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Chapter 1

Limits

§ 1.1. Cluster Points

Definition 1.1.1. Let $c \in \mathbb{R}$ and $\delta > 0$.

(1) The δ-*neighborhood* around c is denoted $V_\delta(c) = (c - \delta, c + \delta)$.

(2) The *deleted* δ-*neighborhood* around c is denoted $\dot{V}_\delta(c) = (c - \delta, c) \cup (c, c + \delta)$.

Lemma 1.1.1. *If* $c \in \mathbb{R}$ *and* $\delta > 0$, *then:*

(1) $x \in V_\delta(c) \iff |x - c| < \delta$;

(2) $x \in \dot{V}_\delta(c) \iff 0 < |x - c| < \delta$.

Definition 1.1.2. Let $D \subseteq \mathbb{R}$. A number $c \in \mathbb{R}$ is a *cluster point of* D if

$$(\forall \epsilon > 0)(\dot{V}_\delta(c) \cap D \neq \emptyset).$$

Example 1.1.1.

(1) The cluster points of $(0, 1)$ are $[0, 1]$.

(2) The cluster points of $\mathbb{Q}$ are $\mathbb{R}$.

(3) If $F \subseteq \mathbb{R}$ is a finite set, then F has no cluster points.

Definition 1.1.3. If $D \subseteq \mathbb{R}$ is a subset, then

$$\overline{D} := \bigcap_{\substack{C \supseteq D \\ C \text{ closed in } \mathbb{R}}} C.$$

Proposition 1.1.2. *Let* $D \subseteq \mathbb{R}$ *and* $c \in \mathbb{R}$.

(1) c *is a cluster point of* D *if and only if there exists a sequence* $(x_n)_n$ *in* D *with* $x_n \neq c$ *and* $(x_n)_n \to c$.

(2) $c \in \overline{D}$ *if and only if there exists a sequence* $(x_n)_n$ *in* D *with* $(x_n)_n \to c$.

Proof. (1) ($\Rightarrow$) Let c be a cluster point of D. By induction, for each $n \geqslant 1$ there exists $x_n \in V_{\frac{1}{n}(c)} \cap D \neq \emptyset$. We obtain a sequence $(x_n)_n$ in D, satisfying $x_n \neq c$ and $|x_n - c| < \frac{1}{n}$. Whence $(x_n)_n \to c$. ($\Leftarrow$) Now suppose such a sequence $(x_n)_n$ exists. Given $\delta > 0$, there exists $N \in \mathbb{N}$ so that $n \geqslant N$ implies $|x_n - c| < \delta$. Whence $x_n \in \dot{V}_\delta(c) \cap D$.

(2) ($\Rightarrow$) Office hours □

§ 1.2. Limits

Definition 1.2.1. Let $f : D \to \mathbb{R}$ and c be a cluster point of D. Then:

$$\begin{aligned}\operatorname*{limit}_{x\to c} f(x) = L \text{ if } (\forall \epsilon > 0)(\exists \delta > 0) &\ni (x \in \dot{V}_\delta(c) \cap D \implies f(x) \in V_\epsilon(L)).\\ &\ni (x \in D, 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon).\end{aligned}$$

Example 1.2.1.

(1) Prove that $\text{limit}_{x\to 2} 3x + 4 = 10$.

Solution. Note that:

$$\begin{aligned}|f(x) - L| &= |3x + 4 - 10|\\ &= |3x - 6|\\ &= 3|x - 2|.\end{aligned}$$

If ϵ is given, pick $\delta = \frac{\epsilon}{3}$. If $|x-2| < \delta$, then $|x-2| < \frac{\epsilon}{3}$, giving $|3x+4-10| < 3|x-2| < \epsilon$.

(2) Prove that $\text{limit}_{x\to 3} x^2 = 9$.

Solution. Note that:

$$\begin{aligned}|f(x) - L| &= |x^2 - 9|\\ &= |x - 3||x + 3|.\end{aligned}$$

If $0 < |x - 3| < 1$ don't get the rest of the examples

Proposition 1.2.1 (Sequential Characterization of a Limit)**.** *Let $f : D \to \mathbb{R}$ and c a cluster point of D. The following are equivalent:*

(1) $\operatorname*{limit}_{x\to c} f(x) = L$;

(2) $(\forall (x_n)_n \in D^{\mathbb{N}})(x_n \neq c \wedge (x_n)_n \to c \implies (f(x_n))_n \to L)$.

Proof. ($\Rightarrow$) Suppose $\text{limit}_{x\to c} f(x) = L$. Let $(x_n)_n$ be in D with $x_n \neq c$ and $(x_n)_n \to c$. Given $\epsilon > 0$, we know there exists $\delta > 0$ such that $x \in D$ and $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$. We know there exists some $N \in \mathbb{N}$ with $n \geqslant N$ implying $|x_n - c| < \delta$. Whence $|f(x_n) - L| < \epsilon$; i.e., $(f(x_n))_n \to L$.

($\Leftarrow$) Towards a contradiction, suppose that for every sequence $(x_n)_n$ in D such that $x_n \neq c$ and $(x_n)_n \to c$, it holds that $(f(x_n))_n \to L$, *yet* $\text{limit}_{x\to c} f(x) \neq L$. Then by definition:

$$(\exists \epsilon_0 > 0)(\forall \delta > 0) \ni (x \in \dot{V}_\delta(c) \cap D \ \wedge \ f(x) \notin V_{\epsilon_0}(L)).$$

So for each $\delta = \frac{1}{n}$, we can find $x_n \in \dot{V}_{\frac{1}{n}}(c) \cap D$ and $f(x_n) \notin V_{\epsilon_0}(L)$, or equivalently $(x_n)_n \to c$ and $(f(x_n))_n \not\to L$. This is a contradiction, since $(x_n)_n \to c$ implies $(f(x_n))_n \to L$. This establishes that $\text{limit}_{x\to c} f(x) = L$. ◻

Theorem 1.2.2 (Sequential Characterization of Divergence I). *Let* $f : D \to \mathbb{R}$ *and* c *a cluster point of* D. *The following are equivalent:*

