

Math 310

Homework 3

Due: 9/27/2024

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Exercise 1. Find $\sup(A)$ and $\inf(A)$ where:

(1) $A_1 = \left\{ 1 - \frac{(-1)^n}{n} \mid n \in \mathbf{N} \right\}.$

(2) $A_2 = \left\{ \frac{1}{n} - \frac{1}{m} \mid m, n \in \mathbf{N} \right\}.$

(3) $A_3 = \left\{ \frac{m}{n} \mid m, n \in \mathbf{N}, m + n \leq 10 \right\}.$

Proof. Claim: $\inf(A_1) = \frac{1}{2}$. Note that $\frac{1}{2}$ is a lowerbound because $\frac{1}{2} \leq a$ for all $a \in A_1$. Let t be a lowerbound of A_1 . If $t \leq \frac{1}{2}$, then we are done. If $t > \frac{1}{2}$, then $t - \frac{1}{2} > 0$. By the Archimedean Property, there exists an element $n \in \mathbf{N}$ with $t - \frac{1}{2} > \frac{1}{n}$. This gives $t > \frac{1}{2} + \frac{1}{n}$, which is a contradiction because $\frac{1}{2} + \frac{1}{n} \in A_1$ for all positive natural numbers. Thus $\inf(A_1) = \frac{1}{2}$. Note that $2 \geq 1 + \left| -\frac{(-1)^n}{n} \right| = 1 + \frac{(-1)^n}{n}$ for all $n \in \mathbf{N}$. Hence 2 is an upper bound. Furthermore, since $2 \in A_1$, it must be the case that $\sup(A_1) = 2$.

(2)

(3) Note that $\frac{1}{9} \leq \frac{m}{n} \leq \frac{9}{1}$ for all $m, n \in \mathbf{N}, m + n \leq 10$. So $\frac{1}{9}$ is a lower bound of A_3 and $\frac{9}{1}$ is an upper bound of A_3 . Since $\frac{1}{9}, \frac{9}{1} \in A_3$, it must be the case that $\inf(A_3) = \frac{1}{9}$ and $\sup(A_3) = \frac{9}{1}$.

□

Exercise 2. Suppose $u = \sup(A)$ such that $u \notin A$. Show that there is a strictly increasing sequence

$$t_1 < t_2 < t_3 < \dots$$

with $t_n \in A$ and $t_n + \frac{1}{n} > u$ for all $n \geq 1$.

Proof. Note that for all $\epsilon > 0$, there exists an a_ϵ with $u - \epsilon < a_\epsilon$. Define:

$$t_1 > u - 1$$

$$t_2 > \max \left\{ t_1, u - \frac{1}{2} \right\}$$

$$t_3 > \max \left\{ t_2, u - \frac{1}{3} \right\}$$

\vdots

$$t_n > \max \left\{ t_{n-1}, u - \frac{1}{n} \right\}.$$

If $\max \left\{ t_{n-1}, u - \frac{1}{n} \right\} = t_{n-1}$, then clearly $t_n > t_{n-1}$. If $\max \left\{ t_{n-1}, u - \frac{1}{n} \right\} = u - \frac{1}{n}$, then $t_n > u - \frac{1}{n} > t_{n-1}$. This gives $t_n + \frac{1}{n} > u$ for all $n \geq 1$, and furthermore, we obtain a strictly increasing sequence:

$$t_1 < t_2 < t_3 < \dots$$

□

Exercise 3. If m is a lower bound for $A \subseteq \mathbf{R}$, show that the following are equivalent:

- (1) $m = \inf(A)$.
- (2) For all $t > m$, there exists $a_t \in A$ with $a_t < t$.
- (3) For all $\epsilon > 0$ there exists $a_\epsilon \in A$ with $m + \epsilon > a_\epsilon$.

Proof. Let $m = \inf(A)$. Assuming $t > m$, suppose towards contradiction there does not exist an $a \in A$ with $a < t$. Then it must be the case that $m < t \leq a$ for all $t > m$. This is a contradiction, because m is the greatest lower bound.

Now assume for all $t > m$, there exists $a_t \in A$ with $a_t < t$. Given $\epsilon > 0$, pick $t = m + \epsilon$. Then by (2) there exists an a_t with $m + \epsilon > a_t$.

