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Last update: 2024 October 3

# 1

# Introduction

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out"

-Emil Artin

## 1.1 Basic Properties of Vector Spaces

**Definition 1.1.1.** Let $F$ be any field. Let $V$ be a nonempty set with binary operations:

$$\begin{aligned} V \times V &\longrightarrow B \\ (v, w) &\longmapsto v + w \end{aligned}$$

called *vector addition* and

$$\begin{aligned} F \times V &\longrightarrow V \\ (c, v) &\longmapsto cv \end{aligned}$$

called *scalar multiplication*. Then $V$ is an *$F$-vector space* if the following properties are satisfied:

(1) $V$ is an abelian group, that is:

  (i) there exists a $0_v \in V$ such that $0_v + v = v = v + 0v$,

  (ii) for every $v \in V$ there exists a $-v \in V$ such that $v + (-v) = 0_v = (-v) + v$,

  (iii) for every $u, v, w \in V$, $(u + v) + w = u + (w + v)$, and

  (iv) $v + w = w + v$ for all $v, w \in V$.

(2) $c(v + w) = cv + cw$ for all $c \in F$, $v, w \in V$,

(3) $(c + d)v = cv + dv$ for all $c, d \in F$, $v \in V$,

(4) $(cd)v = c(dv)$ for all $c, d \in F$, $v \in V$,

(5) there exists a $1_F \in F$ such that $1_F v = v$.

**Example 1.1.1.**

(1) Let $F$ be any field. Define $F^n = \{(a_1, \ldots, a_n) \mid a_i \in F\}$ as *affine $n$-space*. Then $F^n$ is an $F$-vector space.

(2) Let $n \in \mathbf{Z}_{\geqslant 0}$. Define $P_n(F) = \{a_0 + a_1 x + \ldots + a_n x^n \mid a_i \in F\}$. This is an $F$-vector space with polynomial addition and scalar multiplication. Define $F[x] = \bigcup_{n \geqslant 0} P_n(F)$. This is also an $F$-vector space, but either via polynomial addition <u>or</u> polynomial multiplication.

(3) Let $m, n \in \mathbf{Z}_{\geqslant 0}$. Set $V = \mathrm{Mat}_{n,m}(F) = \{\text{all } m \times n \text{ matrices with entries in } F\}$. This is an $F$-vector space with matrix addition and scalar mutliplication. If $m = n$ then write $\mathrm{Mat}_n(F)$ for $\mathrm{Mat}_{n,n}(F)$.

**Lemma 1.1.1.** *Let $V$ be an $F$-vector space.*

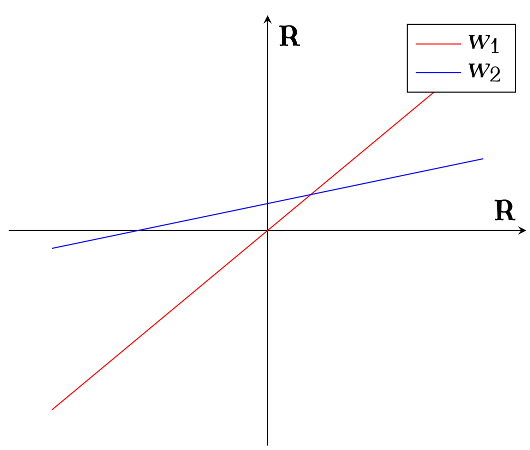
1. *The element $0_v \in V$ is unique,*
2. *$0v = 0_v$ for all $v \in V$,*
3. *$(-1_F)v = -v$ for all $v \in V$.*

*Proof.* (1) Let $0, 0'$ satisfy the following properties: $0 + v = v$ and $0' + v = v$ for all $v \in V$. Observe that $0 = 0' + 0 = 0 + 0' = 0'$. (2) Note that $0_F v = (0_F + 0_F)v = 0_F v + 0_F v$. Subtracting both sides by $0_F v$ yields $0 = 0_F v$. (3) Observe that $(-1_F)v + v = (-1_F)v + 1_F v = (-1_F + 1_F)v = 0_F v = 0$. Hence $(-1_F)v = -v$. $\square$

**Definition 1.1.2.** Let $V$ be an $F$-vector space. We say $W \subseteq V$ is an *F-subspace* (or just *subspace* if $F$ is obvious by context) if $W$ is an $F$-vector space under the same addition and scalar multiplication.

**Example 1.1.2.**

(1) Consider the plane $V = \mathbf{R}^2$. Let $w_1, w_2$ be subsets of $\mathbf{R}^2$ as follows:




Note that $w_2$ is not a subspace, as it does not contain $0_{\mathbf{R}^2}$. On the other hand $w_1$ is a subspace; note that every element of $w_1$ is of the form $(x, ax)$, hence $(x_1, ax_1) + (x_2, ax_2) = (x_1 + x_2, a(x_1 + x_2))$. The other axioms follow similarly.

(2) Let $V = \mathbf{C}$ and $W = \{a + 0i \mid a \in \mathbf{R}\}$. If $F = \mathbf{R}$, then clearly $W$ is an $\mathbf{R}$-subspace. If $F = \mathbf{C}$, then $W$ is not a $\mathbf{C}$-subspace; given $2 \in W$ and $i \in \mathbf{C}$, $2i \notin W$.

(3) $\mathrm{Mat}_2(\mathbf{R})$ is not a subspace of $\mathrm{Mat}_4(\mathbf{R})$, as $\mathrm{Mat}_2(\mathbf{R}) \nsubseteq \mathrm{Mat}_4(\mathbf{R})$.

(4) Let $m, n \in \mathbf{Z}_{\geqslant 0}$. If $m \leqslant n$, then $P_m(F)$ is a subspace of $P_n(F)$.

**Lemma 1.1.2.** *Let $V$ be an $F$-vector space and $W \subseteq V$. Then $W$ is an $F$-subspace of $V$ if:*

*(1) $W$ is nonempty,*

*(2) $W$ is closed under addition, and*

*(3) $W$ is closed under scalar multiplication.*

*Proof.* Let $x, y \in W$ and $\alpha \in F$, then by assumption $x + \alpha y \in W$. Take $\alpha = -1$, then $x - y \in W$ which implies $W$ is an abelian subgroup of $V$. Then by (3) it must be the case that $W$ is an $F$-subspace of $V$. $\square$

**Definition 1.1.3.** Let $V, W$ be $F$-vector spaces. Let $T : V \to W$. We say $T$ is a *linear transformation* (or *linear map*) if for every $v_1, v_2 \in V$ and $c \in F$ we have

$$T(v_1 + cv_2) = T(v_1) + cT(v_2).$$

The collection of all linear maps from $V$ to $W$ is denoted $\mathrm{Hom}_F(V, W)$ (some textbooks write this as $\mathcal{L}(V, W)$).

**Example 1.1.3.**

(1) Let $V$ be an $F$-vector space. Define $\mathrm{id}_v : V \to V$ by $\mathrm{id}_v(v) = v$. This is a linear map; i.e., $\mathrm{id}_v \in \mathrm{Hom}_F(V, V)$ because $\mathrm{id}_v(v_1 + cv_2) = v_1 + cv_2 = \mathrm{id}_v(v_1) + c\,\mathrm{id}_v(v_2)$.

(2) Let $V = \mathbf{C}$. Define $T : V \to V$ by $z \mapsto \overline{z}$. Observe that:

$$\begin{aligned} T(z_1 + cz_2) &= \overline{z_1 + cz_2} = \overline{z_1} + \overline{c}\,\overline{z_2} \\ T(z_1) + cT(z_2) &= \overline{z_1} + c\,\overline{z_2}. \end{aligned}$$

Note that these two are only equal if $c = \overline{c}$. Hence $T \in \mathrm{Hom}_F(\mathbf{C}, \mathbf{C})$ if $F = \mathbf{R}$ but not if $F = \mathbf{C}$.

(3) Let $A \in \mathrm{Mat}_{m,n}(F)$. Define $T_A : F^n \to F^m$ by $x \mapsto Ax$. Then $T_A \in \mathrm{Hom}_F(F^n, F^m)$.

(4) Recall that $C^\infty(\mathbf{R})$ is the set of all smooth functions $f : \mathbf{R} \to \mathbf{R}$ (another way of saying "smooth" is "infinitely differentiable"). Let $V = C^\infty(\mathbf{R})$. This is an $\mathbf{R}$-vector space under pointwise addition and scalar multiplication. If $a \in \mathbf{R}$ then:

- $E_a : V \to \mathbf{R}$ defined by $f \mapsto f(a)$ is an element of $\mathrm{Hom}_{\mathbf{R}}(V, \mathbf{R})$,
- $D : V \to V$ defined by $f \mapsto f'$ is an element of $\mathrm{Hom}_{\mathbf{R}}(V, V)$,
- $I_a : V \to V$ defined by $f \mapsto \int_a^x f(t)dt$ is an element of $\mathrm{Hom}_{\mathbf{R}}(V, V)$, and
- $\tilde{E}_a : V \to V$ defined by $f \mapsto f(a)$ (where $f(a)$ is the constant function) is an element of $\mathrm{Hom}_{\mathbf{R}}(V, V)$.

From this, we can express the fundamental theorem of calculus as follows:

$$\begin{aligned} D \circ I_a &= \mathrm{id}_v \\ I_a \circ D &= \mathrm{id}_v - \tilde{E}_a. \end{aligned}$$

**Proposition 1.1.1.** $\mathrm{Hom}_F(V, W)$ *is an $F$-vector space.*

*Proof.* do this □

**Lemma 1.1.3.** *Let* $T \in \mathrm{Hom}_F(V, W)$*. Then* $T(0_v) = 0_w$*.*

*Proof.* do this □

**Definition 1.1.4.** Let $T \in \mathrm{Hom}_F(V, W)$ be invertible; i.e., there exists a linear transformation $T^{-1} : W \to V$ such that $T \circ T^{-1} = \mathrm{id}_w$ and $T^{-1} \circ T = \mathrm{id}_v$. If this is the case we say $T$ is an *isomorphism* and say $V$ and $W$ are *isomorphic*, written as $V \cong W$.

**Proposition 1.1.2.** *Let* $T \in \mathrm{Hom}_F(V, W)$ *be an isomorphism. Then* $T^{-1} \in \mathrm{Hom}_F(W, V)$*.*

*Proof.* do this □

**Example 1.1.4.**

(1) Let $V = \mathbf{R}^2$ and $W = \mathbf{C}$. Define $T : \mathbf{R}^2 \to \mathbf{C}$ by $(x, y) \mapsto x + iy$. This is an isomorphism: note that $T \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^2, \mathbf{C})$ because

$$\begin{aligned} T((x_1, y_1) + r(x_2, y_2)) &= \text{...fill this out} \\ &= T((x_1, y_1)) + rT((x_2, y_2)). \end{aligned}$$

Defining $T^{-1} : \mathbf{C} \to \mathbf{R}^2$ by $x + iy \mapsto (x, y)$ (and showing it's linear) clearly satisfies $(T \circ T^{-1})(x + iy) = x + iy$ and $(T^{-1} \circ T)((x, y)) = (x, y)$, hence $\mathbf{R}^2 \cong \mathbf{C}$ as $\mathbf{R}$-vector spaces.

(2) Set $V = P_n(F)$ and $W = F^{n+1}$. Define $T : P_n(F) \to F^{n+1}$ by

$$a_0 + a_1 x + \ldots + a_n x^n \mapsto (a_0, a_1, \ldots, a_n).$$

This is an isomorphism; $P_n(F) \cong F^{n+1}$.

**Definition 1.1.5.** Let $T \in \mathrm{Hom}_F(V, W)$. Define the *kernel* of $T$ as:

(1) The *kernel of T* is defined as $\ker(T) = \{v \in V \mid T(v) = 0_W\}$.

(2) The *image of T* is defined as $\mathrm{im}(T) = \{w \in W \mid T(v) = w \text{ for some } v \in V\}$.

**Lemma 1.1.4.** *Let* $T \in \mathrm{Hom}_F(V, W)$*. Then:*

*(1)* $\ker(T)$ *is a subspace of* $V$*,*

*(2)* $\mathrm{im}(T)$ *is a subspace of* $W$*.*

*Proof.* Let $v_1, v_2 \in \ker(T)$ and $\alpha \in F$. Observe that $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2) = 0_w + \alpha 0_w = 0_w$, hence $v_1 + \alpha v_2 \in \ker(T)$ establishing $\ker(T)$ as a subspace of $V$.

Let $w_1, w_2 \in \mathrm{im}(T)$ and $\alpha \in F$. Then there exists $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Observe that $w_1 + \alpha w_2 = T(v_1) + \alpha T(v_2) = T(v_1 + \alpha v_2)$, hence $w_1 + \alpha w_2 \in \mathrm{im}(T)$ establishing $\mathrm{im}(T)$ as a subspace of $W$. □

**Lemma 1.1.5.** *Let* $T \in \operatorname{Hom}_F(V, W)$. $T$ *is injective if and only if* $\ker(T) = \{0_V\}$

*Proof.* Let $T$ be injective. Let $v \in \ker(T)$. Then $T(v) = 0_W = T(0_V)$, and since $T$ is injective $v = 0_V$.

Conversely, assume $\ker(T) = 0_V$. Let $v_1, v_2 \in V$ with $T(v_1) = T(v_2)$. Subtracting both sides by $T(v_2)$ gives $T(v_1) - T(v_2) = 0_W$, and since $T$ is a linear transformation yields $T(v_1 - v_2) = 0_W$. Since the kernel is trivial, it must be the case that $v_1 = v_2$. □

**Example 1.1.5.** Let $m > n$. Define $T : F^m \to F^n$ by

$$(a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_m) \mapsto (a_0, a_1, \ldots, a_n)$$

Then $\operatorname{im}(T) = F^n$ and $\ker(T) = \{(0, \ldots, 0, a_{n+1}, a_{n+2}, \ldots, a_m) \in F^m\} \cong F^{m-n}$.

# 2

# Bases and Dimension

## 2.1 Basic Definitions

Unless otherwise stated assume $V$ to be an $F$-vector space.

