

Contents

Contents	i
1 Orderings and Functions	1
1.1 Basic Notation	1
1.2 Orderings	1
1.3 Functions	3
1.4 Cardinality	6
2 Ordered Fields	13
2.1 Ordering of \mathbb{Z}	13
2.2 Ordering of \mathbb{Q}	13
2.3 Rings and Fields	15
3 The Real Numbers	17
3.1 The Completion of \mathbb{Q}	17
3.2 Ordering of \mathbb{R}	17
3.3 Metrics and Norms on \mathbb{R}^n	20
4 Supremum, Infimum, and Completeness	23
4.1 Supremum and Infimum	23
4.2 Nested Intervals	26
5 Sequences	28
5.1 Basic Definitions and Examples	28
5.2 Convergence	29
5.3 Subsequences	39
5.4 Limit Inferior and Limit Superior	41
5.5 Cauchy Sequences	43
5.6 Sequences of Functions	44

Orderings and Functions

1.1 Basic Notation

Definition 1.1.1.

- (1) The natural numbers are defined as $\mathbf{N} = \{1, 2, 3, \dots\}$,
- (2) The positive integers are defined as $\mathbf{N}_0 = \mathbf{Z}^+ = \{0, 1, 2, 3, \dots\}$,
- (3) The integers are defined as $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$,
- (4) The rational numbers are defined as $\mathbf{Q} = \{\frac{a}{b} \mid a, b \in \mathbf{Z}, b \neq 0\}$,
- (5) The real numbers are "defined" (we will get more into this later) as the set $(-\infty, \infty)$,
- (6) The complex numbers are defined as $\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R}, i^2 = -1\}$.

Example 1.1.1. Note that $\sqrt{2}, \pi, e \notin \mathbf{Q}$, as they cannot be expressed as fractions.

Definition 1.1.2. Let A and B be sets. The cartesian product is defined as $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Definition 1.1.3. A relation from A to B is a subset $R \subseteq A \times B$. Typically, when one says "a relation on A " that means a relation from A to A ; i.e., $R \subseteq A \times A$.

Definition 1.1.4. Let A be a set and R a relation on A . Then R is:

- (1) reflexive if $(a, a) \in R$ for all $a \in A$,
- (2) transitive if $(a, b), (b, c) \in R$ implies $(a, c) \in R$,
- (3) symmetric if $(a, b) \in R$ implies $(b, a) \in R$, and
- (4) antisymmetric if $(a, b), (b, a) \in R$ implies $a = b$.

1.2 Orderings

Definition 1.2.1. Let A be a set. An ordering of A is a relation R on A that is reflexive, transitive, and antisymmetric. If this is the case, we write $(a, b) \in R$ as $a \leq_R b$. If A is an ordered set we write it as the ordered pair (A, \leq_R) (or just A if the ordering is obvious by context).

Example 1.2.1.

- (1) Let $m, n \in \mathbf{Z}$. The algebraic ordering \leq_a is defined as follows: $m \leq_a n$ if and only if there exists an element $k \in \mathbf{N}_0$ with $m + k = n$.
- (2) The set of natural numbers \mathbf{N} equipped with the relation of divisibility form an ordering. Let $m, n \in \mathbf{N}$. Then $m \leq_d n$ if and only if $m \mid n$.
- (3) Let S be any set. The subsets of S (i.e., elements of its power set) equipped with the relation of inclusion form an ordering. Let $A, B \in \mathcal{P}(S)$. Then $A \leq_{\mathcal{P}(S)} B$ if and only if $A \subseteq B$.
- (4) The set of rational numbers \mathbf{Q} form an algebraic ordering as follows: if $\frac{a}{b}, \frac{c}{d} \in \mathbf{Q}$, then $\frac{a}{b} \leq_a \frac{c}{d}$ if and only if $ad \leq_a bc$ (in \mathbf{Z}).

Definition 1.2.2. An ordered set (A, \leq_R) is total (or linear) if for all $a, b \in A$ we have that $a \leq_R b$ or $b \leq_R a$.

Example 1.2.2. The ordered sets (\mathbf{Z}, \leq_a) and (\mathbf{Q}, \leq_a) are total orderings, whereas (\mathbf{N}, \leq_d) and $(\mathcal{P}(S), \leq_{\mathcal{P}(S)})$ are not total orderings.

Definition 1.2.3. Let (X, \leq) be an ordered set. Let $A \subseteq X$.

- (1) A is called bounded above if there exists an element $u \in X$ with $a \leq u$ for all $a \in A$. Such a u (not necessarily unique) is called an upperbound for A .
- (2) A is called bounded below if there exists an element $v \in X$ with $v \leq a$ for all $a \in A$. Such a v (not necessarily unique) is called a lowerbound for A .
- (3) If A admits an upperbound u with $u \in A$, then u is called the greatest element of A .
- (4) If A admits a lowerbound v with $v \in A$, then v is called the least element of A .
- (5) Let A be bounded above. The set of upperbounds of A is defined as $\mathcal{U}_A = \{u \in X \mid u \text{ is an upperbound of } A\}$. If l is the least element of \mathcal{U}_A , we write $l = \sup(A)$ and call it the supremum of A .
- (6) Let A be bounded below. The set of lowerbounds of A is defined as $\mathcal{L}_A = \{v \in X \mid v \text{ is a lowerbound of } A\}$. If g is the greatest element of \mathcal{L}_A , we write $g = \inf(A)$ and call it the infimum of A .
- (7) A maximal element of A is an element $m \in A$ such that if $a \geq m$, then $a = m$ (not necessarily unique).
- (8) A minimal element of A is an element $n \in A$ such that if $a \leq n$, then $a = n$ (not necessarily unique).
- (9) If (A, \leq) is a total ordering, then A is called a chain.

Proposition 1.2.1. Let (X, \leq) be an ordered set and $A \subseteq X$.

- (1) If A admits a greatest element, then it is unique,

- (2) If A admits a least element, then it is unique,
- (3) If A admits a least upper bound, then it is unique,
- (4) If A admits a greatest lower bound, then it is unique.

Proof. Suppose u, u' are greatest elements of A , then $u, u' \in A$. Hence $u \leq u'$ and $u' \leq u$. By antisymmetry, $u = u'$, meaning the greatest element is unique. The proof for least elements being unique is identical, which establishes (1) and (2).

Note that $\mathcal{U}_A \subseteq X$. By definition the least element of \mathcal{U}_A is defined to be the supremum of A , and since least elements are unique the supremum of A must be unique. Similarly, $\mathcal{L}_A \subseteq X$. By definition the greatest element of \mathcal{L}_A is defined to be the infimum of A , and since greatest elements are unique the infimum of A must be unique. This establishes (3) and (4). \square

Lemma 1.2.2 (Zorn's Lemma). *Let X be an ordered set with the property that every chain has an upperbound. Then X contains a maximal element.*

Example 1.2.3. Considered the ordered set (\mathbf{N}, \leq_d) and the subset $A = \{4, 7, 12, 28, 35\}$.

- A is bounded above with $4 \times 7 \times 12 \times 28 \times 35$ as an upperbound.
- The supremum of A is $\text{lcm}(4, 7, 12, 28, 35)$.
- There does not exist a greatest element.
- 12, 28, and 35 are maximal elements (no other element in A divides them).

Definition 1.2.4. Let (X, \leq) be an ordered set and $A \subseteq X$. If A is bounded above and below, then we say A is bounded.

Definition 1.2.5. Let (X, \leq) be an ordered set. Then (X, \leq) is complete if, for every bounded set $A \subseteq X$, $\sup(A)$ and $\inf(A)$ exist.

1.3 Functions

Definition 1.3.1. Let X and Y be sets. A function from X to Y is a relation $f \subseteq X \times Y$ such that for all $x \in X$, there exists a unique $y_x \in Y$ with $(x, y_x) \in f$.

- (1) The set X is the domain of f .
- (2) The set Y is the codomain of f .
- (3) The image of f is defined as $f(X) = \{f(x) \mid x \in X\} \subseteq Y$ (also sometimes denoted $\text{im}(f)$).
- (4) The preimage of f is defined as $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\} \subseteq X$.
- (5) The graph of f is defined as $\text{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

If f is a function, we denote it by $f : X \rightarrow Y$ or $X \xrightarrow{f} Y$.

Example 1.3.1. Let X be a set.

- (1) The *identity map* $\text{id}_X : X \rightarrow X$ is defined by $\text{id}_X(x) = x$.
- (2) If $X \subseteq Y$, the *inclusion map* $\iota : X \rightarrow Y$ is defined by $\iota(x) = x$.
- (3) If $A \subseteq X$ is a set, the *characteristic function* (or *step function*) $\mathbf{1}_A : X \rightarrow \mathbf{R}$ is defined by

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Definition 1.3.2. Given $f, g : X \rightarrow \mathbf{R}$ and $\alpha \in \mathbf{R}$, the pointwise operations on f and g are:

- $(f \pm g)(x) = f(x) \pm g(x)$,
- $(\alpha f)(x) = \alpha f(x)$,
- $(fg)(x) = f(x)g(x)$,
- $(f/g)(x) = f(x)/g(x)$.

Definition 1.3.3. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be maps between sets. The composition of f and g is denoted $g \circ f : X \rightarrow Z$.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & \searrow & & \nearrow & \\ & & g \circ f & & \end{array}$$

Definition 1.3.4. Let $f : X \rightarrow Y$ be a map between sets.

- (1) f is left-invertible if there exists a map $g : Y \rightarrow X$ with $g \circ f = \text{id}_X$.
- (2) f is right-invertible if there exists a map $h : Y \rightarrow X$ with $f \circ h = \text{id}_Y$.
- (3) f is invertible if there exists a map $k : Y \rightarrow X$ with $k \circ f = \text{id}_X$ and $f \circ k = \text{id}_Y$.

Example 1.3.2. The *shift function* is a map $s : \mathbf{N} \rightarrow \mathbf{N}$ defined by $s(n) = n + 1$. Note that this function is left-invertible: define $g : \mathbf{N} \rightarrow \mathbf{N}$ by

$$g(n) = \begin{cases} n - 1, & n \geq 2 \\ n_0, & n = 1, \end{cases}$$

where n_0 is an arbitrary natural number, then $g \circ s = \text{id}_{\mathbf{N}}$.

Suppose that s has a right inverse, that is, there exists a function $h : \mathbf{N} \rightarrow \mathbf{N}$ such that $s \circ h = \text{id}_{\mathbf{N}}$. Observe that:

$$(s \circ h)(1) = s(h(1)) = h(1) + 1 = 1.$$

It must be the case that $h(1) = 0$, which is a contradiction. Hence s is not right-invertible.

Example 1.3.3. The function g defined above is right invertible, but not left invertible.

Proposition 1.3.1. *Let $f : X \rightarrow Y$ be a map between sets. The following are equivalent:*

- (1) f is invertible,
- (2) f is right-invertible and left-invertible.

Proof. Clearly (1) implies (2). Assume f to be left and right-invertible. Then there exists maps $h, g : Y \rightarrow X$ with $g \circ f = \text{id}_X$ and $f \circ h = \text{id}_Y$. Observe that:

$$\begin{aligned} h &= \text{id}_X \circ h \\ &= (g \circ f) \circ h \\ &= g \circ (f \circ h) \\ &= g \circ \text{id}_Y \\ &= g, \end{aligned}$$

establishing the proposition. □

Definition 1.3.5. Let $f : X \rightarrow Y$ be a map between sets.

- (1) f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$,
- (2) f is surjective if $\text{im}(f) = Y$, and
- (3) f is bijective if it is injective and surjective.

Proposition 1.3.2. *Let $f : X \rightarrow Y$ be a map between sets.*

- 1. f is injective if and only if f is left-invertible.
- 2. f is surjective if and only if f is right-invertible.
- 3. f is bijective if and only if f is invertible.

Proof. (1) **Do the forward direction yourself!** Now assume $f : X \rightarrow Y$ is injective. Define $g : Y \rightarrow X$ by

$$g(y) = \begin{cases} x_0, & y \notin \text{im}(f) \\ x_y, & y \in \text{im}(f), \end{cases}$$

where x_y is the unique element in x mapping to y ; i.e., $f(x_y) = y$. By our construction, $(g \circ f)(x) = x$ for all $x \in X$.

(2) **Do the forward direction yourself!** Now assume $f : X \rightarrow Y$ is onto. Note that the preimage of f is nonempty, so we can define $h : Y \rightarrow X$ by $h(y) = x_y$, where $x_y \in f^{-1}(y)$. By our construction $(f \circ h)(y) = f(x_y) = y$ for all $y \in Y$.

(3) **Do this yourself its easy!** □

Corollary 1.3.3. *Let A, B be sets. There exists an injection $A \hookrightarrow B$ if and only if there exists a surjection $B \twoheadrightarrow A$.*

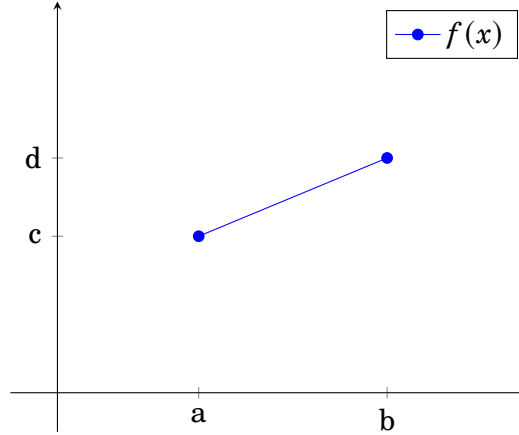
Proof. If $f : A \rightarrow B$ is injective, then f is left invertible, that is, there exists a function $g : B \rightarrow A$ with $g \circ f = \text{id}_A$. But this means g is right invertible, so g is onto. The other direction follows identically. □

1.4 Cardinality

Definition 1.4.1. Let A, B be sets. Then $\text{card}(A) = \text{card}(B)$ if there exists a bijection $A \hookrightarrow B$.

Example 1.4.1.

- (1) Define $f : \mathbf{N}_0 \rightarrow \mathbf{N}$ by $f(n) = n + 1$. This is a bijection, hence $\text{card}(\mathbf{N}_0) = \text{card}(\mathbf{N})$.
- (2) Let $[a, b]$ and $[c, d]$ be intervals with $a < b$ and $c < d$. Define $f : [a, b] \rightarrow [c, d]$ by $f(x) = \left(\frac{d-c}{b-a}\right)(x-a) + c$.



This is a bijection, hence $\text{card}([a, b]) = \text{card}([c, d])$. The result is the same had the intervals been open.

- (3) Recall that $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbf{R}$ is a bijection. Consider the maps $(0, 1) \xrightarrow{g} (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow{\tan} \mathbf{R}$. Since g and \tan are bijective, $\tan \circ g$ is bijective, hence $\text{card}((0, 1)) = \text{card}(\mathbf{R})$.

Definition 1.4.2. A set A is called finite if there exists an $N \in \mathbf{N}$ such that $\text{card}(A) = \text{card}(\{1, \dots, N\})$. If not, then A is called infinite.

Proposition 1.4.1. Given $m, n \in \mathbf{N}$, $m \neq n$, then $\text{card}(\{1, \dots, m\}) \neq \text{card}(\{1, \dots, n\})$.

Proof. Without loss of generality, let $m > n$. Suppose towards contradiction we have a bijection $\{1, \dots, m\} \xrightarrow{f} \{1, \dots, n\}$. By the pigeon-hole principle, it must be the case that —given any $i, j \in \{1, \dots, m\}$ with $i \neq j$, we have that $f(i) = f(j)$. This is a contradiction (f is not injective), hence $\text{card}(\{1, \dots, m\}) \neq \text{card}(\{1, \dots, n\})$. \square

Proposition 1.4.2. \mathbf{N} is infinite.

Proof. Suppose towards contradiction we have a bijection $f : \mathbf{N} \rightarrow \{1, 2, \dots, n\}$, where $n \in \mathbf{N}$. Consider the maps $\{1, 2, \dots, n, n+1\} \xhookrightarrow{\iota} \mathbf{N} \xrightarrow{f} \{1, 2, \dots, n\}$, it must be the case that the composition $f \circ \iota$ is injective. However, we established in Proposition 1.4.1 that this is false. Having reached a contradiction, it must be the case that \mathbf{N} is infinite. \square

Exercise 1.4.1. If A is infinite, there exists an injection $\mathbf{N} \hookrightarrow A$.

Proof. Let $\pi : \mathbf{N} \rightarrow A$ be a map. Pick $a_1 \in A$ and define $\pi(1) = a_1$. Since A is infinite, $A \setminus \{a_1\}$ is also infinite. Pick $a_2 \in A \setminus \{a_1\}$ and define $\pi(2) = a_2$. Inductively, we have an injection $\mathbf{N} \hookrightarrow A$. \square

Example 1.4.2. Define $k : \mathbf{Z} \rightarrow \mathbf{N}$ by $k(n) = (-1)^{n-1} \lfloor \frac{n}{2} \rfloor$. This is a bijection, hence $\text{card}(\mathbf{Z}) = \text{card}(\mathbf{N})$.

