Math 395

Homework 7

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Collaborators:

Exercise 1. *Let V_1, V_2, W_1, and W_2 be F-vector spaces. Let $T_1 \in \operatorname{Hom}_F(V_1, W_1)$ and $T_2 \in \operatorname{Hom}_F(V_2, W_2)$. Prove there is a unique F-linear map $T_1 \otimes T_2$ from $V_1 \otimes_F V_2$ to $W_1 \otimes_F W_2$ satisfying $(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2)$.*

Proof. Define $t : V_1 \otimes V_2 \to W_1 \otimes_F W_2$ by $(v_1, v_2) \mapsto T_1(v_1) \otimes T_2(v_2)$. Observe that:

$$\begin{aligned} t(v_1 + c\tilde{v_1}, v_2) &= T_1(v_1 + c\tilde{v_1}) \otimes T_2(v_2) \\ &= (T_1(v_1) + cT_1(\tilde{v_1})) \otimes T_2(v_2) \\ &= T_1(v_1) \otimes T_2(v_2) + cT_1(\tilde{v_1}) \otimes T_2(v_2) \\ &= t(v_1, v_2) + ct(\tilde{v_1}, v_2) \end{aligned}$$

$$\begin{aligned} t(v_1, v_2 + c\tilde{v_2}) &= T_1(v_1) \otimes T_2(v_2 + c\tilde{v_2}) \\ &= T_1(v_1) \otimes (T_2(v_2) + cT_2(\tilde{v_2})) \\ &= T_1(v_1) \otimes T_2(v_2) + T_1(v_1) \otimes cT_2(\tilde{v_2}) \\ &= t(v_1, v_2) + ct(v_1, \tilde{v_2}). \end{aligned}$$

Thus $t \in \operatorname{Hom}_F(V_1, V_2; W_1 \otimes_F W_2)$. By the universal property, this induces a map $T_1 \otimes T_2 \in \operatorname{Hom}_F(V_1 \otimes_F V_2, W_1 \otimes_F W_2)$ defined by $(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2)$. $\square$

Exercise 2. *Use the definition to compute the determinant of a 3 by 3 matrix over a field F. Check your results agree with the familiar definition of the determinant of a matrix.*

Proof. Let $\mathcal{E}_3 = \{e_1, e_2, e_3\}$ be the standard basis of F^3 and $T \in \operatorname{Hom}_F(F^3, F^3)$ such that:

$$[T]_{\mathcal{E}_3} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Through cofactor expansion, we can determine that $\det([T]_{\mathcal{E}_3}) = aei - afh - bdi + bfg + cdh - ceg$. However, note that:

$$\begin{aligned}
\Lambda^3(T)(e_1 \wedge e_2 \wedge e_3) &= T(e_1) \wedge T(e_2) \wedge T(e_3) \\
&= (ae_1 + de_2 + ge_3) \wedge (be_1 + ee_2 + he_3) \wedge (ce_1 + fe_2 + ie_3) \\
&= (aee_1 \wedge e_2 + ahe_1 \wedge e_3 + dbe_2 \wedge e_1 + dhe_2 \wedge e_3 + gbe_3 \wedge e_1 + gee_3 \wedge e_2) \wedge (ce_1 + fe_2 + ie_3) \\
&= (aeie_1 \wedge e_2 \wedge e_3) - (ahfe_1 \wedge e_2 \wedge e_3) - (dbie_1 \wedge e_2 \wedge e_3) + (dhce_1 \wedge e_2 \wedge e_3) \\
&\quad + (gbfe_1 \wedge e_2 \wedge e_3) - (gece_1 \wedge e_2 \wedge e_3) \\
&= (aei - ahf - dbi + dhc + gbf - gec)e_1 \wedge e_2 \wedge e_3 \\
&= \det([T]_{\mathcal{E}_3})e_1 \wedge e_2 \wedge e_3.
\end{aligned}$$

Hence this result agrees with the familiar definition of the determinant of a matrix. □

Exercise 3. *Let $v_1, \ldots, v_k \in V$. Prove that $v_1 \wedge \ldots \wedge v_k = 0_{\Lambda^k(V)}$ if $v_1, \ldots, v_k$ are linearly dependent.*

Proof. If $v_1, \ldots, v_k$ are linearly dependent, then there is at least one pair $v_i, v_j \in \{v_1, \ldots, v_k\}$ with $i \neq j$ and $v_i = \alpha v_j$, $\alpha \in F$. Since the wedge product is alternating, it must be that $v_1 \wedge \ldots \wedge v_k = 0_{\Lambda^k(V)}$. □

Exercise 4. *Prove that $v \wedge v_1 \wedge v_2 \wedge \ldots \wedge v_k = (-1)^k(v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge v)$.*

Proof. We proceed by induction on k. For $k = 1$, we see that $v \wedge v_1 = (-1)^1 v_1 \wedge v$. Assume our hypothesis is true up to k. For $k + 1$:

$$\begin{aligned}
v \wedge v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge v_{k+1} &= (-1)^k(v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge v \wedge v_{k+1}) \\
&= (-1)^{k+1}(v_1 \wedge v_2 \wedge \ldots \wedge v_k \wedge v_{k+1} \wedge v).
\end{aligned}$$

□