The Riemann-Lebesgue Theorem

Based on *An Introduction to Analysis*, Second Edition, by James R. Kirkwood, Boston: PWS Publishing (1995)

Note. Throughout these notes, we assume that f is a bounded function on the interval $[a, b]$. We follow Chapter 6 of Kirkwood and give necessary and sufficient conditions for such a function to be Riemann integrable on the interval $[a, b]$. We start with the definition of the Riemann integral and its component parts.

Definition. A *partition* $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ is a set such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Definition. Let $x_{i-1}, x_i \in P$, where P is a partition of $[a, b]$. For f a bounded function on $[a, b]$, define

$$m_i(f) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

$$M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

and $\Delta x_i = x_i - x_{i-1}$. Let

$$\overline{S}(f; P) = \sum_{i=1}^{n} M_i(f)\Delta x_i \text{ and } \underline{S}(f; P) = \sum_{i=1}^{n} m_i(f)\Delta x_i.$$

$\overline{S}(f; P)$ and $\underline{S}(f; P)$ are the *upper Riemann sum* and *lower Riemann sum*, respectively, of f on $[a, b]$ with respect to partition P.

Definition. With the notation above, suppose $\overline{x}_i \in [x_{i-1}, x_i]$. Then

$$S(f; P) = \sum_{i=1}^{m} f(\overline{x}_i)\Delta x_i$$

is a *Riemann sum* of f on $[a, b]$ with respect to partition P.

Definition. With the notation above, define

$$\overline{S}(f) = \inf\{\overline{S}(f; P) \mid P \text{ is a partition of } [a, b]\} \text{ and}$$

$$\underline{S}(f) = \sup\{\underline{S}(f; P) \mid P \text{ is a partition of } [a, b]\}.$$

$\overline{S}(f)$ and $\underline{S}(f)$ are the *upper Riemann integral* and *lower Riemann integral*, respectively, of f on $[a, b]$.

Definition. Let f be bounded on $[a, b]$. Then f is said to be *Riemann integrable* on $[a, b]$ if $\overline{S}(f) = \underline{S}(f)$. In this case, $\overline{S}(f)$ is called the *Riemann integral of f on* $[a, b]$, denoted

$$\overline{S}(f) = \int_a^b f(x)\, dx = \int_a^b f.$$

Note. First, let's explore some conditions related to the integrability of f on $[a, b]$. Notice that these conditions are merely restatements of the definition and that the proofs follow from this definition, along with properties of suprema and infima.

Theorem 6-4. Riemann Condition for Integrability.
A bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$ if and only if for all $\varepsilon > 0$, there exists a partition $P(\varepsilon)$ of $[a, b]$ such that

$$\overline{S}(f; P(\varepsilon)) - \underline{S}(f; P(\varepsilon)) < \varepsilon.$$

Definition. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. The *norm* (or *mesh*) of P, denoted $\|P\|$, is

$$\|P\| = \max\{x_i - x_{i-1} \mid i = 1, 2, 3, \ldots, n\}.$$

Theorem 6-6. *A bounded function f is Riemann integrable on $[a, b]$ if and only if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if P is a partition with $\|P\| < \delta(\varepsilon)$ then*

$$\overline{S}(f; P) - \underline{S}(f; P) < \varepsilon.$$

Note. The following result is proved in Calculus 1. In fact, all functions encountered in the setting of integration in Calculus 1 involve continuous functions. We give a proof based on other stated results.

Theorem 6-7. *If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.*

Proof. Since f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$ (Theorem 4-10 of Kirkwood). Let $\varepsilon > 0$. Then by the uniform continuity of f, there exists $\delta(\varepsilon) > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta(\varepsilon)$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ with $\|P\| < \delta(\varepsilon)$. On $[x_{i-1}, x_i]$, f assumes a maximum and a minimum (by the Extreme Value Theorem), say at

x_i' and x_i'' respectively. Thus

$$\overline{S}(f;P) - \underline{S}(f;P) = \sum_{i=1}^{n}(f(x_i') - f(x_i''))\Delta x_i < \frac{\varepsilon}{b-a}\sum_{i=1}^{n}\Delta x_i = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

So by Theorem 6-6, f is Riemann integrable on $[a,b]$. ■

Note. We now introduce a new idea about the "weight" of a set. We will ultimately see that the previous result gives us, in some new sense, a classification of Riemann integrable functions.

Definition. The (Lebesgue) *measure* of an open interval (a,b) is $b-a$. The measure of an unbounded open interval is infinite. The measure of an open interval I is denoted $m(I)$.

