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Chapter 1

Vector Spaces, Algebras, and Normed Spaces

For the entirety of this chapter assume F to be $\mathbf{R}$ or $\mathbf{C}$.

§ 1.1. Vector Spaces

Definition 1.1.1. A *vector space* (or *linear space*) over F is a nonempty set V equipped with two operations:

$$\begin{aligned} V \times V \xrightarrow{+} V &\text{ defined by } (v, w) \mapsto v + w \\ F \times V \to V &\text{ defined by } (\alpha, v) \mapsto \alpha v \end{aligned}$$

satisfying:

(1) $(V, +)$ is an abelian group:

- (i) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$;
- (ii) there exists 0_V such that $v + 0_V = 0_V + v = v$ for all $v \in V$;
- (iii) for all $v \in V$, there exists $w \in V$ satisfying $v + w = w + v = 0_V$;
- (iv) $v + w = w + v$ for all $v, w \in V$;

(2) $(\alpha + \beta)v = \alpha v + \beta v$ for all $\alpha, \beta \in F$, $v \in V$;

(3) $\alpha(\beta v) = (\alpha\beta)v$ for all $\alpha, \beta \in F$, $v \in V$;

(4) $\alpha(v + w) = \alpha v + \alpha w$ for all $\alpha \in F$, $v, w \in V$;

(5) $1_F v = v$ for all $v \in V$.

It can be shown that the vector 0_V is unique, the additive inverse in (iii) is unique (which we denote as $-v$), that $0v = 0_V$, and $(-1)v = -v$.

Exercise 1.1.1. Show (iv) follows from the other axioms.

Exercise 1.1.2. Show $nv = \underbrace{v + v + ... + v}_{n \text{ times}}$ for $n \in \mathbf{Z}_{\geqslant 1}$.

It can be shown that a subspace is a vector space in its own right.

Example 1.1.1. Let $\{W_i\}_{i\in I}$ be a family of vector spaces. Then $\bigcap_{i\in I} W_i$ is also a vector space.

Example 1.1.2. Planes and lines through the origin are subspaces of $\mathbf{R}^3$.

Definition 1.1.2. Let V be a vector space and $S \subseteq V$ a subset.

(1) A *linear combination* from S is a finite sum $\sum_{j=1}^{n} \alpha_j v_j$ with $\alpha_j \in F$, $v_j \in S$.

(2) The *linear span* of S is:

$$\operatorname{span}(S) := \left\{ \sum_{j=1}^{n} \alpha_j v_j \;\middle|\; n \in \mathbf{N}, \alpha_j \in F, v_j \in S \right\}.$$

Exercise 1.1.3. Show that $\operatorname{span}(S) \subseteq V$ is a subspace and:

$$\operatorname{span}(S) = \bigcap \{W \mid S \subseteq W, W \text{ is a subspace }\},$$

that is, $\operatorname{span}(S)$ is the smallest subspace of V containing S.

Definition 1.1.3. Let V be a vector space and $S \subseteq V$ a subset.

(1) S is *spanning* for V if $\operatorname{span}(S) = V$.

(2) S is *independent* if, given $n \in \mathbf{N}$, $\alpha_1, ..., \alpha_n \in F$, $v_1, ..., v_n \in S$, then $\sum_{j=1}^{n} \alpha_j v_j = 0$ implies $\alpha_j = 0$ for all j.

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Our goal is to show that every vector space admits a basis. As such, recall the following definitions from a standard course in Real Analysis.

Definition 1.1.4. An *ordering* on a set X is a relation $R \subseteq X \times X$ on X that is reflexive, transitive, and antisymmetric. We write xRy as $x \leqslant_R y$. The pair $(X, \leqslant_R)$ is called an *ordered set*. An ordering $\leqslant$ on X is called *total* (or *linear*) if for all $x, y \in X$, $x \leqslant y$ or $y \leqslant x$.

Note that if $(X, \leqslant)$ is an ordered set and $Y \subseteq X$ is a subset, then $(Y, \leqslant)$ is an ordered set as well.

Definition 1.1.5. Let $(X, \leqslant)$ be an ordered set and $Y \subseteq X$. An *upper bound* for Y is an element $u \in X$ with $u \geqslant y$ for all $y \in Y$. An element $m \in X$ is called *maximal* if $x \in X$, $x \geqslant m$ implies $x = m$.

Lemma 1.1.1 (Zorn's Lemma)**.** *Let $(X, \leqslant_X)$ be an ordered set. Suppose every subset $Y \subseteq X$ for which $(Y, \leqslant_X)$ is totally ordered has an upper bound in X. Then X admits a maximal element.*

The proof of Zorn's Lemma is outside the interest of this text.

Theorem 1.1.2. *Every vector space admits a basis. Moreover, every independent set is contained in a basis.*

Proof. Let $S \subseteq V$ be linearly independent. Define:

$$\mathfrak{T}(S) = \{T \subseteq V \mid S \subseteq T, T \text{ linearly independent}\}.$$

Let $\mathfrak{C} \subseteq \mathfrak{T}(S)$ be a totally ordered subset. Set $R = \bigcup_{T \in \mathfrak{C}} T$. Clearly $R \supseteq S$. Assume $\sum_{j=1}^{n} \alpha_j v_j = 0$, where $\alpha_j \in F$ and $v_j \in R$. Since $\mathfrak{C}$ is totally ordered, there exists $T_0 \in \mathfrak{C}$ with $v_j \in T_0$ for all $j = 1, ..., n$. Since T_0 is independent, $\alpha_j = 0$ for all $j = 1, ..., n$. Thus R is independent as well. Whence R is an upper bound for $\mathfrak{C}$. By Zorn's Lemma, $\mathfrak{T}(S)$ admits a maximal element, call it B.

Claim: B is a basis for V. Suppose towards contradiction it's not, then there exists $v_0 \in V \setminus \mathrm{span}(B)$. Consider $B \cup \{v_0\}$ and let $\alpha_0 v_0 + \sum_{j=1}^{n} \alpha_j v_j = 0_V$. If $\alpha_0 \neq 0$, then $\sum_{j=1}^{n} \alpha_j v_j = -\alpha_0 v_0$, giving $v_0 \in \mathrm{span}(B)$ which is a contradiction. If $\alpha_0 = 0$, then $\sum_{j=1}^{n} \alpha_j v_j = 0_V$. Since B is independent, $\alpha_j = 0$ for all $j = 1, ..., n$. Thus $B \cup \{v_0\}$ is independent, contradicting the maximality of B. Whence B is a basis for V. □

Theorem 1.1.3. *If B_1 and B_2 are bases for V, then* $\mathrm{card}(B_1) = \mathrm{card}(B_2)$.

Definition 1.1.6. If V is a vector space, its *dimension* is the cardinality of any of its bases.

Corollary 1.1.4. *If B is a basis for V, then every $v \in V$ can be written $v = \sum_{j=1}^{n} \alpha_k \beta_k$, $\alpha_k \in F$, $b_k \in B$ in a unique way.*

Theorem 1.1.5. *Let V be a linear space and $B \subseteq V$ a subset. The following are equivalent:*

(1) B is a basis for V;

(2) B is a maximal element in $\mathfrak{T} = \{T \subseteq V \mid T \text{ independent}\}$;

(3) B is a minimal element in $\mathfrak{S} = \{S \subseteq V \mid S \text{ spans } V\}$;

Definition 1.1.7. Let $\{V_i\}_{i \in I}$ be a family of vector spaces over a field F.

(1) The *product* of $\{V_i\}_{i \in I}$ is denoted:

$$\prod_{i \in I} V_i := \{(v_i)_{i \in I} \mid v_i \in V_i\}.$$

(2) The *co-product* (or *sum*) is denoted

$$\bigoplus_{i \in I} V_i := \left\{(v_i)_{i \in I} \mid v_i \in V_i, \, \mathrm{supp}\big((v_i)_{i \in I}\big) < \infty\right\}.$$

Exercise 1.1.4.

(1) Show that $\prod_{i\in I} V_i$ equipped with pointwise operations:

$$(v_i)_{i\in I} + (w_i)_{i\in I} = (v_i + w_i)_{i\in I}$$
$$\alpha(v_i)_{i\in I} = (\alpha v_i)_{i\in I}$$

is a linear space.

(2) Show that $\bigoplus_{i\in I} V_i$ is a subspace of $\prod_{i\in I} V_i$.

Proposition 1.1.6. *Let V be a vector space over F and $W \subseteq V$. The (additive, abelian) quotient group V/W can be made into a vector space by defining multiplication by scalars as $\alpha(v + W) = \alpha v + W$ for all $\alpha \in F$, $v + W \in V/W$.*

Example 1.1.3.

(1) The set $F^n = \{(x_1, ..., x_n) \mid x_j \in F\}$ with component-wise operations is a vector space.

(2) The set $M_{n,m}(F) = \{(a_{ij}) \mid a_{ij} \in F\}$ with linear operations is a vector space.

(3) Let Ω be a nonempty set. Then $\mathcal{F}(\Omega, F) = \{f \mid f : \Omega \to F\}$ with pointwise operations is a vector space.

(4) The set $\ell_\infty(\Omega, F) = \{f \in \mathcal{F}(\Omega, F) \mid \|f\|_u < \infty\}$ with pointwise operations is a vector space.

Exercise 1.1.5. Show $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(5) Let $f : [a, b] \to \mathbf{R}$ be any function. Let $\mathcal{P} = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$. The *variation of f on $\mathcal{P}$* is defined as:

$$\mathrm{Var}(f; \mathcal{P}) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|.$$

We say f is a *bounded variation* if:

$$\mathrm{Var}(f) := \sup_{\mathcal{P}} \mathrm{Var}(f; \mathcal{P}) < \infty.$$

The set of all functions of bounded variation is defined:

$$\mathrm{BV}([a, b]) = \{f : [a, b] \to \mathbf{R} \mid \mathrm{Var}(f) < \infty\}.$$

This is a vector space by defining addition and scalar multiplication componentwise.

Exercise 1.1.6. Show that BV([a,b]) $\subseteq \ell_\infty([a, b], \mathbf{R})$ is a subspace.

(6) Let $K \subseteq V$ be a convex subset of a vector space V, that is, for all $v, w \in K$ and $t \in [0, 1]$, then $(1-t)v + tw \in K$. A function $f : K \to F$ is said to be *affine* if $x, y \in K$ and $t \in [0, 1]$ implies $f((1 - t)x + ty) = (1 - t)f(x) + tf(y)$. The set $\mathrm{Aff}(K, F) = \{f \in \mathcal{F}(\Omega, F) \mid f \text{ affine}\}$ with pointwise operations is a vector space.

Exercise 1.1.7. Show $\mathrm{Aff}(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(7) The set $C([a,b],F) = \{f : [a,b] \to F \mid f \text{ continuous}\}$ with pointwise operations is a vector space.

Exercise 1.1.8. Explain why $C([a,b],F) \subseteq \ell_\infty([a,b],F)$ is a subspace.

(8) Consider the following sequence spaces:

- $s = \{(a_k)_k \mid a_k \in F\} = \mathcal{F}(\mathbf{N}, F)$;
- $\ell_\infty = \ell_\infty(\mathbf{N}, F) = \{(a_k)_k \mid \sup_{k \geqslant 1} |a_k| < \infty\}$;
- $c = \{(a_k)_k \mid (a_k)_k \text{ converges }\}$;
- $c_0 = \{(a_k)_k \mid (a_k)_k \to 0\}$;
- $c_{00} = \{(a_k)_k \mid \operatorname{supp}\big((a_k)_k\big) < \infty\}$;
- $\ell_1 = \{(a_k)_k \mid \sum_{k=1}^\infty |a_k| < \infty\}$.

These are all vector spaces with pointwise operations. In fact, $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subspaces.

Exercise 1.1.9. Show that $\ell_1 \subseteq c_0$ is a subspace.

(9) Consider the following continuous function spaces on $\mathbf{R}$:

- $C(\mathbf{R}) = \{f : \mathbf{R} \to F \mid f \text{ continuous }\}$;
- $C_b(\mathbf{R}) = C(\mathbf{R}) \cap \ell_\infty(\mathbf{R})$;
- $C_0(\mathbf{R}) = \{f \in C(\mathbf{R}) \mid \operatorname{limit}_{x \to \pm\infty} f(x) = 0\}$;
- Recall that a function is *compactly supported* if for all $\epsilon > 0$, there exists $\alpha > 0$ such that $|x| \geqslant \alpha$ implies $f(x) = 0$. The set of compactly supported functions is denoted $C_c(\mathbf{R}) = \{f \in C(\mathbf{R}) \mid f \text{ compactly supported }\}$.

These are all vector spaces with pointwise operations, and $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq C_b(\mathbf{R}) \subseteq C(\mathbf{R})$ are all subspace inclusion.

Definition 1.1.8. If V and W are linear spaces over a common field F, a map $T : V \to W$ is called *linear* if $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha \in F$.

Example 1.1.4. Let $A \in M_{m,n}(F)$. Then $T_A : F^n \to F^m$ defined by $T_A(v) = Av$ is linear. Let $\{e_1, ..., e_n\}$ be a basis for F^n. If $T : F^n \to F^m$ is linear, set:

$$[T] = \Big(T(e_1)\ \Big|\ T(e_2)\ \Big|\ \ ...\ \ \Big|\ T(e_n)\Big).$$

This gives $T(v) = [T]v$ for all $v \in F^n$. In fact, we also have $[T_A] = A$ and $T_{[T]} = T$.

Example 1.1.5. The *canonical projection* is linear:

$$\pi_j : \prod_{i \in I} V_i \to V_j \ \text{ defined by } \ \pi_j\big((v_i)_i\big) = v_i.$$

We also have that the *coordinate exclusions* are linear:

$$\iota_j : V_j \hookrightarrow \bigoplus_{i \in I} V_i \ \text{ defined by } \ \iota_j(v) = (v_i)_i\,, \text{ where } \ v_i = \begin{cases} 0_v, & i \neq j \\ v_j, & \text{otherwise.} \end{cases}$$

The *evaluation map* is linear as well. For $s \in S$, consider:

$$e_s : \mathcal{F}(S, F) \to F \text{ defined by } e_s(f) = f(s).$$

Proposition 1.1.7. *Let V be a vector space with basis B. Let W be a vector space and suppose $\varphi : B \to W$ is a map. Then there exists a unique linear map $T_\varphi : V \to W$ with $T_\varphi(b) = \varphi(b)$ for all $b \in B$. We have the following diagram.*

$$\begin{tikzcd} B \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & V \arrow[d, dotted, "T_\varphi"] \\ & W \end{tikzcd}$$

Proof. Define $T_\varphi : V \to W$ by:

$$\begin{aligned} T_\varphi(v) &= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\ &= \sum_{j=1}^{n} \alpha_j \varphi(b_j). \end{aligned}$$

Let $v_1, v_2 \in V$ and $c \in F$. We have that:

$$\begin{aligned} T_\varphi(v_1 + cv_2) &= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j + c\sum_{j=1}^{n} \beta_j b_j\right) \\ &= T_\varphi\left(\sum_{j=1}^{n} (\alpha_j + c\beta_j) b_j\right) \\ &= \sum_{j=1}^{n} (\alpha_j + c\beta_j)\varphi(b_j) \\ &= \sum_{j=1}^{n} \alpha_j \varphi(b_j) + c\sum_{j=1}^{n} \beta_j \varphi(b_j) \\ &= T_\varphi(v_1) + cT_\varphi(v_2). \end{aligned}$$

Thus T_φ is linear. Chasing the above diagram makes it clear that $T_\varphi(b) = \varphi(b)$. It remains to show that T_φ is unique. Let T be another linear transformation satisfying $T(b) = \varphi(b)$ for all $b \in B$. Then:

$$\begin{aligned} T(v) &= T\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\ &= \sum_{j=1}^{n} \alpha_j \varphi(b_j) \\ &= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\ &= T_\varphi(v). \end{aligned}$$

Thus T_φ is unique. $\square$

Proposition 1.1.8. *Let $T : V \to W$ be linear.*

(1) $\ker(T) = \{v \in V \mid T(v) = 0_W\}$ *is a linear subspace of V.*

(2) $\operatorname{im}(T) = \{T(v) \mid v \in V\}$ *is a linear subspace of W.*

(3) $\ker(T) = \{0_V\}$ *if and only if T is injective.*

(4) $\operatorname{im}(T) = W$ *if and only if T is surjective.*

Proof. (1) Let $v_1, v_2 \in \ker(T)$ and $\alpha \in F$. Observe that:

$$\begin{aligned} T(v_1 + cv_2) &= T(v_1) + cT(v_2) \\ &= 0. \end{aligned}$$

Thus $v_1 + cv_2 \in \ker(T)$, giving $\ker(T)$ as a linear subspace of V.

(2) Let $w_1, w_2 \in \operatorname{im}(T)$. Then there exists $v_1.v_2 \in V$ with $T(v_1) = w_1$ and $T(v_2) = w_2$. We have:

$$\begin{aligned} w_1 + cw_2 &= T(v_1) + cT(v_2) \\ &= T(v_1 + cv_2). \end{aligned}$$

Whence $w_1 + cw_2 \in \operatorname{im}(T)$, giving $\operatorname{im}(T)$ as a linear subspace of W.

(3) Let $\ker(T) = \{0\}$. Suppose $T(v_1) = T(v_2)$. Then $T(v_1) - T(v_2) = T(v_1 - v_2) = 0_W$. It must be that $v_1 - v_2 = 0_W$, giving $v_1 = v_2$. Thus T is injective. Conversely, suppose T is injective and let $v \in \ker(T)$. Then $T(v) = 0_W = T(0_V)$. Hence $v = 0_V$, establishing $\ker(T) = \{0\}$.

(4) This is by definition of surjectivity. $\square$

Proposition 1.1.9. *If $T : V \to W$ is linear and bijective, then the inverse map $T^{-1} : W \to V$ is linear.*

Proof. We have that:

$$T(T^{-1}(w_1) + \alpha T^{-1}(w_2)) = w_1 + \alpha w_2 = T \circ T^{-1}(w_1 + \alpha w_2).$$

Applying T^{-1} to both sides gives the desired result. $\square$

Proposition 1.1.10 (Vector Spaces are Injective)**.** *Let U, V, W be vector spaces and $0 \to U \xrightarrow{j} V$ be exact (that is, j is injective). Let $\varphi : U \to W$ be linear. There exists a linear map $\Psi : V \to W$ such that $\varphi = \Psi \circ j$; i.e., the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "j"] \arrow[d, "\varphi"'] & V \arrow[dl, dotted, "\Psi"] \\ & W & \end{tikzcd}$$

Proof. Let $\{u_i\}_{i\in I}$ be a basis for U. We must first show that $\{j(u_i)\}_{i\in I}$ is linearly independent. Notice that:

$$\begin{aligned} 0_V &= \sum_{i\in I} \alpha_i j(u_i) \\ &= j\left(\sum_{i\in I} \alpha_i u_i\right). \end{aligned}$$

By the injectivity of j, we have that $\sum_{i\in I} \alpha_i u_i = 0_U$. Thus $\alpha_i = 0$ for all $i \in I$, giving $\{j(u_i)\}_{i\in I}$ as linearly independent.

Since $\{j(u_i)\}_{i\in I}$ is linearly independent in V, we can extend it to a basis $B = \{v_i\}_{i\in J}$ where $I \subseteq J$ and $v_i = j(u_i)$ whenever $i \in I$. Now define $\psi : B \to W$ by:

$$\psi(v_i) = \begin{cases} \varphi(u_i), & i \in I \\ w, & i \in J \setminus I\ , \end{cases}$$

where $w \in W$ is arbitrary. Since this is a map of basis elements, there exists a unique linear map $\Psi : V \to W$ with $\Psi(v_i) = \psi(v_i)$ for all $v_i \in B$. We can finally see that:

$$\begin{aligned} \varphi(u_i) &= \psi(v_i) \\ &= \Psi(v_i) \\ &= \Psi(j(u_i)). \end{aligned}$$

This establishes that $\varphi = \Psi \circ j$. □

Proposition 1.1.11 (Vector Spaces are Projective)**.** *Let U, V, W be vector spaces and $V \xrightarrow{\pi} U \to 0$ be exact (that is, π is onto). Let $\varphi : W \to U$ be linear. There exists a linear map $\Psi : V \to W$ such that $\varphi = \pi \circ \Psi$; i.e., the following diagram commutes:*

$$\begin{tikzcd} & W \arrow[dl, dotted, "\Psi"'] \arrow[d, "\varphi"] & \\ V \arrow[r, "\pi"] & U \arrow[r] & 0 \end{tikzcd}$$

Proof. Let $B = \{w_i\}_{i\in I}$ be a basis for W. Define $\psi : B \to V$ by $\psi(w_i) = \pi^{-1}(\varphi(w_i))$. Since this is a map of basis elements, it extends to a unique (dependent on π^{-1}) linear map $\Psi : W \to V$ with $\Psi(w_i) = \psi(w_i)$ for all $w_i \in B$. Moreover, we have that:

$$\begin{aligned} (\pi \circ \Psi)(w_i) &= (\pi \circ \psi)(w_i) \\ &= (\pi \circ (\pi^{-1} \circ \varphi))(w_i) \\ &= \varphi(w_i). \end{aligned}$$

□

Definition 1.1.9. Let V and W be vector spaces over F. A *linear isomorphism* between V and W is a bijective linear map $T : V \to W$. If such a T exists, we say V and W are *linearly isomorphic*, and write $V \cong W$.

