Math 310

Homework 9

Due: 10/9/2024

Name: Gianluca Crescenzo

Exercise 1. Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function with $f(0) = f(1)$. Show that there is a $c \in \left[0, \frac{1}{2}\right]$ with $f(c) = f\left(c + \frac{1}{2}\right)$. Conclude that there are always antipodal points on the earth's equator with the same temperature. (Hint: consider $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$ on $\left[0, \frac{1}{2}\right]$).

Proof. Let $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$ on $\left[0, \frac{1}{2}\right]$. Note that:

$$\begin{aligned} g(0) &= f(0) - f\left(\tfrac{1}{2}\right) \\ g(\tfrac{1}{2}) &= f(\tfrac{1}{2}) - f(0) = -g(0). \end{aligned}$$

This gives that $g(0)g(\frac{1}{2}) < 0$. By location of roots, there exists $c \in (0, \frac{1}{2})$ with $g(c) = 0$. Equivalently, $f(c) - f(c + \frac{1}{2}) = 0$. Hence $f(c) = f(c + \frac{1}{2})$. □

Exercise 2. Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$ but not on $(0, \infty)$.

Proof. Let $u, v \in [1, \infty)$. We have:

$$\begin{aligned} |f(u) - f(v)| &= \left|\frac{1}{u^2} - \frac{1}{v^2}\right| \\ &= \left|\frac{(u+v)(u-v)}{(uv)^2}\right| \\ &\leqslant \frac{u+v}{(uv)^2}|u - v| \\ &\leqslant 2|u - v|. \end{aligned}$$

Thus f is Lipschitz, giving that f is uniformly continuous on $[1\infty)$.

Now consider $(u_n)_n, (v_n)_n \in (0, \infty)^{\mathbb{N}}$ defined by $u_n = \frac{1}{n}$ and $v_n = \frac{1}{n+1}$. Clearly $(u_n - v_n)_n \to 0$. Moreover, we have:

$$\begin{aligned} |f(u_n) - f(v_n)| &= |n^2 - (n+1)^2| \\ &= |n^2 - n^2 - 2n - 1| \\ &= |-2n - 1| \\ &\geqslant 3. \end{aligned}$$

Let $\epsilon_0 = 3$. By the work above, we've shown there exists sequences $(u_n)_n, (v_n)_n$ with $(u_n - v_n)_n \to 0$ and $|f(u_n) - f(v_n)| \geqslant \epsilon_0$. Thus f is not uniformly continuous on $(0, \infty)$. □

Exercise 3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and vanishes at infinity, that is $\text{limit}_{x \to \pm\infty} f = 0$. Prove that f is uniformly continuous.

Proof. Let $\epsilon > 0$ be given.

Since f vanishes at infinity, there exists $M > 0$ such that $|x| > M$ implies $|f(x)| < \frac{\epsilon}{2}$.

Moreover, since f is continuous on $[-M-1, M+1]$, it is uniformly continuous. In particular, for $u, v \in [-M-1, M+1]$, there exists $\delta > 0$ such that $|u - v| < \delta$ implies $|f(u) - f(v)| < \epsilon$.

Let $\delta_1 = \min\{\delta, 1\}$. Let $u, v \in \mathbb{R}$ and $|u - v| < \delta_1$. We proceed by cases.

Case 1: $u, v \in [-M, M]$. Then f is uniformly continuous by compactness.

Case 2: $u, v \notin [-M, M]$. Then $|f(u) - f(v)| \leqslant |f(u)| + |f(v)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Case 3: Without loss of generality, suppose $u \in [-M, M]$, $v \notin [-M, M]$. Since $|u - v| < \delta_1$, it must be that $v \in [-M-1, M+1]$, furthermore $u \in [-M-1, M+1]$ by inclusion. Then by compactness f is uniformly continuous.

Thus f is uniformly continuous on $\mathbb{R}$. □

Exercise 4. Show that $f(x) = x$ and $g(x) = \sin(x)$ are both uniformly continuous on $\mathbb{R}$, but the product:

$$h(x) = x \sin(x)$$

is not uniformly continuous on $\mathbb{R}$.

Proof. Let $u, v \in \mathbb{R}$. Observe that:

$$|f(u) - f(v)| = |u - v|.$$

Since f is Lipschitz, f is uniformly continuous. Without loss of generality, suppose $u < v$. Apply the Mean Value Theorem to g on $[u, v]$. Then there exists $c \in (u, v)$ with:

$$\frac{\sin(v) - \sin(u)}{v - u} = \sin'(c) = \cos(c).$$

Taking the absolute value of both sides gives:

$$\left|\frac{\sin(v) - \sin(u)}{v - u}\right| = |\cos(c)| \leqslant 1.$$

Whence

$$|\sin(v) - \sin(u)| \leqslant |v - u|.$$

Thus g is Lipschitz, implying that it is uniformly continuous.

