Math 310

Homework 4

Due: 10/9/2024

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Exercise 1. *Prove the following limits:*

(1) $\left(\frac{2n}{n+2}\right)_n \to 2.$

Proof. Let $\epsilon > 0$. There exists $N_\epsilon \in \mathbf{N}$ such that $N_\epsilon > \frac{2}{\epsilon} - 1$. If $n \geqslant N_\epsilon$, then $n > \frac{2}{\epsilon} - 1$ gives:

$$\begin{aligned} \frac{4}{\epsilon} < n+1 &\implies \frac{4}{n+1} < \epsilon \\ &\implies \frac{|2n-2n-4|}{n+1} < \epsilon \\ &\implies \left|\frac{2n-2(n+1)}{n+2}\right| < \epsilon \\ &\implies \left|\frac{2n}{n+2} - 2\right| < \epsilon. \end{aligned}$$

□

(2) $\left(\frac{\sqrt{n}}{n+1}\right)_n \to 0.$

Proof. Observe that:

$$\left|\frac{\sqrt{n}}{n+1}\right| \leqslant \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

Claim: $\left(\frac{1}{\sqrt{n}}\right)_n \to 0$. Let $\epsilon > 0$. There exists $N_\epsilon \in \mathbf{N}$ such that $\frac{1}{\epsilon^2} < N_\epsilon$. If $n \geqslant N_\epsilon$, then $n > \frac{1}{\epsilon^2}$ gives:

$$\begin{aligned} \frac{1}{\epsilon^2} < n &\implies \frac{1}{n} < \epsilon^2 \\ &\implies \frac{1}{\sqrt{n}} < \epsilon \\ &\implies \left|\frac{1}{\sqrt{n}}\right| < \epsilon. \end{aligned}$$

Since $\left(\frac{1}{\sqrt{n}}\right)_n \to 0$, by "Lemma" $\left(\frac{\sqrt{n}}{n+1}\right)_n \to 0$. □

(3) $\left(\frac{(-1)^n}{\sqrt{n+7}}\right)_n \to 0.$

Proof. We have:

$$\left|\frac{(-1)^n}{\sqrt{n+7}}\right| = \frac{1}{\sqrt{n+7}} \leqslant \frac{1}{\sqrt{n}}.$$

Since $\left(\frac{1}{\sqrt{n}}\right)_n \to 0$, by "Lemma" $\left(\frac{(-1)^n}{\sqrt{n+7}}\right)_n \to 0$. □

(4) $(n^k b^n)_n \longrightarrow 0$, *where* $0 \leqslant b < 1$ *and* $k \in \mathbf{N}$.

Proof. We proceed by using the ratio test. Claim: $\left(\left|\frac{(n+1)^k b^{n+1}}{n^k b^n}\right|\right)_n \longrightarrow b$. We have:

$$\begin{aligned}\left|\frac{(n+1)^k b^{n+1}}{n^k b^n} - b\right| &= \left|\frac{\left((n+1)^k - n^k\right) b}{n^k}\right| \\ &= b \cdot \frac{(n+1)^k - n^k}{n^k} \\ &= b\left(\left(\frac{n+1}{n}\right)^k - 1\right) \\ &= b\left(\left(1+\frac{1}{n}\right)^k - 1\right).\end{aligned}$$

Since $(\frac{1}{n})_n \longrightarrow 0$, $\epsilon_n = \left(\left(1+\frac{1}{n}\right)^k - 1\right)_n \longrightarrow 0$. Thus by "Lemma", $\left(\left|\frac{(n+1)^k b^{n+1}}{n^k b^n}\right|\right)_n \longrightarrow b$. Since $0 \leqslant b < 1$, by the ratio test $(n^k b^n)_n \longrightarrow 0$. □

(5) $\left(\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \longrightarrow 3$.

