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# Chapter 1

## Vector Spaces, Algebras, and Normed Spaces

For the entirety of this chapter assume  $F$  to be  $\mathbf{R}$  or  $\mathbf{C}$ .

### § 1.1. Vector Spaces

**Definition 1.1.1.** A *vector space* (or *linear space*) over  $F$  is a nonempty set  $V$  equipped with two operations:

$$\begin{aligned} V \times V &\xrightarrow{+} V \text{ defined by } (v, w) \mapsto v + w \\ F \times V &\rightarrow V \text{ defined by } (\alpha, v) \mapsto \alpha v \end{aligned}$$

satisfying:

(1)  $(V, +)$  is an abelian group:

- (i)  $u + (v + w) = (u + v) + w$  for all  $u, v, w \in V$ ;
- (ii) there exists  $0_V$  such that  $v + 0_V = 0_V + v = v$  for all  $v \in V$ ;
- (iii) for all  $v \in V$ , there exists  $w \in V$  satisfying  $v + w = w + v = 0_V$ ;
- (iv)  $v + w = w + v$  for all  $v, w \in V$ ;

(2)  $\alpha(v + w) = \alpha v + \alpha w$  for all  $\alpha \in F, v, w \in V$ ;

(3)  $\alpha(\beta v) = (\alpha\beta)v$  for all  $\alpha, \beta \in F, v \in V$ ;

(4)  $1_F v = v$  for all  $v \in V$ .

It can be shown that the vector  $0_V$  is unique, the additive inverse in (iii) is unique (which we denote as  $-v$ ), that  $0v = 0_V$ , and  $(-1)v = -v$ .

**Exercise 1.1.1.** Show (iv) follows from the other axioms.

**Exercise 1.1.2.** Show  $nv = \underbrace{v + v + \dots + v}_{n \text{ times}}$  for  $n \in \mathbf{Z}_{\geq 1}$ .

It can be shown that a subspace is a vector space in its own right.

**Example 1.1.1.** Let  $\{W_i\}_{i \in I}$  be a family of vector spaces. Then  $\bigcap_{i \in I} W_i$  is also a vector space.

**Example 1.1.2.** Planes and lines through the origin are subspaces of  $\mathbf{R}^3$ .

**Definition 1.1.2.** Let  $V$  be a vector space and  $S \subseteq V$  a subset.

- (1) A *linear combination* from  $S$  is a finite sum  $\sum_{j=1}^n \alpha_j v_j$  with  $\alpha_j \in F$ ,  $v_j \in S$ .
- (2) The *linear span* of  $S$  is:

$$\text{span}(S) := \left\{ \sum_{j=1}^n \alpha_j v_j \mid n \in \mathbf{N}, \alpha_j \in F, v_j \in S \right\}.$$

**Exercise 1.1.3.** Show that  $\text{span}(S) \subseteq V$  is a subspace and:

$$\text{span}(S) = \bigcap \{W \mid S \subseteq W, W \text{ is a subspace}\},$$

that is,  $\text{span}(S)$  is the smallest subspace of  $V$  containing  $S$ .

**Definition 1.1.3.** Let  $V$  be a vector space and  $S \subseteq V$  a subset.

- (1)  $S$  is *spanning* for  $V$  if  $\text{span}(S) = V$ .
- (2)  $S$  is *independent* if, given  $n \in \mathbf{N}$ ,  $\alpha_1, \dots, \alpha_n \in F$ ,  $v_1, \dots, v_n \in S$ , then  $\sum_{j=1}^n \alpha_j v_j = 0$  implies  $\alpha_j = 0$  for all  $j$ .

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Our goal is to show that every vector space admits a basis. As such, recall the following definitions from a standard course in Real Analysis.

**Definition 1.1.4.** An *ordering* on a set  $X$  is a relation  $R \subseteq X \times X$  on  $X$  that is reflexive, transitive, and antisymmetric. We write  $xRy$  as  $x \leq_R y$ . The pair  $(X, \leq_R)$  is called an *ordered set*. An ordering  $\leq$  on  $X$  is called *total* (or *linear*) if for all  $x, y \in X$ ,  $x \leq y$  or  $y \leq x$ .

Note that if  $(X, \leq)$  is an ordered set and  $Y \subseteq X$  is a subset, then  $(Y, \leq)$  is an ordered set as well.

**Definition 1.1.5.** Let  $(X, \leq)$  be an ordered set and  $Y \subseteq X$ . An *upper bound* for  $Y$  is an element  $u \in X$  with  $u \geq y$  for all  $y \in Y$ . An element  $m \in X$  is called *maximal* if  $x \in X$ ,  $x \geq m$  implies  $x = m$ .

**Lemma 1.1.1** (Zorn's Lemma). *Let  $(X, \leq_X)$  be an ordered set. Suppose every subset  $Y \subseteq X$  for which  $(Y, \leq_X)$  is totally ordered has an upper bound in  $X$ . Then  $X$  admits a maximal element.*

The proof of Zorn's Lemma is outside the interest of this text.

**Theorem 1.1.2.** *Every vector space admits a basis. Moreover, every independent set is contained in a basis.*

*Proof.* Let  $S \subseteq V$  be linearly independent. Define:

$$\mathfrak{T}(S) = \{T \subseteq V \mid S \subseteq T, T \text{ linearly independent}\}.$$

Let  $\mathfrak{C} \subseteq \mathfrak{T}(S)$  be a totally ordered subset. Set  $R = \bigcup_{T \in \mathfrak{C}} T$ . Clearly  $R \supseteq S$ . Assume  $\sum_{j=1}^n \alpha_j v_j = 0$ , where  $\alpha_j \in F$  and  $v_j \in R$ . Since  $\mathfrak{C}$  is totally ordered, there exists  $T_0 \in \mathfrak{C}$  with  $v_j \in T_0$  for all  $j = 1, \dots, n$ . Since  $T_0$  is independent,  $\alpha_j = 0$  for all  $j = 1, \dots, n$ . Thus  $R$  is independent as well. Whence  $R$  is an upper bound for  $\mathfrak{C}$ . By Zorn's Lemma,  $\mathfrak{T}(S)$  admits a maximal element, call it  $B$ .

Claim:  $B$  is a basis for  $V$ . Suppose towards contradiction it's not, then there exists  $v_0 \in V \setminus \text{span}(B)$ . Consider  $B \cup \{v_0\}$  and let  $\alpha_0 v_0 + \sum_{j=1}^n \alpha_j v_j = 0_V$ . If  $\alpha_0 \neq 0$ , then  $\sum_{j=1}^n \alpha_j v_j = -\alpha_0 v_0$ , giving  $v_0 \in \text{span}(B)$  which is a contradiction. If  $\alpha_0 = 0$ , then  $\sum_{j=1}^n \alpha_j v_j = 0_V$ . Since  $B$  is independent,  $\alpha_j = 0$  for all  $j = 1, \dots, n$ . Thus  $B \cup \{v_0\}$  is independent, contradicting the maximality of  $B$ . Whence  $B$  is a basis for  $V$ .  $\square$

**Theorem 1.1.3.** *If  $B_1$  and  $B_2$  are bases for  $V$ , then  $\text{card}(B_1) = \text{card}(B_2)$ .*

**Definition 1.1.6.** If  $V$  is a vector space, its *dimension* is the cardinality of any of its bases.

**Corollary 1.1.4.** *If  $B$  is a basis for  $V$ , then every  $v \in V$  can be written  $v = \sum_{j=1}^n \alpha_k \beta_k$ ,  $\alpha_k \in F$ ,  $\beta_k \in B$  in a unique way.*

**Theorem 1.1.5.** *Let  $V$  be a linear space and  $B \subseteq V$  a subset. The following are equivalent:*

- (1)  $B$  is a basis for  $V$ ;
- (2)  $B$  is a maximal element in  $\mathfrak{T} = \{T \subseteq V \mid T \text{ independent}\}$ ;
- (3)  $B$  is a minimal element in  $\mathfrak{S} = \{S \subseteq V \mid S \text{ spans } V\}$ ;

**Definition 1.1.7.** Let  $\{V_i\}_{i \in I}$  be a family of vector spaces over a field  $F$ .

- (1) The *product* of  $\{V_i\}_{i \in I}$  is denoted:

$$\prod_{i \in I} V_i := \{(v_i)_{i \in I} \mid v_i \in V_i\}.$$

- (2) The *co-product* (or *sum*) is denoted

$$\bigoplus_{i \in I} V_i := \{(v_i)_{i \in I} \mid v_i \in V_i, \text{supp}((v_i)_{i \in I}) < \infty\}.$$

**Exercise 1.1.4.**

- (1) Show that
- $\prod_{i \in I} V_i$
- equipped with pointwise operations:

$$\begin{aligned}(v_i)_{i \in I} + (w_i)_{i \in I} &= (v_i + w_i)_{i \in I} \\ \alpha(v_i)_{i \in I} &= (\alpha v_i)_{i \in I}\end{aligned}$$

is a linear space.

- (2) Show that
- $\bigoplus_{i \in I} V_i$
- is a subspace of
- $\prod_{i \in I} V_i$
- .

**Proposition 1.1.6.** *Let  $V$  be a vector space over  $F$  and  $W \subseteq V$ . The (additive, abelian) quotient group  $V/W$  can be made into a vector space by defining multiplication by scalars as  $\alpha(v + W) = \alpha v + W$  for all  $\alpha \in F$ ,  $v + W \in V/W$ .*

**Example 1.1.3.**

- (1) The set  $F^n = \{(x_1, \dots, x_n) \mid x_j \in F\}$  with component-wise operations is a vector space.
- (2) The set  $M_{n,m}(F) = \{(a_{ij}) \mid a_{ij} \in F\}$  with linear operations is a vector space.
- (3) Let  $\Omega$  be a nonempty set. Then  $\mathcal{F}(\Omega, F) = \{f \mid f : \Omega \rightarrow F\}$  with pointwise operations is a vector space.
- (4) The set  $\ell_\infty(\Omega, F) = \{f \in \mathcal{F}(\Omega, F) \mid \|f\|_u < \infty\}$  with pointwise operations is a vector space.

**Exercise 1.1.5.** Show  $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$  is a subspace.

- (5) Let  $f : [a, b] \rightarrow \mathbf{R}$  be any function. Let  $\mathcal{P} = \{a = x_0 < x_1 < \dots < x_{n-1} < x_n = b\}$  be a partition of  $[a, b]$ . The *variation of  $f$  on  $\mathcal{P}$*  is defined as:

$$\text{Var}(f; \mathcal{P}) = \sum_{k=1}^n |f(x_k) - f(x_{k-1})|.$$

We say  $f$  is a *bounded variation* if:

$$\text{Var}(f) := \sup_{\mathcal{P}} \text{Var}(f; \mathcal{P}) < \infty.$$

The set of all functions of bounded variation is defined:

$$\text{BV}([a, b]) = \{f : [a, b] \rightarrow \mathbf{R} \mid \text{Var}(f) < \infty\}.$$

This is a vector space by defining addition and scalar multiplication componentwise.

**Exercise 1.1.6.** Show that  $\text{BV}([a, b]) \subseteq \ell_\infty([a, b], \mathbf{R})$  is a subspace.

- (6) Let  $K \subseteq V$  be a convex subset of a vector space  $V$ , that is, for all  $v, w \in K$  and  $t \in [0, 1]$ , then  $(1-t)v + tw \in K$ . A function  $f : K \rightarrow F$  is said to be *affine* if  $x, y \in K$  and  $t \in [0, 1]$  implies  $f((1-t)x + ty) = (1-t)f(x) + tf(y)$ . The set  $\text{Aff}(K, F) = \{f \in \mathcal{F}(K, F) \mid f \text{ affine}\}$  with pointwise operations is a vector space.

**Exercise 1.1.7.** Show  $\text{Aff}(K, F) \subseteq \mathcal{F}(K, F)$  is a subspace.

- (7) The set  $C([a, b], F) = \{f : [a, b] \rightarrow F \mid f \text{ continuous}\}$  with pointwise operations is a vector space.

**Exercise 1.1.8.** Explain why  $C([a, b], F) \subseteq \ell_\infty([a, b], F)$  is a subspace.

- (8) Consider the following sequence spaces:

- $s = \{(a_k)_k \mid a_k \in F\} = \mathcal{F}(\mathbf{N}, F)$ ;
- $\ell_\infty = \ell_\infty(\mathbf{N}, F) = \{(a_k)_k \mid \sup_{k \geq 1} |a_k| < \infty\}$ ;
- $c = \{(a_k)_k \mid (a_k)_k \text{ converges}\}$ ;
- $c_0 = \{(a_k)_k \mid (a_k)_k \rightarrow 0\}$ ;
- $c_{00} = \{(a_k)_k \mid \text{supp}((a_k)_k) < \infty\}$ ;
- $\ell_1 = \{(a_k)_k \mid \sum_{k=1}^\infty |a_k| < \infty\}$ .

These are all vector spaces with pointwise operations. In fact,  $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$  are all subspaces.

**Exercise 1.1.9.** Show that  $\ell_1 \subseteq c_0$  is a subspace.

- (9) Consider the following continuous function spaces on  $\mathbf{R}$ :

- $C(\mathbf{R}) = \{f : \mathbf{R} \rightarrow F \mid f \text{ continuous}\}$ ;
- $C_b(\mathbf{R}) = C(\mathbf{R}) \cap \ell_\infty(\mathbf{R})$ ;
- $C_0(\mathbf{R}) = \{f \in C(\mathbf{R}) \mid \lim_{x \rightarrow \pm\infty} f(x) = 0\}$ ;
- Recall that a function is *compactly supported* if for all  $\epsilon > 0$ , there exists  $\alpha > 0$  such that  $|x| \geq \alpha$  implies  $f(x) = 0$ . The set of compactly supported functions is denoted  $C_c(\mathbf{R}) = \{f \in C(\mathbf{R}) \mid f \text{ compactly supported}\}$ .

These are all vector spaces with pointwise operations, and  $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq C_b(\mathbf{R}) \subseteq C(\mathbf{R})$  are all subspace inclusion.

**Definition 1.1.8.** If  $V$  and  $W$  are linear spaces over a common field  $F$ , a map  $T : V \rightarrow W$  is called *linear* if  $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2)$  for all  $v_1, v_2 \in V$  and  $\alpha \in F$ .

**Example 1.1.4.** Let  $A \in M_{m,n}(F)$ . Then  $T_A : F^n \rightarrow F^m$  defined by  $T_A(v) = Av$  is linear. Let  $\{e_1, \dots, e_n\}$  be a basis for  $F^n$ . If  $T : F^n \rightarrow F^m$  is linear, set:

$$[T] = \left( T(e_1) \mid T(e_2) \mid \dots \mid T(e_n) \right).$$

This gives  $T(v) = [T]v$  for all  $v \in F^n$ . In fact, we also have  $[T_A] = A$  and  $T_{[T]} = T$ .

**Example 1.1.5.** The *canonical projection* is linear:

$$\pi_j : \prod_{i \in I} V_i \rightarrow V_j \text{ defined by } \pi_j((v_i)_i) = v_j.$$

We also have that the *coordinate exclusions* are linear:

$$\iota_j : V_j \hookrightarrow \bigoplus_{i \in I} V_i \text{ defined by } \iota_j(v) = (v_i)_i, \text{ where } v_i = \begin{cases} 0_v, & i \neq j \\ v_j, & \text{otherwise.} \end{cases}$$

The *evaluation map* is linear as well. For  $s \in S$ , consider:

$$e_s : \mathcal{F}(S, F) \rightarrow F \text{ defined by } e_s(f) = f(s).$$

**Proposition 1.1.7.** *Let  $V$  be a vector space with basis  $B$ . Let  $W$  be a vector space and suppose  $\varphi : B \rightarrow W$  is a map. Then there exists a unique linear map  $T_\varphi : V \rightarrow W$  with  $T_\varphi(b) = \varphi(b)$  for all  $b \in B$ . We have the following diagram.*

$$\begin{array}{ccc} B & \xhookrightarrow{\iota} & V \\ & \searrow \varphi & \downarrow T_\varphi \\ & & W \end{array}$$

*Proof.* Define  $T_\varphi : V \rightarrow W$  by:

$$\begin{aligned} T_\varphi(v) &= T_\varphi \left( \sum_{j=1}^n \alpha_j b_j \right) \\ &= \sum_{j=1}^n \alpha_j \varphi(b_j). \end{aligned}$$

Let  $v_1, v_2 \in V$  and  $c \in F$ . We have that:

$$\begin{aligned} T_\varphi(v_1 + cv_2) &= T_\varphi \left( \sum_{j=1}^n \alpha_j b_j + c \sum_{j=1}^n \beta_j b_j \right) \\ &= T_\varphi \left( \sum_{j=1}^n (\alpha_j + c\beta_j) b_j \right) \\ &= \sum_{j=1}^n (\alpha_j + c\beta_j) \varphi(b_j) \\ &= \sum_{j=1}^n \alpha_j \varphi(b_j) + c \sum_{j=1}^n \beta_j \varphi(b_j) \\ &= T_\varphi(v_1) + cT_\varphi(v_2). \end{aligned}$$

Thus  $T_\varphi$  is linear. Chasing the above diagram makes it clear that  $T_\varphi(b) = \varphi(b)$ . It remains to show that  $T_\varphi$  is unique. Let  $T$  be another linear transformation satisfying  $T(b) = \varphi(b)$  for all  $b \in B$ . Then:

$$\begin{aligned} T(v) &= T \left( \sum_{j=1}^n \alpha_j b_j \right) \\ &= \sum_{j=1}^n \alpha_j \varphi(b_j) \\ &= T_\varphi \left( \sum_{j=1}^n \alpha_j b_j \right) \\ &= T_\varphi(v). \end{aligned}$$

Thus  $T_\varphi$  is unique. □

**Proposition 1.1.8.** *Let  $T : V \rightarrow W$  be linear.*

- (1)  $\ker(T) = \{v \in V \mid T(v) = 0_W\}$  is a linear subspace of  $V$ .
- (2)  $\operatorname{im}(T) = \{T(v) \mid v \in V\}$  is a linear subspace of  $W$ .
- (3)  $\ker(T) = \{0_V\}$  if and only if  $T$  is injective.
- (4)  $\operatorname{im}(T) = W$  if and only if  $T$  is surjective.

*Proof.* (1) Let  $v_1, v_2 \in \ker(T)$  and  $\alpha \in F$ . Observe that:

$$\begin{aligned} T(v_1 + \alpha v_2) &= T(v_1) + \alpha T(v_2) \\ &= 0. \end{aligned}$$

Thus  $v_1 + \alpha v_2 \in \ker(T)$ , giving  $\ker(T)$  as a linear subspace of  $V$ .

(2) Let  $w_1, w_2 \in \operatorname{im}(T)$ . Then there exists  $v_1, v_2 \in V$  with  $T(v_1) = w_1$  and  $T(v_2) = w_2$ . We have:

$$\begin{aligned} w_1 + \alpha w_2 &= T(v_1) + \alpha T(v_2) \\ &= T(v_1 + \alpha v_2). \end{aligned}$$

Whence  $w_1 + \alpha w_2 \in \operatorname{im}(T)$ , giving  $\operatorname{im}(T)$  as a linear subspace of  $W$ .

(3) Let  $\ker(T) = \{0\}$ . Suppose  $T(v_1) = T(v_2)$ . Then  $T(v_1) - T(v_2) = T(v_1 - v_2) = 0_W$ . It must be that  $v_1 - v_2 = 0_V$ , giving  $v_1 = v_2$ . Thus  $T$  is injective. Conversely, suppose  $T$  is injective and let  $v \in \ker(T)$ . Then  $T(v) = 0_W = T(0_V)$ . Hence  $v = 0_V$ , establishing  $\ker(T) = \{0\}$ .

(4) This is by definition of surjectivity. □

**Proposition 1.1.9.** *If  $T : V \rightarrow W$  is linear and bijective, then the inverse map  $T^{-1} : W \rightarrow V$  is linear.*

*Proof.* We have that:

$$T(T^{-1}(w_1) + \alpha T^{-1}(w_2)) = w_1 + \alpha w_2 = T \circ T^{-1}(w_1 + \alpha w_2).$$

Applying  $T^{-1}$  to both sides gives the desired result. □

**Proposition 1.1.10** (Vector Spaces are Injective). *Let  $U, V, W$  be vector spaces and  $0 \rightarrow U \xrightarrow{j} V$  be exact (that is,  $j$  is injective). Let  $\varphi : U \rightarrow W$  be linear. There exists a linear map  $\Psi : V \rightarrow W$  such that  $\varphi = \Psi \circ j$ ; i.e., the following diagram commutes:*

$$\begin{array}{ccccc} 0 & \longrightarrow & U & \xrightarrow{j} & V \\ & & \downarrow \varphi & \searrow \Psi & \\ & & W & & \end{array}$$



*Proof.* Let  $\{u_i\}_{i \in I}$  be a basis for  $U$ . We must first show that  $\{j(u_i)\}_{i \in I}$  is linearly independent. Notice that:

$$\begin{aligned} 0_V &= \sum_{i \in I} \alpha_i j(u_i) \\ &= j\left(\sum_{i \in I} \alpha_i u_i\right). \end{aligned}$$

By the injectivity of  $j$ , we have that  $\sum_{i \in I} \alpha_i u_i = 0_U$ . Thus  $\alpha_i = 0$  for all  $i \in I$ , giving  $\{j(u_i)\}_{i \in I}$  as linearly independent.

