Math 374

Homework 7

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(b)

(c) From the `Toeplitz_Eigenvalues.nb` file from the drive, we can see:

```
In[1]:= (* Symmetric Matrix *)
        Remove[k, v, T, eig, lambda, lambda1, lambda2, r1, r2, n];
        n = 7;
        r1 = 1;
        r2 = 0;
        v = Table[0, {k, 1, n}];
        v[[1]]=1 -  r1;
        v[[2]]=r1;
        T = ToeplitzMatrix[v];
        eig = Eigenvalues[T]
        T // MatrixForm
        ListPlot[eig, PlotStyle→Blue, GridLines→Automatic];
```

Out[9]= $\{-\sqrt{2+\sqrt{2}},\sqrt{2+\sqrt{2}},-\sqrt{2},\sqrt{2},-\sqrt{2-\sqrt{2}},\sqrt{2-\sqrt{2}},0\}$

Out[10]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since this is a symmetric tri-diagonal Toeplitz matrix, its eigenvalues are given by:

$$\lambda_h(T) = \left\{ 2\cos\left(\frac{h\pi}{8}\right) \;\middle|\; 1 \leqslant h \leqslant 7 \right\}.$$

Moreover, note that $\left|\sqrt{2-\sqrt{2}} - 0\right| < \left|\sqrt{2+\sqrt{2}} - 0\right|$. Since we are sweeping along the unit circle from $\theta = 0$ to $\theta = \pi$, some simple algebra yields:

$$\begin{aligned}
\cos\left(\frac{\pi}{8}\right) &= \frac{1}{2}\sqrt{2+\sqrt{2}} \\
\cos\left(\frac{2\pi}{8}\right) &= \frac{\sqrt{2}}{2} \\
\cos\left(\frac{3\pi}{8}\right) &= \frac{1}{2}\sqrt{2-\sqrt{2}} \\
\cos\left(\frac{4\pi}{8}\right) &= 0 \\
\cos\left(\frac{5\pi}{8}\right) &= -\frac{1}{2}\sqrt{2-\sqrt{2}} \\
\cos\left(\frac{6\pi}{8}\right) &= -\frac{\sqrt{2}}{2} \\
\cos\left(\frac{7\pi}{8}\right) &= -\frac{1}{2}\sqrt{2+\sqrt{2}}
\end{aligned}$$

(d) We are given the matrix:

$$A = \begin{pmatrix} 7 & 8 & 0 & 0 & 0 \\ 2 & 7 & 8 & 0 & 0 \\ 0 & 2 & 7 & 8 & 0 \\ 0 & 0 & 2 & 7 & 8 \\ 0 & 0 & 0 & 2 & 7 \end{pmatrix}$$

Its eigenvalues are given by:

$$\lambda_h(A) = \left\{ 7 + 4\cos\left(\frac{h\pi}{6}\right) \;\middle|\; 1 \leqslant h \leqslant 5 \right\}.$$

The corresponding eigenvector of $\lambda_h(A)$ is v_h, defined as:

$$v_h = \begin{pmatrix} \left(\frac{2}{8}\right)^{\frac{1}{2}} \sin\left(\frac{h\pi}{6}\right) \\ \left(\frac{2}{8}\right)^{\frac{2}{2}} \sin\left(\frac{2h\pi}{6}\right) \\ \left(\frac{2}{8}\right)^{\frac{3}{2}} \sin\left(\frac{3h\pi}{6}\right) \\ \left(\frac{2}{8}\right)^{\frac{4}{2}} \sin\left(\frac{4h\pi}{6}\right) \\ \left(\frac{2}{8}\right)^{\frac{5}{2}} \sin\left(\frac{5h\pi}{6}\right) \end{pmatrix}.$$