Math 310

Homework 7

Due: 10/9/2024

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Exercise 1. Let $D \subseteq \mathbb{R}$ and suppose $c \in \mathbb{R}$. Show that the following are equivalent:

(i) c is a cluster point of D;

(ii) There is a sequence $(x_n)_n$ in $D \setminus \{c\}$ with $(x_n)_n \to c$.

Proof. ($\Rightarrow$) Let c be a cluster point of D. By definition:

$$(\forall \delta > 0)(\dot{V}_\delta(c) \cap D \neq \emptyset).$$

By induction, we have that $x_n \in \dot{V}_{\frac{1}{n}}(c) \cap D$. Hence $x_n \neq c$, $x_n \in D$, and $0 < |x_n - c| < \frac{1}{n}$, giving $(x_n)_n \to c$.

($\Leftarrow$) Let $(x_n)_n \in (D \setminus \{c\})^{\mathbb{N}}$ with $(x_n)_n \to c$. Let $\delta > 0$ be given. Then for N large, $n \geqslant N$ implies $0 < |x_n - c| < \epsilon$. Whence $x_N \in \dot{V}_\delta(c) \cap D$. Thus c is a cluster point. □

Exercise 2. Show that f can have at most one limit at c.

Proof. Suppose towards contradiction that f has more than one limit, that is,

$$\operatorname*{limit}_{x \to c} f = L_1, \quad \text{and}$$
$$\operatorname*{limit}_{x \to c} f = L_2.$$

where $L_1 \neq L_2$. Then for all sequences $(x_n)_n$ in D, $(x_n)_n \to c$ implies $(f(x_n))_n \to L_1$ and $(f(x_n))_n \to L_2$. This is a contradiction, as sequences can only have at most one limit. Thus f must have at most one limit. □

Exercise 3. Show that the following are equivalent:

(i) $\operatorname*{limit}_{x \to c} f = L$;

(ii) For every sequence $(x_n)_n$ in $D \setminus \{c\}$ satisfying $(x_n)_n \to c$, we have $(f(x_n))_n \to L$.

Proof. ($\Rightarrow$) Suppose $\text{limit}_{x \to c} f(x) = L$. Let $(x_n)_n$ be in D with $x_n \neq c$ and $(x_n)_n \to c$. Given $\epsilon > 0$, we know there exists $\delta > 0$ such that $x \in D$ and $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$. We know there exists some $N \in \mathbb{N}$ with $n \geqslant N$ implying $|x_n - c| < \delta$. Whence $|f(x_n) - L| < \epsilon$; i.e., $(f(x_n))_n \to L$.

($\Leftarrow$) Towards a contradiction, suppose that for every sequence $(x_n)_n$ in D such that $x_n \neq c$ and $(x_n)_n \to c$, it holds that $(f(x_n))_n \to L$, *yet* $\text{limit}_{x \to c} f(x) \neq L$. Then by definition:

$$(\exists \epsilon_0 > 0)(\forall \delta > 0) \ni (x \in \dot{V}_\delta(c) \cap D \ \wedge \ f(x) \notin V_{\epsilon_0}(L)).$$

So for each $\delta = \frac{1}{n}$, we can find $x_n \in \dot{V}_{\frac{1}{n}}(c) \cap D$ and $f(x_n) \notin V_{\epsilon_0}(L)$, or equivalently $(x_n)_n \to c$ and $(f(x_n))_n \nrightarrow L$. This is a contradiction, since $(x_n)_n \to c$ implies $(f(x_n))_n \to L$. This establishes that $\operatorname{limit}_{x\to c} f(x) = L$. ☐

Exercise 4. If $\operatorname{limit}_{x\to c} f = L$ exists, show that there is a $\delta > 0$ such that:

$$\sup_{0<|x-c|<\delta} |f(x)| < \infty,$$

that is, f is bounded on a deleted neighborhood of c.

Proof. Let $\epsilon = 1$. Then for all $x \in \dot{V}_\delta(c) \cap D$, $0 < |x - c| < \delta$ implies:

$$\begin{aligned}
|f(x)| &= |f(x) - L + L| \\
&\leqslant |f(x) - L| + |L| \\
&< 1 + |L| \\
&< \infty.
\end{aligned}$$

Whence $\sup_{0<|x-c|<\delta} |f(x)| = 1 + |L| < \infty$. ☐

Exercise 5. Establish the following limits.