Since  $\{j(u_i)\}_{i \in I}$  is linearly independent in  $V$ , we can extend it to a basis  $B = \{v_i\}_{i \in J}$  where  $I \subseteq J$  and  $v_i = j(u_i)$  whenever  $i \in I$ . Now define  $\psi : B \rightarrow W$  by:

$$\psi(v_i) = \begin{cases} \varphi(u_i), & i \in I \\ w, & i \in J \setminus I, \end{cases}$$

where  $w \in W$  is arbitrary. Since this is a map of basis elements, there exists a unique linear map  $\Psi : V \rightarrow W$  with  $\Psi(v_i) = \psi(v_i)$  for all  $v_i \in B$ . We can finally see that:

$$\begin{aligned} \varphi(u_i) &= \psi(v_i) \\ &= \Psi(v_i) \\ &= \Psi(j(u_i)). \end{aligned}$$

This establishes that  $\varphi = \Psi \circ j$ . □

**Proposition 1.1.11** (Vector Spaces are Projective). *Let  $U, V, W$  be vector spaces and  $V \xrightarrow{\pi} U \rightarrow 0$  be exact (that is,  $\pi$  is onto). Let  $\varphi : W \rightarrow U$  be linear. There exists a linear map  $\Psi : V \rightarrow W$  such that  $\varphi = \pi \circ \Psi$ ; i.e., the following diagram commutes:*

$$\begin{array}{ccc} & W & \\ \Psi \swarrow & \downarrow \varphi & \\ V & \xrightarrow{\pi} U & \longrightarrow 0 \end{array}$$

*Proof.* Let  $B = \{w_i\}_{i \in I}$  be a basis for  $W$ . Define  $\psi : B \rightarrow V$  by  $\psi(w_i) = \pi^{-1}(\varphi(w_i))$ . Since this is a map of basis elements, it extends to a unique (dependent on  $\pi^{-1}$ ) linear map  $\Psi : W \rightarrow V$  with  $\Psi(w_i) = \psi(w_i)$  for all  $w_i \in B$ . Moreover, we have that:

$$\begin{aligned} (\pi \circ \Psi)(w_i) &= (\pi \circ \psi)(w_i) \\ &= (\pi \circ (\pi^{-1} \circ \varphi))(w_i) \\ &= \varphi(w_i). \end{aligned} \quad \square$$

**Definition 1.1.9.** Let  $V$  and  $W$  be vector spaces over  $F$ . A *linear isomorphism* between  $V$  and  $W$  is a bijective linear map  $T : V \rightarrow W$ . If such a  $T$  exists, we say  $V$  and  $W$  are *linearly isomorphic*, and write  $V \cong W$ .

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Finite dimensional vector spaces are boring. This is illustrated through the following theorem.

**Theorem 1.1.12.** *Let  $V$  and  $W$  be finite-dimensional vector spaces over  $F$ . Then  $V \cong W$  if and only if  $\dim(V) = \dim(W)$ .*

*Proof.* Suppose  $V \cong W$ . Then there is an isomorphism taking basis of  $V$  to a basis of  $W$ . Therefore they have the same dimension.

Conversely, if  $\dim(V) = \dim(W) = n$ , then they are each isomorphic to  $F^n$ , giving that they are isomorphic to each other.  $\square$

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**Example 1.1.6.** Let  $V$  be a vector space,  $W \subseteq V$  a subspace. The *natural projection*:

$$\pi : V \rightarrow V/W \text{ defined by } \pi(v) = v + W$$

is a linear surjective map.

**Theorem 1.1.13** (First Isomorphism Theorem for Vector Spaces). *Let  $T : V \rightarrow V'$  be a linear map and  $W \subseteq V$  a subspace.*

- (1) *If  $T$  "kills"  $W$  (that is,  $W \subseteq \ker(T)$ ), then there exists a linear map  $\tilde{T} : V/W \rightarrow V'$  with  $\tilde{T} \circ \pi = T$ ; i.e., the following diagram commutes.*

$$\begin{array}{ccc} V & \xrightarrow{T} & V' \\ & \searrow \pi & \nearrow \tilde{T} \\ & V/W & \end{array}$$

- (2) *If  $\ker(T) = W$ , then  $\tilde{T}$  is injective.*  
 (3) *If  $\ker(T) = W$  and  $\text{im}(T) = V'$ , then  $V/W \cong V'$ .*

*Proof.* (1) As stipulated, define  $\tilde{T}(v + W) = T(v)$ . We must show that  $\tilde{T}$  is well-defined: suppose  $v_1 + W = v_2 + W$  for some  $v_1, v_2 \in V$ . Then  $v_1 = v_2 + w$  for some  $w \in W$ . This gives:

$$\begin{aligned} \tilde{T}(v_1 + W) &= \tilde{T}(v_2 + w + W) \\ &= \tilde{T}(v_2 + W). \end{aligned}$$

Whence  $\tilde{T}$  is well-defined. Now given  $v_1 + W, v_2 + W \in V/W$  and  $\alpha \in F$ , observe that:

$$\begin{aligned} \tilde{T}((v_1 + W) + c(v_2 + W)) &= \tilde{T}((v_1 + cv_2) + W) \\ &= T(v_1 + cv_2) \\ &= T(v_1) + cT(v_2) \\ &= \tilde{T}(v_1 + W) + c\tilde{T}(v_2 + W). \end{aligned}$$

Thus  $\tilde{T}$  is linear.

(2) If  $\ker(T) = W$ , then:

$$\begin{aligned}\ker(\tilde{T}) &= \{v + W \mid \tilde{T}(v + W) = 0_{V'}\} \\ &= \{v + W \mid T(v) = 0_{V'}\} \\ &= \{v + W \mid v \in \ker(T)\} \\ &= \{v + W \mid v \in W\} \\ &= \{0\}.\end{aligned}$$

Thus  $\tilde{T}$  is injective.

(3) It remains to show that  $\text{im}(T) = V'$  implies  $\tilde{T}$  is surjective. Observe that:

$$\begin{aligned}\text{im}(\tilde{T}) &= \{\tilde{T}(v + W) \mid v + W \in V/W\} \\ &= \{\tilde{T}(\pi(v)) \mid v \in V\} \\ &= \{T(v) \mid v \in V\} \\ &= \text{im}(T) \\ &= V' .\end{aligned}$$

Thus  $\tilde{T}$  is surjective, which establishes it as a bijection. This gives  $V/W \cong V'$ .  $\square$

**Definition 1.1.10.** Let  $S$  be a nonempty set. The *free vector space* of  $S$  is:

$$\mathbf{F}(S) = \{f : S \rightarrow F \mid \text{supp}(f) < \infty\}.$$

**Exercise 1.1.10.** Show  $\mathbf{F}(S) \subseteq \mathcal{F}(S, F)$  is a subspace.

**Proposition 1.1.14.** The set  $\{\delta_s \mid s \in S\}$  is a basis for  $\mathbf{F}(S)$ , where  $\delta_s : S \rightarrow F$  is defined by:

$$\delta_s(t) = \begin{cases} 1, & t = s \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* If  $f \in \mathbf{F}(S)$  with  $\text{supp}(f) = \{s_1, \dots, s_n\}$ , then  $f = \sum_{k=1}^n f(s_k)\delta_{s_k}$ . If  $\sum_{k=1}^n \alpha_k \delta_{s_k} = 0$ , then for  $j = 1, \dots, n$  we have  $0 = (\sum_{k=1}^n \alpha_k \delta_{s_k})(s_j) = \alpha_j$ .  $\square$

**Theorem 1.1.15.** Given any vector space  $V$  and a map (of sets)  $\varphi : S \rightarrow V$ , there exists a unique linear map  $T_\varphi : \mathbf{F}(S) \rightarrow V$  with  $T_\varphi \circ \iota = \varphi$ , where  $\iota : S \rightarrow \mathbf{F}(S)$  is defined by  $\iota(s) = \delta_s$  for all  $s \in S$ . The following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & \mathbf{F}(S) \\ & \searrow \varphi & \downarrow T_\varphi \\ & & V \end{array}$$

*Proof.* By the previous proposition, we have that  $B = \{\delta_s \mid s \in S\}$  is a basis for  $\mathbf{F}(S)$ . Define  $T : B \rightarrow V$  by  $T(\delta_s) = \varphi(s)$ . Since this is a map of basis elements, there exists

a unique linear map  $T_\varphi : \mathbf{F}(S) \rightarrow V$  with  $T_\varphi(\delta_s) = T(\delta_s)$  for all  $\delta_s \in B$ . The diagram commutes because:

$$\begin{aligned}\varphi(s) &= T(\delta_s) \\ &= T_\varphi(\delta_s) \\ &= T_\varphi(\iota(s)).\end{aligned}$$

Moreover, if  $T'$  satisfies  $\varphi = T' \circ \iota$ , then:

$$\begin{aligned}T'(\delta_s) &= T'(\iota(s)) \\ &= \varphi(s) \\ &= T_\varphi(\iota(s)) \\ &= T_\varphi(\delta_s).\end{aligned}$$

Thus  $T_\varphi$  is unique. □

**Definition 1.1.11.** Let  $V$  and  $W$  be vector spaces. The set of linear transformations between  $V$  and  $W$  is  $\mathcal{L}(V, W) = \{T \mid T : V \rightarrow W \text{ linear}\}$ . The set of linear functionals is  $V' := \mathcal{L}(V, F)$ .

**Exercise 1.1.11.** Show  $\mathcal{L}(V, W)$  is a vector space.

**Exercise 1.1.12.** Show  $M_{m,n}(F) \cong \mathcal{L}(F^m, F^n)$  by  $a \mapsto T_a : (v \mapsto av)$ .

## § 1.2. Algebras

**Definition 1.2.1.** An *algebra* over  $F$  is a linear space  $A$  over  $F$  equipped with a multiplication operation:

$$A \times A \rightarrow A \text{ defined by } (a, b) \mapsto ab$$

satisfying:

- (1)  $(ab)c = a(bc)$  for all  $a, b, c \in A$ ;
- (2)  $(\alpha a)b = \alpha(ab) = a(\alpha b)$  for all  $a, b \in A, \alpha \in F$ ;
- (3)  $a(b + c) = ab + ac$  for all  $a, b, c \in A$ ;
- (4)  $(a + b)c = ac + bc$  for all  $a, b, c \in A$ .

If  $ab = ba$  for all  $a, b \in A$  we say that  $A$  is *commutative*. If there exists  $1_A \in A$  with  $1_A a = a 1_A = a$  for all  $a \in A$  we say  $A$  is *unital*.

**Example 1.2.1.**

- (1)  $M_n(F)$  is a noncommutative unital algebra over  $F$  under the usual matrix multiplication.
- (2) If  $V$  is a vector space over  $F$ ,  $\mathcal{L}(V)$  is a unital algebra over  $F$ . It is noncommutative provided  $\dim(V) > 1$ .
- (3)  $\mathcal{F}(S, F)$  is a unital commutative algebra over  $F$ .

**Definition 1.2.2.** Let  $B$  be a (unital) algebra over  $F$ .

- (1) A (unital) *subalgebra* of  $B$  is a subspace  $A \subseteq B$  ( $1_B \in A$ ) satisfying the property that if  $a, a' \in A$ , then  $aa' \in A$ .
- (2) An *ideal* of  $B$  is a subspace  $I \subseteq B$  with  $b \in B, a \in I$  implying  $ba, ab \in I$ .

**Example 1.2.2.**

- (1)  $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$  is a unital subalgebra.
- (2)  $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$  are all subalgebras. In particular,  $c_0 \subseteq \ell_\infty$  and  $c_{00} \subseteq s$  are ideals.
- (3)  $C([a, b]) \subseteq \ell_\infty([a, b])$  is a unital subalgebra.
- (4)  $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq C_b(\mathbf{R}) \subseteq \ell_\infty(\mathbf{R})$  are all subalgebras. In fact,  $C_b(\mathbf{R}) \subseteq C(\mathbf{R})$  and  $C_b(\mathbf{R}) \subseteq \ell_\infty(\mathbf{R})$  are unital, whereas  $C_0(\mathbf{R}) \subseteq C_b(\mathbf{R})$  and  $C_c(\mathbf{R}) \subseteq C(\mathbf{R})$  are ideals.
- (5) The set  $T_n(F) = \{(a_{ij}) \in M_n(F) \mid a_{ij} = 0, i > j\}$  is a unital subalgebra of  $M_n(F)$ .

**Example 1.2.3** (Group Algebra). Let  $\Gamma$  denote a group (not necessarily abelian). Take the free vector space  $\mathbf{F}(\Gamma)$  and define multiplication as *convolution*: given  $f, g \in \mathbf{F}(\Gamma)$  let:

$$(f * g)(r) = \sum_{\left\{ \begin{array}{l} (s,t) \mid \\ s \in \text{supp}(f), \\ t \in \text{supp}(g), \\ st=r \end{array} \right\}} f(s)g(t).$$

Since  $\text{supp}(f)$  and  $\text{supp}(g)$  are finite, this is a finite sum. We often suppress this notation and write  $(f * g)(r) = \sum_{st=r} f(s)g(t)$ .

We can also make substitutions:

$$\begin{aligned} (f * g)(r) &= \sum_{st=r} f(s)g(t) \\ &= \sum_{t \in \Gamma} f(rt^{-1})g(t) \\ &= \sum_{s \in \Gamma} f(s)g(s^{-1}r). \end{aligned}$$

It is clear that:

$$\begin{aligned} (f + g) * h &= f * h + g * h \\ g * (f + h) &= f * g + g * h \\ \alpha(f * g) &= (\alpha f) * g = f * (\alpha g) \end{aligned}$$

for  $f, g, h \in \mathbf{F}(\Gamma)$ ,  $\alpha \in F$ . Associativity can be similarly shown using the above definition. Rather, we will prove associativity by first show that  $\delta_s * \delta_t = \delta_{st}$ . Given:

$$(\delta_s * \delta_t)(r) = \sum_{q \in \Gamma} \delta_s(rq^{-1})\delta_t(q),$$

notice that:

$$\delta_s(rt^{-1}) = \begin{cases} 1, & s = rt^{-1} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & r = st \\ 0, & \text{otherwise} \end{cases} = \delta_{st}(r).$$

Since  $\{\delta_t \mid t \in \Gamma\}$  is a basis for  $\mathbf{F}(\Gamma)$ , every  $f \in \mathbf{F}(\Gamma)$  looks like:

$$f = \sum_{t \in J} \alpha_t \delta_t, \quad J \subseteq T \text{ finite.}$$

Using distributivity we get:

$$\begin{aligned} \delta_r * (\delta_s * \delta_t) &= \delta_r * \delta_{st} \\ &= \delta_{rst} \\ &= \delta_{rs} * \delta_t \\ &= (\delta_r * \delta_s) * \delta_t. \end{aligned}$$

Whence convolution is associative.

**Exercise 1.2.1.** Let  $\{A_i\}_{i \in I}$  be a family of algebras over  $F$ .

- (1)  $\prod_{i \in I} A_i$  is an algebra under  $(a_i)_i (b_i)_i = (a_i b_i)_i$ .
- (2)  $\bigoplus_{i \in I} A_i \subseteq \prod_{i \in I} A_i$  is an ideal.

**Exercise 1.2.2.** Let  $A$  be an algebra over  $F$  and  $I \subseteq A$  an ideal. Then  $A/I$  is an algebra under  $(a + I)(b + I) = ab + I$ .

## § 1.3. Normed Vector Spaces

To each vector  $v$  in a vector space  $V$ , we want to assign a "length", denoted  $\|v\|$ .

**Definition 1.3.1.** A *norm* on a vector space  $V$  is a map:

$$\|\cdot\| : V \rightarrow [0, \infty), \quad v \mapsto \|v\|$$

satisfying:

- (1)  $\|\alpha v\| = |\alpha| \|v\|$  for all  $\alpha \in F$ ,  $v \in V$  (homogeneity);
- (2)  $\|v + w\| \leq \|v\| + \|w\|$  (triangle inequality);
- (3) If  $\|v\| = 0$ , then  $v = 0_V$  (positive-definite).

If  $\|\cdot\|$  satisfies (1) and (2), it is called a *seminorm*. The pair  $(V, \|\cdot\|)$  is called a *normed space*.

**Definition 1.3.2.** Two norms  $\|\cdot\|$  and  $\|\cdot\|'$  on a vector space  $V$  are called *equivalent* if there exists  $c_1 \geq 0$  and  $c_2 \geq 0$  with  $\|v\| \leq c_1 \|v\|'$  and  $\|v\|' \leq c_2 \|v\|$  for all  $v \in V$ .

**Exercise 1.3.1.** If  $p$  is a seminorm on  $V$ , show that  $|p(v) - p(w)| \leq p(v - w)$ .

**Definition 1.3.3.** Let  $(V, \|v\|)$  be a normed space.

- (1) The *closed unit ball* is denoted  $B_V = \{v \in V \mid \|v\| \leq 1\}$ .

(2) The *open unit ball* is denoted  $U_V = \{v \in V \mid \|v\| < 1\}$ .

(3) The *unit sphere* is denoted  $S_V = \{v \in V \mid \|v\| = 1\}$ .

**Example 1.3.1.** Let  $V = F^n$  and  $x = (x_1, \dots, x_n)$ . We define:

$$\begin{aligned}\|x\|_1 &= \sum_{j=1}^n |x_j|; \\ \|x\|_\infty &= \max_{j=1}^n |x_j|; \\ \|x\|_2 &= \left( \sum_{j=1}^n |x_j|^2 \right)^{\frac{1}{2}}.\end{aligned}$$

For  $p \geq 1$ :

$$\|x\|_p = \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}}.$$

**Exercise 1.3.2.** Show that  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  are norms

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We aim to show that  $\|\cdot\|_p$  is a norm for  $p \in [0, \infty]$ .

**Lemma 1.3.1.** Let  $p, q \in [1, \infty]$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $f : [0, \infty) \rightarrow \mathbf{R}$  be defined by  $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$ . Then  $f(t) \geq 0$  for  $t \geq 0$ .

*Proof.* Note that  $f'(t) = t^{p-1} - 1$ . Since:

$$\begin{aligned}f'(1) &= 0 \\ f'(t) &> 0 \text{ for } t > 1 \\ f'(t) &< 0 \text{ for } 0 \leq t < 1,\end{aligned}$$

we can see that  $f(t) \geq 0$  for all  $t \geq 0$ . □

**Lemma 1.3.2** (Young's Inequality). Let  $p, q \in [0, \infty]$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $x, y \geq 0$ , then  $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ .

*Proof.* By Lemma 1.3.1,  $t \leq \frac{1}{p}t^p + \frac{1}{q}$ . Multiplying both sides by  $y^q$  gives:

$$ty^q \leq \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Let  $t = xy^{1-q}$ . Then:

$$xy^{1-q}y^q \leq \frac{1}{p}x^p y^{p-pq}y^q + \frac{1}{q}y^q.$$

Since  $\frac{1}{p} + \frac{1}{q} = 1$ , we have that  $p - pq = -q$ . Whence:

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

□

**Lemma 1.3.3** (Hölders Inequality). *Let  $p, q \in [0, \infty]$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then for  $x, y \in F^n$ :*

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \|x\|_p \|y\|_q.$$

*Proof.* We proceed by cases.

Case 1:  $p = 1$ . Then:

$$\begin{aligned} \left| \sum_{j=1}^n x_j y_j \right| &\leq \sum_{j=1}^n |x_j| |y_j| \\ &\leq \sum_{j=1}^n |x_j| \|y\|_\infty \\ &= \|x\|_1 \|y\|_\infty. \end{aligned}$$

Case 2:  $p = \infty$ . This follows similarly to Case 1.