**Definition 2.1.1.** Let $\mathcal{B} = \{v_i\}_{i\in I}$ be a subset of $V$ where $I$ is an indexing set (possibly infinite). We say $v \in V$ is an *F-linear combination of* $\mathcal{B}$ (or just *linear combination*) if there is a set $\{a_i\}_{i\in I}$ with $a_i = 0$ for all but finitely many $i$ such that $v = \sum_{i\in I} a_i v_i$. The collection of $F$-linear combinations is denoted $\operatorname{span}_F(\mathcal{B})$.

**Example 2.1.1.** Let $V = P_2(F)$.

(1) Set $\mathcal{B} = \{1, x, x^2\}$. We have $\operatorname{span}_F(\mathcal{B}) = P_2(F)$.

(2) Set $\mathcal{C} = \{1, (x-1), (x-1)^2\}$. We have $\operatorname{span}_F(\mathcal{C}) = P_2(F)$.

**Definition 2.1.2.** Let $\mathcal{B} = \{v_i\}_{i\in I}$ be a subset of $V$. We say $\mathcal{B}$ is *F-linearly independent* (or just *linearly independent*) if whenever $\sum_{i\in I} a_i v_i = 0$ then $a_i = 0$ for all $i \in I$.

**Definition 2.1.3.** Let $\mathcal{B} = \{v_i\}_{i\in I}$ be a subset of $V$. We say $\mathcal{B}$ is an *F-basis* (or just *basis*) of $V$ if:

- $\operatorname{span}_F(\mathcal{B}) = V$, and
- $\mathcal{B}$ is linearly independent.

**Example 2.1.2.** Let $V = F^n$. Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ with

$$\begin{aligned} e_1 &= (1,0,0,\dots,0) \\ e_2 &= (0,1,0,\dots,0) \\ &\vdots \\ e_n &= (0,0,0,\dots,1). \end{aligned}$$

We have that $\mathcal{E}_n$ is a basis of $F^n$ and is referred to as the *standard basis*.

## 2.2 Every Vector Space Admits a Basis

**Definition 2.2.1.** A *relation* from $A$ to $B$ is a subset $R \subseteq A \times B$. Typically, when one says "a relation on $A$" that means a relation from $A$ to $A$; i.e., $R \subseteq A \times A$.

**Definition 2.2.2.** Let $A$ be a set. An *ordering* of $A$ is a relation $R$ on $A$ that is

(1) *reflexive*: $(a,a) \in R$ for all $a \in A$,

(2) *transitive*: $(a,b),(b,c) \in R$ implies $(a,c) \in R$, and

(3) *antisymmetric*: $(a,b),(b,a) \in R$ implies $a = b$.

If this is the case, we write $(a,b) \in R$ as $a \leqslant_R b$. If $A$ is an ordered set we write it as the ordered pair $(A, \leqslant_R)$ (or just $A$ if the ordering is obvious by context).

**Definition 2.2.3.** An ordered set $(X, \leqslant_R)$ is *total* if for all $a, b \in X$ we have that $a \leqslant_R b$ or $b \leqslant_R a$.

**Definition 2.2.4.** Let $(X, \leqslant)$ be an ordered set and $A \subseteq X$ nonempty.

(1) $A$ is called a *chain* if $(A, \leqslant)$ is a total ordering.

(2) $A$ is called *bounded above* if there exists an element $u \in X$ with $a \leqslant u$ for all $a \in A$. Such a $u$ is called an *upperbound* for $A$.

(3) A *maximal element of A* is an element $m \in A$ such that if $a \geqslant m$, then $a = m$.

**Lemma 2.2.1** (Zorn's Lemma). *Let $X$ be an ordered set with the property that every chain has an upperbound. Then $X$ contains a maximal element.*

**Theorem 2.2.1.** *Let $\mathcal{A}$ and $\mathcal{C}$ be subsets of $V$ with $\mathcal{A} \subseteq \mathcal{C}$. Assume $\mathcal{A}$ is linearly independent and $\operatorname{span}_F(\mathcal{C}) = V$. Then there exists a basis $\mathcal{B}$ of $V$ with $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$*[1].

*Proof.* Let $X = \{\mathcal{B}' \subseteq V \mid \mathcal{A} \subseteq \mathcal{B}' \subseteq \mathcal{C},\ \mathcal{B}' \text{ is linearly independent}\}$. We have $\mathcal{A} \in X$, so $X \neq \emptyset$. $X$ is ordered with respect to inclusion, and has an upperbound of $\mathcal{C}$. By Zorn's Lemma we have a maximal element in $X$, call it $\mathcal{B}$.

Claim: $\operatorname{span}_F(\mathcal{B}) = V$. Suppose towards contradiction it's not, then there exists a $v \in \mathcal{C}$ with $v \notin \operatorname{span}_F(\mathcal{B})$. But then $\mathcal{B} \cup \{v\}$ is still linearly independent, and $\mathcal{B} \cup \{v\} \subseteq \mathcal{C}$. This gives $\mathcal{B} \subsetneq \mathcal{B} \cup \{v\}$, which is a contradiction because $\mathcal{B}$ is maximal in $X$. Thus $\operatorname{span}_F(\mathcal{B}) = V$. □

## 2.3 Cardinality and Dimension

**Lemma 2.3.1.** *A homogenous system of $m$ linear equations in $n$ unknowns with $m < n$ has a nonzero solution.*

*Proof.* do this □

**Corollary 2.3.1.** *Let $\mathcal{B} \subseteq V$ with $\operatorname{span}_F(\mathcal{B}) = V$ and $|\mathcal{B}| = m$. Any set with more than $m$ elements cannot be linearly independent.*

[1]Given any linearly-independent set $\mathcal{A}$, we can constructing a basis $\mathcal{B}$ by adding elements. Given any spanning set $\mathcal{C}$, we can construct a basis $\mathcal{B}$ by removing elements.

*Proof.* Let $\mathcal{C} = \{w_1, \dots, w_n\}$ with $n > m$. We will show $\mathcal{C}$ cannot be linearly independent. Write $\mathcal{B} = \{v_1, \dots, v_m\}$ with $\operatorname{span}_F(\mathcal{B}) = V$. For each $i$, write

$$w_i = \sum_{j=1}^{m} a_{ji} v_j \text{ for some } a_{ji} \in F.$$

Consider the equations

$$\sum_{i=1}^{n} a_{ji} x_i = 0.$$

By Lemma 2.3.1 there exists nonzero solutions $(x_1, \dots, x_n) = (c_1, \dots, c_n) \neq (0, \dots, 0)$. We have

$$\begin{aligned} 0 &= \sum_{j=1}^{m} \left( \sum_{i=1}^{n} a_{ji} c_i \right) v_j \\ &= \sum_{i=1}^{n} c_i \left( \sum_{j=1}^{m} a_{ji} v_j \right) \\ &= \sum_{i=1}^{n} c_i w_i. \end{aligned}$$

Thus $\mathcal{C} = \{w_1, \dots, w_n\}$ is not linearly independent. □

**Corollary 2.3.2.** *If $\mathcal{B}$ and $\mathcal{C}$ are both finite bases of $V$, then $|\mathcal{B}| = |\mathcal{C}|$.*

*Proof.* Let $|\mathcal{B}| = m$ and $|\mathcal{C}| = n$. Because $\operatorname{span}_F(\mathcal{B}) = V$ and $\mathcal{C}$ is linearly independent, it must be the case that $n \leqslant m$. But since $\operatorname{span}_F(\mathcal{C}) = V$ and $\mathcal{B}$ is also linearly independent, it must be the case that $m \leqslant n$. By antisymmetry, $n = m$. □

**Definition 2.3.1.** Let $\mathcal{B}$ be a basis of $V$. The <u>*dimension*</u> of $V$, written $\dim_F(V)$, is the cardinality of $\mathcal{B}$; i.e., $\dim_F(V) = |\mathcal{B}|$.

**Theorem 2.3.1.** *Let $V$ be a finite dimensional vector space with $\dim_F(V) = n$. Let $\mathcal{C} \subseteq V$ with $|\mathcal{C}| = m$.*

*(1) If $m > n$, then $\mathcal{C}$ is not linearly independent.*

*(2) If $m < n$, then $\operatorname{span}_F(\mathcal{C}) \neq V$.*

*(3) If $m = n$, then the following are equivalent:*

- *$\mathcal{C}$ is a basis;*
- *$\mathcal{C}$ is linearly independent;*
- $\operatorname{span}_F(\mathcal{C}) = V$.

**Corollary 2.3.3.** *Let $W \subseteq V$ be a subspace. We have $\dim_F(W) \leqslant \dim_F(V)$. If $\dim_F(V) < \infty$, then $V = W$ if and only if $\dim_F(V) = \dim_F(W)$.*

**Example 2.3.1.** Let $V = \mathbf{C}$.

(1) If $F = \mathbf{C}$, then $\mathscr{B} = \{1\}$ is a basis and $\dim_{\mathbf{C}}(\mathbf{C}) = 1$.

(2) If $F = \mathbf{R}$, then $\mathscr{B} = \{1, i\}$ is a basis and $\dim_{\mathbf{R}}(\mathbf{C}) = 2$.

(3) If $F = \mathbf{Q}$, then $|\mathscr{B}| = \mathfrak{c}$ and $\dim_{\mathbf{Q}}(\mathbf{C}) = \mathfrak{c}$ (the *continuum*).

**Example 2.3.2.** Let $V = F[x]$ and let $f(x) \in F[x]$. We can use this polynomial to split $F[x]$ into equivalence classes analogous to how one creates the field $\mathbf{F}_p$. Define $g(x)\ h(x)$ if $f(x) \mid (g(x) - h(x))$. This is an equivalence relation. We let $[g(x)]$ denote the equivalence class containing $g(x) \in F[x]$. Let $F[x]/(f(x)) = \{[g(x)] \mid g(x) \in F[x]\}$ denote the collection of equivalence classes. Define $[g(x)] + [h(x)] = [g(x) + h(x)]$ and $\alpha[g(x)] = [\alpha g(x)]$, this makes $F[x]/(f(x))$ into a vector space.

Set $n = \deg(f(x))$. Let $\mathscr{B} = \{[1], [x], \ldots, [x^{n-1}]\}$. We will show this is a basis for $F[x]/(f(x))$. Suppose there exists $a_0, \ldots, a_{n-1} \in F$ with $a_0[1] + a_1[x] + \ldots + a_{n-1}[x^{n-1}] = [0]$. So $[a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}] = [0]$, hence $f(x) \mid (a_0 + a_1 x + \ldots + a_{n-1}x^{n-1})$. But $\deg(f(x)) = n$, so we must have $a_0 = a_1 = \ldots = 0$ (linear independence).

Let $[g(x)] \in F[x]/(f(x))$. The Euclidean algorithm of polynomials gives $g(x) = f(x)q(x) + r(x)$ for some $q(x), r(x) \in F[x]/(f(x))$ with $r(x) = 0$ or $\deg(r(x)) \leqslant \deg(g(x))$. Observe that $[g(x)] = [f(x)q(x) + r(x)] = [f(x)q(x)] + [r(x)] = [r(x)]$. Since $[r(x)]$ can be written as a linear combination of basis elements from $\mathscr{B}$, we have $[g(x)] \in \operatorname{span}_F(\mathscr{B})$. Note that any element of $\operatorname{span}_F(\mathscr{B})$ is clearly contained in $F[x]/(f(x))$, establishing $\operatorname{span}_F(\mathscr{B}) = F[x]/(f(x))$.

**Lemma 2.3.2.** *Let $V$ be an $F$-vector space and $\mathscr{C} = \{v_i\}_{i \in I}$ be a subset of $V$. Then $\mathscr{C}$ is a basis if and only if each $v \in V$ can be written uniquely as a linear combination of elements of $\mathscr{C}$.*

*Proof.* Suppose $\mathscr{C}$ is a basis. Let $v \in V$ and suppose

$$v = \sum_{i \in I} a_i v_i = \sum_{i \in I} b_i v_i,$$

for some $a_i, b_i \in F$. Observe that:

$$0_v = \sum_{i \in I} (a_i - b_i) v_i.$$

Since $\mathscr{C}$ is a basis, it is linearly independent, so $a_i - b_i = 0$ for all $i$. Thus $a_i = b_i$ for all $i$ establishing that the expansion is unique.

Conversely, suppose every vector $v \in V$ is a unique linear combination of $\mathscr{C}$. Certainly we have $\operatorname{span}_F(\mathscr{C}) = V$. Suppose $0_v = \sum_{i \in I} a_i v_i$ for some $a_i \in F$. We also have that $0_v = \sum_{i \in I} 0 \cdot v_i$. Uniqueness gives $a_i = 0$ for all $i \in I$; i.e., $\mathscr{C}$ is linearly independent. $\square$

**Proposition 2.3.1.** *Let $V, W$ be $F$-vector spaces.*

*(1) Let $T \in \operatorname{Hom}_F(V, W)$. We have that $T$ is determined by what it does to a basis (where it maps it).*

*(2) Let $\mathscr{B} = \{v_i\}_{i \in I}$ be a basis of $V$ and $\mathscr{C} = \{w_i\}_{i \in I}$ be a subset of $V$. If $|\mathscr{B}| = |\mathscr{C}|$, there is a $T \in \operatorname{Hom}_F(V, W)$ such that $T(v_i) = w_i$ for all $i \in I$.*

*Proof.* (1) Let $v \in V$. Let $\mathcal{B} = \{v_i\}_{i\in I}$ be a basis of $V$ and write $v = \sum_{i\in I} a_i v_i$. We have $T(v) = T(\sum_{i\in I} a_i v_i) = \sum_{i\in I} a_i T(v_i)$.

