Contents

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1

Introduction

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out"

-Emil Artin

1.1 Basic Properties of Vector Spaces

Definition 1.1.1. Let F be any field. Let V be a nonempty set with binary operations:

$$V \times V \longrightarrow B$$
$$(v, w) \mapsto v + w$$

called *vector addition* and

$$F \times V \longrightarrow V$$
$$(c, v) \mapsto cv$$

called *scalar multiplication*. Then V is an *F-vector space* if the following properties are satisfied:

(1) V is an abelian group, that is:

  (i) there exists a $0_v \in V$ such that $0_v + v = v = v + 0v$,

  (ii) for every $v \in V$ there exists a $-v \in V$ such that $v + (-v) = 0_v = (-v) + v$,

  (iii) for every $u, v, w \in V$, $(u + v) + w = u + (w + v)$, and

  (iv) $v + w = w + v$ for all $v, w \in V$.

(2) $c(v + w) = cv + cw$ for all $c \in F$, $v, w \in V$,

(3) $(c + d)v = cv + dv$ for all $c, d \in F$, $v \in V$,

(4) $(cd)v = c(dv)$ for all $c, d \in F$, $v \in V$,

(5) there exists a $1_F \in F$ such that $1_F v = v$.

Example 1.1.1.

(1) Let F be any field. Define $F^n = \{(a_1, ..., a_n) \mid a_i \in F\}$ as *affine n-space*. Then F^n is an F-vector space.

(2) Let $n \in \mathbf{Z}_{\geqslant 0}$. Define $P_n(F) = \{a_0 + a_1 x + ... + a_n x^n \mid a_i \in F\}$. This is an F-vector space with polynomial addition and scalar multiplication. Define $F[x] = \bigcup_{n \geqslant 0} P_n(F)$. This is also an F-vector space, but either via polynomial addition *or* polynomial multiplication.

(3) Let $m, n \in \mathbf{Z}_{\geqslant 0}$. Set $V = \mathrm{Mat}_{n,m}(F) = \{$all $m \times n$ matrices with entries in $F\}$. This is an F-vector space with matrix addition and scalar mutliplication. If $m = n$ then write $\mathrm{Mat}_n(F)$ for $\mathrm{Mat}_{n,n}(F)$.

Lemma 1.1.1. *Let V be an F-vector space.*

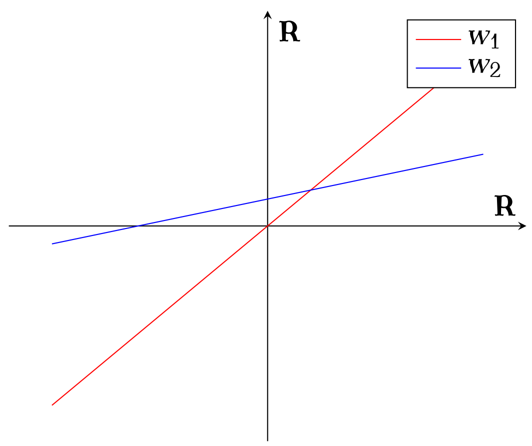
1. *The element $0_v \in V$ is unique,*
2. *$0v = 0_v$ for all $v \in V$,*
3. *$(-1_F)v = -v$ for all $v \in V$.*

Proof. (1) Let $0, 0'$ satisfy the following properties: $0 + v = v$ and $0' + v = v$ for all $v \in V$. Observe that $0 = 0' + 0 = 0 + 0' = 0'$. (2) Note that $0_F v = (0_F + 0_F)v = 0_F v + 0_F v$. Subtracting both sides by $0_F v$ yields $0 = 0_F v$. (3) Observe that $(-1_F)v + v = (-1_F)v + 1_F v = (-1_F + 1_F)v = 0_F v = 0$. Hence $(-1_F)v = -v$. □

Definition 1.1.2. Let V be an F-vector space. We say $W \subseteq V$ is an *F-subspace* (or just *subspace* if F is obvious by context) if W is an F-vector space under the same addition and scalar multiplication.

Example 1.1.2.

(1) Consider the plane $V = \mathbf{R}^2$. Let w_1, w_2 be subsets of $\mathbf{R}^2$ as follows:



Note that w_2 is not a subspace, as it does not contain $0_{\mathbf{R}^2}$. On the other hand w_1 is a subspace; note that every element of w_1 is of the form (x, ax), hence $(x_1, ax_1) + (x_2, ax_2) = (x_1 + x_2, a(x_1 + x_2))$. The other axioms follow similarly.

(2) Let $V = \mathbf{C}$ and $W = \{a + 0i \mid a \in \mathbf{R}\}$. If $F = \mathbf{R}$, then clearly W is an $\mathbf{R}$-subspace. If $F = \mathbf{C}$, then W is not a $\mathbf{C}$-subspace; given $2 \in W$ and $i \in \mathbf{C}$, $2i \notin W$.

(3) $\mathrm{Mat}_2(\mathbf{R})$ is not a subspace of $\mathrm{Mat}_4(\mathbf{R})$, as $\mathrm{Mat}_2(\mathbf{R}) \not\subseteq \mathrm{Mat}_4(\mathbf{R})$.

(4) Let $m, n \in \mathbf{Z}_{\geqslant 0}$. If $m \leqslant n$, then $P_m(F)$ is a subspace of $P_n(F)$.

Lemma 1.1.2. *Let V be an F-vector space and $W \subseteq V$. Then W is an F-subspace of V if:*

(1) *W is nonempty,*

(2) *W is closed under addition, and*

(3) *W is closed under scalar multiplication.*

Proof. Let $x, y \in W$ and $\alpha \in F$, then by assumption $x + \alpha y \in W$. Take $\alpha = -1$, then $x - y \in W$ which implies W is an abelian subgroup of V. Then by (3) it must be the case that W is an F-subspace of V. □

Definition 1.1.3. Let V, W be F-vector spaces. Let $T : V \to W$. We say T is a <u>*linear transformation*</u> (or <u>*linear map*</u>) if for every $v_1, v_2 \in V$ and $c \in F$ we have

$$T(v_1 + cv_2) = T(v_1) + cT(v_2).$$

The collection of all linear maps from V to W is denoted $\operatorname{Hom}_F(V, W)$ (some textbooks write this as $\mathcal{L}(V, W)$).

Example 1.1.3.

(1) Let V be an F-vector space. Define $\mathrm{id}_v : V \to V$ by $\mathrm{id}_v(v) = v$. This is a linear map; i.e., $\mathrm{id}_v \in \operatorname{Hom}_F(V, V)$ because $\mathrm{id}_v(v_1 + cv_2) = v_1 + cv_2 = \mathrm{id}_v(v_1) + c\,\mathrm{id}_v(v_2)$.

(2) Let $V = \mathbf{C}$. Define $T : V \to V$ by $z \mapsto \overline{z}$. Observe that:

$$\begin{aligned} T(z_1 + cz_2) &= \overline{z_1 + cz_2} = \overline{z_1} + \overline{c}\,\overline{z_2} \\ T(z_1) + cT(z_2) &= \overline{z_1} + c\,\overline{z_2}. \end{aligned}$$

Note that these two are only equal if $c = \overline{c}$. Hence $T \in \operatorname{Hom}_F(\mathbf{C}, \mathbf{C})$ if $F = \mathbf{R}$ but not if $F = \mathbf{C}$.

(3) Let $A \in \mathrm{Mat}_{m,n}(F)$. Define $T_A : F^n \to F^m$ by $x \mapsto Ax$. Then $T_A \in \operatorname{Hom}_F(F^n, F^m)$.

(4) Recall that $C^\infty(\mathbf{R})$ is the set of all smooth functions $f : \mathbf{R} \to \mathbf{R}$ (another way of saying "smooth" is "infinitely differentiable"). Let $V = C^\infty(\mathbf{R})$. This is an $\mathbf{R}$-vector space under pointwise addition and scalar multiplication. If $a \in \mathbf{R}$ then:

- $E_a : V \to \mathbf{R}$ defined by $f \mapsto f(a)$ is an element of $\operatorname{Hom}_{\mathbf{R}}(V, \mathbf{R})$,
- $D : V \to V$ defined by $f \mapsto f'$ is an element of $\operatorname{Hom}_{\mathbf{R}}(V, V)$,
- $I_a : V \to V$ defined by $f \mapsto \int_a^x f(t)dt$ is an element of $\operatorname{Hom}_{\mathbf{R}}(V, V)$, and
- $\tilde{E}_a : V \to V$ defined by $f \mapsto f(a)$ (where $f(a)$ is the constant function) is an element of $\operatorname{Hom}_{\mathbf{R}}(V, V)$.

From this, we can express the fundamental theorem of calculus as follows:

$$\begin{aligned} D \circ I_a &= \mathrm{id}_v \\ I_a \circ D &= \mathrm{id}_v - \tilde{E}_a. \end{aligned}$$

Proposition 1.1.3. *$\operatorname{Hom}_F(V, W)$ is an F-vector space.*

Proof. do this □

Lemma 1.1.4. *Let* $T \in \mathrm{Hom}_F(V, W)$*. Then* $T(0_v) = 0_w$*.*

Proof. do this □

Definition 1.1.4. Let $T \in \mathrm{Hom}_F(V, W)$ be invertible; i.e., there exists a linear transformation $T^{-1} : W \to V$ such that $T \circ T^{-1} = \mathrm{id}_w$ and $T^{-1} \circ T = \mathrm{id}_v$. If this is the case we say T is an *isomorphism* and say V and W are *isomorphic*, written as $V \cong W$.

Proposition 1.1.5. *Let* $T \in \mathrm{Hom}_F(V, W)$ *be an isomorphism. Then* $T^{-1} \in \mathrm{Hom}_F(W, V)$*.*

Proof. do this □

Example 1.1.4.

(1) Let $V = \mathbf{R}^2$ and $W = \mathbf{C}$. Define $T : \mathbf{R}^2 \to \mathbf{C}$ by $(x, y) \mapsto x + iy$. This is an isomorphism: note that $T \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^2, \mathbf{C})$ because

$$\begin{aligned} T((x_1, y_1) + r(x_2, y_2)) &= \text{...fill this out} \\ &= T((x_1, y_1)) + rT((x_2, y_2)). \end{aligned}$$

Defining $T^{-1} : \mathbf{C} \to \mathbf{R}^2$ by $x + iy \mapsto (x, y)$ (and showing it's linear) clearly satisfies $(T \circ T^{-1})(x + iy) = x + iy$ and $(T^{-1} \circ T)((x, y)) = (x, y)$, hence $\mathbf{R}^2 \cong \mathbf{C}$ as $\mathbf{R}$-vector spaces.

(2) Set $V = P_n(F)$ and $W = F^{n+1}$. Define $T : P_n(F) \to F^{n+1}$ by

$$a_0 + a_1 x + \ldots + a_n x^n \mapsto (a_0, a_1, \ldots, a_n).$$

This is an isomorphism; $P_n(F) \cong F^{n+1}$.

Definition 1.1.5. Let $T \in \mathrm{Hom}_F(V, W)$. Define the *kernel* of T as:

(1) The *kernel of T* is defined as $\ker(T) = \{v \in V \mid T(v) = 0_W\}$.

(2) The *image of T* is defined as $\mathrm{im}(T) = \{w \in W \mid T(v) = w \text{ for some } v \in V\}$.

Lemma 1.1.6. *Let* $T \in \mathrm{Hom}_F(V, W)$*. Then:*

(1) $\ker(T)$ *is a subspace of* V*,*

(2) $\mathrm{im}(T)$ *is a subspace of* W*.*

Proof. Let $v_1, v_2 \in \ker(T)$ and $\alpha \in F$. Observe that $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2) = 0_w + \alpha 0_w = 0_w$, hence $v_1 + \alpha v_2 \in \ker(T)$ establishing $\ker(T)$ as a subspace of V.

Let $w_1, w_2 \in \mathrm{im}(T)$ and $\alpha \in F$. Then there exists $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Observe that $w_1 + \alpha w_2 = T(v_1) + \alpha T(v_2) = T(v_1 + \alpha v_2)$, hence $w_1 + \alpha w_2 \in \mathrm{im}(T)$ establishing $\mathrm{im}(T)$ as a subspace of W. □

Lemma 1.1.7. *Let* $T \in \mathrm{Hom}_F(V, W)$. T *is injective if and only if* $\ker(T) = \{0_V\}$

Proof. Let T be injective. Let $v \in \ker(T)$. Then $T(v) = 0_W = T(0_V)$, and since T is injective $v = 0_V$.

Conversely, assume $\ker(T) = 0_V$. Let $v_1, v_2 \in V$ with $T(v_1) = T(v_2)$. Subtracting both sides by $T(v_2)$ gives $T(v_1) - T(v_2) = 0_W$, and since T is a linear transformation yields $T(v_1 - v_2) = 0_W$. Since the kernel is trivial, it must be the case that $v_1 = v_2$. □

Example 1.1.5. Let $m > n$. Define $T : F^m \to F^n$ by

$$(a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_m) \mapsto (a_0, a_1, \ldots, a_n)$$

Then $\mathrm{im}(T) = F^n$ and $\ker(T) = \{(0, \ldots, 0, a_{n+1}, a_{n+2}, \ldots, a_m) \in F^m\} \cong F^{m-n}$.

2

Bases and Dimension

2.1 Basic Definitions

Unless otherwise stated assume V to be an F-vector space.

Definition 2.1.1. Let $\mathcal{B} = \{v_i\}_{i\in I}$ be a subset of V where I is an indexing set (possibly infinite). We say $v \in V$ is an *F-linear combination of $\mathcal{B}$* (or just *linear combination*) if there is a set $\{a_i\}_{i\in I}$ with $a_i = 0$ for all but finitely many i such that $v = \sum_{i\in I} a_i v_i$. The collection of F-linear combinations is denoted $\text{span}_F(\mathcal{B})$.

Example 2.1.1. Let $V = P_2(F)$.

(1) Set $\mathcal{B} = \{1, x, x^2\}$. We have $\text{span}_F(\mathcal{B}) = P_2(F)$.

(2) Set $\mathcal{C} = \{1, (x-1), (x-1)^2\}$. We have $\text{span}_F(\mathcal{C}) = P_2(F)$.

Definition 2.1.2. Let $\mathcal{B} = \{v_i\}_{i\in I}$ be a subset of V. We say $\mathcal{B}$ is *F-linearly independent* (or just *linearly independent*) if whenever $\sum_{i\in I} a_i v_i = 0$ then $a_i = 0$ for all $i \in I$.

