Monotone Convergence Theorem

Definitions

(1) A sequence $(x_n)_n$ is *monotone* if it is either increasing, decreasing, strictly increasing, or strictly decreasing.

Theorems/Propositions/Lemmas

(1) A convergent sequence is bounded.

Proof. Suppose $(x_n)_n \to x$. Since $(x_n)_n$ is convergent, we know:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni n \geqslant N \implies |x_n - x| < \epsilon.$$

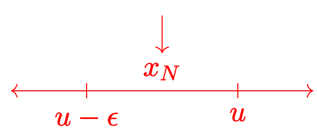
Pick $\epsilon = 1$. Then there exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $x_n \in V_1(x)$. Define:

$$c = \max\{|x_1|, |x_2|, ..., |x_N|, |x-1|, |x+1|\}.$$

If $n \leqslant N$, then $|x_n| \leqslant c$. If $n \geqslant N_1$, then $|x_n| \leqslant c$. $\square$

(2) (Monotone Convergence Theorem) Let $(x_n)_n$ be a monotone sequence. $(x_n)_n$ is convergent if and only if $(x_n)_n$ is bounded. Moreover, If $(x_n)_n$ is increasing and bounded above, then $\lim x_n = \sup\{x_n \mid n \in \mathbf{N}\}$ *or* if $(x_n)_n$ is decreasing and bounded below, then $\lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}$

Proof. $(\Rightarrow)$ This direction was showed in (1). $(\Leftarrow)$ Suppose $(x_n)_n$ is bounded above and increasing. Let $u = \sup\{x_n \mid n \in \mathbf{N}\}$. Supremum property says given $\epsilon > 0$, there exists $N \in \mathbf{N}$ with $u - \epsilon < x_N$.



But for $n \geqslant N$:

$$u - \epsilon < x_N \leqslant x_n \leqslant u < u + \epsilon.$$

Hence $|x_n - u| < \epsilon$, establishing that $(x_n)_n \to u$. Now let $y_n = -x_n$. Then y_n is increasing and bounded above. We get:

$$\begin{aligned} \lim y_n = \sup\{y_n \mid n \in \mathbf{N}\} &\implies -\lim x_n = \sup\{-x_n \mid n \in \mathbf{N}\} \\ &\implies -\lim x_n = -\inf\{x_n \mid n \in \mathbf{N}\} \\ &\implies \quad \lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}. \end{aligned}$$ $\square$

(3) If $(x_n)_n$ is increasing and unbounded, then $(x_n)_n$ diverges properly to $+\infty$.

Proof. Pick M large. Since $(x_n)_n$ is unbounded, there exists $N \in \mathbf{N}$ with $x_N > M$. Hence if $n \geqslant N$, then $x_n \geqslant x_N > M$, establishing $(x_n)_n \to +\infty$. □

Examples

(1) Let $x_1 = 8$ and inductively set $x_{n+1} = \frac{1}{2}x_n + 2$. Show that $(x_n)_n$ converges and find its limit.

Solution. Note that $(x_n)_n = (8, 6, 5, \frac{9}{2}, \ldots)$. We will show this sequence is bounded below by 4 and decreasing. Clearly $x_1 = 8 \geqslant 4$. Now assume $x_n \geqslant 4$. Then:

$$\begin{aligned} x_{n+1} &= \frac{1}{2}x_n + 2 \\ &\geqslant \frac{1}{2}(4) + 2 \\ &= 4. \end{aligned}$$

Moreover,

$$\begin{aligned} x_{n+1} \leqslant x_n &\iff \frac{1}{2}x_n + 2 \leqslant x_n \\ &\iff 4 \leqslant x_n. \end{aligned}$$

Thus $(x_n)_n$ is bounded below by 4 and decreasing. By MCT $(x_n)_n \to L$. Observe that:

$$\begin{aligned} x_{n+1} = \frac{1}{2}x_n + 2 &\overset{n\to\infty}{\iff} L = \frac{1}{2}L + 2 \\ &\iff L = 4. \end{aligned}$$

(2) Let $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Show that $(x_n)_n$ converges.

Solution. Clearly $x_n \leqslant x_{n+1}$. We have:

$$\begin{aligned} x_n &= \sum_{k=1}^{n} \frac{1}{k^2} \\ &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} \\ &\leqslant 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} && \text{Since } k^2 \geqslant k(k-1) \\ &= 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) && \text{Partial fractions} \\ &= 1 + \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] \\ &= 1 + 1 - \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &\leqslant 2. \end{aligned}$$

Since $(x_n)_n$ is increasing and bounded above, by MCT $(x_n)_n \to L$.

