Math 310

Homework 1

Due: 9/9/2024

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Exercise 1. If F is a finite set and $k : F \to F$ is a self map, prove that k is injective if and only if k is surjective.

Proof. Let k be injective. Suppose towards contradiction that k is not surjective. Then $k(F) \subset F$. But then there exists $f_i, f_j \in F$ such that $k(f_i) = k(f_j)$ with $f_i \neq f_j$, contradicting the fact that k is injective. Hence k must also be surjective.

Now suppose k is not injective. Then there exists *at least* two elements f_i, f_j with $k(f_i) = k(f_j)$ and $f_i \neq f_j$. So $k(F) \subset F$, hence k is not surjective. $\square$

Exercise 2. Prove that a set A is infinite if and only if there is a non-surjective injection $f : A \to A$.

Proof. Suppose A is infinite. Then there exists an injection $\pi : \mathbf{N} \hookrightarrow A$ defined by $\pi(n) = a_n$. Define $f : A \to A$ by $f(\pi(n)) = \pi(n+1)$. Suppose $f(\pi(i)) = f(\pi(j))$, then $\pi(i+1) = \pi(j+i)$. Simplifying further yields $i+1 = j+1$, or equivalently $i = j$. Hence $\pi(i) = \pi(j)$, establishing f as an injection. Now suppose there exists a surjection $A \twoheadrightarrow A$. This immediately leads to a contradiction, as there does not exist an element $x \in A$ such that $f(x) = \pi(1)$. Hence f is a non-surjective injection.

Conversely, suppose that A is finite. Then by Exercise 1 there exists a bijection $A \hookrightarrow\!\!\!\!\!\rightarrow A$. $\square$

Exercise 3. Let A, B, and C be sets and suppose $\operatorname{card}(A) < \operatorname{card}(B) \leqslant \operatorname{card}(C)$. Prove that $\operatorname{card}(A) < \operatorname{card}(C)$.

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be functions with g injective and f injective but not surjective. Then $g \circ f$ is injective, but note that $f(A) \subset B$ implies $(g \circ f)(A) \subset C$ (otherwise g would not be a function). Hence there does not exist a surjection $g \circ f : A \to C$, establishing $\operatorname{card}(A) < \operatorname{card}(C)$. $\square$

Exercise 4. If $A \subseteq B$ is an inclusion of sets with A countable and B uncountable, show that $B \setminus A$ is uncountable.

Proof. Suppose towards contradiction that $B \setminus A$ is countable. Since countable unions of countable sets is countable, $(B \setminus A) \cup A = B$ is countable, which is a contradiction. Thus $B \setminus A$ is uncountable. $\square$

Exercise 5. Is the set $\{x \in \mathbf{R} \mid x > 0 \text{ and } x^2 \in \mathbf{Q}\}$ countable?

Proof. Let S be the above set. If $x \in S$, then $x > 0$ and $x^2 \in \mathbf{Q}$. This implies that $x^2 = q$ for some $q \in \mathbf{Q}^+$, hence $x = \sqrt{q}$. Define $f : S \to \mathbf{Q}$ by $\sqrt{q} \mapsto q$. Let $f(\sqrt{m}) = f(\sqrt{n})$ for some $\sqrt{m}, \sqrt{n} \in S$. Then $m = n$, and square-rooting both sides gives $\sqrt{m} = \sqrt{n}$, establishing an injection. Thus S is countable. $\square$

Exercise 6. Consider the set $\mathcal{F}(\mathbf{N})$ of all finite subsets of $\mathbf{N}$. Is $\mathcal{F}(\mathbf{N})$ countable?

Proof. Let $A_n = \{A \subseteq \mathbf{N} \mid \operatorname{card}(A) = n \text{ for some } n \in \mathbf{N}\}$ Note that A_n by our definition is finite, and $\bigcup_{n \in \mathbf{N}} A_n = \mathcal{F}(\mathbf{N})$. Thus $\mathcal{F}(\mathbf{N})$ is countable. $\square$

Exercise 7. Let $k \in \mathbf{N}$.

