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# Chapter 1

# Vector Spaces, Algebras, and Normed Spaces

For the entirety of this chapter assume $\mathrm{F}$ to be $\mathbb{R}$ or $\mathbb{C}$.

## § 1.1. Vector Spaces

**Definition 1.1.1.** A *vector space* (or *linear space*) over $\mathrm{F}$ is a nonempty set $\mathrm{V}$ equipped with two operations:

$$\begin{aligned} \mathrm{V} \times \mathrm{V} \xrightarrow{+} \mathrm{V} &\text{ defined by } (v, w) \mapsto v + w \\ \mathrm{F} \times \mathrm{V} \longrightarrow \mathrm{V} &\text{ defined by } (\alpha, v) \mapsto \alpha v \end{aligned}$$

satisfying:

(1) $(\mathrm{V}, +)$ is an abelian group:

- (i) $u + (v + w) = (u + v) + w$ for all $u, v, w \in \mathrm{V}$;
- (ii) there exists $0_{\mathrm{V}}$ such that $v + 0_{\mathrm{V}} = 0_{\mathrm{V}} + v = v$ for all $v \in \mathrm{V}$;
- (iii) for all $v \in \mathrm{V}$, there exists $w \in \mathrm{V}$ satisfying $v + w = w + v = 0_{\mathrm{V}}$;
- (iv) $v + w = w + v$ for all $v, w \in \mathrm{V}$;

(2) $\alpha(v + w) = \alpha v + \alpha w$ for all $\alpha \in \mathrm{F}$, $v, w \in \mathrm{V}$;

(3) $\alpha(\beta v) = (\alpha\beta)v$ for all $\alpha, \beta \in \mathrm{F}$, $v \in \mathrm{V}$;

(4) $1_{\mathrm{F}} v = v$ for all $v \in \mathrm{V}$.

It can be shown that the vector $0_{\mathrm{V}}$ is unique, the additive inverse in (iii) is unique (which we denote as $-v$), that $0v = 0_{\mathrm{V}}$, and $(-1)v = -v$.

**Exercise 1.1.1.** Show (iv) follows from the other axioms.

**Exercise 1.1.2.** Show $nv = \underbrace{v + v + \ldots + v}_{n \text{ times}}$ for $n \in \mathbb{Z}_{\geqslant 1}$.

It can be shown that a subspace is a vector space in its own right.

**Example 1.1.1.** Let $\{W_i\}_{i\in I}$ be a family of vector spaces. Then $\bigcap_{i\in I} W_i$ is also a vector space.

**Example 1.1.2.** Planes and lines through the origin are subspaces of $\mathbb{R}^3$.

**Definition 1.1.2.** Let $V$ be a vector space and $S \subseteq V$ a subset.

(1) A *linear combination* from $S$ is a finite sum $\sum_{j=1}^{n} \alpha_j v_j$ with $\alpha_j \in F$, $v_j \in S$.

(2) The *linear span* of $S$ is:

$$\operatorname{span}(S) := \left\{ \sum_{j=1}^{n} \alpha_j v_j \;\middle|\; n \in \mathbb{N}, \alpha_j \in F, v_j \in S \right\}.$$

**Exercise 1.1.3.** Show that $\operatorname{span}(S) \subseteq V$ is a subspace and:

$$\operatorname{span}(S) = \bigcap \{W \mid S \subseteq W, W \text{ is a subspace}\},$$

that is, $\operatorname{span}(S)$ is the smallest subspace of $V$ containing $S$.

**Definition 1.1.3.** Let $V$ be a vector space and $S \subseteq V$ a subset.

(1) $S$ is *spanning* for $V$ if $\operatorname{span}(S) = V$.

(2) $S$ is *independent* if, given $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in F$, $v_1, \ldots, v_n \in S$, then $\sum_{j=1}^{n} \alpha_j v_j = 0$ implies $\alpha_j = 0$ for all $j$.

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Our goal is to show that every vector space admits a basis. As such, recall the following definitions from a standard course in Real Analysis.

**Definition 1.1.4.** An *ordering* on a set $X$ is a relation $R \subseteq X \times X$ on $X$ that is reflexive, transitive, and antisymmetric. We write $xRy$ as $x \leqslant_R y$. The pair $(X, \leqslant_R)$ is called an *ordered set.* An ordering $\leqslant$ on $X$ is called *total* (or *linear*) if for all $x, y \in X$, $x \leqslant y$ or $y \leqslant x$.

Note that if $(X, \leqslant)$ is an ordered set and $Y \subseteq X$ is a subset, then $(Y, \leqslant)$ is an ordered set as well.

**Definition 1.1.5.** Let $(X, \leqslant)$ be an ordered set and $Y \subseteq X$. An *upper bound* for $Y$ is an element $u \in X$ with $u \geqslant y$ for all $y \in Y$. An element $m \in X$ is called *maximal* if $x \in X$, $x \geqslant m$ implies $x = m$.

**Lemma 1.1.1** (Zorn's Lemma)**.** *Let* $(X, \leqslant_X)$ *be an ordered set. Suppose every subset* $Y \subseteq X$ *for which* $(Y, \leqslant_X)$ *is totally ordered has an upper bound in* $X$*. Then* $X$ *admits a maximal element.*

The proof of Zorn's Lemma is outside the interest of this text.

**Theorem 1.1.2.** *Every vector space admits a basis. Moreover, every independent set is contained in a basis.*

*Proof.* Let $S \subseteq V$ be linearly independent. Define:

$$\mathfrak{T}(S) = \{T \subseteq V \mid S \subseteq T, T \text{ linearly independent}\}.$$

Let $\mathfrak{C} \subseteq \mathfrak{T}(S)$ be a totally ordered subset. Set $R = \bigcup_{T \in \mathfrak{C}} T$. Clearly $R \supseteq S$. Assume $\sum_{j=1}^{n} \alpha_j v_j = 0$, where $\alpha_j \in F$ and $v_j \in R$. Since $\mathfrak{C}$ is totally ordered, there exists $T_0 \in \mathfrak{C}$ with $v_j \in T_0$ for all $j = 1, ..., n$. Since $T_0$ is independent, $\alpha_j = 0$ for all $j = 1, ..., n$. Thus $R$ is independent as well. Whence $R$ is an upper bound for $\mathfrak{C}$. By Zorn's Lemma, $\mathfrak{T}(S)$ admits a maximal element, call it $B$.

Claim: $B$ is a basis for $V$. Suppose towards contradiction it's not, then there exists $v_0 \in V \setminus \operatorname{span}(B)$. Consider $B \cup \{v_0\}$ and let $\alpha_0 v_0 + \sum_{j=1}^{n} \alpha_j v_j = 0_V$. If $\alpha_0 \neq 0$, then $\sum_{j=1}^{n} \alpha_j v_j = -\alpha_0 v_0$, giving $v_0 \in \operatorname{span}(B)$ which is a contradiction. If $\alpha_0 = 0$, then $\sum_{j=1}^{n} \alpha_j v_j = 0_V$. Since $B$ is independent, $\alpha_j = 0$ for all $j = 1, ..., n$. Thus $B \cup \{v_0\}$ is independent, contradicting the maximality of $B$. Whence $B$ is a basis for $V$. □

**Theorem 1.1.3.** *If* $B_1$ *and* $B_2$ *are bases for* $V$*, then* $\operatorname{card}(B_1) = \operatorname{card}(B_2)$.