Case 3:  $1 < p < \infty$ . Suppose  $\|x\|_p = \|y\|_q = 1$ . Then:

$$\begin{aligned} \left| \sum_{j=1}^n x_j y_j \right| &\leq \sum_{j=1}^n |x_j| |y_j| \\ &\leq \sum_{j=1}^n \left( \frac{1}{p} |x_j|^p + \frac{1}{q} |y_j|^q \right) \\ &= \frac{1}{p} \left( \sum_{j=1}^n |x_j|^p \right) + \frac{1}{q} \left( \sum_{j=1}^n |y_j|^q \right) \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

Whence the inequality holds. Now suppose  $\|x\|_p = 0$  or  $\|y\|_q = 0$ . Then  $x = 0_{F^n}$  or  $y = 0_{F^n}$ , whence the inequality holds. Suppose  $\|x\|_p \neq 0$  and  $\|y\|_p \neq 0$ . Set:

$$\begin{aligned} x' &= \frac{x}{\|x\|_p} \\ y' &= \frac{y}{\|y\|_p}. \end{aligned}$$



Then  $\|x'\|_p = 1 = \|y'\|_p$ . Observe that:

$$\begin{aligned} 1 &\geq \left| \sum_{j=1}^n x'_j y'_j \right| \\ &= \left| \sum_{j=1}^n \frac{x}{\|x\|_p} \frac{y}{\|y\|_p} \right|. \end{aligned}$$

Multiplying both sides by  $\|x\|_p \|y\|_p$  gives the desired result.  $\square$

**Lemma 1.3.4** (Minkowski's Inequality). *Let  $p, q \in [1, \infty]$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . For  $x, y \in F^n$ :*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

*Proof.* The only nontrivial case is for  $1 < p < \infty$ . Observe that:

$$\begin{aligned} (\|x + y\|_p)^p &= \sum_{j=1}^n |x_j + y_j|^p \\ &= \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\ &\leq \sum_{j=1}^n |x_j| |x_j + y_j|^{p-1} + |y_j| |x_j + y_j|^{p-1} \\ &\leq \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} + \left( \sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^{p-1(\frac{p}{p-1})} \right)^{1-\frac{1}{p}} + \left( \sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^{p-1(\frac{p}{p-1})} \right)^{1-\frac{1}{p}} \\ &= \left( \sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^p \right)^{1-\frac{1}{p}} + \left( \sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |x_j + y_j|^p \right)^{1-\frac{1}{p}} \\ &= (\|x\|_p + \|y\|_p) \frac{\|x + y\|_p^p}{\|x + y\|_p}. \end{aligned}$$

Multiplying both sides by  $\frac{\|x + y\|_p}{\|x + y\|_p^p}$  gives the desired inequality.  $\square$

**Theorem 1.3.5.** *Let  $V = F^n$ . Then  $(F^n, \|\cdot\|_p)$  is a normed space.*

*Proof.* Let  $x = (x_1, \dots, x_n) \in F^n$  and  $\alpha \in F$ . Observe that:

$$\begin{aligned}\|\alpha x\|_p &= \left( \sum_{j=1}^n |\alpha x_j|^p \right)^{\frac{1}{p}} \\ &= \left( \sum_{j=1}^n |\alpha|^p |x_j|^p \right)^{\frac{1}{p}} \\ &= |\alpha| \|x\|_p.\end{aligned}$$

This satisfies homogeneity. Moreover, Minkowski's Inequality satisfies the triangle inequality. It remains to show that  $\|\cdot\|_p$  is positive-definite. If  $\|x\|_p = 0$ , then  $x_j = 0$  for all  $1 \leq j \leq n$ . Thus  $x = 0_{F^n}$ .  $\square$

**Corollary 1.3.6.** *Let  $p \in [1, \infty]$ . Then  $\ell_p = \{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k|^p < \infty\}$  with norm  $\|(a_k)_k\|_p = (\sum_{k=1}^{\infty} |a_k|^p)^{\frac{1}{p}}$  is a normed space.*

*Proof.* Homogeneity and positive-definiteness are trivial to prove. Let  $(x_k)_k, (y_k)_k \in \ell_p$ . It is clear that:

$$\begin{aligned}\left( \sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} &\leq \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} \\ &= \|(x_k)_k\|_p + \|(y_k)_k\|_p.\end{aligned}$$

We have that  $\sum_{k=1}^n |x_k + y_k|^p$  is increasing and bounded above by  $(\|(x_k)_k\|_p + \|(y_k)_k\|_p)^p$ . By the Monotone Convergence Theorem  $\lim_{n \rightarrow \infty} \sum_{k=1}^n |x_k + y_k|^p = \sum_{k=1}^{\infty} |x_k + y_k|^p$  exists. Whence  $(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{p}} = \|(x_k)_k + (y_k)_k\|_p \leq \|(x_k)_k\|_p + \|(y_k)_k\|_p$   $\square$

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**Example 1.3.2.**

- (1)  $(\ell_{\infty}(\Omega, F), \|\cdot\|_u)$  where  $\|f\|_u = \sup_{x \in \Omega} |f(x)|$  is a normed space. This includes its subspaces, such as  $C([a, b], F) \subseteq \ell_{\infty}([a, b], F)$  and  $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq \ell_{\infty}(\mathbf{R})$ , all with  $\|\cdot\|_u$ .
- (2) Take  $\Omega = \mathbf{N}$  in the previous example. Then  $(\ell_{\infty}, \|\cdot\|_{\infty})$  is a normed space. This includes its subspaces  $c_{00} \subseteq c_0 \subseteq \ell_{\infty}$  with  $\|\cdot\|_{\infty}$ .
- (3)  $(\ell_1, \|\cdot\|_1)$  is a normed space.
- (4)  $(C([a, b]), \|\cdot\|_1)$  with  $\|f\|_1 = \int_a^b |f(t)| dt$  is a normed space.
- (5)  $(BV([a, b]), \|\cdot\|_{BV})$  where  $\|f\|_{BV} = |f(a)| + \text{Var}(f)$  is a normed space.

- (6) Let  $(V, \|\cdot\|_V)$  and  $(W, \|\cdot\|_W)$  be normed spaces. Then  $(B(V, W), \|\cdot\|_{\text{op}})$  is a normed space, where  $B(V, W) = \{T \in \mathcal{L}(V, W) \mid \|T\|_{\text{op}} < \infty\}$  is the set of bounded linear maps and  $\|T\|_{\text{op}} = \sup_{v \in B_V} \|T(v)\|_W$ . Intuitively,  $\|T\|_{\text{op}}$  measures the radius of the smallest ball which contains  $B_V$ .

**Exercise 1.3.3.** Show that  $V^* := B(V, F)$  is a subspace of  $V'$ .

- (7) Let  $S$  be a nonempty set. Both  $(\mathbf{F}(S), \|\cdot\|_1)$  and  $(\mathbf{F}(S), \|\cdot\|_p)$  are normed spaces, where  $\|f\|_1 = \sum_{s \in S} |f(s)|$  and  $\|f\|_p = \left(\sum_{s \in S} |f(s)|^p\right)^{\frac{1}{p}}$ . Note that since  $f(s) \neq 0$  for finitely many  $s \in S$ , both  $\|\cdot\|_1$  and  $\|\cdot\|_p$  are well-defined.

**Exercise 1.3.4.** Show that  $\|f\|_{\infty} := \sup_{s \in S} |f(s)|$  is a norm on  $\mathbf{F}(S)$ .

## § 1.4. Inner Product Spaces

**Definition 1.4.1.** Let  $V$  be a vector space over  $F$  and  $\varphi : V \times V \rightarrow F$  a map.

- (1) The map  $\varphi$  is said to be a *bilinear form* if it is linear in the first and second variable separately; i.e., for all  $v_1, v_2, v \in V$  and  $c \in F$  we have:

$$(i) \quad \varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$$

$$(ii) \quad \varphi(v, cv_1 + v_2) = c\varphi(v, v_1) + \varphi(v, v_2).$$

- (2) The map  $\varphi$  is said to be a *sesquilinear form* if it is linear in the first variable and conjugate linear in the second variable; i.e., for all  $v_1, v_2, v \in V$  and  $c \in F$  we have:

$$(i) \quad \varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$$

$$(ii) \quad \varphi(v, cv_1 + v_2) = \bar{c}\varphi(v, v_1) + \varphi(v, v_2).$$

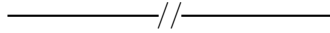
If we wish to keep track of a bilinear form on  $V$  we write  $(V, \varphi)$ .

**Definition 1.4.2.** Let  $V$  be a vector space over  $F$ .

- (1) A bilinear form  $\varphi$  on  $V$  is said to be *symmetric* if  $\varphi(v, w) = \varphi(w, v)$  for all  $v, w \in V$ .
- (2) A sesquilinear form  $\varphi$  on  $V$  is said to be *Hermitian* if  $\varphi(v, w) = \overline{\varphi(w, v)}$  for all  $v, w \in V$ .

**Definition 1.4.3.** Let  $(V, \varphi)$  be a vector space over  $F$  such that if  $\varphi$  is symmetric, then  $F = \mathbf{R}$  or if  $\varphi$  is Hermitian, then  $F = \mathbf{C}$ . We say  $\varphi$  is *positive-definite* if for all nonzero  $v \in V$  we have  $\varphi(v, v) \neq 0$ .

**Definition 1.4.4.** Let  $(V, \varphi)$  be a vector space over  $\mathbf{R}$  with  $\varphi$  a positive-definite symmetric bilinear form or over  $\mathbf{C}$  with  $\varphi$  a positive-definite Hermitian sesquilinear form. Then we say  $\varphi$  is an *inner product* on  $V$  and write  $\varphi$  as  $\langle \cdot, \cdot \rangle$ . We say  $(V, \langle \cdot, \cdot \rangle)$  is an *inner product space*.



**Definition 1.4.5.** If  $V$  is an inner product space we define  $\|v\|_2 = \langle v, v \rangle^{\frac{1}{2}}$ .

**Definition 1.4.6.** Let  $V$  be an inner product space. Two vectors  $v, w \in V$  are *orthogonal* if  $\langle u, v \rangle = 0$ . We denote this as  $u \perp v$ .

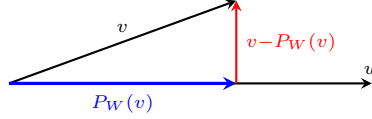
**Theorem 1.4.1** (Pythagorean Theorem). *Let  $v_1, \dots, v_n$  be mutually orthogonal. Then  $\sum_{j=1}^n \|v_j\|_2^2 = \left\| \sum_{j=1}^n v_j \right\|_2^2$ .*

*Proof.* Because  $v_i \perp v_j$  for  $1 \leq i, j \leq n$ , we have  $\langle v_i, v_j \rangle = 0$ . Observe that:

$$\begin{aligned} \left\| \sum_{j=1}^n v_j \right\|_2^2 &= \left\langle \sum_{j=1}^n v_j, \sum_{j=1}^n v_j \right\rangle \\ &= \sum_{j=1}^n \left( \sum_{i=1}^n \langle v_j, v_i \rangle \right) \\ &= \sum_{j=1}^n \langle v_j, v_j \rangle \\ &= \sum_{j=1}^n \|v_j\|_2^2. \end{aligned}$$

□

**Definition 1.4.7.** Let  $V$  be an inner product space and  $w \in V$  nonzero. The *projection* of a vector  $v \in V$  onto  $w$  is a map  $P_W : V \rightarrow V$  defined by  $P_W(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$ .



**Proposition 1.4.2.** \*\*\* Let  $V$  be an inner product space and  $w \in V$  a nonzero vector. Then  $P_w(v) \perp v - P_w(v)$ .

*Proof.*

□

**Corollary 1.4.3.** \*\*\* Let  $V$  be an inner product space and  $w \in W$  a nonzero vector. Then  $\|v\|_2^2 = \|P_w(v)\|_2^2 + \|v - P_w(v)\|_2^2$ .

*Proof.*

□

**Lemma 1.4.4** (Cauchy-Schwartz Inequality). Let  $V$  be an inner product space and  $v, w \in V$ . Then  $|\langle v, w \rangle| \leq \|v\|_2 \|w\|_2$ .

*Proof.* The previous corollary gives  $\|v\|_2 \geq \|P_w(v)\|_2$ . We have that:

$$\begin{aligned} \|v\|_2 &\geq \left\| \frac{\langle v, w \rangle}{\langle w, w \rangle} w \right\|_2 \\ &= \frac{|\langle v, w \rangle|}{\|w\|_2^2} \|w\|_2 \\ &= \frac{\langle v, w \rangle}{\|w\|_2}. \end{aligned}$$

Multiplying both sides by  $\|w\|_2$  gives the desired result.

□

**Theorem 1.4.5.** Let  $V$  be an inner product space. Then  $(V, \|\cdot\|_2)$  is a normed space.

*Proof.* Let  $v, w \in V$  and  $\alpha \in F$ . We have that:

$$\begin{aligned}\|\alpha v\|_2 &= \langle \alpha v, \alpha v \rangle^{\frac{1}{2}} \\ &= (\alpha \bar{\alpha} \langle v, v \rangle)^{\frac{1}{2}} \\ &= (|\alpha|^2 \langle v, v \rangle)^{\frac{1}{2}} \\ &= |\alpha| \|v\|_2.\end{aligned}$$

Thus  $\|\cdot\|_2$  satisfies homogeneity. It follows from the Cauchy-Schwartz Inequality that:

$$\begin{aligned}\|v + w\|_2^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= \|v\|_2^2 + \langle v, w \rangle + \overline{\langle v, w \rangle} + \|w\|_2^2 \\ &= \|v\|_2^2 + 2\Re(\langle v, w \rangle) + \|w\|_2^2 \\ &\leq \|v\|_2^2 + 2|\langle v, w \rangle| + \|w\|_2^2 \\ &\leq \|v\|_2^2 + 2\|v\|_2\|w\|_2 + \|w\|_2^2 \\ &= (\|v\|_2 + \|w\|_2)^2,\end{aligned}$$

where we used the fact that  $2\Re(\langle v, w \rangle) = 2|\langle v, w \rangle|$ . Squaring both sides proves that  $\|\cdot\|_2$  satisfies the triangle inequality. It remains to show positive-definiteness. Suppose  $\|v\|_2 = 0$ . Then  $\langle v, v \rangle = 0$ , but since the inner-product is by definition positive-definite, we get that  $v = 0_V$ .  $\square$

—————//—————

**Example 1.4.1.**

- (1)  $\ell_2^n = F^n$  is an inner product space where  $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle := \sum_{j=1}^n x_j \bar{y}_j$ .
- (2)  $\ell_2$  is an inner product space where  $\langle (a_k)_k, (b_k)_k \rangle := \sum_{k=1}^{\infty} a_k \bar{b}_k$ . Note that:

$$\begin{aligned}\sum_{k=1}^n |a_k \bar{b}_k| &= \sum_{k=1}^n |a_k| |b_k| \\ &\leq \left( \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} |b_k|^2 \right)^{\frac{1}{2}} \\ &= \|(a_k)_k\|_2 \|(b_k)_k\|_2 \\ &< \infty \quad (\text{Because } (a_k)_k, (b_k)_k \in \ell_2).\end{aligned}$$

Since  $\sum_{k=1}^n |a_k \bar{b}_k|$  is increasing and bounded above, the Monotone Convergence Theorem says  $\sum_{k=1}^{\infty} |a_k \bar{b}_k|$  exists and is finite. Whence  $\langle (a_k)_k, (b_k)_k \rangle$  converges.

- (3) Recall that  $\text{Tr} : M_n(\mathbf{C}) \rightarrow \mathbf{C}$  is defined by  $\text{Tr}(a_{ij}) = \sum_{i=1}^n a_{ii}$ . Then  $M_n(\mathbf{C})$  is an inner product space where  $\langle a_{ij}, b_{ij} \rangle := \text{Tr}(b_{ij}^* a_{ij})$ .
- (4)  $C([0, 1])$  is an inner product space where  $\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx$ .

## § 1.5. Normed Algebras

**Definition 1.5.1.** A *normed algebra* is an algebra  $A$  equipped with a norm  $\|\cdot\|_A$  such that  $\|ab\|_A \leq \|a\|_A \|b\|_A$ . If  $A$  is unital, we require  $\|1\|_A = 1$ .

**Example 1.5.1.**

- (1)  $\ell_\infty(\Omega)$  equipped with  $\|\cdot\|_u$  is a normed algebra.
- (2)  $C_c(\mathbf{R})$ ,  $C_0(\mathbf{R})$ , and  $C([0, 1])$  are all normed algebras when equipped with  $\|\cdot\|_u$ .
- (3)  $M_n(F)$  equipped with  $\|\cdot\|_{\text{op}}$  is a normed algebra.
- (4) If  $V$  is a normed space, then  $B(V, V)$  with  $\|\cdot\|_{\text{op}}$  is a normed algebra: for  $T, S \in B(V, V)$  and  $v \in B_V$ , we have that

$$\begin{aligned} \|(T \circ S)(v)\| &\leq \|T\|_{\text{op}} \|S(v)\| \\ &\leq \|T\|_{\text{op}} \|S\|_{\text{op}}. \end{aligned}$$

Taking the supremum over all  $v \in B_V$  gives  $\|T \circ S\|_{\text{op}} \leq \|T\|_{\text{op}} \|S\|_{\text{op}}$ .

- (5) Let  $S$  be a group. Equip the algebra  $\mathbf{F}(S)$  with  $\|\cdot\|_1$ . We get a normed algebra.

**Exercise 1.5.1.** For  $a, b \in \ell_1(\mathbf{Z})$ , define  $a * b : \mathbf{Z} \rightarrow F$  by  $(a * b)(n) = \sum_{k \in \mathbf{Z}} a(n - k)b(k)$ . Show that  $\ell_1(\mathbf{Z})$  with this multiplication is a normed algebra.

# Chapter 2

## Metric Spaces

### § 2.1. Basic Definitions and Examples

**Definition 2.1.1.** A *metric* on a nonempty set  $X$  is a map

$$d : X \times X \rightarrow [0, \infty)$$

satisfying for all  $x, y, z \in X$ :

- (1)  $d(x, y) = d(y, x)$  (symmetry);
- (2)  $d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality);
- (3)  $d(x, x) = 0$ ;
- (4) if  $d(x, y) = 0$  then  $x = y$  (positivity).

If  $d$  satisfies all but (iv), then  $d$  is called a *semi-metric*. The pair  $(X, d)$  is called a *metric space*.

**Definition 2.1.2.**

- (1) Two metrics  $d, \rho$  on  $X$  are called *equivalent* if there exists constants  $c, c'$  with  $d(x, y) \leq c\rho(x, y)$  and  $\rho(x, y) \leq c'd(x, y)$ .
- (2) Let  $\{d_k\}_{k=1}^\infty$  be a family of metrics on  $X$ . If for all  $x, y \in X$  and  $k \in \mathbf{N}$  we have  $d_k(x, y) \leq C$ , then we say the family of metrics is *uniformly bounded*.

**Example 2.1.1.**

- (1) The *discrete metric* on  $X \neq \emptyset$  is:

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

- (2) The *hamming distance* between two bit strings of equal length: given  $X = \{0, 1\}^n$ , then  $d_H : X \times X \rightarrow [0, \infty)$  is defined by  $d_H((x_j)_{j=1}^n, (y_j)_{j=1}^n) = |\{j \mid x_j \neq y_j\}|$ .
- (3) If  $(V, \|\cdot\|)$  is any normed space, then  $d(v, w) = \|v - w\|$  is a metric on  $V$ .

**Exercise 2.1.1.** If  $\|\cdot\|$  and  $\|\cdot\|'$  are norms on a linear space  $V$ , show they are equivalent if and only if their induced metrics are equivalent.

- (4) If  $(X, d)$  is a metric and  $Y \subseteq X$  is a subset, then  $(Y, d)$  is a metric space.



- (5) Let  $(X, \rho)$  be a metric space. Fix  $p \in X$ . Then:

$$d(x, y) := \begin{cases} 0, & x = y \\ \rho(x, p) + \rho(p, y), & x \neq y \end{cases}$$

is a metric.

- (6) It is often beneficial to work with metrics that are bounded. Let  $\rho$  be a (semi)-metric of  $X$ . Set:

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

Defining  $d(x, y)$  as above gives  $0 \leq d(x, y) \leq 1$ . Although  $d$  and  $\rho$  are not equivalent metrics, they are topologically equivalent.