Definition. A set $E \subset \mathbb{R}$ has *measure zero* if for all $\varepsilon > 0$, there is a countable collection of open intervals $\{I_1, I_2, I_3, \ldots\}$ such that

$$E \subset \cup_{i=1}^{\infty} I_i \text{ and } \sum_{i=1}^{\infty} m(I_i) < \varepsilon.$$

Note. The following two results follow from the definition of measure zero.

Theorem 6-8. *A subset of a set of measure zero has measure zero.*

Theorem 6-9. *The union of a countable collection of sets of measure zero is a set of measure zero.*

Note. We give a direct proof of a corollary to Theorem 6-9 which gives an idea of the method of proof of Theorem 6-9.

Corollary 6-9. *A countable set has measure zero.*

Proof. Let $\varepsilon > 0$ and let A be a countable set, say $A = \{a_1, a_2, a_3, \ldots\}$. Define $I_i = \left(a_i - \frac{\varepsilon}{2^{i+1}}, a_i + \frac{\varepsilon}{2^{i+1}}\right)$. Then $A \subset \cup_{i=1}^{\infty} I_i$ and $m(I_i) = \frac{\varepsilon}{2^i}$. Now

$$\sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \varepsilon.$$

So $m(A) = 0$. ∎

Note. It is not the case that cardinality and measure are closely related. The converse of Corollary 6-9, for example, is not true. That is, there exists an uncountable set which is also of measure zero. Such a set, as we will see in more detail later, is the Cantor (ternary) set.

Definition. The *oscillation* of a function f on a set A is

$$\sup\{|f(x) - f(y)| \mid x, y \in A \cap \mathcal{D}(f)\}$$

where $\mathcal{D}(f)$ denotes the domain of f. The *oscillation of f at x* is

$$\lim_{h \to 0^+} \left(\sup\{|f(x') - f(x'')| \mid x', x'' \in (x - h, x + h) \cap \mathcal{D}(f)\}\right),$$

denoted $\text{osc}(f; x)$.

Theorem 6-10. *A function f is continuous at $x \in \mathcal{D}(f)$ if and only if* $\operatorname{osc}(f;x) = 0$.

Proof. Suppose $\operatorname{osc}(f;x) = 0$ and let $\varepsilon > 0$. Then there exists $\delta(\varepsilon) > 0$ such that if $0 < h < \delta$, then

$$\sup\{|f(x') - f(x'')| \mid x', x'' \in (x-h, x+h) \cap \mathcal{D}(f)\} < \varepsilon$$

(from the definition of $\operatorname{osc}(f;x)$ and the definition of limit). So if $|x-y| < \delta$ and $x, y \in \mathcal{D}(f)$, then $|f(x) - f(y)| < \varepsilon$. So, by definition, f is continuous at x.

Conversely, suppose f is continuous at x and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|x-y| < \delta \text{ and } x, y \in \mathcal{D}(f) \text{ implies } |f(x) - f(y)| < \varepsilon/2.$$

So if $x', x'' \in (x-\delta, x+\delta) \cap \mathcal{D}(f)$, then

$$|f(x') - f(x'')| \le |f(x') - f(x)| + |f(x) - f(x'')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore

$$\sup\{|f(x') - f(x'')| \mid x', x'' \in (x-\delta, x+\delta) \cap \mathcal{D}(f)\} \le \varepsilon$$

and so $\operatorname{osc}(f;x) = 0$. ■

Note. Now for our main result, proved in two parts.

Theorem 6-11(a). The Riemann Lebesgue Theorem, Part (a)
Consider a bounded function f defined on $[a,b]$. If f is Riemann integrable on $[a,b]$ then the set of discontinuities of f on $[a,b]$ has measure zero.

Proof. Suppose f is bounded and Riemann integrable on $[a,b]$. Let

$$A = \{x \in [a,b] \mid f \text{ is discontinuous at } x\}.$$

Then by Theorem 6-10,

$$A = \{x \in [a,b] \mid \operatorname{osc}(f;x) > 0\}.$$

For each $k \in \mathbb{N}$, let

$$A_k = \left\{x \in [a,b] \mid \operatorname{osc}(f;x) \geq \frac{1}{k}\right\}. \tag{1}$$

Then $A = \cup_{k=1}^{\infty} A_k$.