Now assume for all $\epsilon > 0$ there exists $a_\epsilon \in A$ with $m + \epsilon > a_\epsilon$. Given that m is a lower bound for A , assume there exists another lower bound for A with $l > m$. Pick $\epsilon = l - m$, then there exists an $a \in A$ with $m + (l - m) > a$. Simplifying yields $l > a$, which contradicts l being a lower bound. Hence $\inf(A) = m$. \square

Exercise 4. Let $A, B \subseteq \mathbf{R}$ be bounded subsets.

- (1) Show that

$$\begin{aligned}\sup(A + B) &= \sup(A) + \sup(B), \\ \inf(A + B) &= \inf(A) + \inf(B).\end{aligned}$$

- (2) If $t > 0$, show that

$$\begin{aligned}\sup(tA) &= t \sup(A), \\ \inf(tA) &= t \inf(A).\end{aligned}$$

Proof. (1) Define $\sup(A) = u$ and $\sup(B) = v$. Then for all $\epsilon > 0$, there exists $a_\epsilon \in A$, $b_\epsilon \in B$ with $u - \epsilon < a_\epsilon$ and $v - \epsilon < b_\epsilon$. Pick $\epsilon = \frac{\epsilon}{2}$. Then adding both inequalities gives $(u + v) - \epsilon < a_\epsilon + b_\epsilon \in A + B$. Hence $\sup(A + B) = u + v = \sup(A) + \sup(B)$. Similarly, define $\inf(A) = m$ and $\inf(B) = n$. Then for all $\epsilon > 0$, there exists $a_\epsilon \in A$, $b_\epsilon \in B$ with $m + \epsilon > a_\epsilon$ and $n + \epsilon > b_\epsilon$. Pick $\epsilon = \frac{\epsilon}{2}$. Then adding both inequalities gives $(m + n) + \epsilon > a_\epsilon + b_\epsilon \in A + B$. Hence $\inf(A + B) = m + n = \inf(A) + \inf(B)$.

(2) Let $\sup(A) = u$. Then $a \leq u$ for all $a \in A$. We have that $u - \epsilon < a$ for some $a \in A$. Pick $\epsilon = \frac{\epsilon}{t}$. Then $tu - \epsilon < ta$ for some $ta \in tA$. Hence $\sup(tA) = tu = t \sup(A)$. Similarly, let $\inf(A) = m$. Then $m \leq a$ for all $a \in A$. We have that $m + \epsilon > a$ for some $a \in A$. Pick $\epsilon = \frac{\epsilon}{t}$. Then $tm + \epsilon > ta$ for some $ta \in tA$. Hence $\inf(tA) = tm = t \inf(A)$. \square

Exercise 5. Let $I = (0, 1)$ denote the open interval and consider the function

$$F : I \times I \rightarrow \mathbf{R} \text{ defined by } F(x, y) = 2x + y.$$

Compute

$$\sup_{y \in I} \left(\inf_{x \in I} F(x, y) \right),$$

and

$$\inf_{y \in I} \left(\sup_{x \in I} F(x, y) \right).$$

Are they equal?

Proof. Observe that:

$$\begin{aligned} \sup_{y \in I} \left(\inf_{x \in I} (2x + y) \right) &= \sup_{y \in I} \left(2 \inf_{x \in I} x + \inf_{x \in I} y \right) \\ &= \sup_{y \in I} y \\ &= 1, \\ \inf_{y \in I} \left(\sup_{x \in I} (2x + y) \right) &= \inf_{y \in I} \left(\sup_{x \in I} 2x + \sup_{x \in I} y \right) \\ &= \inf_{y \in I} (2 + y) \\ &= \inf_{y \in I} 2 + \inf_{y \in I} y \\ &= 2. \end{aligned}$$

□

Exercise 6. Let D be a nonempty set and consider the set of all bounded functions:

$$\ell_\infty(D) := \{f \mid f : D \rightarrow \mathbf{R} \text{ is bounded}\}$$

with point-wise addition and scalar multiplication. Show that

$$d_u(f, g) := \sup_{x \in D} |f(x) - g(x)|$$

defines a metric on $\ell_\infty(D)$. We call d_u the **uniform metric**.