Clearly  $d$  is symmetric and  $d(x, x) = 0$ . Moreover, if  $d(x, y) = 0$ , then  $\rho(x, y) = 0$ , giving  $x = y$  if  $\rho$  is a metric. For the triangle inequality, consider the function  $g : [0, \infty) \rightarrow [0, 1)$  given by  $g(t) = \frac{t}{1+t}$ . We have that  $g'(t) = \frac{(1+t)-t}{(1+t)^2} = \frac{1}{(1+t)^2} > 0$ , whence  $g$  is strictly increasing. Since we know  $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ , observe that:

$$\begin{aligned} d(x, z) &= \frac{\rho(x, z)}{1 + \rho(x, z)} \\ &\leq \frac{\rho(x, y) + \rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \\ &= \frac{\rho(x, y)}{1 + \rho(x, y) + \rho(y, z)} + \frac{\rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \\ &\leq \frac{\rho(x, y)}{1 + \rho(x, y)} + \frac{\rho(y, z)}{1 + \rho(y, z)} \\ &= d(x, y) + d(y, z). \end{aligned}$$

- (7) If  $d_1, \dots, d_n$  are metrics on  $X$  and  $c_1, \dots, c_n > 0$ , then:

$$d(x, y) = \sum_{i=1}^n c_i d_i(x, y)$$

is a metric on  $X$ .

- (8) Let  $\{\rho_k\}_{k=1}^\infty$  be a family of semi-metrics on  $X$ . Assume that the family is *separating*: if  $x, y \in X$  and  $x \neq y$ , then there exists  $k$  such that  $\rho_k(x, y) \neq 0$ . Let  $d_k(x, y) = \frac{\rho_k(x, y)}{1 + \rho_k(x, y)}$ . Then:

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x, y)$$

is a metric on  $X$ . Since  $0 \leq d_k(x, y) \leq 1$ , by comparison  $d(x, y)$  will converge.

- (9) Let  $(X_k, \rho_k)_{k \geq 1}$  be a sequence of metric spaces. For each  $k$  let  $d_k$  be as above. Let  $X = \prod_{k=1}^\infty X_k$ . Then the map  $D : X \times X \rightarrow [0, \infty)$  defined by

$$D(f, g) = \sum_{k=1}^{\infty} 2^{-k} d_k(f(k), g(k))$$

is a metric on  $X$ . Note that we need not make  $d_k$  from  $\rho_k$  if all the  $d_k$  are uniformly bounded.

- (10) Let  $X = \{0, 2\}$  with the discrete metric. The *abstract Cantor set* is  $\Delta := \prod_{k \in \mathbf{N}} X$ . Then the map  $D : \Delta \times \Delta \rightarrow [0, \infty)$  defined by

$$D(f, g) = \sum_{k=1}^{\infty} 3^{-k} |f(k) - g(k)|$$

is a metric on  $\Delta$ .

- (11) Let  $\langle \cdot, \cdot \rangle$  be the standard inner product on  $\mathbf{R}^3$  (or  $\mathbf{R}^n$ ). The unit sphere  $S^2 = \{x \in \mathbf{R}^3 \mid \|x\|_2 = 1\}$  paired with  $d(x, y) := \arccos(\langle x, y \rangle)$  is a metric space.

**Exercise 2.1.2.** Show that  $(S^2, d)$  defined as above is indeed a metric space.

**Definition 2.1.3.**

- (1) Let  $(X, d)$  be a metric space with  $E \subseteq X$ . The *diameter* of  $E$  is  $\text{diam}(E) = \sup_{x, y \in E} d(x, y)$ . We say  $E$  is *bounded* (metrically) if  $\text{diam}(E) < \infty$ .
- (2) If  $\Omega$  is any set and  $(Y, d)$  is a metric space,  $f : \Omega \rightarrow Y$  is *bounded* if  $\text{diam}(f(\Omega)) < \infty$ . The set of bounded functions is  $\text{Bd}(\Omega, Y) := \{f : \Omega \rightarrow Y \mid f \text{ bounded}\}$ .

**Exercise 2.1.3.** If  $(V, \|\cdot\|)$  is a normed space and  $E \subseteq V$ , the following are equivalent:

- (1)  $E$  is bounded;
- (2)  $\sup_{v \in E} \|v\| < \infty$ ;
- (3) there exists  $r > 0$  such that  $E \subseteq B(0, r)$ .

**Example 2.1.2.** The set  $\text{Bd}(\Omega, Y)$  is a metric space with:

$$D_u(f, g) := \sup_{x \in \Omega} d(f(x), g(x)).$$

Clearly  $D_u(f, g) = D_u(g, f)$  and  $D_u(f, f) = 0$ . If  $D_u(f, g) = 0$ , then  $f(x) = g(x)$  for all  $x \in \Omega$ , giving  $f = g$ . Moreover, for every  $x \in \Omega$ :

$$\begin{aligned} d(f(x), h(x)) &\leq d(f(x), g(x)) + d(g(x), h(x)) \\ &\leq D_u(f, g) + D_u(g, h). \end{aligned}$$

Whence  $D_u(f, h) \leq D_u(f, g) + D_u(g, h)$ . Note that if we take the normed space  $(\ell_\infty(\Omega), \|\cdot\|_u)$ , the induced metric is:

$$\begin{aligned} d(f, g) &= \|f - g\|_u \\ &= \sup_{x \in \Omega} |(f - g)(x)| \\ &= \sup_{x \in \Omega} |f(x) - g(x)| \\ &= D_u(f, g). \end{aligned}$$

So as metric spaces,  $\ell_\infty(\Omega) \cong \text{Bd}(\Omega, F)$ . Now consider the subset  $E = \{f \in \text{Bd}(\Omega, F) \mid f(x) \in \{0, 1\}\}$ . We get:

$$\begin{aligned} D_u(f, g) &= \sup_{x \in \Omega} |f(x) - g(x)| \\ &= \begin{cases} 0, & f = g \\ 1, & f \neq g. \end{cases} \end{aligned}$$

So  $(E, D_u)$  is discrete.

## § 2.2. Topology of Metric Spaces

Unless otherwise stated, let  $(X, d)$  be a metric space.

**Definition 2.2.1.** Let  $X$  be a set. A collection  $T$  of subsets of  $X$  is called a *topology on  $X$*  if they satisfy:

- (1)  $\emptyset, X \in T$ ;
- (2) arbitrary unions of elements in  $T$  are in  $T$ ;
- (3) finite intersections of elements in  $T$  are in  $T$ .

**Definition 2.2.2.** Let  $(X, d)$  be a metric space.

- (1) Let  $x_0 \in X$  and  $\delta > 0$ .
  - (i) The *open ball* centered at  $x_0$  of radius  $\delta$  is  $U(x_0, \delta) = \{x \mid d(x, x_0) < \delta\}$ .
  - (ii) The *closed ball* centered at  $x_0$  of radius  $\delta$  is  $B(x_0, \delta) = \{x \mid d(x, x_0) \leq \delta\}$ .
  - (iii) The *sphere* centered at  $x_0$  of radius  $\delta$  is  $S(x_0, \delta) = \{x \mid d(x, x_0) = \delta\}$ .
- (2) A subset  $U \subseteq X$  is *open in  $X$*  if:

$$(\forall x \in U)(\exists \delta > 0) : U(x, \delta) \subseteq U.$$

The collection of open sets is denoted  $\tau_X := \{U \subseteq X \mid U \text{ is open}\}$ .

- (3) A subset  $D \subseteq X$  is *closed in  $X$*  if  $D^c \subseteq X$  is open in  $X$ .
- (4) If  $x \in U \in \tau_X$ , then  $U$  is called an *open neighborhood of  $x$* . If  $x \in U \in \tau_X$  and  $U \subseteq N \subseteq X$ , then  $N$  is called a *neighborhood of  $x$* . The collection of neighborhoods of  $x$  is denoted  $\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$ .
- (5) Let  $A \subseteq X$ .

- (i) The *interior of  $A$*  is:

$$A^\circ := \bigcup \{V \in \tau_X \mid V \subseteq A\}.$$

- (ii) The *closure of  $A$*  is:

$$\overline{A} := \bigcap \{C \mid C \supseteq A, C \text{ closed}\}.$$

- (iii) The *boundary of  $A$*  is  $\partial A := \overline{A} \setminus A^\circ$ .

**Exercise 2.2.1.** Show that  $\overline{A^c} = (A^o)^c$  and  $\overline{A}^c = (A^c)^o$ .

**Proposition 2.2.1.** Let  $(X, d)$  be a metric space. The open sets  $\tau_X$  form a topology.

*Proof.* Both  $\emptyset$  and  $X$  are open by assumption. Let  $\{V_i\}_{i \in I}$  be a family of open sets of  $X$ . Let  $x \in \bigcup_{i \in I} V_i$ . Then  $x \in V_i$  for some  $i$ . Since  $V_i$  is open, there exists  $\delta > 0$  with  $B(x, \delta) \subseteq V_i \subseteq \bigcup_{i \in I} V_i$ . Whence the arbitrary union of open sets is open.

Now let  $\{V_k\}_{k=1}^n$  be a family of open sets. Let  $x \in \bigcap_{k=1}^n V_k$ . Then  $x \in V_k$  for all  $k$ . Since  $V_k$  is open, there exists  $\delta_k > 0$  with  $B(x, \delta_k) \subseteq V_k$ . Pick  $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n\}$ . Then  $B(x, \delta) \subseteq \bigcap_{k=1}^n V_k$ . Whence  $\bigcap_{k=1}^n V_k$  is open.  $\square$

Note that an arbitrary intersection of open sets is not necessarily open. Consider the sequence of intervals  $(I_n)_n = (-\frac{1}{n}, \frac{1}{n})$ . Then  $\bigcap_{n=1}^{\infty} I_n = \{0\}$ , which is closed.

**Exercise 2.2.2.** Let  $(X, d)$  be a metric space and consider a collection  $\mathcal{C}$  of subsets of  $X$  with  $\emptyset, X \in \mathcal{C}$ . Show that:

- (1) if  $\{C_i\}_{i \in I}$  is a family of closed sets, then  $\bigcap_{i \in I} C_i$  is closed;
- (2) if  $\{C_i\}_{i=1}^n$  is a family of closed sets then  $\bigcup_{i=1}^n C_i$  is closed.

**Proposition 2.2.2.** Let  $(X, d)$  be a metric space and  $x \in X$ .

- (1)  $N \in \mathcal{N}_x$  if and only if there exists  $\delta > 0$  such that  $U(x, \delta) \subseteq N$ .
- (2) If  $N \in \mathcal{N}_x$  and  $N \subseteq M$ , then  $M \in \mathcal{N}_x$ .
- (3) If  $N_1, N_2 \in \mathcal{N}_x$ , then  $N_1 \cap N_2 \in \mathcal{N}_x$ .

*Proof.* (1) Let  $N \in \mathcal{N}_x$ . Then there is an open set  $U$  with  $x \in U$  and  $U \subseteq N \subseteq X$ . Since  $U$  is open, there exists  $\delta > 0$  such that  $U(x, \delta) \subseteq U \subseteq N$ . Conversely, suppose there exists  $\delta > 0$  such that  $U(x, \delta) \subseteq N$ . Clearly  $U(x, \delta) \subseteq N \subseteq X$ , whence  $N \in \mathcal{N}_x$ .

(2) If  $N \in \mathcal{N}_x$ , then there is an open set  $U$  with  $x \in U$  and  $U \subseteq N \subseteq X$ . So  $U \subseteq N \subseteq M \subseteq X$ . Whence  $M \in \mathcal{N}_x$ .

(3) If  $N_1, N_2 \in \mathcal{N}_x$ , then there are open sets  $U_1, U_2$  with  $x \in U_1, x \in U_2$  and  $U_1 \subseteq N_1 \subseteq X, U_2 \subseteq N_2 \subseteq X$ . Whence  $U_1 \cap U_2 \subseteq N_1 \cap N_2 \subseteq X$ .  $\square$

**Proposition 2.2.3.** Let  $U \subseteq \mathbf{R}$  be open. Then:

$$U = \bigsqcup_{j \in J} I_j,$$

where  $J$  is countable and  $I_j$  are open intervals.

*Proof.* For each  $x \in U$ , define:

$$I_x := \bigcup \{I \mid x \in I \subseteq U, I \text{ open interval}\}.$$

Clearly  $x \in I_x \subseteq U$ . If  $s, t \in I_x$  with  $s < t$ , then there exists open intervals  $I, I'$  with  $x \in I \subseteq U, x \in I' \subseteq U$ , and  $s \in I, t \in I'$ . Since  $I \cap I' \neq \emptyset, I \cup I'$  is an open interval. Moreover, since  $s, t \in I \cup I'$ , we know  $[s, t] \subseteq I \cup I' \subseteq I_x$ . This shows  $I_x$  is an interval—in particular, since  $I_x$  is the union of open intervals, it must be open.

Now suppose  $x, y \in U$  and  $I_x \cap I_y \neq \emptyset$ . Then there exists  $z \in I_x \cap I_y$ , but  $z \in I_x$  implies  $I_x \subseteq I_z$  and  $z \in I_y$  implies  $I_y \subseteq I_z$ . But we also have  $x \in I_x \subseteq I_z$  which gives  $I_z \subseteq I_x$ ,

and similarly  $y \in I_y \subseteq I_z$  gives  $I_z \subseteq I_y$ . Together, we have  $I_x = I_y$ , which means for any  $x, y \in U$ , then  $I_x \cap I_y = \emptyset$  or  $I_x = I_y$ . Thus there exists  $J \subseteq U$  with  $U = \bigsqcup_{j \in J} I_j$ .

It remains to show that  $J$  is countable. Define  $J \rightarrow \mathbf{Q}$  by  $x \mapsto q_x$ , where  $q_x \in \mathbf{Q} \cap I_x$ . This map is injective, establishing the proposition.  $\square$

**Proposition 2.2.4.** *Let  $A \subseteq X$ .*

- (1)  $x \in A^\circ$  if and only if there exists  $\delta > 0$  such that  $U(x, \delta) \subseteq A$ .
- (2)  $x \in \bar{A}$  if and only if for all  $\delta > 0$ ,  $U(x, \delta) \cap A \neq \emptyset$ .
- (3)  $x \in \partial A$  if and only if for all  $\delta > 0$ ,  $U(x, \delta) \cap A \neq \emptyset$  and  $U(x, \delta) \cap A^c \neq \emptyset$ .

*Proof.* (1)  $x \in A^\circ \iff$  there exists  $V \in \tau_X$  with  $x \in V \subseteq A \iff$  there exists  $\delta > 0$  with  $U(x, \delta) \subseteq V \subseteq A$ .

(2) The converse gives  $x \notin \bar{A} \iff x \in (\bar{A})^c = (A^c)^\circ \iff$  there exists  $\delta > 0$  such that  $U(x, \delta) \subseteq (A^c)^\circ \subseteq A^c \iff U(x, \delta) \cap A = \emptyset$ .

(3)  $x \in \partial A \iff x \in \bar{A} \setminus A^\circ \iff x \in \bar{A} \cap (A^\circ)^c \iff x \in \bar{A} \cap \bar{A}^c \iff$  there exists  $\delta > 0$  with  $U(x, \delta) \cap A \neq \emptyset$  and  $U(x, \delta) \cap A^c \neq \emptyset$   $\square$

**Exercise 2.2.3.** Show that open balls are open, closed balls are closed, and spheres are closed.

**Proposition 2.2.5.** \*\*\* *For any normed space:*

- (1)  $\overline{U(x, \delta)} = B(x, \delta)$
- (2)  $B(x, \delta)^\circ = U(x, \delta)$
- (3)  $\partial U(x, \delta) = \partial B(x, \delta) = S(x, \delta)$ .

*Proof.* (1)

(2) Suppose  $y \in B(x, \delta)^\circ \setminus U(x, \delta)$ . Then there exists  $\delta' > 0$  with  $U(x, \delta') \subseteq B(x, \delta)$ . But this means  $d(x, y) < \delta$ , which is a contradiction. Thus  $B(x, \delta)^\circ \setminus U(x, \delta) = \emptyset$ , establishing  $B(x, \delta)^\circ = U(x, \delta)$ .  $\square$

**Proposition 2.2.6.** \*\*\* *Let  $(X, d)$  be a metric space with  $\{A_i\}_{i \in I}$  a family of subsets. Let  $K \subseteq I$  be finite.*

$$(1) \bigcup_{i \in I} A_i^\circ \subseteq \left( \bigcup_{i \in I} A_i \right)^\circ \quad (\text{Inclusion may be strict}).$$

$$(2) \overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \bar{A}_i \quad (\text{Inclusion may be strict}).$$

$$(3) \bigcap_{i \in K} A_i^\circ = \left( \bigcap_{i \in K} A_i \right)^\circ$$

$$(4) \overline{\bigcup_{i \in K} A_i} = \bigcup_{i \in K} \bar{A}_i.$$

*Proof.*  $\square$

**Proposition 2.2.7.** *Let  $S \subseteq X$ .*

- (1)  $\partial S = \partial S^c$ .
- (2)  $\partial S$  is closed.
- (3)  $\overline{S} = S \cup \partial S$ .
- (4)  $S \setminus \partial S = S^\circ$ .

*Proof.* (1) This follows from the characterization of  $\partial S$ . (2) We have  $\partial S = \overline{S} \setminus S^\circ = \overline{S} \cap (S^\circ)^c$ , which is closed. (3) Clearly  $S \cup \partial S \subseteq \overline{S}$ . Let  $x \in \overline{S}$ . If  $x \in S$  we are done. Otherwise  $x \in \overline{S} \setminus S \subseteq \overline{S} \setminus S^\circ = \partial S$ . (4) Observe that:

$$\begin{aligned}
 S \setminus \partial S &= S \cap (\partial S)^c \\
 &= S \cap (\overline{S} \setminus S^\circ)^c \\
 &= S \cap (\overline{S} \cap (S^\circ)^c)^c \\
 &= S \cap (\overline{S}^c \cup S^\circ) \\
 &= S \cap \overline{S}^c \cup S \cap S^\circ \\
 &= S^\circ.
 \end{aligned}$$

□

**Definition 2.2.3.** Let  $(X, d)$  be a metric space.

- (1) A subset  $A \subseteq X$  is *d-dense* if  $\overline{A} = X$ .
- (2) A subset  $N \subseteq X$  is *nowhere dense* if  $(\overline{N})^\circ = \emptyset$ .
- (3) The space  $(X, d)$  is *separable* if there exists a countable dense subset  $D \subseteq X$ .

**Exercise 2.2.4.** If  $N \subseteq X$  is closed, then  $N$  is nowhere dense if and only if  $N^c$  is dense.

**Proposition 2.2.8.** \*\*\* Let  $A \subseteq X$ . The following are equivalent:

- (1)  $A$  is dense;
- (2)  $(\forall U \in \tau_X), U \cap A \neq \emptyset$ ;
- (3)  $(\forall x \in X)(\forall \epsilon > 0), U(x, \epsilon) \cap A \neq \emptyset$ ;
- (4)  $(\forall x \in X)(\forall \epsilon > 0)(\exists a \in A) : d(a, x) < \epsilon$ .

*Proof.*

□

**Definition 2.2.4.** Let  $(X, d)$  be a metric space.

- (1) A *base* for  $\tau_X$  is a family of open subsets  $\mathcal{B} \subseteq \tau_X$  such that:

$$(\forall U \in \tau_X)(\forall x \in U)(\exists B \in \mathcal{B}) : x \in B \subseteq U.$$

Equivalently, for all  $U \in \tau_X$ , we can write  $U = \bigcup_{i \in I} B_i$ , where  $\{B_i\}_{i \in I} \subseteq \tau_X$ .

- (2)  $X$  is *second countable* if it has a countable base.

Note that this definition can be generalized to any topological space. Clearly  $\mathcal{B} = \{U(x, \epsilon) \mid x \in X, \epsilon > 0\}$  forms a base for any metric space.

**Example 2.2.1.** The set  $\mathcal{B} = \{U(q, \frac{1}{n}) \mid n \geq 1, q \in \mathbf{Q}^d\}$  is a base for  $\mathbf{R}^d$ .

**Proposition 2.2.9.** Let  $(X, d)$  be a metric space.  $X$  is separable if and only if  $X$  is second countable.

*Proof.* Let  $\mathcal{B} = \{U_n\}_{n=1}^\infty$  be a countable base. Choose any  $a_n \in U_n$ . Then  $\{a_n\}_{n=1}^\infty$  is dense. Indeed, given any  $x \in X$  and  $\epsilon > 0$ , there exists  $U_m$  with  $x \in U_m \subseteq U(x, \epsilon)$  (since  $U_m \in \mathcal{B}$ ). Whence  $d(a_m, x) < \epsilon$ .