Definition 2.1.3. Let $\mathcal{B} = \{v_i\}_{i\in I}$ be a subset of V. We say $\mathcal{B}$ is an *F-basis* (or just *basis*) of V if:

- $\text{span}_F(\mathcal{B}) = V$, and
- $\mathcal{B}$ is linearly independent.

Example 2.1.2. Let $V = F^n$. Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ with

$$\begin{aligned} e_1 &= (1, 0, 0, \dots, 0) \\ e_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \dots, 1). \end{aligned}$$

We have that $\mathcal{E}_n$ is a basis of F^n and is referred to as the *standard basis*.

2.2 Every Vector Space Admits a Basis

Definition 2.2.1. A *relation* from A to B is a subset $R \subseteq A \times B$. Typically, when one says "a relation on A" that means a relation from A to A; i.e., $R \subseteq A \times A$.

Definition 2.2.2. Let A be a set. An *ordering* of A is a relation R on A that is

(1) *reflexive*: $(a,a) \in R$ for all $a \in A$,

(2) *transitive*: $(a,b),(b,c) \in R$ implies $(a,c) \in R$, and

(3) *antisymmetric*: $(a,b),(b,a) \in R$ implies $a = b$.

If this is the case, we write $(a,b) \in R$ as $a \leqslant_R b$. If A is an ordered set we write it as the ordered pair $(A, \leqslant_R)$ (or just A if the ordering is obvious by context).

Definition 2.2.3. An ordered set $(X, \leqslant_R)$ is *total* if for all $a, b \in X$ we have that $a \leqslant_R b$ or $b \leqslant_R a$.

Definition 2.2.4. Let $(X, \leqslant)$ be an ordered set and $A \subseteq X$ nonempty.

(1) A is called a *chain* if $(A, \leqslant)$ is a total ordering.

(2) A is called *bounded above* if there exists an element $u \in X$ with $a \leqslant u$ for all $a \in A$. Such a u is called an *upperbound* for A.

(3) A *maximal element of A* is an element $m \in A$ such that if $a \geqslant m$, then $a = m$.

Lemma 2.2.1 (Zorn's Lemma). *Let X be an ordered set with the property that every chain has an upperbound. Then X contains a maximal element.*

Theorem 2.2.2. *Let $\mathcal{A}$ and $\mathcal{C}$ be subsets of V with $\mathcal{A} \subseteq \mathcal{C}$. Assume $\mathcal{A}$ is linearly independent and $\operatorname{span}_F(\mathcal{C}) = V$. Then there exists a basis $\mathcal{B}$ of V with $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$*[1].

Proof. Let $X = \{\mathcal{B}' \subseteq V \mid \mathcal{A} \subseteq \mathcal{B}' \subseteq \mathcal{C},\ \mathcal{B}' \text{ is linearly independent}\}$. We have $\mathcal{A} \in X$, so $X \neq \emptyset$. X is ordered with respect to inclusion, and has an upperbound of $\mathcal{C}$. By Zorn's Lemma we have a maximal element in X, call it $\mathcal{B}$.

Claim: $\operatorname{span}_F(\mathcal{B}) = V$. Suppose towards contradiction it's not, then there exists a $v \in \mathcal{C}$ with $v \notin \operatorname{span}_F(\mathcal{B})$. But then $\mathcal{B} \cup \{v\}$ is still linearly independent, and $\mathcal{B} \cup \{v\} \subseteq \mathcal{C}$. This gives $\mathcal{B} \subsetneq \mathcal{B} \cup \{v\}$, which is a contradiction because $\mathcal{B}$ is maximal in X. Thus $\operatorname{span}_F(\mathcal{B}) = V$. □

2.3 Cardinality and Dimension

Lemma 2.3.1. *A homogenous system of m linear equations in n unknowns with $m < n$ has a nonzero solution.*

Proof. do this □

Corollary 2.3.2. *Let $\mathcal{B} \subseteq V$ with $\operatorname{span}_F(\mathcal{B}) = V$ and $|\mathcal{B}| = m$. Any set with more than m elements cannot be linearly independent.*

[1]Given any linearly-independent set $\mathcal{A}$, we can constructing a basis $\mathcal{B}$ by adding elements. Given any spanning set $\mathcal{C}$, we can construct a basis $\mathcal{B}$ by removing elements.

Proof. Let $\mathcal{C} = \{w_1, \ldots, w_n\}$ with $n > m$. We will show $\mathcal{C}$ cannot be linearly independent. Write $\mathcal{B} = \{v_1, \ldots, v_m\}$ with $\operatorname{span}_F(\mathcal{B}) = V$. For each i, write

$$w_i = \sum_{j=1}^{m} a_{ji} v_j \text{ for some } a_{ji} \in F.$$

Consider the equations

$$\sum_{i=1}^{n} a_{ji} x_i = 0.$$

By Lemma 2.3.1 there exists nonzero solutions $(x_1, \ldots, x_n) = (c_1, \ldots, c_n) \neq (0, \ldots, 0)$. We have

$$\begin{aligned} 0 &= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} c_i \right) v_j \\ &= \sum_{i=1}^{n} c_i \left(\sum_{j=1}^{m} a_{ji} v_j \right) \\ &= \sum_{i=1}^{n} c_i w_i. \end{aligned}$$

Thus $\mathcal{C} = \{w_1, \ldots, w_n\}$ is not linearly independent. □

Corollary 2.3.3. *If $\mathcal{B}$ and $\mathcal{C}$ are both finite bases of V, then $|\mathcal{B}| = |\mathcal{C}|$.*

Proof. Let $|\mathcal{B}| = m$ and $|\mathcal{C}| = n$. Because $\operatorname{span}_F(\mathcal{B}) = V$ and $\mathcal{C}$ is linearly independent, it must be the case that $n \leqslant m$. But since $\operatorname{span}_F(\mathcal{C}) = V$ and $\mathcal{B}$ is also linearly independent, it must be the case that $m \leqslant n$. By antisymmetry, $n = m$. □

Definition 2.3.1. Let $\mathcal{B}$ be a basis of V. The <u>*dimension*</u> of V, written $\dim_F(V)$, is the cardinality of $\mathcal{B}$; i.e., $\dim_F(V) = |\mathcal{B}|$.

Theorem 2.3.4. *Let V be a finite dimensional vector space with $\dim_F(V) = n$. Let $\mathcal{C} \subseteq V$ with $|\mathcal{C}| = m$.*

(1) If $m > n$, then $\mathcal{C}$ is not linearly independent.

(2) If $m < n$, then $\operatorname{span}_F(\mathcal{C}) \neq V$.

(3) If $m = n$, then the following are equivalent:

- *$\mathcal{C}$ is a basis;*
- *$\mathcal{C}$ is linearly independent;*
- *$\operatorname{span}_F(\mathcal{C}) = V$.*

Corollary 2.3.5. *Let $W \subseteq V$ be a subspace. We have $\dim_F(W) \leqslant \dim_F(V)$. If $\dim_F(V) < \infty$, then $V = W$ if and only if $\dim_F(V) = \dim_F(W)$.*

Example 2.3.1. Let $V = \mathbf{C}$.

(1) If $F = \mathbf{C}$, then $\mathcal{B} = \{1\}$ is a basis and $\dim_{\mathbf{C}}(\mathbf{C}) = 1$.

(2) If $F = \mathbf{R}$, then $\mathcal{B} = \{1, i\}$ is a basis and $\dim_{\mathbf{R}}(\mathbf{C}) = 2$.

(3) If $F = \mathbf{Q}$, then $|\mathcal{B}| = \mathfrak{c}$ and $\dim_{\mathbf{Q}}(\mathbf{C}) = \mathfrak{c}$ (the *continuum*).

Example 2.3.2. Let $V = F[x]$ and let $f(x) \in F[x]$. We can use this polynomial to split $F[x]$ into equivalence classes analogous to how one creates the field $\mathbf{F}_p$. Define $g(x)\ h(x)$ if $f(x) \mid (g(x)-h(x))$. This is an equivalence relation. We let $[g(x)]$ denote the equivalence class containing $g(x) \in F[x]$. Let $F[x]/(f(x)) = \{[g(x)] \mid g(x) \in F[x]\}$ denote the collection of equivalence classes. Define $[g(x)] + [h(x)] = [g(x) + h(x)]$ and $\alpha[g(x)] = [\alpha g(x)]$, this makes $F[x]/(f(x))$ into a vector space.

Set $n = \deg(f(x))$. Let $\mathcal{B} = \{[1], [x], \ldots, [x^{n-1}]\}$. We will show this is a basis for $F[x]/(f(x))$. Suppose there exists $a_0, \ldots, a_{n-1} \in F$ with $a_0[1] + a_1[x] + \ldots + a_{n-1}[x^{n-1}] = [0]$. So $[a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}] = [0]$, hence $f(x) \mid (a_0 + a_1 x + \ldots + a_{n-1}x^{n-1})$. But $\deg(f(x)) = n$, so we must have $a_0 = a_1 = \ldots = 0$ (linear independence).

Let $[g(x)] \in F[x]/(f(x))$. The Euclidean algorithm of polynomials gives $g(x) = f(x)q(x) + r(x)$ for some $q(x), r(x) \in F[x]/(f(x))$ with $r(x) = 0$ or $\deg(r(x)) \leqslant \deg(g(x))$. Observe that $[g(x)] = [f(x)q(x) + r(x)] = [f(x)q(x)] + [r(x)] = [r(x)]$. Since $[r(x)]$ can be written as a linear combination of basis elements from $\mathcal{B}$, we have $[g(x)] \in \operatorname{span}_F(\mathcal{B})$. Note that any element of $\operatorname{span}_F(\mathcal{B})$ is clearly contained in $F[x]/(f(x))$, establishing $\operatorname{span}_F(\mathcal{B}) = F[x]/(f(x))$.

Lemma 2.3.6. *Let V be an F-vector space and $\mathcal{C} = \{v_i\}_{i\in I}$ be a subset of V. Then $\mathcal{C}$ is a basis if and only if each $v \in V$ can be written uniquely as a linear combination of elements of $\mathcal{C}$.*

Proof. Suppose $\mathcal{C}$ is a basis. Let $v \in V$ and suppose

$$v = \sum_{i\in I} a_i v_i = \sum_{i \in I} b_i v_i,$$

for some $a_i, b_i \in F$. Observe that:

$$0_v = \sum_{i\in I}(a_i - b_i)v_i.$$

Since $\mathcal{C}$ is a basis, it is linearly independent, so $a_i - b_i = 0$ for all i. Thus $a_i = b_i$ for all i establishing that the expansion is unique.

Conversely, suppose every vector $v \in V$ is a unique linear combination of $\mathcal{C}$. Certainly we have $\operatorname{span}_F(\mathcal{C}) = V$. Suppose $0_v = \sum_{i\in I} a_i v_i$ for some $a_i \in F$. We also have that $0_v = \sum_{i\in I} 0 \cdot v_i$. Uniqueness gives $a_i = 0$ for all $i \in I$; i.e., $\mathcal{C}$ is linearly independent. $\square$

Proposition 2.3.7. *Let V, W be F-vector spaces.*

(1) Let $T \in \operatorname{Hom}_F(V, W)$. We have that T is determined by what it does to a basis (where it maps it).

(2) Let $\mathcal{B} = \{v_i\}_{i\in I}$ be a basis of V and $\mathcal{C} = \{w_i\}_{i\in I}$ be a subset of V. If $|\mathcal{B}| = |\mathcal{C}|$, there is a $T \in \operatorname{Hom}_F(V, W)$ such that $T(v_i) = w_i$ for all $i \in I$.

Proof. (1) Let $v \in V$. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a basis of V and write $v = \sum_{i \in I} a_i v_i$. We have $T(v) = T(\sum_{i \in I} a_i v_i) = \sum_{i \in I} a_i T(v_i)$.

(2) Define $T : V \to W$ by $v \mapsto \sum_{i \in I} a_i w_i$. If $v = \sum_{i \in I} a_i v_i$ this map is linear (show this). $\square$

Corollary 2.3.8. *Let $T \in \mathrm{Hom}_F(V, W)$ with $\mathcal{B} = \{v_i\}_{i \in I}$ a basis of V and $\mathcal{C} = \{w_i = T(v_i)\}_{i \in I}$ a subset of W. We have $\mathcal{C}$ is a basis of W if and only if T is an isomorphism.*

Proof. Suppose $\mathcal{C}$ is a basis of W. Using the result from Proposition 2.3.7, define $S \in \mathrm{Hom}_F(W, V)$ with $S(w_i) = v_i$. Check $T \circ S = \mathrm{id}_W$ and $S \circ T = \mathrm{id}_V$. Thus T is an isomorphism.

Conversely, let T be an isomorphism. Let $w \in W$. As T is surjective, there exists a $v \in V$ such that $T(v) = w$. Using $\mathcal{B}$ as a basis of V, write $v = \sum_{i \in I} a_i v_i$. So observe that:

$$w = T(v) = T\left(\sum_{i \in I} a_i v_i\right) = \sum_{i \in I} a_i T(v_i) \in \mathrm{span}_F(\mathcal{C}),$$

hence $W = \mathrm{span}_F(\mathcal{C})$ (note the other direction is trivial —you never need to show that). Now suppose there exists a collection of elements $a_i \in F$ with $\sum_{i \in I} a_i T(v_i) = 0_W$. Since T is linear, this is equivalent to $T(\sum_{i \in I} a_i v_i) = 0_W$, and since T is injective it must be the case that $\sum_{i \in I} a_i v_i = 0_V$. Since $\mathcal{B}$ is a basis we get $a_i = 0$ for all $i \in I$, establishing that $\mathcal{C}$ is linearly independent. $\square$

Theorem 2.3.9 (Rank-Nullity Theorem). *Let V be an F-vector space with $\dim_F(V) < \infty$. Then:*

$$\dim_F(V) = \dim_F(\ker(T)) + \dim_F(\mathrm{im}(T)).$$

Proof. Let $\dim_F(\ker(T)) = k$ and $\dim_F(V) = n$. Let $\mathcal{A} = \{v_1, \ldots, v_k\}$ be a basis of $\ker(T)$. Extend this to a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V. We'd like to show that $\mathcal{C} = \{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis of $\mathrm{im}(T)$.