(1) $\underset{x\to c}{\text{limit}}\, f(x) \neq L$;

(2) $(\exists (x_n)_n \in D^{\mathbb{N}})\big((x_n \neq c \wedge (x_n)_n \to c) \ \wedge \ (f(x_n))_n \not\to L\big)$

Proof. This follows from negating Proposition 1.2.1. ◻

Theorem 1.2.3 (Sequential Characterization of Divergence II). *Let* $f : D \to \mathbb{R}$ *and* c *a cluster point of* D. *The following are equivalent:*

(1) $\underset{x\to c}{\text{limit}}\, f(x)$ *does not exist;*

(2) $(\exists (x_n)_n \in D^{\mathbb{N}})\big((x_n \neq c \wedge (x_n)_n \to c) \ \wedge \ (f(x_n))_n \ \textit{diverges}\big)$

Proof. ($\Leftarrow$) This direction follows from the converse of Proposition 1.2.1. ($\Rightarrow$) Let $(y_n)_n$ be a sequence in D with $y_n \neq c$ and $(y_n)_n \to c$. We proceed by cases. Case 1: $(f(y_n))_n$ is divergent. Then we are done. Case 2: don't understand this shit at all ◻

Example 1.2.2.

(1) d.n.e. examples, do later

Theorem 1.2.4. *Suppose* $f, g : D \to \mathbb{R}$ *and* c *is a cluster point of* D.

(1) If $\text{limit}_{x\to c} = L_1$ *and* $\text{limit}_{x\to c}\, g(x) = L_2$, *then:*

(i) $\underset{x\to c}{\text{limit}}(f(x) \pm g(x)) = L_1 \pm L_2$;

(ii) $\underset{x\to c}{\text{limit}}(\alpha f(x)) = \alpha L_1$ *for some* $\alpha \in \mathbb{R}$;

(iii) $\underset{x\to c}{\text{limit}}\, f(x)g(x) = L_1 L_2$;

(iv) $\underset{x\to c}{\text{limit}}\, \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$ *if* $L_2 \neq 0$.

(2) $\underset{x\to c}{\text{limit}}\, |f(x)| = |L_1|$.

(3) $\operatorname{limit}_{x\to c} \sqrt{f(x)} = \sqrt{L_1}$ *if* $f(x) \geqslant 0$ *for all* $x \in D$.

(4) *If* $f \in \mathbb{R}[x]$, *then:*

  (1) $\operatorname{limit}_{x\to c} f(x) = f(c)$;

  (2) *If* $f(x) = \frac{p(x)}{q(x)}$ *with* $q(c) \neq 0$, *then* $\operatorname{limit}_{x\to c} f(x) = f(c)$.

Proof. These follow from previous results related to sequences. □

Theorem 1.2.5. *Let* $f : D \to \mathbb{R}$ *and* c *a cluster point of* D.

(1) *If* $f(x) \leqslant b$ *for all* $x \in \dot{V}_\delta(c)$ *and* $\operatorname{limit}_{x\to c} f(x) = L$ *exists, then* $L \leqslant b$.

(2) *If* $f(x) \geqslant a$ *for all* $x \in \dot{V}_\delta(c)$ *and* $\operatorname{limit}_{x\to c} f(x) = L$ *exists, then* $L \geqslant a$.

Proof. (1) Let $(x_n)_n$ be a sequence in $\dot{V}_\delta(c)$ with $(x_n)_n \to c$. We know $(f(x_n))_n \to L$, and since $f(x_n) \leqslant b$ for all n, so must $L \leqslant b$.

(2) This follows similarly. □

Theorem 1.2.6. *Let* $f, g, h : D \to \mathbb{R}$ *and* c *a cluster point of* D. *Suppose* $f(x) \leqslant g(x) \leqslant h(x)$ *with* $x \in \dot{V}_\delta(c)$ *for some* $\delta > 0$. *If* $\operatorname{limit}_{x\to c} f(x) = \operatorname{limit}_{x\to c} h(x) = L$, *then* $\operatorname{limit}_{x\to c} g(x) = L$.

Proof. sequences. □

§ 1.3. Left and Right Limits

Definition 1.3.1. Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$.

(1) Let c be a cluster point of $D \cap (c, \infty)$. Then

$$\begin{aligned}\operatorname{limit}_{x\to c+} f(x) = L \text{ if } (\forall \epsilon > 0)(\exists \delta > 0) &\ni (x \in D \cap (c, c+\delta) \implies f(x) \in V_\epsilon(L)) \\ &\ni (x \in D, 0 < x - c < \delta \implies |f(x) - L| < \epsilon).\end{aligned}$$

(2) Let c be a cluster point of $D \cap (-\infty, c)$. Then:

$$\begin{aligned}\operatorname{limit}_{x\to c^-} f(x) = L \text{ if } (\forall \epsilon > 0)(\exists \delta > 0) &\ni (x \in D \cap (c-\delta, c) \implies f(x) \in V_\epsilon(L)) \\ &\ni (x \in D, 0 < c - x < \delta \implies |f(x) - L| < \epsilon).\end{aligned}$$

Proposition 1.3.1. *Let* $f : D \to \mathbb{R}$ *and* c *a cluster point of* D. *Then:*

$$\operatorname{limit}_{x\to c} f(x) = L \iff \operatorname{limit}_{x\to c^\pm} f(x) = L.$$

Proposition 1.3.2. *Let* $f : D \to \mathbb{R}$ *and* c *a cluster point of* $D \cap (c, \infty)$. *Then* $\text{limit}_{x \to c^+} f(x) = L$ *if and only if:*

$$(\forall (x_n)_n \in (D \cap (c, \infty))^{\mathbb{N}})((x_n)_n \to c \implies (f(x_n))_n \to L).$$

Proposition 1.3.3. *Let* $f : D \to \mathbb{R}$ *and* c *a cluster point of* $D \cap (-\infty, c)$. *Then* $\text{limit}_{x \to c^-} f(x) = L$ *if and only if:*

$$(\forall (x_n)_n \in (D \cap (-\infty, c))^{\mathbb{N}})((x_n)_n \to c \implies (f(x_n))_n \to L).$$

§ 1.4. Infinite Limits

Definition 1.4.1. Let $f : D \to \mathbb{R}$ and c a cluster point of D. Then:

(1) $\underset{x \to c}{\text{limit}} f(x) = +\infty \iff (\forall M > 0)(\exists \delta > 0) \ni (x \in \dot{V}_\delta(c) \implies f(x) > M)$.