**Definition 1.1.6.** If $V$ is a vector space, its *dimension* is the cardinality of any of its bases.

**Corollary 1.1.4.** *If* $B$ *is a basis for* $V$*, then every* $v \in V$ *can be written* $v = \sum_{j=1}^{n} \alpha_k \beta_k$, $\alpha_k \in F$, $b_k \in B$ *in a unique way.*

**Theorem 1.1.5.** *Let* $V$ *be a linear space and* $B \subseteq V$ *a subset. The following are equivalent:*

*(1)* $B$ *is a basis for* $V$*;*

*(2)* $B$ *is a maximal element in* $\mathfrak{T} = \{T \subseteq V \mid T \text{ independent}\}$*;*

*(3)* $B$ *is a minimal element in* $\mathfrak{S} = \{S \subseteq V \mid S \text{ spans } V\}$*;*

**Definition 1.1.7.** Let $\{V_i\}_{i \in I}$ be a family of vector spaces over a field $F$.

(1) The *product* of $\{V_i\}_{i \in I}$ is denoted:

$$\prod_{i \in I} V_i := \{(v_i)_{i \in I} \mid v_i \in V_i\}.$$

(2) The *co-product* (or *sum*) is denoted

$$\bigoplus_{i \in I} V_i := \left\{(v_i)_{i \in I} \mid v_i \in V_i, \operatorname{supp}\big((v_i)_{i \in I}\big) < \infty\right\}.$$

**Exercise 1.1.4.**

(1) Show that $\prod_{i\in I} V_i$ equipped with pointwise operations:

$$(v_i)_{i\in I} + (w_i)_{i\in I} = (v_i + w_i)_{i\in I}$$
$$\alpha(v_i)_{i\in I} = (\alpha v_i)_{i\in I}$$

is a linear space.

(2) Show that $\bigoplus_{i\in I} V_i$ is a subspace of $\prod_{i\in I} V_i$.

**Proposition 1.1.6.** *Let $V$ be a vector space over $F$ and $W \subseteq V$. The (additive, abelian) quotient group $V/W$ can be made into a vector space by defining multiplication by scalars as $\alpha(v + W) = \alpha v + W$ for all $\alpha \in F$, $v + W \in V/W$.*

**Example 1.1.3.**

(1) The set $F^n = \{(x_1, ..., x_n) \mid x_j \in F\}$ with component-wise operations is a vector space.

(2) The set $M_{n,m}(F) = \{(a_{ij}) \mid a_{ij} \in F\}$ with linear operations is a vector space.

(3) Let $\Omega$ be a nonempty set. Then $\mathcal{F}(\Omega, F) = \{f \mid f : \Omega \to F\}$ with pointwise operations is a vector space.

(4) The set $\ell_\infty(\Omega, F) = \{f \in \mathcal{F}(\Omega, F) \mid \|f\|_u < \infty\}$ with pointwise operations is a vector space.

**Exercise 1.1.5.** Show $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(5) Let $f : [a, b] \to \mathbb{R}$ be any function. Let $\mathcal{P} = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$. The *variation of* $f$ *on* $\mathcal{P}$ is defined as:

$$\mathrm{Var}(f; \mathcal{P}) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|.$$

We say $f$ is a *bounded variation* if:

$$\mathrm{Var}(f) := \sup_{\mathcal{P}} \mathrm{Var}(f; \mathcal{P}) < \infty.$$

The set of all functions of bounded variation is defined:

$$BV([a, b]) = \{f : [a, b] \to \mathbb{R} \mid \mathrm{Var}(f) < \infty\}.$$

This is a vector space by defining addition and scalar multiplication componentwise.

**Exercise 1.1.6.** Show that BV([a,b]) $\subseteq \ell_\infty([a, b], \mathbb{R})$ is a subspace.

(6) Let $K \subseteq V$ be a convex subset of a vector space $V$, that is, for all $v, w \in K$ and $t \in [0,1]$, then $(1-t)v + tw \in K$. A function $f : K \to F$ is said to be *affine* if $x, y \in K$ and $t \in [0,1]$ implies $f((1-t)x + ty) = (1-t)f(x) + tf(y)$. The set $\text{Aff}(K, F) = \{f \in \mathcal{F}(\Omega, F) \mid f \text{ affine}\}$ with pointwise operations is a vector space.

**Exercise 1.1.7.** Show $\text{Aff}(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(7) The set $C([a,b], F) = \{f : [a,b] \to F \mid f \text{ continuous}\}$ with pointwise operations is a vector space.

**Exercise 1.1.8.** Explain why $C([a,b], F) \subseteq \ell_\infty([a,b], F)$ is a subspace.

(8) Consider the following sequence spaces:

- $s = \{(a_k)_k \mid a_k \in F\} = \mathcal{F}(\mathbb{N}, F)$;
- $\ell_\infty = \ell_\infty(\mathbb{N}, F) = \{(a_k)_k \mid \sup_{k \geqslant 1} |a_k| < \infty\}$;
- $c = \{(a_k)_k \mid (a_k)_k \text{ converges}\}$;
- $c_0 = \{(a_k)_k \mid (a_k)_k \to 0\}$;
- $c_{00} = \{(a_k)_k \mid \text{supp}\big((a_k)_k\big) < \infty\}$;
- $\ell_1 = \Big\{(a_k)_k \mid \sum_{k=1}^\infty |a_k| \text{ converges}\Big\}$.

These are all vector spaces with pointwise operations. In fact, $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subspaces.

**Exercise 1.1.9.** Show that $\ell_1 \subseteq c_0$ is a subspace.

(9) Consider the following continuous function spaces on $\mathbb{R}$:

- $C(\mathbb{R}) = \{f : \mathbb{R} \to F \mid f \text{ continuous}\}$;
- $C_b(\mathbb{R}) = C(\mathbb{R}) \cap \ell_\infty(\mathbb{R})$;
- $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \text{limit}_{x \to \pm\infty} f(x) = 0\}$;
- Recall that a function is *compactly supported* if for all $\epsilon > 0$, there exists $\alpha > 0$ such that $|x| \geqslant \alpha$ implies $f(x) = 0$. The set of compactly supported functions is denoted $C_c(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid f \text{ compactly supported}\}$.

These are all vector spaces with pointwise operations, and $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C_b(\mathbb{R}) \subseteq C(\mathbb{R})$ are all subspace inclusion.

**Definition 1.1.8.** If $V$ and $W$ are linear spaces over a common field $F$, a map $T : V \to W$ is called *linear* if $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha \in F$.

**Example 1.1.4.** Let $A \in M_{m,n}(F)$. Then $T_A : F^n \longrightarrow F^m$ defined by $T_A(v) = Av$ is linear. Let $\{e_1, ..., e_n\}$ be a basis for $F^n$. If $T : F^n \longrightarrow F^m$ is linear, set:

$$[T] = \left(T(e_1) \,\middle|\, T(e_2) \,\middle|\, \ldots \,\middle|\, T(e_n)\right).$$

This gives $T(v) = [T]v$ for all $v \in F^n$. In fact, we also have $[T_A] = A$ and $T_{[T]} = T$.