Let $\varepsilon > 0$ and let k be a fixed positive integer. Since f is Riemann integrable on $[a,b]$, there is a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ such that

$$\overline{S}(f;P) - \underline{S}(f;P) < \frac{\varepsilon}{k}.$$

For a given i, if $x \in A_k$ and $x \in (x_{i-1}, x_i)$ then

$$(M_i(f) - m_i(f))\Delta x_i \geq \frac{1}{k}\Delta x_i \text{ by (1)}.$$

Now

$$\frac{\varepsilon}{k} > \overline{S}(f;P) - \underline{S}(f;P) = \sum_{i=1}^{n}(M_i(f) - m_i(f))\Delta x_i \geq {\sum}'(M_i(f) - m_i(f))\Delta x_i \tag{2}$$

where $\sum'$ indicates summation over values of i such that $(x_{i-1}, x_i) \cap A_k \neq \varnothing$. Then

$${\sum}'(M_i(f) - m_i(f))\Delta x_i \geq \frac{1}{k}{\sum}'\Delta x_i. \tag{3}$$

Comparing (2) and (3) we get $\varepsilon > \sum' \Delta x_i$. Let I be the set of i values in this $\sum' \Delta x_i$. Then $\cup_I (x_{i-1}, x_i)$ is an open cover of A_k, except possibly for $\{x_0, x_1, x_2, \ldots, x_n\}$. Also,

$$\sum_{i \in I} m((x_{i-1}, x_i)) = {\sum}'\Delta x_i < \varepsilon.$$

Therefore $m\left(A_k \setminus \{x_0, x_1, x_2, \ldots, x_n\}\right) = 0$ and $m(A_k) = 0$ for all k (by Theorem 6-9). Also by Theorem 6-9,

$$m(A) = m\left(\cup_{k=1}^{\infty} A_k\right) = 0. \qquad \blacksquare$$

Note. We need a preliminary result before proving the other half of the Riemann-Lebesgue Theorem (namely, the converse of Theorem 6-11(a)).

Exercise 6.1.8. *Let f be a function with $\mathcal{D}(f) = [a, b]$. Then for any $s > 0$,*

$$A_s = \{x \in [a, b] \mid \ osc(f; x) \geq s\}$$

is compact.

Proof. We will show that A_s^c is open relative to $\mathcal{D}(f) = [a, b]$. Let $x_0 \in A_s^c$. Then $\text{osc}(f; x_0) = t < s$ for some t. So

$$t = \lim_{h \to 0^+} \left(\sup\{|f(x') - f(x'')| \mid x', x'' \in (x_0 - h, x_0 + h) \cap \mathcal{D}(f)\}\right).$$

Therefore, for $\varepsilon = \dfrac{s - t}{2} > 0$, there exists $\delta(\varepsilon) > 0$ such that if $0 < h \leq \delta$ then

$$|\sup\{|f(x') - f(x'')| \mid x', x'' \in (x_0 - h, x_0 + h) \cap \mathcal{D}(f)\} - t| < \varepsilon = \frac{s - t}{2}.$$

That is,

$$\sup\{|f(x') - f(x'')| \mid x', x'' \in (x_0 - \delta, x_0 + \delta) \cap \mathcal{D}(f)\} < t + \varepsilon = \frac{t}{2} + \frac{s}{2}.$$

So for all $y_0 \in (x_0 - \delta, x_0 + \delta)$ we have

$$\begin{aligned} \text{osc}(f; y_0) &= \lim_{h \to 0^+} \left(\sup\{|f(x') - f(x'')| \mid x', x'' \in (y_0 - h, y_0 + h) \cap \mathcal{D}(f)\}\right) \\ &\leq \frac{t}{2} + \frac{s}{2} < s. \end{aligned}$$

That is, $(x_0 - \delta, x_0 + \delta) \cap \mathcal{D}(f) \subset A_s^c$. So A_s^c is open relative to $[a, b]$ and therefore A_s is closed. Also, $A_s \subset [a, b]$ and so A_s is bounded. Therefore A_s is compact by the Heine-Borel Theorem, Theorem 0.8. ∎

Note. With the notation from this exercise, if f is discontinuous at some $x_0 \in [a,b]$, then $x_0 \in A_s$ for some $s > 0$. So for $f : [a,b] \to \mathbb{R}$, the set of discontinuities is $D = \cup_{n=1}^{\infty} A_{1/n}$. That is, the set of discontinuities of $f : [a,b] \to \mathbb{R}$ is a *countable union of closed sets*. Such a set is said to be "F_σ." One can also show that, more generally, if $f : \mathbb{R} \to \mathbb{R}$, then the set of discontinuities is an F_σ set.

