Math 395

Homework 1

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Name: Gianluca Crescenzo

Collaborators: Avinash Iyer, Noah Smith, Carly Venenciano, Ben Langer

For these problems F is assumed to be a field.

Exercise 3. Let V be an F-vector space.

(a) Prove that an arbitrary intersection of subspaces of V is again a subspace of V

(b) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Let $\{W_i\}_{i\in I}$ be an arbitrary collection of subspaces of V. Let $x, y \in \bigcap_{i\in I} W_i$ and $\alpha \in F$. Then $x, y \in W_i$ for all i. Hence $x + \alpha y \in W_i$ for all i which gives $x + \alpha y \in \bigcap_{i\in I} W_i$, establishing (a).

Let U, W be subspaces of V. Let $u, w \in U \cup W$ and $\alpha \in F$. Without loss of generality suppose $U \subseteq W$ with $u \in U$ and $w \in W$. Then it is also the case that $u \in W$, hence $u + \alpha w \in W \subseteq U \cup W$. Conversely, suppose $U \cup W$ is an F-subspace of V. Assume towards contradiction that $U \not\subseteq W$ and $W \not\subseteq U$. Let $u \in U$,$u \notin W$ and $w \in W$, $w \notin U$. Since $U \cup W$ is an F-vector space, $u + w \in U \cup W$. Without loss of generality, let $u + w \in U$, then $(-u) + u + w = w \in U$, a contradiction. Hence $U \subseteq W$ or $W \subseteq U$, establishing (b). □

Exercise 4. Let $T \in \mathrm{Hom}_F(F, F)$. Prove there exists $\alpha \in F$ so that $T(v) = \alpha v$ for every $v \in F$.

Proof. Let $\beta \in F$, $\beta \neq 0$. Then $\{\beta\}$ forms a basis for F as an F-vector space. Let $v \in F$, then $v = \beta v_0$ for some $v_0 \in F$. Observe that:

$$\begin{aligned} T(v) &= T(\beta v_0) \\ &= v_0 T(\beta) \\ &= v\beta^{-1} T(\beta) \\ &= vT(\beta^{-1}\beta) \\ &= vT(1). \end{aligned}$$

From this, "α" is uniquely determined by where $T(1)$ gets mapped to. □

Exercise 9. Let V be a finite dimensional vector space and $T \in \mathrm{Hom}_F(V,V)$ with $T^2 = T$.

(a) Prove that $\mathrm{im}(T) \cap \ker(T) = \{0\}$.

(b) Prove that $V = \mathrm{im}(T) \oplus \ker(T)$.

Proof. Let $v \in \mathrm{im}(T) \cap \ker(T)$. Then $v \in \mathrm{im}(T)$ and $v \in \ker(T)$. So there exists an element $w \in V$ such that $T(w) = v$, and $T(v) = 0$. Observe that $v = T(w) = T(T(w)) = T(v) = 0$.

Let $x + y \in \mathrm{im}(T) + \ker(T)$. Since $x \in \mathrm{im}(T) \subseteq V$ and $y \in \ker(T) \subseteq V$, $x + y \in V$. Now let $v \in V$. Then $T(v) = w$ for some $w \in \mathrm{im}(T)$. Let $k = v - T(w)$. Then $T(k) = T(v - T(w)) = T(v) - T(T(w)) = T(v) - T(v) = 0$, so $k \in \ker(T)$. Hence $v = T(w) + k \in \mathrm{im}(T) + \ker(T)$, which gives $V = \mathrm{im}(T) + \ker(T)$.

We must now show that $\mathrm{im}(T)$ and $\ker(T)$ are independent. If $T(w) + k = 0$, then $k = T(-w)$ implies $k \in \mathrm{im}(T)$. So $k \in \mathrm{im}(T) \cap \ker(T)$, and by (a) it must be that $k = 0$. Similarly, $T(T(w)+k) = 0$ is equivalent to $T(T(w)) + T(k) = 0$, which simplifies to $T(T(w)) = 0$; i.e., $T(w) \in \ker(T)$. So $T(w) \in \mathrm{im}(T) \cap \ker(T)$, which gives that $T(w) = 0$. Thus $\mathrm{im}(T)$ and $\ker(T)$ are independent, giving $V = \mathrm{im}(T) \oplus \ker(T)$. $\square$

Exercise 14. Let V be an F-vector space of dimension n. Let $T \in \mathrm{Hom}_F(V,V)$ so that $T^2 = 0$. Prove that the image of T is contained in the kernel of T and hence the dimension of the image of T is at most $n/2$.