**Example 1.1.5.** The *canonical projection* is linear:

$$\pi_j : \prod_{i \in I} V_i \longrightarrow V_j \text{ defined by } \pi_j\left((v_i)_i\right) = v_i.$$

We also have that the *coordinate exclusions* are linear:

$$\iota_j : V_j \hookrightarrow \bigoplus_{i \in I} V_i \text{ defined by } \iota_j(v) = (v_i)_i\,, \text{ where } v_i = \begin{cases} 0_v, & i \neq j \\ v_j, & \text{otherwise.} \end{cases}$$

The *evaluation map* is linear as well. For $s \in S$, consider:

$$e_s : \mathcal{F}(S, F) \longrightarrow F \text{ defined by } e_s(f) = f(s).$$

**Proposition 1.1.7.** *Let* $V$ *be a vector space with basis* $B$*. Let* $W$ *be a vector space and suppose* $\varphi : B \longrightarrow W$ *is a map. Then there exists a unique linear map* $T_\varphi : V \longrightarrow W$ *with* $T_\varphi(b) = \varphi(b)$ *for all* $b \in B$*. We have the following diagram.*

$$\begin{tikzcd}
B \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & V \arrow[d, dotted, "T_\varphi"] \\
 & W
\end{tikzcd}$$

*Proof.* Define $T_\varphi : V \longrightarrow W$ by:

$$\begin{aligned} T_\varphi(v) &= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\ &= \sum_{j=1}^{n} \alpha_j \varphi(b_j). \end{aligned}$$

Let $v_1, v_2 \in V$ and $c \in F$. We have that:

$$\begin{aligned}
T_\varphi(v_1 + cv_2) &= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j + c\sum_{j=1}^{n} \beta_j b_j\right) \\
&= T_\varphi\left(\sum_{j=1}^{n} (\alpha_j + c\beta_j) b_j\right) \\
&= \sum_{j=1}^{n} (\alpha_j + c\beta_j)\varphi(b_j) \\
&= \sum_{j=1}^{n} \alpha_j \varphi(b_j) + c\sum_{j=1}^{n} \beta_j \varphi(b_j) \\
&= T_\varphi(v_1) + cT_\varphi(v_2).
\end{aligned}$$

Thus $T_\varphi$ is linear. Chasing the above diagram makes it clear that $T_\varphi(b) = \varphi(b)$. It remains to show that $T_\varphi$ is unique. Let $T$ be another linear transformation satisfying $T(b) = \varphi(b)$ for all $b \in B$. Then:

$$\begin{aligned}
T(v) &= T\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\
&= \sum_{j=1}^{n} \alpha_j \varphi(b_j) \\
&= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\
&= T_\varphi(v).
\end{aligned}$$

Thus $T_\varphi$ is unique. □

**Proposition 1.1.8.** *Let* $T : V \to W$ *be linear.*

*(1)* $\ker(T) = \{v \in V \mid T(v) = 0_W\}$ *is a linear subspace of* $V$.

*(2)* $\operatorname{im}(T) = \{T(v) \mid v \in V\}$ *is a linear subspace of* $W$.

*(3)* $\ker(T) = \{0_V\}$ *if and only if* $T$ *is injective.*

*(4)* $\operatorname{im}(T) = W$ *if and only if* $T$ *is surjective.*

*Proof.* (1) Let $v_1, v_2 \in \ker(T)$ and $\alpha \in F$. Observe that:

$$\begin{aligned}
T(v_1 + cv_2) &= T(v_1) + cT(v_2) \\
&= 0.
\end{aligned}$$

Thus $v_1 + cv_2 \in \ker(T)$, giving $\ker(T)$ as a linear subspace of $V$.

(2) Let $w_1, w_2 \in \operatorname{im}(T)$. Then there exists $v_1.v_2 \in V$ with $T(v_1) = w_1$ and $T(v_2) = w_2$. We have:

$$\begin{aligned} w_1 + cw_2 &= T(v_1) + cT(v_2) \\ &= T(v_1 + cv_2). \end{aligned}$$

Whence $w_1 + cw_2 \in \operatorname{im}(T)$, giving $\operatorname{im}(T)$ as a linear subspace of $W$.

(3) Let $\ker(T) = \{0\}$. Suppose $T(v_1) = T(v_2)$. Then $T(v_1) - T(v_2) = T(v_1 - v_2) = 0_W$. It must be that $v_1 - v_2 = 0_W$, giving $v_1 = v_2$. Thus $T$ is injective. Conversely, suppose $T$ is injective and let $v \in \ker(T)$. Then $T(v) = 0_W = T(0_V)$. Hence $v = 0_V$, establishing $\ker(T) = \{0\}$.

(4) This is by definition of surjectivity. □

**Proposition 1.1.9.** *If $T : V \to W$ is linear and bijective, then the inverse map $T^{-1} : W \to V$ is linear.*

*Proof.* We have that:

$$T(T^{-1}(w_1) + \alpha T^{-1}(w_2)) = w_1 + \alpha w_2 = T \circ T^{-1}(w_1 + \alpha w_2).$$

Applying $T^{-1}$ to both sides gives the desired result. □

**Proposition 1.1.10** (Vector Spaces are Injective)**.** *Let $U, V, W$ be vector spaces and $0 \to U \xrightarrow{j} V$ be exact (that is, $j$ is injective). Let $\varphi : U \to W$ be linear. There exists a linear map $\Psi : V \to W$ such that $\varphi = \Psi \circ j$; i.e., the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "j"] \arrow[d, "\varphi"'] & V \arrow[dl, dotted, "\Psi"] \\ & W & \end{tikzcd}$$

*Proof.* Let $\{u_i\}_{i\in I}$ be a basis for $U$. We must first show that $\{j(u_i)\}_{i\in I}$ is linearly independent. Notice that:

$$\begin{aligned} 0_V &= \sum_{i\in I} \alpha_i j(u_i) \\ &= j\left(\sum_{i\in I} \alpha_i u_i\right). \end{aligned}$$

By the injectivity of $j$, we have that $\sum_{i\in I} \alpha_i u_i = 0_U$. Thus $\alpha_i = 0$ for all $i \in I$, giving $\{j(u_i)\}_{i\in I}$ as linearly independent.