Proof. Observe that:

$$\begin{aligned} d_u(f, g) &= \sup_{x \in D} (|f(x) - g(x)|) \\ &= \sup_{x \in D} (|g(x) - f(x)|) \\ &= d_u(g, f). \end{aligned}$$

Thus (ℓ_∞, d_u) is symmetric. We also have that:

$$\begin{aligned} d_u(f, h) &= \sup_{x \in D} (|f(x) - h(x)|) \\ &= \sup_{x \in D} (|f(x) - g(x) + g(x) - h(x)|) \\ &\leq \sup_{x \in D} (|f(x) - g(x)| + |g(x) - h(x)|) \\ &= \sup_{x \in D} (|f(x) - g(x)|) + \sup_{x \in D} (|g(x) - h(x)|) \\ &= d(f, g) + d(g, h). \end{aligned}$$

Hence (ℓ_∞, d_u) satisfies the triangle-inequality. Furthermore:

$$\begin{aligned} d_u(f, f) &= \sup_{x \in D} (|f(x) - f(x)|) \\ &= \sup_{x \in D} 0 \\ &= 0. \end{aligned}$$

Lastly $d_u(f, g) = 0$ implies $\sup_{x \in D} (|f(x) - g(x)|) = 0$. By definition of the absolute value, $|f(x) - g(x)| \geq 0$, so it must be the case that $|f(x) - g(x)| = 0$. Hence $f(x) = g(x)$, establishing that (ℓ_∞, d_u) forms a metric space. \square

Exercise 7. Let $f, g : D \rightarrow \mathbf{R}$ be bounded functions. Show that

- (1) $\sup_{x \in D} (f + g)(x) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x)$.
- (2) $\inf_{x \in D} (f + g)(x) \leq \inf_{x \in D} f(x) + \inf_{x \in D} g(x)$.
- (3) $|\sup_{x \in D} f(x) - \sup_{x \in D} g(x)| \leq \sup_{x \in D} |f(x) - g(x)|$.

Exercise 8. Find $\bigcap_{n=1}^{\infty} I_n$ where

- (1) $I_n = [0, \frac{1}{n}]$,
- (2) $I_n = (0, \frac{1}{n})$,
- (3) $I_n = [n, \infty)$.

Proof. (1) Note that $[0, \frac{1}{n}]$ is closed and bounded for all $n \geq 1$. Note that:

$$\inf \{\text{length}([0, 1/n]) \mid n \geq 1\} = \inf_{n \geq 1} \left(\frac{1}{n} - 0 \right) = 0.$$

By the Nested Interval Theorem:

$$\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n} \right] = \sup_{n \geq 1} 0 = \inf_{n \geq 1} \frac{1}{n} = 0.$$

(2)*** Claim: $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$. Suppose towards contradiction there exists $t \in \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$. Then $t \in (0, \frac{1}{n})$ for all $n \geq 1$. So $t < \frac{1}{n}$ implies $\frac{1}{t} > n$ for all $n \geq 1$, meaning \mathbf{N} is bounded above. This is a contradiction, hence $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$.

(3) Claim: $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$. Suppose towards contradiction there exists $t \in \bigcap_{n=1}^{\infty} [n, \infty)$. Then $t \in [n, \infty)$ for all $n \geq 1$. So $t \geq n$ for all $n \geq 1$. Hence \mathbf{N} is bounded above, which is a contradiction. Thus $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$. \square

Exercise 9. If $x > 0$, show that there is an $n \in \mathbf{N}$ with $\frac{1}{2^n} < x$.

Proof. By the Archimedean Property 2, there exists $n \in \mathbf{N}$ such that $0 < \frac{1}{n} < x$. Claim: $\frac{1}{2^n} < \frac{1}{n}$. It suffices to show that $2^n > n$. Bernoulli's inequality gives $(1 + 1)^n \geq 1 + n$, hence $2^n > n$. \square

Exercise 10. The *Dyadic Rationals* are defined as

$$\mathbf{D} := \left\{ \frac{m}{2^n} \mid m, n \in \mathbf{Z} \right\}.$$

Show that $\mathbf{D} \subseteq \mathbf{R}$ is dense.

Proof. Let $I = (a, b)$. Then $b - a > 0$. By Archimedean Property 2 there exists $n \in \mathbf{N}$ such that $b - a > \frac{1}{n}$. Exercise 9 gives that $b - a > \frac{1}{2^n}$ for some $n \in \mathbf{Z}$. This simplifies to $2^n b > 1 + 2^n a$. Since $2^n a \in \mathbf{R}$, there exists $m \in \mathbf{Z}$ with $m - 1 \leq 2^n a < m$, implying that $a < \frac{m}{2^n}$. Furthermore, we also have that $m \leq 1 + 2^n a < m + 1$, and substituting for $2^n b$ gives $m < 2^n b$. So $\frac{m}{2^n} < b$, which means $\frac{m}{2^n} \in (a, b)$. Thus $I \cap D \neq \emptyset$. \square