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1 Introduction

1.1 Simplicial Homology

Definition 1.1.1. An *n-simplex* is an n-dimensional geometric object with flat sides[1] that is convex hull; i.e., it is the smallest "shape" that encloses "everything" without a concavity.

Example 1.1.1.

1. A 0-dimensional simplex is a point.
2. A 1-dimensional simplex is a line.
3. A 2-dimensional simplex is a triangle.
4. A 3-dimensional simplex is a tetrahedron.
5. A 4-dimensional simplex is a 5-cell.

Note 1.1.1. A simplex, mathematically, does not have any fixed shape, size, or orientation. We are able to rotate, translate, dilate (called *rigid motions*) and stretch a simplex, and it will still count as the same simplex. We cannot however turn an n-simplex into and $(n-1)$-simplex by deforming it.

Definition 1.1.2. Let $\sigma = [v_0, v_1, ..., v_n]$ be an n-dimensional simplex. A *face* of σ is a sub-simplex of σ, namely, the simplex generated by a subset of vertices of σ.

Definition 1.1.3. A *simplicial complex* X is a set of simplexes that satisfies the following conditions:

(1) Every face of a simplex from X is in X, and

(2) The non-empty intersection of any two simplexes $\sigma_1, \sigma_2 \in X$ is a face of both σ_1 and σ_2.

Definition 1.1.4. Let X be a simplicial complex. A *simplicial n-chain* is a finite formal sum

$$\sum_{i=1}^{N} c_i \sigma_i$$

where each c_i is an integer and each σ_i is an oriented simplex. The *group of n-chains on X* is written $C_n(X)$. Furthermore, $C_n(X)$ is a free abelian group whose basis has a one-to-one correspondence with the set of n-simplexes on X.

[1] Another word for this is a *polytope*

Note 1.1.2. In this definition we declare that each oriented simplex is equal to the negative of the simplex with opposite orientation. For example,

$$[v_0, v_1] = -[v_1, v_0]$$

Definition 1.1.5. Let $\sigma = [v_0, v_1, ..., v_n]$ be an oriented n-simplex, viewed as a basis element of $C_n(X)$. The *simplicial boundary map*

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

is a homomorphism defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i [v_0, ..., \hat{v_i}, ..., v_n]$$

where the oriented simplex

$$[v_0, ..., \hat{v_i}, ..., v_n]$$

is the i^{th} face of σ obtained by deleting the i^{th} vertex.

Example 1.1.2. Let $\square$ denote the following rectangle:

$$\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & b \arrow[d, dash] \\ d \arrow[r, dash] & c \end{tikzcd}$$

The triangle $[a, b, c]$ has vertices a, b, c and edges $[b, c], [a, c]$, and $[a, b]$. It's boundary $\partial([a, b, c])$ should be $[b, c] \cup [c, a] \cup [a, b]$. But edges are oriented —we can think of $[c, a]$ as $-[a, c]$; the reverse path from c to a. This aligns exactly with the definition of simplicial boundary maps had we used it: $\partial([a, b, c]) = [b, c] - [a, c] + [a, b]$.

The rectangle $\square$ with vertices a, b, c, d is the union of two triangles, namely $[a, b, c] \cup [a, c, d]$. Since ∂ is a homomorphism, observe that:

$$\begin{aligned} \partial(\square) &= \partial([a, b, c]) + \partial([a, c, d]) \\ &= ([b, c] - [a, c] + [a, b]) + ([c, d] - [a, d] + [a, c]) \\ &= [a, b] + [b, c] + [c, d] - [a, d] \\ &= [a, b] + [b, c] + [c, d] + [d, a]. \end{aligned}$$

Proposition 1.1.6. *Let X be a simplicial complex. For all $n \geqslant 0$, we have that $\partial_{n-1}\partial_n(X) = 0$*

Proof. Observe that $\square$

Definition 1.1.7. Let X be a simplicial complex. For each $n \geqslant 0$,

(1) the subgroup $\ker \partial_n \subseteq C_n(X)$ is denoted by $Z_n(X)$ and its elements are called *simplicial n-cycles*.

(2) the subgroup $\operatorname{im} \partial_{n+1} \subseteq C_n(X)$ is denoted by $B_n(X)$ and its elements are called *simplicial n-boundaries*.

Corollary 1.1.8. *Let X be a simplicial complex. For all n, $B_n(X) \subseteq Z_n(X)$.*

Proof. If $\alpha \in B_n$, then $\alpha = \partial_{n+1}(\beta)$ for some $(n+1)$-chain β. Hence $\partial_n(\alpha) = \partial_n\partial_{n+1}(\beta) = 0$, which gives that $\alpha \in \ker \partial_n = Z_n$. $\square$

Definition 1.1.9. A *chain complex* $(A_\bullet, d_\bullet)$ is a sequence of abelian groups or modules $..., A_0, A_1, A_2, ...$ connected by homomorphisms (called *boundary operators* or *differentials*) $d_n : A_n \to A_{n-1}$ such that the composition of any two consecutive maps is the zero map. In particular, the boundary operators satisfy $d_n d_{n+1} = 0$. The complex may be written out as follows:

$$... \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} ...$$

Example 1.1.3. From Example 1.1.2, Proposition 1.1.6, and Corollary 1.1.8, we have shown that $(C_\bullet(X), \partial_\bullet)$ is a chain complex:

$$... \to C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Definition 1.1.10. The n^{th} *simplicial homology group* of a finite simplicial complex X is

$$H_n(X) = Z_n(X)/B_n(X)$$

Note 1.1.3. What survives in the quotient group $Z_n(X)/B_n(X)$ are the n-dimensional holes, in particular, the n-cycles that are not n-boundaries. I might type up an explicit example but the point of this section was to motivate what "homology" is.

1.2 Module Theory

Definition 1.2.1. Let R be a ring (not necessarily commutative nor with 1).

(1) A *left R-module* is an additive abelian group M with an action of R on M (that is, a map $R \times M \to M$) denoted by $(r, m) \mapsto rm$ such that, for all $m, m' \in M$ and $r, r' \in R$,

   (i) $r(m + m') = rm + rm'$,

   (ii) $(r + r')m = rm + r'm$,

   (iii) $(rr')m = r(r'm)$.

   (iv) $1m = m$ (only if $1 \in M$).

   We often write $_RM$ to denote M being a left R-module

(2) A *right R-module* is an additive abelian group M with an action of R on M (that is, a map $M \times R \to M$) denoted by $(m, r) \mapsto mr$ such that, for all $m, m' \in M$ and $r, r' \in R$,

   (i) $(m + m')r = mr + m'r$,

   (ii) $m(r + r') = mr + mr'$,

   (iii) $m(rr') = (mr)r'$.