(2) $\underset{x \to c}{\text{limit}} f(x) = -\infty \iff (\forall M > 0)(\exists \delta > 0) \ni (x \in \dot{V}_\delta(c) \implies f(x) < -M)$.

Note 1. The definitions for left-handed and right-handed limits follow similarly.

Example 1.4.1. Show that $\underset{x \to 1}{\text{limit}} \dfrac{3}{(x-1)^2} = \infty$.

Solution. Let M be given. Then:

$$\begin{aligned} g(x) > M &\iff \frac{3}{(x-1)^2} > M \\ &\iff \frac{3}{M} > (x-1)^2 \\ &\iff \sqrt{\frac{3}{M}} > |x-1|. \end{aligned}$$

So given M, let $\delta = \sqrt{\frac{3}{M}}$. If $0 < |x - 1| < \delta$, then by above, $g(x) > M$.

Example 1.4.2. Show that $\underset{x \to 3^-}{\text{limit}} \dfrac{-2}{3-x} = -\infty$.

Solution. Let M be given. Then:

$$\begin{aligned} g(x) < -M &\iff \frac{-2}{3-x} < -M \\ &\iff \frac{2}{3-x} > M \\ &\iff \frac{2}{M} > 3 - x \qquad \text{sign did not flip since } x < 3. \end{aligned}$$

So given M, let $\delta = \frac{2}{M}$. For $x \in (3 - \delta, 3)$, we have $3 - x < \delta$. By work above, $g(x) < -M$.

§ 1.5. Limits at Infinity

Definition 1.5.1. Let $f : (a, \infty) \to \mathbb{R}$.

(1) $\underset{x\to\infty}{\text{limit}}\, f(x) = L \iff (\forall\epsilon > 0)(\exists\alpha > 0) \ni (x > \alpha \implies f(x) \in V_\epsilon(L))$.

(2) $\underset{x\to+\infty}{\text{limit}}\, f(x) = \infty \iff (\forall M > 0)(\exists\alpha) \ni (x > \alpha \implies f(x) > M)$.

Example 1.5.1. dont wanna type

Remark. Each of these limits at infinity have sequential expressions. figure it out and reformat.

Proposition 1.5.1. *Let* $f : (a, \infty) \to \mathbb{R}$*. Then:*

(1) $\underset{x\to\infty}{\text{limit}}\, f(x) = L \iff ()$

Proposition 1.5.2. *something about limits of rational functions*

Corollary 1.5.3. *something about polynomials.*

Chapter 2
Continuity

§ 2.1. Continuity

Definition 2.1.1. Let $f : D \to \mathbb{R}$ be a function and let $c \in D$.

(1) f is *continuous at* $x = c$ if:

$$\begin{aligned}(\forall \epsilon > 0)(\exists \delta > 0) &\ni (|x - c| < \delta, x \in D \implies |f(x) - f(c)| < \epsilon)\\ &\ni (x \in D \cap V_\delta(c) \implies f(x) \in V_\epsilon(f(c)))\\ &\ni f(D \cap V_\delta(c)) \subseteq V_\epsilon(f(c)).\end{aligned}$$

(2) f is *continuous on* D if f is continuous at every $c \in D$.

Proposition 2.1.1. *Let* c *be a cluster point of* D *with* $c \in D$*. Then* f *is continuous at* $x = c$ *if and only if* $\text{limit}_{x \to c} f(x) = f(c)$.

Proposition 2.1.2. *Let* $f : D \to \mathbb{R}$ *and* $c \in D$*. Then* f *is continuous at* $x = c$ *if and only if:*

$$(\forall (x_n)_n \in D^{\mathbb{N}})((x_n)_n \to c \implies (f(x_n))_n \to f(c)).$$

Example 2.1.1.

(1) Polynomial and rational functions are continuous.

(2) Given that $\text{limit}_{x \to c} x = c$, we've shown that $\text{limit}_{x \to c} \sqrt{x} = \sqrt{c}$ for $c \geqslant 0$. Whence $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Note 2. The negation of Proposition 2.1.2 is:

$$(\exists (x_n)_n \in D^{\mathbb{N}})((x_n)_n \to c \wedge (f(x_n))_n \nrightarrow f(c)).$$

Example 2.1.2. Show that $\text{sign} : \mathbb{R} \to \mathbb{R}$ is not continuous at $x = 0$.

Solution. Let $x_n = \frac{1}{n}$. Then $(x_n)_n \to 0$. But $(\text{sign}(x_n))_n = (1)_n \to 1 \neq \text{sign}(0)$.

Example 2.1.3. Show that $\mathbb{1}_{\mathbb{Q}}(x)$ is not continuous at every point.

Solution. Fix any $c \in \mathbb{R}$. We proceed by cases.

Case 1: $c \in \mathbb{Q}$. Since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, we can find a sequence $(r_n)_n \in (\mathbb{R} \setminus \mathbb{Q})^{\mathbb{N}}$ with $(r_n)_n \to c$. Then $(\mathbb{1}_{\mathbb{Q}}(r_n))_n = (0)_n \to 0 \neq \mathbb{1}_{\mathbb{Q}}(c)$.

Case 2: $c \in \mathbb{R} \setminus \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a sequence $(r_n)_n \in \mathbb{Q}^{\mathbb{N}}$ with $(r_n)_n \to c$. Then $(\mathbb{1}_{\mathbb{Q}}(r_n))_n = (1)_n \to 1 \neq \mathbb{1}_{\mathbb{Q}}(c)$.