(a) $\displaystyle\operatorname*{limit}_{x\to 1} \frac{3x}{1+x} = \frac{3}{2}.$

Proof. Observe that:

$$\begin{aligned}
|f(x) - L| &= \left|\frac{3x}{1+x} - \frac{3}{2}\right| \\
&= \frac{3}{2}\left|\frac{x-1}{x+1}\right|.
\end{aligned}$$

If $|x-1| < \frac{1}{2}$, then $\frac{1}{2} < x < \frac{3}{2}$ implies $\frac{2}{3} > \frac{1}{x+1} > \frac{2}{5}$. Thus:

$$\begin{aligned}
\frac{3}{2}\left|\frac{x-1}{x+1}\right| &< \frac{3}{2}\cdot\frac{2}{3}|x-1| \\
&= |x-1|.
\end{aligned}$$

Formally, given $\epsilon > 0$, let $\delta = \min\left\{\frac{1}{2}, \epsilon\right\}$. If $0 < |x-1| < \delta$, then by the work above $|f(x) - L| < \epsilon$. ☐

(b) $\displaystyle\operatorname*{limit}_{x\to 6} \frac{x^2-3x}{x+3} = 2.$

Proof. Observe that:

$$\begin{aligned}
|f(x) - L| &= \left|\frac{x^2-3x}{x+3} - 2\right| \\
&= \left|\frac{x^2-5x-6}{x+3}\right| \\
&= \left|\frac{x+1}{x+3}\right| \cdot |x-6|.
\end{aligned}$$

If $|x-6|<1$, then $6<x+1<8$ and $\frac{1}{8}>\frac{1}{x+1}>\frac{1}{11}$. Thus:

$$\begin{aligned}\left|\frac{x+1}{x+3}\right|\cdot|x-6| &< \frac{8}{8}|x-6| \\ &= |x-6|.\end{aligned}$$

Formally, given $\epsilon>0$, let $\delta=\min\{1,\epsilon\}$. If $0<|x-6|<\delta$, then by the work above $|f(x)-L|<\epsilon$. □

(c) $\operatorname*{limit}_{x\to 0} x\mathbb{1}_{\mathbb{Q}}(x)=0$.

Proof. Let $\epsilon>0$ be given. Let $\delta=\epsilon$. If $0<|x-0|<\delta$, then $0<|x|<\epsilon$. Whence $|f(x)=0||f(x)|\leqslant|x|<\epsilon$. □

(d) $\operatorname*{limit}_{x\to 0}\dfrac{x^2}{|x|}=0$.

Proof. Observe that:

$$\begin{aligned}\left|\frac{x^2}{|x|}-0\right| &= \frac{|x|^2}{|x|} \\ &= |x|.\end{aligned}$$

Given $\epsilon>0$, set $\delta=\epsilon$. If $0<|x-0|<\delta$, then $\left|\frac{x^2}{|x|}-0\right|=|x|<\delta=\epsilon$. □

Exercise 6. For which values of $k\in\mathbb{N}_0$ does:

$$\operatorname*{limit}_{x\to 0} x^k\sin\left(\frac{1}{x}\right)$$

exist?

Proof. For $k\geqslant 1$, observe that:

$$-x^k\leqslant x^k\sin\left(\frac{1}{x}\right)\leqslant x^k$$

$$\Longleftrightarrow$$

$$\operatorname*{limit}_{x\to 0}-x^k\leqslant\operatorname*{limit}_{x\to 0}x^k\sin\left(\frac{1}{x}\right)\leqslant\operatorname*{limit}_{x\to 0}x^k$$

$$\Longleftrightarrow$$

$$0\leqslant\operatorname*{limit}_{x\to 0}x^k\sin\left(\frac{1}{x}\right)\leqslant 0.$$

So by the Squeeze Theorem, $\operatorname{limit}_{x\to 0}x^k\sin\left(\frac{1}{x}\right)=0$.

If $k=0$, then $\operatorname{limit}_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist. □

Exercise 7. Assume $f(x) \geqslant 0$ for all $x \in D$ and suppose $\text{limit}_{x\to c} f := L$ exists. Show that $L \geqslant 0$ and that:

$$\underset{x\to c}{\text{limit}} \sqrt{f} = \sqrt{L}.$$

Proof. Since $\text{limit}_{x\to c} f = L$ exists:

$$(\forall (x_n)_n \in (D \setminus \{c\})^{\mathbb{N}})((x_n)_n \to c \implies (f(x_n))_n \to L).$$

Since $f(x) \geqslant 0$ for all x, $L \geqslant 0$. Moreover, $(f(x_n))_n \to L$ implies $\left(\sqrt{f(x_n)}\right)_n \to \sqrt{L}$. Thus $\text{limit}_{x\to c} \sqrt{f} = \sqrt{L}$. □

Exercise 8. Assume $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $\text{limit}_{x\to 0} f := L$ exists, show that $L = 0$ and show that $\text{limit}_{x\to c} f$ exists for all $c \in \mathbb{R}$.

Proof. Since $f(x + y) = f(x) + f(y)$, observe that:

$$\begin{aligned} f(1) &= f\left(n \cdot \frac{1}{n}\right) \\ &= n \cdot f\left(\frac{1}{n}\right). \end{aligned}$$

So $f\left(\frac{1}{n}\right) = \frac{1}{n} f(1)$. Since $\text{limit}_{x\to 0} f = L$ exists, $\left(\frac{1}{n}\right)_n \to 0$ implies $\left(f\left(\frac{1}{n}\right)\right)_n \to L$. But $\left(f\left(\frac{1}{n}\right)\right)_n = \left(\frac{1}{n} f(1)\right)_n \to 0$. Thus $L = 0$.

