Math 397

Homework 3

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Exercise 3. Let $\{X_k, d_k\}_{k\geqslant 1}$ be a sequence of metric spaces with uniformly bounded metrics. Let:

$$X := \prod_{k\geqslant 1} X_k$$

denote the product.

(i) Show that:

$$D : X\times X \to [0,\infty); \quad D(x,y) := \sum_{k=1}^{\infty} 2^{-k} d_k(x_k, y_k)$$

defines a metric on X.

(ii) Consider the special case when $X_k\{0,2\}$ and $d_k(x,y) = |x-y|$ for every $k \geqslant 1$. We get the *abstract Cantor set*:

$$\Delta := \prod_{k\geqslant 1}\{0,2\}; \quad D(x,y) := \sum_{k=1}^{\infty} 3^{-k}|x_k - y_k|.$$

Prove that $D(x,z) = D(y,z)$ implies $x = y$.

Proof. See attached homework. (Sorry for not typing it up, seems overly tedious). □

Exercise 4. Let $(V, \|\cdot\|)$ be a normed space, and suppose $E \subseteq V$. Show that the following are equivalent:

(i) E is bounded, that is, $\operatorname{diam}(E) < \infty$;

(ii) $\sup_{v\in E}\|v\| < \infty$;

(iii) There is an $r > 0$ with $E \subseteq B(0,r)$.

Proof. ($(i) \Rightarrow (ii)$) Let $\operatorname{diam}(E) < \infty$. Note that $\|v-w\| \leqslant \big|\|v\| - \|w\|\big|$ implies:

$$\sup_{v,w\in E}\|v-w\| \leqslant \sup_{v,w\in E}\big|\|v\| - \|w\|\big|.$$

But this is equivalent to:

$$\sup_{v\in E}\|v\| - \inf_{w\in E}\|w\| \leqslant \alpha.$$

Whence $\sup_{v\in E}\|v\| \leqslant \alpha + \inf_{w\in E}\|w\| < \infty$.

($(ii) \Rightarrow (iii)$) Let $\sup_{v\in E}\|v\|$ be finite. Then there exists $r > 0$ such that $\sup_{v\in E}\|v\| = r$. So for all $v \in E$, $\|v\| \leqslant r$, which implies $v \in B(0,r)$. Thus $E \subseteq B(0,r)$.

($(iii) \Rightarrow (i)$) Suppose there exists $r > 0$ with $E \subseteq B(0,r)$. We have:

$$\begin{aligned}\operatorname{diam}(E) &= \sup_{x,y\in E}\|x-y\| \\ &\leqslant \sup_{x,y\in B(0,r)}\|x-y\| \\ &= 2r \\ &< \infty.\end{aligned}$$

$\square$

Exercise 6. In any metric space show that open balls are open, closed balls are closed, and spheres are closed. Moreover, in a normed space, show that $\partial U(v,r) = \partial B(v,r) = S(v,r)$.

Proof. Let $x \in X$ and $\epsilon > 0$. Let $y \in U(x,\epsilon)$. Consider the open ball $U(y, \epsilon - d(x,y))$. If $z \in U(y, \epsilon - d(x,y))$, then $d(y,z) < \epsilon - d(x,y)$. So we have that $\epsilon > d(x,y) + d(y,z) \geqslant d(x,z)$. This gives $z \in U(x,\epsilon)$ establishing $U(y, \epsilon - d(x,y)) \subseteq U(x,\epsilon)$. Thus open balls are open.

Now let $y \in B(x,\epsilon)^c = \{x_0 \mid d(x,x_0) > \epsilon\}$. Consider the open ball $U(y, d(x,y) - \epsilon)$. If $z \in U(y, d(x,y) - \epsilon)$, then $d(y,z) < d(x,y) - \epsilon$. So we have:

$$\begin{aligned}\epsilon &< d(x,y) - d(y,z)\\ &\leqslant d(x,z) + d(z,y) - d(y,z)\\ &= d(x,z).\end{aligned}$$

This gives $z \in U(x,\epsilon)$, establishing $U(y, d(x,y) - \epsilon) \subseteq B(x,\epsilon)^c$. Since $B(x,\epsilon)^c$ is open, $B(x,\epsilon)$ is closed.

Note that $S(x,\epsilon)^c = B(x,\epsilon)^c \cup U(x,\epsilon)$. Since $S(x,\epsilon)^c$ is the union of open sets, it is open. Thus $S(x,\epsilon)$ is closed.

