

Math 310

Homework 5

Due: 10/9/2024

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Exercise 1. Let $x_1 = 1$ and inductively set $x_{n+1} = \sqrt{2 + x_n}$. Show that $(x_n)_n$ converges and find its limit.

Proof. Claim: $x_n \leq 2$ for all n . We prove this by induction on n . Clearly $x_1 = 1 \leq 2$. Now assume our hypothesis is true for n . For $n + 1$ we have:

$$\begin{aligned} x_{n+1} &= \sqrt{2 + x_n} \\ &\leq \sqrt{2 + 2} \\ &= 2. \end{aligned}$$

Claim: $x_n \leq x_{n+1}$. Observe that:

$$\begin{aligned} x_n \leq x_{n+1} &\iff x_n \leq \sqrt{2 + x_n} \\ &\iff x_n \leq \sqrt{2 + 2} \\ &\iff x_n \leq 2. \end{aligned}$$

Since $(x_n)_n$ is increasing and bounded above, by the monotone convergence theorem it has a limit, call it L . Thus:

$$\lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \sqrt{2 + x_n}$$

$$\iff$$

$$L = \sqrt{2 + L}$$

$$\iff$$

$$L^2 - L - 2 = 0$$

$$\iff$$

$$L = 2 \text{ or } L = -1.$$

So it must be the case that $(x_n)_n \rightarrow L$.

□

Exercise 2. Does the following sequence converge?

$$x_n := \sum_{k=n+1}^{2n} \frac{1}{k}.$$

Proof. Claim: $x_n \leq 1$ for all n . Observe that:

$$\begin{aligned}
 x_n &= \sum_{k=n+1}^{2n} \frac{1}{k} \\
 &= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n-2} + \frac{1}{2n-1} + \frac{1}{2n} \\
 &\leq \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} + \dots + \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} \\
 &= \frac{n+1}{n+1} \\
 &= 1.
 \end{aligned}$$

Claim: $x_n \leq x_{n+1}$. We have:

$$\begin{aligned}
 x_n \leq x_{n+1} &\iff \sum_{k=n+1}^{2n} \frac{1}{k} \leq \sum_{k=n+2}^{2n+2} \frac{1}{k} \\
 &\iff \frac{1}{n+1} \leq \frac{1}{2n+1} + \frac{1}{2n+1} \\
 &\iff \frac{1}{n+1} \leq \frac{4n+3}{(2n+1)(2n+1)} \\
 &\iff n+1 \geq \frac{4n^2+6n+2}{4n+3} \\
 &\iff 4n^2+7n+3 \geq 4n^2+6n+2 \\
 &\iff 7n+3 \geq 6n+2
 \end{aligned}$$

which is true for all $n \in \mathbf{N}$. Since $(x_n)_n$ is increasing and bounded above, by the monotone convergence theorem $(x_n)_n$ converges. \square

Exercise 3. Let $(f_n)_n$ denote the Fibonacci sequence and let

$$x_n := \frac{f_{n+1}}{f_n}.$$

Given that $(x_n)_n$ converges, find its limit.

Proof. Note that:

$$\begin{aligned}
 x_n &= \frac{f_{n+1}}{f_n} \\
 &= \frac{f_n + f_{n-1}}{f_n} \\
 &= 1 + \frac{f_{n-1}}{f_n} \\
 &= 1 + \frac{1}{f_n/f_{n-1}} \\
 &= 1 + \frac{1}{x_{n-1}}.
 \end{aligned}$$

If $(x_n)_n \rightarrow L$, then:

$$\begin{aligned}L = 1 + \frac{1}{L} &\iff L^2 = L + 1 \\&\iff L^2 - L - 1 = 0 \\&\iff L = \frac{1 \pm \sqrt{5}}{2}.\end{aligned}$$

Since $\frac{1-\sqrt{5}}{2} < 0$, it must be the case that $(x_n)_n \rightarrow \frac{1+\sqrt{5}}{2}$

□