Definition 2.1.2. A function F is said to be an *extension* of another function f if:

(1) $x \in \mathrm{Dom}(f)$ implies $x \in \mathrm{Dom}(F)$;

(2) $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(F)$;

(3) $F|_{\mathrm{Dom}(f)} = F$.

Example 2.1.4. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = x \sin\left(\frac{1}{x}\right)$. Note that this function is not continuous at $x = 0$ since $0 \notin \mathrm{Dom}(f)$. This discontinuity is removable however; consider the function:

$$\widetilde{f}(x) = \begin{cases} f(x), & x = 0 \\ 0, & x = 0. \end{cases}$$

Then $\widetilde{f}$ is continuous on $\mathbb{R}$ and extends f.

Example 2.1.5. Let $g : (0, \infty) \to \mathbb{R}$ be defined by $g(x) = sin\left(\frac{1}{x}\right)$. Note that g is continuous on $(0, \infty)$. However, we are unable to extend g to a continuous function on $[0, \infty)$.

By way of contradiction, suppose such an $f : [0, \infty) \to \mathbb{R}$ exists. Then:

$$\begin{aligned} f(0) &= \operatorname*{limit}_{x \to 0} f(x) \\ &= \operatorname*{limit}_{x \to 0} g(x) \text{ which does not exists. } \perp \end{aligned}$$

Definition 2.1.3. A function $f : D \to \mathbb{R}$ is *Lipschitz* with constant $c \geqslant 0$ if $|f(x) - f(y)| \leqslant c|x - y|$. When $c = 1$, f is called a *contraction*. When $|f(x) - f(y)| = |x - y|$, f is called an *isometry*.

Proposition 2.1.3. *If f is Lipschitz, then f is continuous*

Proof. Let $c \in \mathrm{Dom}(f)$ and $(x_n)_n \to c$ be any sequence in $\mathrm{Dom}(f)$. By definition:

$$|f(x_n) - f(c)| \leqslant k|x_n - c|.$$

Since $(x_n - c)_n \to 0$, by "Lemma" $(f(x_n))_n \to f(c)$. Whence f is continuous for all $c \in \mathrm{Dom}(f)$. ☐

Theorem 2.1.4 (Extreme Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. We have:*

(1) f *is always bounded;*

(2) There exists x_M, x_m *such that:*

$$\sup_{x \in [a,b]} f(x) = f(x_M)$$
$$\inf_{x \in [a,b]} f(x) = f(x_m).$$

Proof. (1) Suppose towards contradiction that f is not bounded. Then:

$$(\forall n \geqslant 1)(\exists x_n) \ni |f(x_n)| \geqslant n.$$

We inductively obtain a sequence $(x_n)_n \in [a,b]^{\mathbb{N}}$. By the Bolzano-Weierstass theorem, there exists a convergent subsequence $(x_{n_k})_k \longrightarrow x_0 \in [a,b]$. Now since f is continuous, $(f(x_{n_k}))_k \longrightarrow f(x_0)$; i.e., $(f(x_{n_k}))_k$ is bounded. $\perp$ But $|f(x_{n_k})_k| \geqslant n_k$.

(2) Let $u = \sup_{x \in [a,b]} f(x) < \infty$. Note that:

$$(\forall n \in \mathbb{N})(\exists x_n \in [a,b]) \ni \left(u - \frac{1}{n} < f(x_n) \leqslant u\right).$$

By Bolzano-Weierstrass, there exists a subsequence $(x_{n_k})_k \longrightarrow x_0$ for some $x_0 \in [a,b]$. Since f is continuous, $(f(x_{n_k}))_k \longrightarrow f(x_0)$ But since $(f(x_n))_n \longrightarrow u$, it must be that $f(x_0) = u$.

A similar argument follows for $\inf_{x \in [a,b]} f(x) = f(x_m)$. □

Lemma 2.1.5 (Contagion Lemma)**.** *Let* $y = f(x)$ *be continuous at* $x = c$.

(1) If $f(c) > 0$, *then there exists* $\delta > 0$ *such that* $f(x) \geqslant \frac{f(c)}{2} > 0$ *for all* $x \in V_\delta(c)$.

(2) If $f(c) < 0$, *then there exists* $\delta > 0$ *such that* $f(x) \leqslant \frac{f(c)}{2} < 0$ *for all* $x \in V_\delta(c)$.

Proof. (1) Let $\epsilon = \frac{f(c)}{2}$. Then $V_\epsilon(f(c)) = \left(\frac{f(c)}{2}, \frac{3f(c)}{2}\right)$. Since f is continuous, there exists $\delta > 0$ such that $x \in V_\delta(c)$ implies $f(x) \in V_\epsilon(f(c))$. Whence $f(x) > \frac{f(c)}{2}$.

(2) This follows similarly. □

Lemma 2.1.6 (Location of Roots)**.** *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be continuous with* $f(a)f(b) < 0$. *Then there exists* $c \in (a,b)$ *with* $f(c) = 0$.

Proof. Without loss of generality, suppose $f(a) < 0$ and $f(b) > 0$. Let $N := \{x \in [a,b] \mid f(x) < 0\}$. Note that $N \neq \emptyset$ because $a \in N$. Moreover, N is bounded. So $c := \sup N$ exists. By the contagion lemma, there exists $\delta > 0$ so that $f(x) < 0$ on $V_\delta(a)$. Hence $c \neq a$. Similarly, there exists $\eta > 0$ so that $f(x) > 0$ on $V_\eta(b)$. So $c \neq b$. Thus $a < c < b$.