Proof. Observe that:

$$\begin{aligned}\left|\frac{2^{n+1}+3^{n+1}}{2^n+3^n} - 3\right| &= \left|\frac{2^{n+1}+3^{n+1} - 3(2^n+3^n)}{2^n+3^n}\right| \\ &= \left|\frac{2\cdot 2^n - 3\cdot 2^n}{2^n+3^n}\right| \\ &= \frac{2^n}{2^n+3^n} \\ &\leqslant \frac{2^n}{3^n} \\ &= \left(\frac{2}{3}\right)^n.\end{aligned}$$

Since $\left(\left(\frac{2}{3}\right)^n\right)_n \longrightarrow 0$, by "Lemma" $\left(\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \longrightarrow 3$. □

Exercise 2. *Show that the sequence* $(\cos(n))_n$ *does not converge.*

Proof. Suppose towards contradiction that $(\cos(n))_n \longrightarrow L$. Then $(\cos(2n))_n \longrightarrow L$ and $(\cos(4n))_n \longrightarrow L$. Observe that:

$$\begin{aligned}(\cos(2n))_n \longrightarrow L &\iff \lim_{n\to\infty} cos(2n) = \lim_{n\to\infty} 2\cos^2(n) - 1 \\ &\iff L = 2L^2 - 1 \\ &\iff L = -\frac{1}{2} \text{ or } L = 1.\end{aligned}$$

$$\begin{aligned}(\cos(4n))_n \longrightarrow L &\iff \lim_{n\to\infty} cos(4n) = \lim_{n\to\infty} 8\cos^4(n) - 8\cos^2(n) - 1 \\ &\iff L = 8L^4 - 8L^2 - 1 \\ &\iff L = -1 \text{ or } L \approx 1.1028.\end{aligned}$$

This is a contradiction, since $(\cos(2n))_n \neq (\cos(4n))_n$. Hence $(cos(n))_n$ does not converge. □

Exercise 3. *If $(x_n)_n$ is a real sequence converging to x, show that*

$$(|x_n|)_n \longrightarrow |x|.$$

Is the converse true?

Proof. Since $(x_n)_n \to x$ is a convergent sequence, we have:

$$||x_n| - |x|| \leqslant |x_n - x| < \epsilon.$$

Thus $(|x_n|)_n \longrightarrow |x|$. Note that the converse is not true: $(|(-1)^n|)_n \longrightarrow 1$ converges whereas $((-1)^n)_n$ does not. □

Exercise 4. *If $(x_n)_n$ is a real sequence converging to $x > 0$, show that there is an $N \in \mathbf{N}$ and $c > 0$ such that*

$$x_n \geqslant c$$

for all $n \geqslant N$.

Proof. Pick $\epsilon = \frac{x}{2}$. Since $(x_n)_n$ is a convergent sequence, there exists $N_c \in \mathbf{N}$ such that $n \geqslant N_c$ implies $|x_n - x| < \frac{x}{2}$. Simplifying yields $\frac{x}{2} < x_n < \frac{3x}{2}$. Taking $c = \frac{x}{2}$ yields the desired result. □

Exercise 5. *If $(x_n)_n$ is a real sequence of positive terms converging to x, show that $x \geqslant 0$ and*

$$(\sqrt{x_n})_n \longrightarrow \sqrt{x}.$$

Proof. Observe that:

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leqslant \left|\sqrt{x_n} - \sqrt{x}\right|\left|\sqrt{x_n} + \sqrt{x}\right| = |x_n - x| < \epsilon.$$

Hence $(\sqrt{x_n})_n \longrightarrow \sqrt{x}$. If $x < 0$, then $\sqrt{x} \notin \mathbf{R}$, contradicting the definition of a real sequence. □

Exercise 6. *If $(x_n)_n$ and $(y_n)_n$ are sequences with $(x_n)_n \longrightarrow 0$ and $(y_n)_n$ bounded, show that*

$$(x_n y_n)_n \longrightarrow 0.$$

Proof. Since $(y_n)_n$ is bounded, $|y_n| \leqslant c$ for some $c > 0$. We have:

$$|x_n y_n| \leqslant c|x_n|.$$

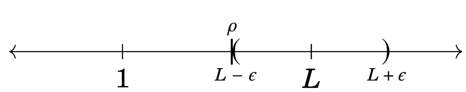
Taking $\epsilon_n = |x_n|$ and using "Lemma" gives $(x_n y_n)_n \longrightarrow 0$. □

Exercise 7. *If $(x_n)_n$ is a sequence of positive terms such that*

$$\left(\frac{x_{n+1}}{x_n}\right)_n \longrightarrow L > 1,$$

show that $(x_n)_n$ is not bounded hence not convergent. If $L = 1$, can we make any conclusion?