Let  $\{a_n\}_{n=1}^\infty$  be dense. Consider:

$$\mathcal{B} = \{U(a_n, \frac{1}{m}) \mid n \geq 1, m \geq 1\}.$$

Clearly  $\mathcal{B}$  is countable—it remains to show that it is a base for  $X$ . Given  $x \in V \in \tau_X$ , find  $\epsilon > 0$  such that  $U(x, \epsilon) \subseteq V$ . Then there exists  $m \geq 1$  with  $\epsilon > \frac{1}{m}$ . Since  $\{a_n\}_{n=1}^\infty$  is dense, there exists  $a_j \in \{a_n\}_{n=1}^\infty$  such that  $d(a_j, x) < \frac{1}{2m}$ . Let  $y \in U(a_j, \frac{1}{2m})$ . Observe that:

$$\begin{aligned} d(x, y) &\leq d(x, a_j) + d(a_j, y) \\ &< \frac{1}{2m} + \frac{1}{2m} \\ &= \frac{1}{m} \\ &< \epsilon. \end{aligned}$$

So  $y \in U(x, \epsilon)$ . Thus  $x \in U(a_j, \frac{1}{2m}) \subseteq U(x, \epsilon) \subseteq V$ , establishing  $\mathcal{B}$  as a base.  $\square$

**Example 2.2.2.**

- (1) The space  $(\mathbf{R}^d, \|\cdot\|_p)$  is separable for any  $1 \leq p \leq \infty$ . Indeed, if  $(r_1, \dots, r_d) \in \mathbf{R}^d$  and  $\epsilon > 0$ , find  $q_j \in \mathbf{Q}$ ,  $j = 1, \dots, d$  with:

$$|r_j - q_j| < \frac{\epsilon}{d}.$$

Then:

$$\|r - q\|_1 = \sum_{j=1}^d |r_j - q_j| < \epsilon.$$

So  $\mathbf{Q}^d$  is  $\|\cdot\|_1$ -dense in  $\mathbf{R}^d$ . For  $1 \leq p \leq \infty$ , let  $C > 0$  be such that  $\|\cdot\|_p \leq C \|\cdot\|_1$ . So given  $\epsilon > 0$ , find  $q \in \mathbf{Q}^d$  with  $\|r - q\|_1 \leq \frac{\epsilon}{C}$ . Then  $\|r - q\|_p < \epsilon$ .

- (2) Similarly,  $\mathbf{C}_{\mathbf{Q}}^d \subseteq \mathbf{C}^d$  is  $\|\cdot\|_p$ -dense, where:

$$\mathbf{C}_{\mathbf{Q}} = \{a + bi \mid a, b \in \mathbf{Q}\}.$$

- (3) Recall that  $c_{00} = \{(z_k)_k \mid \text{supp}((z_k)_k) < \infty\}$ . The space  $(c_{00}, \|\cdot\|_u)$  is separable.

Note that  $c_{00} = \mathbf{C}\text{-span}\{e_k \mid k \in \mathbf{N}\}$ . This space is not countable—clearly  $\mathbf{C}\text{-span}\{e_1\} = \{\alpha e_1 \mid \alpha \in \mathbf{C}\}$  is not countable, so it must be that  $c_{00}$  is also not countable.

Instead, consider:

$$\begin{aligned} \mathbf{C}_{\mathbf{Q}}\text{-span}\{e_k \mid k \in \mathbf{N}\} &= \left\{ \sum_{k=1}^{\infty} t_k e_k \mid t_k \in \mathbf{C}_{\mathbf{Q}}, \right\} \\ &= \bigcup_{k=1}^{\infty} \{C_k \mid C_k = \mathbf{C}_{\mathbf{Q}}\text{-span}\{e_1, \dots, e_k\}\} \end{aligned}$$

Note that  $C_k$  is in bijection with  $\mathbf{Q}^{2k}$ , whence  $\mathbf{C}_{\mathbf{Q}}\text{-span}\{e_k \mid k \in \mathbf{N}\}$  is countable.

Given  $z \in c_{00}$ , let  $z = \sum_{k=1}^N z_k e_k$  and  $\epsilon > 0$ . Find  $t_k \in \mathbf{C}_{\mathbf{Q}}$  with  $|z_k - t_k| < \epsilon$ . Then:

$$\begin{aligned} \|z - t\|_u &= \left\| \sum_{k=1}^N z_k e_k - \sum_{k=1}^K t_k e_k \right\|_u \\ &= \left\| \sum_{k=1}^N (z_k - t_k) e_k \right\|_u \\ &= \sup_{k=1}^N |z_k - t_k| \\ &< \epsilon. \end{aligned}$$

Thus  $\mathbf{C}_{\mathbf{Q}}\text{-span}\{e_k \mid k \in \mathbf{N}\}$  is dense in  $c_{00}$ , whence the space  $(c_{00}, \|\cdot\|_u)$  is separable.

**Proposition 2.2.10.** *If  $(X, d)$  is a separable metric space and  $Y \subseteq X$ , then  $(Y, d)$  is separable.*

*Proof.* Let  $A = \{a_k\}_{k=1}^{\infty}$  be dense in  $X$ . Let:

$$N = \{(m, n) \mid U(a_m, \frac{1}{n}) \cap Y \neq \emptyset\}.$$

For each  $(m, n) \in N$ , choose  $b_{(m,n)} \in Y \cap U(a_m, \frac{1}{n})$ . Claim: the set

$$\{b_{(m,n)} \mid (m, n) \in N\}$$

is dense in  $Y$ . Let  $y \in Y$  and  $\epsilon > 0$ . Then there exists  $n \geq 1$  with  $\frac{\epsilon}{2} > \frac{1}{n}$ . Since  $A$  is dense,  $U(y, \frac{1}{n}) \cap A \neq \emptyset$  (this is because for all  $U \in \tau_X$ , we have  $U \cap A \neq \emptyset$ ). So  $d(a_m, y) < \frac{1}{n}$ . Whence:

$$\begin{aligned} d(b_{(m,n)}, y) &\leq d(b_{(m,n)}, a_m) + d(a_m, y) \\ &< \frac{1}{n} + \frac{1}{n} \\ &< \epsilon. \end{aligned}$$

□

**Example 2.2.3.**

(1) The space  $(\ell_{\infty}, \|\cdot\|_u)$  is not separable. If it were, consider:

$$E = \{(x_k)_k \mid x_k \in \{0, 1\}\} \subseteq \ell_{\infty}.$$



This set is uncountable, and by the previous proposition  $(E, \|\cdot\|_u)$  is also separable. Let  $a, b \in E$ . We have:

$$\begin{aligned} \|(a_k)_k - (b_k)_k\|_u &= \|(a_k - b_k)_k\|_u \\ &= \sup_{k \geq 1} |a_k - b_k| \\ &= \begin{cases} 0, & (a_k)_k = (b_k)_k \\ 1, & (a_k)_k \neq (b_k)_k \end{cases}. \end{aligned}$$

So  $(E, \|\cdot\|_u)$  is discrete. Note that a metric space is discrete if and only if its singletons are open. Whence if  $(E, \|\cdot\|_u)$  is separable, then  $A = \bar{A} = E$ , which contradicts the countability of  $A$ . It must be that  $(\ell_\infty, \|\cdot\|_u)$  is not separable.

- (2) The space  $(\ell_p, \|\cdot\|_p)$  is separable for  $1 \leq p < \infty$ . Let  $a = (a_k)_k \in \ell_p$  and  $\epsilon > 0$ . Find  $N$  large so that  $\sum_{k > N} |a_k|^p < \frac{\epsilon^p}{2}$ . Find  $b_k \in \mathbf{C}_\mathbf{Q}$  with  $|a_k - b_k| < \frac{\epsilon}{(2N)^{\frac{1}{p}}}$ . Let  $b = (b_1, b_2, \dots, b_{N-1}, b_N, 0, 0, \dots)$ . We have:

$$\begin{aligned} \|a - b\|_p^p &= \sum_{k=1}^{\infty} |a_k - b_k|^p \\ &= \sum_{k=1}^N |a_k - b_k|^p + \sum_{k > N} |a_k|^p \\ &< N \cdot \frac{\epsilon^p}{2N} + \frac{\epsilon^p}{2} \\ &= \epsilon. \end{aligned}$$

Whence the set  $\mathbf{C}_\mathbf{Q}$ -span $\{e_k \mid k \in \mathbf{N}\}$  is  $\|\cdot\|_u$ -dense, giving  $(\ell_p, \|\cdot\|_p)$  as separable.

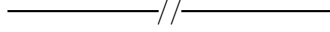
- (3) We will eventually show that the set of polynomial functions:

$$P([0, 1]) = \left\{ \sum_{k=0}^n a_k x^k \mid a_k \in F, n \geq 0 \right\}$$

is  $\|\cdot\|_u$ -dense in  $C([0, 1])$  (note that this set is not countable). With this fact, we can show that  $C([0, 1])$  is separable. Indeed, given  $f \in C([0, 1])$  and  $\epsilon > 0$ , find  $p \in P([0, 1])$  with  $\|f - p\|_u < \frac{\epsilon}{2}$ . Now let  $p(x) = \sum_{k=0}^n a_k x^k$ . Find  $b_k \in \mathbf{C}_\mathbf{Q}$  with  $|a_k - b_k| < \frac{\epsilon}{2(n+1)}$  and define  $q(x) = \sum_{k=0}^n b_k x^k$ . Observe that:

$$\begin{aligned} \|f - q\|_u &= \|f - p + p - q\|_u \\ &\leq \|f - p\|_u + \|p - q\|_u \\ &= \|f - p\|_u + \sum_{k=0}^n |a_k - b_k| \\ &< \frac{\epsilon}{2} + (n+1) \cdot \frac{\epsilon}{2(n+1)} \\ &= \epsilon. \end{aligned}$$

Thus the set  $\mathbf{C}_\mathbf{Q}$ -span $\{x^k \mid k \in \mathbf{N}\}$  is  $\|\cdot\|_u$ -dense in  $C([0, 1])$ . In particular, since it is countable,  $C([0, 1])$  is separable.



We have seen that subsets of metric spaces are metric spaces in their own right. Then what are their open sets?

**Proposition 2.2.11.** *Let  $(X, d)$  be a metric space and let  $Y \subseteq X$  be any subset. Then  $V \subseteq Y$  is open in  $Y$  if and only if there exists an open set  $U \subseteq X$  with  $U \cap Y = V$ . That is,  $\tau_Y = \{U \cap Y \mid U \in \tau_X\}$ .*

*Proof.* Let  $V \subseteq Y$  be open. Then for every  $x \in V$ , there exists  $\delta_x > 0$  with  $U_y(x, \delta_x) \subseteq V$ , where:

$$\begin{aligned} U_y(x, \delta_x) &= \{y \in Y \mid d(y, x) < \delta_x\} \\ &= U(x, \delta_x) \cap Y. \end{aligned}$$

Set  $U = \bigcup_{x \in V} U(x, \delta_x)$ . Then  $U$  is indeed open in  $X$ . Also:

$$\begin{aligned} U \cap Y &= \left( \bigcup_{x \in V} U(x, \delta_x) \right) \cap Y \\ &= \bigcup_{x \in V} (U(x, \delta_x) \cap Y) \\ &= \bigcup_{x \in V} U_y(x, \delta_x) \\ &= V. \end{aligned}$$

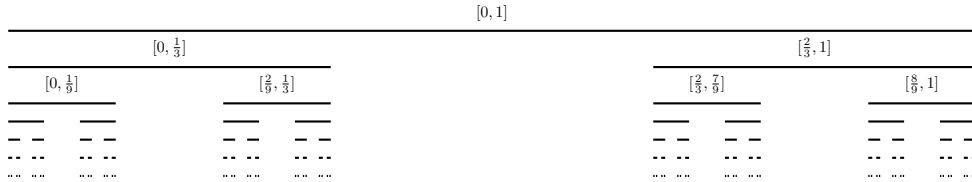
Conversely, suppose  $V = U \cap Y$  for some open  $U \in \tau_X$ . Let  $x \in V$ . Since  $x \in U$  and  $U$  is open, there exists  $\delta > 0$  such that  $U(x, \delta) \subseteq U$ . So  $U_y(x, \delta) = U(x, \delta) \cap Y \subseteq U \cap Y = V$ . Thus  $V$  is open in  $Y$ .  $\square$

**Example 2.2.4.**

- (1)  $[0, \frac{1}{2}]$  is not open in  $\mathbf{R}$ , but it is open in  $[0, 1]$ .
- (2)  $\ell_\infty$  is not a discrete metric space, but  $\{0, 1\}^{\mathbf{N}} \subseteq \ell_\infty$  is.

## § 2.3. The Cantor Set

Given the interval  $[0, 1]$ , start by deleting the open middle third  $(\frac{1}{3}, \frac{2}{3})$ , leaving two line segments  $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . Next, the open middle third of each of these remaining segments is deleted, leaving four line segments  $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ .



We are interested in studying the topological properties of the points which are not deleted at any step of this infinite process. The set of these points have a special name and are defined below.

**Definition 2.3.1.** Let  $C_0 := [0, 1]$  and  $C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$  for  $n \geq 1$ . The *Cantor set* is  $\mathfrak{C} = \bigcap_{n=0}^{\infty} C_n$ .

**Proposition 2.3.1.** *The Cantor set is closed.*

*Proof.* Since the Cantor set is defined as the intersection of closed sets, it must be closed.  $\square$

**Proposition 2.3.2.** *The Cantor set is nowhere dense.*

*Proof.* Suppose towards contradiction its not, that is,  $\overline{\mathfrak{C}}^o \neq \emptyset$ . Then there is some  $x \in \overline{\mathfrak{C}}^o$ . We can find an  $\epsilon > 0$  with  $(x - \epsilon, x + \epsilon) \subseteq \mathfrak{C}$ , in particular  $(x - \epsilon, x + \epsilon) \subseteq C_n$  for all  $n \geq 1$ . Find  $m$  large so that  $\epsilon > \frac{1}{3^m}$  and consider  $(x - \epsilon, x + \epsilon) \subseteq C_m$ . We have that  $C_m = \bigsqcup_{j=1}^{2^m} C_{m,j}$  where  $\text{length}(C_{m,j}) = \frac{1}{3^m}$ . Since each  $C_{m,j}$  is disjoint, it must be the case that  $(x - \epsilon, x + \epsilon) \subseteq C_{m,j}$  for some  $1 \leq j \leq 2^m$ . But the length of  $(x - \epsilon, x + \epsilon)$  is  $2\epsilon$ , which is impossible. It must be that  $\mathfrak{C}$  is nowhere dense.  $\square$

**Proposition 2.3.3.** *The total length of the Cantor set is 0.*

*Proof.* The total length of the removed intervals is:

$$\begin{aligned} \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots &= \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \\ &= \frac{1}{2} \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} \\ &= 1. \end{aligned}$$

Thus  $\text{length}(\mathfrak{C}) = 0$ .  $\square$

**Lemma 2.3.4.** \*\*\*

*Proof.*  $\square$

**Lemma 2.3.5.** \*\*\*

*Proof.*  $\square$

**Lemma 2.3.6.** \*\*\*

*Proof.*  $\square$

**Proposition 2.3.7.** \*\*\*  $\text{card}(\mathfrak{C}) = \mathfrak{c}$ .

*Proof.*  $\square$

## § 2.4. Convergent Sequences

**Definition 2.4.1.** Fix a metric space  $(X, d)$ .

- (1) A *sequence* in  $X$  is a map  $x : \mathbf{N} \rightarrow X$  defined by  $n \mapsto x_n$ . We denote a sequence as  $(x_n)_{n \geq 1}$ ,  $(x_n)_{n=1}^\infty$ , or  $(x_n)_n$ .
- (2) A *natural sequence* is a sequence  $(x_n)_n$  in  $\mathbf{N}$  with  $n_1 < n_2 < n_3 < \dots$
- (3) A *subsequence* of a sequence  $(x_n)_n$  is a sequence  $(x_{n_k})_k$  where  $(n_k)_k$  is a natural sequence. This is equivalent to the composition of maps  $\mathbf{N} \xrightarrow{k \mapsto n_k} \mathbf{N} \xrightarrow{n_k \mapsto x_{n_k}} X$ .

**Definition 2.4.2.** A sequence  $(x_n)_n$  *converges* to  $x \in X$  if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geq N \implies d(x_n, x) < \epsilon).$$

We write  $(x_n)_n \xrightarrow{d} x$  or limit  $x_n = x$ .

**Exercise 2.4.1.** Show that a sequence can have at most one limit.

**Proposition 2.4.1.** *Given  $(x_n)_n$  in  $X$  and  $x \in X$ , the following are equivalent:*

- (1)  $(x_n)_n \rightarrow x$  in  $X$ ;
- (2)  $(d(x_n, x))_n \rightarrow 0$  in  $\mathbf{R}$ ;
- (3)  $(\forall V \in \mathcal{N}_x)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geq N \implies x_n \in V)$ .

**Exercise 2.4.2.** Let  $(X, d)$  be a metric space and  $\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ . Then  $(x_n)_n \xrightarrow{d} x$  if and only if  $(x_n)_n \xrightarrow{\rho} x$ .

**Proposition 2.4.2.** *Convergent sequences are bounded.*

*Proof.* Suppose that  $(x_n)_n \rightarrow x$  and let  $\epsilon = 1$ . Find  $N$  large so that for  $n \geq N$  we have  $d(x_n, x) < 1$ . Then for all  $m, n \geq N$ , we have  $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < 2$ . Set  $C = \max_{1 \leq n, m \leq N} d(x_m, x_n)$ . Now if  $n \geq N$  and  $m \leq N$ , we have:

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_N) + d(x_N, x_m) \\ &\leq 1 + C. \end{aligned}$$

Let  $K = \max\{2, 1 + C, C\}$ . Then  $\text{diam}(\{x_n\}_{n \geq 1}) = \sup_{m, n \geq 1} d(x_n, x_m) \leq K$ . □

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**Definition 2.4.3.** Let  $(f_n)_n$  be a sequence of functions.

- (1)  $(f_n)_n$  *converges pointwise* to  $f \in \mathcal{F}(\Omega, F)$  if:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N_{x, \epsilon} \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geq N \implies |f_n(x) - f(x)| < \epsilon).$$

- (2)  $(f_n)_n$  *converges uniformly* to  $f \in \mathcal{F}(\Omega, F)$  if:

$$(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) : (\forall n \in \mathbf{N})(\forall x \in \Omega)(n \geq N \implies |f_n(x) - f(x)| < \epsilon).$$

**Example 2.4.1.** Recall that a sequence  $(f_n)_n$  converges to  $f$  in  $\text{Bd}(\Omega, Y)$  if:

$$(D(f_n, f))_n \rightarrow 0 \text{ in } \mathbf{R}.$$

But this is equivalent to:

$$\sup_{x \in \Omega} d(f_n(x), f(x)) \rightarrow 0 \text{ in } \mathbf{R}.$$

Again, we can rewrite this as:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N}) \left( n \geq N \implies \sup_{x \in \Omega} d(f_n(x), f(x)) < \epsilon \right).$$

But this definition is equivalent to:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(\forall x \in \Omega) \left( n \geq N \implies d(f_n(x), f(x)) < \epsilon \right).$$

This is precisely the definition of uniform convergence. Thus convergence in  $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq \ell_\infty(\Omega) = \text{Bd}(\Omega, F)$  is uniform.

**Proposition 2.4.3.** Let  $(d_k)_k$  be a separating family of semi-metrics which are uniformly bounded. Define:

$$d(x, y) := \sum_{k=1}^{\infty} 2^{-k} d_k(x, y).$$

Then  $(x_n)_n \xrightarrow{d} x$  if and only if for all  $k$ ,  $(d_k(x_n, x))_n \rightarrow 0$ .

*Proof.* Fix  $k$ . We have:

$$0 \leq 2^{-k} d_k(x_n, x) \leq d(x_n, x).$$

Note that  $(x_n)_n \xrightarrow{d} x$  implies  $(d(x_n, x))_n \rightarrow 0$ . Multiplying both sides of the above equation by  $2^k$  and applying the squeeze theorem gives  $(d_k(x_n, x))_n \rightarrow 0$ .

