Math 310

Homework 2

Due: 9/20/2024

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Exercise 1. *Let F be a field. Show that the following hold:*

(i) $-1(a) = -a$.

(ii) $-(-a) = a$.

(iii) $-(a+b) = (-a) + (-b)$.

(iv) $(-a)^{-1} = -(a^{-1})$.

(v) $(ab)^{-1} = a^{-1}b^{-1}$.

Proof. (i) Note that $(-1+1)a = 0$. Distributing a gives $(-1)a + a = 0$. Hence $(-1)a = -a$.

(ii) $-(-a) = -1(-a)$ by part (i). Adding $1(-a)$ to both sides gives $-(-a) + (-a) = 0$. So $-(-a)$ is the additive inverse of $-a$; we denote this a. Hence $-(-a) = a$.

(iii) $-(a+b) = (-1)(a+b) = (-1)a + (-1)b = (-a) + (-b)$.

(iv) Note that $(-a) \cdot 1 = a$ implies $(-a)(a \cdot a^{-1}) = 1$. Further simplification yields $(-a)(-(a^{-1})) = 1$. So $-(a^{-1})$ is the multiplicative inverse of $-a$, which is denoted $(-a)^{-1}$. Hence $(-a)^{-1} = -(a^{-1})$.

(v) From $ab = ab$, we have that $1 = ab(ab)^{-1}$. Then $a^{-1} = b(ab)^{-1}$, hence $a^{-1}b^{-1} = (ab)^{-1}$.

□

Exercise 2. *Consider the set $K := \{a + b\sqrt{2} \mid a, b \in \mathbf{Q}\}$. Show that:*

(i) If $x, y \in K$, then $x + y \in K$ and $xy \in K$.

(ii) If $x \neq 0$, then $x^{-1} \in K$.

Proof. Let $x, y \in K$. Then $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$. So $x + y = a + b\sqrt{2} + c + d\sqrt{2} = (a+c) + (b+d)\sqrt{2} \in K$ (since $\mathbf{Q}$ is closed under addition). Similarly, $xy = (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \in K$.

Let $x = (a + b\sqrt{2})$. Suppose there exists a $y \neq 0$ such that $y = c + d\sqrt{2}$. Let $(a + b\sqrt{2})(c + d\sqrt{2}) = 1$. Then $ac + 2bd + (bc + ad)\sqrt{2} = 1$. We have the following system of equations:

$$bc + ad = 0$$
$$ac + 2bd = 1.$$

Solving for c and d yields:

$$c = \frac{-a}{-(a^2) + 2b^2} \in \mathbf{Q}$$
$$d = \frac{b}{-(a^2) + 2b^2} \in \mathbf{Q}.$$

Hence $\left(\frac{-a}{-(a^2) + 2b^2}\right) + \left(\frac{b}{-(a^2) + 2b^2}\right)\sqrt{2} = x^{-1} \in K$. □

Exercise 3. *Suppose F is a field admitting a subset $P \subseteq F$ with properties:*

(1) If $x, y \in P$, then $x + y \in P$ and $xy \in P$.

(2) For all $x \in F$, $x \in P$ or $-x \in P$.

(3) If $x, -x \in P$, then $x = 0$.

Show that there is an ordering on F making it into an ordered field.

Proof. Define $a \leqslant_P b$ if and only if $b - a \in P$. This ordering is reflexive because $a \leqslant_P a$ if and only if $a - a = 0 \in P$. The ordering is also transitive as follows: if $x \leqslant_p y \leqslant_p z$, then $y - x \in P$ and $z - y \in P$. Since P is closed under addition, $y - x + (z - y) = z - x \in P$ implies $x \leqslant_p z$. The ordering is antisymmetric as follows: Let $x \leqslant_p y$ and $y \leqslant_p x$. Then $x - y \in P$ and $y - x = -(x - y) \in P$. Hence $x - y = 0$, implying $x = y$. Note that this ordering is total as well: if $x, y \in F$ then $x - y \in F$ by closure of fields under addition. Then either $x - y \in P$ or $-(x - y) = y - x \in P$. Hence $x \leqslant_P y$ or $y \leqslant_P x$. Thus $\leqslant_P$ is a total ordering on F.