Definition 1.4.3. Let X and Y be sets.

- (1) The power set of X is $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.
- (2) The set of functions from X to Y is $Y^X = \{f \mid f : X \rightarrow Y\}$.

Lemma 1.4.3. Let X be a set. There exists a bijection $\mathcal{P}(X) \hookrightarrow 2^X$.

Proof. Let $A \subseteq X$. Define $\varphi : \mathcal{P}(X) \rightarrow 2^X$ by $A \mapsto \mathbf{1}_A$, where

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

is the *characteristic function* defined in Example 1.3.1. Let $\varphi(A) = \varphi(B)$. This is equivalent to $\mathbf{1}_A = \mathbf{1}_B$. Note that functions are equal if and only if $\mathbf{1}_A(x) = \mathbf{1}_B(x)$ for all $x \in X$. Hence $x \in A$ implies $\mathbf{1}_A(x) = 1 = \mathbf{1}_B(x)$, giving $x \in B$. The reverse inclusion is identical, hence $A = B$. Let $f \in 2^X$. Let $A = \{x \in X \mid f(x) = 1\}$. Then $\varphi(A) = \mathbf{1}_A = f$. Thus $\mathcal{P}(X) \hookrightarrow 2^X$. \square

Exercise 1.4.2. Show that $\text{card}(\mathcal{P}(\{1, \dots, n\})) = 2^n$.

Proof. Note that $\text{card}(\mathcal{P}(\{1, \dots, n\})) = \text{card}(2^{\{1, \dots, n\}})$. Let $f \in 2^{\{1, \dots, n\}}$. For each $i \in \{1, \dots, n\}$, there is a choice of two outputs for $f(i)$. Hence by the fundamental principle of counting $\text{card}(\mathcal{P}(\{1, \dots, N\})) = \text{card}(2^{\{1, \dots, n\}}) = 2^n$. \square

Theorem 1.4.4 (Cantor's Diagonal Argument). $\text{card}(\mathbf{N}) < \text{card}((0, 1))$.

Proof. Recall that every $\sigma \in (0, 1)$ has a decimal expansion $\sigma = 0.\sigma_1\sigma_2\dots = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$, where $\sigma_j \in \{0, 1, 2, \dots, 9\}$ which does not terminate in 9's. By way of contradiction, suppose there exists a surjection $r : \mathbf{N} \rightarrow (0, 1)$ defined by $r(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\dots$, where $\sigma_j(n) \in \{0, 1, 2, \dots, 9\}$ is the j^{th} digit in the decimal expansion.

Consider the map $\tau : \mathbf{N} \rightarrow \{0, 1, \dots, 9\}$ defined by:

$$\tau(n) = \begin{cases} 3, & \sigma_n(n) = 2 \\ 2, & \sigma_n(n) = 3, \end{cases}$$

and let $t = 0.\tau(1)\tau(2)\tau(3)\dots$. Observe the following:

$$\begin{aligned} r(1) &= 0.\sigma_1(1)\sigma_2(1)\sigma_3(1)\sigma_4(1)\dots \\ r(2) &= 0.\sigma_1(2)\sigma_2(2)\sigma_3(2)\sigma_4(2)\dots \\ r(3) &= 0.\sigma_1(3)\sigma_2(3)\sigma_3(3)\sigma_4(3)\dots \\ r(4) &= 0.\sigma_1(4)\sigma_2(4)\sigma_3(4)\sigma_4(4)\dots \\ &\vdots \\ r(n) &= 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\sigma_4(n) \dots \sigma_n(n). \end{aligned}$$

Since r is surjective, there is an $m \in \mathbf{N}$ with $r(m) = t$. It follows that:

$$\begin{aligned} r(m) &= 0.\sigma_1(m)\sigma_2(m)\sigma_3(m)\dots\sigma_m(m)\dots \\ &= 0.\tau(1)\tau(2)\tau(3)\dots\tau(m)\dots \end{aligned}$$

which implies that $\sigma_m(m) = \tau(m)$. But recall how we defined $\tau(n)$ —if $\sigma_m(m) = 2$, then $\tau(2) = 3$ and if $\sigma_m(m) \neq 2$, then $\tau(2) = 2$. This is a contradiction, hence there does not exist a surjection $\mathbf{N} \xrightarrow{r} (0, 1)$. \square

Corollary 1.4.5. $\text{card}(\mathbf{N}) \neq \text{card}(\mathbf{R})$

Proof. It follows from Example 1.4.1 that $\text{card}(\mathbf{N}) < \text{card}((0, 1)) = \text{card}(\mathbf{R})$. \square

Definition 1.4.4. Let A and B be sets.

- (1) We write $\text{card}(A) \leq \text{card}(B)$ if there exists an injection $A \hookrightarrow B$.
- (2) We write $\text{card}(A) < \text{card}(B)$ if $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(A) \neq \text{card}(B)$

Example 1.4.3.

- (1) If $A \subseteq B$, then the inclusion map $\iota : A \rightarrow B$ gives $\text{card}(A) \leq \text{card}(B)$.
- (2) If $m > n$, then $\text{card}\{1, \dots, n\} < \text{card}\{1, \dots, m\}$

Proposition 1.4.6. Let A be a set. Then $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

Proof. Define $f : A \rightarrow \mathcal{P}(A)$ by $a \mapsto \{a\}$. This is clearly an injective map. Now suppose towards contradiction that there exists a surjection $g : A \rightarrow \mathcal{P}(A)$ defined by $a \mapsto g(a)$. Then $g(a) \subseteq A$ (by the definition of a power set).

Let $S = \{a \in A \mid a \notin g(a)\}$. Then $S \subseteq A$. Since g is onto, there exists an element $x \in A$ with $g(x) = S$. Case 1: $x \in S$. This implies that $x \notin g(x)$. But $g(x) = S$, so $x \notin S$, a contradiction. Case 2: $x \notin S$. This implies that $x \in g(x)$. But by definition this means $x \in S$, a contradiction. Since we have exhausted all the necessary cases, it must be that there does not exist a surjection from $A \rightarrow \mathcal{P}(A)$. Hence $\text{card}(A) < \text{card}(\mathcal{P}(A))$. \square

Lemma 1.4.7. Let A and B be sets. The following are equivalent:

- (1) $\text{card}(A) \leq \text{card}(B)$;
- (2) *there exists an injection* $A \hookrightarrow B$;
- (3) *there exists a surjection* $B \twoheadrightarrow A$.

Example 1.4.4.

- (1) Define $\mathbf{N} \times \mathbf{Z} \rightarrow \mathbf{Q}$ by $(n, m) \mapsto \frac{m}{n}$. This is surjective, so $\text{card}(\mathbf{Q}) \leq \text{card}(\mathbf{N} \times \mathbf{Z})$.
- (2) Define $\mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ by $(m, n) \mapsto 2^m \cdot 3^n$. Then g is injective by the fundamental theorem of arithmetic. So $\text{card}(\mathbf{N} \times \mathbf{N}) \leq \text{card}(\mathbf{N})$.
- (3) Recall from Example 1.4.2 that $k : \mathbf{N} \rightarrow \mathbf{Z}$ defined by $k(n) = (-1)^{n-1} \lfloor \frac{n}{2} \rfloor$ is a bijection. Define $K : \mathbf{Z} \times \mathbf{N} \rightarrow \mathbf{N} \times \mathbf{N}$ by $(m, n) \mapsto (k^{-1}(m), n)$. This is a bijection, so $\text{card}(\mathbf{Z} \times \mathbf{N}) = \text{card}(\mathbf{N} \times \mathbf{N})$.
- (4) From the previous examples, we've established that:

$$\text{card}(\mathbf{N}) \leq \text{card}(\mathbf{Q}) \leq \text{card}(\mathbf{Z} \times \mathbf{N}) = \text{card}(\mathbf{N} \times \mathbf{N}) \leq \text{card}(\mathbf{N})$$

Theorem 1.4.8. *Let \mathfrak{N} denote the class of cardinals. The pair (\mathfrak{N}, \leq) forms a total ordering —where \leq is defined by $\text{card}(A) \leq \text{card}(B)$ if and only if $A \hookrightarrow B$. In particular, if A, B, C are sets with $\text{card}(A), \text{card}(B), \text{card}(C) \in \text{obj}(\mathfrak{N})$, then we have the following:*

- (1) $\text{card}(A) \leq \text{card}(A)$ (*reflexive*).
- (2) If $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$ (*transitive*).
- (3) If $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A)$, then $\text{card}(A) = \text{card}(B)$ (*antisymmetric*).
- (4) Either $\text{card}(A) \leq \text{card}(B)$ or $\text{card}(B) \leq \text{card}(A)$ (*total*).

Proof. (1) and (2) follow by simply applying definitions. Note that any set bijects into itself, hence $A \hookrightarrow A$ implies $A \hookrightarrow A$, establishing $\text{card}(A) \leq \text{card}(A)$. Similarly, if there are bijections $A \hookrightarrow B \hookrightarrow C$, then clearly there is a bijection $A \hookrightarrow C$. Hence $\text{card}(A) = \text{card}(C)$.

(3) (Cantor-Schröder-Bernstein Theorem) We have injections $A \xhookrightarrow{f}$ and $B \xhookrightarrow{g} A$. Let:

$$\begin{aligned} A_0 &= \text{im}(g)^{\complement} \\ A_1 &= (g \circ f)(A_0) \\ A_2 &= (g \circ f)(A_1) \\ &\vdots \\ A_n &= (g \circ f)(A_{n-1}). \end{aligned}$$

Note that $A_1 \cap A_0 = \emptyset$ because $A_1 \subseteq \text{im}(g)$ and $A_0 = \text{im}(g)^{\complement}$. We similarly have that $A_2 \cap A_0 = \emptyset$. Claim: $A_1 \cap A_2$. **finish this**

(4) Let $A \rightarrow B$ be a map. Let $\mathcal{F} = \{(D, f) \mid D \subseteq A, f : D \hookrightarrow B, f \text{ is injective}\}$. Note that $\mathcal{F} \neq \emptyset$ because $(\emptyset, k) \in \mathcal{F}$ for some map k . Define an ordering on \mathcal{F} as follows: $(D, f) \leq_{\mathcal{F}} (E, g)$ if and only if $D \subseteq E$ and $g|_D = f$. Then \mathcal{F} admits an upperbound of A . By **Zorn's Lemma**, there exists a

maximal element $(M, h) \in \mathcal{F}$. Suppose towards contradiction there are elements $a \in A$, $a \notin M$ and $b \in B$, $b \notin h(M)$. Consider the map:

$$h' : M \cup \{a\} \rightarrow B \text{ defined by } \begin{cases} h'(M) = h(M) \\ h'(a) = b \end{cases}.$$

This set is clearly injective, and furthermore we have that $(M, h) \leq (M \cup \{a\}, h')$. This is a contradiction, hence $M = A$ or $h(M) = B$. If $M = A$, then the injection $A \xrightarrow{h} B$ implies $\text{card}(A) \leq \text{card}(B)$. If $h(M) = B$, then the map $B \hookrightarrow M \hookrightarrow A$ implies $\text{card}(B) \leq \text{card}(A)$. \square

Corollary 1.4.9. $\text{card}(\mathbf{Q}) = \text{card}(\mathbf{N})$.

Proof. This follows directly from Example 1.4.4 and Theorem 1.4.8 \square

Definition 1.4.5. A set A is countable if $\text{card}(A) \leq \text{card}(\mathbf{N})$. Equivalently, there exists an injection $A \hookrightarrow \mathbf{N}$ and a surjection $\mathbf{N} \twoheadrightarrow A$. If A is countable and infinite, A is called denumerable (or more commonly referred to as countably infinity).

Definition 1.4.6. We say $\text{card}(\mathbf{N}) = \text{card}(\mathbf{Z}) = \text{card}(\mathbf{Q}) := \aleph_0$, called aleph naught. We also define $\text{card}(\mathbf{R}) = \mathfrak{c}$, called the continuum.

Example 1.4.5. By Theorem 1.4.4, $\aleph_0 < \mathfrak{c}$.

Corollary 1.4.10. *There does not exist an infinite set A with $\text{card}(A) < \aleph_0$. In particular, if A is infinite and countable, then $\text{card}(A) = \aleph_0$.*

Proof. By Exercise 1.4.1, $\text{card}(\mathbf{N}) \leq \text{card}(A)$, and by definition (since A is countable), $\text{card}(A) \leq \text{card}(\mathbf{N})$. So by Theorem 1.4.8, $\text{card}(A) = \text{card}(\mathbf{N}) = \aleph_0$. \square

Example 1.4.6. $\text{card}(\mathcal{P}(\mathbf{N})) > \text{card}(\mathbf{N}) = \aleph_0$.

Proposition 1.4.11. *The countable union of countable sets is countable. More precisely, if A_i is countable for all $i \in \mathbf{N}$, then $\bigcup_{i=1}^{\infty} A_i$ is countable.*

Proof. By definition, there exist surjections $\pi_i : \mathbf{N} \rightarrow A_i$. Define $\pi : \mathbf{N} \times \mathbf{N} \rightarrow \bigcup_{i=1}^{\infty} A_i$ by $\pi(i, j) = \pi_i(j)$. Claim: π is onto. Let $x \in \bigcup_{i=1}^{\infty} A_i$, then there exists an i_0 with $x \in A_{i_0}$. Since π_{i_0} is onto, there exists a $j_0 \in \mathbf{N}$ with $\pi_{i_0}(j_0) = x$. So $\pi(i_0, j_0) = x$, establishing that π is surjective as well. Therefore $\text{card}(\bigcup_{i=1}^{\infty} A_i) \leq \text{card}(\mathbf{N} \times \mathbf{N}) = \text{card}(\mathbf{N})$. \square

Lemma 1.4.12. $\text{card}([0, 1]) \leq \text{card}(2^{\mathbf{N}})$.

Proof. Recall that every $\sigma \in [0, 1]$ has a binary expansion $\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k}$, where $\sigma_k \in \{0, 1\}$. Consider the map $\varphi : 2^{\mathbf{N}} \rightarrow [0, 1]$ defined by $\varphi(f) = \sum_{k=1}^{\infty} \frac{f(k)}{2^k}$. Letting $f(k) = \sigma_k$ gives φ is surjective. \square

Lemma 1.4.13. $\text{card}(\mathbf{R}) = \text{card}([0, 1])$.

Proof. By inclusion $[0, 1] \hookrightarrow \mathbf{R}$, which implies that $\text{card}([0, 1]) \leq \text{card}(\mathbf{R})$. Recall that $\mathbf{R} \xrightarrow{\tan} (0, 1) \hookrightarrow [0, 1]$, which implies that $\text{card}(\mathbf{R}) \leq \text{card}([0, 1])$. Then Theorem 1.4.8 gives the desired result. \square

Lemma 1.4.14. $\text{card}(2^{\mathbf{N}}) \leq \text{card}([0, 1])$.

Proof. Consider the map $\lambda : 2^{\mathbf{N}} \rightarrow [0, 1]$ defined by $\lambda(f) = \sum_{k=1}^{\infty} \frac{f(k)}{3^k}$. Claim: λ is injective. Let $f, g \in 2^{\mathbf{N}}$ with $f \neq g$. Let k_0 be the *smallest point* k where f and g are different. So in particular:

$$\begin{aligned} f(1) &= g(1) \\ f(2) &= g(2) \\ &\vdots \\ f(k_0 - 1) &= g(k_0 - 1) \\ f(k_0) &\neq g(k_0). \end{aligned}$$

Let:

$$\begin{aligned} t_1 &= \sum_{k > k_0} \frac{f(k)}{3^k} && \text{sum past } k_0 \\ t_2 &= \sum_{k > k_0} \frac{g(k)}{3^k} && \text{sum past } k_0 \\ s_1 &= \sum_{k=1}^{k_0-1} \frac{f(k)}{3^k} && \text{sum before } k_0 \\ s_2 &= \sum_{k=1}^{k_0-1} \frac{g(k)}{3^k} && \text{sum before } k_0 \end{aligned}$$

We have that:

$$\begin{aligned} \lambda(f) &= s_1 + \frac{f(k_0)}{3^{k_0}} + t_1 \\ \lambda(g) &= s_2 + \frac{g(k_0)}{3^{k_0}} + t_2 \end{aligned}$$

Because f and g differ at k_0 , without loss of generality let $f(k_0) = 0$ and $g(k_0) = 1$. Then

$\lambda(g) - \lambda(f) = \frac{1}{3^{k_0}} + t_2 - t_1$. Observe that:

$$\begin{aligned}
 |t_2 - t_1| &= \left| \sum_{k > k_0} \frac{g(k) - f(k)}{3^k} \right| \\
 &\leq \sum_{k > k_0} \frac{|g(k) - f(k)|}{3^k} && \text{By triangle inequality} \\
 &\leq \sum_{k > k_0} \frac{1}{3^k} && \text{By comparison test} \\
 &= \frac{1}{3^{k_0+1}} \sum_{k \geq 0} \frac{1}{3^k} \\
 &= \frac{1}{3^{k_0+1}} \cdot \frac{1}{1 - \frac{1}{3}} \\
 &= \frac{3}{2 \cdot 3^{k_0+1}} \\
 &= \frac{1}{2 \cdot 3^{k_0}} \\
 &< \frac{1}{3^{k_0}}.
 \end{aligned}$$

Since $|t_2 - t_1| < \frac{1}{3^{k_0}}$, $\lambda(g) - \lambda(f) \neq 0$, establishing λ as an injection. Thus $\text{card}(2^{\mathbf{N}}) \leq \text{card}([0, 1])$. \square

Theorem 1.4.15. $\text{card}(2^{\mathbf{N}}) = \text{card}(\mathcal{P}(\mathbf{N})) = \text{card}(\mathbf{R})$.

