Math 395

Homework 4

Due: 10/8/2024

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Exercise 1. *Let $T \in \mathrm{Hom}_F(V,V)$. Prove that the intersection of any collection of T-invariant subspaces of V is T-invariant.*

Proof. Let $\{W_i\}_{i\in I}$ be a collection of T-invariant subspaces of V. Since the intersection of an arbitrary collection of subspaces is also a subspace, it only remains to show that $\bigcap_{i\in I} W_i$ is T-invariant. Let $x \in T\left(\bigcap_{i\in I} W_i\right)$. Then $x \in T(W_i)$ for all $i \in I$. So $x \in W_i$ for all $i \in I$, establishing $x \in \bigcap_{i\in I} W_i$. Thus $T\left(\bigcap_{i\in I} W_i\right) \subseteq \bigcap_{i\in I} W_i$. □

Exercise 2. *Let $T \in \mathrm{Hom}_F(V,V)$ and $v \in V$. Prove that if $T^j(v) \in W = \mathrm{span}_F(v_1, ..., v_n)$ and W is T-invariant, then $T^{j+t}(v) \in W$ for all $t \geqslant 0$.*

Proof. We prove this by induction on t. Let $t = 0$ be the base case, then by assumption $T^j(v) \in W$. Assume our hypothesis to be true up to $t-1$. Then:

$$T^t(T^j(v)) = T(T^{t-1}(T^j(v))).$$

Our induction hypothesis gives $T^{t-1}(T^j(v)) \in W$, and since $T(W) \subseteq W$, we have:

$$T^{j+t}(v) = T(T^{t-1}(T^j(v))) \in W.$$

□

Exercise 3. *Let T satisfy $T^2 = T$. Prove that the only possible eigenvalues of T are 0 and 1.*

Proof. Let $v \neq 0$ be an eigenvector of λ. Then $T^2 = T$ is equivalent to $T^2 - T = 0$. Then:

$$\begin{aligned} 0_V &= (T^2 - T)(v) \\ &= T^2(v) - T(v) \\ &= \lambda^2 v - \lambda v \\ &= (\lambda(1-\lambda))(v) \\ &= \lambda(\lambda - 1). \end{aligned}$$

We have that $\lambda(1-\lambda) = 0_V$, hence $\lambda = 0$ or 1. □

Exercise 4. *Let V be an $\mathbf{R}$-vector space. Let $T \in \mathrm{Hom}_{\mathbf{R}}(V,V)$ satisfy $T^2 + bT + c\,\mathrm{id}_V = 0_{\mathrm{Hom}_{\mathbf{R}}(V,V)}$ for some $b, c \in \mathbf{R}$. Prove that T has an eigenvalue if and only if $b^2 \geqslant 4c$.*

Proof. Let $\lambda \in \mathbf{R}$ be an eigenvalue of T with $v \in E_\lambda^1$, $v \neq 0_V$. Then:

$$\begin{aligned} 0_V &= (T^2 + bT + c\,\mathrm{id}_V)(v) \\ &= T^2(v) + bT(v) + cv \\ &= (\lambda^2 + b\lambda + c)v \\ &= \lambda^2 + b\lambda + c. \end{aligned}$$

Then:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \in \mathbf{R}.$$

It must be the case that $b^2 \geqslant 4c$, otherwise $\lambda \notin \mathbf{R}$, which is a contradiction.

Conversely, let $b^2 \geqslant 4c$. Then $T^2 + bT + c\,\mathrm{id}_V$ factors over $\mathbf{R}$; i.e., $((T - \alpha\,\mathrm{id}_V) \circ (T - \beta\,\mathrm{id}_V)) = 0_{\mathrm{Hom}_{\mathbf{R}}(V,V)}$ for some $\alpha, \beta \in \mathbf{R}$. Let $v \in V$, $v \neq 0$. Then:

$$((T - \alpha\,\mathrm{id}_V) \circ (T - \beta\,\mathrm{id}_V))(v) = 0_V.$$

Write $w = (T - \beta\,\mathrm{id}_V)(v)$. If $w \neq 0$, then:

$$\begin{aligned} 0_V &= (T - \alpha\,\mathrm{id}_V)(w) \\ &= T(w) - \alpha w. \end{aligned}$$

Hence $T(w) = \alpha w$. If $w = 0$, then:

$$\begin{aligned} 0_V &= (T - \beta\,\mathrm{id}_V)(v) \\ &= T(v) - \beta v. \end{aligned}$$

Hence $T(v) = \beta v$, establishing the proof. □