If $f(c) < 0$, then by the contagion lemma there exists $\delta > 0$ so that $f(x) < 0$ on $V_\delta(c)$. So $\sup N > c$. $\perp$

If $f(c) > 0$, then by the contagion lemma there exists $\delta > 0$ so that $f(x) > 0$ on $V_\delta(c)$. By the supremum property, there exists $x \in N$ with $c - \delta < x \leqslant c$. Thus $f(x) < 0$ and $f(x) > 0$. $\perp$

Thus $f(c) = 0$. □

Theorem 2.1.7 (Initial Value Theorem)**.** *Let* I *be an interval and* $f : I \to \mathbb{R}$ *a continuous function. If* $[a, b] \subseteq I$ *and* $k \in \mathbb{R}$ *with* $f(a) < k < f(b)$ *or* $f(a) > k > f(b)$*, then there exists* $c \in (a, b)$ *with* $f(c) = k$*.*

Proof. Let $g(x) = f(x) - k$. Then $g(a)g(b) < 0$. By location of roots, there exists $c \in (a, b)$ with $g(c) = 0$. Thus $f(c) = k$. □

Corollary 2.1.8. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. If* $k \in \left[\inf_{[a,b]} f, \sup_{[a,b]} f\right]$*, then there exists* $c \in [a, b]$ *so that* $f(c) = k$*.*

Proof. By the Extreme Value Theorem, there exists x_m, x_M such that $\inf f = f(x_m)$ and $\sup f = f(x_M)$. Without loss of generality, suppose $x_m \leqslant x_M$. Then applying the Initial Value Theorem on $[x_m, x_M]$ says there exists $c \in (x_m, x_M)$ with $f(c) = k$. □

Corollary 2.1.9. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then there exists* $c \leqslant d$ *with* $f([a, b]) = [c, d]$*.*

Proof. This follows directly from the Extreme Value Theorem and the previous corollary. □

Corollary 2.1.10. *If* I *is any interval and* $f : I \to \mathbb{R}$ *is continuous, then* $f(I)$ *is an interval.*

Proof. Homework. □

Corollary 2.1.11. *Let* $p(x)$ *be a polynomial of odd degree. Then there exists* $z \in \mathbb{R}$ *with* $p(z) = 0$*.*

Proof. Suppose the leading term of $p(x)$ is positive. Since $\deg(p)$ is odd,

$$\operatorname*{limit}_{x \to \infty} p(x) = \infty$$
$$\operatorname*{limit}_{x \to -\infty} p(x) = -\infty.$$

With $M = 1$, there exists α such that $x \geqslant \alpha$ implies $p(x) \geqslant 1$. Similarly, there exists β such that $x \leqslant \beta$ implies $p(x) \leqslant -1$.

We can find $x_1 < x_2$ with $p(x_1)p(x_2) < 0$. Applying the location of roots lemma gives the desired result. □

§ 2.2. Uniform Continuity

Definition 2.2.1. Let $f : D \to \mathbb{R}$ be a function. Then f is *uniformly continuous on* D if:

$$(\forall \epsilon > 0)(\exists \delta > 0) \ni (\forall u, v \in D)(|u - v| < \delta \implies |f(u) - f(v)| < \epsilon).$$

Proposition 2.2.1. *If f is Lipschitz then f is uniformly continuous.*

Proof. We have $|f(x) - f(y)| \leqslant c|x - y|$ for all $x, y \in D$. Given ϵ, let $\delta = \frac{\epsilon}{c}$. If $|u - v| < \delta$, then:

$$\begin{aligned} |f(u) - f(v)| &\leqslant c|u - v| \\ &< c\delta \\ &< \epsilon. \end{aligned}$$

□

Proposition 2.2.2. *If $f : D \to \mathbb{R}$ is uniformly continuous on D, f is continuous on D.*

Proof. Let $c \in D$. We want to show continuity at $x = c$. Let $\epsilon > 0$. Choose $\delta > 0$ as in the definition of uniform continuity. Then $x \in D$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$. ? □

Recall. f is continuous at $x = c$ if:

$$(\forall \epsilon > 0)(\exists \delta > 0) \ni |x - c| < \delta \implies |f(x) - f(x)| < \epsilon.$$

Note that δ might depend on ϵ and c.

$f : D \to \mathbb{R}$ is cts on D if f is continuous at every point $c \in D$.

f is *uniformly conitnuous at* D if:

$$(\forall \epsilon > 0)(\exists \delta > 0) \ni (\forall u, v \in D)(|u - v| < \delta \implies |f(u) - f(v)| < \epsilon).$$

Example 2.2.1. Let $f(x) = \frac{1}{x}$ on $[a, \infty)$, where $a > 0$ is fixed. But notice that:

$$|f(u) - f(v)| = \left|\frac{1}{u} - \frac{1}{v}\right| = \frac{|u - v|}{uv} \leqslant \frac{1}{a^2}|u - v|$$

This function is Lipschitz with constant $\frac{1}{a^2}$. So f is uniformly continuous on $[a, \infty)$.

However, f is <u>not</u> uniformly continuous on $(0, \infty)$.

Proposition 2.2.3. *Let $f : D \to \mathbb{R}$. The following are equivalent:*

(1) f is not uniformly continuous on D;

(2) $(\exists\epsilon_0 > 0)(\forall\delta > 0) \ni (\exists u_\delta, v_\delta \in D)(|u_\delta - v_\delta| \wedge |f(u_\delta) - f(v_\delta)| \geqslant \epsilon_0)$.

(3) $(\exists\epsilon_0 > 0) \ni (\exists (u_n)_n, (v_n)_n \in D^{\mathbb{N}})((u_n - v_n)_n \to 0 \wedge |f(u_n) - f(v_n)| \geqslant \epsilon_0)$

Example 2.2.2. Consider $f(x) = \frac{1}{x}$ on $(0, \infty)$. Let $u_n = \frac{1}{n}$ and $v_n = \frac{1}{n+1}$. We have:

$$|u_n - v_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} \leqslant \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \to 0$, $(u_n - v_n)_n \to 0$. But now:

$$|f(v_n) - f(u_n)| = |(n+1) - n| = 1 := \epsilon_0.$$

So f is not uniformly continuous on $(0, \infty)$.

Theorem 2.2.4 (Compactness Argument)**.** *If* $f : [a, b] \to \mathbb{R}$ *is continuous,* f *is uniformly continuous.*

Proof. By way of contradiction, if not uniformly continuous, we have an $\epsilon_0 > 0$, and sequences $(u_n)_n, (v_n)_n \in [a, b]^{\mathbb{N}}$ with $(u_n - v_n)_n \to 0$ and $|f(u_n) - f(v_n)| \geqslant \epsilon_0$.