Proof. This follows from Lemma 1.4.12, Lemma 1.4.13, and Lemma 1.4.14. \square

2

Ordered Fields

2.1 Ordering of \mathbb{Z}

Definition 2.1.1. Define $\mathbf{Z}^+ = \{n \in \mathbf{Z} \mid n \geq_a 0\}$, where \geq_a is the *algebraic ordering* from Example 1.2.1. We call \mathbf{Z}^+ the cone of positive integers, and they admit the following axioms:

- (1) If $m, n \in \mathbf{Z}^+$, then $m + n \in \mathbf{Z}^+$ and $mn \in \mathbf{Z}^+$.
- (2) For all $m \in \mathbf{Z}$, $m \in \mathbf{Z}^+$ or $-m \in \mathbf{Z}^+$.
- (3) If $m \in \mathbf{Z}^+$ and $-m \in \mathbf{Z}^+$, then $m = 0$.

Proposition 2.1.1 (Properties of \leq_a).

- (1) $m \leq_a n$ if and only if $n - m \in \mathbf{Z}^+$.
- (2) If $m \leq_a n$ and $p \leq_a q$, then $m + p \leq_a n + q$.
- (3) If $m \leq_a n$ and $p \in \mathbf{Z}^+$, then $pm \leq_a pn$.
- (4) If $m \leq_a n$ then $-n \leq_a -m$.
- (5) (\mathbf{Z}, \leq_a) forms a total ordering.
- (6) If $m >_a 0$ and $mn >_a 0$, then $n >_a 0$.
- (7) If $m >_a 0$ and $mn \geq_a mp$, then $n \geq_a p$.

Proof. (5) Let $m, n \in \mathbf{Z}$, since \mathbf{Z} is closed under subtraction $m - n \in \mathbf{Z}$. So either $m - n \in \mathbf{Z}^+$ or $n - m \in \mathbf{Z}^+$. Then by (1) $n \leq_a m$ or $m \leq_a n$. Thus (\mathbf{Z}, \leq_a) is a total ordering.

(6) We have $mn >_a 0$ with $m >_a 0$. If $n = 0$, we are done. So now assume $n \neq 0$. Then either $n \in \mathbf{Z}^+$ or $-n \in \mathbf{Z}^+$. If $-n \in \mathbf{Z}^+$, then $m(-n) = -(mn) \in \mathbf{Z}^+$. But we had assumed $mn >_a 0$; i.e., $mn \in \mathbf{Z}^+$, hence it must be the case that $mn = 0$, a contradiction. Therefore it must be that $n \in \mathbf{Z}^+$. □

2.2 Ordering of \mathbb{Q}

Proposition 2.2.1. Define $\mathbf{Q} := \mathbf{Z} \times \mathbf{N}$. Show that \sim forms an equivalence relation, where $(a, b) \sim (c, d)$ if and only if $ad = bc$.

Proof. I dont wanna do this □

Definition 2.2.1. The set of equivalence classes of \mathbf{Q} is $\mathbf{Q} = \mathbf{Q}/\sim = \{[(a, b)] \mid (a, b) \in \mathbf{Q}\}$. We call this set the rational numbers, and denote the equivalence classes $[(a, b)]$ as $\frac{a}{b}$.

Proposition 2.2.2. *The operations*

$$\begin{aligned} + : \mathbf{Q} \times \mathbf{Q} &\rightarrow \mathbf{Q} \text{ defined by } [(a, b)] + [(c, d)] = [(ad + bc, bd)] \\ \cdot : \mathbf{Q} \times \mathbf{Q} &\rightarrow \mathbf{Q} \text{ defined by } [(a, b)] \cdot [(c, d)] = [(ac, bd)] \end{aligned}$$

are well-defined. Furthermore, $(\mathbf{Q}, +, \cdot)$ forms a field.

Proof. I dont wana □

Lemma 2.2.3. *There is an injective map $\mathbf{Z} \xrightarrow{j} \mathbf{Q}$ defined by $j(n) = \frac{n}{1}$ satisfying the properties*

$$\begin{aligned} j(n + m) &= j(n) + j(m) \\ j(nm) &= j(n)j(m). \end{aligned}$$

Proof. Note that $j(n) = j(m)$ if and only if $\frac{n}{1} = \frac{m}{1}$. By definition this is equivalent to $n = m$, hence j is injective.

Observe that $j(n + m) = \frac{n+m}{1} = \frac{n}{1} + \frac{m}{1} = j(n) + j(m)$ and $j(nm) = \frac{nm}{1} = \frac{n}{1} \cdot \frac{m}{1} = j(n)j(m)$. □

Theorem 2.2.4. (\mathbf{Q}, \leq_Q) is a total ordering, where \leq_Q is a well-defined ordering defined by $\frac{a}{b} \leq_Q \frac{c}{d}$ if and only if $ad \leq_a bc$ in (\mathbf{Z}, \leq_a) . Furthermore, the map $j : \mathbf{Z} \hookrightarrow \mathbf{Q}$ is order preserving, that is, if $n \leq_a m$ in (\mathbf{Z}, \leq_a) , then $j(n) \leq_Q j(m)$ in (\mathbf{Q}, \leq_Q) .

Proof. i REALLY dont □

Definition 2.2.2. Define $\mathbf{Q}_+ := \{q \in \mathbf{Q} \mid q \geq_Q 0\}$ as the cone of positive rationals, and they admit the following axioms:

- (1) If $q_1, q_2 \in \mathbf{Q}^+$, then $q_1 + q_2 \in \mathbf{Q}^+$ and $q_1 q_2 \in \mathbf{Q}^+$.
- (2) For all $q \in \mathbf{Q}$, $q \in \mathbf{Q}^+$ or $-q \in \mathbf{Q}^+$.
- (3) If $q \in \mathbf{Q}^+$ and $-q \in \mathbf{Q}^+$, then $q = 0$.
- (4) $q_1 \leq_Q q_2$ if and only if $q_2 - q_1 \in \mathbf{Q}^+$.

Proposition 2.2.5. *Let $r, s, t, u \in \mathbf{Q}$*

- (1) *If $r \leq_Q s$ and $t \leq_Q u$, then $r + t \leq_Q s + u$.*
- (2) *If $r \leq_Q s$ and $t \geq_Q 0$, then $tr \leq_Q ts$.*

Proof. do this shi later □

2.3 Rings and Fields

Definition 2.3.1. A ring is a non-empty set R equipped with two binary operations:

$$\begin{aligned} R \times R &\xrightarrow{a} R \text{ defined by } a(r, s) = r + s \\ R \times R &\xrightarrow{m} R \text{ defined by } m(r, s) = rs, \end{aligned}$$

such that they admit the following axioms:

- (1) R is an *abelian group* under addition:
 - (i) $r + (s + t) = (r + s) + t$ for all $r, s, t \in R$,
 - (ii) there exists an element $0_R \in R$ with $r + 0_R = r = 0_R + r$ for all $r \in R$,
 - (iii) For all $r \in R$ there exists an $s \in R$ such that $r + s = 0_R = s + r$ (such an s is unique, and is denoted $-r$),
 - (iv) $r + s = s + r$ for all $r, s \in R$.
- (2) $r(st) = (rs)t$ for all $r, s, t \in R$,
- (3) $(r + s)t = rt + rs$ and $r(s + t) = rs + rt$ for all $r, s, t \in R$.

If R contains an element 1_R such that $1_R r = r = r 1_R$, then we say R is unital. If $rs = sr$ for all $r, s \in R$, then we say R is commutative. If R is a unital ring such that $1_R \neq 0_R$ and for all $r \in R$ there exists an $s \in R$ such that $rs = 1_R = sr$ (such an s is unique, and denoted r^{-1}), then we say R is a division ring.

Definition 2.3.2. A field is a commutative division ring.

Example 2.3.1.

- (1) \mathbf{Q} is a field.
- (2) $\mathbf{Z}/p\mathbf{Z}$ is a field.
- (3) $\mathbf{C}_{\mathbf{Q}} = \{r + si \mid r, s \in \mathbf{Q}, i^2 = -1\}$ with addition and multiplication defined by

$$\begin{aligned} (r + si) + (t + ui) &:= (r + t) + (s + u)i \\ (r + si)(t + ui) &:= (rt - su) + (ru + st)i \end{aligned}$$

is a field. We call this set the *complex rationals*.

Definition 2.3.3. An ordered field is a field F equipped with a total ordering \leq_F such that:

- (1) If $x \leq_F y$ and $u \leq_F v$, then $x + u \leq_F y + v$.
- (2) If $x \leq_F y$ and $z \geq_F 0$, then $xz \leq_F zy$.

We similarly define $F^+ = \{x \in F \mid x \geq_F 0\}$ as the cone of positive elements.

Proposition 2.3.1. *Let (F, \leq_F) be an ordered field.*

- (1) *If $x, y \in F^+$, then $x + y \in F^+$ and $xy \in F^+$.*
- (2) *If $x \in F$, then $-x \in F^+$ or $x \in F^+$.*
- (3) *If $x, -x \in F^+$, then $x = 0$.*

Proof. need to do

□

Example 2.3.2.

- (1) \mathbf{Q} is an ordered field.
- (2) Is $\mathbf{C}_{\mathbf{Q}}$ an ordered field?

Proposition 2.3.2. *Let (F, \leq) be an ordered field with $1_F \neq 0_F$.*

- (1) *For all $a \in F$, $a^2 \in F^+$.*
- (2) *$0, 1 \in F^+$.*
- (3) *If $n \in \mathbf{N}$, then $n \cdot 1_F := \underbrace{1_F + 1_F + \dots + 1_F}_{n \text{ times}}$, implying $n \cdot 1_F \in F^+$.*
- (4) *If $x \in F^+$ and $x \neq 0$, then $x^{-1} \in F^+$.*
- (5) *If $xy \in F^+$ and $xy \neq 0$, then $x, y \in F^+$ or $-x, -y \in F^+$.*
- (6) *If $0 < x \leq y$, then $y^{-1} \leq x^{-1}$.*
- (7) *If $x \leq y$, then $-y \leq -x$.*
- (8) *If $x \geq 1_F$, then $x^2 \geq x$.*
- (9) *If $x \leq 1_F$, then $x^2 \leq x$.*

Proof. (1) If $a \in F^+$, then $a \cdot a = a^2 \in F^+$. If $-a \in F^+$, then $(-a) \cdot (-a) = a^2 \in F^+$.

(2) From part (1) we have that $0 = 0 \cdot 0 \in F^+$. Similarly, $1 = 1 \cdot 1 \in F^+$ and $(-1) \cdot (-1) \in F^+$.

(3) Since F^+ is closed under addition, we can inductively show that $n \cdot 1 = 1 + 1 + \dots + 1 \in F^+$.

(4) Suppose towards contradiction $x^{-1} \notin F^+$. Then $-(x^{-1}) \in F^+$, so $(-(x^{-1})) \cdot x = -1(x^{-1} \cdot x) = -1 \in F^+$. But $-1, 1 \in F^+$ implies $1 = 0$, a contradiction. Thus $x^{-1} \in F^+$.

(6) $y \geq x > 0$ implies $x, y \in F^+$. So $x^{-1}, y^{-1} \in F^+$. Then $y^{-1}xx^{-1} \leq y^{-1}yx^{-1}$, and simplifying yields $y^{-1} \leq x^{-1}$. **finish the rest (i'm not going to)** □

3

The Real Numbers

3.1 The Completion of \mathbb{Q}

Definition 3.1.1. A Dedekind cut is a nonempty subset D of \mathbb{Q} with the following properties:

- (1) $D \neq \mathbb{Q}$;
- (2) If $b \in D$, then $a \in D$ for all $a \in \mathbb{Q}$ with $a < b$;
- (3) D does not contain a largest element.

Example 3.1.1. The following examples are Dedekind cuts:

- (1) $\{a \in \mathbb{Q} \mid a < 3\}$ (the set of all rational numbers less than 3).
- (2) $\{a \in \mathbb{Q} \mid a < 0 \text{ or } a^2 < 2\}$ (the set of all rational numbers less than $\sqrt{2}$).
- (3) $\{a \in \mathbb{Q} \mid a < 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \text{ for some } n \in \mathbb{Z}^+\}$ (the set of all rational numbers less than e).

Definition 3.1.2. Let C and D be Dedekind cuts.

will probably not finish this

3.2 Ordering of \mathbb{R}

Axiom 1. \mathbb{R} is an ordered field.

Proposition 3.2.1. $\mathbb{Q}^+ \subseteq \mathbb{R}^+$.

Proof. If $x \in \mathbb{Q}^+$, then $x = \frac{p}{q}$ with $p \in \mathbb{Z}^+$ and $q \in \mathbb{N}$. Write $p = \underbrace{1 + 1 + \dots + 1}_{p \text{ times}}$, then $p \in \mathbb{R}^+$.

Similarly, write $q = \underbrace{1 + 1 + \dots + 1}_{q \text{ times}}$. Then $q \in \mathbb{R}^+$, which implies that $q^{-1} \in \mathbb{R}^+$. Hence $\frac{p}{q} \in \mathbb{R}^+$, establishing $\mathbb{Q}^+ \subseteq \mathbb{R}^+$. □

Proposition 3.2.2. The maps $\mathbb{Z} \xrightarrow{j} \mathbb{Q} \xrightarrow{i} \mathbb{R}$ are order embeddings (defined in Lemma 2.2.3 and Theorem 2.2.4).

Proof. Suppose $i(q_1) \leq_{\mathbb{Q}} i(q_2)$. Then $q_1 \leq_{\mathbb{R}} q_2$, hence $q_2 - q_1 \in \mathbb{R}^+$. Now If $q_2 - q_2 \in \mathbb{Q}^+$, then $q_2 - q_1 \in \mathbb{R}^+$. Hence $q_1 \leq_{\mathbb{R}} q_2$. wtf is this saying? □

Proposition 3.2.3. Let $a, b \in \mathbb{R}$. If $a \leq b$ (or $a < b$), then $a \leq \frac{1}{2}(a + b) \leq b$ (or $a < \frac{1}{2}(a + b) < b$).

Proof. By the order axioms, $a \leq b$ implies $a + a \leq a + b$. So $2a \leq a + b$, which is equivalent to $a \leq \frac{1}{2}(a + b)$. Similarly, $a + b \leq b + b$, which similarly gives $\frac{1}{2}(a + b) \leq b$, establishing the proposition. \square

Corollary 3.2.4. *Given $b > 0$, we have $0 < \frac{1}{2}b < b$.*

Proof. From Proposition 3.2.3, setting $a = 0$ yields the desired result. \square

Proposition 3.2.5. *Suppose $a \in \mathbb{R}$. For all $\epsilon > 0$, if $0 \leq a \leq \epsilon$, then $a = 0$.*

Proof. If $a = 0$ we are done. If $a > 0$, by Corollary 3.2.4 $0 \leq \frac{1}{2}a \leq a$. Pick $\epsilon = \frac{1}{2}a$, then $a \leq \frac{1}{2}a$, a contradiction. Thus $a = 0$. \square

Definition 3.2.1. Let $a_1, a_2, \dots, a_n > 0$. The arithmetic mean is $\frac{1}{2} \left(\sum_{j=1}^n a_j \right)$. The geometric mean is $\left(\prod_{j=1}^n a_j \right)^{\frac{1}{n}}$.

Proposition 3.2.6 (AM-GM Inequality). *For all $a_1, a_2, \dots, a_n \geq 0$, then $\left(\prod_{j=1}^n a_j \right)^{\frac{1}{n}} \leq \frac{1}{2} \left(\sum_{j=1}^n a_j \right)$.*

Proof. We will only prove the $n = 2$ case. Consider the fact that $(a_1 - a_2)^2 \geq 0$, and expanding gives $a_1^2 - 2a_1a_2 + a_2^2$. So $2a_1a_2 \leq a_1^2 + a_2^2$. Adding $2a_1a_2$ to both sides yields $4a_1a_2 \leq a_1^2 + 2a_1a_2 + a_2^2$, which is equivalent to $4a_1a_2 \leq (a_1 + a_2)^2$. Then simplifying yields the desired result of $(a_1a_2)^{\frac{1}{2}} \leq \frac{1}{2}(a_1 + a_2)$. \square

Proposition 3.2.7 (Bernoulli's Inequality). *If $x > -1$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}_0$.*

Proof. We proceed with induction with base case $n = 0$ and $n = 1$; these hold by inspection. Assume the inequality holds true for $n = k$. For $n = k + 1$:

$$\begin{aligned} (1 + x)^{k+1} &= (1 + x)^k (1 + x) \\ &\geq (1 + nx)(1 + x)^1 \\ &= 1 + (n + 1)x + nx^2 \\ &\geq 1 + (n + 1)x. \end{aligned}$$

\square

Proposition 3.2.8 (Cauchy-Schwartz Inequality). *Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}^n$. Then:*

$$\left| \sum_{j=1}^n a_j b_j \right| \leq \left(\sum_{j=1}^n a_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}}.$$

¹Because order is preserved under multiplication by positive elements.