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Finite dimensional vector spaces are boring. This is illustrated through the following theorem.

Theorem 1.1.12. *Let V and W be finite-dimensional vector spaces over F. Then $V \cong W$ if and only if $\dim(V) = \dim(W)$.*

Proof. Suppose $V \cong W$. Then there is an isomorphism taking basis of V to a basis of W. Therefore they have the same dimension.

Conversely, if $\dim(V) = \dim(W) = n$, then they are each isomorphic to F^n, giving that they are isomorphic to eachother. $\square$

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Example 1.1.6. Let V be a vector space, $W \subseteq V$ a subspace. The *natural projection*:

$$\pi : V \to V/W \text{ defined by } \pi(v) = v + W$$

is a linear surjective map.

Theorem 1.1.13 (First Isomorphism Theorem for Vector Spaces)**.** *Let $T : V \to V'$ be a linear map and $W \subseteq V$ a subspace.*

(1) If T "kills" W (that is, $W \subseteq \ker(T)$), then there exists a linear map $\widetilde{T} : V/W \to V'$ with $\widetilde{T} \circ \pi = T$; i.e., the following diagram commutes.

$$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, "\pi"'] & & V' \\ & V/W \arrow[ur, dotted, "\widetilde{T}"'] & \end{tikzcd}$$

(2) If $\ker(T) = W$, then $\widetilde{T}$ is injective.

(3) If $\ker(T) = W$ and $\operatorname{im}(T) = V'$, then $V/W \cong V'$.

Proof. (1) As stipulated, define $\widetilde{T}(v + W) = T(v)$. We must show that $\widetilde{T}$ is well-defined: suppose $v_1 + W = v_2 + W$ for some $v_1, v_2 \in V$. Then $v_1 = v_2 + w$ for some $w \in W$. This gives:

$$\begin{aligned} \widetilde{T}(v_1 + W) &= \widetilde{T}(v_2 + w + W) \\ &= \widetilde{T}(v_2 + W). \end{aligned}$$

Whence $\widetilde{T}$ is well-defined. Now given $v_1 + W, v_2 + W \in V/W$ and $\alpha \in F$, observe that:

$$\begin{aligned} \widetilde{T}\big((v_1 + W) + c(v_2 + W)\big) &= \widetilde{T}\big((v_1 + cv_2) + W\big) \\ &= T(v_1 + cv_2) \\ &= T(v_1) + cT(v_2) \\ &= \widetilde{T}(v_1 + W) + c\widetilde{T}(v_2 + W). \end{aligned}$$

Thus $\widetilde{T}$ is linear.

(2) If $\ker(T) = W$, then:

$$\begin{aligned}\ker(\widetilde{T}) &= \{v + W \mid \widetilde{T}(v + W) = 0_{V'}\}\\ &= \{v + W \mid T(v) = 0_{V'}\}\\ &= \{v + W \mid v \in \ker(T)\}\\ &= \{v + W \mid v \in W\}\\ &= \{0\}.\end{aligned}$$

Thus $\widetilde{T}$ is injective.

(3) It remains to show that $\operatorname{im}(T) = V'$ implies $\widetilde{T}$ is surjective. Observe that:

$$\begin{aligned}\operatorname{im}(\widetilde{T}) &= \{\widetilde{T}(v + W) \mid v + W \in V/W\}\\ &= \{\widetilde{T}(\pi(v)) \mid v \in V\}\\ &= \{T(v) \mid v \in V\}\\ &= \operatorname{im}(T)\\ &= V'.\end{aligned}$$

Thus $\widetilde{T}$ is surjective, which establishes it as a bijection. This gives $V/W \cong V'$. $\square$

Definition 1.1.10. Let S be a nonempty set. The *free vector space* of S is:

$$\mathbf{F}(S) = \{f : S \to F \mid \operatorname{supp}(f) < \infty\}.$$

Exercise 1.1.10. Show $\mathbf{F}(S) \subseteq \mathcal{F}(S, F)$ is a subspace.

Proposition 1.1.14. *The set $\{\delta_s \mid s \in S\}$ is a basis for $\mathbf{F}(S)$, where $\delta_s : S \to F$ is defined by:*

$$\delta_s(t) = \begin{cases} 1, & t = 0\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. If $f \in \mathbf{F}(S)$ with $\operatorname{supp}(f) = \{s_1, ..., s_n\}$, then $f = \sum_{k=1}^n f(s_k)\delta_{s_k}$. If $\sum_{k=1}^n \alpha_k \delta_{s_k} = 0$, then for $j = 1, ..., n$ we have $0 = \left(\sum_{k=1}^n \alpha_k \delta_{s_k}\right)(s_j) = \alpha_j$. $\square$

Theorem 1.1.15. *Given any vector space V and a map (of sets) $\varphi : S \to V$, there exists a unique linear map $T_\varphi : \mathbf{F}(S) \to V$ with $T_\varphi \circ \iota = \varphi$, where $\iota : S \to \mathbf{F}(S)$ is defined by $\iota(s) = \delta_s$ for all $s \in S$. The following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathbf{F}(S) \arrow[d, dotted, "T_\varphi"] \\ & V \end{tikzcd}$$

Proof. By the previous proposition, we have that $B = \{\delta_s \mid s \in S\}$ is a basis for $\mathbf{F}(S)$. Define $T : B \to V$ by $T(\delta_s) = \varphi(s)$. Since this is a map of basis elements, there exists a unique linear map $T_\varphi : \mathbf{F}(S) \to V$ with $T_\varphi(\delta_s) = T(\delta_s)$ for all $\delta_s \in B$. The diagram commutes because:

$$\begin{aligned}\varphi(s) &= T(\delta_s)\\ &= T_\varphi(\delta_s)\\ &= T_\varphi(\iota(s)).\end{aligned}$$

Moreover, if T' satisfies $\varphi = T' \circ \iota$, then:

$$\begin{aligned}T'(\delta_s) &= T'(\iota(s))\\ &= \varphi(s)\\ &= T_\varphi(\iota(s))\\ &= T_\varphi(\delta_s).\end{aligned}$$

Thus T_φ is unique. □

Definition 1.1.11. Let V and W be vector spaces. The set of linear transformations between V and W is $\mathcal{L}(V,W) = \{T \mid T : V \to W \text{ linear}\}$. The set of linear functionals is $V' := \mathcal{L}(V,F)$.

Exercise 1.1.11. Show $\mathcal{L}(V,W)$ is a vector space.

Exercise 1.1.12. Show $M_{m,n}(F) \cong \mathcal{L}(F^m, F^n)$ by $a \mapsto T_a : (v \mapsto av)$.

§ 1.2. Algebras

Definition 1.2.1. An *algebra* over F is a linear space A over F equipped with a multiplication operation:

$$A \times A \to A \text{ defined by } (a,b) \mapsto ab$$

satisfying:

(1) $(ab)c = a(bc)$ for all $a,b,c \in A$;

(2) $(\alpha a)b = \alpha(ab) = a(\alpha b)$ for all $a,b \in A$, $\alpha \in F$;

(3) $a(b+c) = ab + ac$ for all $a,b,c \in A$;

(4) $(a+b)c = ac + bc$ for all $a,b,c \in A$.

If $ab = ba$ for all $a,b \in A$ we say that A is *commutative*. If there exists $1_A \in A$ with $1_A a = a 1_A = a$ for all $a \in A$ we say A is *unital*.

Example 1.2.1.

(1) $M_n(F)$ is a noncommutative unital algebra over F under the usual matrix multiplication.

(2) If V is a vector space over F, $\mathcal{L}(V)$ is a unital algebra over F. It is noncommutative provided $\dim(V) > 1$.

(3) $\mathcal{F}(S,F)$ is a unital commutative algebra over F.

Definition 1.2.2. Let B be a (unital) algebra over F.

(1) A (unital) *subalgebra* of B is a subspace $A \subseteq B$ ($1_B \in A$) satisfying the property that if $a, a' \in A$, then $aa' \in A$.

(2) An *ideal* of B is a subspace $I \subseteq B$ with $b \in B$, $a \in I$ implying $ba, ab \in I$.

Example 1.2.2.

(1) $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a unital subalgebra.

(2) $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subalgebras. In particular, $c_0 \subseteq \ell_\infty$ and $c_{00} \subseteq s$ are ideals.

(3) $C\big([a, b]\big) \subseteq \ell_\infty\big([a, b]\big)$ is a unital subalgebra.

(4) $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq C_b(\mathbf{R}) \subseteq \ell_\infty(\mathbf{R})$ are all subalgebras. In fact, $C_b(\mathbf{R}) \subseteq C(\mathbf{R})$ and $C_b(\mathbf{R}) \subseteq \ell_\infty(\mathbf{R})$ are unital, whereas $C_0(\mathbf{R}) \subseteq C_b(\mathbf{R})$ and $C_c(\mathbf{R}) \subseteq C(\mathbf{R})$ are ideals.

(5) The set $T_n(F) = \{(a_{ij}) \in M_n(F) \mid a_{ij} = 0, i > j\}$ is a unital subalgebra of $M_n(F)$.

Example 1.2.3 (Group Algebra)**.** Let Γ denote a group (not necessarily abelian). Take the free vector space $\mathbf{F}(\Gamma)$ and define multiplication as *convolution*: given $f, g \in \mathbf{F}(\Gamma)$ let:

$$(f * g)(r) = \sum_{\substack{\{(s,t) \mid \\ s \in \operatorname{supp}(f), \\ t \in \operatorname{supp}(g), \\ st = r\}}} f(s)g(t).$$

Since $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are finite, this is a finite sum. We often suppress this notation and write $(f * g)(r) = \sum_{st=r} f(s)g(t)$.

We can also make substitutions:

$$\begin{aligned}(f * g)(r) &= \sum_{st=r} f(s)g(t) \\ &= \sum_{t \in \Gamma} f(rt^{-1})g(t) \\ &= \sum_{s \in \Gamma} f(s)g(s^{-1}r).\end{aligned}$$

It is clear that:

$$\begin{aligned}(f + g) * h &= f * h + g * h \\ g * (g + h) &= f * g + f * h \\ \alpha(f * g) &= (\alpha f) * g = f * (\alpha g)\end{aligned}$$

for $f, g, h \in \mathbf{F}(\Gamma)$, $\alpha \in F$. Associativity can be similarly shown using the above definition. Rather, we will prove associativity by first show that $\delta_s * \delta_t = \delta_{st}$. Given:

$$(\delta_s * \delta_t)(r) = \sum_{q \in \Gamma} \delta_s(rq^{-1})\delta_t(q),$$

notice that:

$$\delta_s(rt^{-1}) = \begin{cases} 1, & s = rt^{-1} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & r = st \\ 0, & \text{otherwise} \end{cases} = \delta_{st}(r).$$

Since $\{\delta_t \mid t \in \Gamma\}$ is a basis for $\mathbf{F}(\Gamma)$, every $f \in \mathbf{F}(\Gamma)$ looks like:

$$f = \sum_{t \in J} \alpha_t \delta_t, \;\; J \subseteq T \text{ finite.}$$

Using distributivity we get:

$$\begin{aligned} \delta_r * (\delta_s * \delta_t) &= \delta_r * \delta_{st} \\ &= \delta_{rst} \\ &= \delta_{rs} * \delta_t \\ &= (\delta_r * \delta_s) * \delta_t. \end{aligned}$$

Whence convolution is associative.

Exercise 1.2.1. Let $\{A_i\}_{i \in I}$ be a family of algebras over F.

(1) $\prod_{i \in I} A_i$ is an algebra under $(a_i)_i (b_i)_i = (a_i b_i)_i$.

(2) $\bigoplus_{i \in I} A_i \subseteq \prod_{i \in I} A_i$ is an ideal.

Exercise 1.2.2. Let A be an algebra over F and $I \subseteq A$ an ideal. Then A/I is an algebra under $(a + I)(b + I) = ab + I$.

§ 1.3. Normed Vector Spaces

To each vector v in a vector space V, we want to assign a "length", denoted $\|v\|$.

Definition 1.3.1. A *norm* on a vector space V is a map:

$$\|\cdot\| : V \to [0, \infty), \;\; v \mapsto \|v\|$$

satisfying:

(1) $\|\alpha v\| = |\alpha| \, \|v\|$ for all $\alpha \in F$, $v \in V$ (homogeneity);

(2) $\|v + w\| \leqslant \|v\| + \|w\|$ (triangle inequality);

(3) If $\|v\| = 0$, then $v = 0_V$ (positive-definite).

If $\|\cdot\|$ satisfies (1) and (2), it is called a *seminorm*. The pair $(V, \|\cdot\|)$ is called a *normed space*.

Definition 1.3.2. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space V are called *equivalent* if there exists $c_1 \geqslant 0$ and $c_2 \geqslant 0$ with $\|v\| \leqslant c_1 \|v\|'$ and $\|v\|' \leqslant c_2 \|v\|$ for all $v \in V$.

Exercise 1.3.1. If p is a seminorm on V, show that $|p(v) - p(w)| \leqslant p(v - w)$.

Definition 1.3.3. Let $(V, \|v\|)$ be a normed space.

(1) The *closed unit ball* is denoted $B_V = \{v \in V \mid \|v\| \leqslant 1\}$.

(2) The *open unit ball* is denoted $U_V = \{v \in V \mid \|v\| < 1\}$.

(3) The *unit sphere* is denoted $S_V = \{v \in V \mid \|v\| = 1\}$.

Example 1.3.1. Let $V = F^n$ and $x = (x_1, ..., x_n)$. We define:

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|;$$
$$\|x\|_\infty = \max_{j=1}^{n} |x_j|;$$
$$\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}.$$

For $p \geqslant 1$:

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}.$$

Exercise 1.3.2. Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms

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We aim to show that $\|\cdot\|_p$ is a norm for $p \in [0, \infty]$.

Lemma 1.3.1. *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [0, \infty) \to \mathbf{R}$ be defined by $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. Then $f(t) \geqslant 0$ for $t \geqslant 0$.*

Proof. Note that $f'(t) = t^{p-1} - 1$. Since:

$$\begin{aligned} &f'(1) = 0 \\ &f'(t) > 0 \ \text{ for } \ t > 1 \\ &f'(t) < 0 \ \text{ for } \ 0 \leqslant t < 1, \end{aligned}$$

we can see that $f(t) \geqslant 0$ for all $t \geqslant 0$. □

Lemma 1.3.2 (Young's Inequality)**.** *Let $p, q \in [0, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $x, y \geqslant 0$, then $xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q$.*

Proof. By Lemma 1.3.1, $t \leqslant \frac{1}{p}t^p + \frac{1}{q}$. Multiplying both sides by y^q gives:

$$ty^q \leqslant \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Let $t = xy^{1-q}$. Then:

$$xy^{1-q}y^q \leqslant \frac{1}{p}x^p y^{p-pq} y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have that $p - pq = -q$. Whence:

$$xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q.$$

$\square$

Lemma 1.3.3 (Hölders Inequality)**.** *Let $p, q \in [0, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for $x, y \in F^n$:*

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leqslant \|x\|_p \|y\|_q.$$

Proof. We proceed by cases.

Case 1: $p = 1$. Then:

$$\begin{aligned} \left| \sum_{j=1}^{n} x_j y_j \right| &\leqslant \sum_{j=1}^{n} |x_j||y_j| \\ &\leqslant \sum_{j=1}^{n} |x_j| \|y\|_\infty \\ &= \|x\|_1 \|y\|_\infty. \end{aligned}$$

Case 2: $p = \infty$. This follows similarly to Case 1.

Case 3: $1 < p < \infty$. Suppose $\|x\|_p = \|y\|_q = 1$. Then:

$$\begin{aligned} \left| \sum_{j=1}^{n} x_j y_j \right| &\leqslant \sum_{j=1}^{n} |x_j||y_j| \\ &\leqslant \sum_{j=1}^{n} \left(\frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q \right) \\ &= \frac{1}{p} \left(\sum_{j=1}^{n} |x_j|^p \right) + \frac{1}{q} \left(\sum_{j=1}^{n} |y_j|^q \right) \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

Whence the inequality holds. Now suppose $\|x\|_p = 0$ or $\|y\|_q = 0$. Then $x = 0_{F^n}$ or $y = 0_{F^n}$, whence the inequality holds. Suppose $\|x\|_p \neq 0$ and $\|y\|_p \neq 0$. Set:

$$\begin{aligned} x' &= \frac{x}{\|x\|_p} \\ y' &= \frac{y}{\|y\|_p}. \end{aligned}$$

Then $\|x'\|_p = 1 = \|y'\|_p$. Observe that:

$$\begin{aligned} 1 &\geqslant \left| \sum_{j=1}^n x_j' y_j' \right| \\ &= \left| \sum_{j=1}^n \frac{x}{\|x\|_p} \frac{y}{\|y\|_p} \right|. \end{aligned}$$

Multiplying both sides by $\|x\|_p \|y\|_q$ gives the desired result. □

Lemma 1.3.4 (Minkowski's Inequality)**.** *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $x, y \in F^n$:*

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p.$$

Proof. The only nontrivial case is for $1 < p < \infty$. Observe that:

$$\begin{aligned} (\|x+y\|_p)^p &= \sum_{j=1}^n |x_j + y_j|^p \\ &= \sum_{j=1}^n |x_j + y_j||x_j + y_j|^{p-1} \\ &\leqslant \sum_{j=1}^n |x_j||x_j + y_j|^{p-1} + |y_j||x_j + y_j|^{p-1} \\ &\leqslant \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{(p-1)q} \right)^{\frac{1}{q}} \\ &= \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{p-1\left(\frac{p}{p-1}\right)} \right)^{1-\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{p-1\left(\frac{p}{p-1}\right)} \right)^{1-\frac{1}{p}} \\ &= \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{1-\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{1-\frac{1}{p}} \\ &= (\|x\|_p + \|y\|_p) \frac{\|x+y\|_p^p}{\|x+y\|_p}. \end{aligned}$$

Multiplying boths sides by $\frac{\|x+y\|_p}{\|x+y\|_p^p}$ gives the desired inequality. □

Theorem 1.3.5. *Let $V = F^n$. Then $(F^n, \|\cdot\|_p)$ is a normed space.*

Proof. Let $x = (x_1, ..., x_n) \in F^n$ and $\alpha \in F$. Observe that:

$$\begin{aligned}
\|\alpha x\|_p &= \left(\sum_{j=1}^{n} |\alpha x_j|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^{n} |\alpha|^p |x_j|^p\right)^{\frac{1}{p}} \\
&= |\alpha| \|x\|_p.
\end{aligned}$$

This satisfies homogeneity. Moreover, Minkowski's Inequality satisifes the triangle inequality. It remains to show that $\|\cdot\|_p$ is positive-definite. If $\|x\|_p = 0$, then $x_j = 0$ for all $1 \leqslant j \leqslant n$. Thus $x = 0_{F^n}$. □

Corollary 1.3.6. *Let $p \in [1, \infty]$. Then $\ell_p = \left\{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k|^p < \infty\right\}$ with norm $\|(a_k)_k\|_p = (\sum_{k=1}^{\infty} |a_k|^p)^{\frac{1}{p}}$ is a normed space.*

Proof. Homogeneity and positive-definiteness are trivial to prove. Let $(x_k)_k, (y_k)_k \in \ell_p$. It is clear that:

$$\begin{aligned}
\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} &\leqslant \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}} \\
&\leqslant \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} \\
&= \|(x_k)_k\|_p + \|(y_k)_k\|_p.
\end{aligned}$$

We have that $\sum_{k=1}^{n} |x_k + y_k|^p$ is increasing and bounded above by $(\|(x_k)_k\|_p + \|(y_k)_k\|_p)^p$. By the Monotone Convergence Theorem $\text{limit}_{n\to\infty} \sum_{k=1}^{n} |x_k + y_k|^p = \sum_{k=1}^{\infty} |x_k + y_k|^p$ exists. Whence $(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{\frac{1}{p}} = \|(x_k)_k + (y_k)_k\|_p \leqslant \|(x_k)_k\|_p + \|(y_k)_k\|_p$ □

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Example 1.3.2.

(1) $\big(\ell_\infty(\Omega, F), \|\cdot\|_u\big)$ where $\|f\|_u = \sup_{x\in\Omega} |f(x)|$ is a normed space. This includes its subspaces, such as $C\big([a, b], F\big) \subseteq \ell_\infty\big([a, b], F\big)$ and $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq \ell_\infty(\mathbf{R})$, all with $\|\cdot\|_u$.

