# Contents



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# Chapter 1

# Normed Vector Spaces

## § 1.1. Vector Spaces

**Definition 1.1.1.** A *vector space* over a field $F$ is a nonempty set $V$ equipped with two operations:

$$V \times V \xrightarrow{+} V \text{ defined by } (v, w) \mapsto v + w$$
$$F \times V \to V \text{ defined by } (\alpha, v) \mapsto \alpha v$$

satisfying:

(1) $(V, +)$ is an abelian group;

(2) $\alpha(v + w) = \alpha v + \alpha w$ for all $\alpha \in F, v, w \in V$;

(3) $\alpha(\beta v) = (\alpha\beta)v$ for all $\alpha, \beta \in F, v \in V$;

(4) $1_F v = v$ for all $v \in V$.

**Definition 1.1.2.** Let $V$ be a vector spaces over $F$. A *subspace* is a nonempty set $W \subseteq V$ satisfying $w_1 + \alpha w_2 \in W$ for all $w_1, w_2 \in W$ and $\alpha \in F$.

**Exercise 1.1.1.** *If $\{W_i\}_{i \in I}$ is a family of subspaces of $V$, then $\bigcap_{i \in I} W_i$ is a subspace of $V$.*

**Exercise 1.1.2.** *If $W_1, W_2 \subseteq V$ are subspaces such that $W_1 \cup W_2$ is a subspace, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.*

**Definition 1.1.3.** Let $S \subseteq V$ be any subset of a vector space $V$. Then:

$$\operatorname{span}(S) = \left\{ \sum_{j=1}^{n} \alpha_j v_j \mid \alpha_j \in F, v_j \in S \right\}.$$

**Note 1.1.1.**

(1) $\operatorname{span}(S) \subseteq V$ is a subspace.

(2) $\operatorname{span}(S) = \bigcap W$, where $S \subseteq W$ and $W \subseteq V$ is a subspace. So $\operatorname{span}(S)$ is the "smallest" subspace containing $S$, or equivalently the subspace generated by $S$.

**Proposition 1.1.1** (Quotient Spaces)**.** *Let $V$ be a vector space and let $W \subseteq V$ be a subspace. Define $u \sim_W v$ if and only if $u - v \in W$.*

*(1)* $\sim_W$ *is an equivalent relation.*

*(2) If* $[v]_W$ *denotes the equivalence classes of* $v$*, then* $[v]_W = v + W = \{v + w \mid w \in W\}$*.*

*(3)* $V/W := \{[v]_W \mid v \in V\}$ *is a vector space with* $[v_1]_W + [v_2]_W = [v_1 + v_2]_W$ *and* $\alpha[v]_W = [\alpha v]_W$*.*

*Proof.* Exercise. □

**Definition 1.1.4.** Let $V$ be a vector space and $S \subseteq V$ be a subset.

(1) $S$ is said to *span* $V$ if $\mathrm{span}(S) = V$.

(2) $S$ is *linearly independent* if, for all $\alpha_j \in F$ and $v_j \in V$, $\sum_{j=1}^{n} \alpha_j v_j = 0_V$ implies $\alpha_j = 0_V$ for all $j$.

(3) $S$ is a *basis* for $V$ if $S$ is linearly independent and spans $V$.

**Proposition 1.1.2.** *Every vector space admits a basis. Moreover, if* $B_0 \subseteq V$ *is a linearly independent set, there exists* $B \subseteq V$ *such that* $B$ *is a basis and* $B_0 \subseteq B$*.*

*Proof.* Let $X = \{D \mid B_0 \subseteq D \subseteq V,\ \ D \text{ linearly independent}\}$. Define an ordering on $X$ as follows: given $D, E \in X$, we have $D \leqslant E$ if and only if $D \subseteq E$. We will show that $X$ admits a maximal element.

Note that $X$ is nonempty because $B_0 \in X$. Let $\{D_i\}_{i \in I}$ be a family of linearly indepedent sets satisfying $D_i \subseteq V$ for all $i$. Suppose $Y = (\{D_i\}_{i\in I}, \leqslant)$ is a totally ordered set. Consider $D = \bigcup_{i \in I} D_i$. Clearly $B_0 \subseteq D \subseteq V$. If $\sum_{j=1}^{n} \alpha_j v_j = 0_V$ with $v_1, ..., v_n \in D$, then since $Y$ is totally ordered, there exists $D_K$ with $v_1, ..., v_n \in D_k$. Since $D_k$ is linearly independent, we have that $\alpha_1 = ... = \alpha_n = 0$. Thus $D$ is linearly independent, whence $D \in X$. Furthermore, $D$ is clearly an uppoerbound of $Y$. By Zorn's Lemma, $X$ has a maximal element $B$.

