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Chapter 1

Vector Spaces, Algebras, and Normed Spaces

For the entirety of this chapter assume F to be $\mathbb{R}$ or $\mathbb{C}$.

§ 1.1. Vector Spaces

Definition 1.1.1. A *vector space* (or *linear space*) over F is a nonempty set V equipped with two operations:

$$V \times V \xrightarrow{+} V \text{ defined by } (v, w) \mapsto v + w$$
$$F \times V \to V \text{ defined by } (\alpha, v) \mapsto \alpha v$$

satisfying:

(1) $(V, +)$ is an abelian group:

  (i) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$;
  (ii) there exists 0_V such that $v + 0_V = 0_V + v = v$ for all $v \in V$;
  (iii) for all $v \in V$, there exists $w \in V$ satisfying $v + w = w + v = 0_V$;
  (iv) $v + w = w + v$ for all $v, w \in V$;

(2) $\alpha(v + w) = \alpha v + \alpha w$ for all $\alpha \in F$, $v, w \in V$;

(3) $\alpha(\beta v) = (\alpha\beta)v$ for all $\alpha, \beta \in F$, $v \in V$;

(4) $1_F v = v$ for all $v \in V$.

It can be shown that the vector 0_V is unique, the additive inverse in (iii) is unique (which we denote as $-v$), that $0v = 0_V$, and $(-1)v = -v$.

Exercise 1.1.1. Show (iv) follows from the other axioms.

Exercise 1.1.2. Show $nv = \underbrace{v + v + \ldots + v}_{n \text{ times}}$ for $n \in \mathbb{Z}_{\geqslant 1}$.

It can be shown that a subspace is a vector space in its own right.

Example 1.1.1. Let $\{W_i\}_{i\in I}$ be a family of vector spaces. Then $\bigcap_{i\in I} W_i$ is also a vector space.

Example 1.1.2. Planes and lines through the origin are subspaces of $\mathbb{R}^3$.

Definition 1.1.2. Let V be a vector space and $S \subseteq V$ a subset.

(1) A *linear combination* from S is a finite sum $\sum_{j=1}^{n} \alpha_j v_j$ with $\alpha_j \in F$, $v_j \in S$.

(2) The *linear span* of S is:

$$\operatorname{span}(S) := \left\{ \sum_{j=1}^{n} \alpha_j v_j \;\middle|\; n \in \mathbb{N}, \alpha_j \in F, v_j \in S \right\}.$$

Exercise 1.1.3. Show that $\operatorname{span}(S) \subseteq V$ is a subspace and:

$$\operatorname{span}(S) = \bigcap \{ W \mid S \subseteq W, W \text{ is a subspace} \},$$

that is, $\operatorname{span}(S)$ is the smallest subspace of V containing S.

Definition 1.1.3. Let V be a vector space and $S \subseteq V$ a subset.

(1) S is *spanning* for V if $\operatorname{span}(S) = V$.

(2) S is *independent* if, given $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in F$, $v_1, \ldots, v_n \in S$, then $\sum_{j=1}^{n} \alpha_j v_j = 0$ implies $\alpha_j = 0$ for all j.

Our goal is to show that every vector space admits a basis. As such, recall the following definitions from a standard course in Real Analysis.

Definition 1.1.4. An *ordering* on a set X is a relation $R \subseteq X \times X$ on X that is reflexive, transitive, and antisymmetric. We write xRy as $x \leqslant_R y$. The pair $(X, \leqslant_R)$ is called an *ordered set*. An ordering $\leqslant$ on X is called *total* (or *linear*) if for all $x, y \in X$, $x \leqslant y$ or $y \leqslant x$.

Note that if $(X, \leqslant)$ is an ordered set and $Y \subseteq X$ is a subset, then $(Y, \leqslant)$ is an ordered set as well.

Definition 1.1.5. Let $(X, \leqslant)$ be an ordered set and $Y \subseteq X$. An *upper bound* for Y is an element $u \in X$ with $u \geqslant y$ for all $y \in Y$. An element $m \in X$ is called *maximal* if $x \in X$, $x \geqslant m$ implies $x = m$.