   (iv) $m = m1$ (only if $1 \in M$).

   We often write M_R to denote M being a right R-module.

(3) If R is a commutative ring, then we dispense the adjectives left and right and say *R-module* instead.

Example 1.2.1.

(1) Every vector space over a field k is k-module.

(2) Let R be any ring. Then $M = R$ is a left R-module. The ring action is just normal multiplication in the ring R. When R is a left module over itself, the submodules of R are the left ideals of R. If R is not commutative its left and right module structure over itself might be different

(3) Let $R = k$ be a field. Define

$$k^n = \{(a_1, a_2, ..., a_n) \mid a_i \in k, n \in \mathbf{Z}^+\}$$

as *affine n-space over k*. We can make k^n into a vector space by defining addition and scalar multiplication componentwise. When $k = \mathbf{R}$ we have the familiar Euclidean n-space.

Example 1.2.2 (Z-Modules)**.** Let $R = \mathbf{Z}$, let A be any abelian group and write the operation of A as $+$. Make A into a $\mathbf{Z}$-module as follows: for any $n \in \mathbf{Z}$ and $a \in A$ define

$$na = \begin{cases} a + a + ... + a \quad (n \text{ times}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -a - a - ... - a \quad (n \text{ times}) & \text{if } n < 0 \end{cases},$$

where 0 is the identity of the additive abelian group A. This definition of $\mathbf{Z}$ acting on A makes A into a $\mathbf{Z}$-module, and furthermore the module axioms show that this is the only action of $\mathbf{Z}$ on A. Thus every abelian group is a $\mathbf{Z}$-module and vice versa.

Definition 1.2.2. Let R be a ring and let M be an R-module. An *R-submodule of M* is a subgroup N of M which is closed under the action of ring elements; i.e., $rn \in N$ for all $r \in R$, $n \in N$.

Proposition 1.2.3 (The Submodule Criterion)**.** *Let R be a ring and let M be an R-module. A subset N of M is a submodule of M if and only if $N \neq \emptyset$ and $x + ry \in N$ for all $r \in R$ and $x, y \in N$.*

Proof. If N is a submodule, then $0 \in N$ so $N \neq \emptyset$. Also N is closed under addition and is sent to itself under the action of elements in R[2].

Conversely, suppose $N \neq \emptyset$ and $x + ry \in N$ for all $r \in R$ and $x, y \in N$. Let $r = -1$, then $x - y \in N$; i.e., N is a subgroup of M. This also gives that $0 \in N$. Let $x = 0$, then $ry \in N$; i.e., N is sent to itself under the action of R. This establishes the proposition. □

Definition 1.2.4. Let R be a ring and let M and N be R-modules.

(1) A map $\varphi : M \to N$ is an *R-module homomorphism* if it respects the R-module structures of M and N; i.e.,

  (a) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M$ and

  (b) $\varphi(rx) = r\varphi(x)$ for all $r \in R$, $x \in M$.

[2] This satisfies axioms (1) and (2) from Definition **??**

(2) An R-module homomorphism is an *isomorphism* (of R-modules) if it is both injective and surjective. The modules M and N are said to be *isomorphic*, denoted $M \cong N$, if there is some R-module isomorphism $\varphi : M \to N$.

(3) If $\varphi : M \to N$ is an R-module homomorphism, let $\ker \varphi = \{m \in M \mid \varphi(m) = 0\}$ and let $\operatorname{im} \varphi = \{n \in N \mid n = \varphi(m) \quad \text{for some } m \in M\}$.

(4) Let M and N be R-modules and define $\operatorname{Hom}_R(M, N)$ to be the set of all R-module homomorphisms from M into N.

Note 1.2.1. Any R-module homomorphism is also a homomorphism of the additive groups, but not every group homomorphism need be a module homomorphism (condition (b) may not be satisfied).

Note 1.2.2. Let H be a subgroup of G. If G is abelian then H is normal. This is relevant for the following proposition.

Proposition 1.2.5. *Let R be a ring, let M be an R-module and let N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by*

$$r(x + N) = rx + N \quad \textit{for all } r \in R,\ x + N \in M/N.$$

The natural projection map $\pi : M \to M/N$ defined by $\pi(x) = x + N$ is an R-module homomorphism with kernel N.

Proof. Since M is an abelian group under $+$ the quotient group M/N is defined and is an abelian group. We must show that the action of the ring element r on the coset $x + N$ is well defined. Suppose $x + N = y + N$; i.e., $x - y \in N$. Since N is a (left) R-module, $r(x - y) \in N$. Thus $rx - ry \in N$; i.e., $rx + N = ry + N$.

Since the operations in M/N are "compatible" with those of M, the axioms for an R-module are easily checked. Likewise, the natural projection map π described as above is, in particular, the natural projection of the abelian group M onto the abelian group M/N hence is a group homomorphism with kernel N. The kernel of any module homomorphism is the same as its kernel when viewed as a homomorphism of the abelian group structures. It remains to show that π is a module homomorphism —which it is: $\pi(rm) = rm + N = r(m + N) = r\pi(m)$. □

Definition 1.2.6. If $\varphi : M \to N$ is a left R-module homomorphism, then $\operatorname{coker} \varphi = N/\operatorname{im} \varphi$.

Example 1.2.3. Consider the additive group $\mathbf{Z}$.

(1) Let $\varphi : \mathbf{Z} \to \mathbf{Z}$ be defined by $\varphi(a) = 3a$. Observe that:

$$\begin{aligned} \operatorname{im} \varphi &= \{b \in \mathbf{Z} \mid b = \varphi(a) \text{ for some } \ a \in \mathbf{Z}\} \\ &= \{b \in \mathbf{Z} \mid b = 3a \text{ for some } \ a \in \mathbf{Z}\} \\ &= 3\mathbf{Z}. \end{aligned}$$

So $\operatorname{coker} \varphi = \mathbf{Z}/\operatorname{im} \varphi = \mathbf{Z}/3\mathbf{Z} = \{[0]_3, [1]_3, [2]_3\}$.

(2) Let $\varphi : \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ be defined by $\varphi(a) = [a]_n$. Clearly $\operatorname{im} \varphi = \mathbf{Z}/n\mathbf{Z}$, hence $\operatorname{coker} \varphi = (\mathbf{Z}/n\mathbf{Z})/(\mathbf{Z}/n\mathbf{Z}) = \{0\}$.

Proposition 1.2.7.

Let $\varphi : M \to N$ be a left R-module homomorphism.

(1) *φ is injective if and only if $\ker \varphi = \{0\}$.*

(2) *φ is surjective if and only if $\operatorname{coker} \varphi = \{0\}$.*

Proof. □

Definition 1.2.8. Let M be an R-module and let $N_1, ..., N_n$ be submodules of M.