(2) Define $T : V \to W$ by $v \mapsto \sum_{i\in I} a_i w_i$. If $v = \sum_{i\in I} a_i v_i$ this map is linear (show this). □

**Corollary 2.3.4.** *Let $T \in \mathrm{Hom}_F(V, W)$ with $\mathcal{B} = \{v_i\}_{i\in I}$ a basis of $V$ and $\mathcal{C} = \{w_i = T(v_i)\}_{i\in I}$ a subset of $W$. We have $\mathcal{C}$ is a basis of $W$ if and only if $T$ is an isomorphism.*

*Proof.* Suppose $\mathcal{C}$ is a basis of $W$. Using the result from Proposition 2.3.1, define $S \in \mathrm{Hom}_F(W, V)$ with $S(w_i) = v_i$. Check $T \circ S = \mathrm{id}_W$ and $S \circ T = \mathrm{id}_V$. Thus $T$ is an isomorphism.

Conversely, let $T$ be an isomorphism. Let $w \in W$. As $T$ is surjective, there exists a $v \in V$ such that $T(v) = w$. Using $\mathcal{B}$ as a basis of $V$, write $v = \sum_{i\in I} a_i v_i$. So observe that:

$$w = T(v) = T\left(\sum_{i\in I} a_i v_i\right) = \sum_{i\in I} a_i T(v_i) \in \mathrm{span}_F(\mathcal{C}),$$

hence $W = \mathrm{span}_F(\mathcal{C})$ (note the other direction is trivial —you never need to show that). Now suppose there exists a collection of elements $a_i \in F$ with $\sum_{i\in I} a_i T(v_i) = 0_W$. Since $T$ is linear, this is equivalent to $T(\sum_{i\in I} a_i v_i) = 0_W$, and since $T$ is injective it must be the case that $\sum_{i\in I} a_i v_i = 0_V$. Since $\mathcal{B}$ is a basis we get $a_i = 0$ for all $i \in I$, establishing that $\mathcal{C}$ is linearly independent. □

**Theorem 2.3.2** (Rank-Nullity Theorem). *Let $V$ be an $F$-vector space with $\dim_F(V) < \infty$. Then:*

$$\dim_F(V) = \dim_F(\ker(T)) + \dim_F(\mathrm{im}(T)).$$

*Proof.* Let $\dim_F(\ker(T)) = k$ and $\dim_F(V) = n$. Let $\mathcal{A} = \{v_1, \dots, v_k\}$ be a basis of $\ker(T)$. Extend this to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of $V$. We'd like to show that $\mathcal{C} = \{T(v_{k+1}), \dots, T(v_n)\}$ is a basis of $\mathrm{im}(T)$.

Let $w \in \mathrm{im}(T)$. So there exists a $v \in V$ with $T(v) = w$. Write $v = \sum_{i=1}^n a_i v_i$. We have:

$$\begin{aligned} w &= T(v) \\ &= T\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i T(v_i) \\ &= \sum_{i=k+1}^n a_i T(v_i) \in \mathrm{span}_F(\mathcal{C}). && \text{b/c } v_1, \dots, v_k \in \ker(T) \end{aligned}$$

Thus $\mathrm{span}_F(\mathcal{C}) = \mathrm{im}(T)$. Now suppose we have $\sum_{i=k+1}^n a_i T(v_i) = 0_W$. Since $T$ is linear we have $T(\sum_{i=1}^n a_i v_i) = 0_w$, which gives $\sum_{i=1}^n a_i v_i \in \ker(T)$. Thus we can write it in terms of the basis $\mathcal{A}$ of $\ker(T)$: there exists $a_1, \dots, a_k$ such that

$$\sum_{i=k+1}^n a_i v_i = \sum_{i=1}^k a_i v_i,$$

which is equivalent to $\sum_{i=1}^k a_i v_i + \sum_{i=k+1}^n a_i v_i = 0_V$. However, $\mathcal{B}$ is a basis of $V$ so $a_1 = \dots = a_n = 0$. □

**Corollary 2.3.5.** *Let $V, W$ be $F$-vector spaces with $\dim_F(V) = n$. Let $V_1 \subseteq V$ be a subspace with $\dim_F(V_1) = k$ and $W_1 \subseteq W$ a subspace with $\dim_F(W_1) = n - k$. Then there exists a $T \in \operatorname{Hom}_F(V, W)$ such that $\ker(T) = V_1$ and $\operatorname{im}(T) = W_1$.*

*Proof.* do it □

**Corollary 2.3.6.** *Let $T \in \operatorname{Hom}_F(V, W)$ with $\dim_F(V) = \dim_F(W) < \infty$. The following are equivalent:*

*(1) $T$ is an isomorphism.*

*(2) $T$ is injective.*

*(3) $T$ is surjective.*

*Proof.* do it □

**Corollary 2.3.7.** *Let $A = \operatorname{Mat}_n(F)$. The following are equivalent:*

*(1) $A$ is invertible.*

*(2) There exists an element $B \in \operatorname{Mat}_n(F)$ such that $BA = 1_n$.*

*(3) There exists an element $B \in \operatorname{Mat}_n(F)$ such that $AB = 1_n$.*

*Proof.* do it □

**Corollary 2.3.8.** *Let $\dim_F(V) = m$ and $\dim_F(W) = n$.*

*(1) If $m < n$ and $T \in \operatorname{Hom}_F(V, W)$, then $T$ is not surjective.*

*(2) If $m > n$ and $T \in \operatorname{Hom}_F(V, W)$, then $T$ is not injective.*

*(3) If $m = n$ then $V \cong W$.*

**Example 2.3.3.** This follows shortly after corollary 2.2.30 (write it down later)

## 2.4 Direct Sums and Quotient Spaces

**Definition 2.4.1.** Let $V$ be an $F$-vector space and $V_1, \dots, V_k$ be subspaces. The <u>sum</u> of $V_1, \dots, V_k$ is

$$V_1 + \dots + V_k = \{v_1 + \dots + v_k \mid v_i \in V_i\}.$$

**Proposition 2.4.1.** *Let $V$ be an $F$-vector space and $V_1, \dots, V_k$ be subspaces. Then $V_1 + \dots + V_k$ is also a subspace of $V$.*

*Proof.* do this □

**Definition 2.4.2.** Let $V_1, \ldots, V_k$ be subspaces of $V$. We say $V_1, \ldots, V_k$ are <u>*independent*</u> if whenever $v_1 + \ldots + v_k = 0_V$ then $v_i = 0_V$.

**Definition 2.4.3.** Let $V_1, \ldots, V_k$ be subspaces of $V$. We say $V$ is the <u>*direct sum*</u> of $V_1, \ldots, V_k$ and write $V = V_1 \oplus \ldots \oplus V_k$ if:

(1) $V = V_1 + \ldots + V_k$, and

(2) $V_1, \ldots, V_k$ are independent.

**Example 2.4.1.**

(1) Let $V = F^2$ with $V_1 = \{(x, 0) \mid x \in F\}$ and $V_2 = \{(0, y) \mid y \in F\}$. Then

$$\begin{aligned} V_1 + V_2 &= \{(x,0) + (0,y) \mid x, y \in F\} \\ &= \{(x,y) \mid x, y \in F\} \\ &= V \end{aligned}$$

If $(x, 0) + (y, 0) = (0, 0)$, then $x = y = 0$ which means $V_1$ and $V_2$ are independent. Hence $F^2 = V_1 \oplus V_2$.

(2) Let $V = F[x]$ and $V_1 = F$, $V_2 = Fx = \{\alpha x \mid \alpha \in F\}$, and $V_3 = P_1(F)$. Note that $P_1(F) = V_1 \oplus V_2$. But $V_1, V_3$ are not independent because $1_F \in V_1$ and $-1_F \in V_3$ and $(-1_F) + 1_F = 0$.

(3) Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of $V$ and $\operatorname{span}_F(v_i) = V_i$. Then $V = V_1 \oplus \ldots \oplus V_n$.

**Lemma 2.4.1.** *Let $V$ be an $F$-vector space with $V_1, \ldots, V_k$ as subspaces. We have $V = V_1 \oplus \ldots \oplus V_k$ if and only if every $v \in V$ can be written uniquely in the form $v = v_1 + \ldots + v_k$ for all $v_i \in V_i$.*

*Proof.* Suppose $V = V_1 \oplus \ldots \oplus V_k$. Let $v \in V$. Suppose $v = v_1 + \ldots + v_k = \tilde{v}_1 + \ldots + \tilde{v}_k$ for $v_i, \tilde{v}_i \in V_i$. Then $0_V = (v_1 - \tilde{v}_1) + \ldots + (v_k - \tilde{v}_k)$. Since $V_1, \ldots, V_k$ are independent and $v_i - \tilde{v}_i \in V$, this gives $v_i - \tilde{v}_i = 0_V$ for all $i$. Thus the expansion for $V$ is unique.

Conversely, suppose every $v \in V$ can be written uniquely in the form $v = v_1 + \ldots + v_k$ with $v_i \in V_i$. Then $V = V_1 + \ldots + V_k$ by definition of sums of subspaces. If $0_V = v_1 + \ldots + v_k$ for some $v_i \in V_i$, and $0_v = 0_v + \ldots + 0_v$, then (by uniqueness) it must be the case that $v_i = 0_V$ for all $i$. □

**Exercise 2.4.1.** *Let $V_1, \ldots, V_k$ be subspaces of $V$. For each $1 \leqslant i \leqslant k$, let $\mathcal{B}_i$ be a basis of $V_i$. Let $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}_i$. Show that:*

*(1) $\mathcal{B}$ spans $V$ if and only if $V = V_1 + \ldots + V_k$.*

*(2) $\mathcal{B}$ is linearly independent if and only if $V_1, \ldots, V_k$ are independent.*

*(3) $\mathcal{B}$ is a basis if and only if $V = V_1 \oplus \ldots \oplus V_k$.*

*Proof.* do this shit □

**Lemma 2.4.2.** *Let $U \subseteq V$ be a subspace. Then $U$ has a complement.*

*Proof.* do this shit □

**Definition 2.4.4.** Let $W \subseteq V$ be a subsapce. Define $v_1\ v_2$ if $v_1 - v_2 \in W$ for some $v_1, v_2 \in V$. This forms an equivalence relation. Denote the equivalence class containing $v$ as $[v]_W = v + W = \{\tilde{v} \in V \mid v\ \tilde{v}\} = \{v + w \mid w \in W\}$. The set containing all equivalence classes over $W$ is denoted $V/W = \{v + W \mid v \in V\}$.

**Proposition 2.4.2.** *Let $v_1 + W, v_2 + W \in V/W$ and $\alpha \in F$. With addition and scalar multiplication defined as follows:*

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ \alpha(v_1 + W) &= \alpha v_1 + W,\end{aligned}$$

*it's operations are well-defined and $V/W$ forms an $F$-vector space.*

*Proof.* Let $v_1 + W = \tilde{v_1} + W$ and $v_2 + W = \tilde{v_2} + W$. Then $v_1 = \tilde{v_1} + w_1$ and $v_2 = \tilde{v_2} + w_2$ for some $w_1, w_2 \in W$. Observe that:

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2 + W) \\ &= (\tilde{v_1} + w_2 + \tilde{v_2} + w_2) + W \\ &= (\tilde{v_1} + \tilde{v_2}) + W \\ &= (\tilde{v_1} + W) + (\tilde{v_2} + W).\end{aligned}$$

$$\begin{aligned}c(v_1 + W) &= cv_1 + W \\ &= c(\tilde{v_1} + w) + W \\ &= c\tilde{v_1} + W \\ &= c(\tilde{v_1} + W).\end{aligned}$$

Hence addition and scalar multiplication are well-defined. show the vector space axioms here. □

**Example 2.4.2.** Let $V = \mathbf{R}^2$ and $W = \{(x, 0) \mid x \in \mathbf{R}\}$. Let $(x_0, y_0) \in V$. We have that $(x_0, y_0) \sim (x, y)$ if $(x_0, y_0) - (x, y) = (x_0 - x, y_0 - y) \in W$. So $(x_0, y_0) + W = \{(x, y_0) \mid x \in \mathbf{R}\}$. Then $V/W$ is a vector space only when $y = 0$.

Define $\tau : \mathbf{R} \to V/W$ by $y \mapsto (0, y) + W$. This is an isomorphism. Let $y_1, y_2, c \in \mathbf{R}$. Observe that:

$$\begin{aligned}\tau(y_1 + cy_2) &= (0, y_1 + cy_2) + W \\ &= ((0, y_1) + (0, cy_2)) + W \\ &= ((0, y_1) + c(0, y_2)) + W \\ &= ((0, y_1) + W) + c((0, y_2) + W) \\ &= \tau(y_1) + c\tau(y_2).\end{aligned}$$

Hence $\tau \in \operatorname{Hom}_F(\mathbf{R}, V/W)$. Let $(x, y) + W \in V/W$. Then $(x, y) + W = (0, y) + W$. So $\tau$ is surjective because $\tau(y) = (0, y) + W$. Now let $y \in \ker(\tau)$. Then $\tau(y) = (0, y) + W = (0, 0) + W$. This implies $y = 0$, meaning the kernel is trivial and so $\tau$ is injective.

Alternatively, it is routine to show that $\tau^{-1} \in \operatorname{Hom}_F(V/W, \mathbf{R})$ with $\tau^{-1} \circ \tau = \mathrm{id}_{\mathbf{R}}$ and $\tau \circ \tau^{-1} = \mathrm{id}_{V/W}$.

**Definition 2.4.5.** Let $W \subseteq V$ be a subspace. The <u>*canonical projection map*</u> $\pi_W : V \to V/W$ is defined by $v \mapsto v + W$. Note that $\pi_W \in \operatorname{Hom}_F(V, V/W)$.

**Note 1.** To define a map $T : V/W \to V'$, you always have to check it is well-defined.

**Theorem 2.4.1** (First Isomorphism Theorem)**.** *Let $T \in \operatorname{Hom}_F(V, W)$. Define $\overline{T} : V/\ker(T) \to W$ by $v + \ker(T) \mapsto T(v)$. Then $\overline{T}$ is a linear map. Moreover, $V/\ker(T) \cong \operatorname{im}(T)$.*

*Proof.* finish this □

## 2.5 Dual Spaces

Note that when one refers to something as "*canonical*", it means the object in question does not depend on a basis.

**Definition 2.5.1.** Let $V$ be an $F$-vector space. The <u>*dual space*</u>, denoted $V^\vee$, is defined to be $V^\vee = \operatorname{Hom}_F(V, F)$.