(3) Given $a > 0$, construct a sequence $(x_n)_n$ which converges to $\sqrt{a}$.

Solution. Let $x_1 = 1$ and inductively set $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. Observe that:

$$\begin{aligned} 2x_{n+1} = x_n + \frac{a}{x_n} &\implies 2x_{n+1}x_n = x_n^2 + a \\ &\implies x_n^2 - 2x_{n+1}x_n + a = 0. \end{aligned}$$

By assumption $(x_n)_n$ converges, hence this polynomial has a real root. So:

$$\begin{aligned} \Delta \geqslant 0 &\implies 4x_{n+1}^2 - 4a \geqslant 0 \\ &\implies x_{n+1}^2 \geqslant a. \end{aligned}$$

Whence $(x_n)_n$ bounded below. It remains to show that $(x_n)_n$ is decreasing. Observe that:

$$\begin{aligned} x_n \geqslant x_{n+1} &\iff x_n \geqslant \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ &\iff 2x_n \geqslant x_n + \frac{a}{x_n} \\ &\iff x_n \geqslant \frac{a}{x_n} \qquad \text{Since } x_n + x_n \geqslant x_n + \frac{a}{x_n} \\ &\iff x_n^2 \geqslant a \\ &\iff x_{n+1}^2 \geqslant a. \qquad \text{Since } a \text{ is a lowerbound} \end{aligned}$$

Hence by MCT, $(x_n)_n \to L$. This gives:

$$\begin{aligned} x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) &\overset{n\to\infty}{\implies} L = \frac{1}{2}\left(L + \frac{a}{L}\right) \\ &\implies L^2 = a \\ &\implies L = \sqrt{a}. \end{aligned}$$

(4) Let $h_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $(h_n)_n \to +\infty$.

Solution. Clearly $(h_n)_n$ is increasing. Observe that:

$$\begin{aligned} h_2 &= 1 + \frac{1}{2} \geqslant 1 + \frac{1}{2} \\ h_{2^2} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geqslant 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + 2\left(\frac{1}{2}\right) \\ h_{2^3} &= \ldots = 1 + 3\left(\frac{1}{2}\right) \end{aligned}$$

Inductively, $h_{2^n} \geqslant 1 + \frac{n}{2}$. Since $(1 + \frac{1}{n})_n \to +\infty$, $(h_n)_n \to +\infty$.

Subsequences

Definitions

(1) A *natural sequence* is a strictly increasing sequence of natural numbers $(n_k)_k$ with $n_k \in \mathbf{N}$.

(2) Let $(x_n)_n$ be a sequence. A *subsequence* of $(x_n)_n$ is a sequence $(x_{n_k})_k$ where $(n_k)_k$ is a natural sequence. Formally, a subsequence is a composition of maps $\mathbf{N} \xrightarrow{k \mapsto n_k} \mathbf{N} \xrightarrow{n_k \mapsto x_{n_k}} X$

(3) If $(x_n)_n$ is a sequence of real numbers, a *peak* of a sequence is a term x_m with $x_m \geqslant x_n$ for all $n \geqslant m$.

Theorems/Propositions/Lemmas

(1) Given a natural sequence $(n_k)_k$, $n_k \geqslant k$ for all k.

Proof. Clearly $n_1 \geqslant 1$. Now assume $n_k \geqslant k$. Then $n_{k+1} \geqslant n_k + 1 \geqslant k + 1$. $\square$

(2) Suppose $(x_n)_n \to x$. For any subsequence $(x_{n_k})_k$, we have $(x_{n_k})_k \to x$.

Proof. Since $(x_n)_n \to x$, $(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni n \geqslant N \implies |x_n - x| < \epsilon$. Consider $K = N$. Then $k \geqslant K$ implies $k \geqslant N$. But by (1) $n_k \geqslant k \geqslant N$. Hence $|x_{n_k} - x| < \epsilon$, establishing $(x_{n_k})_k \to x$. $\square$

(3) Let $(x_n)_n$ be a sequence. Then $(x_n)_n \not\to x$ if and only if there exists $\epsilon_0 > 0$ and a subsequence $(x_{n_k})_k$ such that $d(x_{n_k}, x) \geqslant \epsilon_0$.

