Math 397

Homework 1

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Exercise 1. Let V be a vector space, and suppose $\{W_i\}_{i\in I}$ is a family of subspaces of V.

(1) Show that $\bigcap_{i\in I} W_i$ is the largest subspace of V contained in every W_i.

(2) Show that:

$$\sum_{i\in I} W_i = \left\{ \sum_{i\in F} w_i \mid w_i \in W_i,\ F \subseteq I \text{ finite} \right\}$$

is the smallest subspace containing each W_i.

(3) We say that V is the *internal direct sum* of the family $\{W_i\}_{i\in I}$ and we write $V = \bigoplus_{i\in I} W_i$ if:

(i) $V = \sum_{i\in I} W_i$;

(ii) For each $j \in I$, $W_j \cap \sum_{i\neq j} W_i n\{0\}$.

If $V = \bigoplus_{i\in I} W_i$, show that every $v \in V$ has a unique expression $v = \sum_{i\in F} w_i$ where $F \subseteq I$ is finite and $0 \neq w_i$ for each $w_i \in W_i$.

Proof. (1) Let U be a subspace of V with $U \subseteq W_i$ for each $i \in I$. Then clearly $U \subseteq \bigcap_{i\in I} W_i$.

(2) Let $W = \sum_{i\in I} W_i$ and let U be a subspace of V with $U \supseteq W_i$ for each $i \in I$. If $x \in W$, then $x = \sum_{i\in I} W_i$. But since W_i is a subspace, it is closed under addition. Whence $x \in W_i$ for each $i \in I$. By inclusion then, $x \in U$. Hence $W \subseteq U$.

(3) By the definition of internal direct sums $V = \sum_{i\in I} W_i$, whence each $v \in V$ can be written as $v = \sum_{i\in F} w_i$. It remains to show that this expression is unique. Suppose $v = \sum_{i\in F} w_i = \sum_{i\in F} u_i$ with $w_i, u_i \in W_i$. For each j we have:

$$w_j - u_j = \sum_{\substack{i\in F \\ i\neq j}} (w_i - u_i)$$

But notice that $w_j - u_j \in W_j$ and $\sum_{i\in F, i\neq j}(w_i - u_i) \in \sum_{i\neq j} W_i$. So $w_j - u_j \in W_j \cap \sum_{i\neq j} W_i$. By the definition of internal direct sums this gives $w_j - u_j = 0$, which simplifies to $w_j = u_j$. $\square$

Exercise 2.

Proof. Suppose towards contradiction there exists $c \in \operatorname{span}(S) \setminus \bigcap$... $\square$

Exercise 3. Let V be a vector space with subspaces $W_i \subseteq V$ for $i = 1, 2$. If $W_1 \cup W_2 \subseteq V$ is a subspace, show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. Suppose towards contradiction $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Then there exists $w_1 \in W_1 \setminus W_2$ and $w_2 \in W_2 \setminus W_1$. Let $v = w_1 + w_2$. Then $v \in W_1 \cup W_2$. But this means $w_2 = v - w_1 \in W_2$. Whence $w_1 \in W_2$, which is a contradiction. $\square$

Exercise 4. Let V be a vector space over $\mathbb{F}$ and suppose $W \subset V$ is a subspace.

(1) Show that the quotient space $V/W = \{[v]_w \mid v \in V\}$ is a subspace with operations:

$$[u]_W + [v]_W = [u+v]_W\,; \quad \alpha[v]_W = [\alpha v]_W.$$

(2) Suppose $\|\cdot\|$ is a norm on V. Show that:

$$\|[v]_W\|_{V/W} := \inf_{w \in W} \|v - w\|$$

is a seminorm.