Claim: $B$ is a basis for $V$. Suppose towards contradiction its not, that is, there exists $v \in V$ with $v \notin \mathrm{span}(B)$. Consider $B' = B \cup \{v\}$. Let $\sum_{j=1}^{n} \alpha_j v_j + \alpha v = 0_V$ with $v_1, ..., v_n \in B$. We proceed by cases.

Case 1: $\alpha \neq 0$. Then $\sum_{j=1}^{n} \alpha_j v_j = -\alpha v$, meaning $v \in \mathrm{span}(B)$. $\perp$

Case 2: $\alpha = 0$. Then $\alpha_1 = ... = \alpha_n = 0$. This gives that $B'$ is a linearly independent set, with $B' \in X$ and $B \subsetneq B'$. $\perp$ This contradicts the maximality of $B$.

Thus $\mathrm{span}(B) = V$, giving $B$ as a basis for $V$. □

**Example 1.1.1** (Examples of Vector Spaces)**.**

(1) The set of $n$-dimensional vectors; $F^n = \{(x_1, ..., x_n) \mid x_i \in F\}$ is a vector space by defining addition and scalar multiplication componentwise.

(2) The set of $m \times n$ matrices over a field; $M_{n,m}(F) = \{(a_{ij}) \mid a_{ij} \in F\}$ is a vector space by the usual matrix addition and scalar multiplication.

(3) The set of functions with domain $\Omega$; $\mathcal{F}(\Omega, F) = \{f \mid f : \Omega \to f\}$ is a vector space by defining addition and scalar multiplication pointwise.

(4) The set of bounded functions with domain $\Omega$; $\ell_\infty(\Omega, F) = \{f \in \mathcal{F}(\Omega, F) \mid \|f\|_u < \infty\}$ is a vector space by defining addition and scalar multiplication componentwise.

**Exercise 1.1.3.** *Show that* $(\ell_\infty, \|\cdot\|_u)$ *forms a metric space.*

(5) Continuous functions on a bounded domain:
$C([a, b], F) = \{f : [a, b] \to F \mid \text{ f continuous }\}$ by defining addition and scalar multiplication componentwise.

**Exercise 1.1.4.** *Show that* $C([a, b], F) \subseteq \ell_\infty([a, b], F)$ *is a subspace.*

(6) Let $f : [a, b] \to \mathbb{R}$ be any function. Let $\mathcal{P} = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$. The *variation of* $f$ *on* $\mathcal{P}$ is defined as:

$$\mathrm{Var}(f; \mathcal{P}) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|.$$

We say $f$ is a *bounded variation* if:

$$\mathrm{Var}(f) := \sup_{\mathcal{P}} \mathrm{Var}(f; \mathcal{P}) < \infty.$$

The set of all functions of bounded variation is defined:

$$\mathrm{BV}([a, b]) = \{f : [a, b] \to \mathbb{R} \mid \mathrm{Var}(f) < \infty\}.$$

This is a vector space by defining addition and scalar multiplication componentwise.

**Exercise 1.1.5.** *Show that BV([a,b])* $\subseteq \ell_\infty([a, b], \mathbb{R})$ *is a subspace.*

(7) A subset $K \subseteq V$ of a vector space is *convex* if, for all $v, w \in K$ and $t \in [0, 1]$, then $(1 - t)v + tw \in K$. A function $f : K \to \mathbb{R}$ is *affine* if, for all $v, w \in K$ and $t \in [0, 1]$, then $f((1-t)v+tw) = (1-t)f(v)+tf(w)$. The set of all affine functions over a convex subset $\mathrm{Aff}(K) = \{f : K \to \mathbb{R} \mid \text{ f affine }\}$ is a vector space by defining addition and scalar multiplication componentwise.