(1) The _sum_ of $N_1, ..., N_n$ is the set of all finite sums of elements from the sets N_i: $\{a_1 + a_2 + ... + a_n \mid a_i \in N_i \text{ for all } i\}$. Denote this sum by $N_1 + N_2 + ... + N_n$.

(2) For any subset A of M let

$$RA = \{r_1a_1 + r_2a_2 + ... + r_ma_m \mid r_1, ..., r_m \in R,\ \ a_1, ..., a_m \in A,\ \ m \in \mathbf{Z}^+\}$$

(where by convention $RA = \{0\}$ if $A = \emptyset$). If A is the finite set $\{a_1, a_2, ..., a_n\}$ we shall write $Ra_1 + Ra_2 + ... + Ra_n$ for RA. Call RA the _submodule of M generated by A_. If N is a submodule of M (possibly $N = M$) and $N = RA$ for some subset A of M, we call A a _set of generators_ or _generating set_ for N, and we say N is _generated_ by A.

(3) A submodule N of M (possibly $N = M$) is _finitely generated_ if there is some finite subset A of M such that $N = RA$, that is, if N is generated by some finite subset.

(4) A submodule N of M (possibly $N = M$) is _cyclic_ if there exists an element $a \in M$ such that $N = Ra$, that is, if N is generated by one element:

$$N = RA = \{ra \mid r \in R\}.$$

Definition 1.2.9. Let $M_1, ..., M_k$ be a collection of R-modules. The collection of k-tuples $(m_1, ..., m_k)$ where $m_i \in M_i$ with addition and action of R defined componentwise is called the _direct product_ of $M_1, ..., M_k$, denoted $M_1 \times ... \times M_k$. The direct product of $M_1, ..., M_k$ is also referred to as the (external) _direct sum_ of $M_1, ..., M_k$ and is denoted $M_1 \oplus ... \oplus M_k$.

Proposition 1.2.10. *Let $N_1, N_2, ..., N_k$ be submodules of the R-module M. Then the following are equivalent:*

(1) *The map $\pi : N_1 \times N_2 \times ... \times N_k \to N_1 + N_2 + ... + N_k$ defined by*

$$\pi((a_1, a_2, ..., a_k)) = a_1 + a_2 + ... + a_k$$

is an isomorphism (of R-modules): $N_1 \times N_2 \times ... \times N_k \cong N_1 + N_2 + ... + N_k$.

(2) *$N_j \cap (N_1 + N_2 + ... + N_{j-1} + N_{j+1} + ... + N_k) = 0$ for all $j \in \{1, 2, ..., k\}$.*

(3) *Every $x \in N_1 + N_2 + ... + N_k$ can be written uniquely in the form $a_1 + a_2 + ... + a_k$ with $a_i \in N_i$.*

Proof. To prove that (1) implies (2), suppose for some j (2) fails to hold and let $a_j \in N_j \cap (N_1 + N_2 + ... + N_{j-1} + N_{j+1} + ... + N_k)$ with $a_j \neq 0$. Then $a_j \in N_j$ and $a_j \in N_1 + N_2 + ... + N_{j-1} + N_{j+1} + ... + N_k$, hence $a_j = a_1 + a_2 + ... + a_{j-1} + a_{j+1} + ... + a_k$ for some $a_i \in N_i$. Subtracting a_j from both sides gives $0 = a_1 + a_2 + ... + a_{j-1} - a_j + a_{j+1} + ... + a_k$, which is equivalent to $\pi(0) = (a_1, a_2, ..., a_{j-1}, -a_j, a_{j+1}, ..., a_k)$. Note that this would be a nonzero element of $\ker \pi$, which gives a contradiction.

Assume now that (2) holds. If for some module elements $a_i, b_i \in N_i$ we have:

$$a_1 + a_2 + ... + a_k = b_1 + b_2 + ... + b_k$$

then for each j we have:

$$a_j - b_j = (b_1 - a_1) + (b_2 - a_2) + ... + (b_{j-1} - a_{j-1}) + (b_{j+1} - a_{j+1}) + ... + (b_k - a_k).$$

The left belongs to N_j and the right side belongs to $(N_1 + N_2 + ... + N_{j-1} + N_{j+1} + ... + N_k)$, hence $a_j - b_j \in N_j \cap (N_1 + N_2 + ... + N_{j-1} + N_{j+1} + ... + N_k)$. It must be the case then that $a_j - b_j = 0$; i.e., $a_j = b_j$ for all j. Thus (2) implies (3).

Finally, to see that (3) implies (1), observe first that the map π is clearly a surjective R-module homomorphism. Then (3) simply implies π is injective, hence is an isomorphism, completing the proof. □

Definition 1.2.11. If an R-module M is the sum of submodules $N_1, N_2, ..., N_k$ of M satisfying the conditions of the proposition above, then M is said to be the (internal) *direct sum* of $N_1, N_2, ..., N_k$, written:

$$M = N_1 \oplus N_2 \oplus ... \oplus N_k.$$

Note 1.2.3. Part (1) of Proposition 1.2.10 is the statement that the internal direct sum of $N_1, N_2, ..., N_k$ is isomorphic to their external direct sum (from Definition 1.2.9), which is the reason we identify them and use the same notation for both. In extremely simple terms, "direct sum of submodules $\implies$ internal", "direct sum of modules $\implies$ external" (but who cares).

Definition 1.2.12. An R-module F is said to be *free* on the subset A of F if for every nonzero element x of F, there exist unique nonzero elements $r_1, r_2, ..., r_n$ of R and unique $a_1, a_2, ..., a_n$ in A such that $x = r_1a_1 + r_2a_2 + ... + r_na_n$, for some $n \in \mathbf{Z}^+$. In this situation we say A is a *basis* or *set of free generators* for F. If R is a commutative ring the cardinality of A is called the *rank* of F.

Note 1.2.4. To avoid confusion, we reiterate Definition 1.2.8 and Definition 1.2.12 as follows: An R-module M is called:

- *free* if $M \cong R^n = \bigoplus_{i=1}^n R$. In other words, the map $\phi : R^n \to M$ is an R-module isomorphism. n is called the *rank* of M and it can be infinite.
- *finitely generated* if M has a finite generating set. In other words, the map $\phi : R^n \to M$ is only surjective.

The difference boils down to whether $\ker \phi = 0$ or not. Furthermore, in the case of a direct sum between two modules, the module elements will be unique, whereas in the case of free modules the module elements and ring elements must be unique.