Proof. $(\Leftarrow)$ If $(x_n)_n \to x$, then any subsequence $(x_{n_k})_k$ converges to x. $(\Rightarrow)$ Since $(x_n)_n \not\to x$:

$$(\exists \epsilon_0 > 0)\underline{(\forall N \in \mathbf{N})} \ni (\exists n \in \mathbf{N})(n \geqslant N \;\wedge\; d(x_n - x) \geqslant \epsilon_0).$$

Note that:

$$\begin{aligned}
N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geqslant 1 \;\wedge\; d(x_{n_1}, x) \geqslant \epsilon_0) \\
N = n_1 + 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 > n_1 \;\wedge\; d(x_{n_2}, x) \geqslant \epsilon_0) \\
N = n_2 + 1 &\implies (\exists n_3 \in \mathbf{N})(n_3 > n_2 \;\wedge\; d(x_{n_3}, x) \geqslant \epsilon_0) \\
&\;\vdots \\
N = n_k + 1 &\implies (\exists n_{k+1} \in \mathbf{N})(n_{k+1} > n_k \;\wedge\; d(x_{n_{k+1}}, x) \geqslant \epsilon_0)
\end{aligned}$$

Hence $(x_{n_k})_k$ is a subsequence satisfying $d(x_{n_k}, x) \geqslant \epsilon_0$. $\square$

(4) Let $(x_n)_n$ be a real sequence. There is a subsequence that is monotone.

Proof. We proceed with cases. Case 1: there are infinitely many peaks. Let $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ be an enumeration of peaks. Then $(x_{n_k})_k$ is decreasing by definition. Case 2: there are finitely many peaks. Let $x_{m_1}, x_{m_2}, \dots, x_{m_r}$ be the peaks of our sequence. Then $m_1 < m_2 < \dots < m_r$ by definition. Let $n_1 = m_r + 1$. Since x_{n_1} is not a peak, there exists $n_2 > n_1$ such that $x_{n_3} > x_{n_2}$. Inductively, we obtain a sequence $(x_{n_k})_k$ with $x_{n_k} < x_{n_{k+1}}$. ☐

(5) (Bolzano-Weierstrass Theorem) If $(x_n)_n$ is a real sequence that is bounded, it admits a convergent subsequence.

Proof. Since $(x_n)_n$ is a bounded real sequence it admits a monotone subsequence $(x_{n_k})_k$ which is bounded. By the monotone convergence theorem $(x_{n_k})_k$ converges. ☐

(6) If $(x_n)_n$ is an unbounded sequence of real numbers, show that there is a subsequence $(x_{n_k})_k$ such that $\left(\frac{1}{x_{n_k}}\right)_k \xrightarrow{k\to\infty} 0$.

Proof. Since $(x_n)_n$ is an unbounded real sequence:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(n \geqslant N \;\wedge\; |x_n - 0| \geqslant \epsilon_0).$$

We can construct a subsequence as follows:

$$\begin{aligned} N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geqslant 1 \;\wedge\; |x_{n_1}| \geqslant \epsilon_0) \\ N = n_1 = 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 \geqslant n_1 \;\wedge\; |x_{n_2}| \geqslant \epsilon_0) \\ &\vdots \end{aligned}$$

Inductively, we obtain a sequence $(x_{n_k})_k$ which properly diverges to $+\infty$. Given $\epsilon > 0$, let K be arbitrarily big so that $\epsilon > \frac{1}{n_K}$. Then for $k \geqslant K$, we have $\left|\frac{1}{n_k}\right| < \epsilon$. ☐

(7) Suppose that every subsequence of a sequence $(x_n)_n$ has a subsequence that converges to 0. Show that $(x_n)_n \to 0$.

Proof. Suppose towards contradiction that $(x_n)_n \not\to 0$. Then there exists a subsequence $(x_{n_k})_k \not\to 0$. By definition:

$$(\exists \epsilon_0 > 0)(\forall K \in \mathbf{N}) \ni (\exists k \in \mathbf{N})(k \geqslant K \;\wedge\; d(x_{n_k}, 0) \geqslant \epsilon_0).$$

We will construct a subsequence of $(x_{n_k})_k$ as follows:

$$\begin{aligned} K = 1 &\implies (\exists k_1 \in \mathbf{N})(k_1 \geqslant 1 \;\wedge\; d(x_{n_{k_1}}, 0) \geqslant \epsilon_0) \\ K = k_1 + 1 &\implies (\exists k_2 \in \mathbf{N})(k_2 \geqslant k_1 \;\wedge\; d(x_{n_{k_2}}, 0) \geqslant \epsilon_0) \\ &\vdots \end{aligned}$$

Inductively, we obtain a sequence $(x_{n_{k_j}})_j \not\to 0$. But this contradicts our claim that every subsequence has a subsequence which converges to 0. Hence it must be that $(x_n)_n \to 0$. ☐

(8) If $(x_n)_n$ is a bounded sequence and $s := \sup_n x_n$ is such that $s \notin \{x_n \mid n \geqslant 1\}$, show that there is a subsequence $(x_{n_k})_k$ that converges to s.