Lastly, we can see that:

$$\begin{aligned}\partial U(x,\epsilon) &= \overline{U(x,\epsilon)} \setminus U(x,\epsilon)^o\\ &= B(x,\epsilon) \setminus B(x,\epsilon)^o\\ &= \overline{B(x,\epsilon)} \setminus B(x,\epsilon)^o\\ &= \partial B(x,\epsilon).\end{aligned}$$

$$\begin{aligned}S &= B(x,\epsilon) \setminus U(x,\epsilon)\\ &= \overline{U(x,\epsilon)} \setminus U(x,\epsilon)^o\\ &= \partial U(x,\epsilon).\end{aligned}$$

$\square$

Exercise 7. Let (X,d) be a metric space and suppose $A \subseteq X$. Show that the following are equivalent:

(i) $\overline{A} = X$;

(ii) $(\forall U \in \tau_X) : U \cap A \neq \emptyset$;

(iii) $(\forall x \in X)(\forall \epsilon > 0) : U(x,\epsilon) \cap A \neq \emptyset$;

(iv) $(\forall x \in X)(\forall \epsilon > 0)(\exists a \in A) : d(x,a) < \epsilon$.

Proof. *(i)* $\Rightarrow$ *(ii)* Suppose we can find some $U \in \tau_X$ with $U \cap A = \emptyset$. If $U = X$ or $U = \emptyset$, then clearly $\overline{A} \neq X$. Otherwise, we have $A \subseteq U^c \subsetneq X$. Since A is contained in a closed set, we have $\overline{A} \subseteq U^c \subsetneq X$. Thus $\overline{A} \neq X$.

(ii) $\Rightarrow$ *(iii)* Let $U \in \tau_X$ be arbitrary. If $U \cap A \neq \emptyset$, then we can find some $y \in U \cap A$. Since $\mathcal{B} = \{U(x,\epsilon) \mid x \in X, \epsilon > 0\}$ is a basis, we have $y \in U(x,\epsilon) \subseteq U$. Thus $y \in U(x,\epsilon) \cap A$, so $U(x,\epsilon) \cap A \neq \emptyset$.

(iii) $\Rightarrow$ *(iv)* Since $U(x,\epsilon) \cap A \neq \emptyset$, there exists some $a \in U(x,\epsilon) \cap A$. Thus $d(x,a) < \epsilon$.

$(iv) \Rightarrow (i)$ We can inductively find a sequence $(a_n)_n$ in A with $d(x, a_n) < \frac{1}{n}$. Whence $(a_n)_n \to x$. Given $\epsilon > 0$, find N large so that $d(a_N, x) < \epsilon$. Then $a_N \in U(x, \epsilon) \cap A$. Thus $x \in \overline{A}$. $\square$

Exercise 9. Show that c_0 with $\|\cdot\|_u$ is separable.

Proof. Let $z \in c_0$. Then $z = \sum_{k=1}^{\infty} \alpha_k e_k$. Let $\epsilon > 0$. Fix $t_k \in \mathbf{C_Q}$ with $|\alpha_k - t_k| < \epsilon$. Let $y = \sum_{k=1}^{\infty} t_k e_k$. We have:

$$\begin{aligned} \|x - y\|_u &= \left\| \sum_{k=1}^{\infty} \alpha_k e_k - \sum_{k=1}^{\infty} t_k e_k \right\|_u \\ &= \left\| \sum_{k=1}^{\infty} (\alpha_k - t_k) e_k \right\|_u \\ &= \sup_{k=1}^{\infty} |\alpha_k - t_k| \\ &< \epsilon. \end{aligned}$$

$\square$

Exercise 10. Let $\mathfrak{C}$ denote the Cantor set. Show that $\mathfrak{C}$ is nowhere dense.

Proof. Suppose towards contraction $\overline{\mathfrak{C}}^o \neq \emptyset$. Then there is some $x \in \overline{\mathfrak{C}}^o$. We can find an $\epsilon > 0$ with $(x - \epsilon, x + \epsilon) \subseteq \mathfrak{C}$. In particular, $(x - \epsilon, x + \epsilon) \subseteq C_n$ for all $n \geqslant 1$. Find m sufficiently large so that $\epsilon > \frac{1}{3^m}$ and consider $(x - \epsilon, x + \epsilon) \subseteq C_m$. We have that $C_m = \bigsqcup_{j=1}^{2^m} C_{m,j}$ with $\text{length}(C_{m,j}) = \frac{1}{3^m}$. But the length of $(x - \epsilon, x + \epsilon)$ is 2ϵ, which is impossible. It must be that $\mathfrak{C}$ is nowhere dense. $\square$