Lemma 1.1.1 (Zorn's Lemma). *Let $(X, \leqslant_X)$ be an ordered set. Suppose every subset $Y \subseteq X$ for which $(Y, \leqslant_X)$ is totally ordered has an upper bound in X. Then X admits a maximal element.*

The proof of Zorn's Lemma is outside the interest of this text.

Theorem 1.1.2. *Every vector space admits a basis. Moreover, every independent set is contained in a basis.*

Proof. Let $S \subseteq V$ be linearly independent. Define:

$$\mathfrak{T}(S) = \{T \subseteq V \mid S \subseteq T, T \text{ linearly independent}\}.$$

Let $\mathfrak{C} \subseteq \mathfrak{T}(S)$ be a totally ordered subset. Set $R = \bigcup_{T \in \mathfrak{C}} T$. Clearly $R \supseteq S$. Assume $\sum_{j=1}^{n} \alpha_j v_j = 0$, where $\alpha_j \in F$ and $v_j \in R$. Since $\mathfrak{C}$ is totally ordered, there exists $T_0 \in \mathfrak{C}$ with $v_j \in T_0$ for all $j = 1, \ldots, n$. Since T_0 is independent, $\alpha_j = 0$ for all $j = 1, \ldots, n$. Thus R is independent as well. Whence R is an upper bound for $\mathfrak{C}$. By Zorn's Lemma, $\mathfrak{T}(S)$ admits a maximal element, call it B.

Claim: B is a basis for V. Suppose towards contradiction it's not, then there exists $v_0 \in V \setminus \operatorname{span}(B)$. Consider $B \cup \{v_0\}$ and let $\alpha_0 v_0 + \sum_{j=1}^{n} \alpha_j v_j = 0_V$. If $\alpha_0 \neq 0$, then $\sum_{j=1}^{n} \alpha_j v_j = -\alpha_0 v_0$, giving $v_0 \in \operatorname{span}(B)$ which is a contradiction. If $\alpha_0 = 0$, then $\sum_{j=1}^{n} \alpha_j v_j = 0_V$. Since B is independent, $\alpha_j = 0$ for all $j = 1, \ldots, n$. Thus $B \cup \{v_0\}$ is independent, contradicting the maximality of B. Whence B is a basis for V. □

Theorem 1.1.3. *If* B_1 *and* B_2 *are bases for* V*, then* $\operatorname{card}(B_1) = \operatorname{card}(B_2)$.

Definition 1.1.6. If V is a vector space, its *dimension* is the cardinality of any of its bases.

Corollary 1.1.4. *If* B *is a basis for* V*, then every* $v \in V$ *can be written* $v = \sum_{j=1}^{n} \alpha_k \beta_k$*,* $\alpha_k \in F$*,* $b_k \in B$ *in a unique way.*

Theorem 1.1.5. *Let* V *be a linear space and* $B \subseteq V$ *a subset. The following are equivalent:*

(1) B *is a basis for* V*;*

(2) B *is a maximal element in* $\mathfrak{T} = \{T \subseteq V \mid T \text{ independent}\}$*;*

(3) B *is a minimal element in* $\mathfrak{S} = \{S \subseteq V \mid S \text{ spans } V\}$*;*

Definition 1.1.7. Let $\{V_i\}_{i \in I}$ be a family of vector spaces over a field F.

(1) The *product* of $\{V_i\}_{i \in I}$ is denoted:

$$\prod_{i \in I} V_i := \{(v_i)_{i \in I} \mid v_i \in V_i\}.$$

(2) The *co-product* (or *sum*) is denoted

$$\bigoplus_{i \in I} V_i := \left\{(v_i)_{i \in I} \mid v_i \in V_i, \operatorname{supp}\big((v_i)_{i \in I}\big) < \infty\right\}.$$

Exercise 1.1.4.