Let $w \in \mathrm{im}(T)$. So there exists a $v \in V$ with $T(v) = w$. Write $v = \sum_{i=1}^n a_i v_i$. We have:

$$\begin{aligned} w &= T(v) \\ &= T\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i T(v_i) \\ &= \sum_{i=k+1}^n a_i T(v_i) \in \mathrm{span}_F(\mathcal{C}). \qquad \text{b/c } v_1, \ldots, v_k \in \ker(T) \end{aligned}$$

Thus $\mathrm{span}_F(\mathcal{C}) = \mathrm{im}(T)$. Now suppose we have $\sum_{i=k+1}^n a_i T(v_i) = 0_W$. Since T is linear we have $T(\sum_{i=1}^n a_i v_i) = 0_w$, which gives $\sum_{i=1}^n a_i v_i \in \ker(T)$. Thus we can write it in terms of the basis $\mathcal{A}$ of $\ker(T)$: there exists $a_1, \ldots, a_k$ such that

$$\sum_{i=k+1}^n a_i v_i = \sum_{i=1}^k a_i v_i,$$

which is equivalent to $\sum_{i=1}^k a_i v_i + \sum_{i=k+1}^n a_i v_i = 0_V$. However, $\mathcal{B}$ is a basis of V so $a_1 = \ldots = a_n = 0$. $\square$

Corollary 2.3.10. *Let V, W be F-vector spaces with $\dim_F(V) = n$. Let $V_1 \subseteq V$ be a subspace with $\dim_F(V_1) = k$ and $W_1 \subseteq W$ a subspace with $\dim_F(W_1) = n - k$. Then there exists a $T \in \operatorname{Hom}_F(V, W)$ such that $\ker(T) = V_1$ and $\operatorname{im}(T) = W_1$.*

Proof. do it □

Corollary 2.3.11. *Let $T \in \operatorname{Hom}_F(V, W)$ with $\dim_F(V) = \dim_F(W) < \infty$. The following are equivalent:*

(1) T is an isomorphism.

(2) T is injective.

(3) T is surjective.

Proof. do it □

Corollary 2.3.12. *Let $A = \operatorname{Mat}_n(F)$. The following are equivalent:*

(1) A is invertible.

(2) There exists an element $B \in \operatorname{Mat}_n(F)$ such that $BA = 1_n$.

(3) There exists an element $B \in \operatorname{Mat}_n(F)$ such that $AB = 1_n$.

Proof. do it □

Corollary 2.3.13. *Let $\dim_F(V) = m$ and $\dim_F(W) = n$.*

(1) If $m < n$ and $T \in \operatorname{Hom}_F(V, W)$, then T is not surjective.

(2) If $m > n$ and $T \in \operatorname{Hom}_F(V, W)$, then T is not injective.

(3) If $m = n$ then $V \cong W$.

Example 2.3.3. This follows shortly after corollary 2.2.30 (write it down later)

2.4 Direct Sums and Quotient Spaces

Definition 2.4.1. Let V be an F-vector space and $V_1, \ldots, V_k$ be subspaces. The <u>sum</u> of $V_1, \ldots, V_k$ is

$$V_1 + \ldots + V_k = \{v_1 + \ldots + v_k \mid v_i \in V_i\}.$$

Proposition 2.4.1. *Let V be an F-vector space and $V_1, \ldots, V_k$ be subspaces. Then $V_1 + \ldots + V_k$ is also a subspace of V.*

Proof. do this □

Definition 2.4.2. Let $V_1, \ldots, V_k$ be subspaces of V. We say $V_1, \ldots, V_k$ are *independent* if whenever $v_1 + \ldots + v_k = 0_V$ then $v_i = 0_V$.

Definition 2.4.3. Let $V_1, \ldots, V_k$ be subspaces of V. We say V is the *direct sum* of $V_1, \ldots, V_k$ and write $V = V_1 \oplus \ldots \oplus V_k$ if:

(1) $V = V_1 + \ldots + V_k$, and

(2) $V_1, \ldots, V_k$ are independent.

Example 2.4.1.

(1) Let $V = F^2$ with $V_1 = \{(x, 0) \mid x \in F\}$ and $V_2 = \{(0, y) \mid y \in F\}$. Then

$$\begin{aligned} V_1 + V_2 &= \{(x, 0) + (0, y) \mid x, y \in F\} \\ &= \{(x, y) \mid x, y \in F\} \\ &= V \end{aligned}$$

If $(x, 0) + (y, 0) = (0, 0)$, then $x = y = 0$ which means V_1 and V_2 are independent. Hence $F^2 = V_1 \oplus V_2$.

(2) Let $V = F[x]$ and $V_1 = F$, $V_2 = Fx = \{\alpha x \mid \alpha \in F\}$, and $V_3 = P_1(F)$. Note that $P_1(F) = V_1 \oplus V_2$. But V_1, V_3 are not independent because $1_F \in V_1$ and $-1_F \in V_3$ and $(-1_F) + 1_F = 0$.

(3) Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V and $\operatorname{span}_F(v_i) = V_i$. Then $V = V_1 \oplus \ldots \oplus V_n$.

Lemma 2.4.2. *Let V be an F-vector space with $V_1, \ldots, V_k$ as subspaces. We have $V = V_1 \oplus \ldots \oplus V_k$ if and only if every $v \in V$ can be written uniquely in the form $v = v_1 + \ldots + v_k$ for all $v_i \in V_i$.*

Proof. Suppose $V = V_1 \oplus \ldots \oplus V_k$. Let $v \in V$. Suppose $v = v_1 + \ldots + v_k = \tilde{v}_1 + \ldots + \tilde{v}_k$ for $v_i, \tilde{v}_i \in V_i$. Then $0_V = (v_1 - \tilde{v}_1) + \ldots + (v_k - \tilde{v}_k)$. Since $V_1, \ldots, V_k$ are independent and $v_i - \tilde{v}_i \in V$, this gives $v_i - \tilde{v}_i = 0_V$ for all i. Thus the expansion for V is unique.

Conversely, suppose every $v \in V$ can be written uniquely in the form $v = v_1 + \ldots + v_k$ with $v_i \in V_i$. Then $V = V_1 + \ldots + V_k$ by definition of sums of subspaces. If $0_V = v_1 + \ldots + v_k$ for some $v_i \in V_i$, and $0_v = 0_v + \ldots + 0_v$, then (by uniqueness) it must be the case that $v_i = 0_V$ for all i. □

Exercise 2.4.1. *Let $V_1, \ldots, V_k$ be subspaces of V. For each $1 \leqslant i \leqslant k$, let $\mathcal{B}_i$ be a basis of V_i. Let $\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{B}_i$. Show that:*

(1) $\mathcal{B}$ spans V if and only if $V = V_1 + \ldots + V_k$.

(2) $\mathcal{B}$ is linearly independent if and only if $V_1, \ldots, V_k$ are independent.

(3) $\mathcal{B}$ is a basis if and only if $V = V_1 \oplus \ldots \oplus V_k$.

Proof. do this shit □

Lemma 2.4.3. *Let $U \subseteq V$ be a subspace. Then U has a complement.*

Proof. do this shit □

Definition 2.4.4. Let $W \subseteq V$ be a subsapce. Define $v_1\ v_2$ if $v_1 - v_2 \in W$ for some $v_1, v_2 \in V$. This forms an equivalence relation. Denote the equivalence class containing v as $[v]_W = v + W = \{\tilde{v} \in V \mid v\ \tilde{v}\} = \{v + w \mid w \in W\}$. The set containing all equivalence classes over W is denoted $V/W = \{v + W \mid v \in V\}$.

Proposition 2.4.4. *Let $v_1 + W, v_2 + W \in V/W$ and $\alpha \in F$. With addition and scalar multiplication defined as follows:*

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ \alpha(v_1 + W) &= \alpha v_1 + W,\end{aligned}$$

it's operations are well-defined and V/W forms an F-vector space.

Proof. Let $v_1 + W = \tilde{v_1} + W$ and $v_2 + W = \tilde{v_2} + W$. Then $v_1 = \tilde{v_1} + w_1$ and $v_2 = \tilde{v_2} + w_2$ for some $w_1, w_2 \in W$. Observe that:

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2 + W) \\ &= (\tilde{v_1} + w_2 + \tilde{v_2} + w_2) + W \\ &= (\tilde{v_1} + \tilde{v_2}) + W \\ &= (\tilde{v_1} + W) + (\tilde{v_2} + W).\end{aligned}$$

$$\begin{aligned}c(v_1 + W) &= cv_1 + W \\ &= c(\tilde{v_1} + w) + W \\ &= c\tilde{v_1} + W \\ &= c(\tilde{v_1} + W).\end{aligned}$$

Hence addition and scalar multiplication are well-defined. show the vector space axioms here. □

Example 2.4.2. Let $V = \mathbf{R}^2$ and $W = \{(x, 0) \mid x \in \mathbf{R}\}$. Let $(x_0, y_0) \in V$. We have that $(x_0, y_0) \sim (x, y)$ if $(x_0, y_0) - (x, y) = (x_0 - x, y_0 - y) \in W$. So $(x_0, y_0) + W = \{(x, y_0) \mid x \in \mathbf{R}\}$. Then V/W is a vector space only when $y = 0$.

Define $\tau : \mathbf{R} \to V/W$ by $y \mapsto (0, y) + W$. This is an isomorphism. Let $y_1, y_2, c \in \mathbf{R}$. Observe that:

$$\begin{aligned}\tau(y_1 + cy_2) &= (0, y_1 + cy_2) + W \\ &= ((0, y_1) + (0, cy_2)) + W \\ &= ((0, y_1) + c(0, y_2)) + W \\ &= ((0, y_1) + W) + c((0, y_2) + W) \\ &= \tau(y_1) + c\tau(y_2).\end{aligned}$$

Hence $\tau \in \mathrm{Hom}_F(\mathbf{R}, V/W)$. Let $(x, y) + W \in V/W$. Then $(x, y) + W = (0, y) + W$. So τ is surjective because $\tau(y) = (0, y) + W$. Now let $y \in \ker(\tau)$. Then $\tau(y) = (0, y) + W = (0, 0) + W$. This implies $y = 0$, meaning the kernel is trivial and so τ is injective.

Alternatively, it is routine to show that $\tau^{-1} \in \mathrm{Hom}_F(V/W, \mathbf{R})$ with $\tau^{-1} \circ \tau = \mathrm{id}_{\mathbf{R}}$ and $\tau \circ \tau^{-1} = \mathrm{id}_{V/W}$.

Definition 2.4.5. Let $W \subseteq V$ be a subspace. The *canonical projection map* $\pi_W : V \to V/W$ is defined by $v \mapsto v + W$. Note that $\pi_W \in \mathrm{Hom}_F(V, V/W)$.

Note 1. To define a map $T : V/W \to V'$, you always have to check it is well-defined.

Theorem 2.4.5 (First Isomorphism Theorem). *Let $T \in \mathrm{Hom}_F(V, W)$. Define $\overline{T} : V/\ker(T) \to W$ by $v + \ker(T) \mapsto T(v)$. Then $\overline{T}$ is a linear map. Moreover, $V/\ker(T) \cong \mathrm{im}(T)$.*

Proof. finish this □

2.5 Dual Spaces

Note that when one refers to something as "*canonical*", it means the object in question does not depend on a basis.

Definition 2.5.1. Let V be an F-vector space. The *dual space*, denoted $V^\vee$, is defined to be $V^\vee = \mathrm{Hom}_F(V, F)$.

Theorem 2.5.1. *We have V is isomorphic to a subspace of $V^\vee$. If $\dim_F(V) < \infty$, then $V \cong V^\vee$.*

Proof. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a basis (hence this theorem is not canonical). For each $i \in I$, define:

$$v_i^\vee(v_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

We get $\{v_i^\vee\}_{i \in}$ are elements of $V^\vee$. We obtain $T \in \mathrm{Hom}_F(V, V^\vee)$ by $T(v_i) = v_i^\vee$. To show that V is isomorphic to a subspace of $V^\vee$, it is enough to show T is injective, then by the first isomorphism theorem $V \cong \mathrm{im}(T)$ (a subspace of $V^\vee$).

Let $v \in \ker(T)$, then $T(v) = 0_{V^\vee}$. Write $v = \sum_{i \in I} a_i v_i$. So:

$$\begin{aligned} 0_{V^\vee} &= T(v) \\ &= T\left(\sum_{i \in I} a_i v_i\right) \\ &= \sum_{i \in I} a_i T(v_i) \\ &= \sum_{i \in I} a_i v_i^\vee. \end{aligned}$$

Towards contradiction, pick some j with $a_j \neq 0$. Note that $0_{V^\vee} = \sum_{i \in I} a_i v_i^\vee(v_j) = a_j$ (every term except for $a_j v_j^\vee(v_j)$ equals 0). This is a contradiction, hence T is injective.

Now assume $\dim_F(V) = n$ and write $\mathcal{B} = \{v_1, ..., v_n\}$. Let $v^\vee \in V^\vee$. Define $a_i = v^\vee(v_i)$. Set $v = \sum_{i=1}^n a_i v_i$ and define $S : V^\vee \to V$ by $S(v^\vee) = v = \sum_{i=1}^n v^\vee(v_i)v_i$. We'd like to show that $S \in \mathrm{Hom}_F(V^\vee, V)$ and is the inverse of T. Let $v^\vee, w^\vee \in V^\vee$ and $c \in F$. Set $a_i = v^\vee(v_i)$ and $b_i = w^\vee(v_i)$. Then:

$$\begin{aligned} S(v^\vee + cw^\vee) &= \sum_{i=1}^n \left[(v^\vee + cw^\vee)(v_i)\right] v_i \\ &= \sum_{i=1}^n \left[v^\vee(v_i) + cw^\vee(v_i)\right] v_i \\ &= \sum_{i=1}^n v^\vee(v_i)v_i + c\sum_{i=1}^n w^\vee(v_i)v_i \\ &= S(v^\vee) + cS(w^\vee). \end{aligned}$$

Hence S is linear. Now observe that:

$$\begin{aligned} (S \circ T)(v_j) &= S(T(v_j)) \\ &= S(v_j^\vee) \\ &= \sum_{i=1}^n v_j^\vee(v_i)v_i \\ &= v_j \end{aligned}$$

Let $v^\vee \in V^\vee$. Note that $(T \circ S)(v^\vee)$ is a function, so it will require an input. Observe that

$$\begin{aligned} (T \circ S)(v^\vee)(v_j) &= T(S(v^\vee))(v_j) \\ &= T(\sum_{i=1}^n v^\vee(v_i)v_i)(v_j) \\ &= \left[\sum_{i=1}^n v^\vee(v_i)T(v_i)\right](v_j) \\ &= \sum_{i=1}^n v^\vee(v_i)(v_i^\vee(v_j)) \\ &= v^\vee(v_j). \end{aligned}$$

□

Definition 2.5.2. Let $\mathcal{B} = \{v_1, ..., v_n\}$ be a basis of V. The <u>dual basis</u> for $V^\vee$ is $\mathcal{B}^\vee = \{v_1^\vee, ..., v_n^\vee\}$.