Definition 1.2.13. If (R, α, μ) is a ring, then its *opposite ring* R^{op} is $(R, \alpha, \mu^{\text{o}})$, where $\mu^{\text{o}} : R \times R \to R$ is defined by

$$\mu^{\text{o}}(r, r') = \mu(r', r).$$

Informally, we have reversed the order of multiplication.

1.3 Categories and Functors

Definition 1.3.1. A *class* is a collection of sets (or sometimes other mathematical objects) that can be unambiguously defined by a property that all its members share.

Definition 1.3.2. A *category* $\mathcal{C}$ consists of three ingredients: a class $\text{obj}(\mathcal{C})$ of *objects*, a set of *morphisms* $\text{Hom}(A, B)$ for every ordered pair (A, B) of objects, and *composition* $\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C)$, denoted by

$$(f, g) \mapsto gf,$$

for every ordered tripled A, B, C of objects (we often write $f : A \to B$ or $A \xrightarrow{f} B$ instead of $f \in \text{Hom}(A, B)$). These ingredients are subject to the following axioms:

(1) The Hom sets are pairwise disjoint; i.e., each $f \in \text{Hom}(A, B)$ has a unique *domain* A and a unique *target* B;

(2) for each object A, there is an *identity morphism* $1_A \in \text{Hom}(A, A)$ such that $f1_A = f$ and $1_B f = f$ for all $f : A \to B$;

(3) composition is associative: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then

$$h(gf) = (hg)f.$$

Example 1.3.1.

(1) **Sets**. The objects in this category are sets, morphisms are functions, and composition is the usual composition of functions. It is an axiom of set theory that if A and B are sets, then the class $\text{Hom}(A, B)$ of all functions from A to B is also a set.

(2) **Groups**. Objects are groups, morphisms, are homomorphisms, and composition is the usual composition (as homomorphisms are functions). Part of the verification that **Groups** is a category involves checking that identity functions are homomorphisms and that the composite of two homomorphisms is itself a homomorphism (one needs to know that if $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, C)$, then $gf \in \text{Hom}(A, C)$).

(3) A partially ordered set X can be regarded as the category whose objects are the elements of X, whose Hom sets are either empty or have only one element:

$$\text{Hom}(x, y) = \begin{cases} \emptyset & \text{if } x \not\preceq y, \\ \{\iota_y^x\} & \text{if } x \preceq y \end{cases}$$

(the symbol ι_y^x is the unique element in the Hom set when $x \preceq y$), and whose composition is given by $\iota_z^y \iota_y^x = \iota_z^x$. Note that $1_x = \iota_x^x$, by reflexivity, while composition makes sense because $\preceq$ is transitive. We insisted in the definition of a category that each $\text{Hom}(A, B)$ be a set, but we did not say it was nonempty. This is an example in which this possibility occurs.

(4) **Top**. Objects are topological spaces, morphisms are continous functions, and composition is the usual composition of functions. In checking that **Top** is a category, one must note that identity functions are continuous and that composites of continuous functions are continuous.

(5) The category **Sets**$_*$ of all *pointed sets* has as its objects all ordered pairs (X, x_0), where X is a nonempty set and x_0 is a point in X, called the *basepoint*. A morphism $f : (X, x_0) \to (Y, y_0)$ is called a *pointed map*; it is a function $f : X \to Y$ with $f(x_0) = y_0$. Composition is the usual composition of functions. One defines the category **Top**$_*$ of all *pointed spaces* in a similar way; $\mathrm{obj}\,(\mathbf{Top}_\star)$ consists of all ordered pairs (X, x_0), where X is a nonempty topological space and $x_0 \in X$, and morphisms $f : (X, x_0) \to (Y, y_0)$ are continuous functions $f : X \to Y$ with $f(x_0) = y_0$.

(6) **Ab**. Objects are abelian groups, morphisms are homomorphisms, and composition is the usual composition.

(7) **Rings**. Objects are rings, morphisms are ring homomorphisms, and composition is the usual composition. We assume that all rings R have a unit element 1, but we do not assume that $1 \neq 0$. We agree, as part of the definition, that $\varphi(1) = 1$ for every ring homomorphism φ. Since the inclusion map $S \to R$ of a subring should be a homomorphism, it follows that the unit element 1 in a subring S must be the same as the unit element 1 in R.

(8) **ComRings**. Objects are commutative rings, morphisms are ring homomorphisms, and composition is theusual composition.

(9) The category ${}_R\mathbf{Mod}$ of all *left R-modules* (where R is a ring) has as its objects all left R-modules, its morphisms as all R-module homomorphisms, and as its composition the usual composition of functions. We denote the sets $\mathrm{Hom}\,(A, B)$ in ${}_R\mathbf{Mod}$ by $\mathrm{Hom}_R\,(A, B)$. If $R = \mathbf{Z}$, then ${}_\mathbf{Z}\mathbf{Mod} = \mathbf{Ab}$, for abelian groups are $\mathbf{Z}$-modules and homomorphisms are $\mathbf{Z}$-maps. There is also a category of right R-modules denoted $\mathbf{Mod}_R$.

Definition 1.3.3. A category $\mathcal{S}$ is a *subcategory* of a category $\mathcal{C}$ if

(1) $\mathrm{obj}\,(\mathcal{S}) \subseteq \mathrm{obj}\,(\mathcal{C})$;

(2) $\mathrm{Hom}_\mathcal{S}\,(A, B) \subseteq \mathrm{Hom}_\mathcal{C}\,(A, B)$ for all $A, B \in \mathrm{obj}\,(\mathcal{S})$, where we denote Hom sets in $\mathcal{S}$ by $\mathrm{Hom}_\mathcal{S}\,(\square, \square)$;

(3) if $f \in \mathrm{Hom}_\mathcal{S}\,(A, B)$ and $g \in \mathrm{Hom}_\mathcal{S}\,(B, C)$, then the composite $gf \in \mathrm{Hom}_\mathcal{S}\,(A, C)$ is equal to the composite $gf \in \mathrm{Hom}_\mathcal{C}\,(A, C)$;

(4) if $A \in \mathrm{obj}\,(\mathcal{S})$, then the identity $1_A \in \mathrm{Hom}_\mathcal{S}\,(A, A)$ is equal to the identity $1_A \in \mathrm{Hom}_\mathcal{C}\,(A, A)$.

A subcategory $\mathcal{S}$ of $\mathcal{C}$ is a *full subcategory* if, for all $A, B \in \mathrm{obj}\,(\mathcal{S})$, we have $\mathrm{Hom}_\mathcal{S}\,(A, B) = \mathrm{Hom}_\mathcal{C}\,(A, B)$.

Example 1.3.2.

1. **Ab** is a full subcategory of **Groups**.

