Recall: Given $H \leq G$, $g \in G$

Left coset $gH = \{gh : h \in H\}$

Right coset $Hg = \{hg : h \in H\}$

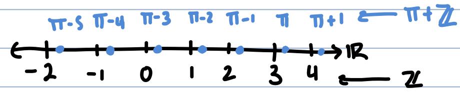
$G/H = \{gH, g \in G\}$

$H \backslash G = \{Hg, g \in G\}$

Example: Let $G = \mathbb{R}$ and let $H = \mathbb{Z}$.

- Operation is addition, G is abelian so $gH = Hg$.
- $r + \mathbb{Z}$ for this example

Fix some $r \in \mathbb{R}$, say $r = \pi$, so $\pi + \mathbb{Z} = \{\pi + n : n \in \mathbb{Z}\}$



Example: Let $G = D_3$, $H_1 = \langle r \rangle = \{e, r, r^2\}$

$\sigma H_1 = \{\sigma, \sigma r, \sigma r^2\}$

$H_1 \sigma = \{\sigma, r\sigma, r^2\sigma\} = \{\sigma, \sigma r^2, \sigma r\} = \sigma H_1$.

Although G not abelian, this particular example has $gH = Hg$.

Let $H_2 = \langle \sigma \rangle = \{e, \sigma\}$

$rH_2 = \{r, r\sigma\}$

$H_2 r = \{r, \sigma r\} = \{r, r^2\sigma\} \neq rH_2$.

Exercise: (1) Let G be a group, $H \leq G$ a subgroup. Is gH a subgroup?

Not in general (see prev. example). NO IDENTITY

When is $e_G \in gH$?

Let $e_G = gh$ for some $h \in H$.

Then $h = g^{-1}$, hence $g^{-1} \in H$.

Since H is a subgroup, $(g^{-1})^{-1} \in H$.

Hence $g \in H$.

Claim: If $g \in H$, then $gH = eH$.

→ Let $gh \in gH$. We want to write $gh = e(\tilde{h})$ for some $\tilde{h} \in H$.

Hence $gh = e(gh)$, so $gH \subseteq eH$.

→ Let $eh \in eH$. We want to write $eh = g\tilde{h}$ for some $\tilde{h} \in H$.

Hence $eh = g(g^{-1}h) = (gg^{-1})h = eh$. So $eH \subseteq gH$.

Thus, the only time gH is a subgroup is if $g = e_G$, i.e. $eH \leq G$.

(2) Let $G = S_3$ and $H = \langle (12) \rangle$. Compute $(123)H$ and $H(123)$.

$(123)H = \{(123)(1), (123)(12)\} = \{(123), (13)\}$.

$H(123) = \{(1)(123), (12)(123)\} = \{(123), (23)\}$.

Proposition: Let G be finite group, $H \leq G$ a subgroup.

(1) $|H| = |gH| = |Hg| \ \forall g \in G$. i.e., there is a bijection between H and gH.

(2) There is an equivalence relation on G defined by $x \sim y$ iff $x^{-1}y \in H$. The equivalence classes are the left cosets of H.

$$G = \bigsqcup_{g \in G} gH$$

(3) $|G/H| = \frac{|G|}{|H|}$

proof: (1) Let $g \in G$. Define a map $L_g : G \to G$, $x \mapsto gx$.

L_g evaluated at h.

Note $L_g|_H : H \to gH$.

Let $gh \in gH$. Note $L_g(h) = gh$. So L_g is surjective.

Let $h_1, h_2 \in H$. Assume $L_g(h_1) = L_g(h_2)$.

We have $gh_1 = gh_2$, $\Leftrightarrow g^{-1}gh_1 = g^{-1}gh_2 \Leftrightarrow h_1 = h_2$.

Since $L_g : H \to gH$ is bijective, $|H| = |gH|$.

(2) • Note $x \sim x$ b/c $x^{-1}x = e_H \in H$ b/c H is a subgroup. Thus $\sim$ reflexive.

• Suppose $x \sim y$, i.e., $x^{-1}y \in H$.

But H is a subgroup, so $(x^{-1}y)^{-1} \in H$.

However $(x^{-1}y)^{-1} = y^{-1}x$. Thus $y \sim x$. Thus $\sim$ is symmetric.

• Suppose $x \sim y$ and $y \sim z$, i.e., $x^{-1}y, y^{-1}z \in H$.

Note $x^{-1}z = x^{-1}yy^{-1}z \in H$. Thus $x \sim y$. Thus $\sim$ is transitive.

• $[y] = \{x \in G : xy^{-1} \in G\}$

$= \{x \in G : xy^{-1} = h \text{ for some } h \in H\}$

$= \{x \in G : y^{-1}x = h^{-1} \text{ for some } h \in H\}$

$= \{x \in G : x = y\tilde{h} \text{ for some } \tilde{h} \in H\}$

$= \{y\tilde{h} : h \in H\}$

$= yH$.

(3) $G/H = \{g_1H, g_2H, \dots, g_mH\}$

$G = \bigsqcup_{i=1}^{m} g_iH$

$$|G| = \left|\bigsqcup_{i=1}^{m} g_iH\right| = \sum_{i=1}^{m} \underbrace{|g_iH|}_{\text{from (1)}} = \sum_{i=1}^{m} |H| = m|H|, \quad m = |G/H|$$

Hence $|G| = |G/H||H| \Rightarrow |G/H| = |G|/|H|$ $\square$

Example: Let $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$.

Equivalence classes are $m + 5\mathbb{Z}$ for $m \in \mathbb{Z}$.

$m + 5\mathbb{Z} = \{m + 5k : k \in \mathbb{Z}\}$.

We have for any integer m, $m = 5q + r$ for some $q, r \in \mathbb{Z}$, $0 \leq r < 5$.

Note $m - r = 5q \in \mathbb{Z}$, so $m \sim r$.

i.e., $m + 5\mathbb{Z} = r + 5\mathbb{Z}$.

So cosets are $0+5\mathbb{Z} = [0]_5$
$1+5\mathbb{Z} = [1]_5$
$2+5\mathbb{Z} = [2]_5$
$3+5\mathbb{Z} = [3]_5$
$4+5\mathbb{Z} = [4]_5$

$G/H = \mathbb{Z}/5\mathbb{Z} = \{\ldots\}$

<u>Example</u>: Let $G = D_3$

$$\left.\begin{aligned} H &= \langle\sigma\rangle = \{e,\sigma\} \\ rH &= \{r, r\sigma\} \\ r^2H &= \{r^2, r^2\sigma\} \end{aligned}\right\} \text{3 cosets}$$

$|G/H| = 3$

$G = H \sqcup rH \sqcup r^2H$

Note: $|H| = |rH| = |r^2H| = 2$

And $|D_3| = |G/H| \cdot |H| = 3 \cdot 2 = 6$

<u>Theorem</u>: (Lagrange's Theorem) Let G be a finite group, $H \leq G$ a subgroup. Then $|H| \mid |G|$.

<u>proof</u>: $\underset{\mathbb{Z}}{\underset{\cap}{|G/H|}} = \frac{|G|}{|H|} \longleftarrow |H| \mid |G|$