Examples

Limit Inferior & Limit Superior

Definitions

(1) Let $X = (x_n)_n$ be a fixed bounded sequence who's limit may not exist. Then $\overline{X} = \{t \in \mathbf{R} \mid t = \lim_{k\to\infty} x_{n_k}, x_{n_k}$ some subsequence$\}$ is the set containing all *subsequential limits* (or *limit points*) of X.

(2) Let $(x_n)_n$ be a bounded sequence.

(i) $l = \lim_{m\to\infty} l_m = \lim_{m\to\infty}(\inf_{n\geqslant m} x_n) := \liminf x_n$

(ii) $u = \lim_{m\to\infty} u_m = \lim_{m\to\infty}(\sup_{n\geqslant m} x_n) := \limsup x_n$.

Theorems/Propositions/Lemmas

(1) Let $X = (x_n)_n$ be a bounded sequence with $l = \liminf x_n$ and $u = \limsup x_n$. If $x \in \overline{X}$, then $x \in [l, u]$.

Proof. Note that:

$$\begin{aligned}
\inf_{n\geqslant n_k} x_n \leqslant x_{n_k} &\implies \lim_{k\to\infty}\big(\inf_{n\geqslant n_k} x_n\big) \leqslant \lim_{k\to\infty} x_{n_k} \\
&\implies l \leqslant x.
\end{aligned}$$

$$\begin{aligned}
\sup_{n\geqslant n_k} x_n \geqslant x_{n_k} &\implies \lim_{k\to\infty}\big(\sup_{n\geqslant n_k} x_n\big) \geqslant \lim_{k\to\infty} x_{n_k} \\
&\implies u \geqslant x.
\end{aligned}$$

□

(2) Let $(x_n)_n = X$ be a bounded sequence. Let $l = \liminf x_n$ and $u = \limsup x_n$. Then $l, u \in \overline{X}$.

Proof. Let $u_m = \sup_{n\geqslant m} x_n$. By the supremum property:

$$\begin{aligned}
N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geqslant 1 \ \wedge\ u_1 - 1 < x_{n_1} \leqslant u_1) \\
N = n_1 + 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 > n_1 \ \wedge\ u_2 - \frac{1}{2} < x_{n_2} \leqslant u_2) \\
N = n_2 + 1 &\implies (\exists n_3 \in \mathbf{N})(n_3 > n_2 \ \wedge\ u_3 - \frac{1}{3} < x_{n_3} \leqslant u_3) \\
&\ \ \vdots
\end{aligned}$$

Inductively:

$$\begin{aligned}
u_k - \frac{1}{k} < x_{n_k} \leqslant u_k &\implies \lim_{k\to\infty} u_k < \lim_{k\to\infty} x_{n_k} \leqslant \lim_{k\to\infty} u_k \\
&\implies u < \lim_{k\to\infty} x_{n_k} \leqslant u.
\end{aligned}$$

By the squeeze theorem, $(x_{n_k})_k \to u$. Now let $l_m = \inf_{n \geqslant m} x_n$. By the infimum property:

$$\begin{aligned}
N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geqslant 1 \;\wedge\; l_1 \leqslant x_{n_1} < l_1 + 1) \\
N = n_1 + 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 > n_1 \;\wedge\; l_2 \leqslant x_{n_2} < l_2 + \frac{1}{2}) \\
N = n_2 + 1 &\implies (\exists n_3 \in \mathbf{N})(n_3 > n_2 \;\wedge\; l_3 \leqslant x_{n_3} < l_3 + \frac{1}{3}) \\
&\vdots
\end{aligned}$$

Inductively:

$$\begin{aligned}
l_k \leqslant x_{n_k} < l_k + \frac{1}{k} &\implies \lim_{k\to\infty} l_k \leqslant \lim_{k\to\infty} x_{n_k} < \lim_{k\to\infty} l_k + \frac{1}{k} \\
&\implies l \leqslant \lim_{k\to\infty} x_{n_k} < l.
\end{aligned}$$

By the squeeze theorem, $(x_{n_k})_k \to l$. Hence $l, u \in \overline{X}$. $\square$

(3) * Let $(x_n)_n$ be bounded.

(i) $\liminf x_n \leqslant \limsup x_n$.

(ii) $(x_n)_n \to x$ if and only if $\liminf x_n = \limsup x_n = x$.

Proof. (i) Note that $l_m \leqslant u_m$ for all $m \geqslant 1$. Taking the limit $m \to \infty$ gives $l \leqslant u$.