(1) Show that $\prod_{i\in I} V_i$ equipped with pointwise operations:

$$\begin{aligned}(v_i)_{i\in I} + (w_i)_{i\in I} &= (v_i + w_i)_{i\in I}\\ \alpha(v_i)_{i\in I} &= (\alpha v_i)_{i\in I}\end{aligned}$$

is a linear space.

(2) Show that $\bigoplus_{i\in I} V_i$ is a subspace of $\prod_{i\in I} V_i$.

Proposition 1.1.6. *Let V be a vector space over F and $W \subseteq V$. The (additive, abelian) quotient group V/W can be made into a vector space by defining multiplication by scalars as $\alpha(v + W) = \alpha v + W$ for all $\alpha \in F$, $v + W \in V/W$.*

Example 1.1.3.

(1) The set $F^n = \{(x_1, ..., x_n) \mid x_j \in F\}$ with component-wise operations is a vector space.

(2) The set $M_{n,m}(F) = \{(a_{ij}) \mid a_{ij} \in F\}$ with linear operations is a vector space.

(3) Let Ω be a nonempty set. Then $\mathcal{F}(\Omega, F) = \{f \mid f : \Omega \to F\}$ with pointwise operations is a vector space.

(4) The set $\ell_\infty(\Omega, F) = \{f \in \mathcal{F}(\Omega, F) \mid \|f\|_u < \infty\}$ with pointwise operations is a vector space.

Exercise 1.1.5. Show $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(5) Let $K \subseteq V$ be a convex subset of a vector space V, that is, for all $v, w \in K$ and $t \in [0, 1]$, then $(1 - t)v + tw \in K$. A function $f : K \to F$ is said to be *affine* if $x, y \in K$ and $t \in [0, 1]$ implies $f((1 - t)x + ty) = (1 - t)f(x) + tf(y)$. The set $\mathrm{Aff}(K, F) = \{f \in \mathcal{F}(\Omega, F) \mid f \text{ affine}\}$ with pointwise operations is a vector space.

Exercise 1.1.6. Show $\mathrm{Aff}(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(6) The set $C([a, b], F) = \{f : [a, b] \to F \mid f \text{ continuous}\}$ with pointwise operations is a vector space.

Exercise 1.1.7. Explain why $C([a, b], F) \subseteq \ell_\infty([a, b], F)$ is a subspace.

(7) Consider the following sequence spaces:

- $s = \{(a_k)_k \mid a_k \in F\} = \mathcal{F}(\mathbb{N}, F)$;
- $\ell_\infty = \ell_\infty(\mathbb{N}, F) = \{(a_k)_k \mid \sup_{k\geqslant 1} |a_k| < \infty\}$;
- $c = \{(a_k)_k \mid (a_k)_k \text{ converges}\}$;
- $c_0 = \{(a_k)_k \mid (a_k)_k \to 0\}$;

- $c_{00} = \{(a_k)_k \mid \operatorname{supp}(a_k)_k < \infty\}$;
- $\ell_1 = \left\{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k| \text{ converges}\right\}$.

These are all vector spaces with pointwise operations. In fact, $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subspaces.

Exercise 1.1.8. Show that $\ell_1 \subseteq c_0$ is a subspace.

(8) Consider the following continuous function spaces on $\mathbb{R}$:

- $C(\mathbb{R}) = \{f : \mathbb{R} \to F \mid f \text{ continuous}\}$;
- $C_b(\mathbb{R}) = C(\mathbb{R}) \cap \ell_\infty(\mathbb{R})$;
- $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \operatorname{limit}_{x\to\pm\infty} f(x) = 0\}$;
- Recall that a function is *compactly supported* if for all $\epsilon > 0$, there exists $\alpha > 0$ such that $|x| \geqslant \alpha$ implies $f(x) = 0$. The set of compactly supported functions is denoted $C_c(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid f \text{ compactly supported}\}$.

These are all vector spaces with pointwise operations, and $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C_b(\mathbb{R}) \subseteq C(\mathbb{R})$ are all subspace inclusion.

Definition 1.1.8. If V and W are linear spaces over a common field F, a map $T : V \to W$ is called *linear* if $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha \in F$.

