Math 395

Homework 6

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Exercise 1. *Let V be an $\mathbf{R}$-vector space. Prove that $\mathbf{C} \otimes_{\mathbf{R}} V \cong V_{\mathbf{C}}$*

Proof. Define $t : V \to V \oplus V$ by $v \mapsto (v, 0_V)$. Clearly $t \in \operatorname{Hom}_{\mathbf{R}}(V, V \oplus V)$. This extends to a map $T \in \operatorname{Hom}_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{R}} V, V_{\mathbf{C}})$ satisfying $t = T \circ \iota$, where $\iota : V \to \mathbf{C} \otimes_{\mathbf{R}} V$ is defined by $v \mapsto 1 \otimes v$. Claim: defining T by $1 \otimes v_1 + i \otimes v_2 \mapsto (v_1, v_2)$ satisfies the universal property. Observe that:

$$T(\iota(v)) = T(1 \otimes v) = (v, 0_V) = t(v).$$

Furthermore, given $a \in \mathbf{C}$ we have:

$$\begin{aligned} T(a \otimes v) &= T(a(1 \otimes v)) \\ &= aT(1 \otimes v) \\ &= a(v, 0_V). \end{aligned}$$

Define $S : V_{\mathbf{C}} \to \mathbf{C} \otimes_{\mathbf{R}} V$ by $(v_1, v_2) \mapsto 1 \otimes v_1 + i \otimes v_2$. Given $v_1, v_2, v_1', v_2' \in V$ and $a + bi \in \mathbf{C}$, observe that:

$$\begin{aligned} S((v_1, v_2) + (a+bi)(v_1', v_2')) &= S((v_1, v_2) + (av_1' - bv_2', bv_1' + av_2')) \\ &= S((v_1 + av_1' - bv_2', v_2 + bv_1' + av_2')) \\ &= 1 \otimes (v_1 + av_1' - bv_2') + i \otimes (v_2 + bv_1' + av_2') \\ &= 1 \otimes v_1 + 1 \otimes (av_1' - bv_2') + i \otimes v_2 + i \otimes (bv_1' + av_2') \\ &= (1 \otimes v_1 + i \otimes v_2) + (a \otimes v_1' + bi \otimes v_1' + ai \otimes v_2' + (-b) \otimes v_2') \\ &= (1 \otimes v_1 + i \otimes v_2) + ((a + bi) \otimes v_1' + (ai - b) \otimes v_2') \\ &= (1 \otimes v_1 + i \otimes v_2) + ((a+bi)(1 \otimes v_1') + (a+bi)(i \otimes v_2')) \\ &= (1 \otimes v_1 + i \otimes v_2) + (a+bi)(1 \otimes v_1' + i \otimes v_2') \\ &= S((v_1, v_2)) + (a+bi)S((v_1', v_2')). \end{aligned}$$

Hence $S \in \operatorname{Hom}_{\mathbf{C}}(V_{\mathbf{C}}, \mathbf{C} \otimes_{\mathbf{R}} V)$. Now consider:

$$\begin{aligned} T(S((v_1, v_2))) &= T(1 \otimes v_1 + i \otimes v_2) \\ &= T(1 \otimes v_1) + T(i \otimes v_2) \\ &= (v_1, 0_V) + (0_V, v_2) \\ &= (v_1, v_2). \end{aligned}$$

Moreover, since $\{1 \otimes v_k, i \otimes v_j\}_{k,j}$ is a basis of $\mathbf{C} \otimes_{\mathbf{R}} V$, it suffices to show:

$$S(T(1 \otimes v_k)) = S((v_k, 0_V)) = 1 \otimes v_k$$
$$S(T(i \otimes v_j)) = S((0_V, v_j)) = i \otimes v_j.$$

Thus $T \circ S = \mathrm{id}_{V_{\mathbf{C}}}$ and $S \circ T = \mathrm{id}_{\mathbf{C} \otimes_{\mathbf{R}} V}$, establishing $\mathbf{C} \otimes_{\mathbf{R}} V \cong V_{\mathbf{C}}$. ∎

Exercise 2. *Let $t : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $t(v,w) = v \times w$. Let $\mathcal{E}_3$ be the standard basis of $\mathbf{R}^3$ and $\mathcal{B} = \{e_i \otimes e_j\}_{1 \leqslant i,j \leqslant 3}$. Let $T \in \operatorname{Hom}_F(\mathbf{R}^3 \otimes_{\mathbf{R}} \mathbf{R}^3, \mathbf{R}^3)$ be the linear map associated to t. Calculate $[T]_{\mathcal{B}}^{\mathcal{E}_3}$.*