Now let  $\epsilon > 0$ . Since each  $d_k$  is uniformly bounded, let  $C = \sup_{k,x,y} d_k(x, y)$ . Find  $K$  large so that:

$$\sum_{k > K} 2^{-k} < \frac{\epsilon}{2C}.$$

We know  $(d_k(x_n, x))_n \rightarrow 0$  for  $k = 1, 2, \dots, K$ . So there exists  $N_1, N_2, \dots, N_K$  with  $n \geq N_k$  implying  $d_k(x_n, x) < \frac{\epsilon}{2}$ . Let  $N = \max_{k=1}^K N_k$ . For  $n \geq N$ , we have:

$$\begin{aligned} d(x_n, x) &= \sum_{k=1}^K 2^{-k} d_k(x_n, x) + \sum_{k > K} 2^{-k} d_k(x_n, x) \\ &= \sum_{k=1}^K 2^{-k} d_k(x_n, x) + \sum_{k > K} 2^{-k} C \\ &< \sum_{k=1}^K 2^{-k} \frac{\epsilon}{2} + C \cdot \frac{\epsilon}{2C} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

**Example 2.4.2.** Let  $X = C(\mathbf{R})$ . How does one define a distance between two functions on this space? For  $f, g \in C(\mathbf{R})$ , one might naively consider the uniform metric:

$$d(f, g) := \sup_{x \in \Omega} |f(x) - g(x)|.$$

This construction does not guarantee  $d(f, g) < \infty$ . We can fix this as follows—for each  $k \geq 1$ , let:

$$P_k(f) = \sup_{x \in [-k, k]} |f(x)|.$$

Then  $\{P_k\}_k = 1^\infty$  is a separating family of semi-norms which generate a separating family of semi-metrics:

$$\rho_k(f, g) := P_k(f - g).$$

These are not uniformly bounded, so make:

$$d_k(f, g) = \frac{\rho_k(f, g)}{1 + \rho_k(f, g)}.$$

Then  $\{d_k\}_{k=1}^\infty$  is a family of uniformly bounded semi-metrics. We can now define the *Fréchet metric*:

$$d_F(f, g) := \sum_{k=1}^{\infty} 2^{-k} d_k(f, g).$$

In  $(C(\mathbf{R}), d_F)$ , observe that:

$$\begin{aligned} (f_n)_n \xrightarrow{d_F} f &\iff \forall k, (d_k(f_n, f))_n \rightarrow 0 \\ &\iff \forall k, (\rho_k(f_n, f))_n \rightarrow 0 \\ &\iff \forall k, \left( \sup_{x \in [-k, k]} |f_n(x) - f(x)| \right)_n \rightarrow 0 \\ &\iff \forall k, (f_n)_n \rightarrow f \text{ uniformly on } [-k, k]. \end{aligned}$$

We've obtained a new type of convergence defined below.

**Definition 2.4.4.** Let  $(X, \tau)$  be a topological space and  $(Y, d_Y)$  be a metric space. A sequence of functions:

$$f_n : X \rightarrow Y; \quad n \in \mathbf{N}$$

converges *compactly* to  $f : X \rightarrow Y$  if, for every compact set  $K \subseteq X$ , the sequence  $(f_n|_K)_n \rightarrow f|_K$  converges uniformly.

**Example 2.4.3.** \*\*\*  $x^n$  does not converge uniformly but it does converge compactly (idk how to do this)

—————//—————

**Example 2.4.4.** Let  $(X, d)$ ,  $(Y, \rho)$  be metric spaces. There are different metrics we may put on  $X \times Y$ , for example:

- $D_1((x, y), (x', y')) = d(x, x') + \rho(y, y')$
- $D_2((x, y), (x', y')) = \sqrt{d(x, x')^2 + \rho(y, y')^2}$
- $D_\infty((x, y), (x', y')) = \max\{d(x, x') + \rho(y, y')\}$ .

These metrics are all equivalent. A sequence of points  $(x_n, y_n)_n$  converges to  $(x, y)$  in  $X \times Y$  with respect to any of these metrics if and only if  $(x_n)_n \xrightarrow{d} x$  and  $(y_n)_n \xrightarrow{\rho} y$ .

**Example 2.4.5.** Let  $\{(X_k, d_k)_k\}$  be a family of metric spaces where the  $d_k$  are uniformly bounded. We looked at the product:

$$X = \prod_{k=1}^{\infty} X_k$$

with:

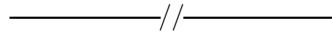
$$d(f, g) = \sum_{k=1}^{\infty} 2^{-k} d_k(f(k), g(k)).$$

We have that  $(f_n)_n \xrightarrow{d} f$  if and only if  $(f_n(k))_n \rightarrow f(k)$  for all  $k$  (pointwise convergence).

**Example 2.4.6.** Let  $(V, \|\cdot\|)$  be a normed space. Then  $(v_n)_n \rightarrow v$  if and only if  $(\|v_n - v\|)_n \rightarrow 0$ .

**Definition 2.4.5.** Let  $(v_k)_k$  be a sequence in  $(V, \|\cdot\|)$ .

- (1) A *sequence of partial sums*  $(s_n)_n$  is defined as  $s_n = \sum_{k=1}^n v_k$ .
- (2) If  $(s_n)_n \rightarrow s$  in  $V$  we say the *series*  $\sum_{k=1}^{\infty} v_k$  converges and write  $\sum_{k=1}^{\infty} v_k = s$ .
- (3) The series  $\sum v_k$  converges *absolutely* if  $\sum \|v_k\|$  converges.



**Proposition 2.4.4.** Let  $(X, d)$  be a metric space and  $A \subseteq X$ . We have  $x \in \overline{A}$  if and only if there exists a sequence  $(a_n)_n$  in  $A$  with  $(a_n)_n \rightarrow x$ .

*Proof.* Let  $x \in \overline{A}$ . For each  $n \geq 1$ , we have  $U(x, \frac{1}{n}) \cap A \neq \emptyset$ , so choose  $a_n \in U(x, \frac{1}{n}) \cap A$ . Then  $d(x, a_n) < \frac{1}{n}$ , so  $(a_n)_n \rightarrow x$ .

Now suppose  $(a_n)_n$  is a sequence in  $A$  and  $(a_n)_n \rightarrow x$ . Given  $\epsilon > 0$ , find  $N$  large so  $d(x, a_N) < \epsilon$ . Then  $a_N \in U(x, \epsilon) \cap A$ . Since  $U(x, \epsilon) \cap A \neq \emptyset$ , we have  $x \in \overline{A}$ .  $\square$

**Proposition 2.4.5.** Let  $(X, d)$  be a metric space and  $A \subseteq X$ . The following are equivalent:

- (1)  $A$  is closed;
- (2) If  $(a_n)_n$  is a sequence in  $A$  which converges to  $x \in X$ , then  $x \in A$ .

*Proof.* Consider the sequence  $(a_n)_n$  in  $A$  with  $(a_n)_n \rightarrow x$ . Then by the previous proposition  $x \in \bar{A}$ . But since  $A$  is closed,  $A = \bar{A}$ . Thus  $x \in A$ .

To show  $A$  is closed we show  $A = \bar{A}$ . Clearly  $A \subseteq \bar{A}$ . Let  $x \in \bar{A}$ . Then there exists a sequence  $(a_n)_n$  in  $A$  with  $(a_n)_n \rightarrow x$ . So  $x \in A$  by (2). Thus  $\bar{A} \subseteq A$ .  $\square$

**Exercise 2.4.3.** Show  $x \in \bar{A}$  if and only if  $x \in A$  or there exists a sequence  $(a_n)_n$  in  $A \setminus \{a\}$  with  $(a_n)_n \rightarrow x$ .

**Example 2.4.7.** \*\*\* The space  $c_0 \subseteq \ell_\infty$  is closed, and furthermore  $\overline{c_{00}} = c_0$ . To see this, consider a sequence  $(z_n)_n$  in  $c_0$  converging to  $f \in \ell_\infty$ . We will show that  $f \in c_0$ .

Let  $\epsilon > 0$  be given. Find  $N$  large so that:

$$\|z_N - f\|_u < \frac{\epsilon}{2}.$$

Since  $z_N \in c_0$ , we know  $\lim_{k \rightarrow \infty} z_N(k) = 0$ . So find  $K$  large so that for  $k \geq K$ :

$$|z_N(k)| < \frac{\epsilon}{2}.$$

Putting this all together, for  $k \geq K$  we have:

$$\begin{aligned} |f(k)| &= |f(k) - z_N(k) + z_N(k)| \\ &\leq |f(k) - z_N(k)| + |z_N(k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus  $f \in c_0$ . By Proposition 2.4.5,  $c_0$  is closed.

**Definition 2.4.6.** Let  $(X, d)$  be a metric space and  $A \subseteq X$ . The *distance* of an element  $x \in X$  to  $A$  is a map  $\text{dist}_A : X \rightarrow [0, \infty)$  defined by  $\text{dist}_A(x) = \inf_{a \in A} d(x, a)$ .

**Proposition 2.4.6.** \*\*\*

## § 2.5. Continuity

**Definition 2.5.1.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A map  $f : X \rightarrow Y$  is *continuous* at  $x_0 \in X$  if any of the equivalent definitions are satisfied:

- (1)  $(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X) \left( d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \epsilon \right)$
- (2)  $(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X) \left( x \in U_X(x_0, \delta) \implies f(x) \in U_Y(f(x_0), \epsilon) \right)$
- (3)  $(\forall \epsilon > 0)(\exists \delta > 0) : f(U_X(x_0, \delta)) \subseteq U_Y(f(x_0), \epsilon)$

**Proposition 2.5.1.** Let  $f : (X, d) \rightarrow (Y, \rho)$  be a map between metric spaces and  $x_0 \in X$ . The following are equivalent:

- (1)  $f$  is continuous at  $x_0$ ;
- (2)  $(\forall V \in \mathcal{N}_{f(x_0)})(\exists U \in \mathcal{N}_{x_0}) : f(U) \subseteq V$ ;



$$(3) (\forall (x_n)_n \in X^{\mathbf{N}})((x_n)_n \rightarrow x_0 \implies (f(x_n))_n \rightarrow f(x_0))$$

*Proof.* (1) $\implies$ (2) follows from Definition 2.5.1.

(1) $\implies$ (3) Let  $(x_n)_n \rightarrow x_0$ . Let  $\epsilon > 0$ . Since  $f$  is continuous, find  $\delta > 0$  so that  $x \in U(x, \delta)$  implies  $f(x) \in U(f(x_0), \epsilon)$ . Whence  $d(f(x_n), f(x_0)) < \epsilon$ , establishing  $(f(x_n))_n \rightarrow f(x_0)$ .

(3) $\implies$ (1) We prove the contrapositive of this statement. If  $f$  is not continuous, choose  $\epsilon_0 > 0$  so that  $d(x_n, x_0) < \frac{1}{n}$  and  $d(f(x_n), f(x_0)) \geq \epsilon_0$ . Whence  $(x_n)_n \rightarrow x_0$  and  $(f(x_n))_n \not\rightarrow f(x_0)$ .  $\square$

**Proposition 2.5.2.** *Let  $f : (X, d) \rightarrow (Y, \rho)$  be a map of metric spaces. The following are equivalent:*

- (1)  $f$  is continuous;
- (2)  $(\forall V \in \tau_Y), f^{-1}(V) \in \tau_X$ ;

*Proof.* Let  $V \subseteq Y$  be open. If  $f^{-1}(V) = \emptyset$ , we're done. If not, let  $x \in f^{-1}(V)$ . Then  $f(x) \in V$ . Since  $V$  is open, find  $\epsilon > 0$  so that  $U(f(x), \epsilon) \subseteq V$ . Since  $f$  is continuous, find  $\delta > 0$  so that  $f(U(x, \delta)) \subseteq U(f(x), \epsilon) \subseteq V$ . Whence  $U(x, \delta) \subseteq f^{-1}(V)$ .

Let  $x \in X$  and  $\epsilon > 0$ . Since  $U(f(x), \epsilon) \in \tau_Y$ , we have  $f^{-1}(U(f(x), \epsilon)) \in \tau_X$ . Note that  $x \in f^{-1}(U(f(x), \epsilon))$ . Since this set is open, find  $\delta > 0$  so that  $U(x, \delta) \subseteq f^{-1}(U(f(x), \epsilon))$ . Thus  $f(U(x, \delta)) \subseteq U(f(x), \epsilon)$ ; i.e.,  $f$  is continuous.  $\square$

**Proposition 2.5.3.** *Let  $(X, d) \xrightarrow{f} (Y, \rho) \xrightarrow{g} (Z, \gamma)$  be maps of metric spaces. If  $f$  is continuous at  $x \in X$  and  $g$  is continuous at  $y = f(x)$ , then  $g \circ f$  is continuous at  $x$ .*

*Proof.* Exercise.  $\square$

**Proposition 2.5.4.** *Let  $(X, d)$  be a metric space with  $A \subseteq X$  dense. Let  $f : X \rightarrow F$  be a continuous and bounded function. Then  $\sup_{x \in A} f(x) = \sup_{x \in X} f(x)$ .*

*Proof.* Since  $A \subseteq X$  we have  $\sup_{x \in A} f(x) \leq \sup_{x \in X} f(x)$ . Conversely, let  $\epsilon > 0$ . Find  $x' \in X$  so that  $\sup_{x \in X} f(x) - \frac{\epsilon}{2} < f(x')$ . Since  $f$  is continuous, find  $\delta > 0$  so that for all  $x \in X$ ,  $d(x, x') < \delta$  implies  $|f(x) - f(x')| < \frac{\epsilon}{2}$ . Since  $A$  is dense, find  $a \in A$  so that  $d(x', a) < \delta$ . This implies  $|f(x') - f(a)| < \frac{\epsilon}{2}$ , or equivalently  $f(x') - \frac{\epsilon}{2} < f(a)$ . This gives:

$$\sup_{x \in X} f(x) - \epsilon < f(a).$$

Hence for  $\epsilon > 0$  we have:

$$\sup_{x \in X} f(x) < \sup_{x \in A} f(x) + \epsilon.$$

Taking  $\epsilon \rightarrow 0$  gives  $\sup_{x \in X} f(x) \leq \sup_{x \in A} f(x)$ .  $\square$

**Definition 2.5.2.** Let  $f : (X, d) \rightarrow (Y, \rho)$  be a map of metric spaces.

- (1)  $f$  is *uniformly continuous* if:

$$(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x, x' \in X)(d(x, x') < \delta \implies \rho(f(x), f(x')) < \epsilon).$$

- (2)  $f$  is *Lipschitz* if there exists  $C > 0$  such that:

$$(\forall x, x' \in X)(\rho(f(x), f(x')) \leq C d(x, x')).$$

If  $C < 1$ , we say  $f$  is *contractive*.

(3)  $f$  is *isometric* (or an *isometry*) if:

$$(\forall x, x' \in X)(\rho(f(x), f(x')) = d(x, x'))$$

**Exercise 2.5.1.** Show that Lipschitz implies uniform continuity. Show that uniform continuity implies continuity. Show that the converse direction fails in general.

**Example 2.5.1.** Let  $(V, \|\cdot\|)$  be a normed space. Then  $V \xrightarrow{\|\cdot\|} [0, \infty)$  is continuous. Indeed, we have  $|\|v\| - \|w\|| \leq \|v - w\|$ , so  $\|\cdot\|$  is Lipschitz.

**Example 2.5.2.** Let  $(X, d)$  be a metric space and equip  $X \times X$  with the product metric  $D_1$ . Claim:  $d : X \times X \rightarrow [0, \infty)$  is continuous. Indeed, given  $(x, y), (x', y') \in X \times X$ , then we have:

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y).$$

So we have  $d(x, y) - d(x', y') \leq d(x, x') + d(y, y')$ . But this is equivalent to  $|d(x, y) - d(x', y')| \leq D_1((x, y), (x', y'))$ . So  $d$  is Lipschitz.

**Example 2.5.3.** If  $(X, d)$  is a metric space and  $A \subseteq X$ , then  $\text{dist}_A : X \rightarrow [0, \infty)$  is continuous. We've shown that  $|\text{dist}_A(x) - \text{dist}_A(y)| \leq d(x, y)$ , so  $\text{dist}_A$  is Lipschitz.

**Definition 2.5.3.** Let  $X$  be a topological space. We say  $X$  is *normal* (or T4) if, for any  $A, B \subseteq X$  closed satisfying  $A \cap B = \emptyset$ , then there exists  $U, V \in \tau_X$  with  $A \subseteq U$ ,  $B \subseteq V$  satisfying  $U \cap V = \emptyset$ .

**Proposition 2.5.5.** *Metric spaces are normal.*

*Proof.* Let  $A, B \subseteq (X, d)$  with  $A \cap B = \emptyset$ . Define  $f : (X, d) \rightarrow \mathbf{R}$  by:

$$f(x) = \frac{\text{dist}_A(x)}{\text{dist}_A(x) + \text{dist}_B(x)}.$$

Then  $f$  is continuous. Moreover, define:

$$\begin{aligned} U &:= f^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) \\ V &:= f^{-1}\left(\left(-\frac{1}{2}, \frac{3}{2}\right)\right) \end{aligned}$$

Then  $U$  and  $V$  open with  $U \cap V = \emptyset$ . □

**Proposition 2.5.6.** *Let  $V$  and  $W$  be normed spaces and  $T : V \rightarrow W$  linear. The following are equivalent:*

- (1)  $T$  is continuous at  $0_V$ ;
- (2)  $T$  is continuous;
- (3)  $T$  is uniformly continuous;
- (4)  $T$  is Lipschitz;
- (5) there exists  $C \geq 0$  such that  $\|Tv\| \leq C\|v\|$  for all  $v \in V$ ;

(6)  $T$  is bounded.

*Proof.* We will show  $(6) \Leftrightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (5)$ . Let  $T$  be bounded. Given  $v \in V$ ,  $v \neq 0$ , we have:

$$\begin{aligned} \|T\|_{\text{op}} &= \sup_{v \in B_V} \|Tv\| \\ &\geq \left\| T \frac{v}{\|v\|} \right\| \quad \text{for all } v \in V \\ &= \frac{1}{\|v\|} \|Tv\| \quad \text{for all } v \in V. \end{aligned}$$

Thus  $\|Tv\| \leq \|T\|_{\text{op}} \|v\|$  for all  $v \in V$ . The converse is clear by inspection.

Suppose there exists  $C \geq 0$  satisfying (5). We have that  $\|Tv - Tv'\| \leq C \|v - v'\|$ . Thus  $T$  is Lipschitz.

Let  $T$  be Lipschitz. Let  $\epsilon > 0$  and find  $\delta = \frac{\epsilon}{C}$ . Then  $\|v - v'\| < \delta$  implies:

$$\begin{aligned} \|Tv - Tv'\| &\leq C \|v - v'\| \\ &< C \frac{\epsilon}{C} \\ &= \epsilon. \end{aligned}$$

Thus  $T$  is uniformly continuous.

Suppose that  $T$  be uniformly continuous. Fix  $x \in V$ . Given  $\epsilon > 0$ , we can find  $\delta > 0$  so that  $\|v - x\| < \delta$  implies  $\|Tv - Tx\| < \epsilon$ . Thus  $T$  is continuous at  $x \in V$ . Since  $x$  was arbitrary,  $T$  is continuous. Moreover,  $T$  will be continuous at  $0_V$ , establishing (1).

We will now show (1) implies (5). Let  $\epsilon = 1$ . We can find a  $\delta > 0$  such that  $T(U(0, \delta)) \subseteq U(0, 1)$ . If  $v \in V$ ,  $v \neq 0$ , then  $\frac{\delta v}{2\|v\|} \in U(0, \delta)$ . Since  $T$  is continuous at 0, we have  $T \frac{\delta v}{2\|v\|} \in U(0, 1)$ <sup>1</sup>. This gives  $\left\| T \frac{\delta v}{2\|v\|} \right\| < 1$ , which is equivalent to  $\|Tv\| < \frac{2}{\delta} \|v\|$ . This establishes (5).  $\square$

**Corollary 2.5.7.** \*\*\* Let  $V$  be a normed space and  $T : \ell_p^n \rightarrow V$  linear. Then  $T$  is uniformly continuous.

---

<sup>1</sup>Recall that  $T(0) = 0$ .