(2) Take $\Omega = \mathbf{N}$ in the previous example. Then $\big(\ell_\infty, \|\cdot\|_\infty\big)$ is a normed space. This includes its subspaces $c_{00} \subseteq c_0 \subseteq \ell_\infty$ with $\|\cdot\|_\infty$.

(3) $\big(\ell_1, \|\cdot\|_1\big)$ is a normed space.

(4) $\big(C\big([a, b]\big), \|\cdot\|_1\big)$ with $\|f\|_1 = \int_a^b |f(t)|dt$ is a normed space.

(5) $\big(\mathrm{BV}\big([a, b]\big), \|\cdot\|_{\mathrm{BV}}\big)$ where $\|f\|_{\mathrm{BV}} = |f(a)| + \mathrm{Var}(f)$ is a normed space.

(6) Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Then $\big(B(V,W), \|\cdot\|_{\text{op}}\big)$ is a normed space, where $B(V,W) = \{T \in \mathcal{L}(V,W) \mid \|T\|_{\text{op}} < \infty\}$ is the set of bounded linear maps and $\|T\|_{\text{op}} = \sup_{v \in B_V} \|T(v)\|_W$. Intuitively, $\|T\|_{\text{op}}$ measures the radius of the smallest ball which contains B_V.

Exercise 1.3.3. Show that $V^* := B(V, F)$ is a subspace of V'.

(7) Let S be a nonempty set. Both $\big(\mathbf{F}(S), \|\cdot\|_1\big)$ and $\big(\mathbf{F}(S), \|\cdot\|_p\big)$ are normed spaces, where $\|f\|_1 = \sum_{s \in S} |f(s)|$ and $\|f\|_p = \big(\sum_{s \in S} |f(s)|^p\big)^{\frac{1}{p}}$. Note that since $f(s) \neq 0$ for finitely many $s \in S$, both $\|\cdot\|_1$ and $\|\cdot\|_p$ are well-defined.

Exercise 1.3.4. Show that $\|f\|_\infty := \sup_{s \in S} |f(s)|$ is a norm on $\mathbf{F}(S)$.

§ 1.4. Inner Product Spaces

Definition 1.4.1. Let V be a vector space over F and $\varphi : V \times V \to F$ a map.

(1) The map φ is said to be a *bilinear form* if is is linear in the first and second variable seperately; i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have:

(i) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$

(ii) $\varphi(v, cv_1 + v_2) = c\varphi(v, v_1) + \varphi(v, v_2)$.

(2) The map φ is said to be a *sesquilinear form* if it is linear in the first variable and conjugate linear in the second variable; i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have:

(i) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$

(ii) $\varphi(v, cv_1 + v_2) = \bar{c}\varphi(v, v_1) + \varphi(v, v_2)$.

If we wish to keep track of a bilinear form on V we write (V, φ).

Definition 1.4.2. Let V be a vector space over F.

(1) A bilinear form φ on V is said to be *symmetric* if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

(2) A sesquilinear form φ on V is said to be *Hermitian* if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.

Definition 1.4.3. Let (V, φ) be a vector space over F such that if φ is symmetric, then $F = \mathbf{R}$ or if φ is Hermitian, then $F = \mathbf{C}$. We say φ is *positive-definite* if for all nonzero $v \in V$ we have $\varphi(v, v) \neq 0$.

Definition 1.4.4. Let (V, φ) be a vector space over $\mathbf{R}$ with φ a positive-definite symmetric bilinear form or over $\mathbf{C}$ with φ a positive-definite Hermitian sesquilinear form. Then we say φ is an *inner product* on V and write φ as $\langle \cdot, \cdot \rangle$. We say $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space*.

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Definition 1.4.5. If V is an inner product space we define $\|v\|_2 = \langle v, v \rangle^{\frac{1}{2}}$.

Definition 1.4.6. Let V be an inner product space. Two vectors $v, w \in V$ are *orthogonal* if $\langle u, v \rangle = 0$. We denote this as $u \perp v$.

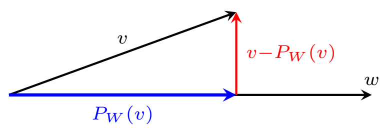
Theorem 1.4.1 (Pythagorean Theorem)**.** *Let $v_1, ..., v_n$ be mutually orthogonal. Then* $\sum_{j=1}^{n} \|v_j\|_2^2 = \left\|\sum_{j=1}^{n} v_j\right\|_2^2$.

Proof. Because $v_i \perp v_j$ for $1 \leqslant i, j \leqslant n$, we have $\langle v_i, v_j \rangle = 0$. Observe that:

$$\begin{aligned}
\left\| \sum_{j=1}^{n} v_j \right\|_2^2 &= \left\langle \sum_{j=1}^{n} v_j, \sum_{j=1}^{n} v_j \right\rangle \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \langle v_j, v_i \rangle \right) \\
&= \sum_{j=1}^{n} \langle v_j, v_j \rangle \\
&= \sum_{j=1}^{n} \|v_j\|_2^2 .
\end{aligned}$$

□

Definition 1.4.7. Let V be an inner product space and $w \in V$ nonzero. The *projection* of a vector $v \in V$ onto w is a map $P_W : V \to V$ defined by $P_W(v) = \frac{\langle v,w \rangle}{\langle w,w \rangle} w$.



Proposition 1.4.2. **** Let V be an inner product space and $w \in V$ a nonzero vector. Then $P_w(v) \perp v - P_w(v)$.*

Proof. □

Corollary 1.4.3. **** Let V be an inner product space and $w \in W$ a nonzero vector. Then $\|v\|_2^2 = \|P_w(v)\|_2^2 + \|v - P_w(v)\|_2^2$.*

Proof. □

Lemma 1.4.4 (Cauchy-Schwartz Inequality)**.** *Let V be an inner product space and $v, w \in V$. Then $|\langle v, w \rangle| \leqslant \|v\|_2 \|w\|_2$.*

Proof. The previous corollary gives $\|v\|_2 \geqslant \|P_w(v)\|_2$. We have that:

$$\begin{aligned}
\|v\|_2 &\geqslant \left\| \frac{\langle v, w \rangle}{\langle w, w \rangle} w \right\|_2 \\
&= \frac{|\langle v, w \rangle|}{\|w\|_2^2} \|w\|_2 \\
&= \frac{\langle v, w \rangle}{\|w\|_2}.
\end{aligned}$$

Multiplying both sides by $\|w\|_2$ gives the desired result. □

Theorem 1.4.5. *Let V be an inner product space. Then $(V, \|\cdot\|_2)$ is a normed space.*

Proof. Let $v, w \in V$ and $\alpha \in F$. We have that:

$$\begin{aligned} \|\alpha v\|_2 &= \langle \alpha v, \alpha v\rangle^{\frac{1}{2}} \\ &= (\alpha\overline{\alpha}\langle v, v\rangle)^{\frac{1}{2}} \\ &= \left(|\alpha|^2\langle v, v\rangle\right)^{\frac{1}{2}} \\ &= |\alpha|\, \|v\|_2 . \end{aligned}$$

Thus $\|\cdot\|_2$ satisfies homogeneity. It follows from the Cauchy-Schwartz Inequality that:

$$\begin{aligned} \|v+w\|_2^2 &= \langle v+w, v+w\rangle \\ &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= \|v\|_2^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} + \|w\|_2^2 \\ &= \|v\|_2^2 + 2\Re\left(\langle v, w\rangle\right) + \|w\|_2^2 \\ &\leqslant \|v\|_2^2 + 2|\langle v, w\rangle| + \|w\|_2^2 \\ &\leqslant \|v\|_2^2 + 2\,\|v\|_2\,\|w\|_2 + \|w\|_2^2 \\ &= \left(\|v\|_2 + \|w\|_2\right)^2 , \end{aligned}$$

where we used the fact that $2\Re(\langle v, w\rangle) = 2|\langle v, w\rangle|$. Squaring both sides proves that $\|\cdot\|_2$ satisfies the triangle inequality. It remains to show positive-definiteness. Suppose $\|v\|_2 = 0$. Then $\langle v, v\rangle = 0$, but since the inner-product is by definition positive-definite, we get that $v = 0_V$. ☐

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Example 1.4.1.

(1) $\ell_2^n = F^n$ is an inner product space where $\langle (x_1, ..., x_n), (y_1, ..., y_n)\rangle := \sum_{j=1}^n x_j\overline{y_j}$.

(2) ℓ_2 is an inner product space where $\langle (a_k)_k, (b_k)_k\rangle := \sum_{k=1}^\infty a_k\overline{b_k}$. Note that:

$$\begin{aligned} \sum_{k=1}^n \left|a_k\overline{b_k}\right| &= \sum_{k=1}^n |a_k||b_k| \\ &\leqslant \left(\sum_{k=1}^n |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^n |b_k|^2\right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{k=1}^\infty |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty |b_k|^2\right)^{\frac{1}{2}} \\ &= \|(a_k)_k\|_2\, \|(b_k)_k\|_2 \\ &< \infty \qquad \text{(Because } (a_k)_k, (b_k)_k \in \ell_2\text{).} \end{aligned}$$

Since $\sum_{k=1}^n |a_k\overline{b_k}|$ is increasing and bounded above, the Monotone Convergence Theorem says $\sum_{k=1}^\infty |a_k\overline{b_k}|$ exists and is finite. Whence $\langle (a_k)_k, (b_k)_k\rangle$ converges.

(3) Recall that $\mathrm{Tr} : M_n(\mathbf{C}) \to \mathbf{C}$ is defined by $\mathrm{Tr}(a_{ij}) = \sum_{i=1}^n a_{ii}$. Then $M_n(\mathbf{C})$ is an inner product space where $\langle a_{ij}, b_{ij}\rangle := \mathrm{Tr}(b_{ij}^* a_{ij})$.

(4) $C([0, 1])$ is an inner product space where $\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}dx$.

§ 1.5. Normed Algebras

Definition 1.5.1. A *normed algebra* is an algebra A equipped with a norm $\|\cdot\|_A$ such that $\|ab\|_A \leqslant \|a\|_A \|b\|_A$. If A is unital, we require $\|1\|_A = 1$.

Example 1.5.1.

(1) $\ell_\infty(\Omega)$ equipped with $\|\cdot\|_u$ is a normed algebra.

(2) $C_c(\mathbf{R})$, $C_0(\mathbf{R})$, and $C([0,1])$ are all normed algebras when equipped with $\|\cdot\|_u$.

(3) $M_n(F)$ equipped with $\|\cdot\|_{\mathrm{op}}$ is a normed algebra.

(4) If V is a normed space, then $B(V,V)$ with $\|\cdot\|_{\mathrm{op}}$ is a normed algebra: for $T, S \in B(V,V)$ and $v \in B_V$, we have that

$$\begin{aligned} \|(T \circ S)(v)\| &\leqslant \|T\|_{\mathrm{op}} \|S(v)\| \\ &\leqslant \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}. \end{aligned}$$

Taking the supremum over all $v \in B_V$ gives $\|T \circ S\|_{\mathrm{op}} \leqslant \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}$.

(5) Let S be a group. Equip the algebra $\mathbf{F}(S)$ with $\|\cdot\|_1$. We get a normed algebra.

Exercise 1.5.1. For $a, b \in \ell_1(\mathbf{Z})$, define $a * b : \mathbf{Z} \to F$ by $(a * b)(n) = \sum_{k \in \mathbf{Z}} a(n-k)b(k)$. Show that $\ell_1(\mathbf{Z})$ with this multiplication is a normed algbera.

Chapter 2

Metric Spaces

§ 2.1. Basic Definitions and Examples

Definition 2.1.1. A *metric* on a nonempty set X is a map

$$d : X \times X \to [0, \infty)$$

satisfying for all $x, y, z \in X$:

(1) $d(x, y) = d(y, x)$ (symmetry);

(2) $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangle inequality);

(3) $d(x, x) = 0$;

(4) if $d(x, y) = 0$ then $x = y$ (positivity).

If d satisfies all but (iv), then d is called a *semi-metric*. The pair (X, d) is called a *metric space*.

Definition 2.1.2.

(1) Two metrics d, ρ on X are called *equivalent* if there exists constants c, c' with $d(x, y) \leqslant c\rho(x, y)$ and $\rho(x, y) \leqslant c' d(x, y)$.

(2) Let $\{d_k\}_{k=1}^{\infty}$ be a family of metrics on X. If for all $x, y \in X$ and $k \in \mathbf{N}$ we have $d_k(x, y) \leqslant C$, then we say the family of metrics is *uniformly bounded*.

Example 2.1.1.

(1) The *discrete metric* on $X \neq \emptyset$ is:

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

(2) The *hamming distance* between two bit strings of equal length: given $X = \{0, 1\}^n$, then $d_H : X \times X \to [0, \infty)$ is defined by $d_H\big((x_j)_{n\geqslant 1}, (y_j)_{n\geqslant 1}\big) = |\{j \mid x_j \neq y_j\}|$.

(3) If $(V, \|\cdot\|)$ is any normed space, then $d(v, w) = \|v - w\|$ is a metric on V.

   Exercise 2.1.1. If $\|\cdot\|$ and $\|\cdot\|'$ are norms on a linear space V, show they are equivalent if and only if their induced metrics are equivalent.

(4) If (X, d) is a metric and $Y \subseteq X$ is a subset, then (Y, d) is a metric space.

(5) Let (X, ρ) be a metric space. Fix $p \in X$. Then:

$$d(x,y) := \begin{cases} 0, & x = y \\ \rho(x,p) + \rho(p,y), & x \neq y \end{cases}$$

is a metric.

(6) It is often beneficial to work with metrics that are bounded. Let ρ be a (semi)-metric of X. Set:

$$d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}.$$

Defining $d(x,y)$ as above gives $0 \leqslant d(x,y) \leqslant 1$. Although d and ρ are not equivalent metrics, they are topologically equivalent.

Clearly d is symmetric and $d(x,x) = 0$. Moreover, if $d(x,y) = 0$, then $\rho(x,y) = 0$, giving $x = y$ if ρ is a metric. For the triangle inequality, consider the function $g : [0,\infty) \to [0,1)$ given by $g(t) = \frac{t}{1+t}$. We have that $g'(t) = \frac{(1+t)-t}{(1+t)^2} = \frac{1}{(1+t)^2} > 0$, whence g is strictly increasing. Since we know $\rho(x,z) \leqslant \rho(x,y) + \rho(y,z)$, observe that:

$$\begin{aligned} d(x,z) &= \frac{\rho(x,z)}{1+\rho(x,z)} \\ &\leqslant \frac{\rho(x,y)+\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\ &= \frac{\rho(x,y)}{1+\rho(x,y)+\rho(y,z)} + \frac{\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\ &\leqslant \frac{\rho(x,y)}{1+\rho(x,y)} + \frac{\rho(y,z)}{1+\rho(y,z)} \\ &= d(x,y) + d(y,z). \end{aligned}$$

(7) If $d_1, ..., d_n$ are metrics on X and $c_1, ..., c_n > 0$, then:

$$d(x,y) = \sum_{i=1}^{n} c_i d_i(x,y)$$

is a metric on X.

(8) Let $\{\rho_k\}_{k=1}^{\infty}$ be a family of semi-metrics on X. Assume that the family is *separating*: if $x, y \in X$ and $x \neq y$, then there exists k such that $\rho_k(x,y) \neq 0$. Let $d_k(x,y) = \frac{\rho_k(x,y)}{1+\rho_k(x,y)}$. Then:

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x,y)$$

is a metric on X. Since $0 \leqslant d_k(x,y) \leqslant 1$, by comparison $d(x,y)$ will converge.

(9) Let $(X_k, \rho_k)_{k \geqslant 1}$ be a sequence of metric spaces. For each k let d_k be as above. Let $X = \prod_{k=1}^{\infty} X_k$. Then the map $D : X \times X \to [0,\infty)$ defined by

$$D(f,g) = \sum_{k=1}^{\infty} 2^{-k} d_k(f(k), g(k))$$

is a metric on X. Note that we need not make d_k from ρ_k if all the d_k are uniformly bounded.

(10) Let $X = \{0, 2\}$ with the discrete metric. The *abstract Cantor set* is $\Delta := \prod_{k \in \mathbf{N}} X$. Then the map $D : \Delta \times \Delta \to [0, \infty)$ defined by

$$D(f, g) = \sum_{k=1}^{\infty} 3^{-k} |f(k) - g(k)|$$

is a metric on Δ.

(11) Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbf{R}^3$ (or $\mathbf{R}^n$). The unit sphere $S^2 = \{x \in \mathbf{R}^3 \mid \|x\|_2 = 1\}$ paired with $d(x, y) := \arccos(\langle x, y \rangle)$ is a metric space.

Exercise 2.1.2. Show that (S^2, d) defined as above is indeed a metric space.

Definition 2.1.3.

(1) Let (X, d) be a metric space with $E \subseteq X$. The *diameter* of E is $\operatorname{diam}(E) = \sup_{x,y \in E} d(x, y)$. We say E is *bounded* (metrically) if $\operatorname{diam}(E) < \infty$.

(2) If Ω is any set and (Y, d) is a metric space, $f : \Omega \to Y$ is *bounded* if $\operatorname{diam}(f(\Omega)) < \infty$. The set of bounded functions is $\operatorname{Bd}(\Omega, Y) := \{f : \Omega \to Y \mid f \text{ bounded}\}$.

Exercise 2.1.3. If $(V, \|\cdot\|)$ is a normed space and $E \subseteq V$, the following are equivalent:

(1) E is bounded;

(2) $\sup_{v \in E} \|v\| < \infty$;

(3) there exists $r > 0$ such that $E \subseteq B(0, r)$.

Example 2.1.2. The set $\operatorname{Bd}(\Omega, Y)$ is a metric space with:

$$D_u(f, g) := \sup_{x \in \Omega} d(f(x), g(x)).$$

Clearly $D_u(f, g) = D_u(g, f)$ and $D_u(f, f) = 0$. If $D_u(f, g) = 0$, then $f(x) = g(x)$ for all $x \in \Omega$, giving $f = g$. Moreover, for every $x \in \Omega$:

$$\begin{aligned} d(f(x), h(x)) &\leqslant d(f(x), g(x)) + d(g(x), h(x)) \\ &\leqslant D_u(f, g) + D_u(g, h). \end{aligned}$$

Whence $D_u(f, h) \leqslant D_u(f, g) + D_u(g, h)$. Note that if we take the normed space $(\ell_\infty(\Omega), \|\cdot\|_u)$, the induced metric is:

$$\begin{aligned} d(f, g) &= \|f - g\|_u \\ &= \sup_{x \in \Omega} |(f - g)(x)| \\ &= \sup_{x \in \Omega} |f(x) - g(x)| \\ &= D_u(f, g). \end{aligned}$$

So as metric spaces, $\ell_\infty(\Omega) \cong \mathrm{Bd}(\Omega, F)$. Now consider the subset $E = \{f \in \mathrm{Bd}(\Omega, F) \mid f(x) \in \{0,1\}\}$. We get:

$$\begin{aligned} D_u(f,g) &= \sup_{x\in\Omega} |f(x) - g(x)| \\ &= \begin{cases} 0, & f = g \\ 1, & f \neq g\,. \end{cases} \end{aligned}$$

So (E, D_u) is discrete.

§ 2.2. Topology of Metric Spaces

Unless otherwise stated, let (X, d) be a metric space.

Definition 2.2.1. Let X be a set. A collection T of subsets of X is called a *topology on* X if they satisfy:

(1) $\emptyset, X \in T$;

(2) arbitrary unions of elements in T are in T;

(3) finite intersections of elements in T are in T.

Definition 2.2.2. Let (X, d) be a metric space.

(1) Let $x_0 \in X$ and $\delta > 0$.

  (i) The *open ball* centered at x_0 of radius δ is $U(x_0, \delta) = \{x \mid d(x, x_0) < \delta\}$.

  (ii) The *closed ball* centered at x_0 of radius δ is $B(x_0, \delta) = \{x \mid d(x, x_0) \leqslant \delta\}$.

  (iii) The *sphere* centered at x_0 of radius δ is $S(x_0, \delta) = \{x \mid d(x, x_0) = \delta\}$.

(2) A subset $U \subseteq X$ is *open in* X if:

$$(\forall x \in U)(\exists \delta > 0) : U(x, \delta) \subseteq U.$$

The collection of open sets is denoted $\tau_X := \{U \subseteq X \mid U \text{ is open}\}$.

(3) A subset $D \subseteq X$ is *closed in* X if $D^c \subseteq X$ is open in X.

(4) If $x \in U \in \tau_X$, then U is called an *open neighborhood of* x. If $x \in U \in \tau_X$ and $U \subseteq N \subseteq X$, then N is called a *neighborhood of* x. The collection of neighborhoods of x is denoted $\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$.

(5) Let $A \subseteq X$.