Proof. We have:

$$\begin{aligned}
T(e_1 \otimes e_1) &= e_1 \times e_1 = 0\\
T(e_1 \otimes e_2) &= e_1 \times e_2 = e_3\\
T(e_1 \otimes e_3) &= e_1 \times e_3 = -e_2\\
T(e_2 \otimes e_1) &= e_2 \times e_1 = -e_3\\
T(e_2 \otimes e_2) &= e_2 \times e_2 = 0\\
T(e_2 \otimes e_3) &= e_2 \times e_3 = e_1\\
T(e_3 \otimes e_1) &= e_3 \times e_1 = e_2\\
T(e_3 \otimes e_2) &= e_3 \times e_2 = -e_1\\
T(e_3 \otimes e_3) &= e_3 \times e_3 = 0.
\end{aligned}$$

Hence:

$$[T]_{\mathcal{B}}^{\mathcal{E}_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

□

Exercise 3. *Let V and W be F-vector spaces. Prove that $V \otimes_F W \cong W \otimes_F V$.*

Proof. Define $t_1 : V \times W \to W \otimes_F V$ by $(v,w) \mapsto w \otimes v$. We have:

$$\begin{aligned}
t_1(v_1 + cv_2, w) &= w \otimes (v_1 + cv_2)\\
&= w \otimes v_1 + c(w \otimes v_2)\\
&= t_1(v_1, w) + ct(v_2, w).
\end{aligned}$$

$$\begin{aligned}
t_1(v, w_1 + cw_2) &= (w_1 + cw_2) \otimes v\\
&= w_1 \otimes v + c(w_2 \otimes v)\\
&= t_1(v, w_1) + ct_1(v, w_2).
\end{aligned}$$

Thus $t_1 \in \operatorname{Hom}_F(V, W; W \otimes_F V)$. This extends to a map $T \in \operatorname{Hom}_F(V \otimes_F W, W \otimes_F V)$ defined by $v \otimes w \mapsto w \otimes v$. Now define $t_2 : W \times V \mapsto V \otimes_F W$ by $(w,v) \mapsto v \otimes w$. We have:

$$\begin{aligned}
t_2(w_1 + cw_2, v) &= v \otimes (w_1 + cw_2)\\
&= v \otimes w_1 + c(v \otimes w_2)\\
&= t_2(w_1, v) + ct(w_1, v).
\end{aligned}$$

$$\begin{aligned}
t_2(w, v_1 + cv_2) &= (v_1 + cv_2) \otimes w\\
&= v_1 \otimes w + c(v_2 \otimes w)\\
&= t_2(w, v_1) + ct(w, v_2).
\end{aligned}$$

Thus $t_2 \in \mathrm{Hom}_F(W, V; V \otimes_F W)$. This extends to a map $S \in \mathrm{Hom}_F(W \otimes_F V, V \otimes_F W)$. Now consider:

$$\begin{aligned} S(T(v \otimes w)) &= S(w \otimes v) = v \otimes w \\ T(S(w \otimes v)) &= T(v \otimes w) = w \otimes v. \end{aligned}$$

Thus $S \circ T = \mathrm{id}_{V \otimes_F W}$ and $T \circ S = \mathrm{id}_{W \otimes_F V}$, establishing $V \otimes_F W \cong W \otimes_F V$. $\square$

Exercise 4.

(a) *Let $\varphi \in V^\vee$ and $\psi \in W^\vee$. Define a map*

$$B_{\varphi,\psi} : V \times W \to F \;\; \textit{by} \;\; (v, w) \mapsto \varphi(v)\psi(w).$$

Show that $B_{\varphi,\psi}$ is a bilinear form.