Proof. Consider the map $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by $F(t) = \sum_{j=1}^n (a_j - b_j t)^2$. Note that $\sum_{j=1}^n (a_j - b_j t)^2 \geq 0$. Observe that:

$$\begin{aligned} \sum_{j=1}^n (a_j - b_j t)^2 &= \sum_{j=1}^n (a_j^2 - 2a_j b_j t + b_j^2 t^2) \\ &= \sum_{j=1}^n a_j^2 - \sum_{j=1}^n 2a_j b_j t + \sum_{j=1}^n b_j^2 t^2. \end{aligned}$$

This is a quadratic equation, and since $F(t) \geq 0$, the discriminant will be less than or equal to 0. Hence:

$$\Delta = \left(\sum_{j=1}^n 2a_j b_j \right)^2 - 4 \left(\sum_{j=1}^n b_j^2 \right) \left(\sum_{j=1}^n a_j^2 \right) \leq 0.$$

Simplifying gives:

$$\left(\sum_{j=1}^n 2a_j b_j \right)^2 \leq 4 \left(\sum_{j=1}^n b_j^2 \right) \left(\sum_{j=1}^n a_j^2 \right).$$

Pulling 2 out from the left-hand side, dividing both sides by 4, and then square-rooting gives the desired result. \square

Question. When do we have equality?

Answer. When $\Delta = 0$, there exists a $t_0 \in \mathbf{R}$ with $F(t_0) = 0$. So $\sum_{j=1}^n (a_j - b_j t_0) = 0$ implies $a_j - b_j t_0 = 0$ for all j . Hence there is equality only when $a_j = b_j t_0$ for all j .

Proposition 3.2.9 (Triangle Inequality). *Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbf{R}^n$. Then:*

$$\left(\sum_{j=1}^n (a_j + b_j)^2 \right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^n a_j^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}}.$$

Proof. Observe that:

$$\begin{aligned} \sum_{j=1}^n (a_j + b_j)^2 &= \sum_{j=1}^n (a_j^2 + 2a_j b_j + b_j^2) \\ &= \sum_{j=1}^n a_j^2 + \sum_{j=1}^n 2a_j b_j + \sum_{j=1}^n b_j^2 \\ &\leq \sum_{j=1}^n a_j^2 + 2 \left(\sum_{j=1}^n a_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}} + \sum_{j=1}^n b_j^2 \\ &= \left(\left(\sum_{j=1}^n a_j^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}} \right)^2. \end{aligned}$$

Squaring both sides gives the desired result. \square

3.3 Metrics and Norms on \mathbb{R}^n

Definition 3.3.1. The absolute value is a function $|\cdot| : \mathbf{R} \rightarrow \mathbf{R}$ defined by:

$$|x| = \begin{cases} x, & x \in \mathbf{R}^+ \\ -x, & -x \in \mathbf{R}^+. \end{cases}$$

Proposition 3.3.1. Let $a, b \in \mathbf{R}$ and $\delta > 0$.

$$(1) \quad |ab| = |a||b|.$$

$$(2) \quad |a|^2 = |a^2|.$$

$$(3) \quad |-a| = |a|.$$

$$(4) \quad |a| \in \mathbf{R}^+.$$

$$(5) \quad -|a| \leq a \leq |a|.$$

$$(6) \quad |a| \leq \delta \text{ if and only if } -\delta \leq a \leq \delta.$$

$$(7) \quad |a + b| \leq |a| + |b|.$$

$$(8) \quad |a - b| \leq |a| + |b|.$$

$$(9) \quad ||a| - |b|| \leq |a - b|.$$

Proof. **do later** □

Lemma 3.3.2. $\pm x \leq \delta$ if and only if $|x| \leq \delta$.

Proof. **do later** □

Lemma 3.3.3. $A \subseteq \mathbf{R}$ is bounded if and only if there exists an $r > 0$ such that $|a| < r$ for all $a \in A$.

Proof. Suppose $A \subseteq \mathbf{R}$ is bounded. Then there exists an $l, u \in \mathbf{R}$ with $l \leq a \leq u$ for all $a \in A$. We have that:

$$-|l| \leq l \leq a \leq u \leq |u|.$$

Let $r = \max\{|l|, |u|\} \geq 0$. So $-r \leq |l| \leq a \leq |u| \leq r$. Thus $|a| \leq r$.

Conversely, suppose there exists an $r > 0$ with $|a| \leq r$ for all $a \in A$. Then $-r \leq a \leq r$ for all $a \in A$, hence A is bounded. □

Definition 3.3.2. A function $f : D \rightarrow \mathbf{R}$ is bounded if $\text{im}(f) \subseteq \mathbf{R}$ is a bounded subset. Equivalently, there exists a $c > 0$ such that $|f(x)| < c$ for all $x \in D$.

Example 3.3.1. Consider the function $f : [3, 7] \rightarrow \mathbf{R}$ defined by $f(x) = \frac{x^2+2x+1}{x-1}$. Since $3 \leq x \leq 7$, observe that:

$$\begin{aligned} |x^2 + 2x + 1| &\leq |x^2| + |2x| + 1 \\ &= |x|^2 + 2|x| + 1 \quad \text{Evaluate at 7} \\ &= 64 \end{aligned}$$

Likewise, $3 \leq x \leq 7$ implies $|x - 1| \geq 2$, hence $\frac{1}{|x-1|} \leq \frac{1}{2}$. Together, we have that:

$$\left| \frac{x^2 + 2x + 1}{x - 1} \right| \leq \frac{64}{2} = 32.$$

Definition 3.3.3. Let $s, t \in \mathbf{R}$. We define the distance between s and t as $d(s, t) = |s - t|$.

Definition 3.3.4. Let X be a nonempty set equipped with a map $d : X \times X \rightarrow \mathbf{R}^+$. We say (X, d) is a semi-metric if for all $x, y, z \in X$,

- (1) $d(x, y) = d(y, x)$,
- (2) $d(x, z) \leq d(x, y) + d(y, z)$, and
- (3) $d(x, x) = 0$.

We say (X, d) is a metric space if it satisfies the additional axiom:

- (4) $d(x, y) = 0$ implies $x = y$.

Proposition 3.3.4.

- (1) $(\mathbf{R}, d_1(s, t) = |s - t|)$ is a metric space.
- (2) $(\mathbf{R}^n, d_1(\vec{x}, \vec{y}) = \sum_{j=1}^n |y_j - x_j|)$ is a metric space.
- (3) $(\mathbf{R}^n, d_\infty(\vec{x}, \vec{y}) = \max_{j=1}^n \{|y_j - x_j|\})$ is a metric space.
- (4) $(\mathbf{R}^n, d_2(\vec{x}, \vec{y}) = \left(\sum_{j=1}^n |y_j - x_j|^2 \right)^{\frac{1}{2}})$ is a metric space.
- (5) $(\mathbf{R}^n, d_p(\vec{x}, \vec{y}) = \left(\sum_{j=1}^n |y_j - x_j|^p \right)^{\frac{1}{p}})$ for some $p \in \mathbf{Q}$ is a metric space.

Proof. (1) We have $d(s, t) = |s - t| = |t - s| = d(t, s)$. Similarly, $d(s, r) = |s - r| = |s - t + t - r| \leq |s - t| + |t - r| = d(s, t) + d(t, r)$. Clearly $d(s, s) = |s - s| = 0$. Lastly, if $d(s, t) = 0$, then $|s - t| = 0$, which is equivalent to $s - t = 0$; i.e., $s = t$. Thus (\mathbf{R}, d_1) is a metric space.

(4) Axioms 2 and 3 of metric spaces are clearly satisfied. If $d_2(\vec{x}, \vec{y}) = 0$ then $|y_j - x_j|^2 = 0$ for all j . Hence $y_j - x_j = 0$; i.e., $y_j = x_j$ for all j , establishing axiom 4. Observe that:

$$\begin{aligned}
 d_2(\vec{x}, \vec{z}) &= \left(\sum_{j=1}^n |z_j - x_j|^2 \right)^{\frac{1}{2}} \\
 &= \left(\sum_{j=1}^n |z_j - y_j + y_j - x_j|^2 \right)^{\frac{1}{2}} \\
 &= \left(\sum_{j=1}^n (z_j - y_j + y_j - x_j)^2 \right)^{\frac{1}{2}} \\
 &\leq \left(\sum_{j=1}^n (z_j - y_j)^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n (y_j - x_j)^2 \right)^{\frac{1}{2}} \\
 &= \left(\sum_{j=1}^n |z_j - y_j|^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n |y_j - x_j|^2 \right)^{\frac{1}{2}} \\
 &= d_2(\vec{x}, \vec{y}) + d_2(\vec{y}, \vec{z}).
 \end{aligned}$$

Thus (\mathbf{R}^n, d_2) is a metric space. □

Definition 3.3.5. Let (X, d) be a metric space.

- (1) The open ball centered at x_0 with radius $\delta > 0$ is $U(x_0, \delta) = \{y \in X \mid d(y, x_0) < \delta\}$.
- (2) The closed ball centered at x_0 with radius $\delta > 0$ is $B(x_0, \delta) = \{y \in X \mid d(y, x_0) \leq \delta\}$.
- (3) A subset $A \subseteq X$ is called open if for all $a \in A$, there exists a $\delta > 0$ such that $U(a, \delta) \subseteq A$.
- (4) A subset $C \subseteq X$ is called closed if $\text{compl}(C) = X \setminus C$ is open.

Example 3.3.2. Consider $X = \mathbf{R}$ and $d(s, t) = |s - t|$. Observe that:

$$\begin{aligned}
 U(t, \delta) &= \{s \in \mathbf{R} \mid d(s, t) < \delta\} \\
 &= \{s \in \mathbf{R} \mid |s - t| < \delta\} \\
 &= \{s \in \mathbf{R} \mid -\delta < s - t < \delta\} \\
 &= \{s \in \mathbf{R} \mid -\delta + t < s < \delta + t\} \\
 &= (t - \delta, t + \delta).
 \end{aligned}$$

It follows similarly that $B(t, \delta) = [t - \delta, t + \delta]$.

Proposition 3.3.5. If I is an open interval, then I is open.

Proof. Let $I = (a, b)$. Let $x \in I$. Let $\delta_x = \min\{x - a, b - x\} > 0$. Now let $t \in V_{\delta}(x)$. Then $t \in (x - \delta, x + \delta)$. Case 1: $\min\{x - a, b - x\} = x - a$. Then $x - (x - a) < t < x + x - a$, **idk how to do this** □

4

Supremum, Infimum, and Completeness

4.1 Supremum and Infimum

Theorem 4.1.1. Let $\emptyset \neq A \subseteq \mathbf{R}$. Let u be an upperbound for A . The following are equivalent:

- (1) $u = \sup(A)$.
- (2) If $t < u$, then there exists an $a_t \in A$ with $t < a_t$.
- (3) For all $\epsilon > 0$, there exists an $a_\epsilon \in A$ such that $u - \epsilon < a_\epsilon$.

Proof. [(1) \implies (2)] Assume $u = \sup(A)$. Let $t < u$. Suppose towards contradiction there does not exist and $a \in A$ with $a > t$. Then $a \leq t$ for all $a \in A$. But this implies t is an upperbound of A less than u , which is a contradiction because u is the least upper bound. [(2) \implies (3)] Given $\epsilon > 0$, let $t = u - \epsilon$. Then applying (2) gives the desired result. [(3) \implies (1)] We know u is an upperbound of A , we aim to show that it is the least upperbound. Let v be an upperbound for A with $v < u$. Pick $\epsilon = u - v > 0$. By (3), there exists an $a_\epsilon \in A$ such that $u - (u - v) < a_\epsilon$. So $v < a_\epsilon$, which is a contradiction (v is an upperbound, how can it be smaller than an element of A ?). \square

Example 4.1.1. Claim: $\sup([0, 1)) = 1$. If $s \in [0, 1)$, by definition $s < 1$, so 1 is an upper bound for $[0, 1)$. Given $t < 1$, set $\delta = 1 - t > 0$. Then $0 < \frac{\delta}{2} < \delta$ **this is not trivial, have to show $\delta - \delta/2$ is positive**. This gives:

$$t < t + \frac{\delta}{2} < t + \delta = 1.$$

Pick $a_t = t + \frac{\delta}{2}$. By (2) of Theorem 4.1.1, $a_t \in [0, 1)$, hence $1 = \sup([0, 1))$.

Proposition 4.1.2. Let $A, B \subseteq \mathbf{R}$ and $a \leq b$ for all $a \in A$ and $b \in B$. Then $\sup(A) \leq \inf(B)$.

Proof. Fix a point $b_0 \in B$. Then $a \leq b_0$ for all $a \in A$. Then b_0 is an upperbound for A . This gives $u := \sup(A) \leq b_0$. But since b_0 was arbitrary, we have $u \leq b$ for all $b \in B$. So u is a lower bound for B , therefore $u \leq \inf(B)$. \square

Axiom 2 (Completeness of \mathbf{R}). Given any nonempty subset $A \subseteq \mathbf{R}$ which is bounded above, $\sup(A)$ exists.

Lemma 4.1.3. For $A \subseteq \mathbf{R}$ which is bounded below, $\sup(-A) = -\inf(A)$.

Proof. If A is bounded below, then $-A$ is bounded above. Then $\sup(-A)$ exists, define it as u . So for all $a \in A$, $-a \leq u$. Hence $-u$ is a lower bound for A . Suppose v is another lower bound for A . Then $v \leq a$ for all $a \in A$. So $-v \geq -a$ for all $a \in A$. Thus $-v$ is an upper bound of $-A$. Therefore, since u is the least upper bound, $-v \geq u$; i.e., $-u \geq v$. Thus $-u = \inf(A)$. \square

Axiom 3 (Well-Ordering Principle). *Every nonempty subset $A \subseteq \mathbf{N}$ contains a least element.*

Proposition 4.1.4 (Arcimedean Property 1). *If $x \in \mathbf{R}$, then there exists $n_x \in \mathbf{N}$ with $x < n_x$.*

Proof. Suppose not. That is, suppose $n \leq x$ for all $n \in \mathbf{N}$. Then x is an upper bound for \mathbf{N} . Thus $\sup(A) := u$ exists. From part (3) of Theorem 4.1.1, take $\epsilon = 1$. Then there exists an $n \in \mathbf{N}$ such that $u - 1 < n$. So $u < n + 1 \in \mathbf{N}$, which is a contradiction. \square

Proposition 4.1.5 (Archimedean Property 2). *If $t > 0$, there exists $n_t \in \mathbf{N}$ with $\frac{1}{n_t} < t$.*

Proof. From **Arcimedean Property 1**, pick $x = \frac{1}{t}$. \square

Corollary 4.1.6. *Given $t > 0$, there exists $m \in \mathbf{N}$ with $\frac{1}{2^m} < t$.*

Proof. By **Archimedean Property 2** there exists an $n \in \mathbf{N}$ with $\frac{1}{n} < t$. Claim: $\frac{1}{2^n} < \frac{1}{n}$. It suffices to show that $2^n > n$. Proposition 1.4.6 gives $\text{card}(\{1, 2, \dots, n\}) < \text{card}(\mathcal{P}(\{1, 2, \dots, n\}))$. Then Exercise 1.4.2 gives:

$$n = \text{card}(\{1, 2, \dots, n\}) < \text{card}(\mathcal{P}(\{1, 2, \dots, n\})) = 2^n.$$

Alternatively, **Bernoulli's Inequality** gives $(1 + 1)^n \geq 1 + n$. Hence $2^n > n$. \square

Example 4.1.2.

(1) Claim: $\inf \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\} = 0$. Note that 0 is indeed a lower bound because $0 < \frac{1}{n}$ for all $n \in \mathbf{N}$. Suppose t is another lower bound. If $t \leq 0$, then we are done. If $t > 0$, by the Archimedean Property there exists an $n_t \in \mathbf{N}$ such that $\frac{1}{n_t} < t$, which is a contradiction (because we asserted that t is a lower bound, and $\frac{1}{n_t} \in \inf \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\}$). Thus $\inf \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\} = 0$.

(2) Claim: $\inf \left\{ \frac{1}{2^m} \mid m \in \mathbf{N} \right\} = 0$. This follows from the above example and previous corollary.

