$ yields $x \leqslant u$. Thus $x \in [l, u]$.*

**Question.** Does $\overline{X} = [l, u]$?

*Answer.* No. Take for example $x_n = (-1)^n$. Then $u = 1$ and $l = -1$ But $\overline{X} = \{-1, 1\} \subset [-1, 1]$.

**Proposition 5.4.2.** *Let $(x_n)_n = X$ be a bounded sequence with $u_m = \sup_{n\geqslant m} x_n$. We have a strictly decreasing sequence $u_1 \geqslant u_2 \geqslant \ldots$ There exists a subsequence $(x_{n_k})_k \to u$. There exists a subsequence $(x_{n_k})_k \to l$. Equivalently, $u, l \in \overline{X}$.*

*Proof.* Recall that $u_m = \sup_{n\geqslant m} x_n$. By the supremum property:

$$\begin{aligned}
\exists n_1 \in \mathbf{N} \ \text{ with } \ & u_1 - 1 < x_{n_1} \leqslant u_1, \\
\exists n_2 \in \mathbf{N} \ \text{ with } \ n_2 > n_1 + 1 > n_1 \ \text{ and } \ & u_{n_1+1} - \frac{1}{2} < x_{n_2} \leqslant u_{n_1+1}, \\
\exists n_3 \in \mathbf{N} \ \text{ with } \ n_3 \geqslant n_2 + 1 > n_2 \ \text{ and } \ & u_{n_2+1} - \frac{1}{3} < x_{n_3} \leqslant u_{n_2+1}.
\end{aligned}$$

Inductively:

$$u_{n_{k-1}+1} - \frac{1}{k} < x_{n_k} \leqslant u_{n_{k-1}+1}.$$

Let $m_k = n_{k-1} + 1$. We can rewrite the above equation as:

$$u_{m_k} - \frac{1}{k} < x_{n_k} \leqslant u_{m_k}.$$

Letting $k \to \infty$ gives:

$$\lim_{k\to\infty} u_{m_k} - \frac{1}{k} < \lim_{k\to\infty} x_{n_k} \leqslant \lim_{k\to\infty} u_{m_k}$$

$$\iff$$

$$u < \lim_{k\to\infty} x_{n_k} \leqslant u.$$

By the squeeze theorem, $(x_{n_k})_k \to u$. □

**Proposition 5.4.3.** *Let $(x_n)_n$ be bounded.*

*(1)* $\liminf x_n \leqslant \limsup x_n$.

*(2)* $(x_n)_n \to x$ *if and only if* $\liminf x_n = \limsup x_n = x$.

*Proof.* We have $l_m \leqslant u_m$ for all $m \geqslant 1$ by the previous motivating example. Letting $m \to \infty$ gives $l \leqslant u$.

I don't know the second part □

## 5.5 Cauchy Sequences

**Definition 5.5.1.** A sequence $(x_n)_n$ is *Cauchy* if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni m, n \geqslant N \implies d(x_n, x_n) < \epsilon.$$

**Example 5.5.1.** Prove $\left(\frac{1}{n}\right)_n$ is Cauchy.

*Solution.* For $n > m$:

$$|x_m - x_n| = \frac{1}{m} - \frac{1}{n} = \frac{n-m}{mn} < \frac{n}{nm} = \frac{1}{m}.$$

We can start the proof. Given $\epsilon > 0$, by Archimedean property 2 there exists $N$ large satisfying $\frac{1}{N} < \epsilon$. For $n > m \geqslant N$, we have $|x_m - x_n| < \frac{1}{m} \leqslant \frac{1}{N} < \epsilon$.