Let $(u_n)_n, (v_n)_n \in \mathbb{R}^{\mathbb{N}}$ defined by $u_n = n\pi$ and $v_n = n\pi + \frac{1}{n}$. Clearly $(u_n - v_n)_n \to 0$. Moreover:

$$\begin{aligned}
|f(u_n) - f(v_n)| &= \left|n\pi\sin(n\pi) - \left(n\pi + \frac{1}{n}\sin\left(n\pi + \frac{1}{n}\right)\right)\right| \\
&= \left|n\pi\cos(n\pi)\sin\left(\frac{1}{n}\right) + \frac{1}{n}\cos(n\pi)\sin\left(\frac{1}{n}\right)\right| \\
&= \left|n\pi(-1)^n\sin\left(\frac{1}{n}\right) + \frac{1}{n}(-1)^n\sin\left(\frac{1}{n}\right)\right| \\
&= \left|n\pi(-1)^n\frac{1}{n} + \frac{1}{n^2}\right|. && \text{(For large } n\text{)}
\end{aligned}$$

So for n large, $|f(u_n) - f(v_n)| \geqslant \frac{\pi}{2}$. Take $\epsilon_0 = \frac{\pi}{2}$. Then by the work above, $h(x) = x\sin(x)$ is not uniformly convergent. □

Exercise 5. If $f : D \to \mathbb{R}$ is uniformly continuous and $|f(x)| \geqslant k > 0$ for some k, show that $\frac{1}{f}$ is uniformly continuous on D.

Proof. Let ϵ be given and $u, v \in D$. Since f is uniformly continuous, there exists δ such that $|u - v| < \delta$ implies $|f(u) - f(v)| < k^2\epsilon$. Moreover, $|u - v| < \delta$ implies:

$$\begin{aligned}
\left|\frac{1}{f(u)} - \frac{1}{f(v)}\right| &= \left|\frac{f(u) - f(v)}{f(u)f(v)}\right| \\
&= \frac{1}{f(u)f(v)}|f(u) - f(v)| \\
&< \frac{1}{k^2}k^2\epsilon \\
&< \epsilon.
\end{aligned}$$

Thus $\frac{1}{f}$ is uniformly continuous. □

Exercise 6. If $D \subseteq \mathbb{R}$ is a bounded set and $f : D \to \mathbb{R}$ is uniformly continuous, show that f is bounded (This gives another proof that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$).

Proof. Suppose towards contradiction f is unbounded. Then for all $n \geqslant 1$, there exists x_n such that $|f(x_n)| \geqslant n$. Since $(x_n)_n \in D^{\mathbb{N}}$, by Bolzano-Weierstrass there exists a convergent subsequence $(x_{n_k})_k \to c$. Since f is continuous, we have that $(f(x_{n_k}))_k \to f(c)$. However, this is a contradiction, as $|f(x_{n_k})| \geqslant n_k$. Thus f is bounded. □

Exercise 7. Prove that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ with:

$$f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}; \quad f(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{Q}.$$

Proof. Note that $f(\mathbb{R}) = f(\mathbb{R} \setminus \mathbb{Q}) \cup f(\mathbb{Q})$. Since $f(\mathbb{R} \setminus \mathbb{Q})$ and $f(\mathbb{Q})$ are countable, it must be that $f(\mathbb{R})$ is countable. Moreover, since $\mathbb{R}$ is an interval, it must be that $f(\mathbb{R})$ is an interval. Hence $f(\mathbb{R}) = \{a\}$ for some $a \in \mathbb{R}$.

But this gives that $f(\mathbb{R} \setminus \mathbb{Q}) = \{a\}$ and $f(\mathbb{Q}) = \{a\}$. Hence $a \in \mathbb{R} \setminus \mathbb{Q}$ and $a \in \mathbb{Q}$, which is a contradiction. □

Exercise 8. Let $n \in \mathbb{N}$ and consider the function:

$$f(x) = \begin{cases} x^n, & x > 0 \\ 0, & x \leqslant 0. \end{cases}$$

For which values of n is f differentiable at $x = 0$?

Proof. Observe that:

$$\operatorname*{limit}_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \operatorname*{limit}_{x \to 0^+} \frac{x^n}{x} = \operatorname*{limit}_{x \to 0^+} x^{n-1} = \begin{cases} 0, & n > 1 \\ 1, & n = 1. \end{cases}$$

$$\operatorname*{limit}_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \operatorname*{limit}_{x \to 0^-} \frac{0}{x} = 0.$$

Thus f is differentiable at $x = 0$ for $n > 1$. □

Exercise 9. Consider the function:

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Show that f is differentiable at $x = 0$ and find $f'(0)$.

Proof. Observe that:

$$\operatorname*{limit}_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \operatorname*{limit}_{x \to 0} \frac{x^2 \mathbb{1}_{\mathbb{Q}}}{x} = \operatorname*{limit}_{x \to 0} x \mathbb{1}_{\mathbb{Q}} = 0.$$

□

Exercise 10. Determine the values of x where $f(x) = x|x|$ is differentiable.