Bolzano-Weierstrass says there exists a convergent subsequence $(u_{n_k})_k \to z \in [a, b]$. Observe that:

$$\begin{aligned} |v_{n_k} - z| &= |v_{n_k} - u_{n_k} + u_{n_k} - z| \\ &\leqslant |v_{n_k} - u_{n_k}| + |u_{n_k} - z|. \end{aligned}$$

Since $(v_{n_k} - u_{n_k})_k \to 0$ and $(u_{n_k} - z)_k \to 0$, we have that $(v_{n_k})_k \to z$.

But since f is continuous, we have that $(f(u_{n_k}))_k \to f(z)$ and $(f(v_{n_k}))_k \to f(z)$.

Hence $(f(u_{n_k}) - f(v_{n_k}))_k \to 0$. $\perp$ Since $|f(u_{n_k}) - f(v_{n_k})| \geqslant \epsilon_0 > 0$ □

Example 2.2.3. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Since f is continuous on $[0, 1]$, it is uniformly continuous. We will show that f is not Lipschitz. Suppose towards contradiction it is. Then:

$$|f(x) - f(y)| \leqslant c|x - y|$$

for all $x, y \in [0, 1]$. Taking $y = 0$ gives:

$$\sqrt{x} \leqslant cx$$

for all $x \in [0, 1]$. But:

$$\frac{1}{\sqrt{x}} \leqslant c$$

for all $x \in [0, 1]$ is a contradiction, as $\frac{1}{\sqrt{x}}$ is blowing up as x approaches 0.

Lemma 2.2.5. *If* $f : D \to \mathbb{R}$ *is uniformly continuous and* $(x_n)_n$ *is Cauchy, then* $(f(x_n))_n$ *is also Cauchy.*

Proof. If $\epsilon > 0$, We want to show that $|f(x_n) - f(x_m)|$ is small for large m.

We know that for $u, v \in D$, $|u - v| < \delta$ implies $|f(u) - f(v)| < \epsilon$.

Now there exists N such that $n, m \geqslant N$ implies $|x_n - x_m| < \delta$.

So if $n, m \geqslant N$, then $|x_n - x_m| < \delta$, which implies that $|f(x_n) - f(x_m)| < \epsilon$.

Thus $(f(x_n))_n$ is Cauchy. □

Theorem 2.2.6. *Let* $f : (a, b) \to \mathbb{R}$. *The following are equivalent:*

(1) f *is uniformly continuous;*

(2) There exists a continuous function $\widetilde{f} : [a, b] \to \mathbb{R}$ *such that* $\widetilde{f}(x) = f(x)$ *for all* $x \in (a, b)$.

Proof. (2) $\Rightarrow$ (1) Since $\tilde{f} : [a, b] \to \mathbb{R}$ is continuous, it is uniformly continuous. Since $\tilde{f} = f$ on $[a, b]$, f must also be uniformly continuous.

(1) $\Rightarrow$ (2) Claim: $\text{limit}_{x \to a^+} f(x)$ exists. Let $(x_n)_n$ be a sequence in (a, b) with $(x_n)_n \to a$.

Since $(x_n)_n$ is convergent, it is Cauchy. By our previous lemma $(f(x_n))_n$ is also Cauchy, hence convergent.

Say $(f(x_n))_n \to L$. Let $(y_n)_n$ be any other sequence in (a, b) with $(y_n)_n \to a$. By the same argument, $(f(y_n))_n \to L'$.

Consider $(z_n)_n = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$. Then $(z_n)_n \to a$. By the same argument again, $(f(z_n))_n \to L''$.

Since $(f(x_n))_n$ and $(f(y_n))_n$ are subsequences of $(f(z_n))_n$, we know that $(f(x_n))_n \to L''$ and $(f(y_n))_n \to L''$.

So $L = L'' = L'$. The claim is proved.

Now simply define:

$$\widetilde{f}(x) = \begin{cases} L, & x = a \\ f(x), & x \in (a, b) \\ \text{limit}_{x \to b^-} f(x), & x = b. \end{cases}$$

The above limit exists by same argument. □

Example 2.2.4. $y = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0, 1)$ because $\text{limit}_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Recall. $f_n(x) = x^n$, $(f_n)_n \to \delta_1$ pointwise but not uniformly.

Proposition 2.2.7. *If* $(f_n : D \to \mathbb{R})_n$ *is a sequence of continuous functions and* $(f_n)_n \to f$ *uniformly on* D*, then* f *is continuous.*

Proof. Let $\epsilon > 0$ be given. Fix $c \in D$.

Since f_n converges uniformly, there exists N large such that for all $n \geqslant N$ we have $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$.

Moreover, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f_N(x) - f_N(c)| < \frac{\epsilon}{3}$.

Then $|x - c| < \delta$ implies:

$$\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\
&\leqslant \dots \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

$\square$

Chapter 3
Differentiation

Throughout, I is an open interval.

§ 3.1. Differentiation

Definition 3.1.1.

(1) Let $f : I \to \mathbb{R}$ and $c \in I$ a cluster point. Then f is *differentiable at* c if:
$$\operatorname*{limit}_{x \to c} \frac{f(x) - f(c)}{x - c} := f'(c)$$
exists and is finite. We say $f'(c)$ is the *derivative of* f *at* $x = c$.

(2) If f is differentiable at every $c \in I$, we say f is differentiable at I.

Example 3.1.1. Let $f(x) = ax + b$. Then:
$$f'(c) = \operatorname*{limit}_{x \to c} \frac{f(x) - f(c)}{x - c} = \operatorname*{limit}_{x \to c} \frac{a(x - c)}{(x - c)} = a.$$

Example 3.1.2. Let $f(x) = |x|$ and $c = 0$. Then:
$$f'(0) = \operatorname*{limit}_{x \to 0} \frac{|x| - |0|}{x - 0} = \operatorname*{limit}_{x \to 0} \frac{|x|}{x}.$$
Since this limit does not exist, f is not differentiable at $c = 0$.