**Theorem 2.5.1.** *We have $V$ is isomorphic to a subspace of $V^\vee$. If $\dim_F(V) < \infty$, then $V \cong V^\vee$.*

*Proof.* Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a basis (hence this theorem is not canonical). For each $i \in I$, define:

$$v_i^\vee(v_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

We get $\{v_i^\vee\}_{i \in}$ are elements of $V^\vee$. We obtain $T \in \operatorname{Hom}_F(V, V^\vee)$ by $T(v_i) = v_i^\vee$. To show that $V$ is isomorphic to a subspace of $V^\vee$, it is enough to show $T$ is injective, then by the first isomorphism theorem $V \cong \operatorname{im}(T)$ (a subspace of $V^\vee$).

Let $v \in \ker(T)$, then $T(v) = 0_{V^\vee}$. Write $v = \sum_{i \in I} a_i v_i$. So:

$$\begin{aligned} 0_{V^\vee} &= T(v) \\ &= T\left(\sum_{i \in I} a_i v_i\right) \\ &= \sum_{i \in I} a_i T(v_i) \\ &= \sum_{i \in I} a_i v_i^\vee. \end{aligned}$$

Towards contradiction, pick some $j$ with $a_j \neq 0$. Note that $0_{V^\vee} = \sum_{i \in I} a_i v_i^\vee(v_j) = a_j$ (every term except for $a_j v_j^\vee(v_j)$ equals 0). This is a contradiction, hence $T$ is injective.

Now assume $\dim_F(V) = n$ and write $\mathcal{B} = \{v_1, \ldots, v_n\}$. Let $v^\vee \in V^\vee$. Define $a_i = v^\vee(v_i)$. Set $v = \sum_{i=1}^n a_i v_i$ and define $S : V^\vee \to V$ by $S(v^\vee) = v = \sum_{i=1}^n v^\vee(v_i) v_i$. We'd like to show that $S \in \mathrm{Hom}_F(V^\vee, V)$ and is the inverse of $T$. Let $v^\vee, w^\vee \in V^\vee$ and $c \in F$. Set $a_i = v^\vee(v_i)$ and $b_i = w^\vee(v_i)$. Then:

$$\begin{aligned} S(v^\vee + cw^\vee) &= \sum_{i=1}^n \left[(v^\vee + cw^\vee)(v_i)\right] v_i \\ &= \sum_{i=1}^n \left[v^\vee(v_i) + cw^\vee(v_i)\right] v_i \\ &= \sum_{i=1}^n v^\vee(v_i) v_i + c \sum_{i=1}^n w^\vee(v_i) v_i \\ &= S(v^\vee) + cS(w^\vee). \end{aligned}$$

Hence $S$ is linear. Now observe that:

$$\begin{aligned} (S \circ T)(v_j) &= S(T(v_j)) \\ &= S(v_j^\vee) \\ &= \sum_{i=1}^n v_j^\vee(v_i) v_i \\ &= v_j \end{aligned}$$

Let $v^\vee \in V^\vee$. Note that $(T \circ S)(v^\vee)$ is a function, so it will require an input. Observe that

$$\begin{aligned} (T \circ S)(v^\vee)(v_j) &= T(S(v^\vee))(v_j) \\ &= T(\sum_{i=1}^n v^\vee(v_i) v_i)(v_j) \\ &= \left[\sum_{i=1}^n v^\vee(v_i) T(v_i)\right](v_j) \\ &= \sum_{i=1}^n v^\vee(v_i)(v_i^\vee(v_j)) \\ &= v^\vee(v_j). \end{aligned}$$

□

**Definition 2.5.2.** Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of $V$. The <u>dual basis</u> for $V^\vee$ is $\mathcal{B}^\vee = \{v_1^\vee, \ldots, v_n^\vee\}$.

**Proposition 2.5.1.** *There is a canonical injective linear map from $V$ to $(V^\vee)^\vee$. If $\dim_F(V) < \infty$, this is an isomorphism.*

*Proof.* Let $v \in V$. Define $\hat{v} : V^\vee \to F$ by $\varphi \mapsto \varphi(v)$[2]. We can easily verify that $\hat{v}$ is linear. Therefore, we have $\hat{v} \in \operatorname{Hom}_F(V^\vee, F) = (V^\vee)^\vee$. We have a map:

$$\Phi : V \to (V^\vee)^\vee \text{ defined by } v \mapsto \hat{v}.$$

We want to verify that $\Phi$ is an injective linear map. Let $v_1, v_2 \in V$ and $c \in F$. Let $\varphi \in V^\vee$, then:

$$\begin{aligned} \Phi(v_1 + cv_2)(\varphi) &= \widehat{v_1 + cv_2}(\varphi) \\ &= \varphi(v_1 + cv_2) \\ &= \varphi(v_1) + c\varphi(v_2) \\ &= \hat{v}_1(\varphi) + c\hat{v}_2(\varphi) \\ &= \Phi(v_1)(\varphi) + c\Phi(v_2)(\varphi). \end{aligned}$$

We will now show that $\Phi$ is injective. Let $v \in V$ and assume $v \neq 0_V$. We will form a basis $\mathcal{B}$ of $V$ that contains v (why is this still canonical?). Let $v^\vee \in V^\vee$, then $v^\vee(v) = 1$ and $v^\vee(w) = 0$ for all $w \in \mathcal{B}$, $w \neq v$. Now assume $v \in \ker(\Phi)$. Then $\Phi(v)(\varphi) = \varphi(v) = 0$ for all $\varphi \in V^\vee$. But picking $\varphi = v^\vee$ gives:

$$\begin{aligned} 0 &= \Phi(v)(v^\vee) \\ &= v^\vee(v) \\ &= 1. \end{aligned}$$

This is a contradiction, hence $\Phi$ is injective. □

**Definition 2.5.3.** Let $T \in \operatorname{Hom}_F(V, W)$. We get an induced map $T^\vee : W^\vee \to V^\vee$ with $T^\vee(\varphi) = \varphi \circ T$. The following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, dashed, "T^\vee(\varphi)"'] & W \arrow[d, "\varphi"] \\ & F. \end{tikzcd}$$

[2]This can be notated as $\operatorname{eval}_v$, but $\hat{v}$ appears more often in literature

# 3

# Linear Transformations and Matrices

## 3.1 Choosing Coordinates

**Example 3.1.1** (Choosing Coordinates)**.** Let $V$ be an $F$-vector space with $\dim_F(V) < \infty$. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for $V$. This basis fixes an isomorphism $V \cong F^n$. Let $v \in V$, write $v = \sum_{i=1}^n a_i v_i$.

$$\text{Define } T_{\mathcal{B}}(v) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n.$$

This is an isomorphism. Given $v \in V$, we write $[v]_{\mathcal{B}} = T_{\mathcal{B}}(v) = \left(\begin{smallmatrix} a_1 \\ \vdots \\ a_n \end{smallmatrix}\right)$. We refer to this as *choosing coordinates* on $V$. asdf

**Example 3.1.2.**

(1) Let $V = \mathbf{Q}^2$ and $\mathcal{B} = \{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\}$. This forms a basis of $V$. Let $v \in V$ with $v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. We have:

$$v = \frac{a+b}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a-b}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ hence } [v]_{\mathcal{B}} = \begin{pmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{pmatrix}.$$

Had we considered the standard basis $\mathcal{E}_2 = \{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$, then $[v]_{\mathcal{E}_2} = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$.

(2) Let $V = P_2(\mathbf{R})$. Let $\mathcal{C} = \{1, (x-1), (x-1)^2\}$. This forms a basis of $V$. Let $f(x) = a + bx + cx^2 \in P_2(\mathbf{R})$. Written in terms of $\mathcal{C}$, we have $f(x) = (a+b+c) + (b+2c)(x-1) + c(x-1)^2$.

$$\text{Thus } [f(x)]_{\mathcal{C}} = \begin{pmatrix} a+b+c \\ b+2c \\ c \end{pmatrix}$$

**Example 3.1.3** (Linear Transformations as Matrices)**.** Recall that given a matrix $A \in \mathrm{Mat}_{m,n}(F)$, we obtain a linear map $T_A \in \mathrm{Hom}_F(F^n, F^m)$ by $T_A(v) = Av$. This process "works in reverse" —given a linear transformation $T \in \mathrm{Hom}_F(F^n, F^m)$, there is a matrix $A$ so that $T = T_A$.

Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ be the standard basis of $F^n$ and $\mathcal{F}_m = \{f_1, \dots, f_m\}$ be the standard basis of $F^m$. We have that $T(e_j) \in F^m$ for each $j$, meaning we have elements $a_{ij} \in F$ with $T(e_j) = \sum_{i=1}^m a_{ij} f_i$. Define $A = (a_{ij}) \in Mat_{m,n}(F)$. Observe that:

$$T_A(e_j) = Ae_j = \sum_{i=1}^m a_{ij} f_i = a_{1j} f_1 + \dots + a_{mj} f_m.$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Working "in reverse", let $T \in \mathrm{Hom}_F(V, W)$ with $\mathcal{B} = \{v_1, \dots, v_n\}$ a basis for $V$ and $\mathcal{C} = \{w_1, \dots, w_m\}$ a basis for W. Define:

$$\begin{aligned} P &= T_{\mathcal{B}} : V \to F^n \text{ by } v \mapsto [v]_{\mathcal{B}} \\ Q &= T_{\mathcal{C}} : W \to F^m \text{ by } w \mapsto [w]_{\mathcal{C}} \end{aligned}$$

From the following diagram:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "P"'] & W \arrow[d, "Q"] \\ F^n \arrow[r, dashed, "Q\circ T\circ P^{-1}"'] & F^m \end{tikzcd}$$

we have that $Q \circ T \circ P^{-1}$ corresponds to a matrix $A \in \mathrm{Mat}_{m,n}(F)$. Write $[T]_{\mathcal{B}}^{\mathcal{C}} = A$, this is the unique matrix that satisfies $[T]_{\mathcal{B}}^{\mathcal{C}}[v]_{\mathcal{B}} = [T(v)]_{\mathcal{C}}$. Given $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$, observe that:

$$[T]_{\mathcal{B}}^{\mathcal{C}} v_j = [T(v_j)]_{\mathcal{C}} = \left[\sum_{i=1}^{m} a_{ij} w_i\right]_{\mathcal{C}} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

So $[T]_{\mathcal{B}}^{\mathcal{C}} v_j$ corresponds to the $j^{\text{th}}$ column of the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$ Thus we have:

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \Big( [T(v_1)]_{\mathcal{C}} \quad | \quad \dots \quad | \quad [T(v_n)]_{\mathcal{C}} \Big)$$

**Example 3.1.4.**

(1) Let $V = P_3(\mathbf{R})$ with $\mathcal{B} = \left\{1, x, x^2, x^3\right\}$. Define $T \in \mathrm{Hom}_{\mathbf{R}}(V, V)$ by $T(f(x)) = f'(x)$. Following Example 3.1.3 gives:

$$\begin{aligned} T(1) &= 0 = 0 \cdot 1 + 0 \cdots x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x^2) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x^3) &= 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3 \end{aligned}$$

$$[T(1)]_{\mathscr{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$[T(x)]_{\mathscr{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\left[T(x^2)\right]_{\mathscr{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$
$$\left[T(x^3)\right]_{\mathscr{B}} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$[T]_{\mathscr{B}}^{\mathscr{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(2) Let $V = P_3(\mathbf{R})$ with $\mathscr{B} = \left\{1, x, x^2, x^3\right\}$ with $\mathscr{C} = \left\{1, (1-x), (1-x)^2, (1-x^3)\right\}$. Then

$$\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2 + 2(x-1) \\ T(x^3) &= -9 - 6(x-1) + 3(x-1)^2 \end{aligned}$$

$$[T(1)]_{\mathscr{C}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$[T(x)]_{\mathscr{C}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\left[T(x^2)\right]_{\mathscr{C}} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$
$$\left[T(x^3)\right]_{\mathscr{C}} = \begin{pmatrix} -9 \\ -6 \\ 3 \\ 0 \end{pmatrix}$$

$$[T]_{\mathscr{B}}^{\mathscr{C}} = \begin{pmatrix} 0 & 1 & 2 & -9 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Exercise 3.1.1.**

(1) *Let $\mathcal{A}$ be a basis of $U$, $\mathcal{B}$ a basis of $V$ and $\mathcal{C}$ a basis of $W$. Let $S \in \mathrm{Hom}_F(U, V)$ and $T \in \mathrm{Hom}_F(V, W)$. Show*

$$[T \circ S]_{\mathcal{A}}^{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} [S]_{\mathcal{A}}^{\mathcal{B}}.$$

(2) *Given $A \in \mathrm{Mat}_{m,k}(F)$ and $B \in \mathrm{Mat}_{n,m}(F)$, we have corresponding linear maps $T_A$ and $T_B$. Show that you can recover the definition of matrix multiplication by using part (1).*

**Note 2.** Instead of $[T]_{\mathcal{B}}^{\mathcal{B}}$ we will write $[T]_{\mathcal{B}}$.

**Example 3.1.5** (Change of Basis)**.** Let $V$ be an $F$-vector space and $\mathcal{B}, \mathcal{B}'$ bases of $V$. Given $V$ expressed in terms of $\mathcal{B}$, we'd like to express it in terms of $\mathcal{B}'$ (or vice versa).