*Proof.* Let  $x \in \ell_p^n$  and  $\mathcal{B} = \{e_1, \dots, e_n\}$  a basis for  $\ell_p^n$ . Find  $\alpha_1, \dots, \alpha_n$  so that:

$$\begin{aligned} \|Tv\| &= \left\| T \left( \sum_{j=1}^n \alpha_j e_j \right) \right\| \\ &= \left\| \sum_{j=1}^n \alpha_j T e_j \right\| \\ &\leq \sum_{j=1}^n |\alpha_j| \|T e_j\| \\ &\leq c \left\| \sum_{j=1}^n \alpha_j e_j \right\|_1 \\ &\leq c \cdot c' \left\| \sum_{j=1}^n \alpha_j e_j \right\|_p, \end{aligned}$$

where  $c = \max_{j=1}^n \|T e_j\|$  and  $\|\cdot\|_1 \leq c' \|\cdot\|_p$ .  $\square$

**Proposition 2.5.8.** *Let  $(X, d)$  be a metric space with  $A \subseteq X$  dense. If  $f, g : X \rightarrow (Y, \rho)$  are continuous with  $f(a) = g(a)$  for all  $a \in A$ , then  $f = g$ .*

*Proof.* If  $x \in X$ , we can find a sequence  $(a_n)_n$  in  $A$  with  $(a_n)_n \xrightarrow{d} x$ . Then:

$$\begin{aligned} (f(a_n))_n &\rightarrow f(x) \\ \parallel \\ (g(a_n))_n &\rightarrow g(x). \end{aligned}$$

Thus  $f(x) = g(x)$ .  $\square$

**Definition 2.5.4.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces and  $f : X \rightarrow Y$ .

- (1)  $f$  is a *homeomorphism* if  $f$  is bijective with  $f$  and  $f^{-1}$  continuous. If such an  $f$  exists, we say  $X \cong Y$  are *homeomorphic*.
- (2)  $f$  is a *uniformism* if  $f$  is bijective with  $f$  and  $f^{-1}$  uniformly continuous. If such an  $f$  exists, we say  $X \cong Y$  are *uniformly isomorphic*.
- (3)  $f$  is an *metric isomorphism* if  $f$  is bijective with  $f$  and  $f^{-1}$  Lipschitz. If such an  $f$  exists, we say  $X \cong Y$  are *metrically isomorphic*.
- (4)  $f$  is an *isometric isomorphism* if  $f$  and  $f^{-1}$  are isometries. We say  $X \cong Y$  are *isometrically isomorphic*.

**Example 2.5.4.**  $(0, 1) \cong \mathbf{R}$  are homeomorphic, but not uniformly isomorphic.

**Example 2.5.5.** \*\*\* Let  $a = (a_k)_k \in \ell_1$ . Define  $\varphi_a : c_0 \rightarrow F$  by  $\varphi_a(z) = \sum_{k \geq 1} a_k z_k$ . This series converges since:

**Definition 2.5.5.** Let  $X$  be a set with two metrics  $d_1$  and  $d_2$ .

- (1)  $d_1$  and  $d_2$  are *metrically equivalent* if  $\text{id} : (X, d_1) \rightarrow (X, d_2)$  and  $\text{id}^{-1} : (X, d_2) \rightarrow (X, d_1)$  are Lipschitz.
- (2)  $d_1$  and  $d_2$  are *uniformly equivalent* if  $\text{id} : (X, d_1) \rightarrow (X, d_2)$  and  $\text{id}^{-1} : (X, d_2) \rightarrow (X, d_1)$  are uniformisms.
- (3)  $d_1$  and  $d_2$  are *topologically equivalent* if  $\text{id} : (X, d_1) \rightarrow (X, d_2)$  and  $\text{id}^{-1} : (X, d_2) \rightarrow (X, d_1)$  are homeomorphisms.

**Example 2.5.6.** \*\*\*

## § 2.6. Completeness

**Definition 2.6.1.** A sequence  $(x_n)_n$  in a metric space  $(X, d)$  is *d-Cauchy* if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall p, q \in \mathbf{N})(p, q \geq N \implies d(x_p, x_q) < \epsilon).$$

**Proposition 2.6.1.** Let  $(x_n)$  be a sequence in  $(X, d)$ .

- (1) If  $(x_n)_n$  converges, then  $(x_n)_n$  is Cauchy.
- (2) If  $(x_n)_n$  is Cauchy, then  $(x_n)_n$  is bounded.

*Proof.* (1) Let  $x \in X$  and suppose  $(x_n)_n \rightarrow x$ . Let  $\epsilon > 0$ . Find  $N$  large so that for  $p \geq N$  we have  $d(x_p, x) < \frac{\epsilon}{2}$ . Then  $p, q \geq N$  implies:

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x) + d(x, x_q) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

(2) Let  $\epsilon = 1$ . Find  $N$  large so  $p, q \geq N$  implies  $d(x_p, x_q) < 1$ . Let  $C = \max_{1 \leq p, q \leq N} d(x_p, x_q)$ . Without loss of generality, if  $p \geq N$  and  $q \leq N$ , then  $d(x_p, x_q) \leq d(x_p, x_N) + d(x_N, x_q) < 1 + C$ . Set  $K = \max\{1, 1 + C\}$ . Then  $\text{diam}(\{x_n\}_{n \geq 1}) = \sup_{p, q \geq 1} d(x_p, x_q) < K$ .  $\square$

**Proposition 2.6.2.** Let  $(x_n)_n$  be a Cauchy sequence in  $X$  and suppose there exists a subsequence  $(x_{n_k})_k$  converging to  $x \in X$ . Then  $(x_n)_n$  converges to  $x$ .

*Proof.* Let  $\epsilon > 0$ . Since  $(x_n)_n$  is Cauchy, there exists  $N$  large so  $n, n_k \geq N$  implies  $d(x_n, x_{n_k}) < \frac{\epsilon}{2}$ . This gives:

$$\begin{aligned} d(x_n, x) &= d(x_n, \lim_{k \rightarrow \infty} x_{n_k}) \\ &= \lim_{k \rightarrow \infty} d(x_n, x_{n_k}) \\ &\leq \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned} \quad \square$$

**Definition 2.6.2.** A metric space is said to be *complete* if every Cauchy sequence converges. A complete normed space is called a *Banach space*. A complete inner product space is called a *Hilbert space*.

**Lemma 2.6.3.** *Let  $f : (X, d) \rightarrow (Y, \rho)$  be uniformly continuous. If  $(x_n)_n$  is  $d$ -Cauchy then  $(f(x_n))_n$  is  $\rho$ -Cauchy.*

*Proof.* Let  $\epsilon > 0$ . Find  $\delta > 0$  so that  $d(x, x') < \delta$  implies  $\rho(f(x), f(x')) < \epsilon$ . Pick  $N$  sufficiently large so that  $p, q \geq N$  implies  $d(x_p, x_q) < \delta$ . This gives  $\rho(f(x_p), f(x_q)) < \epsilon$ , whence  $(f(x_n))_n$  is  $\rho$ -Cauchy.  $\square$

**Corollary 2.6.4.** *If  $f : (X, d) \rightarrow (Y, \rho)$  is a uniformism, then  $(X, d)$  is complete if and only if  $(Y, \rho)$  is complete.*

*Proof.* Let  $(X, d)$  be complete. If  $(y_n)_n$  is  $\rho$ -Cauchy, then  $(f^{-1}(y_n))_n$  is  $d$ -Cauchy in  $X$ . So we can find some  $x \in X$  such that  $(f^{-1}(y_n))_n \rightarrow x$ . Then  $(f(f^{-1}(y_n)))_n = (y_n)_n \rightarrow f(x)$ . The converse follows similarly.  $\square$

**Corollary 2.6.5.** *If  $d_1$  and  $d_2$  are uniformly equivalent metrics on a set  $X$ , then  $(X, d_1)$  is complete if and only if  $(X, d_2)$  is complete.*

*Proof.* Since the map  $\text{id} : (X, d_1) \rightarrow (X, d_2)$  is a uniformism, the previous corollary gives that  $(X, d_1)$  is complete if and only if  $(X, d_2)$  is complete.  $\square$

**Proposition 2.6.6.**  $\ell_p^d$  is a Banach space for  $1 \leq p \leq \infty$ .

*Proof.* We only need to show this for  $\ell_\infty^d$  since all the  $p$ -norms are equivalent. Let  $(x_n)_n$  be  $\|\cdot\|_\infty$ -Cauchy in  $\ell_\infty^d$ . Let  $\epsilon > 0$ . Find  $N$  large so for  $n, m \geq N$  we have  $\|x_n - x_m\|_\infty < \epsilon$ . Observe that:

$$\begin{aligned} |x_n(k) - x_m(k)| &\leq \max_{1 \leq k \leq d} |x_n(k) - x_m(k)| \\ &= \|x_n - x_m\|_\infty \\ &< \epsilon, \end{aligned}$$

where  $x_n(k)$  is the  $k^{\text{th}}$  entry of the  $d$ -tuple  $x_n$ . So for each  $k = 1, \dots, d$ , we know  $(x_n(k))_n$  is Cauchy in  $F$ . Set  $\lim_{n \rightarrow \infty} x_n(k) = x(k)$  for  $k = 1, \dots, d$ . This gives:

$$\|x - x_n\|_\infty = \max_{1 \leq k \leq d} |x(k) - x_n(k)| \xrightarrow{n \rightarrow \infty} 0.$$

Whence  $(x_n)_n \rightarrow x$  in  $\ell_\infty^d$ . Now set  $y = (y_1, \dots, y_n)$ . We have:

$$\begin{aligned} \|x_n - y\|_p &\leq c' \|x_n - y\|_1 \\ &= c' \sum_{j=1}^d |x_n(j) - y_j| \\ &\leq c' \max_{1 \leq j \leq d} |x_n(j) - y_j| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Thus  $\ell_p^d$  is complete.  $\square$

**Proposition 2.6.7.**  $\ell_p$  is a Banach space for  $1 \leq p \leq \infty$ .

*Proof.* Suppose that  $(f_n)_n$  is  $\|\cdot\|_{\ell_p}$ -Cauchy. Observe that

$$\begin{aligned} |f_n(k) - f_m(k)|^p &\leq \sum_{j=1}^{\infty} |f_n(j) - f_m(j)|^p \\ &= \|f_n - f_m\|_{\ell_p}^p \end{aligned}$$

So  $(f_n(k))_n$  is Cauchy in  $F$ . Since this space is complete, define  $\lim_{n \rightarrow \infty} f_n(k) := f(k)$ .

Our goal is to find some function  $f : \mathbf{N} \rightarrow F$  satisfying  $f \in \ell_p$  and  $\|f_n - f\|_{\ell_p} \rightarrow 0$ . The  $f(k)$  we've just obtained will lead us to the most suitable candidate.

Since  $(f_n)_n$  is  $\|\cdot\|_{\ell_p}$ -Cauchy, it is bounded by some constant. Fix  $K \geq 1$  and observe that:

$$\begin{aligned} \sum_{k=1}^K |f(k)|^p &= \sum_{k=1}^K \left| \lim_{n \rightarrow \infty} f_n(k) \right|^p \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^K |f_n(k)|^p \\ &\leq \sup_{n \geq 1} \|f_n\|_{\ell_p}^p \\ &:= C. \end{aligned}$$

Since the sequence  $\left( \sum_{k=1}^K |f(k)|^p \right)_{K=1}^{\infty}$  is increasing and bounded above  $C$ , the Monotone Convergence Theorem says that its limit exists. We obtain:

$$\begin{aligned} \lim_{K \rightarrow \infty} \sum_{k=1}^K |f(k)|^p &= \sum_{k=1}^{\infty} |f(k)|^p \\ &= \|f\|_{\ell_p}^p \\ &< \infty. \end{aligned}$$

Thus  $f \in \ell_p$ .

It remains to show that  $(f_n)_n$  converges to  $f$ . Given  $\epsilon > 0$ , find  $N$  large so that  $n, m \geq N$  implies  $\|f_n - f_m\|_p < \epsilon$ . For every  $n, m \geq N$  we have:

$$\begin{aligned} \sum_{k=1}^K |f_m(k) - f_n(k)|^p &\leq \|f_m - f_n\|_{\ell_p}^p \\ &< \epsilon^p. \end{aligned}$$

Taking the limit as  $m \rightarrow \infty$  and considering all  $n \geq N$  gives:

$$\sum_{k=1}^K |f(k) - f_n(k)|^p < \epsilon^p.$$

Finally, taking the limit as  $K \rightarrow \infty$  and simplifying gives  $\|f - f_n\|_{\ell_p} < \epsilon$ . Thus  $\ell_p$  is complete.  $\square$

**Proposition 2.6.8.** *Let  $(Y, d)$  be a complete metric space. The set of bounded functions  $\text{Bd}(\Omega, Y)$  with  $\|\cdot\|_u$  is complete.*

*Proof.* Let  $(f_n)_n$  be  $D_u$ -Cauchy. Fix  $x, x' \in \Omega$  and let  $\epsilon > 0$ . Find  $N$  large so that  $n, m \geq N$  implies:

$$d(f_n(x), f_m(x)) \leq D_u(f_n, f_m) < \epsilon.$$

Thus  $(f_n(x))_n$  is Cauchy in  $Y$ . Since  $Y$  is complete, define  $\lim_{n \rightarrow \infty} f_n(x) := f(x)$ . Now find  $N_1$  large so  $n \geq N_1$  implies  $d(f(x), f_n(x)) < \epsilon$ . Find  $N_2$  large so  $n \geq N_2$  implies  $d(f(x'), f_n(x')) < \epsilon$ . Observe that:

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), f_{N_1}(x)) + d(f_{N_1}(x), f_{N_2}(x')) + d(f_{N_2}(x'), f(x')) \\ &< 2\epsilon + d(f_{N_1}(x), f_{N_2}(x')) \\ &\leq 2\epsilon + d(f_{N_1}(x), f_{N_2}(x)) + d(f_{N_2}(x), f_{N_2}(x')) \\ &\leq 2\epsilon + \sup_{n, m \geq 1} d(f_n(x), f_m(x)) + \sup_{x, x' \in \Omega} d(f_{N_2}(x), f_{N_2}(x')) \\ &= 2\epsilon + \text{diam}(\{f_n(x)\}_{n \geq 1}) + \text{diam}(f_{N_2}(\Omega)). \end{aligned}$$

Note that the sequence  $(f_n(x))_n$  is bounded because it is Cauchy —so there exists  $C_1 \geq 0$  such that  $\text{diam}(\{f_n(x)\}_{n \geq 1}) < C_1$ . Moreover, since  $f_{N_2} \in \text{Bd}(\Omega, Y)$ , there exists  $C_2 \geq 0$  such that  $\text{diam}(f_{N_2}(\Omega)) < C_2$ . This gives:

$$d(f(x), f(x')) < 2\epsilon + C_1 + C_2.$$

Since this inequality is independent of any  $n$ , and since  $x, x' \in \Omega$  was arbitrary, we have that  $\text{diam}(f(\Omega)) < \infty$ ; i.e.,  $f \in \text{Bd}(\Omega, Y)$ . With the same  $\epsilon$  as above, find  $N_3$  large so that for all  $n, m \geq N_3$  then  $D_u(f_n, f_m) < \frac{\epsilon}{2}$ . We know:

$$d(f_n(x), f_m(x)) \leq D_u(f_n, f_m) < \frac{\epsilon}{2}.$$

Taking  $m \rightarrow \infty$  gives  $d(f_n(x), f(x)) \leq \frac{\epsilon}{2}$  for all  $n \geq N$ . But note that  $N$  does not depend on our fixed  $x \in \Omega$ . It follows that  $D_u(f_n, f) \leq \frac{\epsilon}{2} < \epsilon$ . Thus  $(f_n)_n \rightarrow f$ , establishing  $\text{Bd}(\Omega, Y)$  as complete.  $\square$

**Corollary 2.6.9.**  $\ell_\infty(\Omega)$  is complete.

**Proposition 2.6.10.** *Let  $(X, d)$  be a complete metric space and  $Y \subseteq X$ .  $Y$  is complete if and only if  $Y$  is closed.*

*Proof.* Let  $(y_n)_n$  be a sequence in  $Y$  converging to  $x \in X$ . Then  $(y_n)_n$  is sequence in  $X$ . Since  $X$  is complete,  $(y_n)_n$  is Cauchy. Since  $Y$  is complete,  $(y_n)_n$  must converge to some  $y \in Y$ . Since sequences can have at most one limit, it must be that  $y = x$ . Thus  $x \in Y$ ; i.e.,  $Y$  is closed.

Conversely, if  $(y_n)_n$  is Cauchy in  $Y$ , then it is Cauchy in  $X$ . Since  $X$  is complete, there exists  $x \in X$  with  $(y_n)_n \rightarrow x$ . Since  $Y$  is closed,  $x \in Y$ . Thus  $Y$  is complete.  $\square$

Proposition 2.6.10 and Proposition 2.4.5 are extremely useful tools for showing a space is complete.



**Corollary 2.6.11.** *Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces.*

- (1)  $C_b(X, Y) := C(X, Y) \cap \text{Bd}(X, Y)$  is  $D_u$ -complete.
- (2)  $C_b(X)$  is a  $\|\cdot\|_u$ -Banach space.
- (3)  $C_0(\mathbf{R})$  is a  $\|\cdot\|_u$ -Banach space.

*Proof.* (1) Let  $(f_n)_n$  be a sequence in  $C_b(X, Y)$  converging to  $f \in \text{Bd}(X, Y)$ . Let  $x \in X$  and  $\epsilon > 0$ . Find  $N$  large so that  $D_u(f_N, f) < \frac{\epsilon}{3}$ . Find  $\delta > 0$  so that for all  $x' \in X$ ,  $d(x, x') < \delta$  implies  $\rho(f_N(x), f_N(x')) < \frac{\epsilon}{3}$ . For  $d(x, x') < \delta$ :

$$\begin{aligned} \rho(f(x), f(x')) &\leq \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(x')) + \rho(f_N(x'), f(x')) \\ &\leq 2D_u(f_N, f) + \rho(f_N(x), f_N(x')) \\ &< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Thus  $f \in C_b(X, Y)$  because it is bounded and continuous. Since  $C_b(X, Y) \subseteq \text{Bd}(X, Y)$  is closed, it is complete.

(2) As we've just shown, the space  $C_b(X)$  is complete. It only remains to show it is a vector space.

(3) Let  $(f_n)_n$  be a sequence in  $C_0(\mathbf{R})$  converging to  $f \in C_b(\mathbf{R})$ . Let  $\epsilon > 0$  and find  $N$  large so that  $\|f - f_N\|_u < \frac{\epsilon}{2}$ . Since  $f_N \in C_0(\mathbf{R})$ , we know that  $\lim_{x \rightarrow \pm\infty} f_N(x) = 0$ . So there exists  $M > 0$  with  $|x| \geq M$  implying  $|f_N(x)| < \frac{\epsilon}{2}$ . For  $|x| \geq M$  observe that:

$$\begin{aligned} |f(x)| &\leq |f(x) - f_N(x)| + |f_N(x)| \\ &\leq \|f - f_N\|_u + |f_N(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus  $\lim_{x \rightarrow \infty} f(x) = 0$ ; i.e.,  $f \in C_0(\mathbf{R})$ . Since  $C_0(\mathbf{R}) \subseteq C_b(\mathbf{R})$  is closed, it is complete.  $\square$

**Proposition 2.6.12.** *Let  $V$  be a normed space and  $W$  a Banach space.  $B(V, W)$  with  $\|\cdot\|_{\text{op}}$  is a Banach space.*

*Proof.* Let  $(T_n)_n$  be  $\|\cdot\|_{\text{op}}$ -Cauchy. Let  $v \in V$  and  $\epsilon > 0$ . Find  $N_1$  large so that  $n, m \geq N_1$  implies  $\|T_n - T_m\|_{\text{op}} < \frac{\epsilon}{\|v\|}$ . We can see:

$$\begin{aligned} \|T_n v - T_m v\|_W &= \|(T_n - T_m)v\|_W \\ &\leq \|T_n - T_m\|_{\text{op}} \|v\| \\ &< \frac{\epsilon}{\|v\|} \cdot \|v\| \\ &= \epsilon. \end{aligned}$$

So  $(T_n v)_n$  is  $\|\cdot\|_W$ -Cauchy. Since  $W$  is a Banach space, define  $\lim_{n \rightarrow \infty} T_n v = Tv$ . We must show that  $T$  is linear, bounded, and  $\|T_n - T\|_{\text{op}} \rightarrow 0$ . Given  $v_1, v_2 \in V$  and  $c \in F$  we

can see:

$$\begin{aligned}
 T(v_1 + cv_2) &= \lim_{n \rightarrow \infty} T_n(v_1 + cv_2) \\
 &= \lim_{n \rightarrow \infty} T_n v_1 + c T_n v_2 \\
 &= \lim_{n \rightarrow \infty} T v_1 + c \lim_{n \rightarrow \infty} T v_2 \\
 &= T v_1 + c T v_2.
 \end{aligned}$$

Thus  $T$  is linear. Now since  $(T_n)_n$  is  $\|\cdot\|_{\text{op}}$ -Cauchy, it is bounded, so there exists  $C > 0$  with  $\|T_n\|_{\text{op}} \leq C$  for all  $n \geq 1$ . Using the fact norms are continuous, we have:

$$\begin{aligned}
 \|Tv\|_W &= \left\| \lim_{n \rightarrow \infty} T_n v \right\|_W \\
 &= \lim_{n \rightarrow \infty} \|T_n v\|_W \\
 &\leq \limsup_{n \rightarrow \infty} \|T_n\|_{\text{op}} \|v\| \\
 &\leq C \|v\|.
 \end{aligned}$$

Thus  $T \in B(V, W)$ . With the same epsilon as before, find  $N_2$  so that  $n, m \geq N_2$  implies  $\|T_n - T_m\|_{\text{op}} < \frac{\epsilon}{2}$ . We can show:

$$\|T_n v - T_m v\|_W \leq \|T_n - T_m\|_{\text{op}} < \frac{\epsilon}{2}.$$

Taking  $m \rightarrow \infty$  gives:

$$\|T_n v - T v\|_W \leq \frac{\epsilon}{2}.$$

Taking the supremum over all  $v \in B_V$  gives:

$$\|T_n - T\|_{\text{op}} \leq \frac{\epsilon}{2} < \epsilon.$$

Thus  $B(V, W)$  is complete. □

**Proposition 2.6.13.** *Let  $(V, \|\cdot\|)$  be a normed space. The following are equivalent:*

- (1)  $V$  is a Banach space;
- (2) If  $(v_k)_k$  is a sequence in  $V$  with  $\sum_{k=1}^{\infty} \|v_k\|$  convergent, then  $\sum_{k=1}^{\infty} v_k$  converges.