Corollary 4.1.7. *Let $x \in \mathbf{R}$. Then there exists $n_x \in \mathbf{Z}$ with $n_x - 1 \leq x < n_x$.*

Proof. Case 1: $x \geq 0$. Let $S_x = \{n \in \mathbf{N} \mid x < n\}$. By **Arcimedean Property 1** $S_x \neq \emptyset$. By the **Well-Ordering Principle**, there exists a least element in this set, call it n_x . Since $n_x \in S_x$, it must be the case that $x < n_x$. But since n_x is the least element, $n_x - 1 \notin S_x$. Since S_x is the set of all natural numbers with lower bound x , $n_x - 1$ is not bounded below by x . Whence $n_x - 1 \leq x$.

Case 2: $x < 0$. Define $S_{-x} = \{n \in \mathbf{N} \mid n < -x\}$. As a consequence of the **Well-Ordering Principle**, any subset of the integers which is bounded above admits a greatest element, define it to be $n_{-x} \in \mathbf{Z}$. Then $n_{-x} + 1 \notin S_{-x}$, hence $n_{-x} < -x \leq n_{-x} + 1$. This establishes $-n_{-x} - 1 \leq x < -n_{-x}$. \square

Definition 4.1.1. Let I be an open interval. A subset $D \subseteq \mathbf{R}$ is dense if $I \cap D \neq \emptyset$.

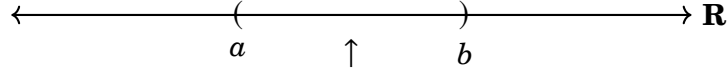
Theorem 4.1.8. $\mathbf{Q} \subseteq \mathbf{R}$ is dense.

Proof. Let I be an open interval. Then there exists $a, b \in \mathbf{R}$ with $(a, b) \subseteq I$. We have that $b - a > 0$. By **Archimedean Property 2** there exists $n \in \mathbf{N}$ with $\frac{1}{n} < b - a$. So $1 + na < nb$. By Corollary 4.1.7, there exists $m \in \mathbf{Z}$ with $m - 1 \leq na < m$. Equivalently, we have that $a < \frac{m}{n}$. We also have that $m \leq na + 1 < nb$, which yields $\frac{m}{n} < b$. Thus $\frac{m}{n} \in (a, b) \cap \mathbf{Q}$. \square

Corollary 4.1.9. $\mathbf{R} \setminus \mathbf{Q} \subseteq \mathbf{R}$ is dense.

Proof. Let $a < b$. Consider $a' = a\sqrt{2}$ and $b' = b\sqrt{2}$. Then $a' < b'$. By Theorem 4.1.8, there exists a $q \in \mathbf{Q}$ with $a' < q < b'$. Thus $a < \frac{q}{\sqrt{2}} < b$. Since $\frac{q}{\sqrt{2}} \notin \mathbf{Q}$, the corollary is established.

Alternatively, observe the following picture:



If there is not an irrational number between (a, b) , then $(a, b) \subseteq \mathbf{Q}$, which is a contradiction. \square

Theorem 4.1.10. There exists a unique positive number x with $x^2 = 2$.

Proof. Consider the set $S = \{t \in \mathbf{R} \mid t > 0, t^2 < 2\}$. Note that $S \neq \emptyset$ because $1 \in S$. If $t \geq 2$, then $t^2 \geq 2t > 4$, meaning it would not be an element of S . So S is bounded above by 2. Hence there exists $u := \sup(S)$.

—————/ /—————

Scratchwork: Assume $u^2 < 2$. Find a sufficiently small n so that $(u + \frac{1}{n})^2 \in S$; i.e., $(u + \frac{1}{n})^2 < 2$. Solving for n yields:

$$u^2 + \frac{2u}{n} + \frac{1}{n^2} < 2$$

$$\iff$$

$$\frac{2u}{n} + \frac{1}{n^2} < 2 - u^2$$

$$\iff$$

$$\frac{1}{n} \left(2u + \frac{1}{n} \right) < 2 - u^2$$

$$\iff$$

$$\frac{1}{n} (2u + 1) < 2 - u^2$$

$$\iff$$

$$\frac{1}{n} < \frac{2 - u^2}{2u + 1} \in \mathbf{R}^+ \setminus \{0\}$$

—————/ /—————

If $u^2 < 2$, then $\frac{2-u^2}{2u+1} > 0$. By **Archimedean Property 2**, there exists an $n \in \mathbf{N}$ with $\frac{1}{n} < \frac{2-u^2}{2u+1}$. Simplifying yields $(u + \frac{1}{n})^2 < 2$, or equivalently $u + \frac{1}{n} \in S$, which is a contradiction. It must be the case that $u^2 \geq 2$; i.e., $u^2 - 2 \geq 0$. Now since $u = \sup(S)$, for all $m \in \mathbf{N}$, there exists $t_m \in S$ with $u - \frac{1}{m} < t_m$. We have that $(u - \frac{1}{m})^2 < t_m^2 < 2$. This simplifies to $u^2 - 2 < \frac{2u}{m} - \frac{1}{m^2} < \frac{2u}{m}$, or equivalently $\frac{u^2-2}{2u} < \frac{1}{m}$. But if $\frac{u^2-2}{2u} < \frac{1}{m}$ for all $m \in \mathbf{N}$, it must be that $\frac{u^2-2}{2u} = 0$, hence $u^2 = 2$.

Lastly we show that u^2 is unique. Suppose $u^2 = 2 = v^2$. Since $u, v \geq 0$, $(u^2 - v^2) = 0$. Then $(u - v)(u + v) = 0$. If $u + v = 0$, then $u = 0$ and $v = 0$, which is a contradiction. So $u - v = 0$ implies $u = v$. \square

Remark. Picking 2 was completely arbitrary, we could have showed $x^2 = a$ for any $a \geq 0$.

Remark. Using the same argument, we have that for all $a > 0$, there exists a unique $b > 0$ with $b^2 = a$. So we have a map:

$$\mathbf{R}^+ \xrightarrow{\sqrt{\cdot}} \mathbf{R}^+,$$

where \sqrt{x} is the unique positive number with $(\sqrt{x})^2 = x$.

Remark. We could have similarly defined S as:

$$S' = \{t \in \mathbf{Q} \mid t > 0, t^2 < 2\},$$

and the proof would not have changed. However, $\sup(S') = \sqrt{2} \notin \mathbf{Q}$, meaning \mathbf{Q} is *not* complete.

4.2 Nested Intervals

Axiom 4. Given any interval I , if $x, y \in I$ with $x < y$, then $[x, y] \in I$.

Theorem 4.2.1. Let $S \subseteq \mathbf{R}$ be any subset containing at least two points. If S satisfies Axiom 4, then S is an interval.

Proof. We proceed with cases. Case 1: S is bounded. Write $a = \inf(S)$ and $b = \sup(S)$. Therefore $S \subseteq [a, b]$. If we show $(a, b) \subseteq S$, then it follows that $S = (a, b]$, or $[a, b)$, or (a, b) or $[a, b]$. We must use that S satisfies Axiom 4 and $a = \inf(S)$ and $b = \sup(S)$. Let $x \in (a, b)$. Since $x > a$, there exists $s_1 \in S$ with $s_1 < x$. Since $x < b$, there exists an $s_2 \in S$ with $x < s_2$. Thus $s_1, s_2 \in S$ and $s_1 < s_2$. By Axiom 4 $[s_1, s_2] \subseteq S$. But $x \in [s_1, s_2]$ implies $x \in S$. Thus $(a, b) \subseteq S$.

Case 2: S is bounded above **do this**.

Case 3: S is bounded below **need to do**. \square

Definition 4.2.1. A sequence of intervals $(I_n)_{n \geq 1}$ is said to be nested if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$

Proposition 4.2.2. $\bigcap_{n \geq 1} [0, \frac{1}{n}) = \{0\}$.

Proof. Note that $0 \in [0, \frac{1}{n})$ for all $n \geq 1$. So $0 \in \bigcap_{n \geq 1} [0, \frac{1}{n})$. Let $a \in \bigcap_{n \geq 1} [0, \frac{1}{n})$. Then $0 \leq a < \frac{1}{n}$ for all $n \geq 1$. Hence $a = 0$. \square

Proposition 4.2.3. $\bigcap_{n \geq 1} [n, \infty) = \emptyset$.

Proof. Suppose towards contradiction there exists a $t \in \bigcap_{n \geq 1} [n, \infty) = \emptyset$. Then $t \in [n, \infty)$ for all $n \geq 1$. So $t \geq n$ for all $n \geq 1$. Hence \mathbf{N} is bounded above, which is a contradiction. \square

Theorem 4.2.4 (Nested Intervals). Let $(I_n)_{n \geq 1}$ be a sequence of closed and bounded nested intervals. Then $\bigcap_{n \geq 1} I_n \neq \emptyset$. Furthermore, if $\inf \{\text{length}(I_n) \mid n \geq 1\} = 0$, then $\bigcap_{n \geq 1} I_n = \{\xi\}$.

Proof. Let $I_n = [a_n, b_n]$. Note that:

$$a_1 \leq a_2 \leq a_3 \leq \dots$$

$$b_1 \geq b_2 \geq b_3 \geq \dots$$

We have that $a_1 \leq a_n \leq b_n \leq b_1$ for all $n \geq 1$. So the set $\{a_n \mid n \geq 1\}$ is bounded above, and similarly $\{b_n \mid n \geq 1\}$ is bounded below. Let

$$\xi = \sup_{n \geq 1} \{a_n\}$$

$$\eta = \inf_{n \geq 1} \{b_n\}.$$

Claim: $\xi \leq b_n$ for all $n \geq 1$. Assume towards contradiction $\xi > b_m$ for some $m \geq 1$. Since $\xi = \sup_{n \geq 1} \{a_n\}$, there exists an a_k with $b_m < a_k \leq \xi$. If $k \geq m$, then $b_m < a_k \leq b_k \leq b_m$, which is a contradiction. If $k < m$, then $a_k \leq a_m \leq b_m < a_k$, which is a contradiction.

Claim: $a_n \leq \xi$ for all $n \geq 1$. Then $\xi \leq \eta$ since $\sup_{n \geq 1} \{a_n\} = \xi$. We have $[\xi, \eta] \subseteq [a_n, b_n]$ for all $n \in \mathbf{N}$. Let $x \in [\xi, \eta]$. Then:

$$a_n \leq \xi \leq x \leq \eta \leq b_n,$$

hence $x \in [a_n, b_n]$; i.e., $[\xi, \eta] \subseteq [a_n, b_n]$ for all $n \geq 1$. Thus $[\xi, \eta] \subseteq \bigcap_{n \geq 1} [a_n, b_n]$. Conversely, let $t \in [a_n, b_n]$ for all $n \geq 1$. Then $a_n \leq t \leq b_n$. This implies t is both an upper bound for $\{a_n\}_{n \geq 1}$ and a lower bound for $\{b_n\}_{n \geq 1}$. Hence $\xi \leq t \leq \eta$, implying $t \in [\xi, \eta]$. This establishes $[\xi, \eta] = \bigcap_{n \geq 1} [a_n, b_n]$.

Now suppose $\inf \{\text{length}(I_n) \mid n \geq 1\} = 0$. Then:

$$0 = \inf_{n \geq 1} (b_n - a_n)$$

$$= \inf_{n \geq 1} b_n - \inf_{n \geq 1} a_n$$

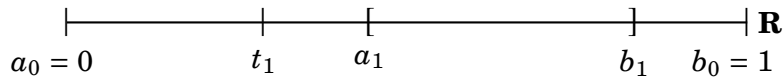
$$= \eta - \xi.$$

Hence $\xi = \eta$, which establishes the theorem.

Alternatively, had we assumed $\xi \neq \eta$, then $\eta - \xi > 0$. So there exists an m such that $b_m - a_m < \eta - \xi$, which is a contradiction since $[\xi, \eta] \subseteq [a_m, b_m]$. \square

Corollary 4.2.5. $[0, 1]$ is uncountable.

Proof. By way of contradiction, suppose $[0, 1] = \{t_1, t_2, t_3, \dots\}$. Consider the following picture:



Find $[a_1, b_1] \subseteq [0, 1]$ with $t_1 \notin [a_1, b_1]$. Find $[a_2, b_2] \subseteq [a_1, b_1]$ with $t_2 \notin [a_2, b_2]$. Inductively, find $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ with $t_n \notin [a_n, b_n]$. Thus $[a_n, b_n]$ is nested. Now let $\xi \in \bigcap_{n \geq 1} [a_n, b_n]$. Then $\xi \in [0, 1]$. But $\xi \neq t_n$ for all n , which is a contradiction. \square

5

Sequences

5.1 Basic Definitions and Examples

Definition 5.1.1. A *sequence* in a metric space X is a map $x : \mathbf{N} \rightarrow X$. We often write $x = (x_n)_n = (x_1, x_2, \dots)$, where $x_n = x(n)$. If $X = \mathbf{R}$, we call x a real sequence.

Example 5.1.1.

(1) Sequences defined explicitly:

(i) Constant sequences: $x_n = t$, $(x_n)_n = (t, t, t, \dots)$

(ii) Sequences defined by a function: $d_n = \left(1 + \frac{1}{n}\right)^n$.

(iii) Geometric sequences: fix $b \in \mathbf{R}$, then $(b^n)_n = (1, b, b^2, \dots)$.

(2) Sequences defined recursively:

(i) Let $a_1 = 1$, $a_{n+1} = 2a_n + 1$. Then $(a_n)_n = (1, 3, 7, 15, \dots)$.

(ii) Let $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$. Then $(f_n)_n = (1, 1, 2, 3, 5, 8, \dots)$. This is the *Fibonacci sequence*.

(iii) Let X be a metric space and $f : X \rightarrow X$ be an endomorphism. Fix $x_0 \in X$. Then define:

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) \\ &\vdots \\ x_n &= f(x_{n-1}). \end{aligned}$$

(3) New sequences from old:

(i) Let $(a_n)_n$ and $(b_n)_n$ be sequences. Define:

$$\begin{aligned} (a_n)_n + (b_n)_n &= (a_n + b_n)_n \\ t(a_n)_n &= (ta_n)_n \\ (a_n)_n \cdot (b_n)_n &= (a_n b_n)_n \\ \frac{(a_n)_n}{(b_n)_n} &= \left(\frac{a_n}{b_n}\right)_n, \quad (b_n)_n \neq 0 \text{ for all } n. \end{aligned}$$

(ii) Given $(x_n)_n$ and $k \in \mathbf{N}$, consider $(x_{n+k})_n = (x_k, x_{k+1}, \dots)$. This is called a *shift* or the k^{th} *tail* of $(x_n)_n$.

(iii) If $(a_n)_n$ is a sequence, $a_n \neq 0$ for all n , consider:

$$r_n = \frac{a_{n+1}}{a_n}.$$

So $(r_n)_n = \left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \dots\right)$. These are called *sequences of ratios*.

(iv) Given a real sequence $(x_k)_k$, consider the sequence $(s_n)_n$ where:

$$s_n = \sum_{k=1}^n x_k = s_{n-1} + x_n.$$

We call these n^{th} *partial sums*. An example of these are geometric sequences and telescoping sequences.

Example 5.1.2. Let F be a field. The set $F^{\mathbf{N}} = \{x \mid x : \mathbf{N} \rightarrow F\}$ is the set of all F -sequences. This forms an F -vector space under componentwise addition and scalar multiplication.

Definition 5.1.2. Let $(x_n)_n$ be a sequence.

- (1) x_n is *increasing* if $x_1 \leq x_2 \leq x_3 \leq \dots$
- (2) x_n is *decreasing* if $x_1 \geq x_2 \geq x_3 \geq \dots$
- (3) x_n is *strictly increasing* if $x_1 < x_2 < x_3 < \dots$
- (4) x_n is *strictly decreasing* if $x_1 > x_2 > x_3 > \dots$

Definition 5.1.3. A sequence is said to *eventually* have a certain property if it does not have the said property across all its ordered instances, but will after some instances have passed.

Definition 5.1.4. A sequence $(x_n)_n$ is *monotone* if it is either increasing, decreasing, strictly increasing, or strictly decreasing.

5.2 Convergence

Definition 5.2.1. Let $(x_n)_n$ be a sequence in a metric space X .

- (1) $(x_n)_n$ *converges* to $x \in X$ if:

$$(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geq N_\epsilon \implies d(x_n, x) < \epsilon).$$

We denote this as $(x_n)_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$.

- (2) $(x_n)_n$ *does not exist* if:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(n \geq N \wedge d(x_n, n) \geq \epsilon_0).$$

We abbreviate this as D.N.E.