2. A category is *discrete* if its only morphisms are identity morphisms. If $\mathcal{S}$ is the discrete category with $\mathrm{obj}\,(\mathcal{S}) = \mathrm{obj}\,(\mathbf{Sets})$, then $\mathcal{S}$ is a subcategory of **Sets** that is not a full subcategory.

Definition 1.3.4. Let $\mathcal{C}$ be any category and $\mathcal{S} \subseteq \operatorname{obj}(\mathcal{C})$. The *full subcategory generated by $\mathcal{S}$*, also denoted by $\mathcal{S}$, is the subcategory with $\operatorname{obj}(\mathcal{S}) = \mathcal{S}$ and with $\operatorname{Hom}_{\mathcal{S}}(A, B) = \operatorname{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \operatorname{obj}(\mathcal{S})$.

Definition 1.3.5. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *covariant functor* $T : \mathcal{C} \to \mathcal{D}$ is a function such that

(1) if $A \in \operatorname{obj}(\mathcal{C})$, then $T(A) \in \operatorname{obj}(\mathcal{D})$,

(2) if $f : A \to A'$ in $\mathcal{C}$, then $T(f) : T(A) \to T(A')$ in $\mathcal{D}$,

(3) if $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{C}$, then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ in $\mathcal{D}$ and

$$T(gf) = T(g)T(f),$$

(4) $T(1_A) = 1_{T(A)}$ for every $A \in \operatorname{obj}(\mathcal{C})$.

Example 1.3.3. If $\mathcal{C}$ is a category and $A \in \operatorname{obj}(\mathcal{C})$, then the **Hom** *(covariant) functor* is a function $T_A : \mathcal{C} \to$ **Sets** defined by

$$T_A(B) = \operatorname{Hom}(A, B) \text{ for all } B \in \operatorname{obj}(\mathcal{C}).$$

The function is usually denoted by $\operatorname{Hom}(A, \square)$. If $f : B \to B'$ is in $\mathcal{C}$, then $T_A(f) : \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B')$ is given by

$$T_A(f) : h \mapsto fh \text{ for each } h \in \operatorname{Hom}(A, B).$$

We call $T_A(f) = \operatorname{Hom}(A, f)$ the *induced map*, and we denote it by f_*; thus,

$$f_* : h \mapsto fh.$$

Because of the importance of this example, we will verify all the axioms of Definition 1.3.5.

(i) The very definition of a category says that $\operatorname{Hom}(A, B)$ is a set. This satisfies Axiom (1).

(ii) From the following diagram:

$$\begin{tikzcd} A \arrow[r, "h"'] \arrow[rr, bend left, "fh"] & B \arrow[r, "f"'] & B' \end{tikzcd}$$

we see that Axiom (2) is satisfied.

(iii) Suppose now that $g : B' \to B''$. We'd like to show that the following two functions are equivalent:

$$(gf)_* : \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B'')$$
$$g_* f_* : \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B'')$$

If $h \in \operatorname{Hom}(A, B)$, then $(gf)_* : h \mapsto (gf)h$. On the other hand, associativity of composition gives that $g_* f_* : h \mapsto fh \mapsto g(fh) = (gf)h$, as desired for Axiom (3).

(iv) Finally, if f is the identity map $1_B : B \to B$, then

$$(1_B)_* : h \mapsto 1_B h = h$$

for all $h \in \operatorname{Hom}(A, B)$, so that $(1_B)_* = 1_{\operatorname{Hom}(A,B)}$.

Definition 1.3.6. A *contravariant functor* $T : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are categories, is a function such that:

(1) If $B \in \operatorname{obj}(\mathcal{C})$, then $T(B) \in \operatorname{obj}(\mathcal{D})$,

(2) if $f : B \to B'$ in $\mathcal{C}$, then $T(f) : T(B') \to T(B)$ in $\mathcal{D}$ (Note the reversal of arrows from Definition 1.3.5),

(3) if $B \xrightarrow{f} B' \xrightarrow{g} B''$ in $\mathcal{C}$, then $T(B'') \xrightarrow{T(g)} T(B') \xrightarrow{T(f)} T(B)$ in $\mathcal{D}$ and

$$T(gf) = T(f)T(g),$$

(4) $T(1_A) = 1_{T(A)}$ for every $A \in \operatorname{obj}(\mathcal{C})$.

Example 1.3.4. If $\mathcal{C}$ is a category and $B \in \operatorname{obj}(\mathcal{C})$, then the ***Hom** (contravariant) functor* is a function $T^B : \mathcal{C} \to$ **Sets** defined by

$$T^B(C) = \operatorname{Hom}(C, B) \quad \text{for all } C \in \operatorname{obj}(\mathcal{C}).$$

The function is usually denoted by $\operatorname{Hom}(\square, B)$. If $f : C \to C'$ is in $\mathcal{C}$, then $T^B(f) : \operatorname{Hom}(C', B) \to \operatorname{Hom}(C, B)$ is given by

$$T^B(f) : h \mapsto hf.$$

We call $T^B(f) = \operatorname{Hom}(f, B)$ the *induced map*, and we denote it by f^*; thus,

$$f^* : h \mapsto hf.$$

Of equal importance to Example 1.3.3, the axioms are verified similarly.

Example 1.3.5. Recall that a *linear functional* on a vector space V over k is a linear transformation $\varphi : V \to k$ (note that k is a one-dimensional vector space over itself). For example, let $V = C([0, 1])$, then integration $f \mapsto \int_0^1 f(t)dt$ is a linear functional on V. If V is a vector space over a field k, then its *dual space* is $V^* = \operatorname{Hom}_k(V, k)$, the set of all linear functionals on V. Note that functions in V^* are closed under pointwise addition —V^* is a vector space over k if we define $af : V \to k$ by $af : v \mapsto a[f(v)]$ for all $f \in V^*$ and $a \in k$. Moreover, if $f : V \to W$ is a linear transformation, then the induced map $f^* : W* \to V*$ is also a linear transformation. The *dual space functor* is $\operatorname{Hom}_k(\square, k) : {}_k\mathbf{Mod} \to {}_k\mathbf{Mod}$.

Definition 1.3.7. If $\mathcal{C}$ is a category, define its *opposite category* $\mathcal{C}^{\text{op}}$ to be the category with $\operatorname{obj}(\mathcal{C}^{\text{op}}) = \operatorname{obj}(\mathcal{C})$, with morphisms $\operatorname{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \operatorname{Hom}_{\mathcal{C}}(B, A)$ and with composition $g^{\text{op}} f^{\text{op}} = f g^{\text{op}}$, where $f^{\text{op}}, g^{\text{op}} \in \operatorname{Hom}_{\mathcal{C}^{\text{op}}}(A, B)$ (note that the composition is the reverse of that in $\mathcal{C}$).