Suppose $x \leqslant_P y$ and $s \leqslant_P t$. Then $y - x \in P$ and $t - s \in P$. Since P is closed under addition, $(y - x) + (t - s) = (y + t) - (x + s) \in P$. Then by our definition $x + s \leqslant_P y + t$. Now consider $z \in P$, $z \neq 0$. Since P is closed under multiplication, $(y - x)z \in P$. Simplifying yields $yz - xz \in P$, or equivalently $xz \leqslant_P yz$. This establishes F as an ordered field. □

Exercise 4. *Let $a, b \in \mathbf{R}$.*

(i) If $0 \leqslant a \leqslant \epsilon$ for all $\epsilon > 0$, then $a = 0$.

(ii) If $a \leqslant b + \epsilon$ for all $\epsilon > 0$, then $a \leqslant b$.

Proof. (i) If $a = 0$ we are done. If $a > 0$, then $0 \leqslant \frac{1}{2}a \leqslant a$. Pick $\epsilon = \frac{1}{2}a$, then $a \leqslant \frac{1}{2}a$, a contradiction. Thus $a = 0$. (ii) If $a \leqslant b$ then we are done. If $a > b$, then $a - b \in \mathbf{R}^+$, hence $\frac{1}{2}(a - b) \in \mathbf{R}^+$. Take $\epsilon = \frac{1}{2}(a - b)$. Then $a \leqslant b + \frac{1}{2}(a - b)$ is equivalent to $a \leqslant \frac{1}{2}(a + b) \leqslant b$, which is a contradiction. Thus $a \leqslant b$. □

Exercise 5. *If $a, b \in \mathbf{R}$, show that:*

$$\left(\frac{1}{2}(a + b)\right)^2 \leqslant \frac{1}{2}(a^2 + b^2)$$

with equality if and only if $a = b$.

Proof. Observe that:

$$\begin{aligned} 0 &\leqslant \frac{1}{4}(a - b)^2 \\ &= \frac{1}{4}(a^2 - 2ab + b^2) \\ &= \frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2. \end{aligned}$$

Adding $\frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2$ to both sides and yields:

$$\frac{1}{4}(a^2 + 2ab + b^2) \leqslant \frac{1}{2}(a^2 + b^2).$$

And upon further simplification we get the desired result:

$$\left(\frac{1}{2}(a + b)\right)^2 \leqslant \frac{1}{2}(a^2 + b^2).$$

Note that we have equality if and only if $a = b$ by the first equation. □

Exercise 6. *For $x \in \mathbf{R}$, show that $\sqrt{x^2} = |x|$.*

Proof. Observe the following maps:

$$\cdot^2 : \mathbf{R} \to \mathbf{R}+ \text{ defined by } x \mapsto x^2$$
$$\sqrt{\cdot} : \mathbf{R}^+ \to \mathbf{R}^+ \text{ defined by } x^2 \mapsto x.$$

Let $x \in \mathbf{R}^+$. Then $\sqrt{x^2} = x$. Now let $-x \in \mathbf{R}^+$. Then $\sqrt{(-x)^2} = -x$. Hence $\sqrt{x^2} = |x|$. □

Exercise 7. *Let $x, y, a, b \in \mathbf{R}$ and $\epsilon > 0$.*

(i) Show that $|x - a| < \epsilon$ if and only if $a - \epsilon < x < a + \epsilon$.

(ii) If $a < x < b$, then $|x| < \max\{|a|, |b|\}$.

(iii) If $a < x < b$ and $a < y < b$, show that $|x - y| < b - a$.

Proof. (i) By definition this is equal to $-\epsilon < x - a < \epsilon$, which is equivalent to $a - \epsilon < x < a + \epsilon$.