Proposition 2.5.2. *There is a canonical injective linear map from V to $(V^\vee)^\vee$. If $\dim_F(V) < \infty$, this is an isomorphism.*

Proof. Let $v \in V$. Define $\hat{v} : V^\vee \to F$ by $\varphi \mapsto \varphi(v)$[2]. We can easily verify that $\hat{v}$ is linear. Therefore, we have $\hat{v} \in \operatorname{Hom}_F(V^\vee, F) = (V^\vee)^\vee$. We have a map:

$$\Phi : V \to (V^\vee)^\vee \text{ defined by } v \mapsto \hat{v}.$$

We want to verify that Φ is an injective linear map. Let $v_1, v_2 \in V$ and $c \in F$. Let $\varphi \in V^\vee$, then:

$$\begin{aligned}
\Phi(v_1 + cv_2)(\varphi) &= \widehat{v_1 + cv_2}(\varphi) \\
&= \varphi(v_1 + cv_2) \\
&= \varphi(v_1) + c\varphi(v_2) \\
&= \hat{v}_1(\varphi) + c\hat{v}_2(\varphi) \\
&= \Phi(v_1)(\varphi) + c\Phi(v_2)(\varphi).
\end{aligned}$$

We will now show that Φ is injective. Let $v \in V$ and assume $v \neq 0_V$. We will form a basis $\mathcal{B}$ of V that contains v (why is this still canonical?). Let $v^\vee \in V^\vee$, then $v^\vee(v) = 1$ and $v^\vee(w) = 0$ for all $w \in \mathcal{B}$, $w \neq v$. Now assume $v \in \ker(\Phi)$. Then $\Phi(v)(\varphi) = \varphi(v) = 0$ for all $\varphi \in V^\vee$. But picking $\varphi = v^\vee$ gives:

$$\begin{aligned}
0 &= \Phi(v)(v^\vee) \\
&= v^\vee(v) \\
&= 1.
\end{aligned}$$

This is a contradiction, hence Φ is injective. $\square$

Definition 2.5.3. Let $T \in \operatorname{Hom}_F(V, W)$. We get an induced map $T^\vee : W^\vee \to V^\vee$ with $T^\vee(\varphi) = \varphi \circ T$. The following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[dr, dashed, "T^\vee(\varphi)"'] & W \arrow[d, "\varphi"] \\
& F.
\end{tikzcd}$$

[2]This can be notated as eval_v, but $\hat{v}$ appears more often in literature

3

Linear Transformations and Matrices

3.1 Choosing Coordinates

Example 3.1.1 (Choosing Coordinates). Let V be an F-vector space with $\dim_F(V) < \infty$. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V. This basis fixes an isomorphism $V \cong F^n$. Let $v \in V$, write $v = \sum_{i=1}^n a_i v_i$.

$$\text{Define } T_{\mathcal{B}}(v) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n.$$

This is an isomorphism. Given $v \in V$, we write $[v]_{\mathcal{B}} = T_{\mathcal{B}}(v) = \left(\begin{smallmatrix} a_1 \\ \vdots \\ a_n \end{smallmatrix}\right)$. We refer to this as *choosing coordinates* on V. asdf

Example 3.1.2.

(1) Let $V = \mathbf{Q}^2$ and $\mathcal{B} = \{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\}$. This forms a basis of V. Let $v \in V$ with $v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. We have:

$$v = \frac{a+b}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a-b}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ hence } [v]_{\mathcal{B}} = \begin{pmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{pmatrix}.$$

Had we considered the standard basis $\mathcal{E}_2 = \{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$, then $[v]_{\mathcal{E}_2} = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$.

(2) Let $V = P_2(\mathbf{R})$. Let $\mathcal{C} = \{1, (x-1), (x-1)^2\}$. This forms a basis of V. Let $f(x) = a + bx + cx^2 \in P_2(\mathbf{R})$. Written in terms of $\mathcal{C}$, we have $f(x) = (a+b+c) + (b+2c)(x-1) + c(x-1)^2$.

$$\text{Thus } [f(x)]_{\mathcal{C}} = \begin{pmatrix} a+b+c \\ b+2c \\ c \end{pmatrix}$$

Example 3.1.3 (Linear Transformations as Matrices). Recall that given a matrix $A \in \mathrm{Mat}_{m,n}(F)$, we obtain a linear map $T_A \in \mathrm{Hom}_F(F^n, F^m)$ by $T_A(v) = Av$. This process "works in reverse" —given a linear transformation $T \in \mathrm{Hom}_F(F^n, F^m)$, there is a matrix A so that $T = T_A$.

Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ be the standard basis of F^n and $\mathcal{F}_m = \{f_1, \dots, f_m\}$ be the standard basis of F^m. We have that $T(e_j) \in F^m$ for each j, meaning we have elements $a_{ij} \in F$ with $T(e_j) = \sum_{i=1}^m a_{ij} f_i$. Define $A = (a_{ij}) \in Mat_{m,n}(F)$. Observe that:

$$T_A(e_j) = Ae_j = \sum_{i=1}^m a_{ij} f_i = a_{1j} f_1 + \dots + a_{mj} f_m.$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Working "in reverse", let $T \in \operatorname{Hom}_F(V, W)$ with $\mathcal{B} = \{v_1, \dots, v_n\}$ a basis for V and $\mathcal{C} = \{w_1, \dots, w_m\}$ a basis for W. Define:

$$\begin{aligned} P &= T_{\mathcal{B}} : V \to F^n \text{ by } v \mapsto [v]_{\mathcal{B}} \\ Q &= T_{\mathcal{C}} : W \to F^m \text{ by } w \mapsto [w]_{\mathcal{C}} \end{aligned}$$

From the following diagram:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "P"'] & W \arrow[d, "Q"] \\ F^n \arrow[r, dashed, "Q\circ T\circ P^{-1}"'] & F^m \end{tikzcd}$$

we have that $Q \circ T \circ P^{-1}$ corresponds to a matrix $A \in \operatorname{Mat}_{m,n}(F)$. Write $[T]_{\mathcal{B}}^{\mathcal{C}} = A$, this is the unique matrix that satisfies $[T]_{\mathcal{B}}^{\mathcal{C}}[v]_{\mathcal{B}} = [T(v)]_{\mathcal{C}}$. Given $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$, observe that:

$$[T]_{\mathcal{B}}^{\mathcal{C}}\, v_j = [T(v_j)]_{\mathcal{C}} = \left[\sum_{i=1}^{m} a_{ij} w_i\right]_{\mathcal{C}} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

So $[T]_{\mathcal{B}}^{\mathcal{C}}\, v_j$ corresponds to the j^{th} column of the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$ Thus we have:

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \Big([T(v_1)]_{\mathcal{C}} \quad | \quad \dots \quad | \quad [T(v_n)]_{\mathcal{C}} \Big)$$

Example 3.1.4.

(1) Let $V = P_3(\mathbf{R})$ with $\mathcal{B} = \{1, x, x^2, x^3\}$. Define $T \in \operatorname{Hom}_{\mathbf{R}}(V, V)$ by $T(f(x)) = f'(x)$. Following Example 3.1.3 gives:

$$\begin{aligned} T(1) &= 0 = 0 \cdot 1 + 0 \cdots x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x^2) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x^3) &= 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3 \end{aligned}$$

$$[T(1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$[T(x)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\left[T(x^2)\right]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$
$$\left[T(x^3)\right]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(2) Let $V = P_3(\mathbf{R})$ with $\mathcal{B} = \left\{1, x, x^2, x^3\right\}$ with $\mathcal{C} = \left\{1, (1-x), (1-x)^2, (1-x^3)\right\}$. Then

$$\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2 + 2(x-1) \\ T(x^3) &= -9 - 6(x-1) + 3(x-1)^2 \end{aligned}$$

$$[T(1)]_{\mathcal{C}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$[T(x)]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\left[T(x^2)\right]_{\mathcal{C}} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$
$$\left[T(x^3)\right]_{\mathcal{C}} = \begin{pmatrix} -9 \\ -6 \\ 3 \\ 0 \end{pmatrix}$$

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 0 & 1 & 2 & -9 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 3.1.1.

(1) *Let $\mathcal{A}$ be a basis of U, $\mathcal{B}$ a basis of V and $\mathcal{C}$ a basis of W. Let $S \in \mathrm{Hom}_F(U, V)$ and $T \in \mathrm{Hom}_F(V, W)$. Show*

$$[T \circ S]^{\mathcal{C}}_{\mathcal{A}} = [T]^{\mathcal{C}}_{\mathcal{B}} [S]^{\mathcal{B}}_{\mathcal{A}}.$$

(2) *Given $A \in \mathrm{Mat}_{m,k}(F)$ and $B \in \mathrm{Mat}_{n,m}(F)$, we have corresponding linear maps T_A and T_B. Show that you can recover the definition of matrix multiplication by using part (1).*

Note 2. Instead of $[T]^{\mathcal{B}}_{\mathcal{B}}$ we will write $[T]_{\mathcal{B}}$.

Example 3.1.5 (Change of Basis)**.** Let V be an F-vector space and $\mathcal{B}, \mathcal{B}'$ bases of V. Given V expressed in terms of $\mathcal{B}$, we'd like to express it in terms of $\mathcal{B}'$ (or vice versa).

Let $\mathcal{B} = \{v_1, ..., v_n\}$ and $\mathcal{B}' = \{v'_1, ..., v'_n\}$. Define:

$$T : V \to F^n \text{ by } v \mapsto [v]_{\mathcal{B}}$$
$$S : V \to F^n \text{ by } v \mapsto [v]_{\mathcal{B}'}.$$

We obtain a diagram similar to Example 3.1.3:

$$\begin{tikzcd} V \arrow[r, "\mathrm{id}_V"] \arrow[d, "T"'] & V \arrow[d, "S"] \\ F^n \arrow[r, "S \circ \mathrm{id}_V \circ T^{-1}"'] & F^n \end{tikzcd}$$

Hence the change of basis matrix is $[\mathrm{id}_V]^{\mathcal{B}'}_{\mathcal{B}}$

Exercise 3.1.2. *Let $\mathcal{B} = \{v_1, ..., v_n\}$. Show that $[\mathrm{id}_V]^{\mathcal{B}'}_{\mathcal{B}} = ([v_1]_{\mathcal{B}'} \mid ... \mid [v_n]_{\mathcal{B}'})$.*

Example 3.1.6.

(1) Let $V = \mathbf{Q}^2$ with $\mathcal{B} = \{e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ and $\mathcal{B}' = \{v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$. Observe that:

$$e_1 = \frac{1}{2}v_1 + \frac{1}{2}v_2$$
$$e_2 = -\frac{1}{2}v_1 + \frac{1}{2}v_2$$

$$[e_1]_{\mathcal{B}} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$
$$[e_2]_{\mathcal{B}} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$[\mathrm{id}_V]^{\mathcal{B}'}_{\mathcal{E}_2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Consider $v = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right) \in \mathbf{Q}^2$. We can express v in terms of $\mathcal{B}'$ by doing the following calculation:

$$\begin{aligned}
[\mathrm{id}_V]_{\mathcal{E}_2}^{\mathcal{B}'}[v_2]_{\mathcal{E}_2} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix} \\
&= [v]_{\mathcal{B}'}.
\end{aligned}$$

(2) Let $V = P_2(\mathbf{R})$ with $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1, (x-2), (x-2)^2\}$. Then:

$$\begin{aligned}
1 &= 1 \cdot 1 + 0 \cdot (x-2) + 0 \cdot (x-2)^2 \\
x &= 2 \cdot 1 + 1 \cdot (x-2) + 0 \cdot (x-2)^2 \\
x^2 &= 4 \cdot 1 + 4 \cdot (x-2) + 1 \cdot (x-2)^2
\end{aligned}$$

$$\begin{aligned}
[1]_{\mathcal{B}'} &= \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) \\
[x]_{\mathcal{B}'} &= \left(\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}\right) \\
[x^2]_{\mathcal{B}'} &= \left(\begin{smallmatrix} 4 \\ 4 \\ 1 \end{smallmatrix}\right)
\end{aligned}$$

$$[\mathrm{id}_V]_{\mathcal{B}}^{\mathcal{B}'} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 3.1.7 (Similar Matrices). Let $A, B \in \mathrm{Mat}_n(F)$. Let $\mathcal{E}_n$ be the standard basis for F^n and $T_A \in \mathrm{Hom}_F(F^n, F^n)$ such that $A = [T_A]_{\mathcal{E}_n}$. We can relate A in terms of an arbitrary basis $\mathcal{B}$ as follows:

$$\begin{tikzcd}
F^n \arrow[r, "T_A"] \arrow[d, "T_{\mathcal{B}}"'] & F^n \arrow[d, "T_{\mathcal{B}}"] \\
F^n \arrow[r, "{[T_A]_{\mathcal{B}}}"'] & F^n.
\end{tikzcd}$$

But by extending our diagram using our change of basis algorithm, we obtain the following:

$$\begin{tikzcd}
F^n \arrow[r, "\mathrm{id}_{F^n}"] \arrow[d, "T_{\mathcal{B}}"'] & F^n \arrow[r, "T_A"] \arrow[d, "T_{\mathcal{E}_n}"'] & F^n \arrow[r, "\mathrm{id}_{F^n}"] \arrow[d, "T_{\mathcal{E}_n}"] & F^n \arrow[d, "T_{\mathcal{B}}"] \\
F^n \arrow[r, "{[\mathrm{id}_{F^n}]_{\mathcal{B}}^{\mathcal{E}_n}}"'] & F^n \arrow[r, "{[T_A]_{\mathcal{E}_n}}"'] & F^n \arrow[r, "{[\mathrm{id}_{F^n}]_{\mathcal{E}_n}^{\mathcal{B}}}"'] & F^n
\end{tikzcd}$$

So $[T_A]_{\mathcal{B}} = [\mathrm{id}_{F^n}]^{\mathcal{E}_n}_{\mathcal{B}} [T_A]_{\mathcal{E}_n} [\mathrm{id}_{F^n}]^{\mathcal{B}}_{\mathcal{E}_n}$. Assigning $P^{-1} = [\mathrm{id}_{F^n}]^{\mathcal{E}_n}_{\mathcal{B}}$ and $P = [\mathrm{id}_{F^n}]^{\mathcal{B}}_{\mathcal{E}_n}$ yields the familiar equation $[T_A]_{\mathcal{B}} = P^{-1}AP$; i.e., $A = P[T_A]_{\mathcal{B}}P^{-1}$. In particular, the matrix $A = [T_A]_{\mathcal{E}_n}$ is similar to $[T_A]_{\mathcal{B}}$ for any basis $\mathcal{B}$.