(ii) ($\Rightarrow$) If $(x_n)_n \to x$, then every subsequence $(x_{n_k})_k \to x$. But we showed in (2) that there exists subsequences which converge to l and u. Whence $x = l = u$. ($\Leftarrow$) If $l = u = x$, then $\overline{X} = [x, x] = \{x\}$. Hence every subsequence $(x_{n_k})_k \to x$. Thus $(x_n)_n \to x$. $\square$

Examples

Cauchy Sequences

Definitions

(1) A sequence $(x_n)_n$ is *Cauchy* if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (m, n \in \mathbf{N})(m, n \geqslant N \implies d(x_n, x_m) < \epsilon)$$

(2) A sequence $(x_n)_n$ is *contractive* if there exists $0 < \rho < 1$ with $|x_{n+1} - x_n| \leqslant \rho |x_n - x_{n-1}|$ for all $n \geqslant 2$. We say ρ is the *constant of contraction*.

Theorems/Propositions/Lemmas

(1) Cauchy sequences are bounded.

Proof. Pick $\epsilon = 1$. Then $(\exists N \in \mathbf{N}) \ni (\forall m, n \in \mathbf{N})(m, n \geqslant N \implies |x_n - x_m| < 1)$. Let $c = \max\{|x_1|, ..., |x_N|\}$. But consider that:

$$n \geqslant N \implies |x_n| = |x_n - x_N + x_N| \leqslant |x_n - x_N| + |x_N| < 1 + |x_N|.$$

So $|x_n| \leqslant c'$, where $c' = \max\{c, 1 + |x_N|\}$. □

(2) If $(x_n)_n$ is Cauchy and there exists a subsequence $(x_{n_k})_k \to x$, then $(x_n)_n \to x$.

Proof. Since $(x_n)_n$ is Cauchy, given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $n, n_k \geqslant N$ implies $d(x_n, x_{n_k}) < \frac{\epsilon}{2}$. Since $(x_{n_k})_k \to x$, given $\epsilon > 0$ there exists $K \in \mathbf{N}$ such that $k \geqslant K$ implies $d(x_{n_k}, x) < \frac{\epsilon}{2}$. Let $J = \max\{K, N\}$. For $n, n_k, k \geqslant J$:

$$|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leqslant |x_n - x_{n_k}| + |x_{n_k} - x| < \epsilon.$$

□

(3) Let $(x_n)_n$ be a sequence. $(x_n)_n$ is Cauchy if and only if $(x_n)_n$ converges.

Proof. $(\Rightarrow)$ Suppose $(x_n)_n \to x$. Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that $n \geqslant N$ implies $|x_n - x| < \frac{\epsilon}{2}$. For $m, n \geqslant N$, we have $|x_n - x_m| = |x_n - x + x - x_m| \leqslant |x_n - x| + |x - x_m| < \epsilon$. $(\Leftarrow)$ If $(x_n)_n$ is Cauchy then $(x_n)_n$ is bounded. Bolzano-Weierstrass theorem says there exists a convergent subsequence. By (2) $(x_n)_n$ converges. □

(4) Contractive Sequences are Cauchy.

Proof. Let $(x_n)_n$ be a contractive sequence. Observe that:

$$\begin{aligned}
&|x_3 - x_2| \leqslant \rho |x_2 - x_1| \\
&|x_4 - x_3| \leqslant \rho |x_3 - x_2| \leqslant \rho^2 |x_2 - x_1| \\
&\qquad\qquad \vdots
\end{aligned}$$

Inductively, $|x_{n+1} - x_n| \leqslant \rho^{n-1}|x_2 - x_1|$. For $m > n$:

$$\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - ... + x_{n+1} - x_n| \\
&\leqslant |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + ... + |x_{n+1} - x_n| \\
&\leqslant \rho^{m-2}|x_2 - x_1| + \rho^{m-3}|x_2 - x_1| + ... + \rho^{n-1}|x_2 - x_1| \\
&= \rho^{n-1}|x_2 - x_1|(1 + \rho + \rho^2 + ... + \rho^{m-n-1}) \\
&= \rho^{n-1}|x_2 - x_1|\frac{1 - \rho^{m-n}}{1 - \rho} \\
&\leqslant \frac{\rho^{n-1}}{1 - \rho}|x_2 - x_1| \\
&= \frac{\rho^n}{\rho(1 - \rho)}|x_2 - x_1|.
\end{aligned}$$

Given $\epsilon > 0$, find N so large that $\frac{\rho^n}{\rho(1-\rho)}|x_2 - x_1| < \epsilon$. When $m, n \geqslant N$, then $|x_m - x_n| \leqslant \frac{\rho^n}{\rho(1-\rho)}|x_2 - x_1| < \epsilon$. $\square$

Examples

Sequences of Functions

Definitions

(1) A *function space* is a set of functions between two fixed sets, denoted $\mathcal{F}(\Omega, X) = \{f \mid f : \Omega \to X\}$.