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ and $\mathcal{B}' = \{v'_1, \dots, v'_n\}$. Define:

$$T : V \to F^n \text{ by } v \mapsto [v]_{\mathcal{B}}$$
$$S : V \to F^n \text{ by } v \mapsto [v]_{\mathcal{B}'}.$$

We obtain a diagram similar to Example 3.1.3:

$$\begin{tikzcd} V \arrow[r, "\mathrm{id}_V"] \arrow[d, "T"'] & V \arrow[d, "S"] \\ F^n \arrow[r, "S \circ \mathrm{id}_V \circ T^{-1}"'] & F^n \end{tikzcd}$$

Hence the change of basis matrix is $[\mathrm{id}_V]_{\mathcal{B}}^{\mathcal{B}'}$

**Exercise 3.1.2.** *Let $\mathcal{B} = \{v_1, \dots, v_n\}$. Show that $[\mathrm{id}_V]_{\mathcal{B}}^{\mathcal{B}'} = ([v_1]_{\mathcal{B}'} \mid \dots \mid [v_n]_{\mathcal{B}'})$.*

**Example 3.1.6.**

(1) Let $V = \mathbf{Q}^2$ with $\mathcal{B} = \{e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ and $\mathcal{B}' = \{v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$. Observe that:

$$e_1 = \frac{1}{2} v_1 + \frac{1}{2} v_2$$
$$e_2 = -\frac{1}{2} v_1 + \frac{1}{2} v_2$$

$$[e_1]_{\mathcal{B}} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$
$$[e_2]_{\mathcal{B}} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$[\mathrm{id}_V]_{\mathcal{E}_2}^{\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Consider $v = \binom{2}{3} \in \mathbf{Q}^2$. We can express $v$ in terms of $\mathcal{B}'$ by doing the following calculation:

$$\begin{aligned}
[\mathrm{id}_V]_{\mathcal{E}_2}^{\mathcal{B}'} [v_2]_{\mathcal{E}_2} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix} \\
&= [v]_{\mathcal{B}'}.
\end{aligned}$$

(2) Let $V = P_2(\mathbf{R})$ with $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1, (x-2), (x-2)^2\}$. Then:

$$\begin{aligned}
1 &= 1 \cdot 1 + 0 \cdot (x-2) + 0 \cdot (x-2)^2 \\
x &= 2 \cdot 1 + 1 \cdot (x-2) + 0 \cdot (x-2)^2 \\
x^2 &= 4 \cdot 1 + 4 \cdot (x-2) + 1 \cdot (x-2)^2
\end{aligned}$$

$$\begin{aligned}
[1]_{\mathcal{B}'} &= \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right) \\
[x]_{\mathcal{B}'} &= \left(\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}\right) \\
[x^2]_{\mathcal{B}'} &= \left(\begin{smallmatrix} 4 \\ 4 \\ 1 \end{smallmatrix}\right)
\end{aligned}$$

$$[\mathrm{id}_V]_{\mathcal{B}}^{\mathcal{B}'} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Example 3.1.7** (Similar Matrices). Let $A, B \in \mathrm{Mat}_n(F)$. Let $\mathcal{E}_n$ be the standard basis for $F^n$ and $T_A \in \mathrm{Hom}_F(F^n, F^n)$ such that $A = [T_A]_{\mathcal{E}_n}$. We can relate $A$ in terms of an arbitrary basis $\mathcal{B}$ as follows:

$$\begin{tikzcd}
F^n \arrow[r, "T_A"] \arrow[d, "T_{\mathcal{B}}"'] & F^n \arrow[d, "T_{\mathcal{B}}"] \\
F^n \arrow[r, "{[T_A]_{\mathcal{B}}}"'] & F^n.
\end{tikzcd}$$

But by extending our diagram using our change of basis algorithm, we obtain the following:

$$\begin{tikzcd}
F^n \arrow[r, "\mathrm{id}_{F^n}"] \arrow[d, "T_{\mathcal{B}}"'] & F^n \arrow[r, "T_A"] \arrow[d, "T_{\mathcal{E}_n}"'] & F^n \arrow[r, "\mathrm{id}_{F^n}"] \arrow[d, "T_{\mathcal{E}_n}"] & F^n \arrow[d, "T_{\mathcal{B}}"] \\
F^n \arrow[r, "{[\mathrm{id}_{F^n}]_{\mathcal{B}}^{\mathcal{E}_n}}"'] & F^n \arrow[r, "{[T_A]_{\mathcal{E}_n}}"'] & F^n \arrow[r, "{[\mathrm{id}_{F^n}]_{\mathcal{E}_n}^{\mathcal{B}}}"'] & F^n
\end{tikzcd}$$

So $[T_A]_{\mathcal{B}} = [\mathrm{id}_{F^n}]^{\mathcal{E}_n}_{\mathcal{B}} [T_A]_{\mathcal{E}_n} [\mathrm{id}_{F^n}]^{\mathcal{B}}_{\mathcal{E}_n}$. Assigning $P^{-1} = [\mathrm{id}_{F^n}]^{\mathcal{E}_n}_{\mathcal{B}}$ and $P = [\mathrm{id}_{F^n}]^{\mathcal{B}}_{\mathcal{E}_n}$ yields the familiar equation $[T_A]_{\mathcal{B}} = P^{-1}AP$; i.e., $A = P[T_A]_{\mathcal{B}}P^{-1}$. In particular, the matrix $A = [T_A]_{\mathcal{E}_n}$ is similar to $[T_A]_{\mathcal{B}}$ for any basis $\mathcal{B}$.

**Example 3.1.8.** Let $A = \begin{pmatrix} 1 & 3 & -5 \\ -2 & -1 & 6 \\ 3 & 2 & 1 \end{pmatrix}$. Let $\mathcal{E}_3 = \{e_1, e_2, e_3\}$ be the standard basis of $F^3$. We have:

$$\begin{aligned} T_A(e_1) &= e_1 - 2e_2 + 3e_3 \\ T_A(e_2) &= 3e_1 - e_2 + 2e_3 \\ T_A(e_3) &= 3e_1 + 2e_2 + e_3. \end{aligned}$$

Now consider $\mathcal{B} = \{v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}\}$. One can check this is indeed a basis. Observe that:

$$\begin{aligned} e_1 &= -2v_1 + -3v_2 + v_3 \\ e_2 &= 3v_1 + 3v_2 - v_3 \\ e_3 &= -2v_1 - 2v_2 + v_3. \end{aligned}$$

So the change of basis matrix from $\mathcal{E}_3$ to $\mathcal{B}$ is given by $P = [\mathrm{id}_{F^3}]^{\mathcal{B}}_{\mathcal{E}_3} = \begin{pmatrix} -2 & 3 & -2 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{pmatrix}$. We have $P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Thus $A$ is similar to the matrix $B = P^{-1}AP = \begin{pmatrix} -29 & 32 & -25 \\ -38 & 45 & -31 \\ -20 & 27 & -15 \end{pmatrix}$.

## 3.2 Row Operations

**Definition 3.2.1.** Let $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(F)$. We say $a_{kl}$ is a *pivot* of $A$ if $a_{kl} \neq 0$ and $a_{ij} = 0$ if $i > k$ or $j < l$.

**Example 3.2.1.** Let $A = \begin{pmatrix} 2 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then 2, 1, and 5 are pivots.

**Definition 3.2.2.** Let $A \in \mathrm{Mat}_{m,n}(F)$. We say $A$ is in *row echelon form* if all its nonzero rows have a pivot and all its zero rows are located below nonzero rows. We say it is *reduced row echelon form* if it is in row echelon form and all of its pivots are 1 and the only nonzero elements in the columns containing pivots.

**Example 3.2.2.** From the previous example, expressing $A = \begin{pmatrix} 2 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in reduced row echelon form yields $A' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

**Example 3.2.3.** Let $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$. Then $T_A : F^4 \to F^4$. Let $\mathcal{B}_4 = \{e_1, e_2, e_3, e_4\}$ and $\mathcal{F}_3 = \{f_1, f_2, f_3\}$. So $A = [T_A]^{\mathcal{F}_3}_{\mathcal{B}_3}$. We have the following set of equations:

$$\begin{aligned} T_A(e_1) &= 3f_1 + f_2 + f_3 \\ T_A(e_2) &= 4f_1 + 2f_2 + f_3 \\ T_A(e_3) &= 5f_1 + 3f_2 + 2f_3 \\ T_A(e_4) &= 6f_1 + 4f_2 + 3f_3. \end{aligned}$$

We are going to perform row operations of $A$ by making substitutions to its basis elements. Consider the operation $R_1 \leftrightarrow R_3$.

$$\mathcal{F}_3^{(2)} = \{f_1^{(2)} = f_3, f_2^{(2)} = f_2, f_3^{(2)} = f_1\}.$$

$$\begin{aligned}
T_A(e_1) &= f_1^{(2)} + f_2^{(2)} + 3f_3^{(2)} \\
T_A(e_2) &= f_1^{(2)} + 2f_2^{(2)} + 4f_3^{(2)} \\
T_A(e_3) &= 2f_1^{(2)} + 3f_2^{(2)} + 5f_3^{(2)} \\
T_A(e_4) &= 3f_1^{(2)} + 4f_2^{(2)} + 6f_3^{(2)}.
\end{aligned}$$

So $[T_A]_{\mathcal{B}_3}^{\mathcal{F}_3^{(2)}} = \left(\begin{smallmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{smallmatrix}\right)$. Now consider the row operation $-R_1 + R_2 \leftrightarrow R_2$.

$$\mathcal{F}_3^{(3)} = \{f_1^{(3)} = f_1^{(2)} + f_2^{(2)}, f_2^{(3)} = f_2^{(2)}, f_3^{(3)} = f_3^{(2)}\}.$$

$$\begin{aligned}
T_A(e_1) &= f_1^{(2)} + f_2^{(2)} + 3f_3^{(2)} \\
&= f_1^{(3)} + 3f_3^{(3)}.
\end{aligned}$$

$$\begin{aligned}
T_A(e_2) &= f_1^{(2)} + 2f_2^{(2)} + 4f_3^{(2)} \\
&= f_1^{(2)} + f_2^{(2)} + f_2^{(2)} + 4f_3^{(2)} \\
&= f_1^{(3)} + f_2^{(3)} + 4f_3^{(3)}.
\end{aligned}$$

$$\begin{aligned}
T_A(e_3) &= \dots \\
T_A(e_4) &= \dots
\end{aligned}$$

So $[T_A]_{\mathcal{B}_3}^{\mathcal{F}_3^{(3)}} = \left(\begin{smallmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 \end{smallmatrix}\right)$. Now consider the row operation $-3R_1 + R_3 \leftrightarrow R_3$.

$$\mathcal{F}_3^{(4)} = \{f_1^{(4)} = f_1^{(3)} + 3f_3^{(3)}, f_2^{(4)} = f_2^{(3)}, f_3^{(4)} = f_3^{(3)}\}.$$

$$\begin{aligned}
T_A(e_1) &= f_1^{(3)} + 3f_3^{(3)} \\
&= f_1^{(4)}
\end{aligned}$$

$$\begin{aligned}
T_A(e_2) &= \dots \\
T_A(e_3) &= \dots \\
T_A(e_4) &= \dots
\end{aligned}$$

The rest of the steps to convert $A$ to reduced row echelon form follow similarly.

**Theorem 3.2.1.** *Let $A \in \mathrm{Mat}_{m,n}(F)$. The matrix $A$ can be put in row echelon form through a series of row operations of the form:*

*(1)* $R_i \leftrightarrow R_j$

*(2)* $R_i \leftrightarrow cR_i$

*(3)* $cR_i + R_J \leftrightarrow R_j$.

**Example 3.2.4.** Instead of directly changing the basis of a matrix, we can use linear maps to perform row operations. Let $\mathcal{C} = \{w_1, ..., w_n\}$ be a basis of $W$.

(1) Define $T_{i,j} : W \to W$ by

$$\begin{aligned} T_{i,j}(w_k) &= w_k \text{ if } k \neq i, j, \\ T_{i,j}(w_i) &= w_j, \\ T_{i,j}(w_j) &= w_i. \end{aligned}$$

Then $E_{i,j} = \left[T_{i,j}\right]_{\mathcal{C}}^{\mathcal{C}}$ corresponds to the identity matrix except the $i^{\text{th}}$ and $j^{\text{th}}$ rows are switched.

(2) Let $c \in F$, $c \neq 0$. Define $T_i^{(c)} : W \to W$ by:

$$\begin{aligned} T_i^{(c)}(w_j) &= w_j \text{ if } j \neq i, \\ T_i^{(c)}(w_i) &= cw_i \end{aligned}$$

Then $E_i^{(c)} = \left[T_i^{(c)}\right]_{\mathcal{C}}^{\mathcal{C}}$ corresponds to the identity matrix with the $i^{\text{th}}$ row multiplied by $c$.

(3) Define $T_{i,j}^{(c)} : W \to W$ by:

$$\begin{aligned} T_{i,j}^{(c)}(w_k) &= w_k \text{ if } k \neq j, \\ T_{i,j}^{(c)}(w_j) &= w_j + cw_i \end{aligned}$$

Then $E_{i,j}^{(c)} = \left[T_{i,j}^{(c)}\right]_{\mathcal{C}}^{\mathcal{C}}$ corresponds to the identity matrix with the what does this mean?

Now let $T_A : F^4 \to F^3$ with $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$ and $\mathcal{E}_4$ and $\mathcal{F}_3$ their respective standard bases. Performing the row operation $R_1 \leftrightarrow R_3$ using the above method yields:

$$\begin{aligned} (T_{1,3} \circ T_A)(e_1) &= T_{1,3}(3f_1 + f_2 + f_3) \\ &= 3T_{1,3}(f_1) + T_{1,3}(f_2) + T_{1,3}(f_3) \\ &= 3f_3 + f_2 + f1 \end{aligned}$$

$$\begin{aligned}\left[T_{1,3} \circ T_{A}{}_{\mathcal{E}_4}^{\mathcal{F}_3}\right] &= \left[T_{1,3}\right]_{\mathcal{F}_3}^{\mathcal{F}_3} [T_A]_{\mathcal{E}_4}^{\mathcal{F}_3} \\ &= E_{1,3}A \\ &= \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{pmatrix}.\end{aligned}$$

The rest of the row operations follow similarly. The reduced-row echelon form of $A$ can then be expressed as:

$$\left[T_{1,3}^{(-1)} \circ T_{2,3}^{(-1)} \circ T_{(3)}^{(\frac{1}{2})} \circ T_{3,2}^{(-1)} \circ T_{3,1}^{(-3)} \circ T_{1,2}^{(-1)} \circ T_{1,3} \circ T_A\right]_{\mathcal{E}_4}^{\mathcal{F}_3}.$$

## 3.3 Column-space and Null-space

**Definition 3.3.1.** Let $A \in \mathrm{Mat}_{m,n}(F)$.

(1) The *column-space* of $A$ is the $F$-span of the column vectors, denoted as $CS(A)$.