  (i) The *interior of* A is:

$$A^o := \bigcup \{V \in \tau_X \mid V \subseteq A\}.$$

  (ii) The *closure of* A is:

$$\overline{A} := \bigcap \{C \mid C \supseteq A,\, C \text{ closed}\}.$$

  (iii) The *boundary of* A is $\partial A := \overline{A} \setminus A^o$.

Exercise 2.2.1. Show that $\overline{A^c} = (A^o)^c$ and $\overline{A}^c = (A^c)^o$.

Proposition 2.2.1. *Let (X, d) be a metric space. The open sets τ_X form a topology.*

Proof. Both $\emptyset$ and X are open by assumption. Let $\{V_i\}_{i\in I}$ be a family of open sets of X. Let $x \in \bigcup_{i\in I} V_i$. Then $x \in V_i$ for some i. Since V_i is open, there exists $\delta > 0$ with $B(x, \delta) \subseteq V_i \subseteq \bigcup_{i\in I} V_i$. Whence the arbitrary union of open sets is open.

Now let $\{V_k\}_{k=1}^n$ be a family of open sets. Let $x \in \bigcap_{k=1}^n V_k$. Then $x \in V_k$ for all k. Since V_k is open, there exists $\delta_k > 0$ with $B(x, \delta_k) \subseteq V_k$. Pick $\delta = \min\{\delta_1, \delta_2, ..., \delta_k\}$. Then $B(x, \delta) \subseteq \bigcap_{k=1}^n V_k$. Whence $\bigcap_{k=1}^n V_k$ is open. □

Note that an arbitrary intersection of open sets is not necessarily open. Consider the sequence of intervals $(I_n)_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^\infty I_n = \{0\}$, which is closed.

Exercise 2.2.2. Let (X, d) be a metric space and consider a collection $\mathcal{C}$ of subsets of X with $\emptyset, X \in \mathcal{C}$. Show that:

(1) if $\{C_i\}_{i\in I}$ is a family of closed sets, then $\bigcap_{i\in I} C_i$ is closed;

(2) if $\{C_i\}_{i=1}^n$ is a family of closed sets then $\bigcup_{i=1}^n C_i$ is closed.

Proposition 2.2.2. *Let (X, d) be a metric space and $x \in X$.*

(1) $N \in \mathcal{N}_x$ if and only if there exists $\delta > 0$ such that $U(x, \delta) \subseteq N$.

(2) If $N \in \mathcal{N}_x$ and $N \subseteq M$, then $M \in \mathcal{N}_x$.

(3) If $N_1, N_2 \in \mathcal{N}_x$, then $N_1 \cap N_2 \in \mathcal{N}_x$.

Proof. (1) Let $N \in \mathcal{N}_x$. Then there is an open set $x \in U$ with $U \subseteq N \subseteq X$. Since U is open, there exists $\delta > 0$ such that $U(x, \delta) \subseteq U \subseteq N$. Conversely, suppose there exists $\delta > 0$ such that $U(x, \delta) \subseteq N$. Clearly $U(x, \delta) \subseteq N \subseteq X$, whence $N \in \mathcal{N}_x$.

(2) If $N \in \mathcal{N}_x$, then there is an open set U with $x \in U$ and $U \subseteq N \subseteq X$. So $U \subseteq N \subseteq M \subseteq X$. Whence $M \in \mathcal{N}_x$.

(3) If $N_1, N_2 \in \mathcal{N}_x$, then there are open sets U_1, U_2 with $x \in U_1$, $x \in U_2$ and $U_1 \subseteq N_1 \subseteq X$, $U_2 \subseteq N_2 \subseteq X$. Whence $U_1 \cap U_2 \subseteq N_1 \cap N_2 \subseteq X$. □

Proposition 2.2.3. *Let $U \subseteq \mathbf{R}$ be open. Then:*

$$U = \bigsqcup_{j\in J} I_j,$$

where J is countable and I_j are open intervals.

Proof. For each $x \in U$, define:

$$I_x := \bigcup \{I \mid x \in I \subseteq U,\ I \text{ open interval}\}.$$

Clearly $x \in I_x \subseteq U$. If $s, t \in I_x$ with $s < t$, then there exists open intervals I, I' with $x \in I \subseteq U$, $x \in I' \subseteq U$, and $s \in I$, $t \in I'$. Since $I \cap I' \neq \emptyset$, $I \cup I'$ is an open interval. Moreover, since $s, t \in I \cup I'$, we know $[s, t] \subseteq I \cup I' \subseteq I_x$. This shows I_x is an interval —in particular, since I_x is the union of open intervals, it must be open.

Now suppose $x, y \in U$ and $I_x \cap I_y \neq \emptyset$. Then there exists $z \in I_x \cap I_y$, but $z \in I_x$ implies $I_x \subseteq I_z$ and $z \in I_y$ implies $I_y \subseteq I_z$. But we also have $x \in I_x \subseteq I_z$ which gives $I_z \subseteq I_x$, and similarly $y \in I_y \subseteq I_z$ gives $I_z \subseteq I_y$. Together, we have $I_x = I_y$, which means for any $x, y \in U$, then $I_x \cap I_y = \emptyset$ or $I_x = I_y$. Thus there exists $J \subseteq U$ with $U = \bigsqcup_{j \in J} I_j$.

It remains to show that J is countable. Define $J \to \mathbf{Q}$ by $x \mapsto q_x$, where $q_x \in \mathbf{Q} \cap I_x$. This map is injective, establishing the proposition. □

Proposition 2.2.4. *Let $A \subseteq X$.*

(1) $x \in A^o$ if and only if there exists $\delta > 0$ such that $U(x, \delta) \subseteq A$.

(2) $x \in \overline{A}$ if and only if for all $\delta > 0$, $U(x, \delta) \cap A \neq \emptyset$.

(3) $x \in \partial A$ if and only if for all $\delta > 0$, $U(x, \delta) \cap A \neq \emptyset$ and $U(x, \delta) \cap A^c \neq \emptyset$.

Proof. (1) $x \in A^o \iff$ there exists $V \in \tau_X$ with $x \in V \subseteq A \iff$ there exists $\delta > 0$ with $U(x, \delta) \subseteq V \subseteq A$.

(2) The converse gives $x \notin \overline{A} \iff x \in (\overline{A})^c = (A^c)^o \iff$ there exists $\delta > 0$ such that $U(x, \delta) \subseteq (A^c)^o \subseteq A^c \iff U(x, \delta) \cap A = \emptyset$.

(3) $x \in \partial A \iff x \in \overline{A} \setminus A^o \iff x \in \overline{A} \cap (A^o)^c \iff x \in \overline{A} \cap \overline{A^c} \iff$ there exists $\delta > 0$ with $U(x, \delta) \cap A \neq \emptyset$ and $U(x, \delta) \cap A^c \neq \emptyset$ □

Exercise 2.2.3. Show that open balls are open, closed balls are closed, and spheres are closed.

Proposition 2.2.5. **** For any normed space:*

(1) $\overline{U(x, \delta)} = B(x, \delta)$

(2) $B(x, \delta)^o = U(x, \delta)$

(3) $\partial U(x, \delta) = \partial B(x, \delta) = S(x, \delta)$.

Proof. (1)

(2) Suppose $y \in B(x, \delta)^o \setminus U(x, \delta)$. Then there exists $\delta' > 0$ with $U(x, \delta') \subseteq B(x, \delta)$. But this means $d(x, y) < \delta$, which is a contradiction. Thus $B(x, \delta)^o \setminus U(x, \delta) = \emptyset$, establishing $B(x, \delta)^o = U(x, \delta)$. □

Proposition 2.2.6. **** Let (X, d) be a metric space with $\{A_i\}_{i \in I}$ a family of subsets. Let $K \subseteq I$ be finite.*

(1) $\displaystyle\bigcup_{i \in I} A_i^o \subseteq \left(\bigcup_{i \in I} A_i\right)^o$ *(Inclusion may be strict).*

(2) $\displaystyle\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$ *(Inclusion may be strict).*

(3) $\displaystyle\bigcap_{i \in K} A_i^o = \left(\bigcap_{i \in K} A_i\right)^o$

(4) $\displaystyle\overline{\bigcup_{i \in K} A_i} = \bigcup_{i \in K} \overline{A_i}.$

Proof. □

Proposition 2.2.7. *Let $S \subseteq X$.*

(1) $\partial S = \partial S^c$.

(2) ∂S *is closed.*

(3) $\overline{S} = S \cup \partial S$.

(4) $S \setminus \partial S = S^o$.

Proof. (1) This follows from the characterization of ∂S. (2) We have $\partial S = \overline{S} \setminus S^o = \overline{S} \cap (S^o)^c$, which is closed. (3) Clearly $S \cup \partial S \subseteq \overline{S}$. Let $x \in \overline{S}$. If $x \in S$ we are done. Otherwise $x \in \overline{S} \setminus S \subseteq \overline{S} \setminus S^o = \partial S$. (4) Observe that:

$$\begin{aligned} S \setminus \partial S &= S \cap (\partial S)^c \\ &= S \cap (\overline{S} \setminus S^o)^c \\ &= S \cap (\overline{S} \cap (S^o)^c)^c \\ &= S \cap (\overline{S}^c \cup S^o) \\ &= S \cap \overline{S}^c \cup S \cap S^o \\ &= S^o. \end{aligned}$$

□

Definition 2.2.3. Let (X, d) be a metric space.

(1) A subset $A \subseteq X$ is *d-dense* if $\overline{A} = X$.

(2) A subset $N \subseteq X$ is *nowhere dense* if $(\overline{N})^o = \emptyset$.

(3) The space (X, d) is *separable* if there exists a countable dense subset $D \subseteq X$.

Exercise 2.2.4. If $N \subseteq X$ is closed, then N is nowhere dense if and only if N^c is dense.

Proposition 2.2.8. **** Let $A \subseteq X$. The following are equivalent:*

(1) A *is dense;*

(2) $(\forall U \in \tau_X), U \cap A \neq \emptyset$;

(3) $(\forall x \in X)(\forall \epsilon > 0), U(x, \epsilon) \cap A \neq \emptyset$;

(4) $(\forall x \in X)(\forall \epsilon > 0)(\exists a \in A) : d(a, x) < \epsilon$.

Proof. □

Definition 2.2.4. Let (X, d) be a metric space.

(1) A *base* for τ_X is a family of open subsets $\mathcal{B} \subseteq \tau_X$ such that:

$$(\forall U \in \tau_X)(\forall x \in U)(\exists B \in \mathcal{B}) : x \in B \subseteq U.$$

Equivalently, for all $U \in \tau_X$, we can write $U = \bigcup_{i \in I} B_i$, where $\{B_i\}_{i \in I} \subseteq \tau_X$.

(2) X is *second countable* if it has a countable base.

Note that this definition can be generalized to any topological space. Clearly $\mathcal{B} = \{U(x, \epsilon) \mid x \in X, \epsilon > 0\}$ forms a base for any metric space.

Example 2.2.1. The set $\mathcal{B} = \{U(q, \frac{1}{n}) \mid n \geqslant 1, q \in \mathbf{Q}^d\}$ is a base for $\mathbf{R}^d$.

Proposition 2.2.9. *Let (X, d) be a metric space. X is separable if and only if X is second countable.*

Proof. Let $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ be a countable base. Choose any $a_n \in U_n$. Then $\{a_n\}_{n=1}^{\infty}$ is dense. Indeed, given any $x \in X$ and $\epsilon > 0$, there exists U_m with $x \in U_m \subseteq U(x, \epsilon)$ (since $U_m \in \mathcal{B}$). Whence $d(a_m, x) < \epsilon$.

Let $\{a_n\}_{n=1}^{\infty}$ be dense. Consider:

$$\mathcal{B} = \left\{U(a_n, \tfrac{1}{m}) \mid n \geqslant 1, m \geqslant 1\right\}.$$

Clearly $\mathcal{B}$ is countable —it remains to show that it is a base for X. Given $x \in V \in \tau_X$, find $\epsilon > 0$ such that $U(x, \epsilon) \subseteq V$. Then there exists $m \geqslant 1$ with $\epsilon > \frac{1}{m}$. Since $\{a_n\}_{n=1}^{\infty}$ is dense, there exists $a_j \in \{a_n\}_{n=1}^{\infty}$ such that $d(a_j, x) < \frac{1}{2m}$. Let $y \in U(a_j, \frac{1}{2m})$. Observe that:

$$\begin{aligned} d(x, y) &\leqslant d(x, a_j) + d(a_j, y) \\ &< \frac{1}{2m} + \frac{1}{2m} \\ &= \frac{1}{m} \\ &< \epsilon. \end{aligned}$$

So $y \in U(x, \epsilon)$. Thus $x \in U(a_j, \frac{1}{2m}) \subseteq U(x, \epsilon) \subseteq V$, establishing $\mathcal{B}$ as a base. □

Example 2.2.2.

(1) The space $(\mathbf{R}^d, \|\cdot\|_p)$ is separable for any $1 \leqslant p \leqslant \infty$. Indeed, if $(r_1, ..., r_d) \in \mathbf{R}^d$ and $\epsilon > 0$, find $q_j \in \mathbf{Q}$, $j = 1, ..., d$ with:

$$|r_j - q_j| < \frac{\epsilon}{d}.$$

Then:

$$\|r - q\|_1 = \sum_{j=1}^{d} |r_j - q_j| < \epsilon.$$

So $\mathbf{Q}^d$ is $\|\cdot\|_1$-dense in $\mathbf{R}^d$. For $1 \leqslant p \leqslant \infty$, let $C > 0$ be such that $\|\cdot\|_p \leqslant C\,\|\cdot\|_1$. So given $\epsilon > 0$, find $q \in \mathbf{Q}^d$ with $\|r - q\|_1 \leqslant \frac{\epsilon}{C}$. Then $\|r - q\|_p < \epsilon$.

(2) Similarly, $\mathbf{C}_{\mathbf{Q}}^d \subseteq \mathbf{C}^d$ is $\|\cdot\|_p$-dense, where:

$$\mathbf{C}_{\mathbf{Q}} = \{a + bi \mid a, b \in \mathbf{Q}\}.$$

(3) Recall that $c_{00} = \left\{(z_k)_k \mid \operatorname{supp}\big((z_k)_k\big) < \infty\right\}$. The space $(c_{00}, \|\cdot\|_u)$ is separable.

Note that $c_{00} = \mathbf{C}\text{-}\operatorname{span}\{e_k \mid k \in \mathbf{N}\}$. This space is not countable —clearly $\mathbf{C}\text{-}\operatorname{span}\{e_1\} = \{\alpha e_1 \mid \alpha \in \mathbf{C}\}$ is not countable, so it must be that c_{00} is also not countable.

Instead, consider:

$$\begin{aligned}\mathbf{C}_{\mathbf{Q}}\text{-}\operatorname{span}\{e_k \mid k \in \mathbf{N}\} &= \left\{\sum_{k=1}^{\infty} t_k e_k \mid t_k \in \mathbf{C}_{\mathbf{Q}},\right\} \\ &= \bigcup_{k=1}^{\infty}\{C_k \mid C_k = \mathbf{C}_{\mathbf{Q}}\text{-}\operatorname{span}\{e_1, ..., e_k\}\}\end{aligned}$$

Note that C_k is in bijection with $\mathbf{Q}^{2k}$, whence $\mathbf{C}_{\mathbf{Q}}\text{-}\operatorname{span}\{e_k \mid k \in \mathbf{N}\}$ is countable. Given $z \in c_{00}$, let $z = \sum_{k=1}^{N} z_k e_k$ and $\epsilon > 0$. Find $t_k \in \mathbf{C}_{\mathbf{Q}}$ with $|z_k - t_k| < \epsilon$. Then:

$$\begin{aligned}\|z - t\|_u &= \left\|\sum_{k=1}^{N} z_k e_k - \sum_{k=1}^{K} t_k e_k\right\|_u \\ &= \left\|\sum_{k=1}^{N}(z_k - t_k)e_k\right\|_u \\ &= \sup_{k=1}^{N}|z_k - t_k| \\ &< \epsilon.\end{aligned}$$

Thus $\mathbf{C}_{\mathbf{Q}}\text{-}\operatorname{span}\{e_k \mid k \in \mathbf{N}\}$ is dense in c_{00}, whence the space $(c_{00}, \|\cdot\|_u)$ is separable.

Proposition 2.2.10. *If (X, d) is a separable metric space and $Y \subseteq X$, then (Y, d) is separable.*

Proof. Let $A = \{a_k\}_{k=1}^{\infty}$ be dense in X. Let:

$$N = \{(m, n) \mid U(a_m, \tfrac{1}{n}) \cap Y \neq \emptyset\}.$$

For each $(m, n) \in N$, choose $b_{(m,n)} \in Y \cap U(a_m, \frac{1}{n})$. Claim: the set

$$\{b_{(m,n)} \mid (m, n) \in N\}$$

is dense in Y. Let $y \in Y$ and $\epsilon > 0$. Then there exists $n \geqslant 1$ with $\frac{\epsilon}{2} > \frac{1}{n}$. Since A is dense, $U(y, \frac{1}{n}) \cap A \neq \emptyset$ (this is because for all $U \in \tau_X$, we have $U \cap A \neq \emptyset$). So $d(a_m, y) < \frac{1}{n}$. Whence:

$$\begin{aligned}d(b_{(m,n)}, y) &\leqslant d(b_{(m,n)}, a_m) + d(a_m, y) \\ &< \frac{1}{n} + \frac{1}{n} \\ &< \epsilon.\end{aligned}$$

□

Example 2.2.3.

(1) The space $(\ell_\infty, \|\cdot\|_u)$ is not separable. If it were, consider:

$$E = \{(x_k)_k \mid x_k \in \{0, 1\}\} \subseteq \ell_\infty.$$

This set is uncountable, and by the previous proposition $(E, \|\cdot\|_u)$ is also separable. Let $a, b \in E$. We have:

$$\begin{aligned}\|(a_k)_k - (b_k)_k\|_u &= \|(a_k - b_k)_k\|_u \\ &= \sup_{k\geqslant 1} |a_k - b_k| \\ &= \begin{cases} 0, & (a_k)_k = (b_k)_k \\ 1, & (a_k)_k \neq (b_k)_k\ . \end{cases}\end{aligned}$$

So $(E, \|\cdot\|_u)$ is discrete. Note that a metric space is discrete if and only if its singletons are open. Whence if $(E, \|\cdot\|_u)$ is separable, then $A = \overline{A} = E$, which contradicts the countability of A. It must be that $(\ell_\infty, \|\cdot\|_u)$ is not separable.

(2) The space $(\ell_p, \|\cdot\|_p)$ is separable for $1 \leqslant p < \infty$. Let $a = (a_k)_k \in \ell_p$ and $\epsilon > 0$. Find N large so that $\sum_{k>N} |a_k|^p < \frac{\epsilon^p}{2}$. Find $b_k \in \mathbf{C}_\mathbf{Q}$ with $|a_k - b_k| < \frac{\epsilon}{(2N)^{\frac{1}{p}}}$. Let $b = (b_1, b_2, ..., b_{N-1}, b_N, 0, 0, ...)$. We have:

$$\begin{aligned}\|a-b\|_p^p &= \sum_{k=1}^{\infty} |a_k - b_k|^p \\ &= \sum_{k=1}^{N} |a_k - b_k|^p + \sum_{k>N} |a_k|^p \\ &< N \cdot \frac{\epsilon^p}{2N} + \frac{\epsilon^p}{2} \\ &= \epsilon.\end{aligned}$$

Whence the set $\mathbf{C}_\mathbf{Q}$- $\operatorname{span}\{e_k \mid k \in \mathbf{N}\}$ is $\|\cdot\|_u$-dense, giving $(\ell_p, \|\cdot\|_p)$ as separable.

(3) We will eventually show that the set of polynomial functions:

$$P\big([0,1]\big) = \left\{ \sum_{k=0}^{n} a_k x^k \mid a_k \in F, n \geqslant 0 \right\}$$

is $\|\cdot\|_u$-dense in $C\big([0,1]\big)$ (note that this set is not countable). With this fact, we can show that $C\big([0,1]\big)$ is separable. Indeed, given $f \in C\big([0,1]\big)$ and $\epsilon > 0$, find $p \in P\big([0,1]\big)$ with $\|f - p\|_u < \frac{\epsilon}{2}$. Now let $p(x) = \sum_{k=0}^{n} a_k x^k$. Find $b_k \in \mathbf{C}_\mathbf{Q}$ with $|a_k - b_k| < \frac{\epsilon}{2(n+1)}$ and define $q(x) = \sum_{k=0}^{n} b_k x^k$. Observe that:

$$\begin{aligned}\|f-q\|_u &= \|f - p + p - q\|_u \\ &\leqslant \|f-p\|_u + \|p-q\|_u \\ &= \|f-p\|_u + \sum_{k=0}^{n} |a_k - b_k| \\ &< \frac{\epsilon}{2} + (n+1) \cdot \frac{\epsilon}{2(n+1)} \\ &= \epsilon.\end{aligned}$$

Thus the set $\mathbf{C}_\mathbf{Q}$- $\operatorname{span}\{x^k \mid k \in \mathbf{N}\}$ is $\|\cdot\|_u$-dense in $C\big([0,1]\big)$. In particular, since it is countable, $C\big([0,1]\big)$ is separable.