(i) Prove that $\mathbf{N}^k := \underbrace{\mathbf{N} \times \mathbf{N} \times ... \times \mathbf{N}}_{k \text{ times}}$ is countable.

Proof. Let $p_1, p_2, ..., p_k$ denote the first k primes. Define $f : \mathbf{N}^k \to \mathbf{N}$ by $(e_1, e_2, ..., e_k) \mapsto p_1^{e_1} \cdot p_2^{e_2} \cdot ... \cdot p_k^{e_k}$. Then $f((e_1, e_2, ..., e_k)) = f((r_1, r_2, ..., r_k))$ is equivalent to $p_1^{e_1} \cdot p_2^{e_2} \cdot ... \cdot p_k^{e_k} = p_1^{r_1} \cdot p_2^{r_2} \cdot ... \cdot p_k^{r_k}$. By the fundamental theorem of arithmetic, every natural number is prime itself or the product of a *unique* combination of prime numbers. Therefore $p_1^{e_1} \cdot p_2^{e_2} \cdot ... \cdot p_k^{e_k} = p_1^{r_1} \cdot p_2^{r_2} \cdot ... \cdot p_k^{r_k}$ implies $e_i = r_i$ for all $1 \leqslant i \leqslant k$. Hence $(e_1, e_2, ..., e_k) = (r_1, r_2, ..., r_k)$, establishing f as an injection into the natural numbers. Thus $\mathbf{N}^k$ is countable. □

(ii) Show that the set

$$\mathbf{N}^\infty := \{(n_k)_{k\geqslant 1} \mid n_k \in \mathbf{N}\}$$

consisting of all sequences of natural numbers is uncountable.

Proof. Note that $2^{\mathbf{N}} = \{f \mid f : \mathbf{N} \to \{0,1\}\} \subseteq \mathbf{N}^\infty$. Since $2^{\mathbf{N}}$ is uncountable, $\mathbf{N}^\infty$ must be uncountable. □

(iii) Prove that the set of **finitely-supported** natural sequences

$$c_c(\mathbf{N}) := \{(n_k)_{k\geqslant 1} \mid n_k \in \mathbf{N}, n_k = 0 \text{ for all but finitely many } k\}$$

is countable.

Proof. Let $c_i(\mathbf{N}) = \{(n_k)_{k\geqslant 1} \mid n_k \in \mathbf{N}, n_k = 0 \text{ for all } \ k > i \in \mathbf{N}\}$. Define $f : c_i(\mathbf{N}) \to \mathbf{N}^i$ by $(n_k)_{k\geqslant 1} \mapsto (n_1, n_2, ..., n_i)$. If $f((n_k)_{k\geqslant 1}) = f((p_k)_{p\geqslant 1})$, then $(n_1, n_2, ..., n_i) = (p_1, p_2, ..., p_i)$; i.e., $n_j = p_j$ for all $1 \leqslant j \leqslant i$. Hence $(n_k)_{k\geqslant 1} = (p_k)_{k\geqslant 1}$. Since f is injective, $c_i(\mathbf{N})$ is countable, therefore $c_c(\mathbf{N}) = \bigcup_{i\in\mathbf{N}} c_i(\mathbf{N})$ is countable. □

(iv) Is the set of decreasing natural sequences

$$D := \{(n_k)_{k\geqslant 1} \mid n_k \in \mathbf{N}, n_{k+1} \leqslant n_k \text{ for all } k \geqslant 1\}$$

countable or uncountable?

Proof. Let $c_{i,j}(\mathbf{N}) = \{(n_k)_{k\geqslant 1} \mid n_k \in \mathbf{N}, n_{k+1} \leqslant n_k \text{ for all } k \geqslant 1 \text{ terminating in } j \text{ for all } k > i \in \mathbf{N}\}$. By (iii) this set is countable, hence $D = \bigcup_{j\in\mathbf{N}} \left(\bigcup_{i\in\mathbf{N}} c_{i,j}(\mathbf{N})\right)$ is countable. □

Exercise 8. Let $f : \mathbf{R} \to \mathbf{R}$ be a function that sends rational numbers to irrational numbers and irrational numbers to rational numbers. Prove that $\mathrm{im}\,(f)$ can't contain any interval.