(2) The *null-space* of $A$ is the $F$-span of vectors $v \in F^n$ such that $Av = 0_V$, denoted as $NS(A)$.

(3) The *rank* of $A$ is $\operatorname{rank} A = \dim_F CS(A)$.

**Example 3.3.1.** Let $T_A \in \mathrm{Hom}_F(F^n, F^m)$ where $\mathcal{E}_n = \{e_1, \ldots, e_n\}$ is the standard basis of $F^n$ and $\mathcal{F}_n = \{f_1, \ldots, f_m\}$ is the standard basis of $F^m$. Since

$$[T_A]_{\mathcal{E}_n}^{\mathcal{F}_m} = A = \Big(T_A(e_1) \mid \ \ldots \ \mid T_A(e_n)\Big),$$

we have that $CS(A) = \operatorname{im}(T_A)$, so $\operatorname{rank} A = \dim_F \operatorname{im}(T_A)$. Recall from an introductory linear algebra course that the column space is calculated by:

(a) Put $A$ into row echelon form,

(b) Look at which columns have pivots,

(c) The same columns in $A$ are then a basis of $CS(A)$.

Why does this work? There exists an isomorphism $E : F^n \to F^m$ so that $[E \circ T_A]_{\mathcal{E}_n}^{\mathcal{F}_m} = [E]_{\mathcal{E}_n}^{\mathcal{F}_m} A$ is in row echelon form. The column space of $[E \circ T_A]_{\mathcal{E}_n}^{\mathcal{F}_m}$ has as its basis the columns containing pivots (denoted $e_{i1}, \ldots, e_{ik}$):

$$\underbrace{[E \circ T_A(e_{i1})]_{\mathcal{F}_m}, \ \ldots \ , [E \circ T_A(e_{ik})]_{\mathcal{F}_m}}_{\text{this is a basis of } CS([E \circ T_A]_{\mathcal{E}_n}^{\mathcal{F}_m})}$$

Since $E$ is an isomorphism, there is an inverse $E^{-1} : F^m \to F^m$ with:

$$\begin{aligned} E^{-1}(w_1) &= [E \circ T_A(e_{i1})]_{\mathcal{F}_m} \\ &\vdots \\ E^{-1}(w_k) &= [E \circ T_A(e_{ik})]_{\mathcal{F}_m} . \end{aligned}$$

These are linearly independent since $E^{-1}$ is an isomorphism. If there is a vector $v \in CS(A)$ with $v \notin \operatorname{span}_F \left([E \circ T_A(e_{i1})]_{\mathcal{F}_m}, \ldots, [E \circ T_A(e_{ik})]_{\mathcal{F}_m}\right)$, then $E(v)$ cannot be in $\operatorname{span}_F(w_1, \ldots, w_k)$. So the columns
$[E \circ T_A(e_{i1})]_{\mathcal{F}_m}, \ldots, [E \circ T_A(e_{ik})]_{\mathcal{F}_m}$ give a basis for the column space of $A$.

**Example 3.3.2.** Let $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$. Rewritten in row echelon form is $A' = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{pmatrix}$. Thus:

$$\begin{aligned} CS(B) &= \operatorname{span}_F \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right) \\ CS(A) &= \operatorname{span}_F \left( \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \right).e \end{aligned}$$

**Example 3.3.3.** We have $v \in NS(A)$ if and only if $Av = 0_{F^m} = T_A(v)$. Note that $T_A(v) = 0_{F^m}$ if and only if $v \in \ker(T_A)$, hence $NS(A) = \ker(T_A)$. In an introductory algebra class, the null space of a matrix $A$ is calculated by:

(1) Putting $A$ into reduced row echelon form,

(2) Solving the equation $A'x = 0_{F^n}$.

This works because given a map $T_A : F^n \to F^m$, row operations change the basis of the codomain, not the domain. So $NS(A) = NS(A')$.

**Example 3.3.4.** Let $A = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -1 & 1 \end{pmatrix}$. The reduce row echelon form of $A$ is $A' = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Solving the equation:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives $x_2 = 0$ and $x_1 = -\frac{1}{2}x_3$. Hence $NS(A) = \operatorname{span}_F \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$.

## 3.4 The Transpose of a Matrix

**Definition 3.4.1.** Let $A \in \mathrm{Mat}_{m,n}(F)$ with $\mathcal{E}_n = \{e_1, \ldots, e_n\}$ and $\mathcal{F}_m = \{f_1, \ldots, f_m\}$ as standard bases. Then $A = [T_A]_{\mathcal{E}^n}^{\mathcal{F}_m}$, and furthermore $T_A \in \operatorname{Hom}_F(F^n, F^m)$ induces a dual map $T_A^\vee \in \operatorname{Hom}_F(F^{m\vee}, F^{n\vee})$. The *transpose* of $A$ is defined as:

$$A^t = \left[T_A^\vee\right]_{\mathcal{F}_m^\vee}^{\mathcal{E}_n^\vee} .$$

**Lemma 3.4.1.** *Let* $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(F)$*. Then* $A^t = (b_{ij}) \in \mathrm{Mat}_n, m(F)$ *with* $b_{ij} = a_{ji}$*.*

*Proof.* We use the same setup as Definition 3.4.1. We have:

$$T_A(e_i) = \sum_{k=1}^{m} a_{ki} f_k$$
$$T_A^\vee(f_j^\vee) = \sum_{k=1}^{n} b_{kj} e_k^\vee.$$

Applying $f_j^\vee$ to $T_A(e_i)$ yields[1]:

$$\begin{aligned}(f_j^\vee \circ T_A)(e_i) &= f_j^\vee\left(\sum_{k=1}^{m} a_{ki} f_k\right)\\ &= \sum_{k=1}^{m} a_{ki} f_j^\vee(f_k)\\ &= a_{ji}.\end{aligned}$$

Evaluating the $T_A^\vee(f_j^\vee)$ at $e_i$ gives:

$$\begin{aligned}T_A^\vee(f_j^\vee)(e_i) &= \sum_{k=1}^{n} b_{kj} e_k^\vee(e_i)\\ &= b_{ij}.\end{aligned}$$

By Definition 2.5.3, we have $(f_j^\vee \circ T_A)(e_i) = T_A^\vee(f_j^\vee)(e_i)$. Hence $a_{ji} = b_{ij}$ □

**Exercise 3.4.1.** *Let* $A_1, A_2 \in \mathrm{Mat}_{m,n}(F)$ *and* $c \in F$*. Show that:*

$$(A_1 + A_2)^t = A_1^t + A_2^t$$
$$(cA_1)^t = cA_1^t.$$

**Lemma 3.4.2.** *Let* $A \in \mathrm{Mat}_{m,n}(F)$ *and* $B \in \mathrm{Mat}_{p,m}(F)$*. Then* $(BA)^t = A^t B^t$*.*

*Proof.* Let $\mathcal{E}_m$, $\mathcal{E}_n$, and $\mathcal{E}_p$ be standard bases with $[T_A]_{\mathcal{E}_n}^{\mathcal{E}_m} = A$ and $[T_B]_{\mathcal{E}_m}^{\mathcal{E}_p} = B$. Then $BA = [T_B \circ T_A]_{\mathcal{E}_n}^{\mathcal{E}_p}$. Thus:

$$\begin{aligned}(BA)^t &= \left[(T_B \circ T_A)^\vee\right]_{\mathcal{E}_p^\vee}^{\mathcal{E}_n^\vee}\\ &= \left[T_A^\vee \circ T_B^\vee\right]_{\mathcal{E}_p^\vee}^{\mathcal{E}_n^\vee}\\ &= \left[T_A^\vee\right]_{\mathcal{E}_m^\vee}^{\mathcal{E}_n^\vee} \left[T_B^\vee\right]_{\mathcal{E}_p^\vee}^{\mathcal{E}_m^\vee}\\ &= A^t B^t.\end{aligned}$$

□

[1] I was really confused about this. In short, given a $T \in \mathrm{Hom}_F(V, V)$ and basis $\mathcal{B}$ we have a matrix representation $[T]_{\mathcal{B}}$. It is natural to wonder what, $[T^\vee]_{\mathcal{B}^\vee}$ looks like, and it turns out to be the "transpose" we were familiar with from 214. Basically, applying $f_j^\vee$ to $T_A(e_i)$ gives us coefficients (by definition of dual basis elements) which correspond to a particular column vector of $[T_A]_{\mathcal{B}}$. Likewise, since we have that fancy property from Definition 2.5.3, naturally we should evaluate $T_A^\vee(f_j^\vee)$ at $e_i$, which gives us coefficients which correspond to column vectors of $[T_A^\vee]_{\mathcal{B}^\vee}$. The rest is self-explanatory.

**Lemma 3.4.3.** *Let $A \in \mathrm{GL}_n(F)$. Then $(A^{-1})^t = (A^t)^{-1}$.*

*Proof.* Let $A = [T_A]_{\mathcal{E}_n}^{\mathcal{E}_n}$. Then $A^{-1} = \left[T_A^{-1}\right]_{\mathcal{E}_n}^{\mathcal{E}_n}$. We have:

$$\begin{aligned}
1_n &= \left[\mathrm{id}_{F^n}^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \\
&= \left[(T_A^{-1} \circ T_A)^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \\
&= \left[T_A^{\vee} \circ (T_A^{-1})^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \\
&= \left[T_A^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \left[(T_A^{-1})^{\vee}\right]_{\mathcal{E}_n^{\vee}}^{\mathcal{E}_n^{\vee}} \\
&= A^t (A^{-1})^t.
\end{aligned}$$

By the uniqueness of inverses, we must have that $(A^{-1})^t = (A^t)^{-1}$ Showing left invertibility follows identically. $\square$

# 4

# Generalized Eigenvectors and Jordan Canonical Form

## 4.1 Diagonalization

**Recall.** We say $A \sim B$ if and only if $A = PBP^{-1}$ for some $P \in \mathrm{GL}_n(F)$. In particular, this means $A = [T]_{\mathcal{A}}$ and $B = [T]_{\mathcal{B}}$ for some bases $\mathcal{A}$ and $\mathcal{B}$ (Example 3.1.7).

**Definition 4.1.1.** We say $A$ is *diagonalizable* if $A \sim D$ for some diagonal matrix $D$. In terms of linear transformations, $A = [T]_{\mathcal{A}}$ is diagonalizable if there is a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}} = D$.

**Example 4.1.1.** If $A \sim B$ then $A$ is diagonalizable if and only if $B$ is diagonalizable. If $A$ and $B$ are diagonalizable, they must be similar to the same diagonal matrix up to reordering the diagonals.

**Example 4.1.2.** Let $V = F^2$ and $T \in \mathrm{Hom}_F(V, V)$. Let $T(e_1) = 3e_1$ and $T(e_2) = -2e_2$. We have that:

$$[T]_{\mathcal{E}_2} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

It follows that $V = V_1 \oplus V_2$, where $V_1 = \mathrm{span}_F(e_1)$ and $V_2 = \mathrm{span}_F(e_2)$. In this case, we have that $T(V_1) \subseteq V_1$ and $T(V_2) \subseteq V_2$, allowing us to write $T$ as a diagonal matrix.

**Example 4.1.3.** Let $V = F^2$ and $T \in \mathrm{Hom}_F(V, V)$. Consider $T(e_1) = 3e_1$ and $T(e_2) = e_1 + 3e_2$. Then:

$$[T]_{\mathcal{E}_2} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Then $V = V_1 \oplus V_2$ with $V_1 = \mathrm{span}_F(e_1)$ and $V_2 = \mathrm{span}_F(e_2)$. But while we have $T(V_1) \subseteq V_1$, we do not have $T(V_2) \subseteq V_2$.

Suppose towards contradiction we have $W_1, W_2 \neq \{0\}$ with $T(W_1) \subseteq W_1$ and $T(W_2) \subseteq W_2$. Write $W_i = \mathrm{span}_F(w_i)$. In particular, this means we can write $T(w_1) = \alpha w_1$ and $T(w_2) = \beta w_2$. For $\mathcal{B} = \{w_1, w_2\}$, we have:

$$[T]_{\mathcal{B}} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Write $w_1 = ae_1 + be_2$ and $w_2 = ce_1 + de_2$. Then:

$$\begin{aligned} \alpha w_1 &= T(w_1) \\ &= aT(e_1) + bT(e_2) \\ &= a(3e_1) + b(e_1 + 3e_2) \\ &= (3a + b)e_1 + (3b)e_2. \end{aligned}$$

Thus, $\alpha(ae_1 + be_2) = (3a + b)e_1 + (3b)e_2$, meaning $\alpha a = 3b + b$ and $\alpha b = 3b$. Either $b = 0$ or $\alpha = 3$. It must be the case that $\alpha = 3$, hence $T(w_1) = 3w_1$. A similar argument for $w_1$ gives:

$$\begin{aligned} \beta w_2 &= T(w_2) \\ &= \ldots \\ &= (3c + d)e_1 + (3d)e_2. \end{aligned}$$

This implies $\beta c = ec + d$ and $\beta d = 3d$. If $\beta = 3$, then this contradicts the first equation. If $w_2 = ce_1$, this contradicts $w_1, w_2$ being a basis.