Example 1.1.4. Let $A \in M_{m,n}(F)$. Then $T_A : F^n \to F^m$ defined by $T_A(v) = Av$ is linear. Let $\{e_1, ..., e_n\}$ be a basis for F^n. If $T : F^n \to F^m$ is linear, set:

$$[T] = \Big(T(e_1) \,\Big|\, T(e_2) \,\Big|\, \dots \,\Big|\, T(e_n)\Big).$$

This gives $T(v) = [T]v$ for all $v \in F^n$. In fact, we also have $[T_A] = A$ and $T_{[T]} = T$.

Example 1.1.5. The *canonical projection* is linear:

$$\pi_j : \prod_{i\in I} V_i \to V_j \;\text{ defined by }\; \pi_j\big((v_i)_i\big) = v_i.$$

We also have that the *coordinate exclusions* are linear:

$$\iota_j : V_j \hookrightarrow \bigoplus_{i\in I} V_i \;\text{ defined by }\; \iota_j(v) = (v_i)_i\,, \;\text{ where }\; v_i = \begin{cases} 0_v, & i \neq j \\ v_j, & \text{otherwise.} \end{cases}$$

The *evaluation map* is linear as well. For $s \in S$, consider:

$$e_s : \mathcal{F}(S, F) \to F \;\text{ defined by }\; e_s(f) = f(s).$$

Proposition 1.1.7. * *Let* V *be a vector space with basis* B. *Let* W *be a vector space and suppose* $\varphi : B \to W$ *is a map. Then there exists a unique linear map* $T_\varphi : V \to W$ *with* $T_\varphi(b) = \varphi(b)$ *for all* $b \in B$.

Proof. □

Proposition 1.1.8. * *Let* $T : V \to W$ *be linear.*

(1) $\ker(T) = \{v \in V \mid T(v) = 0_W\}$ *is a linear subspace of* V.

(2) $\mathrm{im}(T) = \{T(v) \mid v \in V\}$ *is a linear subspace of* W.

(3) $\ker(T) = \{0_V\}$ *if and only if* T *is injective.*

(4) $\mathrm{im}(T) = W$ *if and only if* W *is surjective.*

Proof. (1) Let $v_1, v_2 \in \ker(T)$ and $\alpha \in F$. Observe that:

$$\begin{aligned} T(v_1 + cv_2) &= T(v_1) + cT(v_2) \\ &= 0. \end{aligned}$$

Thus $v_1 + cv_2 \in \ker(T)$, giving $\ker(T)$ as a linear subspace of V.

(2) Let $w_1, w_2 \in \mathrm{im}(T)$. Then there exists $v_1.v_2 \in V$ with $T(v_1) = w_1$ and $T(v_2) = w_2$. We have:

$$\begin{aligned} w_1 + cw_2 &= T(v_1) + cT(v_2) \\ &= T(v_1 + cv_2). \end{aligned}$$

Whence $w_1 + cw_2 \in \mathrm{im}(T)$, giving $\mathrm{im}(T)$ as a linear subspace of W.

(3) Let $\ker(T) = \{0\}$. Suppose $T(v_1) = T(v_2)$. Then $T(v_1) - T(v_2) = T(v_1 - v_2) = 0_W$. It must be that $v_1 - v_2 = 0_W$, giving $v_1 = v_2$. Thus T is injective. Conversely, suppose T is injective and let $v \in \ker(T)$. Then $T(v) = 0_W = T(0_V)$. Hence $v = 0_V$, establishing $\ker(T) = \{0\}$.