Example 3.1.8. Let $A = \begin{pmatrix} 1 & 3 & -5 \\ -2 & -1 & 6 \\ 3 & 2 & 1 \end{pmatrix}$. Let $\mathcal{E}_3 = \{e_1, e_2, e_3\}$ be the standard basis of F^3. We have:

$$T_A(e_1) = e_1 - 2e_2 + 3e_3$$
$$T_A(e_2) = 3e_1 - e_2 + 2e_3$$
$$T_A(e_3) = 3e_1 + 2e_2 + e_3.$$

Now consider $\mathcal{B} = \{v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}\}$. One can check this is indeed a basis. Observe that:

$$e_1 = -2v_1 + -3v_2 + v_3$$
$$e_2 = 3v_1 + 3v_2 - v_3$$
$$e_3 = -2v_1 - 2v_2 + v_3.$$

So the change of basis matrix from $\mathcal{E}_3$ to $\mathcal{B}$ is given by $P = [\mathrm{id}_{F^3}]^{\mathcal{B}}_{\mathcal{E}_3} = \begin{pmatrix} -2 & 3 & -2 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{pmatrix}$. We have $P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Thus A is similar to the matrix $B = P^{-1}AP = \begin{pmatrix} -29 & 32 & -25 \\ -38 & 45 & -31 \\ -20 & 27 & -15 \end{pmatrix}$.

3.2 Row Operations

Definition 3.2.1. Let $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(F)$. We say a_{kl} is a *pivot* of A if $a_{kl} \neq 0$ and $a_{ij} = 0$ if $i > k$ or $j < l$.

Example 3.2.1. Let $A = \begin{pmatrix} 2 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then 2, 1, and 5 are pivots.

Definition 3.2.2. Let $A \in \mathrm{Mat}_{m,n}(F)$. We say A is in *row echelon form* if all its nonzero rows have a pivot and all its zero rows are located below nonzero rows. We say it is *reduced row echelon form* if it is in row echelon form and all of its pivots are 1 and the only nonzero elements in the columns containing pivots.

Example 3.2.2. From the previous example, expressing $A = \begin{pmatrix} 2 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in reduced row echelon form yields $A' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Example 3.2.3. Let $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$. Then $T_A : F^4 \to F^4$. Let $\mathcal{B}_4 = \{e_1, e_2, e_3, e_4\}$ and $\mathcal{F}_3 = \{f_1, f_2, f_3\}$. So $A = [T_A]^{\mathcal{F}_3}_{\mathcal{B}_3}$. We have the following set of equations:

$$T_A(e_1) = 3f_1 + f_2 + f_3$$
$$T_A(e_2) = 4f_1 + 2f_2 + f_3$$
$$T_A(e_3) = 5f_1 + 3f_2 + 2f_3$$
$$T_A(e_4) = 6f_1 + 4f_2 + 3f_3.$$

We are going to perform row operations of A by making substitutions to its basis elements. Consider the operation $R_1 \leftrightarrow R_3$.

$$\mathcal{F}_3^{(2)} = \{f_1^{(2)} = f_3, f_2^{(2)} = f_2, f_3^{(2)} = f_1\}.$$

$$\begin{aligned}
T_A(e_1) &= f_1^{(2)} + f_2^{(2)} + 3f_3^{(2)} \\
T_A(e_2) &= f_1^{(2)} + 2f_2^{(2)} + 4f_3^{(2)} \\
T_A(e_3) &= 2f_1^{(2)} + 3f_2^{(2)} + 5f_3^{(2)} \\
T_A(e_4) &= 3f_1^{(2)} + 4f_2^{(2)} + 6f_3^{(2)}.
\end{aligned}$$

So $[T_A]_{\mathcal{B}_3}^{\mathcal{F}_3^{(2)}} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{pmatrix}$. Now consider the row operation $-R_1 + R_2 \leftrightarrow R_2$.

$$\mathcal{F}_3^{(3)} = \{f_1^{(3)} = f_1^{(2)} + f_2^{(2)}, f_2^{(3)} = f_2^{(2)}, f_3^{(3)} = f_3^{(2)}\}.$$

$$\begin{aligned}
T_A(e_1) &= f_1^{(2)} + f_2^{(2)} + 3f_3^{(2)} \\
&= f_1^{(3)} + 3f_3^{(3)}.
\end{aligned}$$

$$\begin{aligned}
T_A(e_2) &= f_1^{(2)} + 2f_2^{(2)} + 4f_3^{(2)} \\
&= f_1^{(2)} + f_2^{(2)} + f_2^{(2)} + 4f_3^{(2)} \\
&= f_1^{(3)} + f_2^{(3)} + 4f_3^{(3)}.
\end{aligned}$$

$$\begin{aligned}
T_A(e_3) &= \ldots \\
T_A(e_4) &= \ldots
\end{aligned}$$

So $[T_A]_{\mathcal{B}_3}^{\mathcal{F}_3^{(3)}} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 \end{pmatrix}$. Now consider the row operation $-3R_1 + R_3 \leftrightarrow R_3$.

$$\mathcal{F}_3^{(4)} = \{f_1^{(4)} = f_1^{(3)} + 3f_3^{(3)}, f_2^{(4)} = f_2^{(3)}, f_3^{(4)} = f_3^{(3)}\}.$$

$$\begin{aligned}
T_A(e_1) &= f_1^{(3)} + 3f_3^{(3)} \\
&= f_1^{(4)}
\end{aligned}$$

$$\begin{aligned}
T_A(e_2) &= \ldots \\
T_A(e_3) &= \ldots \\
T_A(e_4) &= \ldots
\end{aligned}$$

The rest of the steps to convert A to reduced row echelon form follow similarly.

Theorem 3.2.1. *Let $A \in \mathrm{Mat}_{m,n}(F)$. The matrix A can be put in row echelon form through a series of row operations of the form:*

(1) $R_i \leftrightarrow R_j$

(2) $R_i \leftrightarrow cR_i$

(3) $cR_i + R_J \leftrightarrow R_j$.

Example 3.2.4. Instead of directly changing the basis of a matrix, we can use linear maps to perform row operations. Let $\mathcal{C} = \{w_1, ..., w_n\}$ be a basis of W.

(1) Define $T_{i,j} : W \to W$ by

$$\begin{aligned} T_{i,j}(w_k) &= w_k \text{ if } k \neq i, j, \\ T_{i,j}(w_i) &= w_j, \\ T_{i,j}(w_j) &= w_i. \end{aligned}$$

Then $E_{i,j} = \left[T_{i,j}\right]_{\mathcal{C}}^{\mathcal{C}}$ corresponds to the identity matrix except the i^{th} and j^{th} rows are switched.

(2) Let $c \in F$, $c \neq 0$. Define $T_i^{(c)} : W \to W$ by:

$$\begin{aligned} T_i^{(c)}(w_j) &= w_j \text{ if } j \neq i, \\ T_i^{(c)}(w_i) &= cw_i \end{aligned}$$

Then $E_i^{(c)} = \left[T_i^{(c)}\right]_{\mathcal{C}}^{\mathcal{C}}$ corresponds to the identity matrix with the i^{th} row multiplied by c.

(3) Define $T_{i,j}^{(c)} : W \to W$ by:

$$\begin{aligned} T_{i,j}^{(c)}(w_k) &= w_k \text{ if } k \neq j, \\ T_{i,j}^{(c)}(w_j) &= w_j + cw_i \end{aligned}$$

Then $E_{i,j}^{(c)} = \left[T_{i,j}^{(c)}\right]_{\mathcal{C}}^{\mathcal{C}}$ corresponds to the identity matrix with the what does this mean?

Now let $T_A : F^4 \to F^3$ with $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$ and $\mathcal{E}_4$ and $\mathcal{F}_3$ their respective standard bases. Performing the row operation $R_1 \leftrightarrow R_3$ using the above method yields:

$$\begin{aligned} (T_{1,3} \circ T_A)(e_1) &= T_{1,3}(3f_1 + f_2 + f_3) \\ &= 3T_{1,3}(f_1) + T_{1,3}(f_2) + T_{1,3}(f_3) \\ &= 3f_3 + f_2 + f1 \end{aligned}$$

$$\begin{aligned}\left[T_{1,3}\circ T_{A}{}^{\mathcal{F}_3}_{\mathcal{E}_4}\right] &= \left[T_{1,3}\right]^{\mathcal{F}_3}_{\mathcal{F}_3}\left[T_A\right]^{\mathcal{F}_3}_{\mathcal{E}_4} \\ &= E_{1,3}A \\ &= \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{pmatrix}.\end{aligned}$$

The rest of the row operations follow similarly. The reduced-row echelon form of A can then be expressed as:

$$\left[T^{(-1)}_{1,3}\circ T^{(-1)}_{2,3}\circ T^{(\frac{1}{2})}_{(3)}\circ T^{(-1)}_{3,2}\circ T^{(-3)}_{3,1}\circ T^{(-1)}_{1,2}\circ T_{1,3}\circ T_A\right]^{\mathcal{F}_3}_{\mathcal{E}_4}.$$

3.3 Column-space and Null-space

Definition 3.3.1. Let $A \in \mathrm{Mat}_{m,n}(F)$.

(1) The *column-space* of A is the F-span of the column vectors, denoted as $CS(A)$.

(2) The *null-space* of A is the F-span of vectors $v \in F^n$ such that $Av = 0_V$, denoted as $NS(A)$.

(3) The *rank* of A is $\operatorname{rank} A = \dim_F CS(A)$.

Example 3.3.1. Let $T_A \in \mathrm{Hom}_F(F^n, F^m)$ where $\mathcal{E}_n = \{e_1, \dots, e_n\}$ is the standard basis of F^n and $\mathcal{F}_n = \{f_1, \dots, f_m\}$ is the standard basis of F^m. Since

$$[T_A]^{\mathcal{F}_m}_{\mathcal{E}_n} = A = \Big(T_A(e_1) \mid \ \dots \ \mid T_A(e_n)\Big),$$

we have that $CS(A) = \operatorname{im}(T_A)$, so $\operatorname{rank} A = \dim_F \operatorname{im}(T_A)$. Recall from an introductory linear algebra course that the column space is calculated by:

(a) Put A into row echelon form,

(b) Look at which columns have pivots,

(c) The same columns in A are then a basis of $CS(A)$.

Why does this work? There exists an isomorphism $E : F^n \to F^m$ so that $[E \circ T_A]^{\mathcal{F}_m}_{\mathcal{E}_n} = [E]^{\mathcal{F}_m}_{\mathcal{E}_n} A$ is in row echelon form. The column space of $[E \circ T_A]^{\mathcal{F}_m}_{\mathcal{E}_n}$ has as its basis the columns containing pivots (denoted $e_{i1}, \dots, e_{ik}$):

$$\underbrace{[E\circ T_A(e_{i1})]_{\mathcal{F}_m},\ \dots\ ,[E\circ T_A(e_{ik})]_{\mathcal{F}_m}}_{\text{this is a basis of } CS([E\circ T_A]^{\mathcal{F}_m}_{\mathcal{E}_n})}$$

Since E is an isomorphism, there is an inverse $E^{-1} : F^m \to F^m$ with:

$$\begin{aligned} E^{-1}(w_1) &= [E \circ T_A(e_{i1})]_{\mathcal{F}_m} \\ &\vdots \\ E^{-1}(w_k) &= [E \circ T_A(e_{ik})]_{\mathcal{F}_m} . \end{aligned}$$

These are linearly independent since E^{-1} is an isomorphism. If there is a vector $v \in CS(A)$ with $v \notin \operatorname{span}_F \left([E \circ T_A(e_{i1})]_{\mathcal{F}_m}, \ldots, [E \circ T_A(e_{ik})]_{\mathcal{F}_m}\right)$, then $E(v)$ cannot be in $\operatorname{span}_F(w_1, \ldots, w_k)$. So the columns
$[E \circ T_A(e_{i1})]_{\mathcal{F}_m}, \ldots, [E \circ T_A(e_{ik})]_{\mathcal{F}_m}$ give a basis for the column space of A.

Example 3.3.2. Let $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$. Rewritten in row echelon form is $A' = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{pmatrix}$. Thus:

$$\begin{aligned} CS(B) &= \operatorname{span}_F \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right) \\ CS(A) &= \operatorname{span}_F \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \right).e \end{aligned}$$

Example 3.3.3. We have $v \in NS(A)$ if and only if $Av = 0_{F^m} = T_A(v)$. Note that $T_A(v) = 0_{F^m}$ if and only if $v \in \ker(T_A)$, hence $NS(A) = \ker(T_A)$. In an introductory algebra class, the null space of a matrix A is calculated by:

(1) Putting A into reduced row echelon form,

(2) Solving the equation $A'x = 0_{F^n}$.

This works because given a map $T_A : F^n \to F^m$, row operations change the basis of the codomain, not the domain. So $NS(A) = NS(A')$.

Example 3.3.4. Let $A = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -1 & 1 \end{pmatrix}$. The reduce row echelon form of A is $A' = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Solving the equation:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives $x_2 = 0$ and $x_1 = -\frac{1}{2}x_3$. Hence $NS(A) = \operatorname{span}_F \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$.

