Math 310

Homework 8

Due: 11/26/2024

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Exercise 1. Recall that a subset $U \subseteq \mathbb{R}$ is **open** if:

$$(\forall x \in U)(\exists \epsilon > 0) : V_\epsilon(x) \subseteq U.$$

Prove that the mapping $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(U) \subseteq \mathbb{R}$ is open for every open $U \subseteq \mathbb{R}$.

Proof. ($\Rightarrow$) Let $U \subseteq \mathbb{R}$ be open and suppose $c \in f^{-1}(U)$. Then $f(c) \in U$. Since U is open, there exists $\epsilon > 0$ with $V_\epsilon(f(c)) \subseteq U$. Since f is continuous at c, there exists $\delta > 0$ such that $x \in V_\delta(c)$ implies $f(x) \in V_\epsilon(f(c))$. Thus $f(V_\delta(c)) \subseteq V_\epsilon(f(c))$, whence $V_\delta(c) \subseteq f^{-1}(V_\epsilon(f(c))) \subseteq f^{-1}(U)$. Thus $f^{-1}(U)$ is open.

($\Leftarrow$) Let $c \in \mathbb{R}$ and $\epsilon > 0$. Note that $f^{-1}(V_\epsilon(f(c)))$ is open. So there exists $\delta > 0$ such that $V_\delta(c) \subseteq f^{-1}(V_\epsilon(f(c)))$. Moreover, $U \cap V_\delta(c) \subseteq f^{-1}(V_\epsilon(f(c)))$. Hence $f(U \cap V_\delta(c)) \subseteq V_\epsilon(f(c))$. Thus f is continuous. □

Exercise 2. Let $f, g : D \to \mathbb{R}$ be continuous. Show that the product fg is continuous.

Proof. Since f, g are continuous functions, $(x_n)_n \to c$ implies $(f(x_n))_n \to c$ and $(g(x_n))_n \to c$. Whence $((fg)(x_n))_n = (f(x_n)g(x_n))_n \to f(c)g(c) = (fg)(c)$. □

Exercise 3. Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be continuous mappings with $\operatorname{Ran}(f) \subseteq E$. Show that $g \circ f$ is continuous.

Proof. Let $c \in D$ be arbitrary. Given $\delta_1 > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \delta_1$. Now since g is continuous, it is continuous at $f(c)$. Hence given $\epsilon > 0$, we have $|f(x) - f(c)| < \delta_1$ implies $|g(f(x)) - g(f(c))| < \epsilon$. Thus $g \circ f$ is continuous. □

Exercise 4. Show that the following functions are Lipschitz.

(1) $f : [-M, M] \to \mathbb{R}$ given by $f(x) = x^2$.

(2) $g : [1, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$.

(3) $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \sqrt{x^2 + 4}$.

Proof. Observe that:

$$\begin{aligned} |f(u) - f(v)| &= |u^2 - v^2| \\ &= |(u + v)(u - v)| \\ &\leqslant |u + v||u - v| \\ &\leqslant M^2|u - v|. \end{aligned}$$

$$
\begin{aligned}
|g(u) - g(v)| &= \left|\frac{1}{u} - \frac{1}{v}\right| \\
&= \frac{|u - v|}{uv} \\
&\leqslant |u - v|.
\end{aligned}
$$

$$
\begin{aligned}
|h(u) - h(v)| &= |\sqrt{u^2 + 4} - \sqrt{v^2 + 4}| \\
&= \frac{|\sqrt{u^2 + 4} - \sqrt{v^2 + 4}|\sqrt{u^2 + 4} + \sqrt{v^2 + 4}|}{|\sqrt{u^2 + 4} - \sqrt{v^2 + 4}|} \\
&= \frac{|u^2 + v^2|}{|\sqrt{u^2 + 4} + \sqrt{v^2 + 4}|} \\
&\leqslant \frac{|u - v||u + v|}{|u| + |v|} \\
&\leqslant \frac{|u - v|(|u| + |v|)}{|u| + |v|} \\
&\leqslant |u - v|.
\end{aligned}
$$

□

Exercise 5. Show that the following functions are **not** Lipschitz.

(1) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

(2) $g : (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$.

Proof. (1) Let $u_n = n$ and $v_n = n + \frac{1}{n}$. Then:

$$
|u_n - v_n| = \left|n - \left(n + \frac{1}{n}\right)\right| = \frac{1}{n}.
$$

Hence $(u_n - v_n)_n \to 0$. But observe that:

$$
\begin{aligned}
|f(u_n) - f(v_n)| &= \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| \\
&= \left|n^2 - n^2 - 2 - \frac{1}{n^2}\right| \\
&= 2 + \frac{1}{n^2} \\
&\geqslant 2.
\end{aligned}
$$

Set $\epsilon_0 = 2$, $u_n = n$, and $v_n = n + \frac{1}{n}$. Then by the work above, f is not uniformly continuous. Hence f is not Lipschitz.