—————//—————

We have seen that subsets of metric spaces are metric spaces in their own right. Then what are their open sets?

Proposition 2.2.11. *Let (X, d) be a metric space and let $Y \subseteq X$ be any subset. Then $V \subseteq Y$ is open in Y if and only if there exists an open set $U \subseteq X$ with $U \cap Y = V$. That is, $\tau_Y = \{U \cap Y \mid U \in \tau_X\}$.*

Proof. Let $V \subseteq Y$ be open. Then for every $x \in V$, there exists $\delta_x > 0$ with $U_y(x, \delta_x) \subseteq V$, where:

$$\begin{aligned} U_y(x, \delta_x) &= \{y \in Y \mid d(y, x) < \delta_x\} \\ &= U(x, \delta_x) \cap Y. \end{aligned}$$

Set $U = \bigcup_{x \in V} U(x, \delta_x)$. Then U is indeed open in X. Also:

$$\begin{aligned} U \cap Y &= \left(\bigcup_{x \in V} U(x, \delta_x) \right) \cap Y \\ &= \bigcup_{x \in V} (U(x, \delta_x) \cap Y) \\ &= \bigcup_{x \in V} U_y(x, \delta_x) \\ &= V. \end{aligned}$$

Conversely, suppose $V = U \cap Y$ for some open $U \in \tau_X$. Let $x \in V$. Since $x \in U$ and U is open, there exists $\delta > 0$ such that $U(x, \delta) \subseteq U$. So $U_y(x, \delta) = U(x, \delta) \cap Y \subseteq U \cap Y = V$. Thus V is open in Y. □

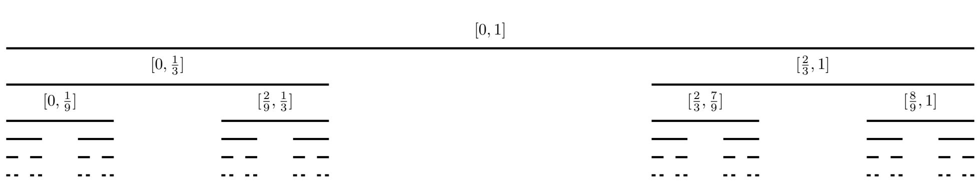
Example 2.2.4.

(1) $[0, \frac{1}{2})$ is not open in $\mathbf{R}$, but it is open in $[0, 1]$.

(2) ℓ_∞ is not a discrete metric space, but $\{0, 1\}^{\mathbf{N}} \subseteq \ell_\infty$ is.

§ 2.3. The Cantor Set

Given the interval $[0, 1]$, start by deleting the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, leaving two line segments $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.



We are interested in studying the topological properties of the points which are not deleted at any step of this infinite process. The set of these points have a special name and are defined below.

Definition 2.3.1. Let $C_0 := [0,1]$ and $C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$ for $n \geqslant 1$. The *Cantor set* is $\mathfrak{C} = \bigcap_{n=0}^{\infty} C_n$.

Proposition 2.3.1. *The Cantor set is closed.*

Proof. Since the Cantor set is defined as the intersection of closed sets, it must be closed. □

Proposition 2.3.2. *The Cantor set is nowhere dense.*

Proof. Suppose towards contradiction its not, that is, $\overline{\mathfrak{C}}^o \neq \emptyset$. Then there is some $x \in \overline{\mathfrak{C}}^o$. We can find an $\epsilon > 0$ with $(x-\epsilon, x+\epsilon) \subseteq \mathfrak{C}$, in particular $(x-\epsilon, x+\epsilon) \subseteq C_n$ for all $n \geqslant 1$. Find m large so that $\epsilon > \frac{1}{3^m}$ and consider $(x-\epsilon, x+\epsilon) \subseteq C_m$. We have that $C_m = \bigsqcup_{j=1}^{2^m} C_{m,j}$ where $\text{length}(C_{m,j}) = \frac{1}{3^m}$. Since each $C_{m,j}$ is disjoint, it must be the case that $(x-\epsilon, x+\epsilon) \subseteq C_{m,j}$ for some $1 \leqslant j \leqslant 2^m$. But the length of $(x-\epsilon, x+\epsilon)$ is 2ϵ, which is impossible. It must be that $\mathfrak{C}$ is nowhere dense. □

Proposition 2.3.3. *The total length of the Cantor set is* 0*.*

Proof. The total length of the removed intervals is:

$$\begin{aligned}
\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots &= \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \\
&= \frac{1}{2} \cdot \frac{2/3}{1 - 2/3} \\
&= 1.
\end{aligned}$$

Thus $\text{length}(\mathfrak{C}) = 0$. □

Lemma 2.3.4. *****

Proof. □

Lemma 2.3.5. *****

Proof. □

Lemma 2.3.6. *****

Proof. □

Proposition 2.3.7. ***** $\text{card}(\mathfrak{C}) = \mathfrak{c}$.

Proof. □

§ 2.4. Convergent Sequences

Definition 2.4.1. Fix a metric space (X, d).

(1) A *sequence* in X is a map $x : \mathbf{N} \to X$ defined by $n \mapsto x_n$. We denote a sequence as $(x_n)_{n\geqslant 1}$, $(x_n)_{n=1}^{\infty}$, or $(x_n)_n$.

(2) A *natural sequence* is a sequence $(x_n)_n$ in $\mathbf{N}$ with $n_1 < n_2 < n_3 < ...$

(3) A *subsequence* of a sequence $(x_n)_n$ is a sequence $(x_{n_k})_k$ where $(n_k)_k$ is a natural sequence. This is equivalent to the composition of maps $\mathbf{N} \xrightarrow{k\mapsto n_k} \mathbf{N} \xrightarrow{n_k\mapsto x_{n_k}} X$.

Definition 2.4.2. A sequence $(x_n)_n$ *converges* to $x \in X$ if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies d(x_n, x) < \epsilon).$$

We write $(x_n)_n \xrightarrow{d} x$ or $\operatorname{limit} x_n = x$.

Exercise 2.4.1. Show that a sequence can have at most one limit.

Proposition 2.4.1. *Given $(x_n)_n$ in X and $x \in X$, the following are equivalent:*

(1) $(x_n)_n \to x$ *in* X*;*

(2) $(d(x_n, x))_n \to 0$ *in* $\mathbf{R}$*;*

(3) $(\forall V \in \mathcal{N}_x)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies x_n \in V)$.

Exercise 2.4.2. Let (X, d) be a metric space and $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Then $(x_n)_n \xrightarrow{d} x$ if and only if $(x_n)_n \xrightarrow{\rho} x$.

Proposition 2.4.2. *Convergent sequences are bounded.*

Proof. Suppose that $(x_n)_n \to x$ and let $\epsilon = 1$. Find N large so that for $n \geqslant N$ we have $d(x_n, x) < 1$. Then for all $m, n \geqslant N$, we have $d(x_m, x_n) \leqslant d(x_m, x) + d(x, x_n) < 2$. Set $C = \max_{1\leqslant n,m\leqslant N} d(x_m, x_n)$. Now if $n \geqslant N$ and $m \leqslant N$, we have:

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_N) + d(x_N, x_m) \\ &\leqslant 1 + C. \end{aligned}$$

Let $K = \max\{2, 1 + C, C\}$. Then $\operatorname{diam}(\{x_n\}_{n\geqslant 1}) = \sup_{m,n\geqslant 1} d(x_n, x_m) \leqslant K$. □

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Definition 2.4.3. Let $(f_n)_n$ be a sequence of functions.

(1) $(f_n)_n$ converges *pointwise* to $f \in \mathcal{F}(\Omega, F)$ if:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N_{x,\epsilon} \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies |f_n(x) - f(x)| < \epsilon).$$

(2) $(f_n)_n$ converges *uniformly* to $f \in \mathcal{F}(\Omega, F)$ if:

$$(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) : (\forall n \in \mathbf{N})(\forall x \in \Omega)(n \geqslant N \implies |f_n(x) - f(x)| < \epsilon).$$

Example 2.4.1. Recall that a sequence $(f_n)_n$ converges to f in $\mathrm{Bd}(\Omega, Y)$ if:

$$(D(f_n, f))_n \to 0 \text{ in } \mathbf{R}.$$

But this is equivalent to:

$$\sup_{x\in\Omega} d(f_n(x), f(x)) \to 0 \text{ in } \mathbf{R}.$$

Again, we can rewrite this as:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})\Big(n \geqslant N \implies \sup_{x\in\Omega} d(f_n(x), f(x)) < \epsilon\Big).$$

But this definition is equivalent to:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(\forall x \in \Omega)\Big(n \geqslant N \implies d(f_n(x), f(x)) < \epsilon\Big).$$

This is precisely the definition of uniform convergence. Thus convergence in $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq \ell_\infty(\Omega) = \mathrm{Bd}(\Omega, F)$ is uniform.

Proposition 2.4.3. *Let $(d_k)_k$ be a separating family of semi-metrics which are uniformly bounded. Define:*

$$d(x, y) := \sum_{k=1}^{\infty} 2^{-k} d_k(x, y).$$

Then $(x_n)_n \xrightarrow{d} x$ if and only if for all k, $(d_k(x_n, x))_n \to 0$.

Proof. Fix k. We have:

$$0 \leqslant 2^{-k} d_k(x_n, x) \leqslant d(x_n, x).$$

Note that $(x_n)_n \xrightarrow{d} x$ implies $(d(x_n, x))_n \to 0$. Multiplying both sides of the above equation by 2^k and applying the squeeze theorem gives $(d_k(x_n, x)) \to 0$.

Now let $\epsilon > 0$. Since each d_k is uniformly bounded, let $C = \sup_{k,x,y} d(x, y)$. Find K large so that:

$$\sum_{k>K} 2^{-k} < \frac{\epsilon}{2C}.$$

We know $(d_k(x_n, x))_n \to 0$ for $k = 1, 2, ..., K$. So there exists $N_1, N_2, ..., N_K$ with $n \geqslant N_k$ implying $d_k(x_n, x) < \frac{\epsilon}{2}$. Let $N = \max_{k=1}^{K} N_k$. For $n \geqslant N$, we have:

$$\begin{aligned} d(x_n, x) &= \sum_{k=1}^{K} 2^{-k} d_k(x_n, x) + \sum_{k>K} 2^{-k} d_k(x_n, x) \\ &= \sum_{k=1}^{K} 2^{-k} d_k(x_n, x) + \sum_{k>K} 2^{-k} C \\ &< \sum_{k=1}^{K} 2^{-k} \frac{\epsilon}{2} + C \cdot \frac{\epsilon}{2C} \\ &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

Example 2.4.2. Let $X = C(\mathbf{R})$. How does one define a distance between two functions on this space? For $f, g \in C(\mathbf{R})$, one might naively consider the uniform metric:

$$d(f,g) := \sup_{x\in\Omega} |f(x) - g(x)|.$$

This construction does not guarantee $d(f,g) < \infty$. We can fix this as follows—for each $k \geqslant 1$, let:

$$P_k(f) = \sup_{x\in[-k,k]} |f(x)|.$$

Then $\{P_k\}_k = 1^\infty$ is a separating family of semi-norms which generate a separating family of semi-metrics:

$$\rho_k(f,g) := P_k(f-g).$$

These are not uniformly bounded, so make:

$$d_k(f,g) = \frac{\rho_k(f,g)}{1+\rho_k(f,g)}.$$

Then $\{d_k\}_{k=1}^\infty$ is a family of uniformly bounded semi-metrics. We can now define the *Fréchet metric*:

$$d_F(f,g) := \sum_{k=1}^\infty 2^{-k} d_k(f,g).$$

In $(C(\mathbf{R}), d_F)$, observe that:

$$\begin{aligned}
(f_n)_n \xrightarrow{d_F} f &\iff \forall k, (d_k(f_n, f))_n \to 0 \\
&\iff \forall k, (\rho_k(f_n, f))_n \to 0 \\
&\iff \forall k, \left(\sup_{x\in[-k,k]} |f_n(x) - f(x)|\right)_n \to 0 \\
&\iff \forall k, (f_n)_n \to f \text{ uniformly on } [-k,k].
\end{aligned}$$

We've obtained a new type of convergence defined below.

Definition 2.4.4. Let (X, τ) be a topological space and (Y, d_Y) be a metric space. A sequence of functions:

$$f_n : X \to Y; \;\; n \in \mathbf{N}$$

converges *compactly* to $f : X \to Y$ if, for every compact set $K \subseteq X$, the sequence $(f_n|_K)_n \to f|_K$ converges uniformly.

Example 2.4.3. *** x^n does not converge uniformly but it does converge compactly (idk how to do this)

—————//—————

Example 2.4.4. Let (X, d), (Y, ρ) be metric spaces. There are different metrics we may put on $X \times Y$, for example:

- $D_1((x, y), (x', y')) = d(x, x') + \rho(y, y')$
- $D_2((x, y), (x', y')) = \sqrt{d(x, x')^2 + \rho(y, y')^2}$
- $D_\infty((x, y), (x', y')) = \max\{d(x, x') + \rho(y, y')\}$.

These metrics are all equivalent. A sequence of points $(x_n, y_n)_n$ converges to (x, y) in $X \times Y$ with respect to any of these metrics if and only if $(x_n)_n \xrightarrow{d} x$ and $(y_n)_n \xrightarrow{\rho} y$.

Example 2.4.5. Let $\{(X_k, d_k)_k\}_k$ be a family of metric spaces where the d_k are uniformly bounded. We looked at the product:

$$X = \prod_{k=1}^{\infty} X_k$$

with:

$$d(f, g) = \sum_{k=1}^{\infty} 2^{-k} d_k(f(k), g(k)).$$

We have that $(f_n)_n \xrightarrow{d} f$ if and only if $(f_n(k))_n \to f(k)$ for all k (pointwise convergence).

Example 2.4.6. Let $(V, \|\cdot\|)$ be a normed space. Then $(v_n)_n \to v$ if and only if $(\|v_n - v\|)_n \to 0$.

Definition 2.4.5. Let $(v_k)_k$ be a sequence in $(V, \|\cdot\|)$.

(1) A *sequence of partial sums* $(s_n)_n$ is defined as $s_n = \sum_{k=1}^n v_k$.

(2) If $(s_n)_n \to s$ in V we say the *series* $\sum_{k=1}^\infty v_k$ converges and write $\sum_{k=1}^\infty v_k = s$.

(3) The series $\sum v_k$ converges *absolutely* if $\sum \|v_k\|$ converges.

——————//——————

Proposition 2.4.4. *Let (X, d) be a metric space and $A \subseteq X$. We have $x \in \overline{A}$ if and only if there exists a sequence $(a_n)_n$ in A with $(a_n)_n \to x$.*

Proof. Let $x \in \overline{A}$. For each $n \geqslant 1$, we have $U(x, \frac{1}{n}) \cap A \neq \emptyset$, so choose $a_n \in U(x, \frac{1}{n}) \cap A$. Then $d(x, a_n) < \frac{1}{n}$, so $(a_n)_n \to x$.

Now suppose $(a_n)_n$ is a sequence in A and $(a_n)_n \to x$. Given $\epsilon > 0$, find N large so $d(x, a_N) < \epsilon$. Then $a_N \in U(x, \epsilon) \cap A$. Since $U(x, \epsilon) \cap A \neq \emptyset$, we have $x \in \overline{A}$. $\square$

Proposition 2.4.5. *Let (X, d) be a metric space and $A \subseteq X$. The following are equivalent:*

(1) A is closed;

(2) If $(a_n)_n$ is a sequence in A which converges to $x \in X$, then $x \in A$.

Proof. Consider the sequence $(a_n)_n$ in A with $(a_n)_n \to x$. Then by the previous proposition $x \in \overline{A}$. But since A is closed, $A = \overline{A}$. Thus $x \in A$.

To show A is closed we show $A = \overline{A}$. Clearly $A \subseteq \overline{A}$. Let $x \in \overline{A}$. Then there exists a sequence $(a_n)_n$ in A with $(a_n)_n \to x$. So $x \in A$ by (2). Thus $\overline{A} \subseteq A$. $\square$

Exercise 2.4.3. Show $x \in \overline{A}$ if and only if $x \in A$ or there exists a sequence $(a_n)_n$ in $A \setminus \{a\}$ with $(a_n)_n \to x$.

Example 2.4.7. *** The space $c_0 \subseteq \ell_\infty$ is closed, and furthermore $\overline{c_{00}} = c_0$. To see this, consider a sequence $(z_n)_n$ in c_0 converging to $f \in \ell_\infty$. We will show that $f \in c_0$.

Let $\epsilon > 0$ be given. Find N large so that:

$$\|z_N - f\|_u < \frac{\epsilon}{2}.$$

Since $z_N \in c_0$, we know $\lim_{k\to\infty} z_N(k) = 0$. So find K large so that for $k \geqslant K$:

$$|z_N(k)| < \frac{\epsilon}{2}.$$

Putting this all together, for $k \geqslant K$ we have:

$$\begin{aligned} |f(k)| &= |f(k) - z_N(k) + z_N(k)| \\ &\leqslant |f(k) - z_N(k)| + |z_N(k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $f \in c_0$. By Proposition 2.4.5, c_0 is closed.

Definition 2.4.6. Let (X, d) be a metric space and $A \subseteq X$. The *distance* of an element $x \in X$ to A is a map $\operatorname{dist}_A : X \to [0, \infty)$ defined by $\operatorname{dist}_A(x) = \inf_{a\in A} d(x, a)$.

Proposition 2.4.6. *****

§ 2.5. Continuity

Definition 2.5.1. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ is *continuous at* $x_0 \in X$ if any of the equivalent definitions are satisfied:

(1) $(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X)\Big(d(x, x_0) < \delta \implies \rho\big(f(x), f(x_0)\big) < \epsilon\Big)$

(2) $(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X)\Big(x \in U_X(x_0, \delta) \implies f(x) \in U_Y\big(f(x_0), \epsilon\big)\Big)$

(3) $(\forall \epsilon > 0)(\exists \delta > 0) : f\big(U_X(x_0, \delta)\big) \subseteq U_Y\big(f(x_0), \epsilon\big)$

Proposition 2.5.1. *Let $f : (X, d) \to (Y, \rho)$ be a map between metric spaces and $x_0 \in X$. The following are equivalent:*

(1) f is continuous at x_0;

(2) $(\forall V \in \mathcal{N}_{f(x_0)})(\exists U \in \mathcal{N}_{x_0}) : f(U) \subseteq V$;

(3) $(\forall (x_n)_n \in X^{\mathbf{N}})\big((x_n)_n \to x_0 \implies (f(x_n))_n \to f(x_0)\big)$

Proof. (1)⇒(2) follows from Definition 2.5.1.

(1)⇒(3) Let $(x_n)_n \to x_0$. Let $\epsilon > 0$. Since f is continuous, find $\epsilon > 0$ do that $x \in U(x, \delta)$ implies $f(x) \in U(f(x_0), \epsilon)$. Whence $d(f(x_n), f(x_0)) < \epsilon$, establishing $(f(x_n))_n \to f(x_0)$.

(3)⇒(1) We prove the contrapositive of this statement. If f is not continuous, choose $\epsilon_0 > 0$ so that $d(x_n, x_0) < \frac{1}{n}$ and $d(f(x_n), f(x_0)) \geqslant \epsilon_0$. Whence $(x_n)_n \to x_0$ and $(f(x_n))_n \not\to f(x_0)$. □

Proposition 2.5.2. *Let $f : (X, d) \to (Y, \rho)$ be a map of metric spaces. The following are equivalent:*

(1) f is continuous;

(2) $(\forall V \in \tau_Y), f^{-1}(V) \in \tau_X$;

Proof. Let $V \subseteq Y$ be open. If $f^{-1}(V) = \emptyset$, we're done. If not, let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since V is open, find $\epsilon > 0$ so that $U(f(x), \epsilon) \subseteq V$. Since f is continuous, find $\delta > 0$ so that $f(U(x, \delta)) \subseteq U(f(x), \epsilon) \subseteq V$. Whence $U(x, \delta) \subseteq f^{-1}(V)$.