**Example 4.1.4.** Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$. Let $F = \mathbf{Q}$. Let $P \in GL_2(\mathbf{Q})$, where $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have:

$$P^{-1}AP = \frac{1}{ad - bc}\begin{pmatrix} ad - 2ab + 2cd - 4bc & -3bd - 3b^2 + 2d^2 \\ 3ac + 3a^2 - 2c^2 & -bc + 3ab - 2cd + 4ad \end{pmatrix}.$$

We must have that $3a^2 + 4ac - 2c^2 = 0$. If $c = 0$, then $a = 0$, which contradicts $P$ being invertible. So $c \neq 0$, meaning we can divide by $c^2$ and set $x = \frac{a}{c}$. Then the roots of $3x^2 + 3x - 2 = 0$ are:

$$x = \frac{-3 \pm \sqrt{33}}{6},$$

which gives:

$$a = \frac{-3 \pm \sqrt{33}}{6}c.$$

Since $c \neq 0$, $a \notin \mathbf{Q}$. Thus we cannot diagonalize $A$ over $\mathbf{Q}$. But if we were to take $F = \mathbf{Q}(\sqrt{33})$, then we have that:

$$\mathcal{B} = \{v_1 = \begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{4} \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{4} \end{pmatrix}\},$$

$$[T]_{\mathcal{B}} = \begin{pmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{pmatrix}.$$

**Definition 4.1.2.** Let $V$ be an $F$-vector space and $T \in \mathrm{Hom}_F(V, V)$. A subspace $W \subseteq V$ is said to be *T-invariant* or *T-stable* if $T(W) \subseteq W$.

**Theorem 4.1.1.** *Let $\dim_F(V) = n$ and $W \subseteq V$ a $k$-dimensional subspace. Let $\mathcal{B}_W = \{v_1, \dots, v_k\}$ be a basis of $W$ and extend to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of $V$. Let $T \in \mathrm{Hom}_F(V, V)$. We have $W$ is $T$-stable if and only if $[T]_{\mathcal{B}}$ is block upper-triangular of the form*

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

*where $A = \left[T|_W\right]_{\mathcal{B}_W}$.*

**Example 4.1.5.** Let $V = \mathbb{Q}^4$ with basis $\mathcal{E}_4 = \{e_1, e_2, \dots, e_4\}$ and define $T$ by:

$$\begin{aligned} T(e_1) &= 2e_1 + 3e_3 \\ T(e_2) &= e_1 + e_4 \\ T(e_3) &= e_1 - e_3 \\ T(e_4) &= 2e_1 - 2e_2 + 5e_3 - 4e_4. \end{aligned}$$

Set $W = \mathrm{span}_{\mathbb{Q}}(e_1, e_3)$, then $W$ is $T$-stable. Since $\mathcal{B}_W = \{e_1, e_3\}$ and $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$, we have:

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 3 & -1 & 0 & 5 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

**Example 4.1.6.** A special case is when $\dim_F W = 1$. If $W = \mathrm{span}_F(w_1)$ and $W$ is $T$-stable, then $T(w_1) \in W_1$; i.e., $T(w_1) = \lambda w_1$ for some $\lambda \in F$ Equivalently, this can be written as $(T - \lambda\,\mathrm{id}_V)(w_1) = 0_V$, meaning $w_1 \in \ker(T - \lambda\,\mathrm{id}_V)$.

## 4.2 Eigenvalues and Eigenvectors

**Definition 4.2.1.** Let $T \in \mathrm{Hom}_F(V, V)$ and $\lambda \in F$. If $\ker(T - \lambda\,\mathrm{id}_V) \neq \{0_V\}$, we say $\lambda$ is an *eigenvalue* of $T$. Any nonzero vector in $\ker(T - \lambda\,\mathrm{id}_V)$ is called a *$\lambda$-eigenvector*. The set $E_\lambda^1 = \ker(T - \lambda\,\mathrm{id}_V)$ is called the *eigenspace* associated with $\lambda$.

**Exercise 4.2.1.** *Show that $E_\lambda^1$ is a subspace.*

**Exercise 4.2.2.** *Let $T \in \mathrm{Hom}_F(V, V)$. If $\lambda_1, \lambda_2 \in F$ with $\lambda_1 \neq \lambda_2$, then $E_{\lambda_1}^1 \cap E_{\lambda_2}^1 = \{0_V\}$.*

**Example 4.2.1.** Let $A = \left(\begin{smallmatrix} 12 & 35 \\ -6 & 17 \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbf{Q})$ and $T_A \in \mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}^2, \mathbf{Q}^2)$. We have:

$$\begin{pmatrix} -12 & 35 \\ -6 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{2}{5} \end{pmatrix} = 2 \begin{pmatrix} 1 \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} -12 & 35 \\ -6 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3}{7} \end{pmatrix} = 3 \begin{pmatrix} 1 \\ \frac{3}{7} \end{pmatrix}$$

So $T_A$ has eigenvalues of 2 and 3. Then

$$E_2^1 = \mathrm{span}_{\mathbf{Q}}\left(v_1 = \left(\begin{smallmatrix} 1 \\ 2/5 \end{smallmatrix}\right)\right)$$
$$E_3^1 = \mathrm{span}_{\mathbf{Q}}\left(v_2 = \left(\begin{smallmatrix} 1 \\ 3/7 \end{smallmatrix}\right)\right)$$

gives:

$$[T_A]_{\{v_1, v_2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

**Example 4.2.2** ($F[x]$-Modules)**.** Let $T \in \mathrm{Hom}_F(V, V)$. Note that $V$ is by definition an $F$-module, but we are able to view $V$ as an $F[x]$-module given some linear transformation $T$. The action $F[x] \times V \to V$ is defined by $(f(x), v) \mapsto f(T)(v)$.

Write $T^m = \underbrace{T \circ T \circ \ldots \circ T}_{m-\text{times}}$. Write $f(x) \in F[x]$ as $f(x) = a_m x^m + \ldots + a_1 x + a_0$. Then

$$f(T) = a_m T^m + \ldots + a_1 T + a_0 \,\mathrm{id}_V \in \mathrm{Hom}_F(V, V).$$

For example, let $g(x) = 2x^2 + 3 \in \mathbf{R}[x]$. Then $g(T) = 2T^2 + 3\,\mathrm{id}_V$ and $g(T)(v) = 2T(T(v)) + 3v$. If $f(x) = g(x)h(x)$ for some $g(x), h(x) \in F[x]$, then $f(T) = g(T) \circ h(T)$. Instead of writing $f(T)(v) = g(T)(h(T)(v))$, we will abuse notation and write $g(T)h(T)(v)$. Normally function composition does not commute, but these do for some reason.

**Theorem 4.2.1.** *Let* $\dim_F(V) = n$ *and* $T \in \mathrm{Hom}_F(V, V)$*. There is a unique monic polynomial* $m_T(x) \in F[x]$ *of lowest degree so that* $m_T(T)(v) = 0_V$ *for all* $v \in V$*. Moreover,* $\deg_{m_T}(T) \leqslant n^2$*.*

*Proof.* Recall that $\mathrm{Hom}_F(V, V)$ is an $F$-vector space. We have $\mathrm{Hom}_F(V, V) \cong \mathrm{Mat}_n(F)$, hence $\dim_F(\mathrm{Hom}_F(V, V)) = n^2$.

Given $T \in \mathrm{Hom}_F(V, V)$, consider the set $\{\mathrm{id}_V, T, T^2, \ldots, T^{n^2}\} \subseteq \mathrm{Hom}_F(V, V)$. This has $n^2 + 1$ elements, so it must be linearly dependent. Let $m$ be the smallest integer so that

$$a_m T^m + \ldots + a_1 T + a_0 \,\mathrm{id}_V = 0_{\mathrm{Hom}_F(V,V)}.$$

We obtain a set $\{\mathrm{id}_V, T, T^2, \ldots, T^m\}$. Since $m$ is minimal, $a_m \neq 0$. Define:

$$m_T(x) = x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0 \in F[x], \text{ where } b_i = \frac{a_i}{a_m}.$$

This gives $m_T(T) = 0_{\mathrm{Hom}_F(V,V)}$; i.e., $m_T(T)(v) = 0_V$ for all $v \in V$. It remains to that $m_T(x)$ is unique. Suppose there exists an $f(x) \in F[x]$ which satisfies $f(T)(v) = 0_V$ for all $v \in V$. Write:

$$f(x) = m_T(x)q(x) + r(x)$$

for some $q(x), r(x) \in F[x]$ with $r(x) = 0$ or $\deg(r(x)) < \deg(m_T(x))$. We have for all $v \in V$:

$$\begin{aligned} 0_V &= f(T)(v) \\ &= q(T)m_T(T)(v) + r(T)(v) \\ &= q(T)(0_V) + r(T)(v) \\ &= r(T)(v) \end{aligned}$$

It must be the case that $r(x) = 0$, otherwise we have a polynomial of lower degree than $m_T(x)$ which kills all vectors. So $f(x) = m_T(x)q(x)$; i.e., $m_T(x) \mid f(x)$. But if $m_T(x)$ and $f(x)$ are both monic and of minimal degree, it must be the case that they are the same degree. This gives $m_T(x) = f(x)$. ☐

**Definition 4.2.2.** The unique monic polynomial $m_T(x)$ is called the *minimal polynomial* of $T$.

**Corollary 4.2.1.** *If $f(x) \in F[x]$ satisfies $f(T)(v) = 0_V$ for all $v \in V$, then $m_T(x) \mid f(x)$.*

*Proof.* wat ☐

**Example 4.2.3.** Let $F = \mathbf{Q}$ and $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$. We can see that:

$$A - a_0 1_2 \neq 0_2 \text{ for any } a_0 \in F.$$

But $A^2 = \left(\begin{smallmatrix} 7 & 10 \\ 15 & 22 \end{smallmatrix}\right)$ gives $A^2 - 5A - 2 \cdot 1_2 = 0_2$. Hence $m_A(x) = x^2 - 5x - 2$. Note the relationship between this example and Example 4.1.4.

**Example 4.2.4.** Let $V = \mathbf{Q}^3$, $\mathcal{E}_3 = \{e_1, e_2, e_3\}$, and

$$[T_A]_{\mathcal{E}_3} = A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $W = \mathrm{span}_{\mathbf{Q}}(e_1)$ Then $T(W) = T(\alpha e_1) = \alpha e_1 \in W$. Hence $T(W) \subseteq W$, meaning $W$ is $T$-stable. This gives 1 as an eigenvalue. On a completely unrelated note, $m_{T_A}(x) = (x-1)^2(x+1)$.

**Theorem 4.2.2.** *Let $V$ be an $F$-vector space and $T \in \mathrm{Hom}_F(V,V)$. We have $\lambda$ is an eigenvalue if and only if $\lambda$ is the root of $m_T(x)$. In particular, if $(x - \lambda) \mid m_T(x)$, then $E_\lambda^1 \neq \{0_V\}$ (i.e., there is a nonzero $v \in V$ such that $T(v) = \lambda v$).*

*Proof.* Let $\lambda$ be an eigenvalue with eigenvector $v$ and write $m_T(x) = x^m + ... + a_1x + a_0$. We have:

$$\begin{aligned} 0_V &= m_T(T)(v) \\ &= (T^m + a_{m-1}T^{m-1} + ... + a_1 T + a_0 \operatorname{id}_V)(v) \\ &= T^m(v) + a_{m-1}T^{m-1}(v) + ... + a_1 T(v) + a_0 v \\ &= \lambda^m v + a_{m-1}\lambda^{m-1} v + ... + a_1\lambda v + a_0 v \\ &= (\lambda^m + a_{m-1}\lambda^{m-1} + ... + a_1\lambda + a_0)v \\ &= m_T(\lambda) \cdot v. \end{aligned}$$

Since $v \neq 0$ and $m_T(\lambda) \in F$, it must be the case that $m_T(\lambda) = 0$. Hence $\lambda$ is a root.

Now suppose $m_T(\lambda) = 0$. This gives $m_T(x) = (x - \lambda)f(x)$ for some $f(x) \in F[x]$. Since $\deg f(x) < \deg m_T(x)$, this gives a nonzero vector $v \in V$ so that $f(T)(v) \neq 0$ why?. Set $w = f(T)(v)$, then:

$$\begin{aligned} 0_V &= (T - \lambda \operatorname{id}_V)f(T) \\ &= (T - \lambda \operatorname{id}_V)w, \end{aligned}$$

which simplifies to $T(w) = \lambda w$. Thus $\lambda$ is an eigenvalue. □

**Corollary 4.2.2.** *Let $\lambda_1, ..., \lambda_n \in F$ be distinct eigenvalues of $T$. For each $i$, let $v_i$ be an eigenvector with eigenvalue $\lambda_i$. The set $\{v_1, ..., v_m\}$ is linearly independent.*

*Proof.* We have $m_T(x) = (x - \lambda_1)(x - \lambda_2)...(x - \lambda_m)f(x)$ for some $f(x) \in F[x]$. Suppose $a_1v_1 + ... + a_m v_m = 0_V$ for some $a_i \in F$. Define $g_1(x) = (x - \lambda_2)...(x - \lambda_m)f(x)$. Note that $g_1(T)(v_i) = 0_V$ for $2 \leqslant i \leqslant m$. Then:

$$\begin{aligned} 0_V &= g_1(T)(0_V) \\ &= \sum_{j=1}^{m} a_j g_1(T)(v_j) \\ &= a_1 g_1(T)(v_1) \\ &= a_1 g_1(\lambda_1) v_1 \end{aligned}$$

But $g_1(\lambda_1) \neq 0$ and $v \neq 0$, so it must be that case that $a_1 = 0$. Inductively, it follows for $2, ..., m$. □

**Corollary 4.2.3.** *If $\deg(m_T(x)) = \dim_F(V)$ and $m_T(x)$ has distinct roots, all of which are in $F$, then we can find a basis $\mathcal{B}$ so that $[T]_{\mathcal{B}}$ is diagonal.*

**Example 4.2.5.** Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. These matrices are not similar, however $m_A(x) = m_B(x) = (x - 1)(x - 2)$. The minimal polynomial is not enough information on the similarity of matrices.

**Example 4.2.6.** Let:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

We have that $m_A(x) = (x-1)^2(x+1)$. Note that $Ae_1 = e_1$, so $E_1^1 \supseteq \mathrm{span}_F(e_1)$ (or, more simply, $e_1 \in E_1^1$). Note that $Ae_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. So $e_2 \notin E_1^1$ (another way of saying this is $(A - 1_3)e_2 \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$). But now consider:

$$(A - 1_3)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 4 \end{pmatrix}.$$

We have $(A - 1_3)^2 e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Thus $e_1, e_2 \in \ker(A - \mathrm{id}_{F^3})^2$.