Since $\{j(u_i)\}_{i\in I}$ is linearly independent in $V$, we can extend it to a basis $B = \{v_i\}_{i\in J}$ where $I \subseteq J$ and $v_i = j(u_i)$ whenever $i \in I$. Now define $\psi : B \to W$ by:

$$\psi(v_i) = \begin{cases} \varphi(u_i), & i \in I \\ w, & i \in J \setminus I\,, \end{cases}$$

where $w \in W$ is arbitrary. Since this is a map of basis elements, there exists a unique linear map $\Psi : V \to W$ with $\Psi(v_i) = \psi(v_i)$ for all $v_i \in B$. We can finally see that:

$$\begin{aligned}\varphi(u_i) &= \psi(v_i) \\ &= \Psi(v_i) \\ &= \Psi(j(u_i)).\end{aligned}$$

This establishes that $\varphi = \Psi \circ j$. □

**Proposition 1.1.11** (Vector Spaces are Projective)**.** *Let* $U, V, W$ *be vector spaces and* $V \xrightarrow{\pi} U \to 0$ *be exact (that is,* $\pi$ *is onto). Let* $\varphi : W \to U$ *be linear. There exists a linear map* $\Psi : V \to W$ *such that* $\varphi = \pi \circ \Psi$*; i.e., the following diagram commutes:*

$$\begin{tikzcd} & W \arrow[dl, dotted, "\Psi"'] \arrow[d, "\varphi"] & \\ V \arrow[r, "\pi"] & U \arrow[r] & 0 \end{tikzcd}$$

*Proof.* Let $B = \{w_i\}_{i \in I}$ be a basis for $W$. Define $\psi : B \to V$ by $\psi(w_i) = \pi^{-1}(\varphi(w_i))$. Since this is a map of basis elements, it extends to a unique (dependent on $\pi^{-1}$) linear map $\Psi : W \to V$ with $\Psi(w_i) = \psi(w_i)$ for all $w_i \in B$. Moreover, we have that:

$$\begin{aligned}(\pi \circ \Psi)(w_i) &= (\pi \circ \psi)(w_i) \\ &= (\pi \circ (\pi^{-1} \circ \varphi))(w_i) \\ &= \varphi(w_i).\end{aligned}$$

□

**Definition 1.1.9.** Let $V$ and $W$ be vector spaces over $F$. A *linear isomorphism* between $V$ and $W$ is a bijective linear map $T : V \to W$. If such a $T$ exists, we say $V$ and $W$ are *linearly isomorphic*, and write $V \cong W$.

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Finite dimensional vector spaces are boring. This is illustrated through the following theorem.

**Theorem 1.1.12.** *Let* $V$ *and* $W$ *be finite-dimensional vector spaces over* $F$*. Then* $V \cong W$ *if and only if* $\dim(V) = \dim(W)$*.*

*Proof.* Suppose $V \cong W$. Then there is an isomorphism taking basis of $V$ to a basis of $W$. Therefore they have the same dimension.

Conversely, if $\dim(V) = \dim(W) = n$, then they are each isomorphic to $F^n$, giving that they are isomorphic to eachother. □

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**Example 1.1.6.** Let $V$ be a vector space, $W \subseteq V$ a subspace. The *natural projection*:

$$\pi : V \to V/W \text{ defined by } \pi(v) = v + W$$

is a linear surjective map.

**Theorem 1.1.13** (First Isomorphism Theorem for Vector Spaces)**.** *Let $T : V \to V'$ be a linear map and $W \subseteq V$ a subspace.*

*(1) If $T$ "kills" $W$ (that is, $W \subseteq \ker(T)$), then there exists a linear map $\widetilde{T} : V/W \to V'$ with $\widetilde{T} \circ \pi = T$; i.e., the following diagram commutes.*

$$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, "\pi"'] & & V' \\ & V/W \arrow[ur, dotted, "\widetilde{T}"'] & \end{tikzcd}$$

*(2) If $\ker(T) = W$, then $\widetilde{T}$ is injective.*

*(3) If $\ker(T) = W$ and $\operatorname{im}(T) = V'$, then $V/W \cong V'$.*

*Proof.* (1) As stipulated, define $\widetilde{T}(v + W) = T(v)$. We must show that $\widetilde{T}$ is well-defined: suppose $v_1 + W = v_2 + W$ for some $v_1, v_2 \in V$. Then $v_1 = v_2 + w$ for some $w \in W$. This gives:

$$\begin{aligned} \widetilde{T}(v_1 + W) &= \widetilde{T}(v_2 + w + W) \\ &= \widetilde{T}(v_2 + W). \end{aligned}$$

Whence $\widetilde{T}$ is well-defined. Now given $v_1 + W, v_2 + W \in V/W$ and $\alpha \in F$, observe that:

$$\begin{aligned} \widetilde{T}\big((v_1 + W) + c(v_2 + W)\big) &= \widetilde{T}\big((v_1 + cv_2) + W\big) \\ &= T(v_1 + cv_2) \\ &= T(v_1) + cT(v_2) \\ &= \widetilde{T}(v_1 + W) + c\widetilde{T}(v_2 + W). \end{aligned}$$

Thus $\widetilde{T}$ is linear.

(2) If $\ker(T) = W$, then:

$$\begin{aligned} \ker(\widetilde{T}) &= \{v + W \mid \widetilde{T}(v + W) = 0_{V'}\} \\ &= \{v + W \mid T(v) = 0_{V'}\} \\ &= \{v + W \mid v \in \ker(T)\} \\ &= \{v + W \mid v \in W\} \\ &= \{0\}. \end{aligned}$$

Thus $\widetilde{T}$ is injective.

(3) It remains to show that $\mathrm{im}(T) = V'$ implies $\widetilde{T}$ is surjective. Observe that:

$$\begin{aligned}\mathrm{im}(\widetilde{T}) &= \{\widetilde{T}(v + W) \mid v + W \in V/W\}\\ &= \{\widetilde{T}(\pi(v)) \mid v \in V\}\\ &= \{T(v) \mid v \in V\}\\ &= \mathrm{im}(T)\\ &= V'.\end{aligned}$$

Thus $\widetilde{T}$ is surjective, which establishes it as a bijection. This gives $V/W \cong V'$. □

**Definition 1.1.10.** Let $S$ be a nonempty set. The *free vector space* of $S$ is:

$$\mathbb{F}(S) = \{f : S \to F \mid \mathrm{supp}(f) < \infty\}.$$

**Exercise 1.1.10.** Show $\mathbb{F}(S) \subseteq \mathcal{F}(S, F)$ is a subspace.

**Proposition 1.1.14.** *The set* $\{\delta_s \mid s \in S\}$ *is a basis for* $\mathbb{F}(S)$*, where* $\delta_s : S \to F$ *is defined by:*

$$\delta_s(t) = \begin{cases} 1, & t = 0 \\ 0, & \textit{otherwise.} \end{cases}$$

*Proof.* If $f \in \mathbb{F}(S)$ with $\mathrm{supp}(f) = \{s_1, \dots, s_n\}$, then $f = \sum_{k=1}^{n} f(s_k)\delta_{s_k}$. If $\sum_{k=1}^{n} \alpha_k \delta_{s_k} = 0$, then for $j = 1, \dots, n$ we have $0 = \left(\sum_{k=1}^{n} \alpha_k \delta_{s_k}\right)(s_j) = \alpha_j$. □

**Theorem 1.1.15.** *Given any vector space* $V$ *and a map (of sets)* $\varphi : S \to V$*, there exists a unique linear map* $T_\varphi : \mathbb{F}(S) \to V$ *with* $T_\varphi \circ \iota = \varphi$*, where* $\iota : S \to \mathbb{F}(S)$ *is defined by* $\iota(s) = \delta_s$ *for all* $s \in S$*. The following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathbb{F}(S) \arrow[d, dotted, "T_\varphi"] \\ & V \end{tikzcd}$$

