Math 310

Homework 5

Due: 10/9/2024

Name: Gianluca Crescenzo

Exercise 1. *Let $x_1 = 1$ and inductively set $x_{n+1} = \sqrt{2 + x_n}$. Show that $(x_n)_n$ converges and find its limit.*

Proof. Claim: $x_n \leqslant 2$ for all n. We prove this by induction on n. Clearly $x_1 = 1 \leqslant 2$. Now assume our hypothesis is true for n. For $n + 1$ we have:

$$\begin{aligned} x_{n+1} &= \sqrt{2 + x_n} \\ &\leqslant \sqrt{2 + 2} \\ &= 2. \end{aligned}$$

Claim: $x_n \leqslant x_{n+1}$. Observe that:

$$\begin{aligned} x_n \leqslant x_{n+1} &\iff x_n \leqslant \sqrt{2 + x_n} \\ &\iff x_n \leqslant \sqrt{2 + 2} \\ &\iff x_n \leqslant 2. \end{aligned}$$

Since $(x_n)_n$ is increasing and bounded above, by the monotone convergence theorem it has a limit, call it L. Thus:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n}$$

$$\iff$$

$$L = \sqrt{2 + L}$$

$$\iff$$

$$L^2 - L - 2 = 0$$

$$\iff$$

$$L = 2 \text{ or } L = -1.$$

So it must be the case that $(x_n)_n \to L$. □

Exercise 2. *Does the following sequence converge?*

$$x_n := \sum_{k=n+1}^{2n} \frac{1}{k}.$$

Proof. Claim: $x_n \leqslant 1$ for all n. Observe that:

$$\begin{aligned}
x_n &= \sum_{k=n+1}^{2n} \frac{1}{k} \\
&= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n-2} + \frac{1}{2n-1} + \frac{1}{2n} \\
&\leqslant \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} + \ldots + \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} \\
&= \frac{n+1}{n+1} \\
&= 1.
\end{aligned}$$

Claim: $x_n \leqslant x_{n+1}$. We have:

$$\begin{aligned}
x_n \leqslant x_{n+1} &\iff \sum_{k=n+1}^{2n} \frac{1}{k} \leqslant \sum_{k=n+2}^{2n+2} \frac{1}{k} \\
&\iff \frac{1}{n+1} \leqslant \frac{1}{2n+1} + \frac{1}{2n+1} \\
&\iff \frac{1}{n+1} \leqslant \frac{4n+3}{(2n+1)(2n+1)} \\
&\iff n+1 \geqslant \frac{4n^2+6n+2}{4n+3} \\
&\iff 4n^2+7n+3 \geqslant 4n^2+6n+2 \\
&\iff 7n+3 \geqslant 6n+2
\end{aligned}$$

which is true for all $n \in \mathbf{N}$. Since $(x_n)_n$ is increasing and bounded above, by the monotone convergence theorem $(x_n)_n$ converges. □

Exercise 3. *Let $(f_n)_n$ denote the Fibonacci sequence and let*

$$x_n := \frac{f_{n+1}}{f_n}.$$

Given that $(x_n)_n$ converges, find its limit.

Proof. Note that:

$$\begin{aligned}
x_n &= \frac{f_{n+1}}{f_n} \\
&= \frac{f_n + f_{n-1}}{f_n} \\
&= 1 + \frac{f_{n-1}}{f_n} \\
&= 1 + \frac{1}{f_n/f_{n-1}} \\
&= 1 + \frac{1}{x_{n-1}}.
\end{aligned}$$

If $(x_n)_n \to L$, then:

$$\begin{aligned} L = 1 + \frac{1}{L} &\iff L^2 = L + 1 \\ &\iff L^2 - L - 1 = 0 \\ &\iff L = \frac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

Since $\frac{1-\sqrt{5}}{2} < 0$, it must be the case that $(x_n)_n \to \frac{1+\sqrt{5}}{2}$ □

Exercise 4. *If $(x_n)_n$ is an unbounded sequence of reals numbers, show that there is a subsequence $(x_{n_k})_k$ such that:*

$$\left(\frac{1}{x_{n_k}}\right)_k \to 0.$$

Proof. Since $(x_n)_n$ is an unbounded real sequence:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(n \geqslant N \ \wedge \ |x_n - 0| \geqslant \epsilon_0).$$

We can construct a subsequence as follows:

$$\begin{aligned} N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geqslant 1 \ \wedge \ |x_{n_1}| \geqslant \epsilon_0) \\ N = n_1 = 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 \geqslant n_1 \ \wedge \ |x_{n_2}| \geqslant \epsilon_0) \\ &\vdots \end{aligned}$$

Inductively, we obtain a sequence $(x_{n_k})_k$ which properly diverges to $+\infty$. Given $\epsilon > 0$, let K be arbitrarily big so that $\epsilon > \frac{1}{n_K}$. Then for $k \geqslant K$, we have $\left|\frac{1}{n_k}\right| < \epsilon$. □

Exercise 5. *Suppose that every subsequence of a sequence $(x_n)_n$ has a subsequence that converges to 0. Show that $(x_n)_n \to 0$.*