Definition 1.3.8. A morphism $f : A \to B$ in a category $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g : B \to A$ in $\mathcal{C}$ with

$$gf = 1_A \text{ and } fg = 1_B.$$

The morphism g is called the *inverse* of f.

Proposition 1.3.9. *Let $T : \mathcal{C} \to \mathcal{D}$ be a functor of either variance. If f is an isomorphism in $\mathcal{C}$, then $T(f)$ is an isomorphism in $\mathcal{D}$.*

Proof. If g is the inverse of f, apply the functor T to the equations $gf = 1$ and $fg = 1$. □

Definition 1.3.10. Let $S, T : \mathcal{A} \to \mathcal{B}$ be covariant functors. A *natural transformation* $\tau : S \to T$ is a one-parameter family of morphisms in $\mathcal{B}$,

$$\tau = (\tau_A : SA \to TA)_{A \in \operatorname{obj}(\mathcal{A})},$$

making the following diagram commute for all $f : A \to A'$ in $\mathcal{A}$:

$$\begin{tikzcd} SA \arrow[r, "\tau_A"] \arrow[d, "Sf"'] & TA \arrow[d, "Tf"] \\ SA' \arrow[r, "\tau_{A'}"'] & TA'. \end{tikzcd}$$

Definition 1.3.11. Let $X, Y \in \operatorname{obj}(\textbf{Sets})$ and $f, g \in \operatorname{Hom}(X, Y)$ The *equaliser of f and g* is the set of elements $x \in X$ such that $f(x)$ is equal to $g(x)$ in Y. Symbolically:

$$\operatorname{Eq}(f, g) = \{x \in X \mid f(x) = g(x)\}.$$

Definition 1.3.12. Let $\mathcal{C}$ be a category with a zero morphism. If $f : X \to Y$ is an arbitrary morphism in $\mathcal{C}$, then a *kernel* of f is an equaliser of f and the zero morphism from X to Y; i.e., $\ker f = \operatorname{Eq}(f, 0_{XY}) = \{x \in X \mid f(x) = 0_Y\}$.

Definition 1.3.13. Include the categorical definition of universal properties here.

Note 1.3.1. This doesn't really make sense if you've never seen universal properties before. Looking at examples gives a better understanding of the definition.

Example 1.3.6.

(1) Let G be a group and N a normal subgroup of G. If K is any group and $\varphi : G \to K$ is a homomorphism which annihilates N (that is, $N \subseteq \ker \varphi$), then there is a unique homomorphism $\Phi : G/N \to K$ such that $\Phi \circ \pi = \varphi$, where π is the natural projection map $\pi : G \to G/N$. The following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & G/N \arrow[d, "\Phi"] \\ & K \end{tikzcd}$$

(2) Let H be a subgroup of G. If G is any group and $\varphi : K \to G$ is a homomorphism whose image is contained in H (that is, $\operatorname{im} \varphi \subseteq H$), then there is a homomorphism $\Phi : K \to H$ such that $\iota \circ \Phi = \varphi$, where ι is the inclusion map $\iota : H \to G$. The following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "\Phi"] \arrow[dr, "\varphi"'] & H \arrow[d, "\iota"] \\ & G \end{tikzcd}$$

(3) Recall Definition 1.2.12. For any set A there is a free R-module $F(A)$ on the set A and $F(A)$ satisfies the following *universal property*: if M is any R-module and $\varphi : A \to M$ is any map of sets, then there is a unique R-module homomorphism $\phi : F(A) \to M$ such that $\phi(a) = \varphi(a)$ for all $a \in A$, that is, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\text{inclusion}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, "\phi"] \\ & M \end{tikzcd}$$

(4) Let $\mathcal{C}$ be a category with a zero morphism. If $f : X \to Y$ is an arbitrary morphism in $\mathcal{C}$ then Definition 1.3.12 gives rise to the following universal property: a kernel of f is an object K together with a morphism $k : K \to X$ such that,

(1) $f \circ k$ is the zero morphism from K to Y,

(2) Given any morphism $k' : K' \to X$ such that $f \circ k'$ is the zero morphism, there is a unique morphism $u : K' \to K$ such that $k \circ u = k'$,

which make the following diagrams commute:

(1)
$$\begin{tikzcd} K \arrow[r, "k"] \arrow[dr, "0_{KY}"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

(2)
$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ & K \arrow[u, "k"] \arrow[r, "0_{KY}"] & Y \\ K' \arrow[uur, "k'"] \arrow[ur, dotted, "u"] \arrow[urr, "0_{K'Y}"'] & & \end{tikzcd}$$

Exercises

Exercise 1.1. Use parts (1) and (2) of Example 1.3.6 to show that if $\varphi : G \to H$ is a group homomorphism, then $G/\ker\varphi \cong \operatorname{im}\varphi$.

Proof. Since $\operatorname{im}\varphi$ is a subgroup of H, the universal property of subgroups gives the correspondence $\varphi = \iota \circ \Phi$, where $\Phi : G \to \operatorname{im}\varphi$ and $\iota : \operatorname{im}\varphi \to H$. Now, since $\ker\Phi$ is a normal subgroup of G, the unversal property of quotient groups gives the correspondence $\Phi = \alpha \circ \pi$, where $\pi : G \to G/\ker\Phi$ and $\alpha : G/\ker\Phi \to \operatorname{im}\varphi$. The following diagram commutes.

$$\begin{tikzcd}
& G/\ker\Phi \arrow[d, "\alpha"] \\
G \arrow[ur, "\pi"] \arrow[r, "\Phi"] \arrow[dr, "\varphi"'] & \operatorname{im}\varphi \arrow[d, "\iota"] \\
& H
\end{tikzcd}$$

Claim: $\ker\Phi = \ker\varphi$. If $k \in \ker\Phi$, then $\Phi(k) = 0_{\operatorname{im}\varphi} = 0_H$, hence $k \in \ker\varphi$. Conversely, if $k \in \ker\varphi$, then k must get mapped to something inside the image of φ; i.e., $\varphi(k) = 0_H = 0_{\operatorname{im}\varphi}$ implies $k \in \ker\Phi$.

Claim: α is an isomorphism. Note that Φ is clearly surjective, hence it must be the case that α is surjective. It remains to show that α is injective: this can be done by showing the map $\alpha : \mathrm{G}/\ker\varphi \to \operatorname{im}\varphi$ has a trivial kernel. Let $g \in G$ be any element, note that $\alpha(g\ker\varphi) = \varphi(g)$ by our definition. Hence when $g \in \ker\varphi$, $\alpha(g\ker\varphi) = \alpha(\ker\varphi) = 0_{\operatorname{im}\varphi}$ (the only element of $G/\ker\varphi$ which maps to $0_{\operatorname{im}\varphi}$ is the identity). This establishes the proof that $G/\ker\varphi \cong \operatorname{im}\varphi$[3]. $\square$

Exercise 1.2.