Let $(x_n)_n \to c$, $x_n \neq c$. Then $(x_n - c)_n \to 0$. Observe that:

$$\begin{aligned} (f(x_n))_n &= (f(x_n - c + c))_n \\ &= (f(x_n - c) + f(c))_n \\ &= (f(x_n - c))_n + (f(c))_n \\ &\xrightarrow{n\to\infty} 0 + f(c) \\ &= f(c). \end{aligned}$$

Thus $\text{limit}_{x\to c} f = f(c)$ exists. □

Exercise 10. Suppose $f(0, \infty) \to \mathbb{R}$. Show that the following are equivalent:

(i) $\underset{x\to\infty}{\text{limit}} f = L$ (where L can be ∞);

(ii) For every sequence $(x_n)_n$ in $(0, \infty)$ with $(x_n)_n \to \infty$ we have $(f(x_n))_n \to L$.

Proof. ($\Rightarrow$) We proceed by cases.

Case 1: $L < \infty$. Let $\epsilon > 0$ be given. Since $\text{limit}_{x\to\infty} f = L$ exists, there exists $\alpha > 0$ such that $x \geqslant \alpha$ implies $|f(x) - L| < \epsilon$. Since $(x_n)_n \to \infty$, there exists $N \in \mathbb{N}$ such that $n \geqslant N$ implies $x_n \geqslant \alpha$. Whence $|f(x_n) - L| < \epsilon$, giving $(f(x_n))_n \to L$.

Case 2: $L = \infty$. Let $M > 0$ be given. Since $\text{limit}_{x\to\infty} f = \infty$ exists, there exists $\alpha > 0$ such that $x \geqslant \alpha$ implies $f(x) \geqslant M$. Since $(x_n)_n \to \infty$, there exists $N \in \mathbb{N}$ such that $n \geqslant N$ implies $x_n \geqslant \alpha$. Whence $f(x_n) \geqslant M$, giving $(f(x_n))_n \to \infty$.

($\Leftarrow$) Suppose towards contradiction $\text{limit}_{x\to\infty} f \neq L$. We proceed by cases.

Case 1: $L < \infty$. Then by the sequential definition of limits:

$$(\exists (x_n)_n \in (0,\infty)^{\mathbb{N}})((x_n)_n \to \infty \ \wedge \ (f(x_n))_n \nrightarrow L).$$

But this contradicts our assumption that for all $(x_n)_n$, $(x_n)_n \to \infty$ implies $(f(x_n))_n \to L$.

Case 2: $L = \infty$. Then by the sequential definition of limits:

$$(\exists (x_n)_n \in (0,\infty)^{\mathbb{N}})((x_n)_n \to \infty \ \wedge \ f(x_n) \leqslant M)$$

But this contradicts our assumption that for all $(x_n)_n$, $(x_n)_n \to \infty$ implies $(f(x_n))_n \to \infty$. □

Exercise 11. If $f : (a,\infty) \to \mathbb{R}$ is such that $\text{limit}_{x\to\infty} xf(x) := L$ exists, show that:

$$\underset{x\to\infty}{\text{limit}} f(x) = 0.$$

Proof. Let $(x_n)_n \to \infty$. We want to show that $(f(x_n))_n \to 0$.

We know that $(x_n f(x_n))_n \to L$, so it is bounded. That is, there exists $c > 0$ such that $|x_n f(x_n)| \leqslant c$ for all n. Observe that:

$$\begin{aligned} |f(x_n)| &= \left|\frac{x_n f(x_n)}{x_n}\right| \\ &\leqslant \frac{c}{|x_n|}. \end{aligned}$$

Since $\left(\frac{c}{|x_n|}\right)_n \to 0$, we have $(f(x_n))_n \to 0$. Thus, by Exercise 10, $\text{limit}_{x\to\infty} f(x) = 0$. □

Exercise 12. Suppose $f, g : (0,\infty) \to \mathbb{R}$ are such that $\text{limit}_{x\to\infty} f := L > 0$, and $\text{limit}_{x\to\infty} g = \infty$. Show that $\text{limit}_{x\to\infty} fg = \infty$. Does this hold if $L = 0$ as well?

Proof. Let $(x_n)_n \to \infty$. Let $M > 0$ be given. There exists N_1 such that $n \geqslant N_1$ implies $f(x) \geqslant \frac{L}{2}$. There exists N_2 such that $n \geqslant N_2$ implies $g(x) \geqslant \frac{2M}{L}$. So for $n \geqslant \max\{N_1, N_2\}$, $f(x_n)g(x_n) \geqslant \frac{L}{2}\frac{2M}{L} = M$.

Note this does not hold for $L = 0$. Take $f(x) = \frac{1}{x}$ and $g(x) = x$. Then $\text{limit}_{x\to\infty} fg = 1$.

□