(3) $(x_n)_n$ diverges properly to $+\infty$ if:

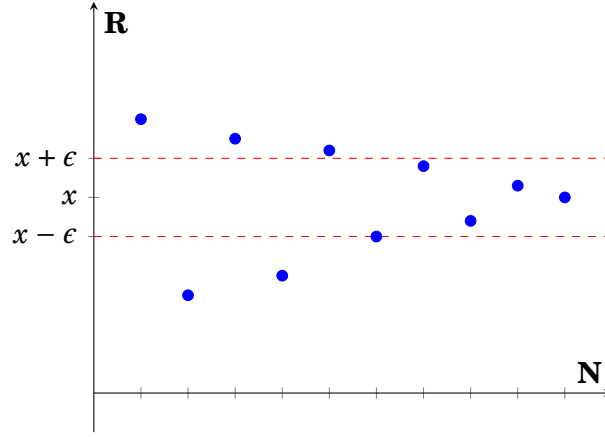
$$(\forall M > 0)(\exists N_M \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geq N_M \implies x_n \geq M).$$

We write $(x_n)_n \rightarrow +\infty$.

(4) $(x_n)_n$ diverges properly to $-\infty$ if:

$$(\forall M < 0)(\exists N_M \in \mathbf{N}) \ni (\forall n \geq N_M)(x_n \leq M).$$

Example 5.2.1. Let $(x_n)_n$ be a real sequence. Then $d(x_n, x) < \epsilon \iff |x_n - x| < \epsilon \iff x_n \in V_\epsilon(x)$. We can visually represent a sequence as follows:



If a sequence is convergent it will eventually be contained between the two dashed lines.

Example 5.2.2.

(1) Prove $(\frac{1}{n})_n \rightarrow 0$.

Solution. Let $\epsilon > 0$. Find $N_\epsilon \in \mathbf{N}$ so that $\frac{1}{N_\epsilon} < \epsilon$. If $n \geq N_\epsilon$, then $\frac{1}{n} \leq \frac{1}{N_\epsilon} < \epsilon$. Hence $\frac{1}{n} = |\frac{1}{n} - 0| < \epsilon$.

(2) Prove $(\frac{5n-1}{3-n})_{n=4}^\infty \rightarrow -5$.

Solution. Note that:

$$|x_n - x| = \left| \frac{5n-1}{3-n} + 5 \right| = \frac{14}{|3-n|} = \frac{14}{n-3}.$$

Let $\epsilon > 0$. Find $N_\epsilon \in \mathbf{N}$ such that $N_\epsilon > \frac{14}{\epsilon} + 3$. If $n \geq N_\epsilon$, then $n > \frac{14}{\epsilon} + 3$ gives:

$$n - 3 > \frac{14}{\epsilon} \implies \frac{14}{n-3} < \epsilon \implies |x_n - x| < \epsilon.$$

Proposition 5.2.1. Let (X, d) be a metric space. Then $(x_n)_n \rightarrow x$ if and only if $(d(x_n, x))_n \rightarrow 0$.

Proof. Suppose $(x_n)_n \rightarrow x$. Given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $d(x_n, x) < \epsilon$. This is equivalent to $|d(x_n, x) - 0| < \epsilon$. The converse follows identically. \square

Theorem 5.2.2. Let $(\epsilon_n)_n \rightarrow 0$ and $(x_n)_n$ be real sequences and $x \in \mathbf{R}$. If for some $c > 0$ and $N \in \mathbf{N}$ we have:

$$|x_n - x| \leq c|\epsilon_n| \quad \text{for all } n \in \mathbf{N} \text{ such that } n \geq N,$$

then $(x_n)_n \rightarrow x$.

Proof. Let $\epsilon > 0$ be given. Since $(\epsilon_n)_n \rightarrow 0$ it follows there exists a natural number K such that if $n \geq K$ then

$$|a_n| = |a_n - 0| < \frac{\epsilon}{c}.$$

If both $n \geq K$ and $n \geq N$, then

$$|x_n - x| \leq c|\epsilon_n| < \epsilon.$$

Thus $(x_n)_n \rightarrow x$. □

Example 5.2.3.

- (1) Prove $\left(\frac{\sin(n^2-1)}{n^2+3}\right)_n \rightarrow 0$.

Solution. Note that:

$$\left|\frac{\sin(n^2-1)}{n^2+3} - 0\right| = \frac{|\sin(n^2-1)|}{n^2+3} \leq \frac{1}{n^2+3} \leq \frac{1}{n^2} \leq \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \rightarrow 0$, we have $\left(\frac{\sin(n^2-1)}{n^2+3}\right)_n \rightarrow 0$.

- (2) Prove $\left(\frac{1}{2^n}\right)_n \rightarrow 0$.

Solution. Note that:

$$\left|\frac{1}{2^n} - 0\right| \leq \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \rightarrow 0$, we have $\left(\frac{1}{2^n}\right)_n \rightarrow 0$.

- (3) Prove $\left(\frac{1}{n} - \frac{1}{n+1}\right)_n \rightarrow 0$.

Solution. Note that:

$$\left|\frac{1}{n} - \frac{1}{n+1} - 0\right| \leq \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \rightarrow 0$, we have $\left(\frac{1}{n} - \frac{1}{n+1}\right)_n \rightarrow 0$.

Proposition 5.2.3. Let $k \geq 1$ be fixed. Given a sequence $(x_n)_n$ in a metric space (X, d) , $(x_n)_n \rightarrow x$ if and only if $(x_{k+n})_n \rightarrow x$.

Proof. (\Rightarrow) Suppose $(x_n)_n \rightarrow x$. Let $\epsilon > 0$. We know there exists $N_\epsilon \in \mathbf{N}$ with $n \geq N_\epsilon$ implying $d(x_n, x) < \epsilon$. But if $n \geq N_\epsilon$, then $n + k \geq N_\epsilon$. Hence $d(x_{n+k}, x) < \epsilon$.

(\Leftarrow) Conversely, assume that $(x_{n+k}) \rightarrow 0$. Let $\epsilon > 0$. We know there exists $N_\epsilon \in \mathbf{N}$ such that $n \geq N_\epsilon$ implies $d(x_{n+k}, x) < \epsilon$. Consider $M = N_\epsilon + k$. Then if $n \geq M$, we have $n \geq N_\epsilon + k$, or equivalently $n - k \geq N_\epsilon$. Hence $d(x_{(n-k)+k}, x) = d(x_n, x) < \epsilon$. \square

Proposition 5.2.4. *If $(x_n)_n$ is a real sequence with $\left(\left|\frac{x_{n+1}}{x_n}\right|\right) \rightarrow L < 1$, then $(x_n)_n \rightarrow 0$.*

Proof. Since $L < 1$, let ρ be a number satisfying $L < \rho < 1$. Pick $\epsilon = \rho - L$. Since $\left(\left|\frac{x_{n+1}}{x_n}\right|\right) \rightarrow L$, there exists $N_\epsilon \in \mathbf{N}$ such that $n \geq N_\epsilon$ implies $\left|\frac{x_{n+1}}{x_n}\right| \in V_\epsilon(L)$, or equivalently $L - \epsilon < \frac{|x_{n+1}|}{|x_n|} < L + \epsilon$. Then $\frac{|x_{n+1}|}{|x_n|} < \rho$, which gives $|x_{n+1}| < \rho|x_n|$. Observe that:

$$\begin{aligned} |x_{N+1}| &< \rho|x_N| \\ |x_{N+2}| &< \rho|x_{N+1}| = \rho^2|x_N| \\ |x_{N+3}| &< \rho|x_{N+2}| = \rho^3|x_N| \\ &\vdots \end{aligned}$$

$$\text{Inductively, } |x_{N+n}| = \rho^n|x_N|.$$

Since $(\rho^n)_n \rightarrow 0$ (and taking $c = |x_N|$), we have that $(x_{N+n})_n \rightarrow 0$. Thus $(x_n)_n \rightarrow 0$. \square

Remark. Consider $(n)_n \rightarrow +\infty$. Then $\left(\frac{n+1}{n}\right)_n \rightarrow 1$. Now consider $\left(\frac{1}{n}\right)_n \rightarrow 0$. Then $\left(\frac{n}{n+1}\right)_n \rightarrow 1$. We gain no information if $L = 1$.

Example 5.2.4.

(1) Prove $((-1)^n)_n$ does not exist.

Solution. Suppose $((-1)^n)_n \rightarrow x$. We want to find some $\epsilon_0 > 0$ such that for all $N \in \mathbf{N}$, we can find an $n \in \mathbf{N}$ satisfying:

$$n \geq N \text{ and } |x_n - x| = |(-1)^n - x| \geq \epsilon_0.$$

Pick $\epsilon_0 = \max\{|x - 1|, |x + 1|\}$. Let $N \in \mathbf{N}$. Set $n = 2N$. This gives:

$$\begin{aligned} (-1)^{2N} &= 1 \\ (-1)^{2N+1} &= -1 \end{aligned}$$

So we have $n \geq N$ and:

$$|(-1)^{2N} - x| = |1 - x| \geq \epsilon_0 \quad \text{or} \quad |(-1)^{2N+1} - x| = |-1 - x| \geq \epsilon_0.$$

(2) Prove $(\sin(n))_n$ does not exist.

Solution.

Proposition 5.2.5. *Let (X, d) be a metric space. A sequence $(x_n)_n$ can have at most one limit.*

Proof. Suppose $(x_n)_n \rightarrow L_1$ and $(x_n)_n \rightarrow L_2$. Set $\epsilon = \frac{|L_1 - L_2|}{2}$. Then $V_\epsilon(L_1) \cap V_\epsilon(L_2) = \emptyset$. Since $(x_n)_n \rightarrow L_1$, there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $x_n \in V_\epsilon(L_1)$. Likewise, since $(x_n)_n \rightarrow L_2$, there exists $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $x_n \in V_\epsilon(L_2)$. Pick $N = \max\{N_1, N_2\}$. Then $x_N \in V_\epsilon(L_1) \cap V_\epsilon(L_2)$, which is a contradiction. \square

Lemma 5.2.6. *If $(x_n)_n \rightarrow x$, then $(|x_n|)_n \rightarrow |x|$.*

Proof. Since $(x_n)_n \rightarrow x$, then there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - x| < \epsilon$. The triangle inequality gives:

$$||x_n| - |x|| \leq |x_n - x| < \epsilon,$$

hence $(|x_n|)_n \rightarrow |x|$. Note that the converse does not hold in general, as:

$$(|(-1)^n|)_n \rightarrow 1 \text{ while } ((-1)^n)_n \text{ does not exist.}$$

\square

Lemma 5.2.7. *Let $(t_n)_n$ be a sequence in (X, d) . $(t_n)_n \rightarrow 0$ if and only if $(|t_n|)_n \rightarrow 0$.*

Proof. (\Rightarrow) The forward direction follows from Lemma 5.2.6. (\Leftarrow) Suppose $(|t_n|)_n \rightarrow 0$. We have that:

$$||t_n| - 0| \leq$$

\square

Lemma 5.2.8. *If $(x_n)_n \rightarrow x \in \mathbf{R}$ with $x_n \geq 0$, then $(\sqrt{x_n})_n \rightarrow \sqrt{x}$.*

Proof. Case 1: $x = 0$. Let $\epsilon > 0$ be given. Since $(x_n)_n \rightarrow 0$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $0 \leq x_n = |x_n - 0| < \epsilon^2$. Hence $0 \leq \sqrt{x_n} < \epsilon$. Since $\epsilon > 0$, was arbitrary, $(\sqrt{x_n})_n \rightarrow 0$.

Case 2: $x > 0$. Then $\sqrt{x} > 0$, and:

$$|\sqrt{x_n} - \sqrt{x}| = \left| (\sqrt{x_n} - \sqrt{x}) \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \right| = \left| \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \right| \leq \left(\frac{1}{\sqrt{x}} \right) |x_n - x|.$$

Hence the convergence $(\sqrt{x_n})_n$ is a consequence of $(x_n)_n \rightarrow x$. \square

Example 5.2.5.

(1) Prove $(\sqrt{n})_n \rightarrow +\infty$.

Solution. Let $M > 0$ be given. Find N_M so that $N_M = \lceil M^2 \rceil$. Hence $N_M \geq M^2$. Then $n \geq N_M$ implies $n \geq M^2$, or equivalently $\sqrt{n} \geq M$.

(2) Prove $(n - \sqrt{n})_n \rightarrow +\infty$.

Solution. Write $(n - \sqrt{n})_n = (n)_n \underbrace{\left(1 - \frac{1}{\sqrt{n}}\right)}_{\rightarrow 1}_n = (n)_n$. Since $(n)_n$ trivially converges to $+\infty$, we have $(n - \sqrt{n})_n \rightarrow +\infty$.

(3) Prove:

$$(b^n)_{n=0}^\infty \rightarrow \begin{cases} 0, & |b| < 1 \\ 1, & b = 1 \\ +\infty, & b > 1 \\ \text{D.N.E.}, & b \leq -1 \end{cases}$$

Solution. Cases $b = 0$ and $b = 1$ are trivial. We showed case $b = -1$ in Example 5.2.4.

Case 1: $0 < b < 1$. Then $b < 1$ implies $\frac{1}{b} > 1$. We have $\frac{1}{b} = 1 + a$ for some $a > 0$, now observe that:

$$\left(\frac{1}{b}\right)^n = (1 + a)^n \geq 1 + na.$$

This gives:

$$|b^n - 0| \leq \frac{1}{1 + na} \leq \frac{1}{na} = \left(\frac{1}{a}\right) \frac{1}{n}.$$

Since $\left(\frac{1}{n}\right)_n \rightarrow 0$, we have $(b^n)_n \rightarrow 0$.

Case 2: $-1 < b < 0$. Since $(|b^n|)_n = (|b|^n)_n$, case 1 gives $(b^n)_n \rightarrow 0$ when $-1 < b < 0$.

Case 3: $b > 1$. Then $b = 1 + a$ for some $a > 0$. We have:

$$b^n = (1 + a)^n \geq 1 + na \geq na.$$

Let $M > 0$ be given. Pick $N_M = \frac{\lceil M \rceil}{a}$. Then $N_M \geq \frac{M}{a}$. If $n \geq N_M$, then $n \geq \frac{M}{a}$, which simplifies to $na \geq M$. Hence $b^n \geq na \geq M$ gives $(b^n)_n \rightarrow +\infty$.

Case 4: $b < -1$. We prove that $(b^n)_n$ does not exist by contradiction. Suppose $(b^n)_n \rightarrow L$ for some $L \in \mathbf{R}$. Then $(|b^n|)_n \rightarrow |L|$. But this is a contradiction via the $b > 1$ case. Now if $(b^n)_n \rightarrow +\infty$, there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $b^n \geq 1$. But for n odd, $b^n < 0$, which is a contradiction. Assuming $(b^n)_n \rightarrow -\infty$ leads to a similar contradiction, establishing the proof.

Example 5.2.6.

(1) Prove if $c > 0$, $(c^{\frac{1}{n}})_n \rightarrow 1$.

Solution. If $c = 1$, then clearly $(1^{\frac{1}{n}})_n \rightarrow 1$. Suppose $c > 1$, then $c^{\frac{1}{n}} > 1$. Write $c^{\frac{1}{n}} = 1 + a_n$, where $a_n > 0$ for all $n \in \mathbf{N}$. We have:

$$c = (c^{\frac{1}{n}})^n = (1 + a_n)^n \geq 1 + na_n \geq na_n.$$

So $0 < na_n \leq c$, giving $a_n \leq \frac{c}{n}$. We have:

$$|c^{\frac{1}{n}} - 1| = a_n \leq \frac{c}{n}.$$

Since $(\frac{1}{n})_n \rightarrow 0$, $(c^{\frac{1}{n}})_n \rightarrow 1$. Now suppose $0 < c < 1$, then $c^{\frac{1}{n}} < 1$. Write $c^{\frac{1}{n}} = 1 + (-a_n)$ with $-1 < -a_n < 0$ for all n . Then:

$$c = (c^{\frac{1}{n}})^n = (1 + (-a_n))^n \geq 1 + n(-a_n) \geq n(-a_n).$$

So $n(-a_n) \leq c$, giving $-a_n \leq \frac{c}{n}$. We have:

$$|c^{\frac{1}{n}} - 1| = -a_n \leq \frac{c}{n}.$$

Since $(\frac{1}{n})_n \rightarrow 0$, $(c^{\frac{1}{n}})_n \rightarrow 1$.

(2) Prove $(n^{\frac{1}{n}})_n \rightarrow 1$.