(2) A sequence $(f_n)_n$ in $\mathcal{F}(\Omega, \mathbf{R})$ converges *pointwise* to $f \in \mathcal{F}(\Omega, \mathbf{R})$ if $(\forall x \in \Omega)((f_n(x))_n \to f(x))$. In particular:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geqslant N \implies d(f_n(x), f(x)) < \epsilon).$$

(3) A sequence $(f_n)_n$ in $\mathcal{F}(\Omega, \mathbf{R})$ converges *uniformly* to $f \in \mathcal{F}(\Omega, \mathbf{R})$ if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(\forall x \in \Omega)(n \geqslant N \implies d(f_n(x), f(x)) < \epsilon).$$

Equivalently (and sometimes preferably):

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geqslant N \implies \sup_{x \in \Omega} |f_n(x) - f(x)| < \epsilon).$$

Theorems/Propositions/Lemmas

(1) Let $(f_n)_n \in \mathcal{F}(\Omega, \mathbf{R})^{\mathbf{N}}$.

(i) If $(f_n)_n \to f$ uniformly, then $(f_n)_n \to f$ pointwise.

(ii) If $(f_n)_n \to f$ and $(f_n)_n \to f'$ pointwise, then $f = f'$.

Proof. (i) Let $x \in \Omega$ be given. We have that $|f_n(x) - f(x)| \leqslant d_u(f_n, f)$. Since $(f_n)_n$ converges uniformly, $(d_u(f_n, f))_n \to 0$. Hence by "Lemma", $(f_n(x))_n \to f(x)$; i.e., $(f_n)_n$ converges pointwise.

(ii) Let $\epsilon > 0$ be given. Since $(f_n)_n \to f$, for all $x \in \Omega$ there exists $N_1 \in \mathbf{N}$ such that $n \geqslant N_1$ implies $|f_n(x) - f(x)| \leqslant \frac{\epsilon}{2}$. Since $(g_n)_n \to f$, for all $x \in \Omega$ there exists $N_2 \in \mathbf{N}$ such that $n \geqslant N_2$ implies $|g_n(x) - g(x)| \leqslant \frac{\epsilon}{2}$. For $n \geqslant \max\{N_1, N_2\}$:

$$\begin{aligned} |f(x) - g(x)| &= |f(x) - f_n(x) + f_n(x) - g(x)| \\ &\leqslant |f(x) - f_n(x)| + |f_n(x) - g(x)| \\ &< \epsilon. \end{aligned}$$

This holds for all $\epsilon > 0$, hence $|f(x) - g(x)| = 0$; i.e., $f(x) = g(x)$. Since f and g are equal on every x, $f = g$. $\square$

(2) $(f_n)_n \not\to f$ uniformly if and only if:

$$(\exists \epsilon > 0)(\exists (f_{n_k})_k)(\exists (x_k)_k) \ni |f_{n_k}(x_k) - f(x_k)| \geqslant \epsilon_0.$$

Proof. ($\Rightarrow$) By definition, $(f_n)_n \not\to f$ uniformly if:

$$(\exists \epsilon_0 > 0)(\forall k \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(\exists x \in \Omega)(n \geqslant N \ \wedge \ |f_n(x) - f(x)| \geqslant \epsilon_0).$$

Using this $\epsilon_0 > 0$:

$$\begin{aligned} k = 1 &\implies (\exists x_1 \in \Omega)(\exists n_1 \in \mathbf{N})(n_1 \geqslant 1 \ \wedge \ |f_{n_1}(x_1) - f(x_1)| \geqslant \epsilon_0) \\ k = n_1 + 1 &\implies (\exists x_2 \in \Omega)(\exists n_2 \in \mathbf{N})(n_2 \geqslant n_1 \ \wedge \ |f_{n_2}(x_2) - f(x_2)| \geqslant \epsilon_0) \\ &\vdots \end{aligned}$$

Inductively, we obtain sequences $(x_k)_k$ and $(f_{n_k})_k$ with $|f_{n_k}(x_k) - f(x_k)| \geqslant \epsilon_0$. ($\Leftarrow$). $\square$

(3) Let $(f_n)_n$ and $(g_n)_n$ be sequences in $\ell_\infty(\Omega)$ with $(f_n)_n \to f$ and $(g_n)_n \to g$ uniformly in Ω. Prove that $(f_n g_n)_n \to fg$ uniformly on Ω.