Example 3.1.3. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Is f differentiable at $c = 0$?

Solution. Observe that:
$$f'(0) = \operatorname*{limit}_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \operatorname*{limit}_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$
Hence f is differentiable at $c = 0$.

Proposition 3.1.1. *If* f *is differentiable at* $x = c$, *then* f *is continuous at* $x = c$.

Proof. Let $(x_n)_n$ be a sequence with $(x_n)_n \to c$, $x_n \neq c$. Note that:
$$|f(x_n) - f(c)| = \left| \frac{f(x_n) - f(c)}{x_n - c}(x_n - c) \right| = f'(c)|x_n - c|.$$
Since $f'(c)$ is a constant and $(x_n - c)_n \to 0$, by "Lemma" $(f(x_n))_n \to f(c)$. □

Theorem 3.1.2. *Let* $f, g : I \to \mathbb{R}$ *be differentiable at* $x = c$.

(1) $(\alpha f + g)'(c) = \alpha f'(c) + g'(c)$;

(2) $(f \cdot g)'(c) = f'(c)g(c) = f(c)g'(c)$;

(3) $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$ *provided* $g(c) \neq 0$.

Proof. (2) Let $(x_n)_n \in I^{\mathbb{N}}$ with $(x_n)_n \to c$, $x_n \neq c$. We have:

$$\begin{aligned}
\frac{fg(x_n) - fg(c)}{x_n - c} &= \frac{f(x_n)g(x_n) - f(c)g(c)}{x_n - c} \\
&= \frac{f(x_n)g(x_n) - f(x_n)g(c) + f(x_n)g(c) - f(c)g(c)}{x_n - c} \\
&= f(x_n)\left(\frac{g(x_n) - g(c)}{x_n - c}\right) + g(c)\left(\frac{f(x_n) - f(c)}{x_n - c}\right) \\
&\xrightarrow{n\to\infty} f(c)g'(c) + g(c)f'(c).
\end{aligned}$$

□

Proposition 3.1.3 (Power Rule)**.**

(1) If $f(x) = x^n$ *for* $n \in \mathbb{N}_0$, *then* $f'(x) = nx^{n-1}$.

(2) If $f(x) = x^n$ *for* $n \in \mathbb{Z}$, *then* $f'(x) = nx^{n-1}$.

(3) If $f(x) = x^r$ *for* $r \in \mathbb{Q}$ *then* $f'(x) = rx^{r-1}$.

Proof. (1) Induction and product rule. (2) Induction and quotient rule. (3) Inverse function theorem. □

Proposition 3.1.4 (Chain Rule)**.** *Let* $I \xrightarrow{f} J \xrightarrow{g} \mathbb{R}$ *and* $\mathrm{Ran}(f) \subseteq J$. *Then* $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$ *whenever* f *is differentiable at* c *and* g *is differentiable at* $f(c)$.

Proof. Apply Careterodry's Theorem. □

§ 3.2. The Pillars of Differentiation

Definition 3.2.1. Let I be an open interval and $f : I \to \mathbb{R}$.

(1) f has a *local minimum* if $(\exists\delta > 0) \ni (\forall x \in V_\delta(c))(f(x) \geqslant f(c))$

(2) f has a *local maximum* if $(\exists\delta > 0) \ni (\forall x \in V_\delta(c))(f(x) \leqslant f(c))$

Theorem 3.2.1 (Fermat's Theorem)**.** *If* $f(x) = y$ *has a local minimum or maximum at* $x = c$, *then* $f'(c) = 0$ *or* $f'(c)$ *does not exist.*

Proof. If $f'(c)$ does not exist, we are done. Assume $f'(c)$ exists and is finite. Assume $f'(c)$ is a local maximum. For n large enough, $x_n \in V_\delta(c)$. Then:

$$\exists \delta > 0 \ni f(x) \leqslant f(c) \forall x \in V_\delta(c).$$

Let $(x_n)_n$ be a decreasing sequence with $(x_n)_n \to c$, $x_n \neq c$. Then:

$$\frac{f(x_n) - f(c)}{x_n - c} \leqslant 0,$$

which implies $f'(c) \leqslant 0$. Now let $(x_n)_n$ be an increasing sequence with $(x_n)_n \to c$, $x_n \neq c$. For n large enough, $x_n \in V_\delta(c)$. Then:

$$\frac{f(x_n) - f(c)}{x_n - c} \geqslant 0,$$

which implies $f'(c) \geqslant 0$. By antisymmetry, $f'(c) = 0$. □

Theorem 3.2.2 (Rolle's Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous (on* $[a, b]$*) and differentiable on* (a, b)*. Suppose that* $f(a) = f(b)$*. Then there exists* $c \in (a, b)$ *with* $f'(c) = 0$*.*

Proof. By the Extreme Value Theorem, there exists $x_M \in [a, b]$ with $\sup_{x \in [a,b]} f(x) = f(x_M)$. Again by the Extreme Value Theorem, there exists $x_m \in [a, b]$ with $\inf_{x \in [a,b]} f(x) = f(x_m)$.

If $x_M \neq a, b$, then by Fermat's Theorem, $f'(x_M) = 0$.

If $x_m \neq a, b$, then by Fermat's Theorem, $f'(x_m) = 0$.

If both of the above cases fail, then by our condition $f(x_m) = f(x_M)$. So $f(x) = K$ for some $k \in \mathbb{R}$. Then clearly $f'(c) = 0$ for all $c \in (a, b)$. □

Theorem 3.2.3 (Mean Value Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous and differentiable on* (a, b)*. There exists* $c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(a)}{b-a}$*.*

Proof. Consider $g : [a, b] \to \mathbb{R}$ defined by:

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

Then g is certainly continuous on $[a, b]$ and differentiable on (a, b) because f is.