*Proof.* By the previous proposition, we have that $B = \{\delta_s \mid s \in S\}$ is a basis for $\mathbb{F}(S)$. Define $T : B \to V$ by $T(\delta_s) = \varphi(s)$. Since this is a map of basis elements, there exists a unique linear map $T_\varphi : \mathbb{F}(S) \to V$ with $T_\varphi(\delta_s) = T(\delta_s)$ for all $\delta_s \in B$. The diagram commutes because:

$$\begin{aligned}\varphi(s) &= T(\delta_s)\\ &= T_\varphi(\delta_s)\\ &= T_\varphi(\iota(s)).\end{aligned}$$

Moreover, if $T'$ satisfies $\varphi = T' \circ \iota$, then:

$$\begin{aligned} T'(\delta_s) &= T'(\iota(s)) \\ &= \varphi(s) \\ &= T_\varphi(\iota(s)) \\ &= T_\varphi(\delta_s). \end{aligned}$$

Thus $T_\varphi$ is unique. □

**Definition 1.1.11.** Let $V$ and $W$ be vector spaces. The set of linear transformations between $V$ and $W$ is $\mathcal{L}(V, W) = \{T \mid T : V \to W \text{ linear}\}$. The set of linear functionals is $V' := \mathcal{L}(V, F)$.

**Exercise 1.1.11.** Show $\mathcal{L}(V, W)$ is a vector space.

**Exercise 1.1.12.** Show $M_{m,n}(F) \cong \mathcal{L}(F^m, F^n)$ by $a \mapsto T_a : (v \mapsto av)$.

## § 1.2. Algebras

**Definition 1.2.1.** An *algebra* over $F$ is a linear space $A$ over $F$ equipped with a multiplication operation:

$$A \times A \to A \text{ defined by } (a, b) \mapsto ab$$

satisfying:

(1) $(ab)c = a(bc)$ for all $a, b, c \in A$;

(2) $(\alpha a)b = \alpha(ab) = a(\alpha b)$ for all $a, b \in A$, $\alpha \in F$;

(3) $a(b + c) = ab + ac$ for all $a, b, c \in A$;

(4) $(a + b)c = ac + bc$ for all $a, b, c \in A$.

If $ab = ba$ for all $a, b \in A$ we say that $A$ is *commutative*. If there exists $1_A \in A$ with $1_A a = a 1_A = a$ for all $a \in A$ we say $A$ is *unital*.

**Example 1.2.1.**

(1) $M_n(F)$ is a noncommutative unital algebra over $F$ under the usual matrix multiplication.

(2) If $V$ is a vector space over $F$, $\mathcal{L}(V)$ is a unital algebra over $F$. It is noncommutative provided $\dim(V) > 1$.

(3) $\mathcal{F}(S, F)$ is a unital commutative algebra over $F$.

**Definition 1.2.2.** Let $B$ be a (unital) algebra over $F$.

(1) A (unital) *subalgebra* of $B$ is a subspace $A \subseteq B$ ($1_B \in A$) satisfying the property that if $a, a' \in A$, then $aa' \in A$.

(2) An *ideal* of $B$ is a subspace $I \subseteq B$ with $b \in B$, $a \in I$ implying $ba, ab \in I$.

**Example 1.2.2.**

(1) $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a unital subalgebra.

(2) $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subalgebras. In particular, $c_0 \subseteq \ell_\infty$ and $c_{00} \subseteq s$ are ideals.

(3) $C\big([a,b]\big) \subseteq \ell_\infty\big([a,b]\big)$ is a unital subalgebra.

(4) $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C_b(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subalgebras. In fact, $C_b(\mathbb{R}) \subseteq C(\mathbb{R})$ and $C_b(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are unital, whereas $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R})$ and $C_c(\mathbb{R}) \subseteq C(\mathbb{R})$ are ideals.

(5) The set $T_n(F) = \{(a_{ij}) \in M_n(F) \mid a_{ij} = 0, i > j\}$ is a unital subalgebra of $M_n(F)$.

**Example 1.2.3** (Group Algebra)**.** Let $\Gamma$ denote a group (not necessarily abelian). Take the free vector space $\mathbb{F}(\Gamma)$ and define multiplication as *convolution*: given $f, g \in \mathbb{F}(\Gamma)$ let:

$$(f * g)(r) = \sum_{\left\{\substack{(s,t)\,\mid \\ s\in\mathrm{supp}(f), \\ t\in\mathrm{supp}(g), \\ st=r}\right\}} f(s)g(t).$$

Since supp(f) and supp(g) are finite, this is a finite sum. We often suppress this notation and write $(f * g)(r) = \sum_{st=r} f(s)g(t)$.

We can also make substitutions:

$$\begin{aligned}(f * g)(r) &= \sum_{st=r} f(s)g(t) \\ &= \sum_{t\in\Gamma} f(rt^{-1})g(t) \\ &= \sum_{s\in\Gamma} f(s)g(s^{-1}r).\end{aligned}$$

It is clear that:

$$\begin{aligned}(f + g) * h &= f * h + g * h \\ g * (g + h) &= f * g + f * h \\ \alpha(f * g) &= (\alpha f) * g = f * (\alpha g)\end{aligned}$$

for $f, g, h \in \mathbb{F}(\Gamma)$, $\alpha \in F$. Associativity can be similarly shown using the above definition. Rather, we will prove associativity by first show that $\delta_s * \delta_t = \delta_{st}$. Given:

$$(\delta_s * \delta_t)(r) = \sum_{q\in\Gamma} \delta_s(rq^{-1})\delta_t(q),$$

notice that:

$$\delta_s(rt^{-1}) = \begin{cases} 1, & s = rt^{-1} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & r = st \\ 0, & \text{otherwise} \end{cases} = \delta_{st}(r).$$

Since $\{\delta_t \mid t \in \Gamma\}$ is a basis for $\mathbb{F}(\Gamma)$, every $f \in \mathbb{F}(\Gamma)$ looks like:

$$f = \sum_{t \in J} \alpha_t \delta_t, \ J \subseteq T \text{ finite.}$$

Using distributivity we get:

$$\begin{aligned} \delta_r * (\delta_s * \delta_t) &= \delta_r * \delta_{st} \\ &= \delta_{rst} \\ &= \delta_{rs} * \delta_t \\ &= (\delta_r * \delta_s) * \delta_t. \end{aligned}$$

Whence convolution is associative.

**Exercise 1.2.1.** Let $\{A_i\}_{i \in I}$ be a family of algebras over $F$.

(1) $\prod_{i \in I} A_i$ is an algebra under $(a_i)_i(b_i)_i = (a_i b_i)_i$.

(2) $\bigoplus_{i \in I} A_i \subseteq \prod_{i \in I} A_i$ is an ideal.

**Exercise 1.2.2.** Let $A$ be an algebra over $F$ and $I \subseteq A$ an ideal. Then $A/I$ is an algebra under $(a + I)(b + I) = ab + I$.