Proof. Let:

$$\begin{aligned} |f_n(x)| &\leqslant U \\ |g(x)| &\leqslant P. \end{aligned}$$

Given $\epsilon > 0$:

$$\begin{aligned} &(\exists N_1 \in \mathbf{N}) \ni (\forall x \in \Omega)(\forall n \in \mathbf{N}) \left(n \geqslant N_1 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2P} \right), \\ &(\exists N_2 \in \mathbf{N}) \ni (\forall x \in \Omega)(\forall n \in \mathbf{N}) \left(n \geqslant N_2 \implies |g_n(x) - g(x)| < \frac{\epsilon}{2U} \right). \end{aligned}$$

Then for all $n \geqslant \max\{N_1, N_2\}$ and $x \in \Omega$:

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\ &\leqslant |f_n(x)(g_n(x) - g(x))| + |g(x)(f_n(x) - f(x))| \\ &\leqslant |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ &< U\frac{\epsilon}{2U} + P\frac{\epsilon}{2P} \\ &= \epsilon. \end{aligned}$$

$\square$

Examples

(1) Let $(f_n)_n \in \mathcal{F}([0,1], \mathbf{R})^{\mathbf{N}}$ be defined by $f_n(x) = x^n$. Determine if f converges pointwise.

Solution. Given $x \in [0, 1)$, note that $(f_n(x))_n = (x^n)_n \to 0$ (geometric). Given $x = 1$, note that $(f_n(x))_n = (1^n)_n \to 1$. Define $f : [0,1] \to \mathbf{R}$ by $f = \mathbf{1}_{\{1\}}$. Then $(f_n)_n \to f$.

(2) Let $(f_n)_n \in \mathcal{F}(\mathbf{R}, \mathbf{R})^{\mathbf{N}}$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}$. Determine if $(f_n)_n$ converges pointwise.

Solution. If $x = 0$, then $(f_n(0))_n \to (0)_n \to 0$. If $x \neq 0$, note that:

$$\begin{aligned} |f_n(x)| &= \left| \frac{nx}{1+n^2x^2} \right| \\ &\leqslant \frac{n|x|}{n^2|x^2|} \\ &= \frac{1}{n|x|}. \end{aligned}$$

Since $\left(\frac{1}{n|x|}\right) \to 0$, $(f_n(x))_n \to 0$. Hence $(f_n)_n \to \mathbf{0}$.

(3) Let $(h_n)_n \in \mathcal{F}([0,\infty), \mathbf{R})^{\mathbf{N}}$ be defined by $h_n(x) = x^{\frac{1}{n}}$. Determine if $(h_n)_n$ converges pointwise.

Solution. If $x > 0$, $(h_n(x)) = (x^{\frac{1}{n}})_n \to 1$. If $x = 0$, $(h_n(0))_n \to (0^{\frac{1}{n}})_n \to 0$. Define $h : [0,\infty) \to \mathbf{R}$ by $h = \mathbf{1}_{(0,\infty)}$. Then $(h_n)_n \to h$ pointwise.

(4) Let $(h_n)_n \in \mathcal{F}([0,\infty), \mathbf{R})^{\mathbf{N}}$ be defined by $h_n(x) = e^{-nx}$ We have that $(h_n)_n \to \mathbf{1}_{\{0\}}$ pointwise. Does it converge uniformly?

Solution.

<u>Series</u>

Definitions

(1) Let $(x_k)_k$ be a sequence of real numbers.

  (i) The <u>*sequence of partial sums*</u> $(s_n)_n$ is $s_n := \sum_{k=1}^n x_k$.

  (ii) If $(s_n)_n \to s$ in $\mathbf{R}$, we say the <u>*infinite series*</u> $\sum_{k=1}^\infty x_k$ converges and we write $\sum_{k=1}^\infty x_k = s$ or $\sum_{k=1}^\infty x_k < \infty$.

  (iii) If $(s_n)_n$ diverges we say that the infinite series $\sum_{k=1}^n x_k$ diverges. If $(s_n)_n$ properly diverges to $\pm\infty$, we may write $\sum_{k=1}^\infty x_k = \pm\infty$.

(2) A series $\sum x_k$ converges <u>*absolutely*</u> if $\sum |x_k| < \infty$.

(3) An <u>*alternating series*</u> is an infinite series of the form $\sum_k (-1)^k b_k$, $b_k \geqslant 0$.