**Exercise 1.1.6.** *Show that Aff*$(K) \subseteq \mathcal{F}(K, \mathbb{R})$ *is a subspace.*

(8) By (3), the set of all sequences $\mathcal{F}(\mathbb{N}, F) = \{(a_k)_k \mid a_k \in F\}$ is a vector space. Define:

$$\begin{aligned} c_{00} &= \{(a_k)_k \mid \operatorname{supp}((a_k)_k) < \infty\} \\ c_0 &= \{(a_k)_k \mid (a_k)_k \to 0\} \\ c &= \{(a_k)_k \mid (a_k)_k \text{ converges}\} \\ \ell_\infty(\mathbb{N}, F) &= \{(a_k)_k \mid \|(a_k)_k\|_u < \infty\} \\ \ell_1(\mathbb{N}, F) &= \left\{(a_k)_k \;\middle|\; \sum_{k=1}^{\infty} |a_k| < \infty\right\}. \end{aligned}$$

These are similarly vector spaces with addition and scalar mutliplcation defined componentwise.

**Exercise 1.1.7.** *Show that the above vector spaces are subspaces of* $\mathcal{F}(\mathbb{N}, F)$.

(9) Recall that a function $f : \mathbb{R} \to F$ is *compactly supported* if there exists $[a, b] \subseteq \mathbb{R}$ such that $x \notin [a, b]$ implies $f(x) = 0$. Then $C_c(\mathbb{R}) := \{f : \mathbb{R} \to F \mid f \text{ compactly supported}\}$ is a vector space with addition and scalar multiplication defined pointwise.

(10) The set of functions which vanish at infinity;
$C_0(\mathbb{R}) = \{f : \mathbb{R} \to F \mid f \text{ continuous}, \ \operatorname{limit}_{x\to\pm\infty} f = 0\}$ is a vector space with addition and scalar multiplication defined pointwise.

**Exercise 1.1.8.** *Show that* $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ *are subspaces.*

(11) Let $\Gamma$ be a nonempty set. The free vector space $\mathbb{F}(\Gamma) = \{f : \Gamma \to F \mid \operatorname{supp}(f) < \infty\}$ is a vector space with addition and scalar multiplication defined pointwise.

Fix $t \in \Gamma$. Recall that $\delta_t : \Gamma \to F$ is defined by:

$$\delta_t(s) = \begin{cases} 1, & s = t \\ 0, & s \neq t \end{cases}.$$

We have that $\delta_t \in \mathcal{F}(\Gamma, F)$, and furthermore $\operatorname{supp}(\delta_t) = \{t\}$. If $f \in \mathcal{F}(\Gamma, F)$ has finite support, then $\operatorname{supp}(f) = \{t_1, ..., t_n\}$ for some $t_i \in F$. If $f(t_i) \neq 0$ for all $1 \leqslant i \leqslant n$, then we can write $f = \sum_{j=1}^{n} f(t_j)\delta_{t_j}$.

Define $\iota : \Gamma \to \mathbb{F}(\Gamma)$ by $\iota(x) = \delta_x$. We have the following universal property: if $W$ is any vector space and $t : \Gamma \to W$ is a map of sets, there is a unique $T \in \operatorname{Hom}_F(\mathbb{F}(\Gamma), W)$ such that $T(x) = t(x)$ for every $x \in \Gamma$; i.e., the following diagram commutes:

$$\begin{tikzcd} \Gamma \arrow[r, hook, "\iota"] \arrow[dr, "t"'] & \mathbb{F}(\Gamma) \arrow[d, "T"] \\ & W \end{tikzcd}.$$

**Exercise 1.1.9.**

(1) *Show that* $\mathbb{F}(\Gamma) \subseteq \mathcal{F}(\Gamma, \mathrm{F})$ *is a subspace.*

(2) *Show that* $\{\delta_t\}_{t\in\Gamma}$ *is a basis for* $\mathbb{F}(\Gamma)$.

(3) *Prove the above universal property.*

(4) *Suppose* $V$ *is a vector space over* $\mathrm{F}$ *with basis* $B$. *Show that* $\mathbb{F}(B) \cong V$.

## § 1.2. Normed Spaces

**Definition 1.2.1.** A *norm* on a vector space V is a map:

$$\|\cdot\| : V \to \mathbb{R}^+ \text{ defined by } v \mapsto \|v\| \geqslant 0.$$

satisfying:

(1) $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in F, v \in V$ (homogeneity);

(2) $\|v + w\| \leqslant \|v\| + \|w\|$ for all $v, w \in V$ (triangle inequality);

(3) If $\|v\| = 0$, then $v = 0_V$ (positive definiteness).

