Math 310

Homework 6

Due: 10/9/2024

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Exercise 1. *Let $(x_k)_k$ be a sequence of strictly positive numbers such that*

$$(kx_k)_k \to L > 0.$$

Show that the infinite series $\sum_k x_k$ diverges.

Proof. By the limit comparison test,

$$\liminf\left(\frac{x_k}{\frac{1}{k}}\right) = \liminf(kx_k) = L > 0.$$

Hence $\sum_k x_k$ diverges. □

Exercise 2. *Let $(x_k)_k$ be a sequence of strictly positive numbers.*

(i) If $\limsup_{n\to\infty} \frac{x_{k+1}}{x_k} < 1$, then the infinite series $\sum_k x_k$ converges.

(ii) If $\liminf_{n\to\infty} \frac{x_{k+1}}{x_k} > 1$, then the infinite series $\sum_k x_k$ diverges.

Proof. (i) Let $\limsup \frac{x_{k+1}}{x_k} = u$ and $r = \frac{u+1}{2}$. Then $u < r < 1$. There exists K large so that $r > \sup_{k\geqslant K} \frac{x_{k+1}}{x_k}$. Hence $rx_k > x_{k+1}$ for all $k \geqslant K$. Inductively, $r^j x_K > x_{K+j}$. Hence:

$$\sum_{k\geqslant K} x_k = \sum_{j=1}^{\infty} x_{K+j} < x_K \sum_{j=1}^{\infty} r^j < \infty.$$

Thus $\sum_k x_k$ converges.

(ii) Let $\liminf \frac{x_{k+1}}{x_k} = l$ and $r = \frac{l+1}{2}$. Then $1 < r < l$. There exists K large so that $r < \inf_{k\geqslant K} \frac{x_{k+1}}{x_k}$. Hence $rx_k < x_{k+1}$ for all $k \geqslant K$. Inductively, $r^j x_K < x_{K+j}$. Hence:

$$\sum_{k\geqslant K} x_k = \sum_{j=1}^{\infty} x_{K+j} > x_K \sum_{j=1}^{\infty} r^j = \infty.$$

Thus $\sum_k x_k$ diverges. □

Exercise 3. *Consider the sequence of functions:*

$$f_n : \mathbf{R} \to \mathbf{R}; \quad f_n(x) = \arctan(nx).$$

(i) Show that $(f_n)_n \to \frac{\pi}{2}$sgn point-wise.

(ii) Show that the convergence in (i) is nonuniform on $(0, \infty)$.

(iii) Show that the convergence in (i) is uniform on $[a, \infty)$ for a fixed $a > 0$.

Proof. Note that:

$$(\arctan(n))_n \to \frac{\pi}{2},$$
$$(\arctan(-n))_n \to -\frac{\pi}{2}.$$

So given $x > 0$, there exists $N_x \in \mathbf{N}$ such that $n \geqslant N_x$ implies $|\arctan(nx) - \frac{\pi}{2}| < \epsilon$. Similarly, given $x < 0$, there exists $N_x \in \mathbf{N}$ such that $n \geqslant N_x$ implies $|\arctan(nx) + \frac{\pi}{2}| < \epsilon$. For $x = 0$, we have that $(\arctan(0))_n = (0)_n \to 0_{\mathcal{F}(\mathbf{R},\mathbf{R})}$. Hence $(\arctan(nx))_n \to \operatorname{sign} \frac{\pi}{2}$.

Let $(x_k)_k = \frac{1}{k}$ and $n_k = k$. Observe that:

$$\begin{aligned}|f_{n_k}(x_k) - f(x_k)| &= |\arctan\left(k \cdot \frac{1}{k}\right) - \operatorname{sign}\left(\frac{1}{k}\right) \cdot \frac{\pi}{2}| \\ &= \arctan(1).\end{aligned}$$

Picking $\epsilon_0 = \arctan(1)$ gives that $(\arctan(nx))_n$ does not converge uniformly on $(0, \infty)$.