Theorems/Propositions/Lemmas

(1) Let $(x_k)_k$ be a sequence and let $k_0 \in \mathbf{N}$. Then $\sum_{k=1}^\infty x_k$ converges if and only if $\sum_{k>k_0}^\infty x_k$ converges. In the case of convergence, $\sum_{k=1}^\infty x_k = \sum_{k=1}^{k_0} x_k + \sum_{k>k_0} x_k$.

Proof. ($\Rightarrow$) Suppose $\sum_{k=1}^\infty x_n = s$. Then $\sum_{k=1}^\infty x_n = \sum_{k=1}^{k_0} x_k + \sum_{k=k_0+1}^\infty x_k = s$. Rearranging gives $\sum_{k=k_0+1}^\infty x_k = s - \sum_{k=1}^{k_0} x_k$. Since $\sum_{k=1}^{k_0} < \infty$, it must be that $\sum_{k=k_0+1}^\infty x_n < \infty$. ($\Leftarrow$) Now suppose $\sum_{k=k_0+1}^\infty x_k = s$. Since $\sum_{k=1}^{k_0} x_k < \infty$, we have that $\sum_{k=1}^\infty x_k = s + \sum_{k=1}^{k_0} x_k$; i.e., the infinite series is convergent. □

(2) (Divergence Test) If $\sum_{k=1}^\infty x_k$ converges then $(x_k)_k \to 0$.

Proof. Suppose $\sum_{k=0}^\infty x_k = s$. Then $(s_n)_n \to s$. We have $x_n = s_n - s_{n-1}$. Taking the limit on both sides gives $(x_n)_n \to 0$. □

(3) Let $(x_k)_k$ be a sequence. The following are equivalent:

  (i) $\sum_{k=1}^\infty x_k$ converges.

  (ii) $(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\exists m, n \in \mathbf{N})(m > n \geqslant N \implies |\sum_{k=n+1}^m| < \epsilon)$.

  (iii) $(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni |\sum_{k>N} x_k| < \epsilon$.

  (iv) $(\sum_{k>n} x_k)_n \to 0$.

Proof. (1) $\iff$ (2). Let $s_n = \sum_{k=1}^{n} x_k$. Note that $s_m - s_n = \sum_{k=n+1}^{m} x_k$. So $\sum_{k=1}^{\infty}$ converges if and only if $(s_n)_n$ converges if and only if $(s_n)_n$ is Cauchy. (3) $\iff$ (4) This follows from definitions. (1) $\implies$ (3) Suppose $(s_n)_n \to s$. Then:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geqslant N \implies |s_n - s| < \epsilon).$$

But $s = s_n + \sum_{k>N} x_k$. So $|s - s_n| < \epsilon$ is equivalent to $|\sum_{k>N} x_k| < \epsilon$. (3) $\implies$ (1) Since $|\sum_{k>N} x_k| < \epsilon$, it converges. This is a tail, hence $\sum_{k=1}^{\infty} x_k$ converges. □

(4) Let $s_n = \sum_{k=1}^{\infty} x_k$ with $x_k \geqslant 0$ for all k. Then $\sum_{k=1}^{\infty} x_k$ converges if and only if $(s_n)_n$ is bounded.

Proof. ($\Rightarrow$) If $\sum_{k=1}^{\infty} x_k$ converges then $(s_n)_n$ converges, hence $(s_n)_n$ is bounded. ($\Leftarrow$) If $(s_n)_n$ is bounded and increasing, then by MCT $(s_n)_n$ converges, hence $\sum_{k=1}^{\infty} x_k$ converges. □

(5) (Comparison Test) Let $(x_k)_k$ and $(y_k)_k$ be sequences with $0 \leqslant x_k \leqslant y_k$.

(i) If $\sum_{k=1}^{\infty} y_k < \infty$, then $\sum_{k=1}^{\infty} x_k < \infty$ with $\sum_{k=1}^{\infty} x_k \leqslant \sum_{k=1}^{\infty} y_k$.

(ii) If $\sum_{k=1}^{\infty} x_k = \infty$, then $\sum_{k=1}^{\infty} = \infty$.

Proof. `https://www.math.uci.edu/~ndonalds/math2b/notes/11-4.pdf` □

(6) * (Limit Comparison) Let $(x_k)_k$ and $(y_k)_k$ be sequences of positive terms.

(i) If $\sum y_k < \infty$ and $\limsup \frac{x_k}{y_k} < \infty$, then $\sum x_k < \infty$.

(ii) If $\sum y_k = \infty$ and $\liminf \frac{x_k}{y_k} > 0$, then $\sum x_k = \infty$.

Proof. □