If $\|\cdot\|$ satisifes only (1) and (2), then we say it is a *seminorm*. The pair $(V, \|\cdot\|)$ is called a *normed space*.

**Definition 1.2.2.** Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called *equivalent* if there exists $\alpha, \beta \geqslant 0$ satisfying:

$$\|v\|_1 \leqslant \alpha\|v\|_2$$
$$\|v\|_2 \leqslant \beta\|v\|_1$$

for all $v \in V$.

**Note 1.2.1.** On $\mathbb{R}^n$, all norms are equivalent.

**Exercise 1.2.1.** *Let* $v, w \in V$. *If* $p$ *is any seminorm on* $V$, *then* $|p(v) - p(w)| \leqslant p(v - w)$.

**Definition 1.2.3.** Let V be any normed space.

(1) The *open ball of radius* $r$ is denoted $U_V = \{v \in V \mid \|v\| < r\}$.

(2) The *closed ball of radius* $r$ is denoted $B_V = \{v \in V \mid \|v\| \leqslant r\}$.

**Example 1.2.1** (Examples of Norms)**.** Given $V = F^n$ and $x = (x_1, ..., x_n)$, we have the following norms:

(1) $\|x\|_1 = \sum_{j=1}^{n} |x_j|$;

(2) $\|x\|_\infty = \max\limits_{1 \leqslant j \leqslant n} |x_j|$;

(3) $\|x\|_2 = \left( \sum\limits_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}}$.

**Exercise 1.2.2.** *Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms.*

**Lemma 1.2.1.** *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. Then $f(t) \geqslant 0$ for $t \geqslant 0$.*

*Proof.* Note that $f'(t) = t^{p-1} - 1$. Since:

$$\begin{aligned} &f'(1) = 0 \\ &f'(t) > 0 \;\text{ for }\; t > 1 \\ &f'(t) < 0 \;\text{ for }\; 0 \leqslant t < 1, \end{aligned}$$

we can see that $f(t) \geqslant 0$ for all $t \geqslant 0$. □

**Lemma 1.2.2.** *Let $p, q \in [0, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $x, y \geqslant 0$, then $xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q$.*

*Proof.* By Lemma 1.2.1, $t \leqslant \frac{1}{p}t^p + \frac{1}{q}$. Multiplying both sides by $y^q$ gives:

$$ty^q \leqslant \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Let $t = xy^{1-q}$. Then:

$$xy^{1-q}y^q \leqslant \frac{1}{p}x^p y^{p-pq} y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have that $p - pq = -q$. Whence:

$$xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q.$$

□

**Definition 1.2.4.** Let $V = F^n$, $x = (x_1, ..., x_n)$, and $p \geqslant 1$. We define:

$$\|x\|_p := \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}}.$$

**Lemma 1.2.3** (Hölders Inequality)**.** *Let $p, q \in [0, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for $x, y \in F^n$:*

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leqslant \|x\|_p \|y\|_q.$$

*Proof.* We proceed by cases.

Case 1: $p = 1$. Then:

$$\begin{aligned} \left|\sum_{j=1}^{n} x_j y_j\right| &\leqslant \sum_{j=1}^{n} |x_j||y_j| \\ &\leqslant \sum_{j=1}^{n} |x_j|\|y\|_\infty \\ &= \|x\|_1\|y\|_\infty. \end{aligned}$$

Case 2: $p = \infty$. This follows similarly to Case 1.

Case 3: $1 < p < \infty$. Suppose $\|x\|_p = \|y\|_q = 1$. Then:

$$\begin{aligned} \left|\sum_{j=1}^{n} x_j y_j\right| &\leqslant \sum_{j=1}^{n} |x_j||y_j| \\ &\leqslant \sum_{j=1}^{n} \left(\frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q\right) \\ &= \frac{1}{p}\left(\sum_{j=1}^{n} |x_j|^p\right) + \frac{1}{q}\left(\sum_{j=1}^{n} |y_j|^q\right) \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

Whence the inequality holds. Now suppose $\|v\|_p = 0$ or $\|y\|_q = 0$. Then $x = 0_{F^n}$ or $y = 0_{F^n}$, whence the inequality holds. Suppose $\|x\|_p \neq 0$ and $\|y\|_p \neq 0$. Set:

$$\begin{aligned} x' &= \frac{x}{\|x\|_p} \\ y' &= \frac{y}{\|y\|_p}. \end{aligned}$$

