Math 395

Homework 6

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Name: Gianluca Crescenzo

Collaborators: Noah Smith, Carly Venenciano, Avinash Iyer

Exercise 1. *Let V be an $\mathbf{R}$-vector space. Prove that $\mathbf{C} \otimes_{\mathbf{R}} V \cong V_{\mathbf{C}}$*

Proof. Define $t : V \to V \oplus V$ by $v \mapsto (v, 0_V)$. Clearly $t \in \mathrm{Hom}_{\mathbf{R}}(V, V \oplus V)$. Define $\iota_V : V \to \mathbf{C} \otimes_{\mathbf{R}} V$ by $v \mapsto 1 \otimes v$; by the universal property of tensor products there exists a unique $T \in \mathrm{Hom}_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{R}} V, V_{\mathbf{C}})$ satisfying $t = T \circ \iota_V$.

Claim: Defining $T : \mathbf{C} \otimes_{\mathbf{R}} V \to V_{\mathbf{C}}$ by $1 \otimes v_1 + i \otimes v_2 \mapsto (v_1, v_2)$ satisfies the universal property. We have:

$$\begin{aligned} T(\iota_V(v)) &= T(1 \otimes v) \\ &= (v, 0_V) \\ &= t(v). \end{aligned}$$

Now let $S : V_{\mathbf{C}} \to \mathbf{C} \otimes_{\mathbf{R}} V$ be defined by $(v_1, v_2) \mapsto 1 \otimes v_1 + i \otimes v_2$. Given $v_1, v_2, v_1', v_2' \in V$ and $a + bi \in \mathbf{C}$, observe that:

$$\begin{aligned} S((v_1, v_2) + (a+bi)(v_1' + v_2')) &= S((v_1 + av_1' - bv_2', v_2 + bv_1' + av_2')) \\ &= 1 \otimes (v_1 = av_1' - bv_2') + i \otimes (v_2 = bv_1' + av_2') \\ &= \dots \\ &= S((v_1, v_2)) + (a+bi)S((v_1', v_2')). \end{aligned}$$

Hence $S \in \mathrm{Hom}_{\mathbf{C}}(V_{\mathbf{C}}, \mathbf{C} \otimes_{\mathbf{R}} V)$. This gives:

$$T(S((v_1, v_2))) =$$

$$S\left(T\left(\sum_{\text{finite}} (a+bi) \otimes v_j\right)\right) =$$

Thus $\mathbf{C} \otimes_{\mathbf{R}} V \cong V_{\mathbf{C}}$. $\square$