Theorem 6-11(b). The Riemann Lebesgue Theorem, Part (b)
Consider a bounded function f defined on $[a,b]$. If the set of discontinuities of f on $[a,b]$ has measure zero, then f is Riemann integrable on $[a,b]$.

Proof. Suppose f is bounded on $[a,b]$ and that the set of discontinuities is set A where $m(A) = 0$. Let

$$M = \sup\{f(x) \mid x \in [a,b]\} \text{ and } m = \inf\{f(x) \mid x \in [a,b]\}$$

and assume $M > m$ (otherwise, f is constant and the result holds). Define

$$A_s = \{x \in [a,b] \mid \operatorname{osc}(f;x) \geq s\}$$

where $s > 0$. Then $A_s \subset A$ and so $m(A_s) = 0$ for all s by Theorem 6-8.

Let $\varepsilon > 0$. Then there exists a set of open intervals $\{I_1, I_2, I_3, \ldots\}$ such that

$$A_{\frac{\varepsilon}{2(b-a)}} \subset \cup_{i=1}^{\infty} I_i \text{ and } \sum_{i=1}^{\infty} m(I_i) < \frac{\varepsilon}{2(M-m)}$$

since $m\left(A_{\frac{\varepsilon}{2(b-a)}}\right) = 0$. By Exercise 6.1.8, A_s is compact for all s, so there is a finite subcover $\{I_1, I_2, I_3, \ldots, I_N\}$ of $A_{\frac{\varepsilon}{2(b-a)}}$.

If

$$x \in [a,b] \setminus \left(\cup_{i=1}^{N} I_i\right) \subset [a,b] \setminus A_{\frac{\varepsilon}{2(b-a)}}$$

then $\operatorname{osc}(f;x) < \dfrac{\varepsilon}{2(b-a)}$. So for any such x, there exists $\delta_x > 0$ such that if $x', x'' \in (x - \delta_x, x + \delta_x)$ then $|f(x') - f(x'')| < \dfrac{\varepsilon}{2(b-a)}$ by the definition of $\operatorname{osc}(f;x)$.

Now, the set $[a,b] \setminus \left(\cup_{i=1}^N I_i\right)$ is closed and bounded and therefore compact by the Heine-Borel Theorem, Theorem 0.8. So, since

$$\left\{(x-\delta_x, x+\delta_x) \mid x \in [a,b] \setminus \left(\cup_{i=1}^N I_i\right)\right\}$$

is an open cover of this set, then there exists a finite subcover, say

$$\{(\overline{x}_1-\delta_1, \overline{x}_1+\delta_1), (\overline{x}_2-\delta_2, \overline{x}_2+\delta_2), \cdots, (\overline{x}_k-\delta_k, \overline{x}_k+\delta_k)\}.$$

Now we will construct a partition P for which $\overline{S}(f;P) - \underline{S}(f;P) < \varepsilon$. Notice that

$$(\overline{x}_1-\delta_1, \overline{x}_1+\delta_1), (\overline{x}_2-\delta_2, \overline{x}_2+\delta_2), \cdots, (\overline{x}_k-\delta_k, \overline{x}_k+\delta_k), I_1, I_2, \ldots, I_N$$

is a finite cover of $[a,b]$. Let $P = \{x_0, x_1, x_2, \ldots, x_j\}$ be a partition of $[a,b]$ such that $[x_{i-1}, x_i]$ is completely contained in one interval of this cover.

Now, we partition the subintervals determined by P into a set C_1 of intervals which are contained in some I_i, and a set C_2 of the remaining subintervals. Then

$$\overline{S}(f;P) - \underline{S}(f;P) = \sum_{C_1}(M_i(f) - m_i(f))\Delta x_i + \sum_{C_2}(M_i(f) - m_i(f))\Delta x_i.$$

Now,

$$\begin{aligned}\sum_{C_1}(M_i(f) - m_i(f))\Delta x_i &\le (M-m)\sum_{C_1}\Delta x_i \le (M-m)\sum_{i=1}^N m(I_i) \\ &< (M-m)\frac{\varepsilon}{2(M-m)} = \frac{\varepsilon}{2}\end{aligned}$$

(this part of the sum takes advantage of the fact that the I_i's are "small" sets). Next, if $[x_{i-1}, x_i] \subset (\overline{x}_r - \delta_r, \overline{x}_r + \delta_r)$ then

$$|f(x') - f(x'')| < \frac{\varepsilon}{2(b-a)} \text{ for all } x', x'' \in [x_{i-1}, x_i]$$

(this part takes advantage of the fact that the oscillation of f is small on the $(\overline{x}-\delta, \overline{x}+\delta)$'s). Then

$$\sum_{C_2}(M_i(f) - m_i(f))\Delta x_i \le \frac{\varepsilon}{2(b-a)}\sum_{C_2}\Delta x_i < \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.$$

So $\overline{S}(f;P) - \underline{S}(f;P) < \varepsilon$ and therefore f is Riemann integrable on $[a,b]$ by Theorem 6-6. ■

Note. Combining Parts (a) and (b), we have:

Theorem 6-11. The Riemann Lebesgue Theorem

Let f be a bounded function defined on $[a,b]$. f is Riemann integrable on $[a,b]$ if and only if the set of discontinuities of f on $[a,b]$ has measure zero.