Then $\|x'\|_p = 1 = \|y'\|_p$. Observe that:

$$\begin{aligned} 1 &\geqslant \left|\sum_{j=1}^{n} x'_j y'_j\right| \\ &= \left|\sum_{j=1}^{n} \frac{x}{\|x\|_p}\frac{y}{\|y\|_p}\right|. \end{aligned}$$

Multiplying both sides by $\|x\|_p\|y\|_q$ gives the desired result. □

**Lemma 1.2.4** (Minkowski's Inequality). *Let* $p, q \in [1, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $x, y \in F^n$*:*

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p.$$

*Proof.* The only nontrivial case is for $1 < p < \infty$. Observe that:

$$\begin{aligned}
\left(\|x+y\|_p\right)^p &= \sum_{j=1}^{n} |x_j + y_j|^p \\
&= \sum_{j=1}^{n} |x_j + y_j||x_j + y_j|^{p-1} \\
&\leqslant \sum_{j=1}^{n} |x_j||x_j + y_j|^{p-1} + |y_j||x_j + y_j|^{p-1} \\
&\leqslant \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{p-1\left(\frac{p}{p-1}\right)}\right)^{1-\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{p-1\left(\frac{p}{p-1}\right)}\right)^{1-\frac{1}{p}} \\
&= \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{p}\right)^{1-\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{p}\right)^{1-\frac{1}{p}} \\
&= (\|x\|_p + \|y\|_p)\frac{\|x+y\|_p^p}{\|x+y\|_p}.
\end{aligned}$$

Multiplying boths sides by $\frac{\|x+y\|_p}{\|x+y\|_p^p}$ gives the desired inequality. ☐

**Theorem 1.2.5.** *Let* $V = F^n$. *Then* $(F^n, \|\cdot\|_p)$ *is a normed space. In particular,* $\|\cdot\|_p$ *is a norm.*

*Proof.* Let $x = (x_1, ..., x_n) \in F^n$ and $\alpha \in F$. Observe that:

$$\begin{aligned}
\|\alpha x\|_p &= \left(\sum_{j=1}^{n} |\alpha x_j|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^{n} |\alpha|^p |x_j|^p\right)^{\frac{1}{p}} \\
&= |\alpha|\|x\|_p.
\end{aligned}$$

This satisfies homogeneity. Moreover, Minkowski's Inequality satisifes the triangle inequality. It remains to show that $\|\cdot\|_p$ is positive-definite. If $\|x\|_p = 0$, then $x_j = 0$ for all $1 \leqslant j \leqslant n$. Thus $x = 0_{F^n}$. ☐

**Example 1.2.2** (Examples of Normed Spaces)**.**

(1) $(\ell_\infty(\Omega, F), \|\cdot\|_u)$ is a normed space. Moreover, subspaces of $\ell_\infty(\Omega, F)$ inherit the norm, such as $C([a, b], F)$.

(2) Let $f \in C([a, b])$. Define:

$$\|f\|_1 = \int_a^b |f(x)| dx.$$

Then $(C([a, b]), \|\cdot\|_1)$ is a normed space.

(3) Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Given $T \in \mathrm{Hom}_F(V, W)$, define:

$$\|T\|_{op} := \sup_{\|v\|_V \leqslant 1} \|T(v)\|_W.$$

Informally, this measures the maximum factor by which T "lengthens" a vector. If $\|T\|_{op} < \infty$, we say it is a *bounded linear operator*. The space of bounded linear operators is denoted:

$$B_F(V, W) = \{T \in \mathrm{Hom}_F(V, W) \mid \|T\|_{op} < \infty\}.$$

Then $(B_F(V, W), \|\cdot\|_{op})$ is a normed space.

**Exercise 1.2.3.** *Show that* $\mathrm{Hom}_F(F^n, F^m) = B_F(\ell_2^n, \ell_2^m)$.

# § 1.3. Inner Product Spaces

**Definition 1.3.1.** Let V be a vector space over F and $\varphi : V \times V \to F$ a map.

(1) The map $\varphi$ is said to be a *bilinear form* if is is linear in the first and second variable seperately; i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have:

   (i) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$

   (ii) $\varphi(v, cv_1 + v_2) = \varphi(v, v_1) + c\varphi(v, v_2)$.