Let $x \in X$ and $\epsilon > 0$. Since $U(f(x), \epsilon) \in \tau_Y$, we have $f^{-1}(U(f(x)), \epsilon) \in \tau_X$. Note that $x \in f^{-1}(U(f(x)), \epsilon)$. Since this set is open, find $\delta > 0$ so that $U(x, \delta) \subseteq f^{-1}(U(f(x)), \epsilon)$. Thus $f(U(x, \delta)) \subseteq U(f(x), \epsilon)$; i.e., f is continuous. □

Proposition 2.5.3. *Let $(X, d) \xrightarrow{f} (Y, \rho) \xrightarrow{g} (Z, \gamma)$ be maps of metric spaces. If f is continuous at $x \in X$ and g is continuous at $y = f(x)$, then $g \circ f$ is continuous at x.*

Proof. Exercise. □

Proposition 2.5.4. *Let (X, d) be a metric space with $A \subseteq X$ dense. Let $f : X \to F$ be a continuous and bounded function. Then $\sup_{x\in A} f(x) = \sup_{x\in X} f(x)$.*

Proof. Since $A \subseteq X$ we have $\sup_{x\in A} f(x) \leqslant \sup_{x\in X} f(x)$. Conversely, let $\epsilon > 0$. Find $x' \in X$ so that $\sup_{x\in X} f(x) - \frac{\epsilon}{2} < f(x')$. Since f is continuous, find $\delta > 0$ so that for all $x \in X$, $d(x, x') < \delta$ implies $|f(x) - f(x')| < \frac{\epsilon}{2}$. Since A is dense, find $a \in A$ so that $d(x', a) < \delta$. This implies $|f(x') - f(a)| < \frac{\epsilon}{2}$, or equivalently $f(x') - \frac{\epsilon}{2} < f(a)$. This gives:

$$\sup_{x\in X} f(x) - \epsilon < f(a).$$

Hence for $\epsilon > 0$ we have:

$$\sup_{x\in X} f(x) < \sup_{x\in A} f(x) + \epsilon.$$

Taking $\epsilon \to 0$ gives $\sup_{x\in X} f(x) \leqslant \sup_{x\in A} f(x)$. □

Definition 2.5.2. Let $f : (X, d) \to (Y, \rho)$ be a map of metric spaces.

(1) f is *uniformly continuous* if:

$$(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x, x' \in X)(d(x, x') < \delta \implies \rho(f(x), f(x')) < \epsilon).$$

(2) f is *Lipschitz* if there exists $C > 0$ such that:

$$(\forall x, x' \in X)(\rho(f(x), f(x')) \leqslant Cd(x, x')).$$

If $C < 1$, we say f is *contractive.*

(3) f is *isometric* (or an *isometry*) if:

$$(\forall x, x' \in X)(\rho(f(x), f(x')) = d(x, x'))$$

Exercise 2.5.1. Show that Lipschitz implies uniform continuity. Show that uniform continuity implies continuity. Show that the converse direction fails in general.

Example 2.5.1. Let $(V, \|\cdot\|)$ be a normed space. Then $V \xrightarrow{\|\cdot\|} [0, \infty)$ is continuous. Indeed, we have $|\|v\| - \|w\|| \leqslant \|v - w\|$, so $\|\cdot\|$ is Lipschitz.

Example 2.5.2. Let (X, d) be a metric space and equip $X \times X$ with the product metric D_1. Claim: $d : X \times X \to [0, \infty)$ is continuous. Indeed, given $(x, y), (x', y') \in X \times X$, then we have:

$$d(x, y) \leqslant d(x, x') + d(x', y') + d(y', y).$$

So we have $d(x, y) - d(x', y') \leqslant d(x, x') + d(y, y')$. But this is equivalent to $|d(x, y) - d(x', y')| \leqslant D_1((x, y), (x', y'))$. So d is Lipschitz.

Example 2.5.3. If (X, d) is a metric space and $A \subseteq X$, then $\operatorname{dist}_A : X \to [0, \infty)$ is continuous. We've shown that $|\operatorname{dist}_A(x) - \operatorname{dist}_A(y)| \leqslant d(x, y)$, so dist_A is Lipschitz.

Definition 2.5.3. Let X be a topological space. We say X is *normal* (or T4) if, for any $A, B \subseteq X$ closed satisfying $A \cap B = \emptyset$, then there exists $U, V \in \tau_X$ with $A \subseteq U$, $B \subseteq V$ satisfying $U \cap V = \emptyset$.

Proposition 2.5.5. *Metric spaces are normal.*

Proof. Let $A, B \subseteq (X, d)$ with $A \cap B = \emptyset$. Define $f : (X, d) \to \mathbf{R}$ by:

$$f(x) = \frac{\operatorname{dist}_A(x)}{\operatorname{dist}_A(x) + \operatorname{dist}_B(x)}.$$

Then f is continuous. Moreover, define:

$$U := f^{-1}\left((-1/2, 1/2)\right)$$
$$V := f^{-1}\left((-1/2, 3/2)\right)$$

Then U and V open with $U \cap V = \emptyset$. $\square$

Proposition 2.5.6. *Let V and W be normed spaces and $T : V \to W$ linear. The following are equivalent:*

(1) T is continuous at 0_V;

(2) T is continuous;

(3) T is uniformly continuous;

(4) T is Lipschitz;

(5) there exists $C \geqslant 0$ such that $\|Tv\| \leqslant C\,\|v\|$ for all $v \in V$;

(6) T is bounded.

Proof. We will show (6)⇔(5)⇒(4)⇒(3)⇒(2)⇒(1)⇒(5). Let T be bounded. Given $v \in V$, $v \neq 0$, we have:

$$\begin{aligned} \|T\|_{\text{op}} &= \sup_{v \in B_V} \|Tv\| \\ &\geqslant \left\| T \frac{v}{\|v\|} \right\| \quad \text{for all } v \in V \\ &= \frac{1}{\|v\|} \|Tv\| \quad \text{for all } v \in V. \end{aligned}$$

Thus $\|Tv\| \leqslant \|T\|_{\text{op}} \|v\|$ for all $v \in V$. The converse is clear by inspection.

Suppose there exists $C \geqslant 0$ satisfying (5). We have that $\|Tv - Tv'\| \leqslant C \|v - v'\|$. Thus T is Lipschitz.

Let T be Lipschitz. Let $\epsilon > 0$ and find $\delta = \frac{\epsilon}{c}$. Then $\|v - v'\| < \delta$ implies:

$$\begin{aligned} \|Tv - Tv'\| &\leqslant C \|v - v'\| \\ &< c\frac{\epsilon}{c} \\ &= \epsilon. \end{aligned}$$

Thus T is uniformly continuous.

Suppose that T be uniformly continuous. Fix $x \in V$. Given $\epsilon > 0$, we can find $\delta > 0$ so that $\|v - x\| < \delta$ implies $\|Tv - Tx\| < \epsilon$. Thus T is continuous at $x \in V$. Since x was arbitrary, T is continuous. Moreover, T will be continuous at 0_V, establishing (1).

We will now show (1) implies (5). Let $\epsilon = 1$. We can find a $\delta > 0$ such that $T\big(U(0,\delta)\big) \subseteq U(0,1)$. If $v \in V$, $v \neq 0$, then $\frac{\delta v}{2\|v\|} \in U(0,\delta)$. Since T is continuous at 0, we have $T\frac{\delta v}{2\|v\|} \in U(0,1)$[1]. This gives $\left\|T\frac{\delta v}{2\|v\|}\right\| < 1$, which is equivalent to $\|Tv\| < \frac{2}{\delta}\|v\|$. This establishes (5). □

Corollary 2.5.7. **** Let V be a normed space and $T : \ell_p^n \to V$ linear. Then T is uniformly continuous.*

[1]Recall that $T(0) = 0$.

Proof. Let $x \in \ell_p^n$ and $\mathcal{B} = \{e_1, ..., e_n\}$ a basis for ℓ_p^n. Find $\alpha_1, ..., \alpha_n$ so that:

$$\begin{aligned}
\|Tv\| &= \left\| T\left(\sum_{j=1}^{n} \alpha_j e_j\right) \right\| \\
&= \left\| \sum_{j=1}^{n} \alpha_j T e_j \right\| \\
&\leqslant \sum_{j=1}^{n} |\alpha_j| \, \|Te_j\| \\
&\leqslant c \left\| \sum_{j=1}^{n} \alpha_j e_j \right\|_1 \\
&\leqslant c \cdot c' \left\| \sum_{j=1}^{n} \alpha_j e_j \right\|_p ,
\end{aligned}$$

where $c = \max_{j=1}^{n} \|Te_j\|$ and $\|\cdot\|_1 \leqslant c' \|\cdot\|_p$. □

Proposition 2.5.8. *Let (X, d) be a metric space with $A \subseteq X$ dense. If $f, g : X \to (Y, \rho)$ are continuous with $f(a) = g(a)$ for all $a \in A$, then $f = g$.*

Proof. If $x \in X$, we can find a sequence $(a_n)_n$ in A with $(a_n)_n \xrightarrow{d} x$. Then:

$$\begin{array}{c} (f(a_n))_n \to f(x) \\ \shortparallel \\ (g(a_n))_n \to g(x). \end{array}$$

Thus $f(x) = g(x)$. □

Definition 2.5.4. Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$.

(1) f is a *homeomorphism* if f is bijective with f and f^{-1} continuous. If such an f exists, we say $X \cong Y$ are *homeomorphic*.

(2) f is a *uniformism* if f is bijective with f and f^{-1} uniformly continuous. If such an f exists, we say $X \cong Y$ are *uniformly isomorphic*.

(3) f is an *metric isomorphism* if f is bijective with f and f^{-1} Lipschitz. If such an f exists, we say $X \cong Y$ are *metrically isomorphic*.

(4) f is an *isometric isomorphism* if f and f^{-1} are isometries. We say $X \cong Y$ are *isometrically isomorphic*.

Example 2.5.4. $(0, 1) \cong \mathbf{R}$ are homeomorphic, but not uniformly isomorphic.

Example 2.5.5. *** Let $a = (a_k)_k \in \ell_1$. Define $\varphi_a : c_0 \to F$ by $\varphi_a(z) = \sum_{k \geqslant 1} a_k z_k$. This series converges since:

Definition 2.5.5. Let X be a set with two metrics d_1 and d_2.

(1) d_1 and d_2 are *metrically equivalent* if $\mathrm{id} : (X, d_1) \to (X, d_2)$ and $\mathrm{id}^{-1} : (X, d_2) \to (X, d_1)$ are Lipschitz.

(2) d_1 and d_2 are *uniformly equivalent* if $\mathrm{id} : (X, d_1) \to (X, d_2)$ and $\mathrm{id}^{-1} : (X, d_2) \to (X, d_1)$ are uniformisms.

(3) d_1 and d_2 are *topologically equivalent* if $\mathrm{id} : (X, d_1) \to (X, d_2)$ and $\mathrm{id}^{-1} : (X, d_2) \to (X, d_1)$ are homeomorphisms.

Example 2.5.6. ***

§ 2.6. Completeness

Definition 2.6.1. A sequence $(x_n)_n$ in a metric space (X, d) is *d-Cauchy* if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall p, q \in \mathbf{N})(p, q \geqslant N \implies d(x_p, x_q) < \epsilon).$$

Proposition 2.6.1. *Let (x_n) be a sequence in (X, d).*

(1) If $(x_n)_n$ converges, then $(x_n)_n$ is Cauchy.

(2) If $(x_n)_n$ is Cauchy, then $(x_n)_n$ is bounded.

Proof. (1) Let $x \in X$ and suppose $(x_n)_n \to x$. Let $\epsilon > 0$. Find N large so that for $p \geqslant N$ we have $d(x_p, x) < \frac{\epsilon}{2}$. Then $p, q \geqslant N$ implies:

$$\begin{aligned} d(x_p, x_q) &\leqslant d(x_p, x) + d(x, x_q) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

(2) Let $\epsilon = 1$. Find N large so $p, q \geqslant N$ implies $d(x_p, x_q) < 1$. Let $C = \max_{1 \leqslant p,q \leqslant N} d(x_p, x_q)$. Without loss of generality, if $p \geqslant N$ and $q \leqslant N$, then $d(x_p, x_q) \leqslant d(x_p, x_N) + d(x_N, x_q) < 1 + C$. Set $K = \max\{1, 1 + C\}$. Then $\mathrm{diam}(\{x_n\}_{n \geqslant 1}) = \sup_{p,q \geqslant 1} d(x_p, x_q) < K$. ☐

Proposition 2.6.2. *Let $(x_n)_n$ be a Cauchy sequence in X and suppose there exists a subsequence $(x_{n_k})_k$ converging to $x \in X$. Then $(x_n)_n$ converges to x.*

Proof. Let $\epsilon > 0$. Since $(x_n)_n$ is Cauchy, there exists N large so $n, n_k \geqslant N$ implies $d(x_n, x_{n_k}) < \frac{\epsilon}{2}$. This gives:

$$\begin{aligned} d(x_n, x) &= d(x_n, \operatorname*{limit}_{k \to \infty} x_{n_k}) \\ &= \operatorname*{limit}_{k \to \infty} d(x_n, x_{n_k}) \\ &\leqslant \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

☐

Definition 2.6.2. A metric space is said to be *complete* if every Cauchy sequence converges. A complete normed space is called a *Banach space*. A complete inner product space is called a *Hilbert space*.

Lemma 2.6.3. *Let $f : (X,d) \to (Y,\rho)$ be uniformly continuous. If $(x_n)_n$ is d-Cauchy then $(f(x_n))_n$ is ρ-Cauchy.*

Proof. Let $\epsilon > 0$. Find $\delta > 0$ so that $d(x,x') < \delta$ implies $\rho(f(x), f(x')) < \epsilon$. Pick N sufficiently large so that $p, q \geqslant N$ implies $d(x_p, x_q) < \delta$. This gives $\rho(f(x_p), f(x_q)) < \epsilon$, whence $(f(x_n))_n$ is ρ-Cauchy. $\square$

Corollary 2.6.4. *If $f : (X,d) \to (Y,\rho)$ is a uniformism, then (X,d) is complete if and only if (Y,ρ) is complete.*

Proof. Let (X,d) be complete. If $(y_n)_n$ is ρ-Cauchy, then $(f^{-1}(y_n))_n$ is d-Cauchy in X. So we can find some $x \in X$ such that $(f^{-1}(y_n))_n \to x$. Then $(f(f^{-1}(y_n)))_n = (y_n)_n \to f(x)$. The converse follows similarly. $\square$

Corollary 2.6.5. *If d_1 and d_2 are uniformly equivalent metrics on a set X, then (X,d_1) is complete if and only if (X,d_2) is complete.*

Proof. Since the map $\mathrm{id} : (X,d_1) \to (X,d_2)$ is a uniformism, the previous corollary gives that (X,d_1) is complete if and only if (X,d_2) is complete. $\square$

Proposition 2.6.6. *ℓ_p^d is a Banach space for $1 \leqslant p \leqslant \infty$.*

Proof. We only need to show this for ℓ_∞^d since all the p-norms are equivalent. Let $(x_n)_n$ be $\|\cdot\|_\infty$-Cauchy in ℓ_∞^d. Let $\epsilon > 0$. Find N large so for $n, m \geqslant N$ we have $\|x_n - x_m\|_\infty < \epsilon$. Observe that:

$$\begin{aligned} |x_n(k) - x_m(k)| &\leqslant \max_{1 \leqslant k \leqslant d} |x_n(k) - x_m(k)| \\ &= \|x_n - x_m\|_\infty \\ &< \epsilon, \end{aligned}$$

where $x_n(k)$ is the k^{th} entry of the d-tuple x_n. So for each $k = 1, ..., d$, we know $(x_n(k))_n$ is Cauchy in F. Set $\text{limit}_{n\to\infty}\, x_n(k) = x(k)$ for $k = 1, .., d$. This gives:

$$\|x - x_n\|_\infty = \max_{1 \leqslant k \leqslant d} |x(k) - x_n(k)| \xrightarrow{n\to\infty} 0.$$

Whence $(x_n)_n \to x$ in ℓ_∞^d. Now set $y = (y_1, ..., y_n)$. We have:

$$\begin{aligned} \|x_n - y\|_p &\leqslant c' \|x_n - y\|_1 \\ &= c' \sum_{j=1}^{d} |x_n(j) - y_j| \\ &\leqslant c' \max_{1 \leqslant j \leqslant d} |x_n(j) - y_j| \xrightarrow{n\to\infty} 0. \end{aligned}$$

Thus ℓ_p^d is complete. $\square$

Proposition 2.6.7. *ℓ_p is a Banach space for $1 \leqslant p \leqslant \infty$.*

Proof. Suppose that $(f_n)_n$ is $\|\cdot\|_{\ell_p}$-Cauchy. Observe that

$$\begin{aligned}|f_n(k) - f_m(k)|^p &\leqslant \sum_{j=1}^{\infty} |f_n(j) - f_m(j)|^p \\ &= \|f_n - f_m\|_{\ell_p}^p\end{aligned}$$

So $(f_n(k))_n$ is Cauchy in F. Since this space is complete, define $\operatorname{limit}_{n\to\infty} f_n(k) := f(k)$.

Our goal is to find some function $f : \mathbf{N} \to F$ satisfying $f \in \ell_p$ and $\|f_n - f\|_{\ell_p} \to 0$. The $f(k)$ we've just obtained will lead us to the most suitable candidate.

Since $(f_n)_n$ is $\|\cdot\|_{\ell_p}$-Cauchy, it is bounded by some constant. Fix $K \geqslant 1$ and observe that:

$$\begin{aligned}\sum_{k=1}^{K} |f(k)|^p &= \sum_{k=1}^{K} \left|\operatorname*{limit}_{n\to\infty} f_n(k)\right|^p \\ &= \operatorname*{limit}_{n\to\infty} \sum_{k=1}^{K} |f_n(k)|^p \\ &\leqslant \sup_{n\geqslant 1} \|f_n\|_{\ell_p}^p \\ &:= C.\end{aligned}$$

Since the sequence $\left(\sum_{k=1}^{K} |f(k)|^p\right)_{K=1}^{\infty}$ is increasing and bounded above C, the Monotone Convergence Theorem says that it's limit exists. We obtain:

$$\begin{aligned}\operatorname*{limit}_{K\to\infty} \sum_{k=1}^{K} |f(k)|^p &= \sum_{k=1}^{\infty} |f(k)|^p \\ &= \|f\|_{\ell_p}^p \\ &< \infty.\end{aligned}$$

Thus $f \in \ell_p$.

It remains to show that $(f_n)_n$ converges to f. Given $\epsilon > 0$, find N large so that $n, m \geqslant N$ implies $\|f_n - f_m\|_p < \epsilon$. For every $n, m \geqslant N$ we have:

$$\begin{aligned}\sum_{k=1}^{K} |f_m(k) - f_n(k)|^p &\leqslant \|f_m - f_n\|_{\ell_p}^p \\ &< \epsilon^p.\end{aligned}$$

Taking the limit as $m \to \infty$ and considering all $n \geqslant N$ gives:

$$\sum_{k=1}^{K} |f(k) - f_n(k)|^p < \epsilon^p.$$

Finally, taking the limit as $K \to \infty$ and simplifying gives $\|f - f_n\|_{\ell_p} < \epsilon$. Thus ℓ_p is complete. $\square$

Proposition 2.6.8. *Let (Y, d) be a complete metric space. The set of bounded functions* $\mathrm{Bd}(\Omega, Y)$ *with* $\|\cdot\|_u$ *is complete.*

Proof. Let $(f_n)_n$ be D_u-Cauchy. Fix $x, x' \in \Omega$ and let $\epsilon > 0$. Find N large so that $n, m \geqslant N$ implies:

$$d(f_n(x), f_m(x)) \leqslant D_u(f_n, f_m) < \epsilon.$$

Thus $(f_n(x))_n$ is Cauchy in Y. Since Y is complete, define $\text{limit}_{n\to\infty} f_n(x) := f(x)$. Now find N_1 large so $n \geqslant N_1$ implies $d(f(x), f_n(x)) < \epsilon$. Find N_2 large so $n \geqslant N_2$ implies $d(f(x'), f_n(x')) < \epsilon$. Observe that:

$$\begin{aligned} d(f(x), f(x')) &\leqslant d(f(x), f_{N_1}(x)) + d(f_{N_1}(x), f_{N_2}(x')) + d(f_{N_2}(x'), f(x')) \\ &< 2\epsilon + d(f_{N_1}(x), f_{N_2}(x')) \\ &\leqslant 2\epsilon + d(f_{N_1}(x), f_{N_2}(x)) + d(f_{N_2}(x), f_{N_2}(x')) \\ &\leqslant 2\epsilon + \sup_{n,m\geqslant 1} d(f_n(x), f_m(x)) + \sup_{x,x'\in\Omega} d(f_{N_2}(x), f_{N_2}(x')) \\ &= 2\epsilon + \mathrm{diam}(\{f_n(x)\}_{n\geqslant 1}) + \mathrm{diam}(f_{N_2}(\Omega)). \end{aligned}$$

Note that the sequence $(f_n(x))_n$ is bounded because it is Cauchy —so there exists $C_1 \geqslant 0$ such that $\mathrm{diam}(\{f_n(x)\}_{n\geqslant 1}) < C_1$. Moreover, since $f_{N_2} \in \mathrm{Bd}(\Omega, Y)$, there exists $C_2 \geqslant 0$ such that $\mathrm{diam}(f_{N_2}(\Omega)) < C_2$. This gives:

$$d(f(x), f(x')) < 2\epsilon + C_1 + C_2.$$

Since this inequality is independent of any n, and since $x, x' \in \Omega$ was arbitrary, we have that $\mathrm{diam}(f(\Omega)) < \infty$; i.e., $f \in \mathrm{Bd}(\Omega, Y)$. With the same ϵ as above, find N_3 large so that for all $n, m \geqslant N_3$ then $D_u(f_n, f_m) < \frac{\epsilon}{2}$. We know:

$$d(f_n(x), f_m(x)) \leqslant D_u(f_n, f_m) < \frac{\epsilon}{2}.$$

Taking $m \to \infty$ gives $d(f_n(x), f(x)) \leqslant \frac{\epsilon}{2}$ for all $n \geqslant N$. But note that N does not depend on our fixed $x \in \Omega$. It follows that $D_u(f_n, f) \leqslant \frac{\epsilon}{2} < \epsilon$. Thus $(f_n)_n \to f$, establishing $\mathrm{Bd}(\Omega, Y)$ as complete. ☐

Corollary 2.6.9. *$\ell_\infty(\Omega)$ is complete.*

Proposition 2.6.10. *Let (X, d) be a complete metric space and $Y \subseteq X$. Y is complete if and only if Y is closed.*

Proof. Let $(y_n)_n$ be a sequence in Y converging to $x \in X$. Then $(y_n)_n$ is sequence in X. Since X is complete, $(y_n)_n$ is Cauchy. Since Y is complete, $(y_n)_n$ must converge to some $y \in Y$. Since sequences can have at most one limit, it must be that $y = x$. Thus $x \in Y$; i.e., Y is closed.

