Math 310

Homework 4

Due: 10/9/2024

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**Exercise 1.** *Prove the following limits:*

(1) $\left(\frac{2n}{n+1}\right)_n \to 2.$

(2) $\left(\frac{\sqrt{n}}{n+1}\right)_n \to 0.$

(3) $\left(\frac{(-1)^n}{\sqrt{n+7}}\right)_n \to 0.$

(4) $(n^k b^n)_n \to 0$, *where* $0 \leqslant b < 1$ *and* $k \in \mathbf{N}$.

(5) $\left(\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \to 3.$

**Exercise 2.** *Show that the sequence* $(\cos(n))_n$ *does not converge.*

**Exercise 3.** *If* $(x_n)_n$ *is a real sequence converging to* $x$*, show that*

$$(|x_n|)_n \to |x|.$$

*Is the converse true?*

*Proof.* Since $(x_n)_n \to x$ is a convergent sequence, we have:

$$||x_n| - |x|| \leqslant |x_n - x| < \epsilon.$$

Thus $(|x_n|)_n \to |x|$. Note that the converse is not true: $(|(-1)^n|)_n \to 1$ converges whereas $((-1)^n)_n$ does not. □

**Exercise 4.** *If* $(x_n)_n$ *is a real sequence converging to* $x > 0$*, show that there is an* $N \in \mathbf{N}$ *and* $c > 0$ *such that*

$$x_n \geqslant c$$

*for all* $n \geqslant N$.

**Exercise 5.** *If* $(x_n)_n$ *is a real sequence of positive terms converging to* $x$*, show that* $x \geqslant 0$ *and*

$$(\sqrt{x_n})_n \to \sqrt{x}.$$

*Proof.* Observe that:

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leqslant \left|\sqrt{x_n} - \sqrt{x}\right|\left|\sqrt{x_n} + \sqrt{x}\right| = |x_n - x| < \epsilon.$$

Hence $(\sqrt{x_n})_n \to \sqrt{x}$. If $x < 0$, then $\sqrt{x} \notin \mathbf{R}$, contradicting the definition of a real sequence. □

**Exercise 6.** *If* $(x_n)_n$ *and* $(y_n)_n$ *are sequences with* $(x_n)_n \to 0$ *and* $(y_n)_n$ *bounded, show that*

$$(x_n y_n)_n \to 0.$$

*Proof.* Since $(y_n)_n$ is bounded, $|y_n| \leqslant c$ for some $c > 0$. We have:

$$|x_n y_n| \leqslant c|x_n|.$$

Taking $\epsilon_n = |x_n|$ and using "Lemma" gives $(x_n y_n)_n \to 0$. □

**Exercise 7.** *If* $(x_n)_n$ *is a sequence of positive terms such that*

$$\left(\frac{x_{n+1}}{x_n}\right)_n \to L > 1,$$

*show that* $(x_n)_n$ *is not bounded hence not convergent. If* $L = 1$*, can we make any conclusion?*

**Exercise 8.** *Let* $a$ *and* $b$ *be positive numbers. Show that*

$$\left((a^n + b^n)^{\frac{1}{n}}\right)_n \to \max\{a, b\}.$$