Proof. Let $v \in \mathrm{im}(T)$. Then there exists a $w \in V$ such that $T(w) = v$. But observe that $T(v) = T(T(w)) = T^2(w) = 0$, hence $v \in \ker(T)$ which establishes the containment $\mathrm{im}(T) \subseteq \ker(T)$. By the rank-nullity theorem $n = \dim_F(\mathrm{im}(T)) + \dim_F(\ker(T))$. If $\mathrm{im}(T) = \ker(T)$ then it must be the case that $\dim_F(\mathrm{im}(T)) = \dim_F(\ker(T)) = n/2$, otherwise (when $\mathrm{im}(T) \subset \ker(T)$) $\dim_F(\mathrm{im}(T)) < n/2$. $\square$

Exercise 15. Let W be a subspace of a finite dimensional vector space V. Let $T \in \mathrm{Hom}_F(V,V)$ so that $T(W) \subset W$. Show that T induces a linear transformation $\overline{T} \in \mathrm{Hom}_F(V/W, V/W)$. Prove that T is nonsingular (i.e., injective) on V if and only if T is restricted to W and $\overline{T}$ on V/W are both nonsingular.

Proof. Define $\overline{T} : V/W \to V/W$ by $v + W \mapsto T(v) + W$. We must first show that $\overline{T}$ is well-defined. Suppose $v_1 + W = v_2 + W$, then $v_1 = v_2 + w$ for some $w \in W$. Observe that:

$$\begin{aligned}
\overline{T}(v_1 + W) &= T(v_1) + W \\
&= T(v_2 + w) + W \\
&= T(v_2) + T(w) + W && \text{Since } T \in \mathrm{Hom}_F(V,V) \\
&= T(v_2) + W && \text{Since } T(W) \subset W \\
&= \overline{T}(v_2 + W).
\end{aligned}$$

Let $v_1 + W, v_2 + W \in V/W$, and $\alpha \in F$. Then:

$$\begin{aligned}
\overline{T}((v_1 + W) + \alpha(v_2 + W)) &= \overline{T}((v_1 + W) + (\alpha v_2 + W)) \\
&= \overline{T}((v_1 + \alpha v_2) + W) \\
&= T(v_1 + \alpha v_2) + W \\
&= T(v_1) + \alpha T(v_2) + W \\
&= (T(v_1) + W) + \alpha(T(v_2) + W) \\
&= \overline{T}(v_1 + W) + \alpha \overline{T}(v_2 + W),
\end{aligned}$$

hence $\overline{T} \in \mathrm{Hom}_F(V/W, V/W)$. Now consider the maps $V \xrightarrow{T} V \xrightarrow{\pi} V/W$, where $\pi : V \to V/W$ is the projection map. It can be proved that $\pi \circ T = \overline{T} \circ \pi$ as follows: let $v \in V$ and observe that $\pi(T(v)) = T(v) + W$, which is equivalent to $\overline{T}(\pi(v)) = \overline{T}(v + W) = T(v) + W$. We've established that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ V/W & \xrightarrow[\overline{T}]{} & V/W \end{array}.$$

Now assume T is injective —by inspection one can see that $T\mid_W : W \to V$ is injective. Since $\overline{T}$ is defined by $v + W \mapsto T(v) + W$, it must be injective as well. Conversely, let $T\mid_W$ and $\overline{T}$ be injective. Let $v \in V$ with $v \in \ker(T)$. Then $T(v) = 0$ is equivalent to $\pi(T(v)) = 0 + W$. Using the fact that the above diagram commutes, we can write $\overline{T}(\pi(v)) = 0 + W$. Since $\overline{T}$ is injective, its kernel is trivial, hence it must be the case that $\pi(v) = 0 + W$. Thus $v \in W$, giving $T\mid_W (v) = 0$. Again, since $T\mid_W$ is injective, its kernel is trivial, establshing that $v = 0$. Thus T is injective. $\square$