Proof. (1) Since V is an abelian group and $W \subseteq V$ is normal, V/W is an abelian group. It only remains to show that $\alpha[v]_W = [\alpha v]_W$ satisfies the vector space axioms. We have that:

$$\begin{aligned}
\alpha\left([u]_W + [v]_W\right) &= \alpha[u+v]_W \\
&= [\alpha(u+v)]_W \\
&= [\alpha u + \alpha v]_W \\
&= [\alpha u]_W + [\alpha v]_W, \\
\alpha\left(\beta[v]_W\right) &= \alpha[\beta v]_W \\
&= [\alpha(\beta v)]_W \\
&= [(\alpha\beta)v]_W \\
&= (\alpha\beta)[v]_W, \\
1_{\mathbb{F}}[v]_W &= [1_{\mathbb{F}} v]_W \\
&= [v]_W.
\end{aligned}$$

Whence V/W is a vector space.

(2) We must first show that $\|\cdot\|_{V/W} : V/W \to \mathbb{F}$ is well-defined. Let $[v_1]_W = [v_2]_W$. Then $v_2 - v_1 \in W$. Observe that:

$$\begin{aligned}
\|[v_1]_W\|_{V/W} &= \inf_{w \in W} \|v_1 - w\| \\
&= \inf_{w-(v_2-v_1) \in W} \left\|v_1 - \left(w - (v_2 - v_1)\right)\right\| \\
&= \inf_{w-(v_2-v_1) \in W} \|v_1 - w + v_2 - v_1\| \\
&= \inf_{w \in W} \|v_2 - w\| \\
&= \|[v_2]_W\|_{V/W}.
\end{aligned}$$

We also have that:

$$\begin{aligned}
\|\alpha[v]_W\|_{V/W} &= \|[\alpha v]_W\|_{V/W} \\
&= \inf_{w \in W} \|\alpha v - w\| \\
&= \inf_{w' \in W} \|\alpha v - \alpha w'\| \\
&= \inf_{w' \in W} \|\alpha(v - w')\| \\
&= |\alpha| \inf_{w' \in W} \|v - w'\| \\
&= |\alpha|\, \|[v]_W\|_{V/W}.
\end{aligned}$$

Whence $\|\cdot\|_{V/W}$ is homogenous. Finally, we can see that:

$$\begin{aligned}
\|[u]_W + [v]_W\|_{V/W} &= \|[u+v]_W\|_{V/W} \\
&= \inf_{w\in W} \|u+v-w\| \\
&= \inf_{w,w'\in W} \|u+v-(w+w')\| \\
&= \inf_{w,w'\in W} \|u-w+v-w'\| \\
&\leqslant \inf_{w,w'\in W} (\|u-w\| + \|v-w'\|) \\
&= \inf_{w\in W} \|u-w\| + \inf_{w'\in W} \|v-w'\| \\
&= \|[u]_W\|_{V/W} + \|[v]_W\|_{V/W} .
\end{aligned}$$

Thus $\|\cdot\|_{V/W}$ is a seminorm. □

Exercise 5. Show that the quantity:

$$\|f\|_1 := \int_0^1 |f(t)|\,dt$$

defines a norm on $C([0,1])$ with $\|f\|_1 \leqslant \|f\|_u$. Are $\|\cdot\|_1$ and $\|\cdot\|_u$ equivalent norms?

Proof. $\|\cdot\|_1$ is homogenous because:

$$\begin{aligned}
\|\alpha f\|_1 &= \int_0^1 |(\alpha f)(t)|\,dt \\
&= \int_0^1 |\alpha f(t)|\,dt \\
&= |\alpha| \int_0^1 |f(t)|\,dt \\
&= |\alpha|\,\|f\|_1 .
\end{aligned}$$

Note that $|f(t)+g(t)| \leqslant |f(t)| + |g(t)|$. Integrating both sides gives:

$$\begin{aligned}
\int_0^1 |f(t)+g(t)|\,dt &= \int_0^1 |(f+g)(t)|\,dt \\
&= \|f+g\|_1 \\
&\leqslant \int_0^1 (|f(t)| + |g(t)|)\,dt \\
&= \int_0^1 |f(t)|\,dt + \int_0^1 |g(t)|\,dt \\
&= \|f\|_1 + \|g\|_1 .
\end{aligned}$$

Whence our norm satisfies the triangle inequality. □