(2) The map $\varphi$ is said to be a *sesquilinear form* if it is linear in the first variable and conjugate linear in the second variable; i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have:

   (i) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$

   (ii) $\varphi(v, cv_1 + v_2) = \overline{c}\varphi(v, v_1) + \varphi(v, v_2)$.

If we wish to keep track of a bilinear form on V we write $(V, \varphi)$.

**Definition 1.3.2.** Let V be a vector space over F.

(1) A bilinear form $\varphi$ on V is said to be *symmetric* if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

(2) A sesquilinear form $\varphi$ on $V$ is said to be *Hermitian* if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.

**Definition 1.3.3.** Let $(V, \varphi)$ be a vector space over $F$ such that if $\varphi$ is symmetric, then $\mathbb{Q} \subset F \subset \mathbb{R}$ **or** if $\varphi$ is Hermitian, then $\mathbb{Q} \subset F \subset \mathbb{C}$. We say $\varphi$ is *positive definite* if $\varphi(v, v) > 0$ for all nonzero $v \in V$.

**Definition 1.3.4.** Let $(V, \varphi)$ be a vector space over $\mathbb{R}$ with $\varphi$ a positive-definite symmetric bilinear form **or** $\mathbb{C}$ with $\varphi$ a positive-definite Hermitian sesquilinear form. Then we say $\varphi$ is an *inner product* on $V$ and write $\varphi$ as $\langle \cdot, \cdot \rangle$. We say $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space*.

**Proposition 1.3.1.** *Every inner product space is a normed vector space, with its canonical norm defined as:*

$$\|v\| := \sqrt{\langle v, v \rangle}.$$

*In particular, for all $v, w \in V$ and $\alpha \in F$ we have:*

*(1)* $\|\alpha v\| = |\alpha| \, \|v\|$;

*(2)* $|\langle v, w \rangle| \leqslant \|v\| \, \|w\|$ *(Cauchy-Schwartz Inequality);*

*(3)* $\|v + w\| \leqslant \|v\| + \|w\|$;

*(4) if* $\|v\| = 0$ *then* $v = 0_V$.

*Proof.* □

**Example 1.3.1.**

(1) Note that $\ell_2^n = F^n$. Then $\langle \cdot, \cdot \rangle : \ell_2^n \times \ell_2^n \to F$ given by

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$$

is an inner product.

(2) Recall that $\ell_2 = \{(a_j)_j \in F^{\mathbb{N}} \mid \sum_{j=1}^{\infty} |a_j|^2 < \infty\}$. Consider $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to F$ given by:

$$\langle (a_j)_j, (b_j)_j \rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}.$$

For finite $n$, the Cauchy-Schwartz inequality gives:

$$\begin{aligned} \left| \sum_{j=1}^{n} a_j \overline{b_j} \right| &\leqslant \left( \sum_{j=1}^{n} a_j \overline{a_j} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{n} b_j \overline{b_j} \right)^{\frac{1}{2}} \\ &= \left( \sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{n} |b_j|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Taking the limit as $n$ approaches infinity yields:

$$\begin{aligned}\left|\sum_{j=1}^{\infty} a_j \overline{b_j}\right| &\leqslant \left(\sum_{j=1}^{\infty} |a_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} |b_j|^2\right)^{\frac{1}{2}} \\ &< \infty.\end{aligned}$$

Thus $\langle (a_j)_j, (b_j)_j \rangle$ is always convergent.

# Chapter 2

# Metric Spaces

## § 2.1. Introduction

**Definition 2.1.1.** Let $X$ be a nonempty set. A *metric* on $X$ is a map:

$$d : X \times X \to \mathbb{R}^+$$

satisfying for all $x, y, z \in X$:

(1) $d(x, y) = d(y, x)$;

(2) $d(x, z) \leqslant d(x, y) + d(y, z)$;

(3) $d(x, x) = 0$;

(4) If $d(x, y) = 0$, then $x = y$.

If $d$ only satisfies (1), (2), and (3), then $d$ is called a *semi-metric*. We call the pair $(X, d)$ a *metric space* (or *semi-metric space*).

**Definition 2.1.2.** Two metrics $d, \rho$ on $X$ are called *equivalent* if there exists $c_1, c_2 \geqslant 0$ such that, for all $x, y \in X$:

$$\begin{aligned} d(x, y) &\leqslant c_1 \rho(x, y); \\ \rho(x, y) &\leqslant c_2 d(x, y). \end{aligned}$$