(i) Prove, in every category $\mathcal{C}$, that each object $A \in \mathcal{C}$ has a unique identity morphism.

(ii) If f is an isomorphism in this category, prove that its inverse is unique.

Proof. Let $f : A \to B$. Suppose $1_A, 1'_A \in \operatorname{Hom}(A, A)$ such that $f1_A = f$ and $f1'_A = f$. Take $B = A$ and $f = 1'_A$, then $1'_A 1_A = 1'_A$. Now consider $g : B \to A$. Then $1'_A g = g$. Take $B = A$ and $g = 1_A$, then $1'_A 1_A = 1_A$. Together, this gives $1'_A = 1'_A 1_A = 1_A$. Hence the identity morphism is unique.

Let $f : A \to B$ be an isomorphism. Suppose $g, g' : B \to A$ are inverses of f. Then $g = 1_A g = (g'f)g = g'(fg) = g'1_B = g'$. Hence inverses are unique. $\square$

Exercise 1.3. If $T : \mathcal{A} \to \mathcal{B}$ is a functor, define $T^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{B}$ by $T^{\text{op}}(A) = T(A)$ for all $A \in \operatorname{obj}(\mathcal{A})$ and $T^{\text{op}}(f^{\text{op}}) = T(f)$ for all morphisms f in $\mathcal{A}$. Prove that T^{op} is a functor having variance opposite to the variance of T.

[3]We can write $\varphi = \iota \circ \alpha \circ \pi$. In other words, *every* momorphism takes the form of a quotient map, followed by an isomorphism, followed by an inclusion.

Exercise 1.4.

(i) If X is a set, define FX to be the free group having basis X, that is, the elements of FX are reduced words on the alphabet X and multiplication is juxtaposition followed by cancellation. If $\varphi : X \to Y$ is a function, prove that there is a unique homomorphism $F\varphi : FX \to FY$ such that $(F\varphi) \mid X = \varphi$.

(ii) Prove that $F : \mathbf{Sets} \to \mathbf{Groups}$ is a functor (F is called the *free functor*).

Exercise 1.5.

(i) Define $\mathcal{C}$ to have objects all ordered pairs (G, H), where G is a group and H is a normal subgroup of G, and to have morphisms $\varphi_* : (G, H) \to (G_1, H_1)$, where $\varphi : G \to G_1$ is a homomorphism with $\varphi(H) \subseteq H_1$. Prove that $\mathcal{C}$ is a category if composition in $\mathcal{C}$ is defined to be ordinary composition.

(ii) Construct a functor $Q : \mathcal{C} \to \mathbf{Groups}$ with $Q(G, H) = G/H$.

(iii) Prove that there is a functor $\mathbf{Groups} \to \mathbf{Ab}$ taking each group G to G/G', where G' is its commutator subgroup.

Exercise 1.6. Let R be a ring. An (additive) abelian group M is an *almost left R-module* if there is a function $R \times M \to M$ which satisfies all of the left R-module axioms except for Axiom (4): we do not assume that $1m = m$ for all $m \in M$. Prove that if M is an almost left R-module then $M = M_1 \oplus M_0$, where $M_1 = \{m \in M : 1m = 1\}$ and $M_0 = \{m \in M : rm = 0 \text{ for all } r \in R\}$ are subgroups of M that are almost left R-modules (in fact, M_1 is a left R-module).

Proof. □

Exercise 1.7. Prove that every right R-module is a left R^{op}-module and vice versa.

Exercise 1.8. Let M be a left R-module.

(i) Prove that $\mathrm{Hom}_R(M, M)$ is a ring with 1 under pointwise addition and composition as multiplication.

(ii) The ring $\mathrm{Hom}_R(M, M)$ is called the *endomorphism ring of M* and is denoted $\mathrm{End}_R(M)$. Elements of $\mathrm{End}_R(M)$ are called *endomorphisms*. Prove that M is a left $\mathrm{End}_R(M)$-module.

(iii) If a ring R is regarded as a left R-module, prove that $\mathrm{End}_R(R) \cong R^{\mathrm{op}}$ as rings.

Proof. The details of $\mathrm{Hom}_R(M, M)$ being an abelian group are left out —associativity and commutativity are shown easily, the identity is the zero map 0_{MM} and inverses follow from this. Let $\varphi, \psi, \gamma \in \mathrm{Hom}_R(M, M)$. Note that function composition is automatically associative. "1" in this case is the identity morphism: $(\varphi \circ \mathrm{id}_M)(m) = \varphi(\mathrm{id}_M(m)) = \varphi(m)$ and $(\mathrm{id}_M \circ \varphi)(m) = \mathrm{id}_M(\varphi(m)) = \varphi(m)$. Function composition distributes over pointwise addition as follows:

$$
\begin{aligned}
(\varphi \circ (\psi + \gamma))(m) &= \varphi((\psi + \gamma)(m)) \\
&= \varphi(\psi(m) + \gamma(m)) \\
&= \varphi(\psi(m)) + \varphi(\psi(m)) \\
&= (\varphi \circ \psi)(m) + (\varphi \circ \gamma)(m) \\
&= ((\varphi \circ \psi) + (\varphi \circ \gamma))(m).
\end{aligned}
$$

Distribution from the right follows similary, hence $\mathrm{Hom}_R(M, M)$ is a unital ring.

$\square$

2

Hom and Tensor

2.1 Constructs in $_R\mathbf{Mod}$

Note 2.1.1. Recall from Exercise **??** that

$$\mathrm{End}_{\mathbf{Z}}(M) = \{\text{homomorphisms } \varphi : M \to M\}$$

is a ring under pointwise addition ($\varphi+\psi : m \mapsto \varphi(m)+\psi(m)$) and composition as multiplication.

Definition 2.1.1. A *representation* of a ring R is a ring homomorphism $\varphi : R \to \mathrm{End}_{\mathbf{Z}}(M)$ for some abelian group M.