(b) *Prove that there is a natural isomorphism between $(V \otimes_F W)^\vee$ and $V^\vee \otimes_F W^\vee$ (note that a natural isomorphism means it does not depend on a choice of basis).*

Proof. Given $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $c \in F$, we have:

$$\begin{aligned} B_{\varphi,\psi}(v_1 + cv_2, w) &= \varphi(v_1 + cv_2)\psi(w) \\ &= \varphi(v_1)\psi(w) + c\varphi(v_2)\psi(w) \\ &= B_{\varphi,\psi}(v_1, w) + cB_{\varphi,\psi}(v_2, w). \end{aligned}$$

$$\begin{aligned} B_{\varphi,\psi}(v, w_1 + cw_2) &= \varphi(v)\psi(w_1 + cw_2) \\ &= \varphi(v)\psi(w_1) + c\varphi(v)\psi(w_2) \\ &= B_{\varphi,\psi}(v, w_1) + cB_{\varphi,\psi}(v, w_2). \end{aligned}$$

Thus $B_{\varphi,\psi} \in \mathrm{Hom}_F(V, W; F)$. This induces a unique $T_{\varphi,\psi} \in \mathrm{Hom}_F(V \otimes_F W, F)$ defined by $T_{\varphi,\psi}(v \otimes w) = \varphi(v)\psi(w)$. Define $s : V^\vee \times W^\vee \to (V \otimes_F W)^\vee$ by $t(\varphi, \psi) \mapsto T_{\varphi,\psi}$. Given $\varphi, \varphi_1, \varphi_2 \in V^\vee$, $\psi, \psi_1, \psi_2 \in W^\vee$, and $c \in F$, we have:

$$\begin{aligned} s(\varphi, \psi_1 + c\psi_2) &= T_{\varphi,\psi_1 + c\psi_2}(v \otimes w) \\ &= \varphi(v)(\psi_1 + c\psi_2)(w) \\ &= \varphi(v)\psi_1(w) + c\varphi(v)\psi_2(w) \\ &= T_{\varphi,\psi_1}(v \otimes w) + cT_{\varphi,\psi_2}(v \otimes w) \\ &= s(\varphi, \psi_1) + cs(\varphi, \psi_2). \end{aligned}$$

$$\begin{aligned} s(\varphi_1 + c\varphi_2, \psi) &= T_{\varphi_1 + c\varphi_2,\psi}(v \otimes w) \\ &= (\varphi_1 + c\varphi_2)(v)\psi(w) \\ &= \varphi_1(v)\psi(w) + c\varphi_2(v)\psi(w) \\ &= T_{\varphi_1,\psi}(v \otimes w) + cT_{\varphi_2,\psi}(v \otimes w) \\ &= s(\varphi_1, \psi) + cs(\varphi_2, \psi). \end{aligned}$$

Thus $s \in \mathrm{Hom}_F(V^\vee, W^\vee; (V \otimes_F W)^\vee)$. This induces a unique $S \in \mathrm{Hom}_F(V^\vee \otimes_F W^\vee, (V \otimes_F W)^\vee)$ defined by $\varphi \otimes \psi \mapsto T_{\varphi,\psi}$.

Let $\{v_i\}_{i\in I}$ be a basis for V and $\{w_j\}_{j\in I}$ a basis for W. We have that $\{v_i^\vee \otimes w_j^\vee\}_{i,j}$ is a basis for $V^\vee \otimes_F W^\vee$ and $\{(v_i \otimes w_j)^\vee\}_{i,j}$ is a basis for $(V \otimes_F W)^\vee$. Given $v \in V$ and $w \in W$, we have:

$$\begin{aligned} S(v_i^\vee \otimes w_j^\vee)(v \otimes w) &= T_{v_i^\vee, w_j^\vee}(v \otimes w) \\ &= v_i^\vee(v) w_j^\vee(w) \\ &= \begin{cases} 1, & v_i = v \text{ and } w_j = w \\ 0, & \text{otherwise} \end{cases} \\ &= (v_i \otimes w_j)^\vee (v \otimes w) \end{aligned}$$

we have that $\mathrm{im}(S) \supseteq \{(v_i \otimes w_j)\}_{i,j}$, which implies that S is surjective. Now suppose:

$$S\left(\sum_{\text{finite}} a_{i,j}(v_i^\vee \otimes w_j^\vee)\right) = 0.$$

Then:

$$\sum_{\text{finite}} a_{i,j}(v_i \otimes w_j)^\vee = 0,$$

which implies that $a_{i,j} = 0$ for all i, j since $\{(v_i \otimes w_j)^\vee\}_{i,j}$ is a basis. Hence $\sum_{\text{finite}} a_{i,j}(v_i^\vee \otimes w_j^\vee) = 0$. Thus $V^\vee \otimes_F W^\vee \cong (V \otimes_F W)^\vee$. $\square$