Proof. Note that $\mathbf{R} = (\mathbf{R} \setminus \mathbf{Q}) \cup \mathbf{Q}$. Then $f(\mathbf{R}) = f((\mathbf{R} \setminus \mathbf{Q}) \cup \mathbf{Q}) = f(\mathbf{R} \setminus \mathbf{Q}) \cup f(\mathbf{Q})$. Since $f(\mathbf{R} \setminus \mathbf{Q}) \subseteq \mathbf{Q}$, it is countable. Likewise, since $\mathbf{Q}$ has a countable number of elements, $f(\mathbf{Q})$ must get mapped to a countable subset of $\mathbf{R} \setminus \mathbf{Q}$ (otherwise f would not be a function). Therefore $f(\mathbf{Q})$ is countable, establishing $f(\mathbf{R} \setminus \mathbf{Q}) \cup f(\mathbf{Q})$ as countable. Then $\mathrm{card}(\mathrm{im}\,(f)) \leqslant \mathrm{card}(\mathbf{N}) < \mathrm{card}(0, 1)$, and Exercise 3 gives $\mathrm{card}(\mathrm{im}\,(f)) < \mathrm{card}(0, 1)$. Hence $\mathrm{im}\,(f)$ cannot contain any interval. □

Exercise 9. Prove that the set

$$\mathbf{Q}[x] = \left\{ \sum_{k=0}^{n} a_k x^k \mid n \in \mathbf{N}_0, a_k \in \mathbf{Q} \right\},$$

consisting of all polynomials with rational coefficients, is countable.

Proof. Let $P_n(\mathbf{Q}) = \{a_0 + a_1 x + ... + a_n x^n \mid a_i \in \mathbf{Q}\}$ be the set of all polynomials of degree n. Define $f : P_n(\mathbf{Q}) \to \mathbf{Q}^{n+1}$ by $a_0 + a_1 x + ... + a_n x^n \mapsto (a_0, a_1, ..., a_n)$. Let $f(a_0 + a_1 x + ... a_n x^n) = f(b_0 + b_1 x + ... b_n x^n)$. We have that $(a_0, a_1, ..., a_n) = (b_0, b_1, ..., b_n)$, hence $a_i = b_i$ for all $0 \leqslant i \leqslant n$. This gives $a_0 + a_1 x + ... + a_n x^n = b_0 + b_1 x + ... b_n x^n$, establishing that f is injective. Therefore $P_n(\mathbf{Q})$ is countable, and since $\bigcup_{k\in\mathbf{N}} P_k(\mathbf{Q}) = \mathbf{Q}[x]$, we can conclude $\mathbf{Q}[x]$ is countable. □

Exercise 10. A real number t is called **algebraic** if there is a nonzero polynomial p with rational coefficients such that $p(t) = 0$. If $t \in \mathbf{R}$ is not algebraic, it is called **transcendental**. For example, $\sqrt{2}$ is algebraic, but π is transcendental. Show that the set of algebraic numbers is countable, and conclude that there are uncountably many transcendental numbers.

Proof. The set containing all such algebraic numbers is denoted $\overline{\mathbf{Q}}$, referred to as the *algebraic closure of* $\mathbf{Q}$ *in* $\mathbf{R}$. Let $A_n = \{t \mid p(t) = 0 \text{ for some } p(x) \in P_n(\mathbf{Q})\}$. Since a polynomial of finite degree has a finite number of roots, A_n is countable. Then $\bigcup_{k \in \mathbf{N}} A_k = \overline{\mathbf{Q}}$, establishing that the algebraic closure of $\mathbf{Q}$ over $\mathbf{R}$ is countable. Since $\overline{\mathbf{Q}} \subseteq \mathbf{R}$, Exercise 4 gives that the transcendental numbers $\mathbf{R} \setminus \overline{\mathbf{Q}}$ are uncountable. $\square$