Note. Theorems 6-4 and 6-6 also give necessary and sufficient conditions for the Riemann integrability of a bounded function. However there is, in a sense, no new information in these results since they are really just restatements of the definition of Riemann integral. On the other hand, the Riemann-Lebesgue Theorem cleanly classifies Riemann integrable functions. It gives a condition *on the function, in terms of properties of the function* without any reference to partitions or Riemann sums (directly, at least).

Note. We know from Theorem 6-7 a continuous function f on $[a,b]$ is Riemann integrable on $[a,b]$. So, perhaps, it is not surprising that necessary and sufficient conditions for Riemann integrability of f involve the "level of discontinuity" of f. Informally, interpret the Riemann-Lebesgue Theorem as saying that a function is Riemann integrable if and only if the function is not *too badly discontinuous.*

Example. Probably the most exotic function you encountered in your undergraduate analysis class is the following:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1/q & \text{if } x = p/q \in \mathbb{Q} \end{cases}$$

where p/q are in reduced terms. It is straightforward to show that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$. Therefore the set of discontinuities of f has measure zero. So f is Riemann integrable and, in fact, $\int_{\mathbb{R}} f = 0$.

Example. You have also seen the *Dirichlet function*:

$$D(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases}$$

D is bounded (on $[0,1]$, say) but discontinuous on $[0,1]$. We will see that the measure of $[0,1]$ is 1 (we already know that the measure of $(0,1)$ is 1 but, technically, we cannot *prove* anything about the measure of a closed interval, since we have not yet dealt with measure theory). So D is not Riemann integrable on $[0,1]$. Notice that D is 0 except on the rationals and we know that the rationals are a measure zero set. So, if we can define $\int_0^1 D$, then it *should* be 0. We take this as a first motivation to study another type of integration—one which makes use of the measure of sets.

Note. In applied math (though I hesitate to motivate new mathematical topics by resorting to applications), it is very common to write a function as a series of some sort. If you want to integrate the function, then you may resort to integrating the series representation of the function (and recall that a series is the *limit* of a sequence of partial sums). But it is not always the case that the integral of the limit of a sequence of functions is the limit of the integrals of the sequence of functions (in fact, one of these limits may exist while the other does not). That is, we require "certain conditions" to conclude the following:

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ on } [a,b] \text{ implies } \int_a^b f = \int_a^b (\lim f_n) = \lim \left(\int_a^b f_n \right).$$

These "certain conditions" will lead us to study results which we will call *convergence theorems.* The only convergence theorem you likely encountered in undergraduate analysis involves uniform convergence.

Definition. Let $\{f_n\}$ be a sequence of functions defined on a set $E \subset \mathbb{R}$ and suppose that $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in E$ (that is, suppose f is the pointwise limit of $\{f_n\}$). Then $\{f_n\}$ *converges uniformly* to f if for any $\varepsilon > 0$, there is a number $N(\varepsilon)$ such that if $n > N(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon \text{ for every } x \in E.$$

Note. The following is the convergence theorem associated with Riemann integration:

Theorem 8-3. *Suppose $\{f_n\}$ is a sequence of Riemann integrable functions on $[a,b]$. If $\{f_n\}$ converges uniformly to f on $[a,b]$, then f is Riemann integrable on $[a,b]$ and*

$$\lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \int_a^b f(x)\,dx.$$

Note. So a second motivation to study another type of integration is to get a *stronger theory of integration*—one which has more convergence theorems. That is, a form of integration that allows us to keep the conclusion of Theorem 8-3 while softening the restriction of uniform convergence.

Note. To summarize, we have two motivations to explore a new type of integration:

1. To integrate a broader class of functions than those which have discontinuities on sets of measure zero, and
2. To have a stronger theory of integration which yields more convergence theorems.

We will accomplish both of these with *Lebesgue integration*. But first, we must study *Lebesgue measure* and measure theory.

Revised: 8/22/2020