Note that $g(a) = f(a)$ and $g(b) = f(b)$. By Rolle's Theorem, there exists $c \in (a, b)$ such that:

$$\begin{aligned} 0 &= g'(c) \\ &= f'(c) - \left(\frac{f(b) - f(a)}{b - a}\right) \end{aligned}$$

Whence $f'(c) = \frac{f(b)-f(a)}{b-a}$. □

Corollary 3.2.4. ****Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be continuous and differentiable on* (a, b) *with* $f'(x) = 0$ *for all* $x \in (a, b)$. *Then* f *is constant.*

Proof. Let $x_1, x_2 \in [a, b]$ with $x_1 \neq x_2$. Without loss of generality, suppose $x_1 < x_2$.

Apply the Mean Value Theorem to f on $[x_1, x_2]$. Then there exists $c \in (x_1, x_2)$ with $0 = f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$.

Simplifying the above equation gives $f(x_1) = f(x_2)$. Whence f is constant. □

Theorem 3.2.5. *Let* I *be an open interval and* $f : I \longrightarrow \mathbb{R}$ *differentiable.*

(1) f *is increasing on* I *if and only if* $f' \geqslant 0$*;*

(2) f *is decreasing on* I *if and only if* $f' \leqslant 0$.

Proof. $(\overset{(1)}{\Longrightarrow})$ Let $c \in I$. Since f is differentiable, the limit $f'(c)$ is defined, allowing us to write:

$$f'(c) = \underset{x \to c^+}{\text{limit}} \frac{f(x) - f(c)}{x - c}.$$

Since we are approaching c from the right, and since f is increasing, it must be that $f'(c) = \text{limit}_{x \to c^+} \frac{f(x)-f(c)}{x-c} \geqslant 0$.

$(\overset{(1)}{\Longleftarrow})$ Let $x_1, x_2 \in I$ with $x_1 < x_2$. Apply the Mean Value Theorem to f on $[x_1, x_2] \subseteq I$. Then there exists $c \in (x_1, x_2)$ with $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \geqslant 0$. Since $x_2 - x_1 \geqslant 0$, it must be that $f(x_2) - f(x_1) \geqslant 0$. Thus $f(x_2) \geqslant f(x_1)$, establishing that f is increasing.

$(\overset{(2)}{\Longrightarrow})$ This direction follows similarly.

$(\overset{(2)}{\Longleftarrow})$ This direction follows similarly. □

Example 3.2.1. Show that $\sin : \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz.
Solution. Pick $x, y \in \mathbb{R}$ and suppose without loss of generality that $x < y$.

Apply the Mean Value Theorem to sin on $[x, y]$. Then there exists $c \in (x, y)$ with:

$$\frac{\sin(y) - \sin(x)}{y - x} = \sin'(c) = \cos(c).$$

Applying the absolute value to both sides gives:

$$\left|\frac{\sin(y) - \sin(x)}{y - x}\right| = |\cos(c)| \leqslant 1.$$

Multiplying $|y - x|$ on both sides gives the desired result.

Exercise 3.2.1. *If* $f : I \to \mathbb{R}$ *is differentiable with* f' *bounded on* I*, then* f *is Lipschitz.*

Lemma 3.2.6. *Let* $I \subseteq \mathbb{R}$ *be an interval,* $f : I \to \mathbb{R}$ *continuous, and* $c \in I$ *with* f *differentiable at* $x = c$.

(1) If $f'(c) > 0$*, then there exists* $\delta > 0$ *such that* $f(x) > f(c)$ *for all* $x \in (c, c + \delta)$.

(2) If $f'(c) < 0$*, then there exists* $\delta > 0$ *such that* $f(x) > f(c)$ *for all* $x \in (c - \delta, c)$.

Proof. (1) $0 < f'(c) = \text{limit}_{x\to c} \frac{f(x)-f(c)}{x-c} = \text{limit}_{x\to c^+} \frac{f(x)-f(c)}{x-c}$ (if the limit exists then the one-sided limit exists). Setting $\epsilon = \frac{f'(c)}{2}$, there exists $\delta > 0$ such that for $x \in (c, c + \delta)$, $\frac{f(x)-f(c)}{x-c} \in V_\epsilon(f'(c))$. Equivalently, $\frac{f(x)-f(c)}{x-c} > \frac{f'(c)}{2} > 0$. Since $x - c > 0$ for $x \in (c, c + \delta)$, we get that $f(x) - f(c) > 0$ for $x \in (c, c + \delta)$.

(2) This follow similarly. □

Theorem 3.2.7 (Darboux's Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be differentiable. Let* k *be a number strictly between* $f'(a)$ *and* $f'(b)$*. Then there exists* $c \in (a, b)$ *with* $f'(c) = k$.

Proof. Let $h(x) = kx - f(x)$ on $[a, b]$. Then h is continuous on $[a, b]$.

By the Extreme Value Theorem, h attains its supremum; i.e., there exists $c \in [a, b]$ with $h(c) \geqslant h(x)$ for all $x \in [a, b]$.

$h'(a) = k - f'(a)$ and $h'(b) = k - f'(b)$. Without loss of generality, suppose:

$$h'(a) = k - f'(a) > 0$$
$$h'(b) = k - f'(b) < 0.$$

By the previous lemma, there exists $\delta > 0$ such that $h(x) > h(a)$ on $(a, a + \delta)$.

By the previous lemma, there exists $\eta > 0$ such that $h(x) > h(b)$ on $(b - \eta, b)$.

So $c \neq a$ and $c \neq b$. So $c \in (a, b)$. Thus $h'(c) = 0$ by Fermat's theorem.

Thus $f'(c) = k$. □

Question. Does there exists $f : \mathbb{R} \to \mathbb{R}$ differentiable with $f'(x) = \text{sign}(x)$?

Answer. No! $\text{sign}(x)$ does not satisfy the Intermediate Value Theorem on $[-1, 1]$.

Corollary 3.2.8. *Let* $f : I \to \mathbb{R}$ *be differentiable and* $f' \neq 0$ *on* I*. Then* f *is monotone.*

Proof. Homework □