Fix $a > 0$. Since $(d_u(f_n, f))_n = \left(\sup_{x\in[a,\infty)} \left|\arctan(nx) - \operatorname{sign}(x)\frac{\pi}{2}\right|\right)_n$, we have:

$$\begin{aligned}\left|\sup_{x\in[a,\infty)} \left|\arctan(nx) - \operatorname{sign}(x)\frac{\pi}{2}\right|\right| &\leqslant \sup_{x\in[a,\infty)} \left|\arctan(nx) - \operatorname{sign}(x)\frac{\pi}{2}\right| \\ &= \sup_{x\in[a,\infty)} \left|\arctan(nx) - \frac{\pi}{2}\right| \\ &= 0.\end{aligned}$$

Thus $(f_n)_n$ converges uniformly on $[a, \infty)$. □

Exercise 4. *Consider the sequence of functions:*

$$f_n : [0, \infty) \to \mathbf{R}; \quad f_n(x) = \frac{\sin(nx)}{1 + nx}.$$

(i) *Show that* $(f_n)_n \to 0$ *point-wise.*

(ii) *Show that the convergence in (i) is nonuniform on* $(0, \infty)$.

(iii) *Show that the convergence in (i) is uniform on* $[a, \infty)$ *for a fixed* $a > 0$.

Proof. For $x = 0$, we have that $(f_n(0))_n = 0_{\mathcal{F}([0,\infty),\mathbf{R})}$. For $x > 0$:

$$\left|\frac{\sin(nx)}{1 + nx}\right| \leqslant \frac{1}{1 + n|x|} \leqslant \frac{1}{n|x|}.$$

Since $\left(\frac{1}{n|x|}\right)_n \to 0$, we have that $\left(\frac{\sin(nx)}{1+nx}\right)_n \to 0$. Hence $(f_n)_n \to 0_{\mathcal{F}([0,\infty),\mathbf{R})}$ pointwise.

Consider $x_k = \frac{\pi}{2k}$ and $n_k = k$. We have:

$$\begin{aligned}|f_{n_k}(x_k) - f(x_k)| &= \left|\frac{\sin(kx_k)}{1 + kx_k}\right| \\ &= \frac{\sin\left(k \cdot \frac{\pi}{2k}\right)}{1 + k \cdot \frac{\pi}{2k}} \\ &= \frac{\sin\left(\frac{\pi}{2}\right)}{1 + \frac{\pi}{2}} \\ &= \frac{1}{1 + \frac{\pi}{2}}.\end{aligned}$$

Picking $\epsilon_0 = \frac{1}{1+\frac{\pi}{2}}$ gives that $(f_n)_n$ does not converge uniformly on $(0, \infty)$.

Fix $a > 0$. Since $(d_u(f_n, f))_n = \left(\sup_{x\in[a,\infty)} \left|\frac{\sin(nx)}{1+nx}\right|\right)$, we have:

$$\begin{aligned}
\left|\sup_{x\in[a,\infty)} \left|\frac{\sin(nx)}{1+nx}\right|\right| &\leqslant \sup_{x\in[a,\infty)} \left|\frac{\sin(nx)}{1+nx}\right| \\
&\leqslant \sup_{x\in[a,\infty)} \left|\frac{1}{1+nx}\right| \\
&= \frac{1}{1+na} \\
&\leqslant \frac{1}{na}.
\end{aligned}$$

Since $\left(\frac{1}{na}\right)_n \to 0$, then $(d_u(f_n, f))_n \to 0$. Thus f_n converges uniformly on $[a, \infty)$. □