## § 1.3. Normed Vector Spaces

To each vector $v$ in a vector space $V$, we want to assign a "length", denoted $\|v\|$.

**Definition 1.3.1.** A *norm* on a vector space $V$ is a map:

$$\|\cdot\| : V \to [0, \infty), \ v \mapsto \|v\|$$

satisfying:

(1) $\|\alpha v\| = |\alpha| \, \|v\|$ for all $\alpha \in F$, $v \in V$ (homogeneity);

(2) $\|v + w\| \leqslant \|v\| + \|w\|$ (triangle inequality);

(3) If $\|v\| = 0$, then $v = 0_V$ (positive-definite).

If $\|\cdot\|$ satisfies (1) and (2), it is called a *seminorm*. The pair $(V, \|\cdot\|)$ is called a *normed space*.

**Definition 1.3.2.** Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space $V$ are called *equivalent* if there exists $c_1 \geqslant 0$ and $c_2 \geqslant 0$ with $\|v\| \leqslant c_1 \|v\|'$ and $\|v\|' \leqslant c_2 \|v\|$ for all $v \in V$.

**Exercise 1.3.1.** If $p$ is a seminorm on $V$, show that $|p(v) - p(w)| \leqslant p(v - w)$.

**Definition 1.3.3.** Let $(V, \|v\|)$ be a normed space.

(1) The *closed unit ball* is denoted $B_V = \{v \in V \mid \|v\| \leqslant 1\}$.

(2) The *open unit ball* is denoted $U_V = \{v \in V \mid \|v\| < 1\}$.

(3) The *unit sphere* is denoted $S_V = \{v \in V \mid \|v\| = 1\}$.

**Example 1.3.1.** Let $V = F^n$ and $x = (x_1, ..., x_n)$. We define:

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|;$$
$$\|x\|_\infty = \max_{j=1}^{n} |x_j|;$$
$$\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}.$$

For $p \geqslant 1$:

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}.$$

**Exercise 1.3.2.** Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms

———————//———————

We aim to show that $\|\cdot\|_p$ is a norm for $p \in [0, \infty]$.

**Lemma 1.3.1.** *Let* $p, q \in [1, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $f : [0, \infty) \to \mathbb{R}$ *be defined by* $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. *Then* $f(t) \geqslant 0$ *for* $t \geqslant 0$.

*Proof.* Note that $f'(t) = t^{p-1} - 1$. Since:

$$f'(1) = 0$$
$$f'(t) > 0 \ \text{ for } \ t > 1$$
$$f'(t) < 0 \ \text{ for } \ 0 \leqslant t < 1, f65qed'; lkjhgfdsa$$

we can see that $f(t) \geqslant 0$ for all $t \geqslant 0$. □

**Lemma 1.3.2** (Young's Inequality). *Let* $p, q \in [0, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $x, y \geqslant 0$, *then* $xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q$.

*Proof.* By Lemma 1.3.1, $t \leqslant \frac{1}{p}t^p + \frac{1}{q}$. Multiplying both sides by $y^q$ gives:

$$ty^q \leqslant \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Let $t = xy^{1-q}$. Then:

$$xy^{1-q}y^q \leqslant \frac{1}{p}x^p y^{p-pq}y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have that $p - pq = -q$. Whence:

$$xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q.$$

□

**Lemma 1.3.3** (Hölders Inequality)**.** *Let* $p, q \in [0, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then for* $x, y \in F^n$*:*

$$\left|\sum_{j=1}^{n} x_j y_j\right| \leqslant \|x\|_p \|y\|_q.$$

*Proof.* We proceed by cases.

Case 1: $p = 1$. Then:

$$\begin{aligned}\left|\sum_{j=1}^{n} x_j y_j\right| &\leqslant \sum_{j=1}^{n} |x_j||y_j| \\ &\leqslant \sum_{j=1}^{n} |x_j|\|y\|_\infty \\ &= \|x\|_1\|y\|_\infty.\end{aligned}$$

Case 2: $p = \infty$. This follows similarly to Case 1.

Case 3: $1 < p < \infty$. Suppose $\|x\|_p = \|y\|_q = 1$. Then:

$$\begin{aligned}
\left|\sum_{j=1}^{n} x_j y_j\right| &\leqslant \sum_{j=1}^{n} |x_j||y_j| \\
&\leqslant \sum_{j=1}^{n} \left(\frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q\right) \\
&= \frac{1}{p}\left(\sum_{j=1}^{n} |x_j|^p\right) + \frac{1}{q}\left(\sum_{j=1}^{n} |y_j|^q\right) \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

Whence the inequality holds. Now suppose $\|x\|_p = 0$ or $\|y\|_q = 0$. Then $x = 0_{F^n}$ or $y = 0_{F^n}$, whence the inequality holds. Suppose $\|x\|_p \neq 0$ and $\|y\|_p \neq 0$. Set:

$$\begin{aligned}
x' &= \frac{x}{\|x\|_p} \\
y' &= \frac{y}{\|y\|_p}.
\end{aligned}$$

Then $\|x'\|_p = 1 = \|y'\|_p$. Observe that:

$$\begin{aligned}
1 &\geqslant \left|\sum_{j=1}^{n} x_j' y_j'\right| \\
&= \left|\sum_{j=1}^{n} \frac{x}{\|x\|_p} \frac{y}{\|y\|_p}\right|.
\end{aligned}$$

Multiplying both sides by $\|x\|_p\|y\|_q$ gives the desired result. □

**Lemma 1.3.4** (Minkowski's Inequality). *Let* $p, q \in [1, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $x, y \in F^n$*:*

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p.$$

*Proof.* The only nontrivial case is for $1 < p < \infty$. Observe that:

$$\begin{aligned}
\left(\|x+y\|_p\right)^p &= \sum_{j=1}^n |x_j + y_j|^p \\
&= \sum_{j=1}^n |x_j + y_j||x_j + y_j|^{p-1} \\
&\leqslant \sum_{j=1}^n |x_j||x_j + y_j|^{p-1} + |y_j||x_j + y_j|^{p-1} \\
&\leqslant \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{j=1}^n |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{p-1\left(\frac{p}{p-1}\right)}\right)^{1-\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{p-1\left(\frac{p}{p-1}\right)}\right)^{1-\frac{1}{p}} \\
&= \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{p}\right)^{1-\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{p}\right)^{1-\frac{1}{p}} \\
&= (\|x\|_p + \|y\|_p)\frac{\|x+y\|_p^p}{\|x+y\|_p}.
\end{aligned}$$

Multiplying boths sides by $\frac{\|x+y\|_p}{\|x+y\|_p^p}$ gives the desired inequality. □

**Theorem 1.3.5.** *Let* $V = F^n$*. Then* $(F^n, \|\cdot\|_p)$ *is a normed space.*

*Proof.* Let $x = (x_1, ..., x_n) \in F^n$ and $\alpha \in F$. Observe that:

$$\begin{aligned}
\|\alpha x\|_p &= \left(\sum_{j=1}^n |\alpha x_j|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^n |\alpha|^p |x_j|^p\right)^{\frac{1}{p}} \\
&= |\alpha|\|x\|_p.
\end{aligned}$$