3.4 The Transpose of a Matrix

Definition 3.4.1. Let $A \in \mathrm{Mat}_{m,n}(F)$ with $\mathcal{E}_n = \{e_1, \ldots, e_n\}$ and $\mathcal{F}_m = \{f_1, \ldots, f_m\}$ as standard bases. Then $A = [T_A]_{\mathcal{E}^n}^{\mathcal{F}_m}$, and furthermore $T_A \in \operatorname{Hom}_F(F^n, F^m)$ induces a dual map $T_A^\vee \in \operatorname{Hom}_F(F^{m\vee}, F^{n\vee})$. The <u>*transpose*</u> of A is defined as:

$$A^t = \left[T_A^\vee\right]_{\mathcal{F}_m^\vee}^{\mathcal{E}_n^\vee} .$$

Lemma 3.4.1. *Let $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(F)$. Then $A^t = (b_{ij}) \in \mathrm{Mat}_n, m(F)$ with $b_{ij} = a_{ji}$.*

Proof. We use the same setup as Definition 3.4.1. We have:

$$T_A(e_i) = \sum_{k=1}^{m} a_{ki} f_k$$
$$T_A^\vee(f_j^\vee) = \sum_{k=1}^{n} b_{kj} e_k^\vee.$$

Applying $f_j^\vee$ to $T_A(e_i)$ yields[1]:

$$\begin{aligned}(f_j^\vee \circ T_A)(e_i) &= f_j^\vee\left(\sum_{k=1}^{m} a_{ki} f_k\right)\\ &= \sum_{k=1}^{m} a_{ki} f_j^\vee(f_k)\\ &= a_{ji}.\end{aligned}$$

Evaluating the $T_A^\vee(f_j^\vee)$ at e_i gives:

$$\begin{aligned}T_A^\vee(f_j^\vee)(e_i) &= \sum_{k=1}^{n} b_{kj} e_k^\vee(e_i)\\ &= b_{ij}.\end{aligned}$$

By Definition 2.5.3, we have $(f_j^\vee \circ T_A)(e_i) = T_A^\vee(f_j^\vee)(e_i)$. Hence $a_{ji} = b_{ij}$ □

Exercise 3.4.1. *Let $A_1, A_2 \in \mathrm{Mat}_{m,n}(F)$ and $c \in F$. Show that:*

$$(A_1 + A_2)^t = A_1^t + A_2^t$$
$$(cA_1)^t = cA_1^t.$$

Lemma 3.4.2. *Let $A \in \mathrm{Mat}_{m,n}(F)$ and $B \in \mathrm{Mat}_{p,m}(F)$. Then $(BA)^t = A^t B^t$.*

Proof. Let $\mathscr{E}_m$, $\mathscr{E}_n$, and $\mathscr{E}_p$ be standard bases with $[T_A]_{\mathscr{E}_n}^{\mathscr{E}_m} = A$ and $[T_B]_{\mathscr{E}_m}^{\mathscr{E}_p} = B$. Then $BA = [T_B \circ T_A]_{\mathscr{E}_n}^{\mathscr{E}_p}$. Thus:

$$\begin{aligned}(BA)^t &= \left[(T_B \circ T_A)^\vee\right]_{\mathscr{E}_p^\vee}^{\mathscr{E}_n^\vee}\\ &= \left[T_A^\vee \circ T_B^\vee\right]_{\mathscr{E}_p^\vee}^{\mathscr{E}_n^\vee}\\ &= \left[T_A^\vee\right]_{\mathscr{E}_m^\vee}^{\mathscr{E}_n^\vee} \left[T_B^\vee\right]_{\mathscr{E}_p^\vee}^{\mathscr{E}_m^\vee}\\ &= A^t B^t.\end{aligned}$$

□

[1]I was really confused about this. In short, given a $T \in \mathrm{Hom}_F(V, V)$ and basis $\mathscr{B}$ we have a matrix representation $[T]_{\mathscr{B}}$. It is natural to wonder what, $[T^\vee]_{\mathscr{B}^\vee}$ looks like, and it turns out to be the "transpose" we were familiar with from 214. Basically, applying $f_j^\vee$ to $T_A(e_i)$ gives us coefficients (by definition of dual basis elements) which correspond to a particular column vector of $[T_A]_{\mathscr{B}}$. Likewise, since we have that fancy property from Definition 2.5.3, naturally we should evaluate $T_A^\vee(f_j^\vee)$ at e_i, which gives us coefficients which correspond to column vectors of $[T_A^\vee]_{\mathscr{B}^\vee}$. The rest is self-explanatory.

Lemma 3.4.3. *Let $A \in \mathrm{GL}_n(F)$. Then $(A^{-1})^t = (A^t)^{-1}$.*

Proof. Let $A = [T_A]_{\mathcal{E}_n}^{\mathcal{E}_n}$. Then $A^{-1} = \left[T_A^{-1}\right]_{\mathcal{E}_n}^{\mathcal{E}_n}$. We have:

$$\begin{aligned}
\mathbb{1}_n &= \left[\mathrm{id}_{F^n}^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \\
&= \left[(T_A^{-1} \circ T_A)^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \\
&= \left[T_A^{\vee} \circ (T_A^{-1})^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \\
&= \left[T_A^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \left[(T_A^{-1})^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \\
&= A^t (A^{-1})^t.
\end{aligned}$$

By the uniqueness of inverses, we must have that $(A^{-1})^t = (A^t)^{-1}$ Showing left invertibility follows identically. $\square$

4

Generalized Eigenvectors and Jordan Canonical Form

4.1 Diagonalization

Recall. We say $A \sim B$ if and only if $A = PBP^{-1}$ for some $P \in \mathrm{GL}_n(F)$. In particular, this means $A = [T]_{\mathcal{A}}$ and $B = [T]_{\mathcal{B}}$ for some bases $\mathcal{A}$ and $\mathcal{B}$ (Example 3.1.7).

Definition 4.1.1. We say A is _diagonalizable_ if $A \sim D$ for some diagonal matrix D. In terms of linear transformations, $A = [T]_{\mathcal{A}}$ is diagonalizable if there is a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}} = D$.

Example 4.1.1. If $A \sim B$ then A is diagonalizable if and only if B is diagonalizable. If A and B are diagonalizable, they must be similar to the same diagonal matrix up to reordering the diagonals.

Example 4.1.2. Let $V = F^2$ and $T \in \mathrm{Hom}_F(V, V)$. Let $T(e_1) = 3e_1$ and $T(e_2) = -2e_2$. We have that:

$$[T]_{\mathcal{E}_2} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

It follows that $V = V_1 \oplus V_2$, where $V_1 = \mathrm{span}_F(e_1)$ and $V_2 = \mathrm{span}_F(e_2)$. In this case, we have that $T(V_1) \subseteq V_1$ and $T(V_2) \subseteq V_2$, allowing us to write T as a diagonal matrix.

Example 4.1.3. Let $V = F^2$ and $T \in \mathrm{Hom}_F(V, V)$. Consider $T(e_1) = 3e_1$ and $T(e_2) = e_1 + 3e_2$. Then:

$$[T]_{\mathcal{E}_2} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Then $V = V_1 \oplus V_2$ with $V_1 = \mathrm{span}_F(e_1)$ and $V_2 = \mathrm{span}_F(e_2)$. But while we have $T(V_1) \subseteq V_1$, we do not have $T(V_2) \subseteq V_2$.

Suppose towards contradiction we have $W_1, W_2 \neq \{0\}$ with $T(W_1) \subseteq W_1$ and $T(W_2) \subseteq W_2$. Write $W_i = \mathrm{span}_F(w_i)$. In particular, this means we can write $T(w_1) = \alpha w_1$ and $T(w_2) = \beta w_2$. For $\mathcal{B} = \{w_1, w_2\}$, we have:

$$[T]_{\mathcal{B}} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Write $w_1 = ae_1 + be_2$ and $w_2 = ce_1 + de_2$. Then:

$$\begin{aligned} \alpha w_1 &= T(w_1) \\ &= aT(e_1) + bT(e_2) \\ &= a(3e_1) + b(e_1 + 3e_2) \\ &= (3a + b)e_1 + (3b)e_2. \end{aligned}$$

Thus, $\alpha(ae_1 + be_2) = (3a + b)e_1 + (3b)e_2$, meaning $\alpha a = 3b + b$ and $\alpha b = 3b$. Either $b = 0$ or $\alpha = 3$. It must be the case that $\alpha = 3$, hence $T(w_1) = 3w_1$. A similar argument for w_1 gives:

$$\begin{aligned} \beta w_2 &= T(w_2) \\ &= \ldots \\ &= (3c + d)e_1 + (3d)e_2. \end{aligned}$$

This implies $\beta c = ec + d$ and $\beta d = 3d$. If $\beta = 3$, then this contradicts the first equation. If $w_2 = ce_1$, this contradicts w_1, w_2 being a basis.

Example 4.1.4. Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$. Let $F = \mathbf{Q}$. Let $P \in GL_2(\mathbf{Q})$, where $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have:

$$P^{-1}AP = \frac{1}{ad - bc} \begin{pmatrix} ad - 2ab + 2cd - 4bc & -3bd - 3b^2 + 2d^2 \\ 3ac + 3a^2 - 2c^2 & -bc + 3ab - 2cd + 4ad \end{pmatrix}.$$

We must have that $3a^2 + 4ac - 2c^2 = 0$. If $c = 0$, then $a = 0$, which contradicts P being invertible. So $c \neq 0$, meaning we can divide by c^2 and set $x = \frac{a}{c}$. Then the roots of $3x^2 + 3x - 2 = 0$ are:

$$x = \frac{-3 \pm \sqrt{33}}{6},$$

which gives:

$$a = \frac{-3 \pm \sqrt{33}}{6} c.$$

Since $c \neq 0$, $a \notin \mathbf{Q}$. Thus we cannot diagonalize A over $\mathbf{Q}$. But if we were to take $F = \mathbf{Q}(\sqrt{33})$, then we have that:

$$\mathcal{B} = \{v_1 = \begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{4} \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{4} \end{pmatrix}\},$$

$$[T]_{\mathcal{B}} = \begin{pmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{pmatrix}.$$

Definition 4.1.2. Let V be an F-vector space and $T \in \mathrm{Hom}_F(V, V)$. A subspace $W \subseteq V$ is said to be *T-invariant* or *T-stable* if $T(W) \subseteq W$.

Theorem 4.1.1. *Let $\dim_F(V) = n$ and $W \subseteq V$ a k-dimensional subspace. Let $\mathcal{B}_W = \{v_1, \ldots, v_k\}$ be a basis of W and extend to a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of V. Let $T \in \operatorname{Hom}_F(V, V)$. We have W is T-stable if and only if $[T]_{\mathcal{B}}$ is block upper-triangular of the form*

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where $A = \left[T|_W\right]_{\mathcal{B}_W}$.

Example 4.1.5. Let $V = \mathbf{Q}^4$ with basis $\mathcal{E}_4 = \{e_1, e_2, \ldots, e_4\}$ and define T by:

$$\begin{aligned} T(e_1) &= 2e_1 + 3e_3 \\ T(e_2) &= e_1 + e_4 \\ T(e_3) &= e_1 - e_3 \\ T(e_4) &= 2e_1 - 2e_2 + 5e_3 - 4e_4. \end{aligned}$$

Set $W = \operatorname{span}_{\mathbf{Q}}(e_1, e_3)$, then W is T-stable. Since $\mathcal{B}_W = \{e_1, e_3\}$ and $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$, we have:

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 3 & -1 & 0 & 5 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Example 4.1.6. A special case is when $\dim_F W = 1$. If $W = \operatorname{span}_F(w_1)$ and W is T-stable, then $T(w_1) \in W_1$; i.e., $T(w_1) = \lambda w_1$ for some $\lambda \in F$ Equivalently, this can be written as $(T - \lambda \operatorname{id}_V)(w_1) = 0_V$, meaning $w_1 \in \ker(T - \lambda \operatorname{id}_V)$.

4.2 Eigenvalues and Eigenvectors

Definition 4.2.1. Let $T \in \operatorname{Hom}_F(V, V)$ and $\lambda \in F$. If $\ker(T - \lambda \operatorname{id}_V) \neq \{0_V\}$, we say λ is an *eigenvalue* of T. Any nonzero vector in $\ker(T - \lambda \operatorname{id}_V)$ is called a *λ-eigenvector*. The set $E_\lambda^1 = \ker(T - \lambda \operatorname{id}_V)$ is called the *eigenspace* associated with λ.

Exercise 4.2.1. *Show that E_λ^1 is a subspace.*

Exercise 4.2.2. *Let $T \in \operatorname{Hom}_F(V, V)$. If $\lambda_1, \lambda_2 \in F$ with $\lambda_1 \neq \lambda_2$, then $E_{\lambda_1}^1 \cap E_{\lambda_2}^1 = \{0_V\}$.*

Example 4.2.1. Let $A = \left(\begin{smallmatrix} 12 & 35 \\ -6 & 17 \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbf{Q})$ and $T_A \in \mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}^2, \mathbf{Q}^2)$. We have:

$$\begin{pmatrix} -12 & 35 \\ -6 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{2}{5} \end{pmatrix} = 2 \begin{pmatrix} 1 \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} -12 & 35 \\ -6 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3}{7} \end{pmatrix} = 3 \begin{pmatrix} 1 \\ \frac{3}{7} \end{pmatrix}$$

So T_A has eigenvalues of 2 and 3. Then

$$E_2^1 = \mathrm{span}_{\mathbf{Q}}\left(v_1 = \left(\begin{smallmatrix} 1 \\ 2/5 \end{smallmatrix}\right)\right)$$
$$E_3^1 = \mathrm{span}_{\mathbf{Q}}\left(v_2 = \left(\begin{smallmatrix} 1 \\ 3/7 \end{smallmatrix}\right)\right)$$

gives:

$$[T_A]_{\{v_1, v_2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Example 4.2.2 ($F[x]$-Modules)**.** Let $T \in \mathrm{Hom}_F(V, V)$. Note that V is by definition an F-module, but we are able to view V as an $F[x]$-module given some linear transformation T. The action $F[x] \times V \to V$ is defined by $(f(x), v) \mapsto f(T)(v)$.

Write $T^m = \underbrace{T \circ T \circ \ldots \circ T}_{m-\text{times}}$. Write $f(x) \in F[x]$ as $f(x) = a_m x^m + \ldots + a_1 x + a_0$. Then

$$f(T) = a_m T^m + \ldots + a_1 T + a_0 \,\mathrm{id}_V \in \mathrm{Hom}_F(V, V).$$

For example, let $g(x) = 2x^2 + 3 \in \mathbf{R}[x]$. Then $g(T) = 2T^2 + 3\,\mathrm{id}_V$ and $g(T)(v) = 2T(T(v)) + 3v$. If $f(x) = g(x)h(x)$ for some $g(x), h(x) \in F[x]$, then $f(T) = g(T) \circ h(T)$. Instead of writing $f(T)(v) = g(T)(h(T)(v))$, we will abuse notation and write $g(T)h(T)(v)$. Normally function composition does not commute, but these do for some reason.

Theorem 4.2.1. *Let $\dim_F(V) = n$ and $T \in \mathrm{Hom}_F(V, V)$. There is a unique monic polynomial $m_T(x) \in F[x]$ of lowest degree so that $m_T(T)(v) = 0_V$ for all $v \in V$. Moreover, $\deg_{m_T}(T) \leqslant n^2$.*

Proof. Recall that $\mathrm{Hom}_F(V, V)$ is an F-vector space. We have $\mathrm{Hom}_F(V, V) \cong \mathrm{Mat}_n(F)$, hence $\dim_F(\mathrm{Hom}_F(V, V)) = n^2$.