**Proposition 5.5.1.** *Cauchy sequences are bounded.*

*Proof.* Pick $\epsilon = 1$. There exists $N \in \mathbf{N}$ such that $m, n \geqslant N$ implies $|x_n - x_m| < 1$. Let $c = \max\{|x_1|, \ldots, |x_N|\}$. For $n \geqslant N$, we have $|x_n| = |x_n - x_N + x_N| \leqslant |x_n - x_N| + |x_N| \leqslant 1 + |x_N|$. So $|x_n| \leqslant c'$, where $c' = \max\{c, 1 + |x_N|\}$. □

**Exercise 5.5.1.** *If $(x_n)_n$ is Cauchy and there exists a subsequence $(x_{n_k})_k$ with $(x_{n_k})_k \to x$, then $(x_n)_n \to x$.*

**Theorem 5.5.2.** *Let $(x_n)_n$ be a sequence. $(x_n)_n$ is Cauchy if and only if $(x_n)_n$ converges.*

*Proof.* ($\Rightarrow$) Suppose $(x_n)_n \to x$. Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $|x_n - x| < \frac{\epsilon}{2}$. For $m, n \geqslant N$, we have $|x_n - x_m| = |x_n - x + x_m - x| \leqslant |x_n - x| + |x_m - x| < \epsilon$.

($\Leftarrow$) If $(x_n)_n$ is Cauchy then $(x_n)_n$ is bounded. The Bolzano-Weierstass theorem says gives there exists some convergent subsequence $(x_{n_k})_k$. By Exercise 5.5.1, $(x_n)_n \to x$. □

**Example 5.5.2.** We will show that $x_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ is a convergent sequence. This sequence is not monotone, so we are unable to use the monotone convergence theorem. Instead, we will show that this sequence is Cauchy, which implies that it is convergent. For $m > n$, observe that:

$$\begin{aligned}
|x_m - x_n| &= \left| \sum_{k=n+1}^{m} \frac{(-1)^k}{k!} \right| \\
&\leqslant \sum_{k=n+1}^{m} \left| \frac{(-1)^k}{k!} \right| \\
&= \sum_{k=n+1}^{m} \frac{1}{k!} \\
&\leqslant \sum_{k=n+1}^{m} \frac{1}{2^{k-1}} \\
&= \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{m-1}} \\
&= \frac{1}{2^n}\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-n-1}}\right) \\
&\leqslant \frac{1}{2^n} \cdot 2 \\
&= \frac{1}{2^{n-1}}.
\end{aligned}$$

If $\epsilon > 0$ is given, choose $N$ large so that $\frac{1}{2^{N-1}} < \epsilon$. If $m > n > N$, then $|x_m - x_n| < \epsilon$. Thus $x_n$ is Cauchy, implying that it is convergent.

**Definition 5.5.2.** A sequence $(x_n)_n$ is <u>*contractive*</u> if there exists $0 < \rho < 1$ with $|x_{n+1} - x_n| \leqslant \rho|x_n - x_{n-1}|$ for all $n \geqslant 2$. We say $\rho$ is the <u>*constant of contraction*</u>.

**Proposition 5.5.3.** *Contractive sequences are Cauchy.*

*Proof.* Observe that:

$$|x_3 - x_2| \leqslant \rho|x_2 - x_1|$$
$$|x_4 - x_3| \leqslant \rho|x_2 - x_1| \leqslant \rho^2|x_2 - x_1|$$

Inductively, $|x_{n+1} - x_n| \leqslant \rho^{n-1}|x_2 - x_1|$. For $m > n$, we have:

$$|x_m - x_n| =$$

finish later □

**Example 5.5.3.** fibonacci finish later

## 5.6 Sequences of Functions

**Definition 5.6.1.** Let $\Omega$ be a nonempty set. The set of all functions $\Omega \xrightarrow{f} X$ is denoted $\mathcal{F}(\Omega, X) = \{f \mid f : \Omega \to X\}$.

**Exercise 5.6.1.** *Show that $\mathcal{F}(\Omega, X)$ is an algebra under pointwise operations.*

**Definition 5.6.2.** A sequence in $(f_n)_n : \mathbf{N} \to \mathcal{F}(\Omega, \mathbf{R})$n converges *pointwise* to $f \in \mathcal{F}(\Omega, \mathbf{R})$ if for all $x \in \Omega$, $(f_n(x))_n \to f(x)$. In particular:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N_{x,\epsilon} \in \mathbf{N}) \ni n \geqslant N \implies |f_n(x) - f(x)| < \epsilon.$$

**Example 5.6.1.**

(1) Let $f_n \in \mathcal{F}([0,1], \mathbf{R})$ be defined by $f_n(x) = x^n$. Given $x \in [0,1)$, we can see that $(f_n(x))_n \to 0$. If $x = 1$, then $(f_n(x))_n \to 1$. So let $f : [0,1] \to \mathbf{R}$ be defined as $f = \delta_1$. Then $(f_n)_n \to f$ converges pointwise.