Exercise 5. *Show that the sequence of functions:*

$$f_n : [0,\infty) \to \mathbf{R}; \quad f_n(x) = x^2 e^{-nx}$$

converges uniformly to 0*.*

Proof. Note that $(f_n)_n \to 0_{\mathcal{F}([0,\infty),\mathbf{R})}$. We have that $(d_u(f_n, f))_n = \left(\sup_{x\in[0,\infty)} \left|x^2e^{-nx}\right|\right)_n$. Observe that:

$$\begin{aligned}
\left|\sup_{x\in[0,\infty)} \left|x^2e^{-nx}\right|\right| &\leqslant \sup_{x\in[0,\infty)} |x^2e^{-nx}| \\
&\leqslant \sup_{x\in[0,\infty)} \left|\frac{x^2}{1+x+\frac{n^2x^2}{2}}\right| \\
&\leqslant \sup_{x\in[0,\infty)} \left|\frac{x^2}{\frac{n^2}{2}x^2}\right| \\
&= \sup_{x\in[0,\infty)} \left|\frac{2}{n^2}\right| \\
&= \frac{2}{n^2}.
\end{aligned}$$

Since $\left(\frac{2}{n^2}\right)_n \to 0$, $(d_u(f_n, f))_n \to 0$. Thus $(f_n)_n$ converges uniformly on $[0, \infty)$. □

Exercise 6. *Let $f_n = \mathbf{1}_{[n,n+1]}$. Show that $(f_n)_n \to 0$ point-wise on $\mathbf{R}$. Is the convergence uniform?*

Proof. Let $x \in \mathbf{R}$. Given $\epsilon > 0$, find N large so that $N > x$. Then if $n \geqslant N$, $|f_n(x) - \mathbf{0}(x)| = |\mathbf{1}_{[n,n+1]}(x)| = 0$. Thus $(f_n)_n \to \mathbf{0}$. However, $(f_n)_n \nrightarrow \mathbf{0}$ uniformly, because $\sup_{x\in\mathbf{R}} |f_n(x) - \mathbf{0}(x)| = 1$ no matter how large n is. □

Exercise 7. *Let $(f_n)_n$ and $(g_n)_n$ be sequences in $\ell_\infty(\Omega)$ with $(f_n)_n \to f$ and $(g_n)_n \to g$ uniformly on Ω. Prove that $(f_ng_n)_n \to fg$ uniformly on Ω.*

Proof. Since $(f_n)_n, (g_n)_n \in \ell_\infty(\Omega)$, let:

$$\sup_{x\in\Omega} |f_n(x)| \leqslant U_1$$
$$\sup_{x\in\Omega} |g(x)| \leqslant U_2.$$

Since $(f_n)_n \to f$ uniformly,

$$(\exists N_1 \in \mathbf{N}) \text{ s.t. } n \geqslant N_1 \implies \sup_{x\in\Omega} |f_n(x) - f(x)| < \frac{\epsilon}{2U_2}.$$

Since $(g_n)_n \to g$ uniformly,

$$(\exists N_2 \in \mathbf{N}) \text{ s.t. } n \geqslant N_2 \implies \sup_{x\in\Omega} |g_n(x) - g(x)| < \frac{\epsilon}{2U_1}.$$

If $n \geqslant \max\{N_1, N_2\}$, we have:

$$\begin{aligned}
\sup_{x\in\Omega} |f_n(x)g_n(x) - f(x)g(x)| &= \sup_{x\in\Omega} |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
&\leqslant \sup_{x\in\Omega} |f_n(x)(g_n(x) - g(x))| + \sup_{x\in\Omega} |g(x)(f_n(x) - f(x))| \\
&\leqslant \sup_{x\in\Omega} |f_n(x)| \sup_{x\in\Omega} |g_n(x) - g(x)| + \sup_{x\in\Omega} |g(x)| \sup_{x\in\Omega} |f_n(x) - f(x)| \\
&< U_1 \cdot \frac{\epsilon}{2U_1} + U_2 \cdot \frac{\epsilon}{2U_2} \\
&= \epsilon.
\end{aligned}$$

Thus $(f_n g_n)_n \to fg$ uniformly on Ω. □

Exercise 8. *Find a sequence of functions $(f_n)_n$ defined on $[0, \infty)$ such that $\|f_n\|_u \geqslant n$ but with $(f_n)_n \to 0$ point-wise.*