Given $T \in \mathrm{Hom}_F(V, V)$, consider the set $\{\mathrm{id}_V, T, T^2, \ldots, T^{n^2}\} \subseteq \mathrm{Hom}_F(V, V)$. This has $n^2 + 1$ elements, so it must be linearly dependent. Let m be the smallest integer so that

$$a_m T^m + \ldots + a_1 T + a_0 \,\mathrm{id}_V{}^{1} = 0_{\mathrm{Hom}_F(V, V)}.$$

We obtain a set $\{\mathrm{id}_V, T, T^2, \ldots, T^m\}$. Since m is minimal, $a_m \neq 0$. Define:

$$m_T(x) = x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0 \in F[x], \text{ where } b_i = \frac{a_i}{a_m}.$$

[1]This seems kind of out of nowhere, so think of it like this: Let $I_T = \{p \in F[x] \mid p(T)(v) = 0_V \text{ for all } v \in V\}$. $F[x]$ is a P.I.D., so every ideal is generated by a single element. The minimal polynomial $m_T(x)$ is the generator of this ideal.

This gives $m_T(T) = 0_{\mathrm{Hom}_F(V,V)}$; i.e., $m_T(T)(v) = 0_V$ for all $v \in V$. It remains to that $m_T(x)$ is unique. Suppose there exists an $f(x) \in F[x]$ which satisfies $f(T)(v) = 0_V$ for all $v \in V$. Write:

$$f(x) = m_T(x)q(x) + r(x)$$

for some $q(x), r(x) \in F[x]$ with $r(x) = 0$ or $\deg(r(x)) < \deg(m_T(x))$. We have for all $v \in V$:

$$\begin{aligned} 0_V &= f(T)(v) \\ &= q(T)m_T(T)(v) + r(T)(v) \\ &= q(T)(0_V) + r(T)(v) \\ &= r(T)(v) \end{aligned}$$

It must be the case that $r(x) = 0$, otherwise we have a polynomial of lower degree than $m_T(x)$ which kills all vectors. So $f(x) = m_T(x)q(x)$; i.e., $m_T(x) \mid f(x)$. But if $m_T(x)$ and $f(x)$ are both monic and of minimal degree, it must be the case that they are the same degree. This gives $m_T(x) = f(x)$. □

Definition 4.2.2. The unique monic polynomial $m_T(x)$ is called the *minimal polynomial* of T.

Corollary 4.2.2. *If $f(x) \in F[x]$ satisfies $f(T)(v) = 0_V$ for all $v \in V$, then $m_T(x) \mid f(x)$.*

Proof. wat □

Example 4.2.3. Let $F = \mathbf{Q}$ and $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$. We can see that:

$$A - a_0 1_2 \neq 0_2 \text{ for any } a_0 \in F.$$

But $A^2 = \left(\begin{smallmatrix} 7 & 10 \\ 15 & 22 \end{smallmatrix}\right)$ gives $A^2 - 5A - 2 \cdot 1_2 = 0_2$. Hence $m_A(x) = x^2 - 5x - 2$. Note the relationship between this example and Example 4.1.4.

Example 4.2.4. Let $V = \mathbf{Q}^3$, $\mathcal{E}_3 = \{e_1, e_2, e_3\}$, and

$$[T_A]_{\mathcal{E}_3} = A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $W = \mathrm{span}_{\mathbf{Q}}(e_1)$ Then $T(W) = T(\alpha e_1) = \alpha e_1 \in W$. Hence $T(W) \subseteq W$, meaning W is T-stable. This gives 1 as an eigenvalue. On a completely unrelated note, $m_{T_A}(x) = (x-1)^2(x+1)$.

Theorem 4.2.3. *Let V be an F-vector space and $T \in \mathrm{Hom}_F(V,V)$. We have λ is an eigenvalue if and only if λ is the root of $m_T(x)$. In particular, if $(x-\lambda) \mid m_T(x)$, then $E_\lambda^1 \neq \{0_V\}$ (i.e., there is a nonzero $v \in V$ such that $T(v) = \lambda v$).*

Proof. Let λ be an eigenvalue with eigenvector v and write $m_T(x) = x^m + \ldots + a_1x + a_0$. We have:

$$\begin{aligned} 0_V &= m_T(T)(v) \\ &= (T^m + a_{m-1}T^{m-1} + \ldots + a_1T + a_0\,\mathrm{id}_V)(v) \\ &= T^m(v) + a_{m-1}T^{m-1}(v) + \ldots + a_1T(v) + a_0v \\ &= \lambda^m v + a_{m-1}\lambda^{m-1}v + \ldots + a_1\lambda v + a_0 v \\ &= (\lambda^m + a_{m-1}\lambda^{m-1} + \ldots + a_1\lambda + a_0)v \\ &= m_T(\lambda)\cdot v. \end{aligned}$$

Since $v \neq 0$ and $m_T(\lambda) \in F$, it must be the case that $m_T(\lambda) = 0$. Hence λ is a root.

Now suppose $m_T(\lambda) = 0$. This gives $m_T(x) = (x - \lambda)f(x)$ for some $f(x) \in F[x]$. Since $\deg f(x) < \deg m_T(x)$, this gives a nonzero vector $v \in V$ so that $f(T)(v) \neq 0$ (since $m_T(x)$ is the smallest polynomial that satisfies $m_T(T)(v) = 0_V$, it must be the case that there is a nonzero $v \in V$ that satisfies $f(T)(v) \neq 0$). Set $w = f(T)(v)$, then:

$$\begin{aligned} 0_V &= (T - \lambda\,\mathrm{id}_V)f(T) \\ &= (T - \lambda\,\mathrm{id}_V)w, \end{aligned}$$

which simplifies to $T(w) = \lambda w$. Thus λ is an eigenvalue. □

Corollary 4.2.4. *Let $\lambda_1, \ldots, \lambda_n \in F$ be distinct eigenvalues of T. For each i, let v_i be an eigenvector with eigenvalue λ_i. The set $\{v_1, \ldots, v_m\}$ is linearly independent.*

Proof. We have $m_T(x) = (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_m)f(x)$ for some $f(x) \in F[x]$. Suppose $a_1v_1 + \ldots + a_mv_m = 0_V$ for $a_i \in F$. Define $g_1(x) = (x - \lambda_2)\ldots(x - \lambda_m)f(x)$. Note that $g_1(T)(v_i) = 0_V$ for $2 \leqslant i \leqslant m$. Then:

$$\begin{aligned} 0_V &= g_1(T)(0_V) \\ &= \sum_{j=1}^{m} a_j g_1(T)(v_j) \\ &= a_1 g_1(T)(v_1) \\ &= a_1 g_1(\lambda_1)v_1 \end{aligned}$$

But $g_1(\lambda_1) \neq 0$ and $v \neq 0$, so it must be that case that $a_1 = 0$. Inductively, it follows for $2, \ldots, m$. □

Corollary 4.2.5. *If* $\deg(m_T(x)) = \dim_F(V)$ *and $m_T(x)$ has distinct roots, all of which are in F, then we can find a basis $\mathcal{B}$ so that $[T]_{\mathcal{B}}$ is diagonal.*

Example 4.2.5. Let $A = \begin{pmatrix} 1&0&0\\0&2&0\\0&0&2 \end{pmatrix}$ and $B = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&2 \end{pmatrix}$. These matrices are not similar, however $m_A(x) = m_B(x) = (x-1)(x-2)$. The minimal polynomial is not enough information on the similarity of matrices.

Example 4.2.6. Let:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

We have that $m_A(x) = (x-1)^2(x+1)$. Note that $Ae_1 = e_1$, so $E_1^1 \supseteq \operatorname{span}_F(e_1)$ (or, more simply, $e_1 \in E_1^1$). Note that $Ae_2 = \left(\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}\right)$. So $e_2 \notin E_1^1$ (another way of saying this is $(A - 1_3)e_2 \neq \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$). But now consider:

$$(A - 1_3)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 4 \end{pmatrix}.$$

We have $(A - 1_3)^2 e_2 = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$. Thus $e_1, e_2 \in \ker(A - \mathrm{id}_{F^3})^2$.

Definition 4.2.3. Let $T \in \operatorname{Hom}_F(V, V)$. For $k \geqslant 1$, the <u>k^{th} generalized eigenspace</u> of T associated to λ is $E_\lambda^k = \ker(T - \lambda\,\mathrm{id}_V)^k = \{v \in V \mid (T - \lambda\,\mathrm{id}_V)^k v = 0_V\}$. Elements of E_λ^k are called <u>*generalized eigenvectors*</u>. Set $E_\lambda^\infty = \bigcup_{k \geqslant 1} E_\lambda^k$.

Example 4.2.7. Continuing Example 4.2.6, let $\alpha e_1 + \beta e_2 \in \operatorname{span}_F(e_1, e_2)$. Then:

$$(A - 1_3)^2(\alpha e_1 + \beta e_2) = \alpha(A - 1_3)^2 e_1 + \beta(A - 1_3)^2 e_2 = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right).$$

So $\operatorname{span}_F(e_1, e_2) \subseteq E_1^2$. We also have -1 as an eigenvalue with eigenvector $v_3 = \left(\begin{smallmatrix} \frac{1}{2} \\ -2 \\ 1 \end{smallmatrix}\right)$. Check that $v_3 \notin E_1^2$. So $\dim_F(E_1^2) \leqslant 2$; i.e., $E_1^2 = \operatorname{span}_F(e_1, e_2)$. why does $v_3 \notin E_1^2$ imply the dimension which implies containment in the other direction.

Lemma 4.2.6. *Let V be a finite dimensional F-vector space, $\dim_F(V) = n$, and $T \in \operatorname{Hom}_F(V, V)$. There exists m with $1 \leqslant m \leqslant n$ such that $\ker(T^{\cdot}) = \ker(T^{m+1})$. Moreover, for such an m, $\ker(T^m) = \ker(T^{m+j})$ for all $j \geqslant 0$.*

Proof. We have $\ker(T^1) \subseteq \ker(T^2) \subseteq \ldots$ If these containments are always strict, then the dimension increases indefinitely, which contradicts $\dim_F(V) = n$. Hence we have an m with $1 \leqslant m \leqslant n$ and $\ker(T^m) = \ker(T^{m+1})$.

Let m be the smallest value where $\ker(T^m) = \ker(T^{m+1})$. We use induction on j. Base case of $j = 1$ is what defines m. Assume $\ker(T^m) = \ker(T^{m+j})$ for all $1 \leqslant j \leqslant N$. Let $v \in \ker(T^{m+N+1})$. This gives:

$$\begin{aligned} 0_V &= T^{m+N+1}(v) \\ &= T^{m+1}(T^N(v)). \end{aligned}$$

So $T^N(v) \in \ker(T^{m+1})$. However $\ker(T^{m+1}) = \ker(T^m)$, so $T^N(v) \in \ker(T^m)$. Hence:

$$\begin{aligned} 0_V &= T^m(T^n(v)) \\ &= T^{m+N}(v), \end{aligned}$$

so $v \in \ker(T^{m+N})$. Induction hypothesis gives $\ker(T^{m+N}) = \ker(T^m)$, giving $v \in \ker(T^m)$. Thus $\ker(T^{m+N+1}) \subseteq \ker(T^m)$. The other direction of containment is trivial. $\square$

Example 4.2.8. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V and $T \in \mathrm{Hom}_F(V, V)$, $\lambda \in F$ such that:

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

In other words, $[T]_{\mathcal{B}}$ contains λ along the diagonal and 1 along the super-diagonal. Let $A = [T]_{\mathcal{B}}$. Consider:

$$(A - \lambda 1_n) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We get:

$$\begin{aligned} (A - \lambda 1_n)v_1 &= 0_V \\ (A - \lambda 1_n)v_2 &= v_1 \\ &\vdots \\ (A - \lambda 1_n)v_n &= v_{n-1}. \end{aligned}$$

This gives $E_\lambda^1 = \mathrm{span}_F(v_1)$ (by the first equation). Now observe:

$$(A - \lambda 1_n)^2 v_1 = 0_V$$

$$\begin{aligned} (A - \lambda 1_n)^2 v_2 &= (A - \lambda 1_n)(A - \lambda 1_n)v_2 \\ &= (A - \lambda 1_n)v_1 \\ &= 0_V \end{aligned}$$

$$\begin{aligned} (A - \lambda 1_n)^2 v_3 &= v_1 \\ &\vdots \\ (A - \lambda 1_n)^2 v_n &= v_{n-2}. \end{aligned}$$

So $E_\lambda^2 = \operatorname{span}_F(v_1, v_2)$. In general, we have that $E_\lambda^k = \operatorname{span}_F(v_1, \dots, v_k)$. Moreover, Lemma 4.2.6 gives $E_\lambda^1 \subseteq E_\lambda^2 \subseteq \dots \subseteq E_\lambda^k$.