Conversely, if $(y_n)_n$ is Cauchy in Y, then it is Cauchy in X. Since X is complete, there exists $x \in X$ with $(y_n)_n \to x$. Since Y is closed, $x \in Y$. Thus Y is complete. ☐

Proposiiton 2.6.10 and Proposition 2.4.5 are extremely useful tools for showing a space is complete.

Corollary 2.6.11. *Let (X, d) and (Y, ρ) be metric spaces.*

(1) $C_b(X, Y) := C(X, Y) \cap \mathrm{Bd}(X, Y)$ is D_u-complete.

(2) $C_b(X)$ is a $\|\cdot\|_u$-Banach space.

(3) $C_0(\mathbf{R})$ is a $\|\cdot\|_u$-Banach space.

Proof. (1) Let $(f_n)_n$ be a sequence in $C_b(X, Y)$ converging to $f \in \mathrm{Bd}(X, Y)$. Let $x \in X$ and $\epsilon > 0$. Find N large so that $D_u(f_N, f) < \frac{\epsilon}{3}$. Find $\delta > 0$ so that for all $x' \in X$, $d(x, x') < \delta$ implies $\rho(f_N(x), f_N(x')) < \frac{\epsilon}{3}$. For $d(x, x') < \delta$:

$$\begin{aligned}\rho(f(x), f(x')) &\leqslant \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(x')) + \rho(f_N(x'), f(x')) \\ &\leqslant 2D_u(f_n, f) + \rho(f_N(x), f_N(x')) \\ &< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Thus $f \in C_b(X, Y)$ because it is bounded and continuous. Since $C_b(X, Y) \subseteq \mathrm{Bd}(X, Y)$ is closed, it is complete.

(2) As we've just shown, the space $C_b(X)$ is complete. It only remains to show it is a vector space.

(3) Let $(f_n)_n$ be a sequence in $C_0(\mathbf{R})$ converging to $f \in C_b(\mathbf{R})$. Let $\epsilon > 0$ and find N large so that $\|f - f_N\|_u < \frac{\epsilon}{2}$. Since $f_N \in C_0(\mathbf{R})$, we know that $\operatorname{limit}_{x\to\pm\infty} f_N(x) = 0$. So there exists $M > 0$ with $|x| \geqslant M$ implying $|f_N(x)| < \frac{\epsilon}{2}$. For $|x| \geqslant M$ observe that:

$$\begin{aligned}|f(x)| &\leqslant |f(x) - f_N(x)| + |f_N(x)| \\ &\leqslant \|f - f_N\|_u + |f_N(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Thus $\operatorname{limit}_{x\to\infty} f(x) = 0$; i.e., $f \in C_0(\mathbf{R})$. Since $C_0(\mathbf{R}) \subseteq C_b(\mathbf{R})$ is closed, it is complete. ☐

Proposition 2.6.12. *Let V be a normed space and W a Banach space. $B(V, W)$ with $\|\cdot\|_{\mathrm{op}}$ is a Banach space.*

Proof. Let $(T_n)_n$ be $\|\cdot\|_{\mathrm{op}}$-Cauchy. Let $v \in V$ and $\epsilon > 0$. Find N_1 large so that $n, m \geqslant N_1$ implies $\|T_n - T_m\|_{\mathrm{op}} < \frac{\epsilon}{\|v\|}$. We can see:

$$\begin{aligned}\|T_n v - T_m v\|_W &= \|(T_n - T_m)v\|_W \\ &\leqslant \|T_n - T_m\|_{\mathrm{op}} \|v\| \\ &< \frac{\epsilon}{\|v\|} \cdot \|v\| \\ &= \epsilon.\end{aligned}$$

So $(T_n v)_n$ is $\|\cdot\|_W$-Cauchy. Since W is a Banach space, define $\operatorname{limit}_{n\to\infty} T_n v = Tv$. We must show that T is linear, bounded, and $\|T_n - T\|_{\mathrm{op}} \to 0$. Given $v_1, v_2 \in V$ and $c \in F$ we can see:

$$\begin{aligned} T(v_1 + cv_2) &= \operatorname*{limit}_{n\to\infty} T_n(v_1 + cv_2) \\ &= \operatorname*{limit}_{n\to\infty} T_n v_1 + cT_n v_2 \\ &= \operatorname*{limit}_{n\to\infty} T v_1 + c \operatorname*{limit}_{n\to\infty} T v_2 \\ &= Tv_1 + cTv_2. \end{aligned}$$

Thus T is linear. Now since $(T_n)_n$ is $\|\cdot\|_{\text{op}}$-Cauchy, it is bounded, so there exists $C > 0$ with $\|T_n\|_{\text{op}} \leqslant C$ for all $n \geqslant 1$. Using the fact norms are continuous, we have:

$$\begin{aligned} \|Tv\|_W &= \left\| \operatorname*{limit}_{n\to\infty} T_n v \right\|_W \\ &= \operatorname*{limit}_{n\to\infty} \|T_n v\|_W \\ &\leqslant \limsup_{n\to\infty} \|T_n\|_{\text{op}} \|v\| \\ &\leqslant C \|v\|. \end{aligned}$$

Thus $T \in B(V, W)$. With the same epsilon as before, find N_2 so that $n, m \geqslant N_2$ implies $\|T_n - T_m\|_{\text{op}} < \frac{\epsilon}{2}$. We can show:

$$\|T_n v - T_m v\|_W \leqslant \|T_n - T_m\|_{\text{op}} < \frac{\epsilon}{2}.$$

Taking $m \to \infty$ gives:

$$\|T_n v - Tv\|_W \leqslant \frac{\epsilon}{2}.$$

Taking the supremum over all $v \in B_V$ gives:

$$\|T_n - T\|_{\text{op}} \leqslant \frac{\epsilon}{2} < \epsilon.$$

Thus $B(V, W)$ is complete. □

Proposition 2.6.13. *Let $(V, \|\cdot\|)$ be a normed space. The following are equivalent:*

(1) V is a Banach space;

(2) If $(v_k)_k$ is a sequence in V with $\sum_{k=1}^{\infty} \|v_k\|$ convergent, then $\sum_{k=1}^{\infty} v_k$ converges.

Proof. Suppose V is a Banach space. Let $s_n = \sum_{k=1}^{n} v_k$ and $t_n = \sum_{k=1}^{n} \|v_k\|$. For $p > q > 1$:

$$\begin{aligned} \|s_p - s_q\| &= \left\| \sum_{k=q+1}^{p} v_k \right\| \\ &\leqslant \sum_{k=q+1}^{p} \|v_k\| \\ &= |t_p - t_k|. \end{aligned}$$

Since $(t_n)_n$ is convergent, it is Cauchy. So $(s_n)_n$ is Cauchy, implying it is convergent. Thus $\sum_{k=1}^{\infty} v_k$ converges.

Now let $(v_n)_n$ be Cauchy. Find $n_1 \in \mathbf{N}$ such that $p, q \geqslant n_1$ implies $\|v_p - v_q\| < 2^{-1}$. Find $n_2 > n_1$ such that $p, q \geqslant n_2$ implies $\|v_p - v_q\| < 2^{-2}$. Inductively, find $n_k > n_{k-1}$ such that $p, q \geqslant n_k$ implies $\|v_p - v_q\| < 2^{-k}$. Consider the sequence $(v_{n_{k+1}} - v_{n_k})_k$. Then:

$$\sum_{k=1}^{\infty} \left\| v_{n_{k+1}} - v_{n_k} \right\| \leqslant \sum_{k=1}^{\infty} 2^{-k} = 1.$$

By our hypothesis, $\sum_{k=1}^{\infty}(v_{n_{k+1}} - v_{n_k})$ converges. So the sequence of partial sums:

$$\begin{aligned} w_m &= \sum_{k=1}^{m} v_{n_{k+1}} - v_{n_k} \\ &= v_{n_m} - v_{n_1} \end{aligned}$$

also converges to some $w \in V$ as $m \to \infty$. However, notice:

$$\begin{aligned} (v_{n_m})_m &= (v_{n_m} - v_{n_1})_m + v_{n_1} \\ &\xrightarrow{m \to \infty} w + v_{n_1}. \end{aligned}$$

Since $(v_n)_n$ is a Cauchy sequence which admits a convergent subsequence, $(v_n)_n$ converges. Thus V is a Banach space. □

Example 2.6.1. *** Let $\mathcal{H}$ be a Hilbert space. Suppose $(e_n)_n$ is an orthonormal sequence in $\mathcal{H}$ and $(t_k)_k \in \ell_2$. We will show $\sum_{k=1}^{\infty} t_k e_k$ converges in $\mathcal{H}$, but not absolutely in general.

Let $s_n = \sum_{k=1}^{n} t_k e_k$. For $n > m > 1$:

$$\begin{aligned} \|s_n - s_m\|^2 &= \left\| \sum_{k=m+1}^{n} t_k e_k \right\|^2 \\ &= \sum_{k=m+1}^{n} |t_k|^2. \end{aligned}$$

Since $\left(\sum_{k=1}^{n} |t_k|^2\right)_n$ is Cauchy, then $(s_n)_n$ is Cauchy, whence $\sum_{k=1}^{\infty} t_k e_k$ converges. But notice $\|s_n\|^2 = \sum_{k=1}^{n} |t_k|^2$. As $n \to \infty$, we see $\|\sum_{k=1}^{\infty} t_k e_k\|^2 = \sum_{k=1}^{\infty} |t_k|^2$.

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Recall that if $f : (X, d) \to (Y, \rho)$ is a uniformly continuous map between metric spaces and $(x_n)_n$ is Cauchy, then $(f(x_n))_n$ is Cauchy. Complete spaces have the unique property that, given a map from a dense subset $A \to Y$, we can define an extension of such function.

Theorem 2.6.14. *Let (X, d) and (Y, ρ) be metric spaces with Y complete. Suppose $A \subseteq X$ is dense and $f : A \to Y$ is uniformly continuous. There exists a unique uniformly continuous map $\widetilde{f} : X \to Y$ with $\widetilde{f}(x) = f(x)$ for all $x \in A$.*

Proof. Let $x \in X$. We know there exists a sequence $(a_n)_n$ in A with $(a_n)_n \to x$. Since $(a_n)_n$ is convergent, it is Cauchy, so $(f(a_n))_n$ is also Cauchy. By the completeness of Y, $(f(a_n))_n$ converges. Define $\widetilde{f}(x) := \text{limit}_{n \to \infty} f(a_n)$.

We must show this extension is well-defined. Suppose $(b_n)_n$ is another sequence in A with $(b_n)_n \to x$. Then the mixed sequence $(a_1, b_1, a_2, b_2, ...)$ will converge to x. The same reasoning as above tells us $(f(a_1), f(b_1), f(a_2), f(b_2), ...)$ converges in Y. The two subsequences $(f(a_n))_n$ and $(f(b_n))_n$ must then converge to the same limit.

We will now show that $\widetilde{f}$ is uniformly continuous. Let $\epsilon > 0$. Find $\delta > 0$ so that for all $a, b \in A$, $d(a, b) < \delta$ implies $\rho(f(a), f(b)) < \frac{\epsilon}{2}$. Now let $x, x' \in X$ with $d(x, x') < \frac{\delta}{4}$. Find sequences $(a_n)_n$ and $(b_n)_n$ in A with $(a_n)_n \to x$ and $(b_n)_n \to x'$. Find N large so that $n \geqslant N$ implies $d(a_n, x) < \frac{\delta}{4}$ and $d(b_n, x) < \frac{\delta}{4}$. The triangle inequality gives $d(a_n, b_n) < \frac{3\delta}{4} < \delta$. So $\rho(f(a_n), f(b_n)) < \frac{\epsilon}{2}$ for all $n \geqslant N$. Observe that:

$$\begin{aligned}
\rho(\widetilde{f}(x), \widetilde{f}(x')) &= \operatorname*{limit}_{n\to\infty} \rho(f(a_n), f(b_n)) \\
&\leqslant \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

Thus $\widetilde{f}$ is uniformly continuous. It remains to show that $\widetilde{f}$ is unique. Suppose $g : X \to Y$ is also a continuous extension of f. Then $g(x) = f(x) = \widetilde{f}(x)$ for all $x \in A$. Since g and $\widetilde{f}$ agree on elements of a dense set, we must have $g = \widetilde{f}$. □

Proposition 2.6.15. *Let (X, d) and (Y, ρ) be metric spaces with $A \subseteq X$ dense, Y complete, and $f : A \to Y$ an isometry. Then the continuous extension $\widetilde{f} : X \to Y$ is an isometry.*

Proof. Let $x, x' \in X$. Let $(a_n)_n$ and $(b_n)_n$ be sequences in A with $(a_n)_n \to x$ and $(b_n)_n \to x'$. We have:

$$\begin{aligned}
\rho(\widetilde{f}(x), \widetilde{f}(x')) &= \operatorname*{limit}_{n\to\infty} \rho(f(a_n), f(b_n)) \\
&= \operatorname*{limit}_{n\to\infty} d(a_n, b_n) \\
&= d(x, x').
\end{aligned}$$
□

Corollary 2.6.16. *Let V be a normed space, W a Banach space, and $U \subseteq V$ a dense linear subspace. Let $T_0 \in B(U, W)$. There exists a unique $T \in B(V, W)$ with $\|T\|_{\text{op}} = \|T_0\|_{\text{op}}$. Moreover, if T_0 is isometric, then so if T.*

Proof. We only need to show T is linear and $\|T\|_{\text{op}} = \|T_0\|_{\text{op}}$. Let $v, v' \in V$ and $\alpha \in F$. Let $(x_n)_n$ and $(y_n)_n$ be sequences in U with $(x_n)_n \to v$ and $(y_n)_n \to v'$. Observe that:

$$\begin{aligned}
T(v + \alpha v') &= \operatorname*{limit}_{n\to\infty} T_0(x_n + \alpha y_n) \\
&= \operatorname*{limit}_{n\to\infty} T_0(x_n) + \alpha \operatorname*{limit}_{n\to\infty} T_0(y_n) \\
&= T(v) + \alpha T(v').
\end{aligned}$$

Note that the composition $V \xrightarrow{T} W \xrightarrow{\|\cdot\|_W} F$ will be continuous and bounded, so by

Proposition **??** we have:

$$\begin{aligned}\|T\|_{\text{op}} &= \sup_{v\in B_V} \|T(v)\| \\ &= \sup_{v\in B_U} \|T(v)\| \\ &= \sup_{v\in B_U} \|T_0(v)\| \\ &= \|T_0\|_{\text{op}}\,.\end{aligned}$$

Thus $T \in B(V, W)$. ☐

Example 2.6.2. *** (Something about Hilbert spaces...)

————//————

Definition 2.6.3. Let (X, d) be a metric space. A *completion* of (X, d) is a pair $\big((Z, \rho), \iota\big)$ where:

(1) (Z, ρ) is a complete metric space;

(2) $\iota : X \hookrightarrow Z$ is an isometry;

(3) $\overline{\iota(X)}^{\rho} = Z$.

Example 2.6.3. $\big(([0, 1], \|\cdot\|), \iota(t) = t\big)$ is a completion of $(0, 1)$.

Lemma 2.6.17. *Let $f : (X, d) \to (Y, \rho)$ be an isometry between metric spaces. If X is complete, then $f(X) \subseteq Y$ is closed. In particular, if $f(X)$ is dense, then f is onto.*

Proof. If $(f(x_n))_n \to y$ in Y, then $(f(x_n))_n$ is ρ-Cauchy. Since f is an isometry, $d(x_n, x_m) = \rho(f(x_n), f(x_m))$, so $(x_n)_n$ is d-Cauchy. Let $x \in X$ such that $(x_n)_n \to x$. Since f is continuous, we have $(f(x_n))_n \to f(x)$. It must be that $y = f(x) \in f(X)$.

Since we've just shown that $f(X)$ is closed, if it were also dense then $f(X) = \overline{f(X)}^{\rho} = Y$, whence f is surjective. ☐

Theorem 2.6.18. *Let (X, d) be a metric space. If $\big((Z, \rho), \iota\big)$ and $\big((Z', \rho'), \jmath\big)$ are completions of X, there exists a unique isometric isomorphism $\varphi : Z \to Z'$ with $\varphi \circ \iota = \jmath$; that is, the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\jmath"'] & Z \arrow[d, dotted, "\varphi"] \\ & Z' \end{tikzcd}$$

Proof. Let $z \in Z$. Since $\iota(X)$ is dense in Z, there exists a sequence $(\iota(x_n))_n$ in $\iota(X)$ such that $(\iota(x_n))_n \to z$. It is clear that $(\iota(x_n))_n$ is ρ-Cauchy, and furthermore we have:

$$\begin{aligned}\rho'(\jmath(x_n), \jmath(x_m)) &= d(x_n, x_m) \\ &= \rho(\iota(x_n), \iota(x_m)).\end{aligned}$$

So if $(\iota(x_n))_n$ is ρ-Cauchy, then $(\jmath(x_n))_n$ is ρ'-Cauchy. Since $\jmath(X)$ is complete, define $\varphi(z) := \text{limit}_{n\to\infty}\, \jmath(x_n)$.

We will show that $\varphi : Z \to Z'$ is well-defined, that is, it does not depend on our particular choice of sequence. Let (y_n) be another sequence in X with $(y_n)_n \to z$. We can see that:

$$\begin{aligned} d(x_n, y_n) &= \rho(\iota(x_n), \iota(y_n)) \\ &\leqslant \rho(\iota(x_n), z) + \rho(z, \iota(y_n)) \\ &\to 0, \end{aligned}$$

which gives:

$$\begin{aligned} \rho'(\jmath(y_n), \varphi(z)) &\leqslant \rho'(\jmath(y_n), \jmath(x_n)) + \rho'(\jmath(x_n), \varphi(z)) \\ &= d(y_n, x_n) + \rho'(\jmath(x_n), \varphi(z)) \\ &\to 0. \end{aligned}$$

Thus $(\jmath(y_n))_n \to \varphi(z)$, so φ is well-defined.

We will now show that φ is an isometric isomorphism. Given $z_1, x_2 \in Z$, let $(\iota(x_n))_n \to z_1$ and $(\iota(y_n))_n \to z_2$, corresponding to $(\jmath(x_n))_n \to \varphi(z_1)$ and $(\jmath(y_n))_n \to \varphi(z_2)$. Observe that:

$$\begin{aligned} \rho'(\varphi(z_1), \varphi(z_2)) &= \operatorname*{limit}_{n\to\infty} \rho'(\jmath(x_n), \jmath(y_n)) \\ &= \operatorname*{limit}_{n\to\infty} d(x_n, y_n) \\ &= \operatorname*{limit}_{n\to\infty} \rho(\iota(x_n), \iota(y_n)) \\ &= \rho(z_1, z_2). \end{aligned}$$

Since φ is an isometry, it is injective. Clearly $\varphi \circ \iota = \jmath$ holds, so we have $\jmath(X) = \varphi(\iota(X)) \subseteq \varphi(Z)$. Then $Z' = \overline{\jmath(X)}^{\rho} \subseteq \overline{\varphi(Z)}$. Since Z is complete, by the previous lemma $\varphi(Z)$ is closed, whence $Z' \subseteq \overline{\varphi(Z)} = \varphi(Z)$. Thus φ is onto.