(ii) Let $a < x < b$. Suppose $x < 0$. Then $a < 0$. So $|x| = -x < -a = |a| \leqslant \max\{|a|, |b|\}$. Thus $|x| < |a|$. Now suppose $x > 0$. Then $b > 0$. So $|x| = x < b = |b| \leqslant \max\{|a|, |b|\}$. Hence $|x| < \max\{|a|, |b|\}$.

(iii) Note that $a < x < b$ is equivalent to $-b < -x < -a$. Hence $a - b < y - x < b - a$. Similarly, $a < y < b$ is equivalent to $-b < -y < -a$. Hence $a - b < x - y < b - a$. Thus $|x - y| < b - a$. □

Exercise 8. *Find all $x \in \mathbf{R}$ that satisfy*

$$4 < |x + 2| + |x - 1| < 5.$$

Proof. We proceed with cases. Case 1: $x + 2 \in \mathbf{R}^+$ and $x - 1 \in \mathbf{R}^+$. Then:

$$4 < x + 2 + x - 1 < 5,$$
$$4 < 2x - 1 < 5,$$
$$\frac{5}{2} < x < 3.$$

Case 2: $-(x + 2) \in \mathbf{R}^+$ and $x - 1 \in \mathbf{R}^+$. Then:

$$4 < -x - 2 + x + 1 < 5,$$
$$4 < -3 < 5,$$

which is a contradiction. Case 3: $x + 2 \in \mathbf{R}^+$ and $-(x - 1) \in \mathbf{R}^+$. Then:

$$4 < x + 2 - x + 1 < 5,$$
$$4 < 3 < 5,$$

which is a contradiction. Case 4: $-(x + 2) \in \mathbf{R}^+$ and $-(x - 1) \in \mathbf{R}^+$. Then:

$$4 < -x - 2 - x + 1 < 5,$$
$$5 < -2x < 6,$$
$$-5 > 2x > -6,$$
$$-\frac{5}{2} > x > -3.$$

. Thus $x \in (-3, -\frac{5}{2}) \cup (\frac{5}{2}, 3)$. □

Exercise 9. *Let $a, b \in \mathbf{R}$. Show that:*

$$\max\{a, b\} = \frac{1}{2}(a + b + |a - b|) \text{ and } \min\{a, b\} = \frac{1}{2}(a + b - |a - b|).$$

Proof. Without loss of generality, suppose $a \leqslant b$. Then $b - a \in \mathbf{R}^{+}$. Observe that:

$$\begin{aligned}
\max\{a, b\} = b &= \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}a + \frac{1}{2}b \\
&= \frac{1}{2}(a + b - a + b) \\
&= \frac{1}{2}(a + b + |a - b|).
\end{aligned}$$

$$\begin{aligned}
\min\{a, b\} = a &= \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}b \\
&= \frac{1}{2}(a + b + a - b) \\
&= \frac{1}{2}(a + b - (b - a)) \\
&= \frac{1}{2}(a + b - |a - b|).
\end{aligned}$$

□

Exercise 10. *If $x \neq y \in \mathbf{R}$, show that there is a $\delta > 0$ such that $V_\delta(x) \cap V_\delta(y) = \emptyset$.*

Proof. Without loss of generality suppose $x < y$. Pick $\delta = \frac{|x-y|}{3}$. Suppose towards contradiction $t \in V_\delta(x) \cap V_\delta(y)$. Then $t \in V_\delta(x)$ and $t \in V_\delta(y)$. Hence $t \in (x - \frac{|x-y|}{3}, x + \frac{|x-y|}{3})$ and $t \in (y - \frac{|x-y|}{3}, y + \frac{|x-y|}{3})$. But this gives:

$$t < x + \frac{|x - y|}{3} < y - \frac{|x - y|}{3} < t,$$

which is a contradiction. Hence $V_\delta(x) \cap V_\delta(y) = \emptyset$. □