**Definition 4.2.3.** Let $T \in \mathrm{Hom}_F(V, V)$. For $k \geqslant 1$, the <u>$k^{th}$ generalized eigenspace</u> of $T$ associated to $\lambda$ is $E_\lambda^k = \ker(T - \lambda\,\mathrm{id}_V)^k = \{v \in V \mid (T - \lambda\,\mathrm{id}_V)^k v = 0_V\}$. Elements of $E_\lambda^k$ are called <u>*generalized eigenvectors*</u>. Set $E_\lambda^\infty = \bigcup_{k \geqslant 1} E_\lambda^k$.

**Example 4.2.7.** Continuing Example 4.2.6, let $\alpha e_1 + \beta e_2 \in \mathrm{span}_F(e_1, e_2)$. Then:

$$(A - 1_3)^2(\alpha e_1 + \beta e_2) = \alpha(A - 1_3)^2 e_1 + \beta(A - 1_3)^2 e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So $\mathrm{span}_F(e_1, e_2) \subseteq E_1^2$. We also have $-1$ as an eigenvalue with eigenvector $v_3 = \begin{pmatrix} \frac{1}{2} \\ -2 \\ 1 \end{pmatrix}$. Check that $v_3 \notin E_1^2$. So $\dim_F(E_1^2) \leqslant 2$; i.e., $E_1^2 = \mathrm{span}_F(e_1, e_2)$. why does $v_3 \notin E_1^2$ imply the dimension which implies containment in the other direction.

**Lemma 4.2.1.** *Let $V$ be a finite dimensional $F$-vector space, $\dim_F(V) = n$, and $T \in \mathrm{Hom}_F(V, V)$. There exists $m$ with $1 \leqslant m \leqslant n$ such that $\ker(T^{\cdot}) = \ker(T^{m+1})$. Moreover, for such an $m$, $\ker(T^m) = \ker(T^{m+j})$ for all $j \geqslant 0$.*

*Proof.* We have $\ker(T^1) \subseteq \ker(T^2) \subseteq \ldots$ If these containments are always strict, then the dimension increases indefinitely, which contradicts $\dim_F(V) = n$. Hence we have an $m$ with $1 \leqslant m \leqslant n$ and $\ker(T^m) = \ker(T^{m+1})$.

Let $m$ be the smallest value where $\ker(T^m) = \ker(T^{m+1})$. We use induction on $j$. Base case of $j = 1$ is what defines $m$. Assume $\ker(T^m) = \ker(T^{m+j})$ for all $1 \leqslant j \leqslant N$. Let $v \in \ker(T^{m+N+1})$. This gives:

$$\begin{aligned} 0_V &= T^{m+N+1}(v) \\ &= T^{m+1}(T^N(v)). \end{aligned}$$

So $T^N(v) \in \ker(T^{m+1})$. However $\ker(T^{m+1}) = \ker(T^m)$, so $T^N(v) \in \ker(T^m)$. Hence:

$$\begin{aligned} 0_V &= T^m(T^n(v)) \\ &= T^{m+N}(v), \end{aligned}$$

so $v \in \ker(T^{m+N})$. Induction hypothesis gives $\ker(T^{m+N}) = \ker(T^m)$, giving $v \in \ker(T^m)$. Thus $\ker(T^{m+N+1}) \subseteq \ker(T^m)$. The other direction of containment is trivial. $\square$

**Example 4.2.8.** Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of $V$ and $T \in \mathrm{Hom}_F(V, V)$, $\lambda \in F$ such that:

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

In other words, $[T]_{\mathcal{B}}$ contains $\lambda$ along the diagonal and $1$ along the super-diagonal. Let $A = [T]_{\mathcal{B}}$. Consider:

$$(A - \lambda 1_n) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We get:

$$\begin{aligned} (A - \lambda 1_n)v_1 &= 0_V \\ (A - \lambda 1_n)v_2 &= v_1 \\ &\vdots \\ (A - \lambda 1_n)v_n &= v_{n-1}. \end{aligned}$$

This gives $E_\lambda^1 = \mathrm{span}_F(v_1)$ (by the first equation). Now observe:

$$\begin{aligned} (A - \lambda 1_n)^2 v_1 &= 0_V \\[2ex] (A - \lambda 1_n)^2 v_2 &= (A - \lambda 1_n)(A - \lambda 1_n)v_2 \\ &= (A - \lambda 1_n)v_1 \\ &= 0_V \\[2ex] (A - \lambda 1_n)^2 v_3 &= v_1 \\ &\vdots \\ (A - \lambda 1_n)^2 v_n &= v_{n-2}. \end{aligned}$$

So $E_\lambda^2 = \mathrm{span}_F(v_1, v_2)$. In general, we have that $E_\lambda^k = \mathrm{span}_F(v_1, \ldots, v_k)$. Moreover, Lemma 4.2.1 gives $E_\lambda^1 \subseteq E_\lambda^2 \subseteq \ldots \subseteq E_\lambda^k$.

**Corollary 4.2.4.** *If $\dim_F(V) = n$ and $T \in \mathrm{Hom}_F(V, V)$, there exists an $m$ with $1 \leqslant m \leqslant n$ so that for any $\lambda \in F$, $E_\lambda^\infty = E_\lambda^m$.*

**Theorem 4.2.3.** *Let $T \in \mathrm{Hom}_F(V, V)$, and $\lambda \in F$ with $(x-\lambda)^j \mid m_T(x)$. We have:*

$$\dim_F(E_\lambda^j) \geqslant j.$$

*Proof.* Write $m_T(x) = (x-\lambda)^k f(x)$ where $f(x) \in F[x]$, $f(\lambda) \neq 0$. Define $g_j(x) = (x-\lambda)^j$. We have that $(x-\lambda)^{k-1} f(x) = g_{k-1}(x) f(x)$ is *not* the minimal polynomial. So there is a $v \in V$ with $v \neq 0_V$ such that:

$$g_{k-1}(T) f(T)(v) \neq 0_V.$$

Set $v_k = f(T)(v)$. Observe that:

$$\begin{aligned}(T - \lambda\,\mathrm{id}_V)^k (v_k) &= (T-\lambda\,\mathrm{id}_V)^k f(T)(v) \\ &= m_T(T)(v) \\ &= 0_V.\end{aligned}$$

So $v_k \in E_\lambda^k$. Moreover, by our construction:

$$\begin{aligned}(T - \lambda\,\mathrm{id}_V)^{k-1}(v_k) &= g_{k-1}(T)(v_k) \\ &= g_{k-1}(T) f(T)(v) \\ &\neq 0_V.\end{aligned}$$

Hence $v_k \in E_\lambda^k \setminus E_\lambda^{k-1}$. Now set $v_{k-1} = (T-\lambda\,\mathrm{id}_V)v_k = (T-\lambda\,\mathrm{id}_V)f(T)(v)$. Note:

$$\begin{aligned}(T - \lambda\,\mathrm{id}_V)^{k-1}(v_{k-1}) &= (T-\lambda\,\mathrm{id}_V)^{k-1}(T-\lambda\,\mathrm{id}_V)(v_k) \\ &= (T-\lambda\,\mathrm{id}_V)^k(v_k) \\ &= (T-\lambda\,\mathrm{id}_V)^k f(T)(v) \\ &= m_T(T)(v) \\ &= 0_V.\end{aligned}$$

So $v_{k-1} \in E_\lambda^{k-1}$. Again, by our construction:

$$\begin{aligned}(T - \lambda\,\mathrm{id}_V)^{k-2}(v_{k-1}) &= (T-\lambda\,\mathrm{id}_V)^{k-2}(T-\lambda\,\mathrm{id}_V)(v_k) \\ &= (T-\lambda\,\mathrm{id}_V)^{k-1}(v_k) \\ &\neq 0_V.\end{aligned}$$

So $v_{k-1} \in E_\lambda^{k-1} \setminus E_\lambda^{k-2}$. Setting $v_{k-2} = (T-\lambda\,\mathrm{id}_V)^2 v_k$ gives a similar result. By this construction, we obtain a set $\{v_k, v_{k-1}, \dots, v_2, v_1\}$. Claim: this set is linearly independent. Suppose towards contradiction it's not, that is, $a_1 v_1 + \dots + a_k v_k = 0_V$ does not imply $a_1 = \dots = a_k = 0$. This gives $v_k = \frac{-1}{a_k}(a_1 v_1 + \dots + a_{k-1} v_{k-1}) \in E_\lambda^{k-1}$, which is a contradiction. It follows that $a_1 = \dots = a_k = 0$, hence $\{v_k, v_{k-1}, \dots, v_2, v_1\}$ is linearly independent. but why does this establish the theorem. □

**Example 4.2.9.** Let $T_A \in \mathrm{Hom}_F(F^3, F^3)$ be defined by:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

We have that $m_T(x) = (x-2)^3$. Now observe:

$$(A - 2 \cdot 1_3)^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note $(A-2\cdot 1_3)^2 e_3 = 4e_3 \neq 0_F^3$, but $(A-2\cdot 1_3)^3 e_3 = 0_{F^3}$. Set $v_3 = e_3$, we have $v_3 \in E_2^3$. Now observe:

$$\begin{aligned} v_2 &= (A - 2 \cdot 1_3)(v_3) \\ &= \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}. \end{aligned}$$

Similarly:

$$\begin{aligned} v_1 &= (A - 2 \cdot 1_3)(v_2) \\ &= \ldots \\ &= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence:

$$\begin{aligned} E_2^3 &= \operatorname{span}_F \left( \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 \\ 4 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 \\ 0 \\ 0 \end{smallmatrix}\right) \right) \\ E_2^2 &= \operatorname{span}_F \left( \left(\begin{smallmatrix} 3 \\ 4 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 \\ 0 \\ 0 \end{smallmatrix}\right) \right) \\ E_2^1 &= \operatorname{span}_F \left( \left(\begin{smallmatrix} 4 \\ 0 \\ 0 \end{smallmatrix}\right) \right). \end{aligned}$$

Setting $\mathcal{B} = \{v_1, v_2, v_3\}$, we have:

$$[T_A]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

## 4.3 Characteristic Polynomials

**Definition 4.3.1.** Let $A \in \mathrm{Mat}_n(F)$. The <u>*characteristic polynomial*</u> is $c_A(x) = \det(x1_n - A)$.

**Definition 4.3.2.** Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in F[x]$. The <u>*companion matrix*</u> of $f(x)$ is given by:

$$C(f(x)) = \begin{pmatrix} 1 \end{pmatrix}$$

The companion matrix shows that any polynomial $f(x) \in F[x]$ can be realized as the characteristic polynomial of a matrix.

**Lemma 4.3.1.** *If $A = C(f(x))$, then $c_A(x) = f(x)$.*

**Lemma 4.3.2.** *Let $A, B \in \mathrm{Mat}_n(F)$ be similar matrices. Then $c_A(x) = c_B(x)$.*

*Proof.* Let $A = PBP^{-1}$ for some $P \in GL_n(F)$. We have:

$$\begin{aligned} c_A(x) &= \det(x1_n - A) \\ &= \det(x1_n - PBP^{-1}) \\ &= \det(P(x1_n)P^{-1} - PBP^{-1}) \\ &= \det(P(x1_n - B)P^{-1}) \\ &= \det(P)\det(x1_n - B)\det(P^{-1}) \\ &= \det(x1_n - B) \\ &= c_B(x). \end{aligned}$$

□

**Definition 4.3.3.** For $T \in \mathrm{Hom}_F(V, V)$, let $\mathcal{B}$ be a basis of $V$ and set $c_T(x) = c_{[T]_{\mathcal{B}}}(x)$.

**Theorem 4.3.1.** *Let $v \in V$, $v \neq 0_V$. Let $\dim_F(V) = n$. Then there is a unique monic polynomial $m_{T,v}(x) \in F[x]$ so that $m_{T,v}(T)(v) = 0_V$. Moreover, if $f(x) \in F[x]$ with $f(T)(v) = 0_V$, then $m_{T,v}(x) \mid f(x)$.*

*Proof.* Consider the set $\{v, T(v), T^2(v), \dots, T^n(V)\}$. Since this set contains $n+1$ elements and the dimension of $V$ is $n$, the set must be linearly dependent. Write:

$$a_m T^m(v) + \dots + a_1 T(v) + a_0 = 0_V$$

for some $m \leqslant n$ of minimal order and $a_i \neq 0$ for all $i$. Set:

$$p(x) = x^m + \frac{a_{m-1}}{a_m}x^{m-1} + \dots + \frac{a_1}{a_m}x + \frac{a_0}{a_m} \in F[x].$$

By construction $p(T)(v) = 0_V$. Set $I_v = \{g(x) \in F[x] \mid g(T)(v) = 0_V\}$. We have that $p(x)$ is a monic nonzero polynomial in $I_v$ of minimal degree. Set $m_{T,v}(x) = p(x)$.

Let $f(x) \in I_v$. We'd like to show that $m_{T,v}(x) \mid f(x)$. Write:

$$f(x) = q(x)m_{T,v}(x) + r(x),$$

with $q(x), r(x) \in F[x]$ and $\deg(r(x)) = 0$ or $deg(r) < \deg(m_{T,v}(x))$. Observe that:

$$\begin{aligned} r(T)(v) &= f(T)(v) - q(T)m_{T,v}(T)(v) \\ &= 0_V - q(T)0_V \\ &= 0_V. \end{aligned}$$

So $r(x) \in I_v$. But $m_{T,v}(x)$ had minimal degree, so it must be the case that $r(x) = 0$. Thus $f(x) = q(x)m_{T,v}(x)$, implying $m_{T,v}(x) \mid f(x)$[1]. Now suppose $h(x) \in I_v$ with $\deg(h(x)) = \deg(m_{T,v}(x))$. Since both polynomials are monic and of equal degree, if $m_{T,v}(x) \mid h(x)$ then $m_{T,v}(x) = h(x)$. $\square$

**Definition 4.3.4.** We refer to $m_{T,v}(x)$ as the *T-annihilator* of $v$.

**Example 4.3.1.**

[1]The proof of $F[x]$ being a P.I.D. follows identically. Instead of considering $I_v$ we would consider an arbitrary polynomial in $F[x]$.