This satisfies homogeneity. Moreover, Minkowski's Inequality satisifes the triangle inequality. It remains to show that $\|\cdot\|_p$ is positive-definite. If $\|x\|_p = 0$, then $x_j = 0$ for all $1 \leqslant j \leqslant n$. Thus $x = 0_{F^n}$. □

**Corollary 1.3.6.** *Let* $p \in [1, \infty]$*. Then* $\ell_p = \left\{(a_k)_k \mid \sum_{k=1}^\infty |a_k|^p < \infty\right\}$ *with norm* $\|(a_k)_k\|_p = \left(\sum_{k=1}^\infty |a_k|^p\right)^{\frac{1}{p}}$ *is a normed space.*

*Proof.* Homogeneity and positive-definiteness are trivial to prove. Let $(x_k)_k, (y_k)_k \in \ell_p$. It is clear that:

$$\begin{aligned}\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} &\leqslant \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}} \\ &\leqslant \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} \\ &= \|(x_k)_k\|_p + \|(y_k)_k\|_p .\end{aligned}$$

We have that $\sum_{k=1}^{n} |x_k + y_k|^p$ is increasing and bounded above by $(\|(x_k)_k\|_p + \|(y_k)_k\|_p)^p$. By the Monotone Convergence Theorem $\mathrm{limit}_{n\to\infty} \sum_{k=1}^{n} |x_k + y_k|^p = \sum_{k=1}^{\infty} |x_k+y_k|^p$ exists. Whence $\left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{\frac{1}{p}} = \|(x_k)_k + (y_k)_k\|_p \leqslant \|(x_k)_k\|_p + \|(y_k)_k\|_p$ □

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**Example 1.3.2.**

(1) $\left(\ell_\infty(\Omega, \mathbb{F}), \|\cdot\|_u\right)$ where $\|f\|_u = \sup_{x\in\Omega} |f(x)|$ is a normed space. This includes its subspaces, such as $C\left([a, b], \mathbb{F}\right) \subseteq \ell_\infty\left([a, b], \mathbb{F}\right)$ and $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$, all with $\|\cdot\|_u$.

(2) Take $\Omega = \mathbb{N}$ in the previous example. Then $\left(\ell_\infty, \|\cdot\|_\infty\right)$ is a normed space. This includes its subspaces $c_{00} \subseteq c_0 \subseteq \ell_\infty$ with $\|\cdot\|_\infty$.

(3) $\left(\ell_1, \|\cdot\|_1\right)$ is a normed space.

(4) $\left(C\left([a, b]\right), \|\cdot\|_1\right)$ with $\|f\|_1 = \int_a^b |f(t)|dt$ is a normed space.

(5) $\left(BV\left([a, b]\right), \|\cdot\|_{BV}\right)$ where $\|f\|_{BV} = |f(a)| + \mathrm{Var}(f)$ is a normed space.

(6) Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Then $\left(B(V, W), \|\cdot\|_{op}\right)$ is a normed space, where $B(V, W) = \{T \in \mathcal{L}(V, W) \mid \|T\|_{op} < \infty\}$ is the set of bounded linear maps and $\|T\|_{op} = \sup_{v\in B_V} \|T(v)\|_W$. Intuitively, $\|T\|_{op}$ measures the radius of the smallest ball which contains $B_V$.

   **Exercise 1.3.3.** Show that $V^* := B(V, \mathbb{F})$ is a subspace of $V'$.

(7) Let $S$ be a nonempty set. Both $\left(\mathbb{F}(S), \|\cdot\|_1\right)$ and $\left(\mathbb{F}(S), \|\cdot\|_p\right)$ are normed spaces, where $\|f\|_1 = \sum_{s\in S} |f(s)|$ and $\|f\|_p = (\sum_{s\in S} |f(s)|^p)^{\frac{1}{p}}$. Note that since $f(s) \neq 0$ for finitely many $s \in S$, both $\|\cdot\|_1$ and $\|\cdot\|_p$ are well-defined.

   **Exercise 1.3.4.** Show that $\|f\|_\infty := \sup_{s\in S} |f(s)|$ is a norm on $\mathbb{F}(S)$.

# § 1.4. Inner Product Spaces

Before defining what an inner product space is, we must introduce new terminology.

**Definition 1.4.1.** Let $V$ be a vector space over $F$ and $\varphi : V \times V \to F$ a map.

(1) The map $\varphi$ is said to be a *bilinear form* if is is linear in the first and second variable seperately; i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have:

   (i) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$
   (ii) $\varphi(v, cv_1 + v_2) = \varphi(v, v_1) + c\varphi(v, v_2)$.

(2) The map $\varphi$ is said to be a *sesquilinear form* if it is linear in the first variable and conjugate linear in the second variable; i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have:

   (i) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$
   (ii) $\varphi(v, cv_1 + v_2) = \overline{c}\varphi(v, v_1) + \varphi(v, v_2)$.

If we wish to keep track of a bilinear form on $V$ we write $(V, \varphi)$.

**Definition 1.4.2.** Let $V$ be a vector space over $F$.

(1) A bilinear form $\varphi$ on $V$ is said to be *symmetric* if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

(2) A sesquilinear form $\varphi$ on $V$ is said to be *Hermitian* if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.

**Definition 1.4.3.** Let $(V, \varphi)$ be a vector space over $F$ such that if $\varphi$ is symmetric, then $F = \mathbb{R}$ or if $\varphi$ is Hermitian, then $F = \mathbb{C}$. We say $\varphi$ is *positive-definite* if for all nonzero $v \in V$ we have $\varphi(v, v) > 0$.

**Definition 1.4.4.** Let $(V, \varphi)$ be a vector space over $\mathbb{R}$ with $\varphi$ a positive-definite symmetric bilinear form or over $\mathbb{C}$ with $\varphi$ a positive-definite Hermitian sesquilinear form. Then we say $\varphi$ is an *inner product* on $V$ and write $\varphi$ as $\langle \cdot, \cdot \rangle$. We say $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space.*

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With that out of the way, we will now show that every inner product space induces a norm.

**Definition 1.4.5.** If $V$ is an inner product space we define $\|v\|_2 = \langle v, v \rangle^{\frac{1}{2}}$.

**Definition 1.4.6.** Let $V$ be an inner product space. Two vectors $v, w \in V$ are *orthogonal* if $\langle u, v\rangle = 0$. We denote this as $u \perp v$.

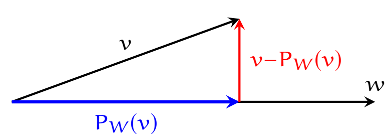
**Theorem 1.4.1** (Pythagorean Theorem)**.** *Let $v_1, \ldots, v_n$ be mutually orthogonal. Then $\sum_{j=1}^{n} \|v_j\|_2^2 = \left\|\sum_{j=1}^{n} v_j\right\|_2^2$.*

*Proof.* Because $v_i \perp v_j$ for $1 \leqslant i, j \leqslant n$, we have $\langle v_i, v_j\rangle = 0$. Observe that:

$$\begin{aligned}
\left\|\sum_{j=1}^{n} v_j\right\|_2^2 &= \left\langle \sum_{j=1}^{n} v_j, \sum_{j=1}^{n} v_j \right\rangle \\
&= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \langle v_j, v_i \rangle \right) \\
&= \sum_{j=1}^{n} \langle v_j, v_j \rangle \\
&= \sum_{j=1}^{n} \|v_j\|_2^2 .
\end{aligned}$$

□

**Definition 1.4.7.** Let $V$ be an inner product space and $w \in V$ nonzero. The *projection* of a vector $v \in V$ onto $w$ is a map $P_W : V \to V$ defined by $P_W(v) = \frac{\langle v, w\rangle}{\langle w, w\rangle} w$.