(4) □

Proposition 1.1.9. *If* $T : V \to W$ *is linear and bijective, then the inverse map* $T^{-1} : W \to V$ *is linear.*

Proof. We have that:

$$T(T^{-1}(w_1) + \alpha T^{-1}(w_2)) = w_1 + \alpha w_2 = T \circ T^{-1}(w_1 + \alpha w_2).$$

Applying T^{-1} to both sides gives the desired result. □

Proposition 1.1.10 (Vector Spaces are Injective). * *Let* U, V, W *be vector spaces and* $0 \to U \xrightarrow{j} V$ *be exact (that is,* j *is injective). Let* $\varphi : U \to W$ *be linear. There exists a linear map* $\psi : V \to W$ *such that* $\varphi = \psi \circ j$*; i.e., the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "j"] \arrow[d, "\varphi"'] & V \arrow[dl, dotted, "\psi"] \\ & W & \end{tikzcd}$$

Proof. Let $\{u_i\}_{i\in I}$ be a basis for U. Claim: $\{j(u_i)\}_{i\in I}$ is linearly independent. Observe that:

$$\begin{aligned} 0 &= \sum_{i\in I} \alpha_i j(u_i) \\ &= j\left(\sum_{i\in I} \alpha_i u_i\right). \end{aligned}$$

By the injectivivity of j, we have that $\sum_{i=1}^{n} \alpha_i u_i = 0$, giving $\alpha_i = 0$ for all $i \in I$. Thus $\{j(u_i)\}_{i\in I}$ is linearly independent. We can extend this set to a basis of V as follows: let □

Proposition 1.1.11 (Vector Spaces are Projective)**.** ** Let U, V, W be vector spaces and $V \xrightarrow{\pi} U \to 0$ be exact (that is, π is onto). Let $\varphi : W \to U$ be linear. There exists a linear map $\phi : V \to W$ such that $\varphi = \pi \circ \psi$; i.e., the following diagram commutes:*

$$\begin{tikzcd} & W \arrow[d, "\varphi"] \arrow[dl, dotted, "\psi"'] & \\ V \arrow[r, "\pi"] & U \arrow[r] & 0 \end{tikzcd}$$

Proof. □

Definition 1.1.9. Let V and W be vector spaces over F. A *linear isomorphism* between V and W is a bijective linear map $T : V \to W$. If such a T exists, we say V and W are *linearly isomorphic*, and write $V \cong W$.

Finite dimensional vector spaces are boring. This is illustrated through the following theorem.

Theorem 1.1.12. *Let V and W be finite-dimensional vector spaces over F. Then $V \cong W$ if and only if $\dim(V) = \dim(W)$.*

Proof. Suppose $V \cong W$. Then there is an isomorphism taking basis of V to a basis of W. Therefore they have the same dimension.

Conversely, if $\dim(V) = \dim(W) = n$, then they are each isomorphic to F^n, giving that they are isomorphic to eachother. □

Example 1.1.6. Let V be a vector space, $W \subseteq V$ a subspace. The *natural projection*:

$$\pi : V \to V/W \text{ defined by } \pi(v) = v + W$$

is a linear surjective map.

Theorem 1.1.13 (First Isomorphism Theorem for Vector Spaces)**.** * *Let* $T : V \to V'$ *be a linear map and* $W \subseteq V$ *a subspace.*

(1) If T *"kills"* W *(that is,* $W \subseteq \ker(T)$*), then there exists a linear map* $\widetilde{T} : V/W \to V'$ *with* $\widetilde{T} \circ \pi = T$*; i.e., the following diagram commutes.*

$$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, "\pi"'] & & V' \\ & V/W \arrow[ur, dotted, "\widetilde{T}"'] & \end{tikzcd}$$

(2) If $\ker(T) = W$*, then* $\widetilde{T}$ *is injective.*

(3) If $\ker(T) = W$ *and* $\operatorname{im}(T) = V'$*, then* $V/W \cong V'$*.*

Proof. (1) As stipulated, define $\widetilde{T}(v + W) = T(v)$. We must show that $\widetilde{T}$ is well-defined: suppose $v_1 + W = v_2 + W$ for some $v_1, v_2 \in V$. Then $v_1 = v_2 + w$ for some $w \in W$. This gives:

$$\begin{aligned} \widetilde{T}(v_1 + W) &= \widetilde{T}(v_2 + w + W) \\ &= \widetilde{T}(v_2 + W). \end{aligned}$$

