Math 310

Homework 4

Due: 10/9/2024

Name: Gianluca Crescenzo

Exercise 1. *Prove the following limits:*

(1) $\left(\frac{2n}{n+1}\right)_n \to 2$.

(2) $\left(\frac{\sqrt{n}}{n+1}\right)_n \to 0$.

(3) $\left(\frac{(-1)^n}{\sqrt{n+7}}\right)_n \to 0$.

(4) $(n^k b^n)_n \to 0$, *where* $0 \leqslant b < 1$ *and* $k \in \mathbf{N}$.

(5) $\left(\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\right)_n \to 3$.

Proof. (1) Let $\epsilon > 0$. There exists $N_\epsilon \in \mathbf{N}$ such that $N_\epsilon > \frac{2}{\epsilon} - 1$. If $n \geqslant N_\epsilon$, then $n > \frac{2}{\epsilon} - 1$ gives:

$$\begin{aligned} \frac{2}{\epsilon} < n+1 &\implies \frac{2}{n+1} < \epsilon \\ &\implies \frac{|2n-2n-2|}{n+1} < \epsilon \\ &\implies \left|\frac{2n-2(n+1)}{n+1}\right| < \epsilon \\ &\implies \left|\frac{2n}{n+1} - 2\right| < \epsilon. \end{aligned}$$

(2) Observe that:

$$\left|\frac{\sqrt{n}}{n+1}\right| \leqslant \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

Since $\left(\frac{1}{\sqrt{n}}\right)_n \to 0$, by "Lemma" $\left(\frac{\sqrt{n}}{n+1}\right)_n \to 0$.

(3) We have:

$$\left|\frac{(-1)^n}{\sqrt{n+7}}\right| = \frac{1}{\sqrt{n+7}} \leqslant \frac{1}{\sqrt{n}}.$$

$\square$

Exercise 2. *Show that the sequence* $(\cos(n))_n$ *does not converge.*

Exercise 3. *If* $(x_n)_n$ *is a real sequence converging to* x*, show that*

$$(|x_n|)_n \to |x|.$$

Is the converse true?

Proof. Since $(x_n)_n \to x$ is a convergent sequence, we have:

$$||x_n| - |x|| \leqslant |x_n - x| < \epsilon.$$

Thus $(|x_n|)_n \to |x|$. Note that the converse is not true: $(|(-1)^n|)_n \to 1$ converges whereas $((-1)^n)_n$ does not. □

Exercise 4. *If $(x_n)_n$ is a real sequence converging to $x > 0$, show that there is an $N \in \mathbf{N}$ and $c > 0$ such that*

$$x_n \geqslant c$$

for all $n \geqslant N$.

Proof. Pick $\epsilon = \frac{x}{2}$. Since $(x_n)_n$ is a convergent sequence, there exists $N_\epsilon \in \mathbf{N}$ such that $n \geqslant N_\epsilon$ implies $|x_n - x| < \frac{x}{2}$. Simplifying yields $\frac{x}{2} < x_n < \frac{3x}{2}$. Taking $c = \frac{x}{2}$ yields the desired result. □

Exercise 5. *If $(x_n)_n$ is a real sequence of positive terms converging to x, show that $x \geqslant 0$ and*

$$(\sqrt{x_n})_n \to \sqrt{x}.$$

Proof. Observe that:

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leqslant \left|\sqrt{x_n} - \sqrt{x}\right|\left|\sqrt{x_n} + \sqrt{x}\right| = |x_n - x| < \epsilon.$$

Hence $(\sqrt{x_n})_n \to \sqrt{x}$. If $x < 0$, then $\sqrt{x} \notin \mathbf{R}$, contradicting the definition of a real sequence. □

Exercise 6. *If $(x_n)_n$ and $(y_n)_n$ are sequences with $(x_n)_n \to 0$ and $(y_n)_n$ bounded, show that*

$$(x_n y_n)_n \to 0.$$

Proof. Since $(y_n)_n$ is bounded, $|y_n| \leqslant c$ for some $c > 0$. We have:

$$|x_n y_n| \leqslant c|x_n|.$$

Taking $\epsilon_n = |x_n|$ and using "Lemma" gives $(x_n y_n)_n \to 0$. □

Exercise 7. *If $(x_n)_n$ is a sequence of positive terms such that*

$$\left(\frac{x_{n+1}}{x_n}\right)_n \to L > 1,$$

show that $(x_n)_n$ is not bounded hence not convergent. If $L = 1$, can we make any conclusion?

Exercise 8. *Let a and b be positive numbers. Show that*

$$\left((a^n + b^n)^{\frac{1}{n}}\right)_n \to \max\{a, b\}.$$

Proof. Case 1: $\max\{a,b\} = a$. Then $b < a$. We have:

$$\begin{aligned} (a^n)^{\frac{1}{n}} &\leqslant (a^n + b^n)^{\frac{1}{n}} \leqslant (2a^n)^{\frac{1}{n}} \\ \implies a &\leqslant (a^n + b^n)^{\frac{1}{n}} \leqslant (2^{\frac{1}{n}})a. \end{aligned}$$

Hence $\left((a^n + b^n)^{\frac{1}{n}}\right)_n \to a$. Case 2: $\max\{a,b\} = b$. Then $a < b$. We have:

$$\begin{aligned} (b^n)^{\frac{1}{n}} &\leqslant (a^n + b^n)^{\frac{1}{n}} \leqslant (2b^n)^{\frac{1}{n}} \\ \implies b &\leqslant (a^n + b^n)^{\frac{1}{n}} \leqslant (2^{\frac{1}{n}})b. \end{aligned}$$

Hence $\left((a^n + b^n)^{\frac{1}{n}}\right)_n \to b$. □