Proof. Note that $n^{\frac{1}{n}} > 1$ for all $n > 1$. Write $n^{\frac{1}{n}} = 1 + a_n$. Then:

$$n = (1 + a_n)^n = \sum_{k=0}^n \binom{n}{k} a_n^k \geq \binom{n}{0} + \binom{n}{2} a_n^2 = 1 + \frac{n(n-1)}{2} a_n^2.$$

We have:

$$n - 1 \geq \frac{n(n-1)}{2} a_n^2,$$

which simplifies to:

$$\frac{2}{n} \geq a_n^2.$$

Hence $a_n \leq \sqrt{2} \frac{1}{n}$, thus by our lemma $(a_n)_n^\infty \rightarrow 0$. Therefore:

$$|n^{\frac{1}{n}} - 1| = d_n,$$

establishing that $(n^{\frac{1}{n}})_n \rightarrow 1$. □

Proposition 5.2.9. *A convergent sequence is bounded.*

Proof. Suppose $(x_n)_n \rightarrow x$. Since $(x_n)_n$ is convergent, we know for all $\epsilon > 0$ that $|x_n - x| < \epsilon$. Pick $\epsilon = 1$. Eventually the entire sequence will be contained in $V_1(x)$. More formally, there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $x_n \in V_1(x)$. Define:

$$c = \max \{|x_1|, |x_2|, \dots, |x_{N_1}|, |x - 1|, |x + 1|\}.$$

If $n \leq N_1$, then $|x_n| \leq c$. If $n \geq N_1$, then $x - 1 < x_n < x + 1$; i.e., $|x_n| \leq c$. □

Theorem 5.2.10. *Let x_n, y_n, z_n be convergent sequences with $(x_n)_n \rightarrow x$, $(y_n)_n \rightarrow y$, and $(z_n)_n \rightarrow z$ and $t \in \mathbf{R}$. Moreover, let $z_n \neq 0$ for all n and $z \neq 0$. We have:*

(1) $(x_n \pm y_n)_n \rightarrow x \pm y$.

(2) $(tx_n)_n \rightarrow tx$.

$$(3) \quad (x_n y_n)_n \rightarrow xy.$$

$$(4) \quad \left(\frac{1}{z_n} \right)_n \rightarrow \frac{1}{z}.$$

$$(5) \quad \left(\frac{x_n}{z_n} \right)_n \rightarrow \frac{x}{z}.$$

Proof. (3) We have:

$$\begin{aligned} |x_n y_n - x y| &= |x_n y_n - x y_n + x y_n - x y| \\ &= |(x_n - x) y_n + x (y_n - y)| \\ &\leq |(x_n - x) y_n| + |x (y_n - y)| \\ &= |x_n - x| |y_n| + |x| |y_n - y|. \end{aligned}$$

Since y_n is convergent, it is bounded. So there exists a $c > 0$ with $|y_n| \leq c$ for all $n \geq 1$. Hence:

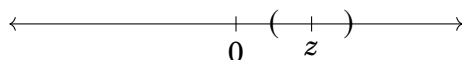
$$|x_n - x||y_n| + |x||y_n - y| \leq \overset{\rightarrow 0}{|x_n - x|} c + |x| \overset{\rightarrow 0}{|y_n - y|}.$$

Thus $(|x_n y_n - xy|)_n \rightarrow 0$, which implies $(x_n y_n)_n \rightarrow xy$.

(4) We have:

$$\left| \frac{1}{z_n} - \frac{1}{z} \right| = \frac{|z - z_n|}{|z||z_n|}.$$

Since $z \neq 0$, it won't be "near" zero. We have the following picture:



Let $\delta = \frac{|z|}{2} > 0$. There exists $N \in \mathbf{N}$ such that $n \geq N$ implies $z_n \in V_\delta(z)$. We have:

$$\begin{aligned} z - \delta &< z_n < z + \delta \\ \implies z - \frac{|z|}{2} &< z_n \\ \implies \frac{|z|}{2} &< |z_n|. \end{aligned}$$

Since $|z_n| \geq \frac{|z|}{2}$, we have $\frac{1}{|z_n|} < \frac{2}{|z|}$. So for $n \geq N$,

$$\left| \frac{1}{z_n} - \frac{1}{z} \right| = \frac{|z - z_n|}{|z||z_n|} \leq \frac{2}{|z|^2} |z - z_n|.$$

Thus $\left(\frac{1}{z_n}\right)_n \rightarrow \frac{1}{z}$.

Theorem 5.2.11. Suppose $(x_n) \rightarrow x$ and $(y_n)_n \rightarrow y$ with $x_n \leq y_n$ for all n . Then $x \leq y$.

Proof. We have that $(y_n - x_n)_n \rightarrow y - x$, and $y_n - x_n \geq 0$ for all n . Thus $y - x \geq 0$.

Corollary 5.2.12. *If $(x_n)_n \rightarrow x$ and $a \leq x_n \leq b$, then $a \leq x \leq b$.*

Proof. Taking $(y_n)_n = (a, a, a, \dots)$ and $(y_n)_n = (b, b, b, \dots)$ gives the desired result. \square

Theorem 5.2.13 (Squeeze Theorem). *Let $(x_n)_n$, $(y_n)_n$, and $(z_n)_n$ be sequences with $(x_n)_n \leq (y_n)_n \leq (z_n)_n$ for all $n \geq 1$. If $\lim x_n = \lim z_n = L$, then $(y_n)_n \rightarrow L$.*

Proof. Let $\epsilon > 0$. There exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $x_n \in V_\epsilon(L)$. Likewise, there exists $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $z_n \in V_\epsilon(L)$. So for $n \geq \max\{N_1, N_2\} := N$, both $x_n, z_n \in V_\epsilon(L)$. We have:

$$L - \epsilon < x_n \leq y_n < z_n \leq L + \epsilon.$$

Thus $y_n \in V_\epsilon(L)$ for $n \geq N$. \square

Theorem 5.2.14 (Monotone Convergence Theorem). *Let $(x_n)_n$ be a monotone sequence. $(x_n)_n$ is convergent if and only if $(x_n)_n$ is bounded. Moreover,*

(a) *If $(x_n)_n$ is increasing and bounded above, $\lim x_n = \sup\{x_n \mid n \in \mathbf{N}\}$.*

(b) *If $(x_n)_n$ is decreasing and bounded below, $\lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}$.*

Proof. (\Rightarrow) We showed this direction in Proposition 5.2.9. (\Leftarrow) (a) Suppose $(x_n)_n$ is bounded above and increasing. Let $u = \sup\{x_n \mid n \in \mathbf{N}\}$. Given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $u - \epsilon < x_N$. But for $n \geq N$, $u - \epsilon < x_N \leq x_n \leq u < u + \epsilon$. Hence $x_n \in V_\epsilon(u)$, establishing that $(x_n)_n \rightarrow u$.

(b) Consider $y_n = -x_n$, we get y_n is increasing and bounded above. By (a), we get:

$$\begin{aligned} \lim y_n = \sup\{y_n \mid n \in \mathbf{N}\} &\implies -\lim x_n = \sup\{-x_n \mid n \in \mathbf{N}\} \\ &\implies -\lim x_n = -\inf\{x_n \mid n \in \mathbf{N}\} \\ &\implies \lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}. \end{aligned}$$

\square

Example 5.2.7.

(1) Consider the recursively defined sequence $x_1 = 8$, $x_{n+1} = \frac{1}{2}x_n + 2$. We will show by induction that it is bounded below by 4. Clearly $x_1 = 8 \geq 4$. Now assume $x_k \geq 4$. Then:

$$\begin{aligned} x_{k+1} &= \frac{1}{2}x_k + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4. \end{aligned}$$

Therefore $(x_n)_n$ is bounded below by 4. Now observe that:

$$\begin{aligned} x_{n+1} \leq x_n &\iff \frac{1}{2}x_n + 2 \leq x_n \\ &\iff 4 \leq x_n. \end{aligned}$$

Hence $(x_n)_n$ is decreasing. By the **Monotone Convergence Theorem**, $(x_n)_n \rightarrow L$. Now observe that:

$$\begin{aligned} (x_{n+1})_n = \left(\frac{1}{2}x_n + 2\right)_n &\iff L = \frac{1}{2}L + 2 \\ &\iff L = 4. \end{aligned}$$

- (2) Let $x_n = \sum_{k=1}^n \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9}$. We will show that this sequence converges. Clearly $x_n \leq x_{n+1}$, so it is increasing. We will use the fact that $k^2 \geq k(k-1)$ as follows:

$$\begin{aligned} x_n &= \sum_{k=1}^n \frac{1}{k^2} \\ &= 1 + \sum_{k=2}^n \frac{1}{k^2} \\ &\leq 1 + \sum_{k=2}^n \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ &= 1 + \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \right] \\ &= 1 + 1 - \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &\leq 2. \end{aligned}$$

So $(x_n)_n$ is increasing and bounded above, hence it has a limit.

- (3) Given $a > 0$, we will find a sequence $(x_n)_n$ which converges to \sqrt{a} . Consider the recursively defined sequence $x_1 = 1$, $x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$. Claim: $x_n^2 \geq a$ for all $n \geq 2$. Note that:

$$\begin{aligned} 2x_{n+1} &= x_n + \frac{a}{x_n} \implies 2x_{n+1}x_n = x_n^2 + a \\ &\implies 0 = x_n^2 - 2x_{n+1}x_n = a. \end{aligned}$$

This polynomial has a real root, hence $\Delta \geq 0$. We get:

$$\Delta = 4x_{n+1}^2 - 4a \geq 0 \implies x_{n+1}^2 \geq a$$

We will now show that $(x_n)_n$ is eventually decreasing. Observe that:

$$\begin{aligned} x_n \geq x_{n+1} &\iff x_n \geq \frac{1}{2} \left(x_n + \frac{a}{x_n} \right) \\ &\iff 2x_n \geq x_n + \frac{a}{x_n} \\ &\iff x_n \geq \frac{a}{x_n} \\ &\iff x_n^2 \geq a. \end{aligned}$$

By the **Monotone Convergence Theorem**, $(x_n)_n \rightarrow L$. We have:

$$\begin{aligned} x_{n+1} &= \frac{1}{2} \left(x_n + \frac{a}{x_n} \right) \implies L = \frac{1}{2} \left(L + \frac{a}{L} \right) \\ &\implies L^2 = a \\ &\implies L = \sqrt{a}. \end{aligned}$$

Example 5.2.8 (Euler's Number). **I will do this later.**

Proposition 5.2.15. *If $(x_n)_n$ is increasing and unbounded, then $(x_n)_n$ diverges properly to $+\infty$.*

Proof. Let M be arbitrarily big. Since $(x_n)_n$ is unbounded, there exists $N \in \mathbf{N}$ with $x_N > M$. Hence if $n \geq M$, $x_n \geq x_N > M$ because $(x_n)_n$ is increasing. \square

Example 5.2.9. We will show that $h_n = \sum_{k=1}^n \frac{1}{k}$ diverges properly to $+\infty$. **do this later**

5.3 Subsequences

Definition 5.3.1. A natural sequence is a strictly increasing sequence of natural numbers: $(n_k)_{k=1}^\infty$ with $n_k \in \mathbf{N}$, $n_1 < n_2 < \dots$

Example 5.3.1.

$$(1) \quad (2k+1)_k = (3, 5, 7, \dots)$$

$$(2) \quad (k^2)_k = (1, 4, 9, \dots)$$

Exercise 5.3.1. *Given a natural sequence $(n_k)_k$, prove $n_k \geq k$.*

Definition 5.3.2. Let $(x_n)_n$ be a sequence. A subsequence of $(x_n)_n$ is a sequence $(x_{n_k})_{k=1}^\infty$ where $(n_k)_k$ is a natural sequence. Formally, a subsequence is a composition of maps:

$$\mathbf{N} \xrightarrow[k \mapsto n_k]{} \mathbf{N} \xrightarrow[n_k \mapsto x_{n_k}]{} X.$$

Example 5.3.2.

$$(1) \quad \text{Consider } (x_n)_n \rightarrow \frac{1}{n}. \text{ Let } n_k = 2k. \text{ Then } (x_{n_k})_k = \left(\frac{1}{2k}\right)_k = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right).$$

$$(2) \quad \text{Consider } (x_n)_n = (-1)^n. \text{ Then } (x_{2k})_k = (1, 1, 1, \dots) \text{ and } (x_{2k+1})_k = (-1, -1, -1, \dots)$$

Proposition 5.3.1. *Suppose $(x_n)_n \rightarrow x$. For any subsequence $(x_{n_k})_k$, we have $(x_{n_k})_k \rightarrow x$.*

Proof. Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - x| < \epsilon$. Take $K = N$. Then $k \geq K$ implies $k \geq N$. But by Exercise ??, $n_k \geq k \geq N$. Hence $|x_{n_k} - x| < \epsilon$. \square

Example 5.3.3. We give an alternate proof of $(b^n)_n \rightarrow 0$ for $0 < b < 1$. Clearly $b^{n+1} < b^n$ if and only if $b < 1$. So b^n is decreasing and bounded below by 0. By the **Monotone Convergence Theorem**, $(b^n)_n \rightarrow L$ for some L . But we also have that $(b^{2k})_k \rightarrow L$. So we have:

$$\begin{aligned} (b^{2k})_k = (b^k)_k^2 &\iff L = L^2 \\ &\iff L(1 - L) = 0. \end{aligned}$$

Since $L \neq 1$, it must be that $L = 0$.

Proposition 5.3.2. Let $(x_n)_n$ be a sequence. Then $(x_n)_n \rightarrow x$ if and only if there exists an $\epsilon_0 > 0$ and subsequence $(x_{n_k})_k$ such that $d(x_{n_k}, x) > \epsilon_0$.

Proof. (\Leftarrow) If $(x_n)_n \rightarrow x$, then any subsequence $(x_{n_k})_k$ converges to x .

(\Rightarrow) Since $(x_n)_n \not\rightarrow x$, we have:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N})(\exists n \geq N) \ni (x_n \notin V_{\epsilon_0}(x)).$$

With this ϵ_0 , we will construct our subsequence x_{n_k} . Note that:

$$\begin{aligned} N = 1 &\implies (\exists n_1 \geq 1) \ni (x_{n_1} \notin V_{\epsilon_0}(x)) \\ N = n_1 + 1 &\implies (\exists n_2 \geq n_1) \ni (x_{n_2} \notin V_{\epsilon_0}(x)) \\ N = n_2 + 1 &\implies (\exists n_3 \geq n_2) \ni (x_{n_3} \notin V_{\epsilon_0}(x)) \\ &\vdots \\ \text{Inductively, } N = n_k + 1 &\implies (\exists n_{k+1} \geq n_k) \ni (x_{n_{k+1}} \notin V_{\epsilon_0}(x)) \end{aligned}$$

Thus $(x_{n_k})_k$ is a subsequence with $x_{n_k} \notin V_{\epsilon_0}(x)$, so $|x_{n_k} - x| \geq \epsilon_0$ for all $k = 1, 2, 3, \dots$ □

Definition 5.3.3. If $(x_n)_n$ is a sequence of real numbers, a peak of the sequence is a term x_m satisfying $x_m \geq x_n$ for all $n \geq m$.

Proposition 5.3.3. Let $(x_n)_n$ be a real sequence. There is a subsequence that is monotone.

Proof. Case 1: There are infinitely many peaks. Let $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ be an enumeration of peaks. Then $(x_{n_k})_k$ is decreasing by definition.

Case 2: There are finitely many peaks. Let $x_{m_1}, x_{m_2}, \dots, x_{m_r}$ be the peaks of our sequence where $m_1 < m_2 < \dots < m_r$. Let $n_1 = m_r + 1$. Since x_{n_1} is not a peak, there exists $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Since x_{n_2} is not a peak, there exists $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$. Inductively, we obtain a sequence $(x_{n_k})_k = (x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ with $x_{n_k} < x_{n_{k+1}}$. □

Theorem 5.3.4 (Bolzano-Weierstass Theorem). If $(x_n)_n$ is a real sequence that is bounded, it admits a convergent subsequence.

Proof. By Proposition 5.3.3, there exists a subsequence $(x_{n_k})_k$ which is monotone and bounded. By the **Monotone Convergence Theorem**, $(x_{n_k})_k$ converges. □

5.4 Limit Inferior and Limit Superior

Definition 5.4.1. Let $X = (x_n)_n$ be a fixed bounded sequence whose limit may not exist. Then

$$\overline{X} = \{t \in \mathbf{R} \mid t = \lim_{k \rightarrow \infty} x_{n_k}, x_{n_k} \text{ some subsequence}\}$$

is the set containing all *subsequential limits* (or *limit points*) of X .

Example 5.4.1. Let $X = ((-1)^n)_n$. Then $\overline{X} = \{-1, 1\}$.