Proposition 2.1.2. *Let R be a ring and let M be an abelian group. If $\varphi : R \to \mathrm{End}_{\mathbf{Z}}(M)$ is a representation, define $\sigma : R \times M \to M$ by $\sigma(r,m) = \varphi_r(m)$, where we write $\varphi(r) = \varphi_r$; then σ is a scalar multiplier making M into a left R-module. Conversely, if M is a left R-module, then the function $\psi : R \to \mathrm{End}_{\mathbf{Z}}(M)$, given by $\psi(r) : m \mapsto rm$, is a representation.*

Proof. □

Definition 2.1.3. A functor $T : {}_R\mathbf{Mod} \to \mathbf{Ab}$ of either variance is called an *additive functor* if, for every pair of R-module homomorphisms $f, g : A \to B$, we have

$$T(f+g) = T(f) + T(g).$$

Lemma 2.1.4. *If $A, B \in \mathrm{obj}\,({}_R\mathbf{Mod})$, then the set $\mathrm{Hom}_R(A,B)$ is an abelian group. Moreover, if $p : A' \to A$ and $q : B \to B'$ are R-module homomorphisms, then*

$$(\varphi+\psi)p = \varphi p + \psi p \quad \textit{and} \quad q(\varphi+\psi) = q\varphi + q\psi.$$

Proof. Let $\varphi, \psi \in \mathrm{Hom}_R(A,B)$ and $r \in R$, $x, y \in A$. Observe that

$$\begin{aligned}(\varphi+\psi)(x+y) &= \varphi(x+y) + \psi(x+y)\\ &= \varphi(x) + \varphi(y) + \psi(x) + \psi(y)\\ &= \varphi(x) + \psi(x) + \varphi(y) + \psi(y)\\ &= (\varphi+\psi)(x) + (\varphi+\psi)(y).\end{aligned}$$

Hence $\varphi + \psi$ is an R-module homomorphism. The identity element of $\mathrm{Hom}_R(A,B)$ is the zero-map : $(\varphi + 0_{AB})(x) = \varphi(x) + 0_{AB}(x) = \varphi(x) + 0_B = \varphi(x)$ ($(0_{AB} + \varphi)(x)$ holds similarly). Given any R-module homomorphism φ, the inverse of φ is $-\varphi : x \mapsto -(\varphi(x))$: observe that $(\varphi + (-\varphi))(x) = \varphi(x) + (-\varphi)(x) = \varphi(x) - \varphi(x) = 0_B$. It is routine to show that addition is associative, and likewise $(\varphi+\psi)(x) = \varphi(x)+\psi(x) = \psi(x)+\varphi(x) = (\psi+\varphi)(x)$. Hence $\mathrm{Hom}_R(A,B)$ is an additive abelian group.

Let $a' \in A'$ and $b \in B$. Observe that

$$\begin{aligned}(\varphi + \psi)(p)(a') &= (\varphi + \psi)(p(a')) \\ &= \varphi(p(a')) + \psi(p(a')) \\ &= (\varphi p)(a') + (\psi p)(a') \\ &= (\varphi p + \psi p)(a')\end{aligned}$$

and

$$\begin{aligned}(q)(\varphi + g)(b) &= q((\varphi + g)(b)) \\ &= q(\varphi(b) + g(b)) \\ &= q(\varphi(b)) + q(g(b)) \\ &= (q\varphi)(b) + (q\psi)(b) \\ &= (q\varphi + q\psi)(b).\end{aligned}$$

This establishes the lemma. □

Proposition 2.1.5. *Let R be a ring, and let A, B, B' be left R-modules.*

(1) $\operatorname{Hom}_R(A, \square)$ *is an additive functor* ${}_R\mathbf{Mod} \to \mathbf{Ab}$.

Proof. Lemma 2.1.4 says that $\operatorname{Hom}_R(A, B)$ is an abelian group. This satisfies axiom (1). Define $\operatorname{Hom}_R(A, q) : \operatorname{Hom}_R(A, B) \to \operatorname{Hom}_R(A, B')$ by $f \mapsto qf$, this clearly satisfies axiom (2).Let $p : B' \to B''$ be an R-module homomorphism. We'd like to show that the following two functions are equivalent:

$$\operatorname{Hom}_R(A, pq) : \operatorname{Hom}_R(A, B) \to \operatorname{Hom}_R(A, B'')$$
$$\operatorname{Hom}_R(A, p)\operatorname{Hom}_R(A, q) : \operatorname{Hom}_R(A, B) \to \operatorname{Hom}_R(A, B'')$$

If $f \in \operatorname{Hom}_R(A, B)$, then $\operatorname{Hom}_R(A, pq) : f \mapsto (pq)f$. On the other hand, associativity of composition gives that $\operatorname{Hom}_R(A, p)\operatorname{Hom}_R(A, q) : f \mapsto qf \mapsto p(qf) = (pq)f$, as desired for axiom (3). If $1_B : B \to B$ is the identity map, then

$$\operatorname{Hom}_R(A, 1_B) : f \mapsto 1_B f = f$$

for all $f \in \operatorname{Hom}_R(A, B)$—hence $\operatorname{Hom}_R(A, 1_B) = 1_{\operatorname{Hom}_R(A,B)}$ satisfying axiom (4). Finally, Lemma 2.1.4 also showed that $\operatorname{Hom}_R(A, q)$ is additive: $\operatorname{Hom}_R(A, q) : f + g \mapsto q(f + g) = qf + qg$. This establishes the proposition. □

Proposition 2.1.6. *Let R be a ring and let $A, A', andB$ be left R-modules.*

(1) $\operatorname{Hom}_R(\square, B)$ *is an additive (contravariant) functor* ${}_R\mathbf{Mod} \to \mathbf{Ab}$.

Proof. Similar to the proof of Proposition 2.1.5 □

Proposition 2.1.7. *Let $T : {}_R\mathbf{Mod} \to \mathbf{Ab}$ be an additive functor of either varience.*

(1) If $0_{AB} : A \to B$ is the zero map, then $T(0) = 0$.

(2) $T(\{0\}) = \{0\}$

2.2 d and f

Throughout this section all rings contain a 1.

Definition 2.2.1.

(1) The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be *exact* (at Y) if $\operatorname{im}\alpha = \ker\beta$.

(2) A sequence $\ldots \to X_{n-1} \to X_n \to X_{n+1} \to \ldots$ of homomorphisms is said to be an *exact sequnce* if it is exact at every X_n between a pair of homomorphisms.

Proposition 2.2.2. *Let A, B, and C be R-modules over some ring R. Then*

(1) The sequence $0 \to A \xrightarrow{\psi} B$ is exact (at A) if and only if ψ is injective.

(2) The sequence $B \xrightarrow{\varphi} C \to 0$ is exact (at C) if and only if φ is surjective.

Proof. The (uniquely defined) homomorphism $0 \to A$ has image 0 in A. This will be the kernel of ψ if and only if ψ is injective.

Similarly, the kernel of the (uniquely defined) zero homomorphism $C \to 0$ is all of C, which is the image of φ if and only if φ is surjective. $\square$