(2) Let $f_n \in \mathcal{F}(\mathbf{R}, \mathbf{R})$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}$. Note that:

$$\begin{aligned} |f_n(x)| &= \left|\frac{nx}{1+n^2x^2}\right| \\ &= \frac{n|x|}{1+n^2x^2} \\ &\leqslant \frac{|x|}{nx^2} \\ &= \frac{1}{n|x|}. \end{aligned}$$

Claim: $(f_n)_n \to 0_{\mathcal{F}(\mathbf{R},\mathbf{R})}$. If $x = 0$, then $(f_n(0))_n = (0)_n \to 0$. If $x \neq 0$, then $(f_n(x))_n \to 0$ because $|f_n(x) - 0| \leqslant \frac{1}{|x|} \cdot \frac{1}{n}$. Since $x$ is fixed and $\left(\frac{1}{n}\right)_n \to 0$, by "Lemma" $(f_n(x))_n \to 0_{\mathcal{F}(\mathbf{R},\mathbf{R})}$.

(3) Let $h_n \in \mathcal{F}([0,\infty), \mathbf{R})$ be defined by $h_n(x) = x^{\frac{1}{n}}$. If $x > 0$, then $(h_n(x))_n \to 1$. If $x = 0$, then $(h_n(x))_n \to 0$. Then $(h_n(x))_n \to \mathbf{1}_{(0,\infty)}$.

(4) Let $g_n \in \mathcal{F}([0,\infty), \mathbf{R})$ be defined by $g_n(x) = e^{-nx}$. If $x = 0$, then $(g_n(x))_n \to 1$. If $x > 0$, then $(g_n(x))_n \to 0$. So $(g_n(x))_n \to \delta_0$.

**Example 5.6.2.** Our goal in this example is to construct a function $f_n \in \mathcal{F}([0,\infty), \mathbf{R}$ satisfying $(f_n(x))_n \to 0_{\mathcal{F}([0,\infty),\mathbf{R})}$ and $\int_{0,\infty} f_n(x)dx = 1$. Let:

$$f_n = \begin{cases} 4n^2x, & x < \frac{1}{2n} \\ 4n - 4n^2x, & \frac{1}{2} \leqslant x < \frac{1}{n} \\ 0, & x \geqslant \frac{1}{n}. \end{cases}$$

Given $x = 0$, $(f_n(0)) = (0)_n \to 0$. Given $x > 0$, then there exists $N \in \mathbf{N}$ such that $\frac{1}{N} < x$ (AP1). If $n \geqslant N$, then $f_n(x) = 0$, hence $(f_n(x))_n \to 0$. Thus $(f_n)_n \to 0_{\mathcal{F}([0,\infty),\mathbf{R})}$. Furthermore:

$$\begin{aligned} \int_0^\infty f_n(x)dx &= \int_0^{\frac{1}{2n}} f_n(x)dx + \int_{\frac{1}{2n}}^{\frac{1}{n}} f_n(x)dx \\ &= 1/2 + 1/2 \\ &= 1. \end{aligned}$$

**Definition 5.6.3.** A sequence $(f_n)_n \in \mathcal{F}(\Omega, \mathbf{R})$ converges *uniformly* to $f \in \mathcal{F}(\Omega, \mathbf{R})$ if:

$$\begin{aligned}(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) \ni n \geqslant N &\implies |f_n(x) - f(x)| < \epsilon \ \ \forall x \in \Omega \\ &\implies \sup_{x \in \Omega} |f_n(x) - f(x)| \leqslant \epsilon.\end{aligned}$$