Proof. Consider the following picture:



Since $\left(\frac{x_{n+1}}{x_n}\right)_n \to L$, we know there exists some $N \in \mathbf{N}$ such that $n \geqslant N$ implies $L - \epsilon < \frac{x_{n+1}}{x_n} < L + \epsilon$. Pick $\rho = L - \epsilon > 1$, then $\frac{x_{n+1}}{x_n} \geqslant \rho$. This gives $x_{n+1} \geqslant \rho x_n$. But inductively we have that:

$$\begin{aligned} x_{N+1} &\geqslant \rho x_N \\ x_{N+2} &\geqslant \rho x_{N+1} \geqslant \rho^2 x_N \\ &\vdots \\ x_{N+n} &\geqslant \rho^n x_N. \end{aligned}$$

Note that x_{N+n} is a tail of $(x_n)_n$, and since $(\rho^n)_n \to +\infty$, it must be the case that $(x_n)_n \to +\infty$.

Now consider

$$\begin{aligned} (n)_n \to +\infty, \quad & \left(\frac{n+1}{n}\right)_n \to 1, \\ \left(\frac{1}{n}\right)_n \to 0, \quad & \left(\frac{n}{n+1}\right)_n \to 1. \end{aligned}$$

Hence if $L = 1$, we cannot make any conclusion. □

Exercise 8. *Let a and b be positive numbers. Show that*

$$\left((a^n + b^n)^{\frac{1}{n}}\right)_n \to \max\{a, b\}.$$

Proof. Case 1: $\max\{a, b\} = a$. Then $b < a$. We have:

$$\begin{aligned} & (a^n)^{\frac{1}{n}} \leqslant (a^n + b^n)^{\frac{1}{n}} \leqslant (2a^n)^{\frac{1}{n}} \\ \implies & a \leqslant (a^n + b^n)^{\frac{1}{n}} \leqslant (2^{\frac{1}{n}})a. \end{aligned}$$

Hence $\left((a^n + b^n)^{\frac{1}{n}}\right)_n \to a$ by the squeeze theorem. Case 2: $\max\{a, b\} = b$. Then $a < b$. We have:

$$\begin{aligned} & (b^n)^{\frac{1}{n}} \leqslant (a^n + b^n)^{\frac{1}{n}} \leqslant (2b^n)^{\frac{1}{n}} \\ \implies & b \leqslant (a^n + b^n)^{\frac{1}{n}} \leqslant (2^{\frac{1}{n}})b. \end{aligned}$$

Hence $\left((a^n + b^n)^{\frac{1}{n}}\right)_n \to b$ by the squeeze theorem. □

Exercise 9. *Let $(x_n)_n$ be a sequence of positive terms such that:*

$$(x_n^{1/n})_n \to L < 1.$$

Prove that $(x_n)_n \to 0$. If $L = 1$ can we make any conclusion? What about $L > 1$?

Proof. Since $(x_n^{1/n})_n$ is a convergent sequence, we have that $L - \epsilon < x_n^{1/n} < L + \epsilon$.

Case 1: $L < 1$. Then $\rho := L + \epsilon < 1$. Hence $x_n^{1/n} < \rho$; i.e., $x_n = |x_n| < \rho^n$. Since $(\rho^n)_n \to 0$, we have that $(x_n)_n \to 0$.

Case 2: $L > 1$. Then $\rho := L - \epsilon > 1$. Hence $x_n^{1/n} \geqslant \rho$; i.e., $x_n \geqslant \rho^n$. Since $(\rho^n)_n \to +\infty$, we have that $(x_n^{1/n})_n \to +\infty$.

Case 3: $L = 1$. Observe that:

$$\begin{aligned} & (a)_n \to a, \quad (a^{1/n})_n \to 1 \text{ for some } a > 1, \\ & (n)_n \to +\infty, \quad (n^{1/n})_n \to 1. \end{aligned}$$

Therefore we cannot make any conclusion if $L = 1$. □