Whence $\widetilde{T}$ is well-defined. Now given $v_1 + W, v_2 + W \in V/W$ and $\alpha \in F$, observe that:

$$\begin{aligned} \widetilde{T}\big((v_1 + W) + c(v_2 + W)\big) &= \widetilde{T}\big((v_1 + cv_2) + W\big) \\ &= T(v_1 + cv_2) \\ &= T(v_1) + cT(v_2) \\ &= \widetilde{T}(v_1 + W) + c\widetilde{T}(v_2 + W). \end{aligned}$$

Thus $\widetilde{T}$ is linear. □

Definition 1.1.10. Let S be a nonempty set. The *free vector space* of S is:

$$\mathbb{F}(S) = \{f : S \to F \mid \operatorname{supp}(f) < \infty\}.$$

Exercise 1.1.9. Show $\mathbb{F}(S) \subseteq \mathcal{F}(S, F)$ is a subspace.

Proposition 1.1.14. *The set* $\{\delta_s \mid s \in S\}$ *is a basis for* $\mathbb{F}(S)$*, where* $\delta_s : S \to F$ *is defined by:*

$$\delta_s(t) = \begin{cases} 1, & t = 0 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. If $f \in \mathbb{F}(S)$ with $\mathrm{supp}(f) = \{s_1, ..., s_n\}$, then $f = \sum_{k=1}^{n} f(s_k)\delta_{s_k}$. If $\sum_{k=1}^{n} \alpha_k \delta_{s_k} = 0$, then for $j = 1, ..., n$ we have $0 = \left(\sum_{k=1}^{n} \alpha_k \delta_{s_k}\right)(s_j) = \alpha_j$. □

Theorem 1.1.15. ** Given any vector space* V *and a map (of sets)* $\varphi : S \to V$*, there exists a unique linear map* $T_\varphi : \mathbb{F}(S) \to V$ *with* $T_\varphi \circ \iota = \varphi$*, where* $\iota : S \to \mathbb{F}(S)$ *is defined by* $\iota(s) = \delta_s$ *for all* $s \in S$*. In other words, the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathbb{F}(S) \arrow[d, dotted, "T_\varphi"] \\ & V \end{tikzcd}$$

Proof. □

Definition 1.1.11. Let V and W be vector spaces. The set of linear transformations between V and W is $\mathcal{L}(V, W) = \{T \mid T : V \to W \text{ linear}\}$. The set of linear functionals is $V' := \mathcal{L}(V, F)$.

Exercise 1.1.10. Show $\mathcal{L}(V, W)$ is a vector space.

Exercise 1.1.11. Show $M_{m,n}(F) \cong \mathcal{L}(F^m, F^n)$ by $a \mapsto T_a : (v \mapsto av)$.

§ 1.2. Algebras

Definition 1.2.1. An *algebra* over F is a linear space A over F equipped with a multiplication operation:

$$A \times A \to A \text{ defined by } (a, b) \mapsto ab$$

satisfying:

(1) $(ab)c = a(bc)$ for all $a, b, c \in A$;

(2) $(\alpha a)b = \alpha(ab) = a(\alpha b)$ for all $a, b \in A$, $\alpha \in F$;

(3) $a(b + c) = ab + ac$ for all $a, b, c \in A$;

(4) $(a + b)c = ac + bc$ for all $a, b, c \in A$.

If $ab = ba$ for all $a, b \in A$ we say that A is *commutative*. If there exists $1_A \in A$ with $1_A a = a 1_A = a$ for all $a \in A$ we say A is *unital*.

Example 1.2.1.

(1) $M_n(F)$ is a noncommutative unital algebra over F under the usual matrix multiplication.

(2) If V is a vector space over F, $\mathcal{L}(V)$ is a unital algebra over F. It is noncommutative provided $\dim(V) > 1$.

(3) $\mathcal{F}(S, F)$ is a unital commutative algebra over F.

Definition 1.2.2. Let B be a (unital) algebra over F.

(1) A (unital) *subalgebra* of B is a subspace $A \subseteq B$ ($1_B \in A$) satisfying the property that if $a, a' \in A$, then $aa' \in A$.

(2) An *ideal* of B is a subspace $I \subseteq B$ with $b \in B$, $a \in I$ implying $ba, ab \in I$.