It remains to show that φ is unique. Let $\psi : Z \to Z'$ be another isometric isomorphism satisfying $\psi \circ \iota = \jmath$. Given $z \in Z$, we can find a sequence $(\iota(x_n))_n$ in $\iota(X)$ such that $(\iota(x_n))_n \to z$. Observe that:

$$\begin{aligned} \varphi(z) &= \operatorname*{limit}_{n\to\infty} \varphi(\iota(x_n)) \\ &= \operatorname*{limit}_{n\to\infty} \jmath(x_n) \\ &= \operatorname*{limit}_{n\to\infty} \psi(\iota(x_n)) \\ &= \psi(z). \end{aligned}$$

Since $z \in Z$ was arbitrary, this proves $\varphi = \psi$. $\square$

Lemma 2.6.19. *If (X, d) is a metric space and $i : (X, d) \to (Y, \rho)$ is an isometry into a complete metric space (Y, ρ), then $\left(\left(\overline{i(X)}^{\rho}, \rho\right), i\right)$ is a completion of X.*

Proof. The space $\left(\overline{i(X)}^{\rho}, \rho\right)$ is complete by Proposition 2.6.10. Clearly $i : X \to \overline{i(X)}^{\rho}$ is an isometry because $\overline{i(X)}^{\rho} \subseteq Y$. $\square$

Theorem 2.6.20. *Every metric space admits a unique completion up to isometric isomorphism.*

Proof. Uniqueness was shown in Theorem 2.6.18. Given (X, d), consider the Banach space $(C_b(X), \|\cdot\|_u)$. By the previous lemma we only need to construct an isometry $X \overset{i}{\hookrightarrow} C_b(X)$.

Fix any $x_0 \in X$. Define $f_x : X \to F$ by $f_x(t) = d(t,x) - d(t,x_0)$. Clearly f_x is continuous, and it is bounded because $|f_x(t)| = |d(t,x) - d(t,x_0)| \leqslant d(x,x_0)$. So $f_x \in C_b(X)$. Now define $i : X \to C_b(X)$ by $x \mapsto f_x$. Observe that:

$$\begin{aligned} \|f_x - f_y\|_u &= \sup_{t \in X} |f_x(t) - f_y(t)| \\ &= \sup_{t \in X} |d(t,x) - d(t,x_0) - d(t,y) + d(t,x_0)| \\ &= \sup_{t \in X} |d(t,x) - d(t,y)| \\ &= d(x,y). \end{aligned}$$

Thus i is an isometry, making $\Big(\big(\overline{i(X)}^{\|\cdot\|_u}, \|\cdot\|_u\big), i\Big)$ a completion of X. ☐

Theorem 2.6.21. *Let (X, d) be a metric space with completion $\big((Z,\rho), i\big)$. If $f : X \to Y$ is a uniformly continuous function into a complete metric space Y, then there exists a unique uniformly continuous function $\widetilde{f} : Z \to Y$ such that $\widetilde{f} \circ i = f$; i.e., the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[d, dotted, "\widetilde{f}"] \\ & Y \end{tikzcd}$$

Proof. Define $g : i(X) \to Z$ by $g = f \circ i^{-1}$. Since g is uniformly continuous and $i(X) \subseteq Z$ is dense, Theorem 2.6.14 says there exists a unique uniformly continuous map $\widetilde{f} : Z \to Y$. Clearly $\widetilde{f} \circ i = f$. ☐

Theorem 2.6.22. **** Let $(V, \|\cdot\|)$ be a normed space. The completion of V is a Banach space.*

Proof. Let $\big((W,\rho), i\big)$ be the completion of V. We must first verify that W is a vector space. In doing so, we have to define what the vector space operations of W are. Let $w \in W$ and $\alpha \in F$. We know there exists a sequence $(i(v_n))_n$ in $i(V)$ converging to w. Since this sequence is convergent, it is ρ-Cauchy. From the following:

$$\begin{aligned} \rho(i(\alpha v_n), i(\alpha v_m)) &= \|\alpha v_n - \alpha v_m\| \\ &= |\alpha|\,\|v_n - v_m\| \\ &= |\alpha|\rho(i(v_n), i(v_m)), \end{aligned}$$

we can see $(i(\alpha v))_n$ is also ρ-Cauchy, whence it is convergent. Define $s : F \times W \to W$ by $s(\alpha, w) = \text{limit}_{n\to\infty}\, i(\alpha v_n) := \alpha w$. We will first show that this action is well-defined. Let $(i(u_n))_n$ be another sequence in $i(V)$ converging to w. As above, we have:

$$\begin{aligned} \rho(i(\alpha v_n), i(\alpha u_n)) &= |\alpha|\rho(i(v_n), i(u_n)) \\ &\leqslant \alpha\big(\rho(i(v_n), w) + \rho(w, i(u_n))\big) \\ &\to 0. \end{aligned}$$

This gives:

$$\begin{aligned}\rho(i(\alpha u_n), \alpha w) &\leqslant \rho(i(\alpha u_n), i(\alpha v_n)) + \rho(i(\alpha v_n), \alpha w) \\ &\to 0.\end{aligned}$$

Thus $(i(\alpha u_n))_n \to \alpha w$, meaning scalar multiplication is well-defined. Now let $w_1, w_2 \in W$. Let $(i(v_n))_n$ and $(i(u_n))_n$ be sequences in $i(V)$ converging respectively to w_1 and w_2. Since these sequences are convergent, they are ρ-Cauchy. From the fact that:

$$\begin{aligned}\rho(i(v_n + u_n), i(v_m + u_m)) &= \|v_n + u_n - v_m - u_m\| \\ &\leqslant \|v_n - v_m\| + \|u_n - u_m\| \\ &= \rho(i(v_n), i(v_m)) + \rho(i(u_n), i(u_m)),\end{aligned}$$

we can see $(i(v_n + u_n))_n$ is also ρ-Cauchy, hence it is convergent. Define $a : W \times W \to W$ by $a(w_1, w_2) = \operatorname{limit}_{n\to\infty} i(v_n + u_n) := w_1 + w_2$. We will show this binary operation is well-defined. Let $(i(x_n))_n$ and $(i(y_n))_n$ be sequences in $i(V)$ also converging to w_1 and w_2 respectively. Note that:

$$\begin{aligned}\rho(i(v_n + u_n), i(x_n + y_n)) &= \|v_n + u_n - x_n - y_n\| \\ &\leqslant \|v_n - x_n\| + \|u_n - y_n\| \\ &= \rho(i(v_n), i(x_n)) + \rho(i(u_n), i(y_n)) \\ &\leqslant \rho(i(v_n), w_1) + \rho(w_1, (x_n)) + \rho(i(u_n), w_2) + \rho(w_2, i(y_n)) \\ &\to 0.\end{aligned}$$

This gives:

$$\begin{aligned}\rho(i(x_n + y_n), w_1 + w_2) &\leqslant \rho(i(x_n + y_n), i(v_n + u_n)) + \rho(i(v_n + u_n), w_1 + w_2) \\ &\to 0.\end{aligned}$$

Thus $(i(x_n + y_n))_n \to w_1 + w_2$, meaning vector addition is well-defined.

Before showing W paired with the above operations is a vector space, we need to verify that the isometry $i : V \to W$ is linear. If $v, v' \in V$ and $\alpha \in F$, by taking $v_n = v$ and $u_n = v'$ for all $n \geqslant 1$, we can see that:

$$\begin{aligned}i(v + \alpha v') &= \operatorname*{limit}_{n\to\infty} i(v + \alpha v') \\ &= i(v) + \alpha i(v').\end{aligned}$$

With this fact, we can show that W is a vector space —this is left as an exercise. Moreover, we have that $i(V) \subseteq W$ is a ρ-dense linear subspace[2].

It remains to show that W is a normed space. Exactly as before, if $w \in W$, we can find a sequence $(i(v_n))_n$ in $i(V)$ converging to w. Since this sequence is Cauchy, observe that:

$$\begin{aligned}\Big| \|v_n\| - \|v_m\| \Big| &\leqslant \|v_n - v_m\| \\ &= \rho(i(v_n), i(v_m)).\end{aligned}$$

So $\big(\|v_n\|\big)_n$ is Cauchy, hence it is convergent. Define $\|w\|_W := \operatorname{limit}_{n\to\infty} \|v_n\|$. Showing this definition is well-defined, and that is satisfies the properties of a norm are left as an exercise. $\square$

[2]If $T : V \to W$ is injective, then $V \cong T(V)$

——————//——————

There is a softer proof of Theorem 2.6.22, but it requires heavier machinery. If V is a vector space over F, recall that

$$V' = \{\varphi \mid \varphi : V \to F \text{ linear}\}$$

is the linear space of all linear functionals on V. By Zorn's Lemma, $V' \neq \emptyset$. If V is a normed space, then

$$V^* = \{\varphi \in V' \mid \varphi \text{ continuous}\}$$

is called the *continuous dual space*. This is in fact a Banach space with norm $\|\varphi\|_{\text{op}} = \sup_{v \in B_V} |\varphi(v)|$. However, is $V^* \neq \emptyset$?

Theorem 2.6.23 (Hahn-Banach)**.** *Let V be a normed space. For any nonzero $v_0 \in V$, there is a $\varphi_{v_0} \in V^*$ with $\varphi_{v_0}(v_0) = \|v_0\|$. Such a φ_{v_0} is called a norming functional.*

Corollary 2.6.24. *Let V be a normed space and $v \in V$. Then $\|v\| = \sup_{\varphi \in B_{V^*}} |\varphi(v)|$.*

Proof. If $\varphi \in B_{V^*}$, then $|\varphi(v)| \leqslant \|\varphi\|_{\text{op}} \|v\| \leqslant \|v\|$. Choose $\varphi_v \in V^*$. Since $\varphi_v(v) = \|v\|$, we have $\sup_{v \in B_{V^*}} |\varphi(v)| \geqslant |\varphi_v(V)| = \|v\|$. □

Corollary 2.6.25. *There is a linear isometry $V \hookrightarrow (V^*)^*$.*

Proof. Let $v \in V$. Define $\widehat{v} : V^* \to F$ by $\varphi \mapsto \varphi(v)$. We can easily verify that $\widehat{v} \in (V^*)'$. By the above corollary:

$$\begin{aligned} \|\widehat{v}\| &= \sup_{\varphi \in B_{V^*}} |\widehat{v}(\varphi)| \\ &= \sup_{\varphi \in B_{V^*}} |\varphi(v)| \\ &= \|v\|. \end{aligned}$$

Whence $\widehat{v} \in (V^*)^*$. Define $i_V : V \to (V^*)^*$ by $v \mapsto \widehat{v}$. Note that this is an isometry since $\|i_V(v)\| = \|\widehat{v}\| = \|v\|$. Given $v_1, v_2 \in V$, $c \in F$, and $\varphi \in V^*$ we can see:

$$\begin{aligned} i_V(v_1 + cv_2)(\varphi) &= \widehat{v_1 + cv_2}(\varphi) \\ &= \varphi(v_1 + cv_2) \\ &= \varphi(v_1) + c\varphi(v_2) \\ &= \widehat{v_1}(\varphi) + c\widehat{v_2}(\varphi) \\ &= i_V(v_1)(\varphi) + ci_V(v_2)(\varphi). \end{aligned}$$

Thus i_V is linear. □

Using the tools above, we can now demonstrate in a cleaner way that the completion of any normed space forms a Banach space.

Alternative proof of Theorem 2.6.22. If V is a normed space, then $(V^*)^* = B(V^*, F)$ is complete by Proposition 2.6.12. Then $\overline{i_V(V)}^{\|\cdot\|_{\text{op}}} \subseteq (V^*)^*$ is complete by Proposition 2.6.10. Clearly $i_V : V \to \overline{i_V(V)}^{\|\cdot\|_{\text{op}}}$ is an isometry because $\overline{i_V(V)}^{\|\cdot\|_{\text{op}}} \subseteq (V^*)^*$. Thus the completion of V is $\left(\left(\overline{i_V(V)}^{\|\cdot\|_{\text{op}}}, \|\cdot\|_{\text{op}}\right), i_V\right)$ □

Lemma 2.6.26. *Let $T : V \to W$ be continuous and linear. There is an induced continous and linear map $T^* : W^* \to V^*$ with $T^*(\psi) = \psi \circ T$; i.e., the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "T^*(\psi)"'] & W \arrow[d, "\psi"] \\ & F \end{tikzcd}$$

Proof. Let $\psi_1\, \psi_2 \in W^*$ and $\alpha \in F$. We have:

$$\begin{aligned} T^*(\psi_1 + \alpha\psi_2) &= (\psi_1 + \alpha\psi_2) \circ T \\ &= \psi_1 \circ T + \alpha(\psi_2 \circ T) \\ &= T^*(\psi_1) + \alpha T^*(\psi_2). \end{aligned}$$

Thus T^* is linear. Moreover:

$$\begin{aligned} \|T^*\|_{\text{op}} &= \sup_{\|\psi\|_{\text{op}} \leqslant 1} \|T^*(\psi)\|_{\text{op}} \\ &= \sup_{\|\psi\|_{\text{op}} \leqslant 1} \|\psi \circ T\|_{\text{op}} \\ &\leqslant \sup_{\|\psi\|_{\text{op}} \leqslant 1} \left(\|\psi\|_{\text{op}} \|T\|_{\text{op}}\right) \\ &= \|T\|_{\text{op}} \cdot \sup_{\|\psi\|_{\text{op}} \leqslant 1} \|\psi\|_{\text{op}} \\ &\leqslant \|T\|_{\text{op}}. \end{aligned}$$

Thus T^* is continuous. □

Theorem 2.6.27. *Let V and W be normed spaces with completions $\widetilde{V} = \overline{i_V(V)}$ and $\widetilde{W} = \overline{i_W(W)}$. If $T : V \to W$ is a continuous and linear map, there exists a unique continuous and linear map $\widetilde{T} : \widetilde{V} \to \widetilde{W}$ such that $\widetilde{T} \circ i_V = i_W \circ T$; i.e., the following diagram commutes:*

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\widetilde{T}"] & \widetilde{W} \\ V \arrow[u, "i_V"] \arrow[r, "T"'] & W \arrow[u, "i_W"'] \end{tikzcd}$$

Proof. Since T is continuous, we can induce the map $T^* : W^* \to V^*$ where $T^*(\psi) = \psi \circ T$. Since T^* is continuous, we can induce the map $T^{**} : V^{**} \to W^{**}$ where $T^{**}(\xi) = \xi \circ T^*$. If $v \in V$ and $\psi \in V^*$, note that:

$$\begin{aligned} (\widehat{v} \circ T^*)(\psi) &= (T^* \circ \psi)(v) \\ &= (\psi \circ T)(v) \\ &= \widehat{T(v)}(\psi). \end{aligned}$$

This allows us to show:

$$\begin{aligned} T^{**} \circ i_V(v) &= \widehat{v} \circ T^* \\ &= \widehat{T(v)} \\ &= i_W(T(v)). \end{aligned}$$

Whence $T^{**}(i_V(V)) \subseteq i_W(W)$. Since T^{**} is continuous:

$$\begin{aligned} T^{**}\left(\widetilde{V}\right) &= T^{**}\left(\overline{i_V(V)}\right) \\ &\subseteq \overline{T^{**}(i_V(V))} \\ &\subseteq \overline{i_W(W)} \\ &= \widetilde{W}. \end{aligned}$$

Thus $T^{**}|_{\widetilde{V}} : \widetilde{V} \to \widetilde{W}$ is the desired linear extension. We obtain the following diagram:

$$\begin{tikzcd}
V^{**} \arrow[r, "T^{**}"] & W^{**} \\
\widetilde{V} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "T^{**}|_{\widetilde{V}}"] & \widetilde{W} \arrow[u, phantom, sloped, "\subseteq"] \\
i_V(V) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "T^*|_{i_V(V)}"] & i_W(W) \arrow[u, phantom, sloped, "\subseteq"] \\
V \arrow[u, hook, "i_V"] \arrow[r, "T"'] & W \arrow[u, hook, "i_W"']
\end{tikzcd}$$

$\square$

Exercise 2.6.1. Show that $\left\|\widetilde{T}\right\|_{\text{op}} = \|T\|_{\text{op}}$.

Definition 2.6.4. If V is a normed space and $i_V : V \to (V^*)^*$ is surjective, then V is called *reflexive*.

Example 2.6.4. Hilbert spaces are reflexive by the Riesz representation theorem.

§ 2.7. Baire's Theorem

Recall that $A \subseteq X$ is *nowhere dense* if $\overline{A}^o = \emptyset$. For example, if $f : \mathbf{R} \to \mathbf{R}$ is any map, the set $\{(x, y) \in \mathbf{R}^2 \mid y = f(x)\}$ is nowhere dense.

Proposition 2.7.1. *For a metric space (X, d) and $A \subseteq X$, the following are equivalent:*

(1) A is nowhere dense;

(2) There exists a closed subset $F \subseteq X$ such that $F^o = \emptyset$ and $F \supseteq A$;

(3) There exists an open and dense subset $U \subseteq X$ such that $U \subseteq A^c$.

Proof. (1) $\Rightarrow$ (2) Take $F = \overline{A}$.

(2) $\Rightarrow$ (1) Let F be such a set. Then $\overline{A} \subseteq \overline{F}$. So $\overline{A}^o \subseteq \overline{F}^o = \emptyset$.

(2) $\Rightarrow$ (3) Let F be such a set. Let $U = F^c$. Then $\overline{U} = \overline{F^c} = (F^o)^c = \emptyset^c = X$. We also have $U = F^c \subseteq A^c$.

(3) $\Rightarrow$ (2) Let U be such a set. Take $F = U^c$. Then $F^o = (U^c)^o = (\overline{U})^c = X^c = \emptyset$. Also $F = U^c \supseteq (A^c)^c = A$. $\square$

Definition 2.7.1. A point $x \in X$ is *isolated* if there exists an $\epsilon > 0$ such that $U(x, \epsilon) = \{x\}$.

Proposition 2.7.2. *Let (X, d) be a metric space.*

(1) If $A \subseteq X$ is nowhere dense and $B \subseteq A$, then B is nowhere dense.

(2) If $A \subseteq X$ is nowhere dense, then $\overline{A}$ is nowhere dense.

(3) If $A_1, A_2, ..., A_n$ are nowhere dense, then $\bigcup_{k=1}^n A_k$ is nowhere dense.

(4) If X has no isolated points, then every finite subset is nowhere dense.

Proof. (1) If $B \subseteq A$ then $\overline{B} \subseteq \overline{A}$, so $(\overline{B})^o \subseteq (\overline{A})^o = \emptyset$.

(2) Note that $\overline{A} = \overline{\overline{A}}$. So $(\overline{\overline{A}})^o = (\overline{A})^o = \emptyset$.

(3) We show this for $n = 2$. Let A_1 and A_2 be nowhere dense. By Proposition 2.7.1, $A_2^c \supseteq U_1$, where U_1 is open and dense. Similarly, $A_2^c \supseteq U_2$, where U_2 is open and dense. Then:

$$\begin{aligned}(A_1 \cup A_2)^c &= A_1^c \cap A_2^c \\ &\supseteq U_1 \cap U_2.\end{aligned}$$

Clearly $U_1 \cap U_2$ is open by Proposition 2.2.1. Claim: $U_1 \cap U_2$ is dense. Let $x \in X$ and $\epsilon > 0$. We'd like to show, by Proposition 2.2.8, that $(U_1 \cap U_2) \cap U(x, \epsilon) \neq \emptyset$. Since U_1 is dense, we know $U_1 \cap U(x, \epsilon) \neq \emptyset$. Let $z \in U_1 \cap U(x, \epsilon)$. Since $U_1 \cap U(x, \epsilon)$ is open, by Definition 2.2.2 there exists a $\delta > 0$ such that $U(z, \delta) \subseteq U_1 \cap U(x, \epsilon)$. Now since U_2 is dense, $U(z, \delta) \cap U_2 \neq \emptyset$. Therefore:

$$\begin{aligned}\emptyset &= U(z, \delta) \cap U_2 \\ &\subseteq U(x, \epsilon) \cap (U_1 \cap U_2).\end{aligned}$$

Thus $U_1 \cap U_2$ is dense.

(4) Since X has no isolated points, $\{x\}$ is closed but not open. Then $(\overline{\{x\}})^o = \{x\}^o = \emptyset$. So $\{x\}$ is nowhere dense. By (3), any finite set $\{x_1, x_2, ..., x_n\} = \{x_1\} \cup \{x_2\} \cup ... \cup \{x_n\}$ is nowhere dense. $\square$

Note that $\mathbf{Q}$ is the *countable* union of open nowhere dense sets, but $\mathbf{Q}$ is not nowhere dense. Regardless, there is something to be said about the union of countable dense sets.