Proof. Note that:

$$f(x) = x|x| = \begin{cases} x^2, & x \geqslant 0 \\ -x^2, & x < 0. \end{cases}$$

Clearly f is differentiable at $c \neq 0$. So observe that:

$$\operatorname*{limit}_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \operatorname*{limit}_{x \to 0^+} \frac{x^2}{x} = \operatorname*{limit}_{x \to 0^+} x = 0.$$

$$\operatorname*{limit}_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \operatorname*{limit}_{x \to 0^-} \frac{-x^2}{x} = \operatorname*{limit}_{x \to 0^-} -x = 0.$$

Thus f is differentiable everywhere. □

Exercise 11. Let I be an interval and suppose $f : I \to \mathbb{R}$ is differentiable with $f'(x) < 0$ for all $x \in I$. Show that f is strictly decreasing on I.

Proof. Let $x_1, x_2 \in I$ with $x_1 < x_2$. Apply the Mean Value Theorem to f on $[x_1, x_2]$. Then there exists $c \in (x_1, x_2)$ with:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0.$$

Since $x_2 - x_1 > 0$, it must be that $f(x_2) - f(x_1) < 0$. Thus $f(x_2) < f(x_1)$, establishing f to be strictly decreasing on I. □

Exercise 12. Prove that the function $f(x) = x^3 + e^x$ has a unique real root.

Proof. Note that $f(1) = 1 + e > 0$ and $f(-1) = \frac{1-e}{e} < 0$. By the Intermediate Value Theorem, there exists $c \in (-1, 1)$ with $f(c) = 0$. Since $f'(x) > 0$ on $\mathbb{R}$, we have that f is strictly increasing. Hence c is unique. □

Exercise 13. Show that $\log(x) \leqslant x - 1$ for all $x > 0$.

Proof. We proceed by cases.

Case 1: $0 < x < 1$. Apply the Mean Value Theorem to $\log(x)$ on $[x, 1]$. Then there exists $c \in (x, 1)$ such that:

$$\frac{\log(1) - \log(x)}{1 - x} = \frac{1}{c} \geqslant 1.$$

Whence $-\log(x) \geqslant 1 - x$; i.e., $\log(x) \leqslant x - 1$.

Case 2: $x = 1$. Then clearly $\log(x) = x - 1$.

Case 3: $x > 1$. Apply the Mean Value Theorem to $\log(x)$ on $[1, x]$. Then there exists $c \in (1, x)$ such that:

$$\frac{\log(x) - \log(1)}{x - 1} = \frac{1}{c} \leqslant 1.$$

Whence $\log(x) \leqslant x - 1$ for all $x > 0$. □

Exercise 14. Suppose $f : [0,2] \to \mathbb{R}$ is continuous on $[0,2]$ and differentiable on $(0,2)$ and satisfies $f(0) = 0$, $f(1) = 1$, and $f(2) = 1$.

(i) Show that there is a $c_1 \in (0,1)$ with $f'(c_1) = 1$.

Proof. Apply the Mean Value Theorem to f on $[0,1]$. Then there exists $c_1 \in (0,1)$ so that $f'(c_1) = \frac{f(1)-f(0)}{1-0} = 1$. □

(ii) Show that there is a $c_2 \in (1,2)$ with $f'(c_2) = 0$.

Proof. Apply the Mean Value Theorem to f on $[1,2]$. Then there exists $c_2 \in (1,2)$ so that $f'(c_2) = \frac{f(2)-f(1)}{2-1} = 0$. □

(iii) Show that there is a $c_3 \in (0,2)$ with $f'(c_3) = \frac{1}{3}$.

Proof. Apply Darboux's Theorem to f on $[c_1, c_2]$. Note that $f'(c_2) = 0 < \frac{1}{3} < 1 = f'(c_2)$. So there exists $c_3 \in [c_1, c_2]$ so that $f'(c_3) = \frac{1}{3}$. □

Exercise 15. Suppose $f, g : \mathbb{R} \to (0, \infty)$ are everywhere differentiable with $f' = f$ and $g' = g$. Prove that $f = \alpha g$ for some constant $\alpha > 0$.

Proof. Observe that:

$$\left(\frac{f}{g}\right)' = \frac{fg' - f'g}{g^2} = \frac{fg - fg}{g^2} = 0.$$

Thus $\frac{f}{g} = \alpha$ for some $\alpha > 0$. Whence $f = \alpha g$. □

Exercise 16. Let $h = \mathbb{1}_{[0,\infty)}$. Prove that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ for which $f' = h$ on $\mathbb{R}$.

Proof. The converse of Darboux's Theorem says:

$$(\exists k \in (f'(a), f'(b)))(\forall c \in (a,b)) : f'(c) \neq k \implies f \text{ is not differentiable.}$$

Let $k = \frac{1}{2}$ Notice that for all $c \in \mathbb{R}$, we have that $f'(c) \neq \frac{1}{2}$. Whence f is not differentiable. □