**Proposition 1.4.2.** ** Let $V$ be an inner product space and $w \in V$ a nonzero vector. Then $P_w(v) \perp v - P_w(v)$.*

*Proof.* □

**Corollary 1.4.3.** ** Let $V$ be an inner product space and $w \in W$ a nonzero vector. Then $\|v\|_2^2 = \|P_w(v)\|_2^2 + \|v - P_w(v)\|_2^2$.*

*Proof.* □

**Lemma 1.4.4** (Cauchy-Schwartz Inequality)**.** *Let $V$ be an inner product space and $v, w \in V$. Then $|\langle v, w\rangle| \leqslant \|v\|_2 \|w\|_2$.*

*Proof.* The previous corollary gives $\|v\|_2 \geqslant \|P_w(v)\|_2$. We have that:

$$\begin{aligned}
\|v\|_2 &\geqslant \left\| \frac{\langle v, w\rangle}{\langle w, w\rangle} w \right\|_2 \\
&= \frac{|\langle v, w\rangle|}{\|w\|_2^2} \|w\|_2 \\
&= \frac{\langle v, w\rangle}{\|w\|_2}.
\end{aligned}$$

Multiplying both sides by $\|w\|_2$ gives the desired result. □

**Theorem 1.4.5.** *Let* $V$ *be an inner product space. Then* $(V, \|\cdot\|_2)$ *is a normed space.*

*Proof.* Let $v, w \in V$ and $\alpha \in F$. We have that:

$$\begin{aligned}
\|\alpha v\|_2 &= \langle \alpha v, \alpha v\rangle^{\frac{1}{2}} \\
&= (\alpha\overline{\alpha}\langle v, v\rangle)^{\frac{1}{2}} \\
&= \left(|\alpha|^2\langle v, v\rangle\right)^{\frac{1}{2}} \\
&= |\alpha| \, \|v\|_2 \, .
\end{aligned}$$

Thus $\|\cdot\|_2$ satisfies homogeneity. It follows from the Cauchy-Schwartz Inequality that:

$$\begin{aligned}
\|v + w\|_2^2 &= \langle v + w, v + w\rangle \\
&= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\
&= \|v\|_2^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} + \|w\|_2^2 \\
&= \|v\|_2^2 + 2\Re\left(\langle v, w\rangle\right) + \|w\|_2^2 \\
&\leqslant \|v\|_2^2 + 2|\langle v, w\rangle| + \|w\|_2^2 \\
&\leqslant \|v\|_2^2 + 2\|v\|_2\|w\|_2 + \|w\|_2^2 \\
&= (\|v\|_2 + \|w\|_2)^2 \, ,
\end{aligned}$$

where we used the fact that $2\Re(\langle v, w\rangle) = 2|\langle v, w\rangle|$. Squaring both sides proves that $\|\cdot\|_2$ satisfies the triangle inequality. It remains to show positive-definiteness. Suppose $\|v\|_2 = 0$. Then $\langle v, v\rangle = 0$, but since the inner-product is by definition positive-definite, we get that $v = 0_V$. □

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**Example 1.4.1.**

(1) $\ell_2^n = F^n$ is an inner product space where $\langle (x_1, ..., x_n), (y_1, ..., y_n)\rangle := \sum_{j=1}^n x_j \overline{y_j}$.

(2) $\ell_2$ is an inner product space where $\langle (a_k)_k, (b_k)_k \rangle := \sum_{k=1}^{\infty} a_k \overline{b_k}$. Note that:

$$\begin{aligned}\sum_{k=1}^{n} \left| a_k \overline{b_k} \right| &= \sum_{k=1}^{n} |a_k||b_k| \\ &\leqslant \left( \sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} |b_k|^2 \right)^{\frac{1}{2}} \\ &\leqslant \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} |b_k|^2 \right)^{\frac{1}{2}} \\ &= \|(a_k)_k\|_2 \, \|(b_k)_k\|_2 \\ &< \infty \qquad \text{(Because } (a_k)_k, (b_k)_k \in \ell_2\text{).}\end{aligned}$$

Since $\sum_{k=1}^{n} |a_k \overline{b_k}|$ is increasing and bounded above, the Monotone Convergence Theorem says $\sum_{k=1}^{\infty} |a_k \overline{b_k}|$ exists and is finite. Whence $\langle (a_k)_k, (b_k)_k \rangle$ converges.

(3) Recall that $\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}$ is defined by $\mathrm{Tr}(a_{ij}) = \sum_{i=1}^{n} a_{ii}$. Then $M_n(\mathbb{C})$ is an inner product space where $\langle a_{ij}, b_{ij} \rangle := \mathrm{Tr}(b_{ij}^* a_{ij})$.

(4) $C([0,1])$ is an inner product space where $\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}dx$.

## § 1.5. Normed Algebras

**Definition 1.5.1.** A *normed algebra* is an algebra $A$ equipped with a norm $\|\cdot\|_A$ such that $\|ab\|_A \leqslant \|a\|_A \|b\|_A$. If $A$ is unital, we require $\|1\|_A = 1$.

**Example 1.5.1.**

(1) $\ell_\infty(\Omega)$ equipped with $\|\cdot\|_u$ is a normed algebra.

(2) $C_c(\mathbb{R})$, $C_0(\mathbb{R})$, and $C([0,1])$ are all normed algebras when equipped with $\|\cdot\|_u$.

(3) $M_n(F)$ equipped with $\|\cdot\|_{op}$ is a normed algebra.

(4) If $V$ is a normed space, then $B(V,V)$ with $\|\cdot\|_{op}$ is a normed algebra: for $T, S \in B(V,V)$ and $v \in B_V$, we have that

$$\begin{aligned}\|(T \circ S)(v)\| &\leqslant \|T\|_{op} \|S(v)\| \\ &\leqslant \|T\|_{op} \|S\|_{op}.\end{aligned}$$

Taking the supremum over all $v \in B_V$ gives $\|T \circ S\|_{op} \leqslant \|T\|_{op} \|S\|_{op}$.

(5) Let $S$ be a group. Equip the algebra $\mathbb{F}(S)$ with $\|\cdot\|_1$. We get a normed algebra.

**Exercise 1.5.1.** For $a, b \in \ell_1(\mathbb{Z})$, define $a * b : \mathbb{Z} \to F$ by $(a * b)(n) = \sum_{k \in \mathbb{Z}} a(n-k)b(k)$. Show that $\ell_1(\mathbb{Z})$ with this multiplication is a normed algbera.