Corollary 4.2.7. *If $\dim_F(V) = n$ and $T \in \operatorname{Hom}_F(V, V)$, there exists an m with $1 \leqslant m \leqslant n$ so that for any $\lambda \in F$, $E_\lambda^\infty = E_\lambda^m$.*

Theorem 4.2.8. *Let $T \in \operatorname{Hom}_F(V, V)$, and $\lambda \in F$ with $(x-\lambda)^k \mid m_T(x)$. We have:*

$$\dim_F(E_\lambda^k) \geqslant k.$$

Proof. Write $m_T(x) = (x-\lambda)^k f(x)$ where $f(x) \in F[x]$, $f(\lambda) \neq 0$. Define $g_k(x) = (x-\lambda)^k$. We have that $(x-\lambda)^{k-1}f(x) = g_{k-1}(x)f(x)$ is *not* the minimal polynomial. So there is a $v \in V$ with $v \neq 0_V$ such that:

$$g_{k-1}(T)f(T)(v) \neq 0_V.$$

Set $v_k = f(T)(v)$. Observe that:

$$\begin{aligned}(T - \lambda \operatorname{id}_V)^k (v_k) &= (T - \lambda \operatorname{id}_V)^k f(T)(v)\\ &= m_T(T)(v)\\ &= 0_V.\end{aligned}$$

So $v_k \in E_\lambda^k$. Moreover, by our construction:

$$\begin{aligned}(T - \lambda \operatorname{id}_V)^{k-1}(v_k) &= g_{k-1}(T)(v_k)\\ &= g_{k-1}(T)f(T)(v)\\ &\neq 0_V.\end{aligned}$$

Hence $v_k \in E_\lambda^k \setminus E_\lambda^{k-1}$. Now set $v_{k-1} = (T - \lambda \operatorname{id}_V)v_k = (T - \lambda \operatorname{id}_V)f(T)(v)$. Note:

$$\begin{aligned}(T - \lambda \operatorname{id}_V)^{k-1}(v_{k-1}) &= (T - \lambda \operatorname{id}_V)^{k-1}(T - \lambda \operatorname{id}_V)(v_k)\\ &= (T - \lambda \operatorname{id}_V)^k(v_k)\\ &= (T - \lambda \operatorname{id}_V)^k f(T)(v)\\ &= m_T(T)(v)\\ &= 0_V.\end{aligned}$$

So $v_{k-1} \in E_\lambda^{k-1}$. Again, by our construction:

$$\begin{aligned}(T - \lambda \operatorname{id}_V)^{k-2}(v_{k-1}) &= (T - \lambda \operatorname{id}_V)^{k-2}(T - \lambda \operatorname{id}_V)(v_k)\\ &= (T - \lambda \operatorname{id}_V)^{k-1}(v_k)\\ &\neq 0_V.\end{aligned}$$

So $v_{k-1} \in E_\lambda^{k-1} \setminus E_\lambda^{k-2}$. Setting $v_{k-2} = (T - \lambda \operatorname{id}_V)^2 v_k$ gives a similar result. By this construction, we obtain a set $\{v_k, v_{k-1}, \dots, v_2, v_1\}$. Claim: this set is linearly independent. Suppose towards contradiction it's not, that is, $a_1 v_1 + \dots + a_k v_k = 0_V$ does not imply $a_1 = \dots = a_k = 0$. This gives $v_k = \frac{-1}{a_k}(a_1 v_1 + \dots + a_{k-1}v_{k-1}) \in E_\lambda^{k-1}$, which is a contradiction. It follows that $a_1 = \dots = a_k = 0$, hence $\{v_k, v_{k-1}, \dots, v_2, v_1\}$ is linearly independent. but why does this establish the theorem. □

Example 4.2.9. Let $T_A \in \mathrm{Hom}_F(F^3, F^3)$ be defined by:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

We have that $m_T(x) = (x-2)^3$. Now observe:

$$(A - 2 \cdot 1_3)^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note $(A - 2 \cdot 1_3)^2 e_3 = 4e_3 \neq 0_F^3$, but $(A - 2 \cdot 1_3)^3 e_3 = 0_{F^3}$. Set $v_3 = e_3$, we have $v_3 \in E_2^3$. Now observe:

$$\begin{aligned} v_2 &= (A - 2 \cdot 1_3)(v_3) \\ &= \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}. \end{aligned}$$

Similarly:

$$\begin{aligned} v_1 &= (A - 2 \cdot 1_3)(v_2) \\ &= \dots \\ &= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence:

$$\begin{aligned} E_2^3 &= \mathrm{span}_F \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right) \\ E_2^2 &= \mathrm{span}_F \left(\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right) \\ E_2^1 &= \mathrm{span}_F \left(\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right). \end{aligned}$$

Setting $\mathcal{B} = \{v_1, v_2, v_3\}$, we have:

$$[T_A]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

4.3 Characteristic Polynomials

Definition 4.3.1. Let $A \in \mathrm{Mat}_n(F)$. The *characteristic polynomial* is $c_A(x) = \det(x1_n - A)$.

Definition 4.3.2. Let $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \in F[x]$. The *companion matrix* of $f(x)$ is given by:

$$C(f(x)) = \begin{pmatrix} -a_0 & 0 & 0 & \dots & 0 \\ -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \dots & 1 \end{pmatrix}$$

The companion matrix shows that any polynomial $f(x) \in F[x]$ can be realized as the characteristic polynomial of a matrix.

Lemma 4.3.1. *If $A = C(f(x))$, then $c_A(x) = f(x)$.*

Lemma 4.3.2. *Let $A, B \in \mathrm{Mat}_n(F)$ be similar matrices. Then $c_A(x) = c_B(x)$.*

Proof. Let $A = PBP^{-1}$ for some $P \in GL_n(F)$. We have:

$$\begin{aligned} c_A(x) &= \det(x1_n - A) \\ &= \det(x1_n - PBP^{-1}) \\ &= \det(P(x1_n)P^{-1} - PBP^{-1}) \\ &= \det(P(x1_n - B)P^{-1}) \\ &= \det(P)\det(x1_n - B)\det(P^{-1}) \\ &= \det(x1_n - B) \\ &= c_B(x). \end{aligned}$$

□

Definition 4.3.3. For $T \in \mathrm{Hom}_F(V, V)$, let $\mathcal{B}$ be a basis of V and set $c_T(x) = c_{[T]_{\mathcal{B}}}(x)$.

Theorem 4.3.3. *Let $v \in V$, $v \neq 0_V$. Let $\dim_F(V) = n$. Then there is a unique monic polynomial $m_{T,v}(x) \in F[x]$ so that $m_{T,v}(T)(v) = 0_V$. Moreover, if $f(x) \in F[x]$ with $f(T)(v) = 0_V$, then $m_{T,v}(x) \mid f(x)$.*

Proof. Consider the set $\{v, T(v), T^2(v), ..., T^n(v)\}$. Since this set contains $n + 1$ elements and the dimension of V is n, the set must be linearly dependent. Write:

$$a_mT^m(v) + ... + a_1T(v) + a_0 = 0_V$$

for some $m \leqslant n$ of minimal order and $a_i \neq 0$ for all i. Set:

$$p(x) = x^m + \frac{a_{m-1}}{a_m}x^{m-1} + \dots + \frac{a_1}{a_m}x + \frac{a_0}{a_m} \in F[x].$$

By construction $p(T)(v) = 0_V$. Set $I_v = \{g(x) \in F[x] \mid g(T)(v) = 0_V\}$. We have that $p(x)$ is a monic nonzero polynomial in I_v of minimal degree. Set $m_{T,v}(x) = p(x)$.

Let $f(x) \in I_v$. We'd like to show that $m_{T,v}(x) \mid f(x)$. Write:

$$f(x) = q(x)m_{T,v}(x) + r(x),$$

with $q(x), r(x) \in F[x]$ and $\deg(r(x)) = 0$ or $deg(r) < \deg(m_{T,v}(x))$. Observe that:

$$\begin{aligned} r(T)(v) &= f(T)(v) - q(T)m_{T,v}(T)(v) \\ &= 0_V - q(T)0_V \\ &= 0_V. \end{aligned}$$

So $r(x) \in I_v$. But $m_{T,v}(x)$ had minimal degree, so it must be the case that $r(x) = 0$. Thus $f(x) = q(x)m_{T,v}(x)$, implying $m_{T,v}(x) \mid f(x)$[2]. Now suppose $h(x) \in I_v$ with $\deg(h(x)) = \deg(m_{T,v}(x))$. Since both polynomials are monic and of equal degree, if $m_{T,v}(x) \mid h(x)$ then $m_{T,v}(x) = h(x)$. □

Definition 4.3.4. We refer to $m_{T,v}(x)$ as the *T-annihilator* of v.

Example 4.3.1. Let $V = F^n$ and $\mathcal{E}_n = \{e_1, \dots, e_n\}$. Define $T \in \mathrm{Hom}_F(V, V)$ by:

$$\begin{aligned} T(e_1) &= 0_v \\ T(e_j) &= e_{j-1} \text{ for } 2 \leqslant j \leqslant n. \end{aligned}$$

Consider $f(x) = x$. Then $f(T)(e_1) = T(e_1) = 0_V$. Hence $m_{T,e_1}(x) \mid x$. So either $m_{T,e_1}(x) = 1$ or $m_{T,e_1}(x) = x$. But $\mathrm{id}_V(e_1) = e_1 \neq 0_V$, hence it must be the case that $m_{T,e_1}(x) = x$.

Now consider $g(x) = x^2$. Then $g(T)(e_2) = T^2(e_2) = T(T(e_2)) = T(e_1) = 0_V$. Hence $m_{T,e_2}(x) \mid x^2$. So $m_{T,e_2}(x) = 1$ or x or x^2. If $m_{T,e_2}(x) = 1$, then $\mathrm{id}_V(e_2) = e_2 \neq 0_V$. If $m_{T,e_2}(x) = x$, then $T(e_2) = e_1 \neq 0$. So $m_{T,e_2}(x) = x^2$. It follows for $i \leqslant j \leqslant n$, $m_{T,e_j}(x) = x^j$.

Example 4.3.2. Let $V = \mathbf{Q}^2$. Define $T \in \mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}^2, \mathbf{Q}^2)$ by:

$$\begin{aligned} T(e_1) &= e_1 + 3e_2 \\ T(e_2) &= 2e_1 + 4e_2. \end{aligned}$$

We are trying to find $m_{T,e_1}(x)$. Since V is two-dimensional, $\deg(m_{T,e_1}(x)) = 1$ or 2. Write $m_{T,e_1}(x) = x + a$. Then:

$$\begin{aligned} m_{T,e_1}(T)(e_1) &= T(e_1) + ae_1 \\ &= e_1 + 3e_2 + ae_1 \\ &\neq 0_V. \end{aligned}$$

[2]The proof of $F[x]$ being a P.I.D. follows identically. Instead of considering I_v we would consider an arbitrary polynomial in $F[x]$.

So it must be that $\deg(m_{T,e_1}(x)) = 2$. Note that:

$$\begin{aligned} T^2(e_1) &= T(e_1 + 3e_2) \\ &= T(e_1) + 3T(e_2) \\ &= 7e_1 + 15e_2. \end{aligned}$$

Now let:

$$T^2(e_1) + bT(e_1) + ce_1 = 0_V,$$

for some $b, c \in \mathbf{Q}$. This will yield a system of equations, and solving for it gives:

$$\begin{aligned} b &= -5 \\ c &= -2. \end{aligned}$$

Hence $m_{T,e_1}(x) = x^2 - 5x - 2$.

Exercise 4.3.1.

1. *Show* $m_{T,e_2}(x) = x^2 - 5x - 2$.

2. *Calculate* $m_{T,e_1}(x)$ *and* $m_{T,e_2}(x)$ *of* $F = \mathbf{F}_3$.

Theorem 4.3.4. *Let* $\dim_F(V) = n$ *and* $\mathcal{B} = \{v_1, \ldots, v_n\}$ *be a basis of* V. *Let* $T \in \operatorname{Hom}_F(V, V)$. *We have:*

$$m_T(x) = \operatorname*{lcm}_{1 \leqslant i \leqslant n} m_{T,v_i}(x).$$

Proof. Let $f(x) = \operatorname*{lcm}_{1 \leqslant i \leqslant n} m_{T,v_i}(x)$. Note that $m_T(T)(v_i) = 0_V$, so $m_{T,v_i}(x) \mid m_T(x)$ for each i. Hence $f(x) \mid m_T(x)$.

Now let $v \in V$. Write $v = \sum_{i=1}^n a_i v_i$. We have:

$$\begin{aligned} f(T)(v) &= f(T)(\sum_{i=1}^n a_i v_i) \\ &= \sum_{i=1}^n a_i f(T)(v_i) \\ &= 0_V, \end{aligned}$$

because $m_{T,v_i}(x) \mid f(x)$ for all i. Hence $m_T(x) \mid f(x)$. i dont quite get this number theory stuff □

Lemma 4.3.5. *Let* $T \in \operatorname{Hom}_F(V, V)$. *Let* $v_1, \ldots, v_k \in V$, *and set* $p_i(x) = m_{T,v_i}(x)$. *Suppose* $p_i(x)$ *are pairwise relatively prime. Set* $v = v_1 + \ldots + v_k$. *Then:*

$$m_{T,v}(x) = p_1(x) \ldots p_k(x).$$

Proof. We prove this for $k \geqslant 2$; i.e., $m_{T,v_1+v_2}(x) = m_{T,v_1}(x)m_{T,v_2}(x)$. Since $p_1(x)$ and $p_2(x)$ are relatively prime, there exists $q_1(x), q_2(x) \in F[x]$ so that $1 = p_1(x)q_1(x) + p_2(x)q_2(x)$. In particular, $\mathrm{id}_V = p_1(T)q_1(T) + p_2(T)q_2(T)$. Set $v = v_1 + v_2$. We have:

$$\begin{aligned} v &= \mathrm{id}_V(v) \\ &= (p_1(T)q_1(T) + p_2(T)q_2(T))(v) \\ &= p_1(T)q_1(T)(v) + p_2(T)q_2(T)(v) \\ &= p_1(T)q_1(T)(v_1 + v_2) + p_2(T)q_2(T)(v_1 + v_2) \\ &= p_1(T)q_1(T)(v_2) + p_2(T)q_2(T)(v_2). \end{aligned}$$

Write $w_1 = p_1(T)q_1(T)(v_2)$ and $w_2 = p_2(T)q_2(T)(v_1)$. This means $v = w_1 + w_2$. Note:

$$\begin{aligned} p_1(T)(w_1) &= p_1(T)p_2(T)q_2(T)(v_1) \\ &= q_2(T)p_2(T)\underbrace{p_1(T)(v_1)} \\ &= 0_V. \end{aligned}$$

Hence $w_1 \in \ker(p_1(T))$. It follows similarly that $w_1 \in \ker(p_2(T))$. Let $r(x) \in F[x]$ with $r(T)(v) = 0_V$. We have $v = w_1 + w_2$ and $w_2 \in \ker(p_2(T))$, so:

$$\begin{aligned} p_2(T)(v) &= p_2(T)(w_1 + w_2) \\ &= p_2(T)(w_1). \end{aligned}$$

Thus:

$$\begin{aligned} 0_V &= p_2(T)q_2(T)(0_V) \\ &= p_2(T)q_2(T)r(T)(v) \\ &= r(T)p_2(T)q_2(T)(v) \\ &= r(T)p_2(T)q_2(T)(w_1). \end{aligned}$$

We also know $r(T)q_1(T)p_1(T)(w_1) = 0_V$ because $w_1 \in \ker(p_1(T))$. Hence:

$$\begin{aligned} 0_V &= r(T)p_2(T)q_2(T)(w_1) + r(T)p_1(T)q_1(T)(w_1) \\ &= r(T)\underbrace{(p_2(T)q_2(T) + p_1(T)q_1(T))}_{\mathrm{id}_V}(w_1) \\ &= r(T)(w_1). \end{aligned}$$

So:

$$\begin{aligned} 0_V &= r(T)(w_1) \\ &= r(T)p_2(T)q_2(T)(v_1). \end{aligned}$$

□