Proof. Suppose towards contradiction that $(x_n)_n \nrightarrow 0$. Then there exists a subsequence $(x_{n_k})_k \nrightarrow 0$. By definition:

$$(\exists \epsilon_0 > 0)(\forall K \in \mathbf{N}) \ni (\exists k \in \mathbf{N})(k \geqslant K \ \wedge \ d(x_{n_k}, 0) \geqslant \epsilon_0).$$

We will construct a subsequence of $(x_{n_k})_k$ as follows:

$$\begin{aligned} K = 1 &\implies (\exists k_1 \in \mathbf{N})(k_1 \geqslant 1 \ \wedge \ d(x_{n_{k_1}}, 0) \geqslant \epsilon_0) \\ K = k_1 + 1 &\implies (\exists k_2 \in \mathbf{N})(k_2 \geqslant k_1 \ \wedge \ d(x_{n_{k_2}}, 0) \geqslant \epsilon_0) \\ &\vdots \end{aligned}$$

Inductively, we obtain a sequence $(x_{n_{k_j}})_j \nrightarrow 0$. But this contradicts our claim that every subsequence has a subsequence which converges to 0. Hence it must be that $(x_n)_n \to 0$. □

Exercise 6. *Let $(I_n)_n$ be a nested sequence of closed and bounded intervals. For each $n \in \mathbf{N}$ let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem for the sequence $(x_n)_n$ to give a proof of the Nested Intervals Property, that is to show that*

$$\bigcap_{n \geqslant 1} I_n \neq \emptyset.$$

Proof. For each $n \in \mathbf{N}$, let $x_n \in I_n$. Then $(x_n)_n$ is bounded. By the Bolzano-Weirstrass theorem, $(x_n)_n$ admits a convergent subsequence $(x_{n_k})_k \to x$. Given $\epsilon > 0$, there exists a natural number N such that $n_k \geqslant k \geqslant N$ implies $x_{n_k} \in V_\epsilon(x)$. Moreover, for $n = n_k$, we have $V_\epsilon(x) \subseteq I_n$, implying that $x \in I_n \subseteq I_{n-1} \subseteq \ldots$ Thus $x \in \bigcap_{n\geqslant 1} I_n$, establishing that the intersection is nonempty. □

Exercise 7. *If $(x_n)_n$ is a bounded sequence and $s := \sup_n x_n$ is such that $s \notin \{x_n | n \geqslant 1\}$, show that there is a subsequence $(x_{n_k})_k$ that converges to s.*

Proof. Using the supremum property, we will construct a subsequence as follows:

$$k = 1 \implies \exists n_1 \ni s - 1 < x_{n_1} < u$$

Because $(s-1, s) \cap \{x_n \mid n \in \mathbf{N}\}$ is infinite, we have:

$$k = 2 \implies \exists n_2 \ni s - \frac{1}{2} < x_{n_2} < u$$

$$\vdots$$

Inductively, we obtain $s - \frac{1}{k} < x_{n_k} < s$. By the squeeze theorem, $(x_{n_k}) \to s$. □

Exercise 8. *Let $(x_n)_n$ and $(y_n)_n$ be bounded sequences. Show that*

$$\limsup((x_n + y_n)_n) \leqslant \limsup(x_n)_n + \limsup(y_n)_n$$

Proof. We have $x_k + y_k \leqslant \sup_{m\geqslant n} x_m + \sup_{m\geqslant n} y_n$ for $k \geqslant n$. This implies that $\sup_{m\geqslant n} x_m + \sup_{m\geqslant n} y_n$ is an upperbound of $x_k + y_k$. Whence $\sup_{k\geqslant n}(x_k + y_k) \leqslant \sup_{m\geqslant n} x_m + \sup_{m\geqslant n} y_n$. Taking the limit of both sides gives $\limsup(x_n + y_n) \leqslant \limsup x_n + \limsup y_n$. □

Exercise 9. *Let $(x_n)_n$ be a bounded sequence. Show that:*

$$\liminf(x_n)_n = \sup\{t \in \mathbf{R} \mid \{n \mid x_n < t\} \textit{ is finite } \}.$$

Proof. Let $l = \liminf(x_n)_n = \sup_{m\geqslant 1}(\inf_{n\geqslant m} x_n)$. By the supremum property, we have $l - \epsilon < (\inf_{n\geqslant m} x_n)_n \leqslant l$. So given any $M \in \mathbf{N}$, there are only finitely many x_n with $x_n < \inf_{m\geqslant M}$, hence $\{n \mid x_n < l - \epsilon\}$ is finite. Setting $t = l - \epsilon$ gives $l = \sup\{t \in \mathbf{R} \mid \{n \mid x_n < t\} \text{ is finite } \}$. □

Exercise 10. *Let $(x_n)_n$ be a bounded sequence. Show that*

$$\liminf(-x_n)_n = -\limsup(x_n)_n.$$

Proof. We have:

$$\begin{aligned}
-\limsup x_n &= -\inf_{m\geqslant 1}(\sup_{n\geqslant m} x_n) \\
&= \sup_{m\geqslant 1}(-\sup_{n\geqslant m} x_n) \\
&= \sup_{m\geqslant 1}(\inf_{n\geqslant m} -x_n) \\
&= \liminf -x_n.
\end{aligned}$$

□