Recall: Let G be a group, $N \trianglelefteq G$ a normal subgroup.
$G/N = \{gN : g \in G\}$. If $N \trianglelefteq G$, then $aNbN = abN$ is well-defined.

Theorem: Let $N \trianglelefteq G$. Then G/N is a group. We call G/N a quotient group.

proof. We have $G/N \neq \emptyset$ b/c $e_G N \in G/N$. Moreover, G/N is closed under multiplication. Claim: $e_{G/N} = eN$.

Id Let $gN \in G/N$. We have $gNeN = geN = gN$ and $eNgN = egN = gN$.

Inv Observe $gNg^{-1}N = gg^{-1}N = eN$ and $g^{-1}NgN = g^{-1}gN = eN$.

Hence $(gN)^{-1} = g^{-1}N$.

Let $g_1N, g_2N, g_3N \in G/N$.

$$\begin{aligned}
(g_1Ng_2N)g_3N &= g_1g_2Ng_3N \\
&= (g_1g_2)g_3N \\
&= g_1(g_2g_3)N \\
&= g_1Ng_2g_3N \\
&= g_1N(g_2Ng_3N).
\end{aligned}$$

$g_1N = g_2N$ and $g_3N = g_4N$ iff $g_1g_2^{-1} \in N$ and $g_3g_4^{-1} \in N$ $\square$

Example: Let $G = D_3$ and $N = \langle r \rangle = \{e, r, r^2\}$. We saw before that $\langle r \rangle \trianglelefteq D_3$. This gives $D_3/\langle r \rangle$ is a group.

$D_3/\langle r \rangle = \{\langle r \rangle, \sigma\langle r \rangle\}$

	$\langle r \rangle$	$\sigma\langle r \rangle$
$\langle r \rangle$	$\langle r \rangle$	$\sigma\langle r \rangle$
$\sigma\langle r \rangle$	$\sigma\langle r \rangle$	$\langle r \rangle$

	$[0]_2$	$[1]_2$
$[0]_2$	$[0]_2$	$[1]_2$
$[1]_2$	$[1]_2$	$[0]_2$

$D_3/\langle r \rangle \cong \mathbb{Z}/2\mathbb{Z}$
$\langle r \rangle \mapsto [0]_2$
$\sigma\langle r \rangle \mapsto [1]_2$

Example: Set $G = D_4$ and $N = \{e, r^2\}$.
$D_4 = \{e, r, r^2, r^3, \sigma, \sigma r, \sigma r^2, \sigma r^3\}$, recall $r\sigma = \sigma r^3$. We can check all the cosets to see if $N \trianglelefteq D_4$.

- $eN = \{e, r^2\} = Ne$ *
- $rN = \{r, r^3\} = Nr$ *
- $r^2N = \{r^2, e\} = N = Nr^2$
- $r^3N = \{r^3, r^5\} = Nr = Nr^3$
- $\sigma N = \{\sigma, \sigma r^2\}$
  - $\rightarrow N\sigma = \{\sigma, r^2\sigma\} = \{\sigma, \sigma r^2\} = \sigma N$ *
- $\sigma r N = \{\sigma r, \sigma r^3\}$
  - $\rightarrow N\sigma r = \{\sigma r, r^2\sigma r\} = \{\sigma r, \sigma r^3\} = \sigma r N$ *

$\bigcap_{g \in D_4} gN = D_4$, hence last two subgroups are non-distinct.

- $\sigma r^2 N = \sigma N = N\sigma = N\sigma r^2$
- $\sigma r^3 N = \sigma r N = N\sigma r = N\sigma r^3$

Thus $D_4/N = \{N, rN, \sigma N, \sigma r N\}$.

	N	rN	σN	$\sigma r N$
N	N	rN	σN	$\sigma r N$
rN	rN	N	$\sigma r N$	σN
σN	σN	$\sigma r N$	N	rN
$\sigma r N$	$\sigma r N$	σN	rN	N

* D_4 NOT abelian, but D_4/N is abelian. Note that our cayley table is symmetric along diagonal.

If G is order 4, the elements of G must have 1, 2, 4.
If G has an element of order 4, G is cyclic. So $G \cong \mathbb{Z}/4\mathbb{Z}$.
If G has no elements of order 4, $G = \{e, g_1, g_2, g_3\}$ w/ $|g_1| = |g_2| = |g_3| = |g_4| = 2$.
↳ In this case, you can show $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Hence, from our cayley table, every element has order 2. Thus $D_4/N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Example: Consider $G = \mathbb{Z}/12\mathbb{Z}$ and $N = \{[0]_{12}, [4]_{12}, [8]_{12}\} \subseteq G$.

1) Show $N \trianglelefteq G$.

$G = \{0,1,2,3,4,5,6,7,8,9,10,11\}$

$N = \{0,4,8\}$

$0+N = \{0,4,8\} = N+0 = N+4 = N+8$

$1+N = \{1,5,9\} = N+1 = N+5 = N+9$

$2+N = \{2,6,10\} = N+2 = N+6 = N+10$

$3+N = \{3,7,11\} = N+3 = N+7 = N+11$

Thus $N \trianglelefteq G$.

Let $\varphi: \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, $[m]_{12} \mapsto [m]_4$

Let $[a]_{12}, [b]_{12} \in \mathbb{Z}/12\mathbb{Z}$. $[a]_{12} = [b]_{12}$ iff $a \equiv b \pmod{12}$; i.e., $a = b + 12k$.

- $\varphi([a]_{12}) = [a]_4 = [b+12k]_4 = [b]_4 + [4 \cdot 12k]_4 = [b]_4 = \varphi([b]_{12})$
- $\varphi([a]_{12} + [b]_{12}) = \varphi([a+b]_{12}) = [a+b]_4 = [a]_4 + [b]_4 = \varphi([a]_4) + \varphi([b]_4)$.
- Let $[a]_{12} \in \ker\varphi$. Then $\varphi([a]_{12}) = [0]_4 \Rightarrow [a]_4 = [0]_4 \Rightarrow a = 4k,\ k \in \mathbb{Z}$.
  $\Rightarrow [4k]_{12} \in \ker\varphi$
  $\Rightarrow 4\mathbb{Z}/12\mathbb{Z} \subseteq \ker\varphi$.

2) Give the cosets of G/N.

$G/N = \{N, 1+N, 2+N, 3+N\}$

3) Write out a cayley table for G/N. Identify a familiar group that is isomorphic to G/N.

+	N	1+N	2+N	3+N
N	N	1+N	2+N	3+N
1+N	1+N	2+N	3+N	N
2+N	2+N	3+N	N	1+N
3+N	3+N	N	1+N	2+N

$\cong \mathbb{Z}/4\mathbb{Z}$