*Proof.* Suppose  $V$  is a Banach space. Let  $s_n = \sum_{k=1}^n v_k$  and  $t_n = \sum_{k=1}^n \|v_k\|$ . For  $p > q > 1$ :

$$\begin{aligned}
 \|s_p - s_q\| &= \left\| \sum_{k=q+1}^p v_k \right\| \\
 &\leq \sum_{k=q+1}^p \|v_k\| \\
 &= |t_p - t_q|.
 \end{aligned}$$

Since  $(t_n)_n$  is convergent, it is Cauchy. So  $(s_n)_n$  is Cauchy, implying it is convergent. Thus  $\sum_{k=1}^{\infty} v_k$  converges.

Now let  $(v_n)_n$  be Cauchy. Find  $n_1 \in \mathbf{N}$  such that  $p, q \geq n_1$  implies  $\|v_p - v_q\| < 2^{-1}$ . Find  $n_2 > n_1$  such that  $p, q \geq n_2$  implies  $\|v_p - v_q\| < 2^{-2}$ . Inductively, find  $n_k > n_{k-1}$  such that  $p, q \geq n_k$  implies  $\|v_p - v_q\| < 2^{-k}$ . Consider the sequence  $(v_{n_{k+1}} - v_{n_k})_k$ . Then:

$$\sum_{k=1}^{\infty} \|v_{n_{k+1}} - v_{n_k}\| \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

By our hypothesis,  $\sum_{k=1}^{\infty} (v_{n_{k+1}} - v_{n_k})$  converges. So the sequence of partial sums:

$$\begin{aligned} w_m &= \sum_{k=1}^m v_{n_{k+1}} - v_{n_k} \\ &= v_{n_m} - v_{n_1} \end{aligned}$$

also converges to some  $w \in V$  as  $m \rightarrow \infty$ . However, notice:

$$\begin{aligned} (v_{n_m})_m &= (v_{n_m} - v_{n_1})_m + v_{n_1} \\ &\xrightarrow{m \rightarrow \infty} w + v_{n_1}. \end{aligned}$$

Since  $(v_n)_n$  is a Cauchy sequence which admits a convergent subsequence,  $(v_n)_n$  converges. Thus  $V$  is a Banach space.  $\square$

**Example 2.6.1.** \*\*\* Let  $\mathcal{H}$  be a Hilbert space. Suppose  $(e_n)_n$  is an orthonormal sequence in  $\mathcal{H}$  and  $(t_k)_k \in \ell_2$ . We will show  $\sum_{k=1}^{\infty} t_k e_k$  converges in  $\mathcal{H}$ , but not absolutely in general.

Let  $s_n = \sum_{k=1}^n t_k e_k$ . For  $n > m > 1$ :

$$\begin{aligned} \|s_n - s_m\|^2 &= \left\| \sum_{k=m+1}^n t_k e_k \right\|^2 \\ &= \sum_{k=m+1}^n |t_k|^2. \end{aligned}$$

Since  $(\sum_{k=1}^n |t_k|^2)_n$  is Cauchy, then  $(s_n)_n$  is Cauchy, whence  $\sum_{k=1}^{\infty} t_k e_k$  converges. But notice  $\|s_n\|^2 = \sum_{k=1}^n |t_k|^2$ . As  $n \rightarrow \infty$ , we see  $\|\sum_{k=1}^{\infty} t_k e_k\|^2 = \sum_{k=1}^{\infty} |t_k|^2$ .

—————/—————

Recall that if  $f : (X, d) \rightarrow (Y, \rho)$  is a uniformly continuous map between metric spaces and  $(x_n)_n$  is Cauchy, then  $(f(x_n))_n$  is Cauchy. Complete spaces have the unique property that, given a map from a dense subset  $A \rightarrow Y$ , we can define an extension of such function.

**Theorem 2.6.14.** *Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces with  $Y$  complete. Suppose  $A \subseteq X$  is dense and  $f : A \rightarrow Y$  is uniformly continuous. There exists a unique uniformly continuous map  $\tilde{f} : X \rightarrow Y$  with  $\tilde{f}(x) = f(x)$  for all  $x \in A$ .*

*Proof.* Let  $x \in X$ . We know there exists a sequence  $(a_n)_n$  in  $A$  with  $(a_n)_n \rightarrow x$ . Since  $(a_n)_n$  is convergent, it is Cauchy, so  $(f(a_n))_n$  is also Cauchy. By the completeness of  $Y$ ,  $(f(a_n))_n$  converges. Define  $\tilde{f}(x) := \lim_{n \rightarrow \infty} f(a_n)$ .

We must show this extension is well-defined. Suppose  $(b_n)_n$  is another sequence in  $A$  with  $(b_n)_n \rightarrow x$ . Then the mixed sequence  $(a_1, b_1, a_2, b_2, \dots)$  will converge to  $x$ . The same reasoning as above tells us  $(f(a_1), f(b_1), f(a_2), f(b_2), \dots)$  converges in  $Y$ . The two subsequences  $(f(a_n))_n$  and  $(f(b_n))_n$  must then converge to the same limit.

We will now show that  $\tilde{f}$  is uniformly continuous. Let  $\epsilon > 0$ . Find  $\delta > 0$  so that for all  $a, b \in A$ ,  $d(a, b) < \delta$  implies  $\rho(f(a), f(b)) < \frac{\epsilon}{2}$ . Now let  $x, x' \in X$  with  $d(x, x') < \frac{\delta}{4}$ . Find sequences  $(a_n)_n$  and  $(b_n)_n$  in  $A$  with  $(a_n)_n \rightarrow x$  and  $(b_n)_n \rightarrow x'$ . Find  $N$  large so that  $n \geq N$  implies  $d(a_n, x) < \frac{\delta}{4}$  and  $d(b_n, x') < \frac{\delta}{4}$ . The triangle inequality gives  $d(a_n, b_n) < \frac{3\delta}{4} < \delta$ . So  $\rho(f(a_n), f(b_n)) < \frac{\epsilon}{2}$  for all  $n \geq N$ . Observe that:

$$\begin{aligned} \rho(\tilde{f}(x), \tilde{f}(x')) &= \lim_{n \rightarrow \infty} \rho(f(a_n), f(b_n)) \\ &\leq \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

Thus  $\tilde{f}$  is uniformly continuous. It remains to show that  $\tilde{f}$  is unique. Suppose  $g : X \rightarrow Y$  is also a continuous extension of  $f$ . Then  $g(x) = f(x) = \tilde{f}(x)$  for all  $x \in A$ . Since  $g$  and  $\tilde{f}$  agree on elements of a dense set, we must have  $g = \tilde{f}$ .  $\square$

**Proposition 2.6.15.** *Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces with  $A \subseteq X$  dense,  $Y$  complete, and  $f : A \rightarrow Y$  an isometry. Then the continuous extension  $\tilde{f} : X \rightarrow Y$  is an isometry.*

*Proof.* Let  $x, x' \in X$ . Let  $(a_n)_n$  and  $(b_n)_n$  be sequences in  $A$  with  $(a_n)_n \rightarrow x$  and  $(b_n)_n \rightarrow x'$ . We have:

$$\begin{aligned} \rho(\tilde{f}(x), \tilde{f}(x')) &= \lim_{n \rightarrow \infty} \rho(f(a_n), f(b_n)) \\ &= \lim_{n \rightarrow \infty} d(a_n, b_n) \\ &= d(x, x'). \end{aligned} \quad \square$$

**Corollary 2.6.16.** *Let  $V$  be a normed space,  $W$  a Banach space, and  $U \subseteq V$  a dense linear subspace. Let  $T_0 \in B(U, W)$ . There exists a unique  $T \in B(V, W)$  with  $\|T\|_{\text{op}} = \|T_0\|_{\text{op}}$ . Moreover, if  $T_0$  is isometric, then so is  $T$ .*

*Proof.* We only need to show  $T$  is linear and  $\|T\|_{\text{op}} = \|T_0\|_{\text{op}}$ . Let  $v, v' \in V$  and  $\alpha \in F$ . Let  $(x_n)_n$  and  $(y_n)_n$  be sequences in  $U$  with  $(x_n)_n \rightarrow v$  and  $(y_n)_n \rightarrow v'$ . Observe that:

$$\begin{aligned} T(v + \alpha v') &= \lim_{n \rightarrow \infty} T_0(x_n + \alpha y_n) \\ &= \lim_{n \rightarrow \infty} T_0(x_n) + \alpha \lim_{n \rightarrow \infty} T_0(y_n) \\ &= T(v) + \alpha T(v'). \end{aligned}$$

Note that the composition  $V \xrightarrow{T} W \xrightarrow{\|\cdot\|_W} F$  will be continuous and bounded, so by

Proposition ?? we have:

$$\begin{aligned}
 \|T\|_{\text{op}} &= \sup_{v \in B_V} \|T(v)\| \\
 &= \sup_{v \in B_U} \|T(v)\| \\
 &= \sup_{v \in B_U} \|T_0(v)\| \\
 &= \|T_0\|_{\text{op}}.
 \end{aligned}$$

Thus  $T \in B(V, W)$ . □

**Example 2.6.2.** \*\*\* (Something about Hilbert spaces...)

—————/—————

**Definition 2.6.3.** Let  $(X, d)$  be a metric space. A *completion* of  $(X, d)$  is a pair  $((Z, \rho), \iota)$  where:

- (1)  $(Z, \rho)$  is a complete metric space;
- (2)  $\iota : X \hookrightarrow Z$  is an isometry;
- (3)  $\overline{\iota(X)}^\rho = Z$ .

**Example 2.6.3.**  $(([0, 1], \|\cdot\|), \iota(t) = t)$  is a completion of  $(0, 1)$ .

**Lemma 2.6.17.** Let  $f : (X, d) \rightarrow (Y, \rho)$  be an isometry between metric spaces. If  $X$  is complete, then  $f(X) \subseteq Y$  is closed. In particular, if  $f(X)$  is dense, then  $f$  is onto.

*Proof.* If  $(f(x_n))_n \rightarrow y$  in  $Y$ , then  $(f(x_n))_n$  is  $\rho$ -Cauchy. Since  $f$  is an isometry,  $d(x_n, x_m) = \rho(f(x_n), f(x_m))$ , so  $(x_n)_n$  is  $d$ -Cauchy. Let  $x \in X$  such that  $(x_n)_n \rightarrow x$ . Since  $f$  is continuous, we have  $(f(x_n))_n \rightarrow f(x)$ . It must be that  $y = f(x) \in f(X)$ .

Since we've just shown that  $f(X)$  is closed, if it were also dense then  $f(X) = \overline{f(X)}^\rho = Y$ , whence  $f$  is surjective. □

**Theorem 2.6.18.** Let  $(X, d)$  be a metric space. If  $((Z, \rho), \iota)$  and  $((Z', \rho'), j)$  are completions of  $X$ , there exists a unique isometric isomorphism  $\varphi : Z \rightarrow Z'$  with  $\varphi \circ \iota = j$ ; that is, the following diagram commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{\iota} & Z \\
 & \searrow j & \downarrow \varphi \\
 & & Z'
 \end{array}$$

*Proof.* Let  $z \in Z$ . Since  $\iota(X)$  is dense in  $Z$ , there exists a sequence  $(\iota(x_n))_n$  in  $\iota(X)$  such that  $(\iota(x_n))_n \rightarrow z$ . It is clear that  $(\iota(x_n))_n$  is  $\rho$ -Cauchy, and furthermore we have:

$$\begin{aligned}
 \rho'(j(x_n), j(x_m)) &= d(x_n, x_m) \\
 &= \rho(\iota(x_n), \iota(x_m)).
 \end{aligned}$$

So if  $(\iota(x_n))_n$  is  $\rho$ -Cauchy, then  $(j(x_n))_n$  is  $\rho'$ -Cauchy. Since  $j(X)$  is complete, define  $\varphi(z) := \lim_{n \rightarrow \infty} j(x_n)$ .

We will show that  $\varphi : Z \rightarrow Z'$  is well-defined, that is, it does not depend on our particular choice of sequence. Let  $(y_n)$  be another sequence in  $X$  with  $(y_n)_n \rightarrow z$ . We can see that:

$$\begin{aligned} d(x_n, y_n) &= \rho(\iota(x_n), \iota(y_n)) \\ &\leq \rho(\iota(x_n), z) + \rho(z, \iota(y_n)) \\ &\rightarrow 0, \end{aligned}$$

which gives:

$$\begin{aligned} \rho'(j(y_n), \varphi(z)) &\leq \rho'(j(y_n), j(x_n)) + \rho'(j(x_n), \varphi(z)) \\ &= d(y_n, x_n) + \rho'(j(x_n), \varphi(z)) \\ &\rightarrow 0. \end{aligned}$$

Thus  $(j(y_n))_n \rightarrow \varphi(z)$ , so  $\varphi$  is well-defined.

We will now show that  $\varphi$  is an isometric isomorphism. Given  $z_1, z_2 \in Z$ , let  $(\iota(x_n))_n \rightarrow z_1$  and  $(\iota(y_n))_n \rightarrow z_2$ , corresponding to  $(j(x_n))_n \rightarrow \varphi(z_1)$  and  $(j(y_n))_n \rightarrow \varphi(z_2)$ . Observe that:

$$\begin{aligned} \rho'(\varphi(z_1), \varphi(z_2)) &= \lim_{n \rightarrow \infty} \rho'(j(x_n), j(y_n)) \\ &= \lim_{n \rightarrow \infty} d(x_n, y_n) \\ &= \lim_{n \rightarrow \infty} \rho(\iota(x_n), \iota(y_n)) \\ &= \rho(z_1, z_2). \end{aligned}$$

Since  $\varphi$  is an isometry, it is injective. Clearly  $\varphi \circ \iota = j$  holds, so we have  $j(X) = \varphi(\iota(X)) \subseteq \varphi(Z)$ . Then  $Z' = \overline{j(X)}^\rho \subseteq \overline{\varphi(Z)}$ . Since  $Z$  is complete, by the previous lemma  $\varphi(Z)$  is closed, whence  $Z' \subseteq \varphi(Z) = \varphi(Z)$ . Thus  $\varphi$  is onto.

It remains to show that  $\varphi$  is unique. Let  $\psi : Z \rightarrow Z'$  be another isometric isomorphism satisfying  $\psi \circ \iota = j$ . Given  $z \in Z$ , we can find a sequence  $(\iota(x_n))_n$  in  $\iota(X)$  such that  $(\iota(x_n))_n \rightarrow z$ . Observe that:

$$\begin{aligned} \varphi(z) &= \lim_{n \rightarrow \infty} \varphi(\iota(x_n)) \\ &= \lim_{n \rightarrow \infty} j(x_n) \\ &= \lim_{n \rightarrow \infty} \psi(\iota(x_n)) \\ &= \psi(z). \end{aligned}$$

Since  $z \in Z$  was arbitrary, this proves  $\varphi = \psi$ . □

**Lemma 2.6.19.** *If  $(X, d)$  is a metric space and  $i : (X, d) \rightarrow (Y, \rho)$  is an isometry into a complete metric space  $(Y, \rho)$ , then  $((i(X)^\rho), \rho, i)$  is a completion of  $X$ .*

*Proof.* The space  $(i(X)^\rho, \rho)$  is complete by Proposition 2.6.10. Clearly  $i : X \rightarrow i(X)^\rho$  is an isometry because  $\overline{i(X)}^\rho \subseteq Y$ . □

**Theorem 2.6.20.** *Every metric space admits a unique completion up to isometric isomorphism.*

*Proof.* Uniqueness was shown in Theorem 2.6.18. Given  $(X, d)$ , consider the Banach space  $(C_b(X), \|\cdot\|_u)$ . By the previous lemma we only need to construct an isometry  $X \xrightarrow{i} C_b(X)$ .

Fix any  $x_0 \in X$ . Define  $f_x : X \rightarrow F$  by  $f_x(t) = d(t, x) - d(t, x_0)$ . Clearly  $f_x$  is continuous, and it is bounded because  $|f_x(t)| = |d(t, x) - d(t, x_0)| \leq d(x, x_0)$ . So  $f_x \in C_b(X)$ . Now define  $i : X \rightarrow C_b(X)$  by  $x \mapsto f_x$ . Observe that:

$$\begin{aligned} \|f_x - f_y\|_u &= \sup_{t \in X} |f_x(t) - f_y(t)| \\ &= \sup_{t \in X} |d(t, x) - d(t, x_0) - d(t, y) + d(t, x_0)| \\ &= \sup_{t \in X} |d(t, x) - d(t, y)| \\ &= d(x, y). \end{aligned}$$

Thus  $i$  is an isometry, making  $\left(\overline{(i(X))^{\|\cdot\|_u}}, \|\cdot\|_u, i\right)$  a completion of  $X$ .  $\square$

**Theorem 2.6.21.** *Let  $(X, d)$  be a metric space with completion  $((Z, \rho), i)$ . If  $f : X \rightarrow Y$  is a uniformly continuous function into a complete metric space  $Y$ , then there exists a unique uniformly continuous function  $\tilde{f} : Z \rightarrow Y$  such that  $\tilde{f} \circ i = f$ ; i.e., the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & \searrow f & \downarrow \tilde{f} \\ & & Y \end{array}$$

*Proof.* Define  $g : i(X) \rightarrow Z$  by  $g = f \circ i^{-1}$ . Since  $g$  is uniformly continuous and  $i(X) \subseteq Z$  is dense, Theorem 2.6.14 says there exists a unique uniformly continuous map  $\tilde{f} : Z \rightarrow Y$ . Clearly  $\tilde{f} \circ i = f$ .  $\square$

**Theorem 2.6.22.** *Let  $(V, \|\cdot\|)$  be a normed space. The metric space completion  $((W, \rho), i)$  has the structure of a Banach space with  $\rho(w_1, w_2) = \|w_1 - w_2\|_W$ .*

*Proof.* We have the first define what the vector space operations of  $W$  are. Let  $w \in W$  and  $\alpha \in F$ . We know there exists a sequence  $(i(v_n))_n$  in  $i(V)$  converging to  $w$ . Since this sequence is convergent, it is  $\rho$ -Cauchy. From the following:

$$\begin{aligned} \rho(i(\alpha v_n), i(\alpha v_m)) &= \|\alpha v_n - \alpha v_m\| \\ &= |\alpha| \|v_n - v_m\| \\ &= |\alpha| \rho(i(v_n), i(v_m)), \end{aligned}$$

we can see  $(i(\alpha v_n))_n$  is also  $\rho$ -Cauchy, whence it is convergent. Define scalar multiplication by  $s : F \times W \rightarrow W$  with  $s(\alpha, w) = \lim_{n \rightarrow \infty} i(\alpha v_n) := \alpha w$ . We will first show that this action is well-defined. Let  $(i(u_n))_n$  be another sequence in  $i(V)$  converging to  $w$ . As above, we have:

$$\begin{aligned} \rho(i(\alpha v_n), i(\alpha u_n)) &= |\alpha| \rho(i(v_n), i(u_n)) \\ &\leq \alpha(\rho(i(v_n), w) + \rho(w, i(u_n))) \\ &\rightarrow 0. \end{aligned}$$

This gives:

$$\rho(i(\alpha u_n), \alpha w) \leq \rho(i(\alpha u_n))$$

□