Example 5.4.2. Fix a bounded sequence $(x_n)_n$. Let

$$\begin{aligned} u_1 &= \sup_{n \geq 1} (x_n), \\ l_1 &= \inf_{n \geq 1} (x_n). \end{aligned}$$

If a subsequence $(x_{n_k})_k \rightarrow x$, we know $x \in [l_1, u_1]$ because $l_1 \leq x \leq u_1$. Hence $l_1 \leq x_{n_k} \leq u_1$. Now let

$$\begin{aligned} u_2 &= \sup_{n \geq 2} (x_n), \\ l_2 &= \inf_{n \geq 1} (x_n). \end{aligned}$$

We have $u_2 \leq u_1$ (we know u_1 is an upper bound for all $n \geq 2$, hence u_2 must be the least upper bound) and $l_1 \leq l_2$. Similarly, if $(x_{n_k})_k \rightarrow x$ for some subsequence, then $x \in [l_2, u_2]$ because $l_2 \leq x_{n_k} \leq u_2$ for k large enough. Inductively:

$$\begin{aligned} u_m &= \sup_{n \geq m} x_n, \\ l_m &= \inf_{n \geq m} x_n. \end{aligned}$$

We get:

$$l_1 \leq l_2 \leq \dots \leq l_m \leq u_m \leq \dots \leq u_2 \leq u_1.$$

This holds for all $m \geq 1$. Let $I_m = [l_m, u_m]$. Then $(I_m)_m$ is a sequence of closed and bounded nested intervals. So

$$\bigcap_{m \geq 1} I_m = [l, u]$$

where

$$\begin{aligned} l &= \sup_{m \geq 1} l_m = \sup_{m \geq 1} \left(\inf_{n \geq m} x_n \right), \\ u &= \inf_{m \geq 1} u_m = \inf_{m \geq 1} \left(\sup_{n \geq m} x_n \right). \end{aligned}$$

Note that:

$$\begin{aligned} \sup_{m \geq 1} l_m &= \lim_{m \rightarrow \infty} l_m \\ \inf_{m \geq 1} u_m &= \lim_{m \rightarrow \infty} u_m. \end{aligned}$$

This follows from the **Monotone Convergence Theorem**, as $(l_m)_m$ is an increasing sequence bounded above and $(u_m)_m$ is a decreasing sequence bounded below.

Definition 5.4.2. Let $(x_n)_n$ be a bounded sequence.

$$(1) \quad l = \lim_{m \rightarrow \infty} l_m = \lim_{m \rightarrow \infty} \left(\inf_{n \geq m} x_n \right) := \liminf x_n.$$

$$(2) \quad u = \lim_{m \rightarrow \infty} u_m = \lim_{m \rightarrow \infty} \left(\sup_{n \geq m} x_n \right) := \limsup x_n.$$

Proposition 5.4.1. Let $X = (x_n)_n$ be a bounded sequence with $l = \liminf x_n$ and $u = \limsup x_n$. If $x \in X$, then $x \in [l, u]$. We have:

$$l_{n_k} = \inf_{n \geq n_k} x_n \leq x_{n_k}.$$

Taking the limit as $k \rightarrow \infty$ yields $l \leq x$. Similarly, we have:

$$u_{n_k} = \sup_{n \geq n_k} x_n \geq x_{n_k}.$$

Taking the limit as $k \rightarrow \infty$ yields $x \leq u$. Thus $x \in [l, u]$.

Question. Does $\overline{X} = [l, u]$?

Answer. No. Take for example $x_n = (-1)^n$. Then $u = 1$ and $l = -1$ But $\overline{X} = \{-1, 1\} \subset [-1, 1]$.

Proposition 5.4.2. Let $(x_n)_n = X$ be a bounded sequence with $u_m = \sup_{n \geq m} x_n$. We have a strictly decreasing sequence $u_1 \geq u_2 \geq \dots$. There exists a subsequence $(x_{n_k})_k \rightarrow u$. There exists a subsequence $(x_{n_k})_k \rightarrow l$. Equivalently, $u, l \in \overline{X}$.

Proof. Recall that $u_m = \sup_{n \geq m} x_n$. By the supremum property:

$$\begin{aligned} \exists n_1 \in \mathbf{N} \text{ with } u_1 - 1 < x_{n_1} \leq u_1, \\ \exists n_2 \in \mathbf{N} \text{ with } n_2 > n_1 + 1 > n_1 \text{ and } u_{n_1+1} - \frac{1}{2} < x_{n_2} \leq u_{n_1+1}, \\ \exists n_3 \in \mathbf{N} \text{ with } n_3 \geq n_2 + 1 > n_2 \text{ and } u_{n_2+1} - \frac{1}{3} < x_{n_3} \leq u_{n_2+1}. \end{aligned}$$

Inductively:

$$u_{n_{k-1}+1} - \frac{1}{k} < x_{n_k} \leq u_{n_{k-1}+1}.$$

Let $m_k = n_{k-1} + 1$. We can rewrite the above equation as:

$$u_{m_k} - \frac{1}{k} < x_{n_k} \leq u_{m_k}.$$

Letting $k \rightarrow \infty$ gives:

$$\lim_{k \rightarrow \infty} u_{m_k} - \frac{1}{k} < \lim_{k \rightarrow \infty} x_{n_k} \leq \lim_{k \rightarrow \infty} u_{m_k}$$

$$\Longleftrightarrow$$

$$u < \lim_{k \rightarrow \infty} x_{n_k} \leq u.$$

By the squeeze theorem, $(x_{n_k})_k \rightarrow u$. □

Proposition 5.4.3. *Let $(x_n)_n$ be bounded.*

$$(1) \liminf x_n \leq \limsup x_n.$$

$$(2) (x_n)_n \rightarrow x \text{ if and only if } \liminf x_n = \limsup x_n = x.$$

Proof. We have $l_m \leq u_m$ for all $m \geq 1$ by the previous motivating example. Letting $m \rightarrow \infty$ gives $l \leq u$. □

I don't know the second part

5.5 Cauchy Sequences

Definition 5.5.1. A sequence $(x_n)_n$ is Cauchy if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni m, n \geq N \implies d(x_n, x_m) < \epsilon.$$

Example 5.5.1. Prove $(\frac{1}{n})_n$ is Cauchy.

Solution. For $n > m$:

$$|x_m - x_n| = \frac{1}{m} - \frac{1}{n} = \frac{n - m}{mn} < \frac{n}{nm} = \frac{1}{m}.$$

We can start the proof. Given $\epsilon > 0$, by Archimedean property 2 there exists N large satisfying $\frac{1}{N} < \epsilon$. For $n > m \geq N$, we have $|x_m - x_n| < \frac{1}{m} \leq \frac{1}{N} < \epsilon$.

Proposition 5.5.1. *Cauchy sequences are bounded.*

Proof. Pick $\epsilon = 1$. There exists $N \in \mathbf{N}$ such that $m, n \geq N$ implies $|x_n - x_m| < 1$. Let $c = \max\{|x_1|, \dots, |x_N|\}$. For $n \geq N$, we have $|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| \leq 1 + |x_N|$. So $|x_n| \leq c'$, where $c' = \max\{c, 1 + |x_N|\}$. □

Exercise 5.5.1. *If $(x_n)_n$ is Cauchy and there exists a subsequence $(x_{n_k})_k$ with $(x_{n_k})_k \rightarrow x$, then $(x_n)_n \rightarrow x$.*

Theorem 5.5.2. *Let $(x_n)_n$ be a sequence. $(x_n)_n$ is Cauchy if and only if $(x_n)_n$ converges.*

Proof. (\implies) Suppose $(x_n)_n \rightarrow x$. Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - x| < \frac{\epsilon}{2}$. For $m, n \geq N$, we have $|x_n - x_m| = |x_n - x + x_m - x| \leq |x_n - x| + |x_m - x| < \epsilon$.

(\impliedby) If $(x_n)_n$ is Cauchy then $(x_n)_n$ is bounded. The Bolzano-Weierstrass theorem says there exists some convergent subsequence $(x_{n_k})_k$. By Exercise 5.5.1, $(x_n)_n \rightarrow x$. □

Example 5.5.2. We will show that $x_n = \sum_{k=0}^n \frac{(-1)^k}{k!}$ is a convergent sequence. This sequence is not monotone, so we are unable to use the monotone convergence theorem. Instead, we will show that this sequence is Cauchy, which implies that it is convergent. For $m > n$, observe that:

$$\begin{aligned}
 |x_m - x_n| &= \left| \sum_{k=n+1}^m \frac{(-1)^k}{k!} \right| \\
 &\leq \sum_{k=n+1}^m \left| \frac{(-1)^k}{k!} \right| \\
 &= \sum_{k=n+1}^m \frac{1}{k!} \\
 &\leq \sum_{k=n+1}^m \frac{1}{2^{k-1}} \\
 &= \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{m-1}} \\
 &= \frac{1}{2^n} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{m-n-1}} \right) \\
 &\leq \frac{1}{2^n} \cdot 2 \\
 &= \frac{1}{2^{n-1}}.
 \end{aligned}$$

If $\epsilon > 0$ is given, choose N large so that $\frac{1}{2^{N-1}} < \epsilon$. If $m > n > N$, then $|x_m - x_n| < \epsilon$. Thus x_n is Cauchy, implying that it is convergent.

Definition 5.5.2. A sequence $(x_n)_n$ is contractive if there exists $0 < \rho < 1$ with $|x_{n+1} - x_n| \leq \rho|x_n - x_{n-1}|$ for all $n \geq 2$. We say ρ is the constant of contraction.

Proposition 5.5.3. *Contractive sequences are Cauchy.*

Proof. Observe that:

$$\begin{aligned}
 |x_3 - x_2| &\leq \rho|x_2 - x_1| \\
 |x_4 - x_3| &\leq \rho|x_3 - x_2| \leq \rho^2|x_2 - x_1|
 \end{aligned}$$

Inductively, $|x_{n+1} - x_n| \leq \rho^{n-1}|x_2 - x_1|$. For $m > n$, we have:

$$|x_m - x_n| =$$

finish later

□

Example 5.5.3. fibonacci finish later

5.6 Sequences of Functions

Definition 5.6.1. Let Ω be a nonempty set. The set of all functions $\Omega \xrightarrow{f} X$ is denoted $\mathcal{F}(\Omega, X) = \{f \mid f : \Omega \rightarrow X\}$.

Exercise 5.6.1. Show that $\mathcal{F}(\Omega, X)$ is an algebra under pointwise operations.

Definition 5.6.2. A sequence in $(f_n)_n : \mathbf{N} \rightarrow \mathcal{F}(\Omega, \mathbf{R})$ converges pointwise to $f \in \mathcal{F}(\Omega, \mathbf{R})$ if, for all $x \in \Omega$, $(f_n(x))_n \rightarrow f(x)$. In particular:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N_{x,\epsilon} \in \mathbf{N}) \ni n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

Example 5.6.1.

- (1) Let $f_n \in \mathcal{F}([0, 1], \mathbf{R})$ be defined by $f_n(x) = x^n$. Given $x \in [0, 1)$, we can see that $(f_n(x))_n \rightarrow 0$ (since f^n is geometric). If $x = 1$, then $(f_n(x))_n \rightarrow 1$. So let $f : [0, 1] \rightarrow \mathbf{R}$ be defined as $f = \delta_1$. Then $(f_n)_n \rightarrow f$ converges pointwise.
- (2) Let $f_n \in \mathcal{F}(\mathbf{R}, \mathbf{R})$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}$. Note that:

$$\begin{aligned} |f_n(x)| &= \left| \frac{nx}{1+n^2x^2} \right| \\ &= \frac{n|x|}{1+n^2x^2} \\ &\leq \frac{|x|}{nx^2} \\ &= \frac{1}{n|x|}. \end{aligned}$$

Claim: $(f_n)_n \rightarrow 0_{\mathcal{F}(\mathbf{R}, \mathbf{R})}$. If $x = 0$, then $(f_n(0))_n = (0)_n \rightarrow 0$. If $x \neq 0$, then $(f_n(x))_n \rightarrow 0$ because $|f_n(x) - 0| \leq \frac{1}{|x|} \cdot \frac{1}{n}$. Since x is fixed and $(\frac{1}{n})_n \rightarrow 0$, by "Lemma" $(f_n(x))_n \rightarrow 0_{\mathcal{F}(\mathbf{R}, \mathbf{R})}$.

- (3) Let $h_n \in \mathcal{F}([0, \infty), \mathbf{R})$ be defined by $h_n(x) = x^{\frac{1}{n}}$. If $x > 0$, then $(h_n(x))_n \rightarrow 1$. If $x = 0$, then $(h_n(x))_n \rightarrow 0$. Then $(h_n(x))_n \rightarrow \mathbf{1}_{(0, \infty)}$.
- (4) Let $g_n \in \mathcal{F}([0, \infty), \mathbf{R})$ be defined by $g_n(x) = e^{-nx}$. If $x = 0$, then $(g_n(x))_n \rightarrow 1$. If $x > 0$, then $(g_n(x))_n \rightarrow 0$. So $(g_n(x))_n \rightarrow \delta_0$.

Example 5.6.2. Our goal in this example is to construct a function $f_n \in \mathcal{F}([0, \infty), \mathbf{R})$ satisfying $(f_n(x))_n \rightarrow 0_{\mathcal{F}([0, \infty), \mathbf{R})}$ and $\int_{0, \infty} f_n(x) dx = 1$. Let:

$$f_n = \begin{cases} 4n^2x, & x < \frac{1}{2n} \\ 4n - 4n^2x, & \frac{1}{2} \leq x < \frac{1}{n} \\ 0, & x \geq \frac{1}{n}. \end{cases}$$

Given $x = 0$, $(f_n(0))_n = (0)_n \rightarrow 0$. Given $x > 0$, then there exists $N \in \mathbf{N}$ such that $\frac{1}{N} < x$ (AP1). If $n \geq N$, then $f_n(x) = 0$, hence $(f_n(x))_n \rightarrow 0$. Thus $(f_n)_n \rightarrow 0_{\mathcal{F}([0, \infty), \mathbf{R})}$. Furthermore:

$$\begin{aligned} \int_0^\infty f_n(x) dx &= \int_0^{\frac{1}{2n}} f_n(x) dx + \int_{\frac{1}{2n}}^{\frac{1}{n}} f_n(x) dx \\ &= 1/2 + 1/2 \\ &= 1. \end{aligned}$$

Definition 5.6.3. A sequence $(f_n)_n \in \mathcal{F}(\Omega, \mathbf{R})$ converges uniformly to $f \in \mathcal{F}(\Omega, \mathbf{R})$ if:

$$\begin{aligned} (\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) \ni n \geq N &\implies |f_n(x) - f(x)| < \epsilon \quad \forall x \in \Omega \\ &\implies \sup_{x \in \Omega} |f_n(x) - f(x)| \leq \epsilon. \end{aligned}$$

Example 5.6.3. Consider the sequence of functions $f_n : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f_n = \mathbf{1}_{[n, n+1]}$ for $n \in \mathbf{N}_0$. Claim: $(f_n)_n \rightarrow 0_{\mathcal{F}(\mathbf{R}, \mathbf{R})}$ point-wise. Let $x \in \mathbf{R}$. Given $\epsilon > 0$, find N large so that $N > x$ (AP1). Then if $n \geq N$, $|f_n(x) - 0_{\mathcal{F}(\mathbf{R}, \mathbf{R})}| = |f_n(x)| = |\mathbf{1}_{[n, n+1]}(x)| = 0$.

However, $(f_n)_n \not\rightarrow 0_{\mathcal{F}(\mathbf{R}, \mathbf{R})}$ uniformly. Notice that $\sup_{x \in \mathbf{R}} |f_n(x) - 0_{\mathcal{F}(\mathbf{R}, \mathbf{R})}| = 1$ no matter how large N is.

Proposition 5.6.1.

- (1) $(f_n)_n \rightarrow f$ uniformly implies $(f_n)_n \rightarrow f$ point-wise.
- (2) If $(f_n)_n \rightarrow f$ point-wise and $(f_n)_n \rightarrow g$ point-wise, then $f = g$.

Note. Recall that $(\ell_\infty(\Omega), d_u)$ where $d_u(f, g) = \sup_{x \in \Omega} |f(x) - g(x)|$ forms a metric space, where d_u is called the *uniform norm*. Note that $(f_n)_n \rightarrow f$ uniformly if and only if $(f_n)_n \rightarrow f$ in $(\ell_\infty(\Omega), d_u)$; i.e., $d_u(f_n, f) \rightarrow 0$.

Example 5.6.4. Consider the sequence of functions $h_n : [0, \infty) \rightarrow \mathbf{R}$ defined by $h_n(x) = e^{-nx}$. We saw that $(h_n)_n \rightarrow \delta_0$. Claim: $\sup_{x \geq 0} |h_n(x) - \delta_0(x)| = 1$. It suffices to show for $x > 0$, because:

$$\begin{aligned} \sup_{x > 0} |e^{-nx} - \delta_0(x)| &= \sup_{x > 0} |e^{-nx} - 0| \\ &= \sup_{x > 0} \frac{1}{e^{nx}} \\ &= 1. \end{aligned}$$

We prove this rigorously as follows: Given $0 < t < 1$, we want to find $x > 0$ such that $t < \frac{1}{e^{nx}}$. This gives:

$$\begin{aligned} e^{nx} < \frac{1}{t} &\iff nx < \ln\left(\frac{1}{t}\right) \\ &\iff x < \frac{1}{n} \ln\left(\frac{1}{t}\right). \end{aligned}$$

Given $0 < t < 1$, by AP1 we have that $x < \frac{1}{n} \ln\left(\frac{1}{t}\right)$. This gives ...

Lemma 5.6.2. $(f_n)_n \not\rightarrow f$ uniformly if and only if:

$$(\exists \epsilon_0 > 0)(\exists (f_{n_k})_k)(\exists (x_k)_k) \text{ s.t. } |f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0.$$

Remark. It suffices to show that $|f_{n_k}(x_k) - f(x_k)| > 0$ by constructing a sequence