Math 310

Homework 5

Due: 10/9/2024

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**Exercise 1.** *Let $x_1 = 1$ and inductively set $x_{n+1} = \sqrt{2+x_n}$. Show that $(x_n)_n$ converges and find its limit.*

*Proof.* Claim: $x_n \leqslant 2$ for all $n$. We prove this by induction on $n$. Clearly $x_1 = 1 \leqslant 2$. Now assume our hypothesis is true for $n$. For $n+1$ we have:

$$\begin{aligned} x_{n+1} &= \sqrt{2+x_n} \\ &\leqslant \sqrt{2+2} \\ &= 2. \end{aligned}$$

Claim: $x_n \leqslant x_{n+1}$. Observe that:

$$\begin{aligned} x_n \leqslant x_{n+1} &\iff x_n \leqslant \sqrt{2+x_n} \\ &\iff x_n \leqslant \sqrt{2+2} \\ &\iff x_n \leqslant 2. \end{aligned}$$

Since $(x_n)_n$ is increasing and bounded above, by the monotone convergence theorem it has a limit, call it $L$. Thus:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2+x_n}$$

$$\iff$$

$$L = \sqrt{2+L}$$

$$\iff$$

$$L^2 - L - 2 = 0$$

$$\iff$$

$$L = 2 \ \text{ or } \ L = -1.$$

So it must be the case that $(x_n)_n \to L$. □

**Exercise 2.** *Does the following sequence converge?*

$$x_n := \sum_{k=n+1}^{2n} \frac{1}{k}.$$

*Proof.* Claim: $x_n \leqslant 1$ for all $n$. Observe that:

$$\begin{aligned}
x_n &= \sum_{k=n+1}^{2n} \frac{1}{k} \\
&= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n-2} + \frac{1}{2n-1} + \frac{1}{2n} \\
&\leqslant \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} + \ldots + \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} \\
&= \frac{n+1}{n+1} \\
&= 1.
\end{aligned}$$

Claim: $x_n \leqslant x_{n+1}$. We have:

$$\begin{aligned}
x_n \leqslant x_{n+1} &\iff \sum_{k=n+1}^{2n} \frac{1}{k} \leqslant \sum_{k=n+2}^{2n+2} \frac{1}{k} \\
&\iff \frac{1}{n+1} \leqslant \frac{1}{2n+1} + \frac{1}{2n+1} \\
&\iff \frac{1}{n+1} \leqslant \frac{4n+3}{(2n+1)(2n+1)} \\
&\iff n+1 \geqslant \frac{4n^2+6n+2}{4n+3} \\
&\iff 4n^2+7n+3 \geqslant 4n^2+6n+2 \\
&\iff 7n+3 \geqslant 6n+2
\end{aligned}$$

which is true for all $n \in \mathbf{N}$. Since $(x_n)_n$ is increasing and bounded above, by the monotone convergence theorem $(x_n)_n$ converges. □

**Exercise 3.** *Let $(f_n)_n$ denote the Fibonacci sequence and let*

$$x_n := \frac{f_{n+1}}{f_n}.$$

*Given that $(x_n)_n$ converges, find its limit.*

*Proof.* Note that:

$$\begin{aligned}
x_n &= \frac{f_{n+1}}{f_n} \\
&= \frac{f_n + f_{n-1}}{f_n} \\
&= 1 + \frac{f_{n-1}}{f_n} \\
&= 1 + \frac{1}{f_n/f_{n-1}} \\
&= 1 + \frac{1}{x_{n-1}}.
\end{aligned}$$

If $(x_n)_n \to L$, then:

$$\begin{aligned} L = 1 + \frac{1}{L} &\iff L^2 = L + 1 \\ &\iff L^2 - L - 1 = 0 \\ &\iff L = \frac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

Since $\frac{1-\sqrt{5}}{2} < 0$, it must be the case that $(x_n)_n \to \frac{1+\sqrt{5}}{2}$ □

**Exercise 4.** *If $(x_n)_n$ is an unbounded sequence of reals numbers, show that there is a subsequence $(x_{n_k})_k$ such that:*

$$\left(\frac{1}{x_{n_k}}\right)_k \to 0.$$

*Proof.* □