Math 395

Homework 4

Due: 10/3/2024

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Exercise 1. *Let $T \in \operatorname{Hom}_F(V,V)$. Prove that the intersection of any collection of T-invariant subspaces of V is T-invariant.*

Proof. Let $\{W_i\}_{i\in I}$ be a collection of T-stable subspaces of V. Let $x \in T\left(\bigcap_{i\in I} W_i\right)$. Then $x \in T(W_i)$ for all $i \in I$. So $x \in W_i$ for all $i \in I$, establishing $x \in \bigcap_{i\in I} W_i$. Thus $T\left(\bigcap_{i\in I} W_i\right) \subseteq \bigcap_{i\in I} W_i$. □

Exercise 2. *Let $T \in \operatorname{Hom}_F(V,V)$ and $v \in V$. Prove that if $T^j(v) \in W = \operatorname{span}_F(v_1,\dots,v_n)$ and W is T-invariant, then $T^{j+t}(v) \in W$ for all $t \geqslant 0$.*

Proof. We prove this by induction on t. Let $t = 0$ be the base case, then by assumption $T^j(v) \in W$. Assume our hypothesis to be true up to $t-1$. Then:

$$T^t(T^j(v)) = T(T^{t-1}(T^j(v))).$$

Our induction hypothesis gives $T^{t-1}(T^j(v)) \in W$, and since $T(W) \subseteq W$, we have:

$$T^{j+t}(v) = T(T^{t-1}(T^j(v))) \in W.$$

□