(2) Let $u_n = \frac{1}{n}$ and $v_n = \frac{1}{n+1}$. Then:

$$
|u_n - v_n| = \left|\frac{1}{n} - \frac{1}{n + 1}\right| = \frac{1}{n(n + 1)} \leqslant \frac{1}{n}.
$$

Since $\left(\frac{1}{n}\right)_n \to 0$, we have $(u_n - v_n)_n \to 0$. But observe that:

$$|f(u_n) - f(v_n)| = |n - (n+1)| = 1.$$

Set $\epsilon_0 = 1$, $u_n = \frac{1}{n}$, and $v_n = \frac{1}{n+1}$. Then by the work above, f is not uniformly continuous. Hence f is not Lipschitz. □

Exercise 6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and for some $C \geqslant 0$ we have $|f(q)| \leqslant C$ for all $q \in \mathbb{Q}$. Show that $\|f\|_{\mathbb{R}} \leqslant C$.

Proof. Let $\alpha \in \mathbb{R}$ be arbitrary. By the density of $\mathbb{Q}$, there exists a sequence $(q_n)_n$ in $\mathbb{Q}$ with $(q_n)_n \to \alpha$. Since f is continuous, $(q_n)_n \to \alpha$ implies $(f(q_n))_n \to f(\alpha)$. Moreover, $(|f(q_n)|)_n \to |f(\alpha)|$. Since $|f(q)| \leqslant C$ for all $q \in \mathbb{Q}$, it must be that $|f(\alpha)| \leqslant C$. Hence $\|f\|_{\mathbb{R}} \leqslant C$. □

Exercise 9. Let p be a polynomial of odd degree. Show that p has a real root.

Proof. Without loss of generality, let the leading term of p be positive. Since deg(p) is odd:

$$\operatorname*{limit}_{x \to \infty} p(x) = \infty$$
$$\operatorname*{limit}_{x \to -\infty} p(x) = -\infty.$$

For $M = 1$, there exists α such that $x \geqslant \alpha$ implies $p(x) \geqslant 1$. Similarly, there exists β such that $x \leqslant \beta$ implies $p(x) \leqslant -1$. So there exists x_1, x_2 with $x_1 < x_2$ and $p(x_1)p(x_2) < 0$. By the Location of Roots lemma, there exists c such that $p(c) = 0$. Hence p has a real root. □

Exercise 10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that **vanishes at infinity**, that is,

$$\operatorname*{limit}_{x \to \pm\infty} f(x) = 0.$$

Show that f is bounded.

Proof. Let ϵ be given. Since $\operatorname{limit}_{x \to -\infty} f(x) = 0$, there exists α_1 such that for all $x \in \mathbb{R}$, $x < \alpha_1$ implies $|f(x)| < \epsilon$. Since $\operatorname{limit}_{x \to \infty} f(x) = 0$, there exists α_2 such that for all $x \in \mathbb{R}$, $x > \alpha_2$ implies $|f(x)| < \epsilon$.

Since f is continuous, it is bounded on $[\alpha_1, \alpha_2]$. So there exists c such that $|f(x)| \leqslant c$ for all $x \in [\alpha_1, \alpha_2]$.

Let $M = \max\{\epsilon, c\}$. Then $|f(x)| \leqslant M$ for all $x \in \mathbb{R}$. Hence f is bounded. □

Exercise 12. Let $f : [a, b] \to \mathbb{R}$ be a continuous function satisfying the following property:

$$(\forall x \in [a, b])(\exists y \in [a, b]) : |f(y)| \leqslant \frac{1}{2}|f(x)|.$$

Show that there is a $c \in [a, b]$ with $f(c) = 0$.

Proof. Let $x \in [a, b]$ be given. By the above property, we can inductively obtain a sequence $(y_n)_n$ such that $|f(y_n)| \leqslant \frac{1}{2^n}|f(x)|$. Whence $(f(y_n))_n \to 0$.

Moreover, since $(y_n)_n \in [a, b]^{\mathbb{N}}$, by Bolzano-Weierstass there exists a convergent subsequence $(y_{n_k})_k \to c$. Since f is continuous, we have that $(f(y_{n_k}))_k \to f(c)$. Whence $f(c) = 0$. □