Example 1.2.2.

(1) $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a unital subalgebra.

(2) $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subalgebras. In particular, $c_0 \subseteq \ell_\infty$ and $c_{00} \subseteq s$ are ideals.

(3) $C\big([a, b]\big) \subseteq \ell_\infty\big([a, b]\big)$ is a unital subalgebra.

(4) $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C_b(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are all subalgebras. In fact, $C_b(\mathbb{R}) \subseteq C(\mathbb{R})$ and $C_b(\mathbb{R}) \subseteq \ell_\infty(\mathbb{R})$ are unital, whereas $C_0(\mathbb{R}) \subseteq C_b(\mathbb{R})$ and $C_c(\mathbb{R}) \subseteq C(\mathbb{R})$ are ideals.

(5) The set $T_n(F) = \{(a_{ij}) \in M_n(F) \mid a_{ij} = 0, i > j\}$ is a unital subalgebra of $M_n(F)$.

Example 1.2.3 (Group Algebra)**.** Let Γ denote a group (not necessarily abelian). Take the free vector space $\mathbb{F}(\Gamma)$ and define multiplication as *convolution*: given $f, g \in \mathbb{F}(\Gamma)$ let:

$$(f * g)(r) = \sum_{\substack{\big\{(s,t) \mid \\ s \in \operatorname{supp}(f), \\ t \in \operatorname{supp}(g), \\ st = r\big\}}} f(s)g(t).$$

Since $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are finite, this is a finite sum. We often suppress this notation and write $(f * g)(r) = \sum_{st=r} f(s)g(t)$.

We can also make substitutions:

$$\begin{aligned}(f * g)(r) &= \sum_{st=r} f(s)g(t) \\ &= \sum_{t \in \Gamma} f(rt^{-1})g(t) \\ &= \sum_{s \in \Gamma} f(s)g(s^{-1}r).\end{aligned}$$

It is clear that:

$$\begin{aligned}(f+g)*h &= f*h+g*h\\ g*(g+h) &= f*g+f*h\\ \alpha(f*g) &= (\alpha f)*g = f*(\alpha g)\end{aligned}$$

for $f, g, h \in \mathbb{F}(\Gamma)$, $\alpha \in F$. Associativity can be similarly shown using the above definition. Rather, we will prove associativity by first showing that $\delta_s * \delta_t = \delta_{st}$. Given:

$$(\delta_s * \delta_t)(r) = \sum_{q\in\Gamma} \delta_s(rq^{-1})\delta_t(q),$$

notice that:

$$\delta_s(rt^{-1}) = \begin{cases}1, & s = rt^{-1}\\ 0, & \text{otherwise}\end{cases} = \begin{cases}1, & r = st\\ 0, & \text{otherwise}\end{cases} = \delta_{st}(r).$$

Since $\{\delta_t \mid t \in \Gamma\}$ is a basis for $\mathbb{F}(\Gamma)$, every $f \in \mathbb{F}(\Gamma)$ looks like:

$$f = \sum_{t\in J} \alpha_t\delta_t,\ \ J \subseteq T \text{ finite.}$$

Using distributivity we get:

$$\begin{aligned}\delta_r * (\delta_s * \delta_t) &= \delta_r * \delta_{st}\\ &= \delta_{rst}\\ &= \delta_{rs} * \delta_t\\ &= (\delta_r * \delta_s) * \delta_t.\end{aligned}$$

Whence convolution is associative.

Proposition 1.2.1. ** Let $\{A_i\}_{i\in I}$ be a family of algebras over F.*

(1) $\prod_{i\in I} A_i$ *is an algebra under* $(a_i)_i(b_i)_i = (a_ib_i)_i$.

(2) $\bigoplus_{i\in I} \subseteq \prod_{i\in I} A_i$ *is an ideal.*

Proof. □

Proposition 1.2.2. ** Let A be an algebra over F and $I \subseteq A$ an ideal. Then A/I is an algebra under* $(a+I)(b+I) = ab + I$.

Proof. □