Contents

Preface

These are the notes I took for a second semester analysis course.

Urgent Things That I Need To Fix

(1) Any corollary, lemma, proposition, example, or theorem with three asterisks (***) means I don't understand the proof, or there is no proof altogether. There are 18 instances of these throughout the notes.

(2) Turn every "Exercise" into a proposition and do the proofs. Any proposition whose proof is an "exercise" needs to be filled in.

(3) The end of § 2.4 Convergent Sequences is supposed to include one or two propositions related to the distance of an element $x \in X$ to a set $A \subseteq X$.

(4) § 2.3 The Cantor Set is incomplete.

(5) The notes on applications of meager sets and the Baire Category Theorem need to be included at the end of § 2.6 Completeness.

(6) § 2.2 Topology of Metric Spaces, § 2.4 Convergent Sequences, and § 2.5 Continuity are kind of messy. Can't put my finger on why but it doesn't feel as nice as some other parts of these notes.

Less Urgent Things That I'm Probably Gonna Do Instead

(1) Clean up § 2.6 Completeness and § 2.7 Compactness. Finish § 2.8 Connectedness.

(2) Start Chapter 3. First section will be on Riemann Integration, will probably copy most of it from Bartle's *An Introduction to Real Analysis*. Second section will be on measure theory. Third section will be on Lebesgue integration.

Things That I'm Really Happy With

(1) Besides some stuff about inner-products, Chapter 1 is almost perfect.

Last update: 2025 July 10

Chapter 1

Vector Spaces, Algebras, and Normed Spaces

For the entirety of this chapter assume F to be $\mathbf{R}$ or $\mathbf{C}$.

§ 1.1. Vector Spaces

Definition 1.1.1. A *vector space* (or *linear space*) over F is a nonempty set V equipped with two operations:

$$\begin{aligned} V \times V \xrightarrow{+} V &\text{ defined by } (v, w) \mapsto v + w \\ F \times V \to V &\text{ defined by } (\alpha, v) \mapsto \alpha v \end{aligned}$$

satisfying:

(1) $(V, +)$ is an abelian group:

- (i) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$;
- (ii) there exists 0_V such that $v + 0_V = 0_V + v = v$ for all $v \in V$;
- (iii) for all $v \in V$, there exists $w \in V$ satisfying $v + w = w + v = 0_V$;
- (iv) $v + w = w + v$ for all $v, w \in V$;

(2) $(\alpha + \beta)v = \alpha v + \beta v$ for all $\alpha, \beta \in F$, $v \in V$;

(3) $\alpha(\beta v) = (\alpha\beta)v$ for all $\alpha, \beta \in F$, $v \in V$;

(4) $\alpha(v + w) = \alpha v + \alpha w$ for all $\alpha \in F$, $v, w \in V$;

(5) $1_F v = v$ for all $v \in V$.

It can be shown that the vector 0_V is unique, the additive inverse in (iii) is unique (which we denote as $-v$), that $0v = 0_V$, and $(-1)v = -v$.

Exercise 1.1.1. Show (iv) follows from the other axioms.

Exercise 1.1.2. Show $nv = \underbrace{v + v + ... + v}_{n \text{ times}}$ for $n \in \mathbf{Z}_{\geqslant 1}$.

It can be shown that a subspace is a vector space in its own right.

Example 1.1.1. Let $\{W_i\}_{i\in I}$ be a family of vector spaces. Then $\bigcap_{i\in I} W_i$ is also a vector space.

Example 1.1.2. Planes and lines through the origin are subspaces of $\mathbf{R}^3$.

Definition 1.1.2. Let V be a vector space and $S \subseteq V$ a subset.

(1) A *linear combination* from S is a finite sum $\sum_{j=1}^{n} \alpha_j v_j$ with $\alpha_j \in F$, $v_j \in S$.

(2) The *linear span* of S is:

$$\operatorname{span}(S) := \left\{ \sum_{j=1}^{n} \alpha_j v_j \;\middle|\; n \in \mathbf{N}, \alpha_j \in F, v_j \in S \right\}.$$

Exercise 1.1.3. Show that $\operatorname{span}(S) \subseteq V$ is a subspace and:

$$\operatorname{span}(S) = \bigcap \{W \mid S \subseteq W, W \text{ is a subspace}\},$$

that is, $\operatorname{span}(S)$ is the smallest subspace of V containing S.

Definition 1.1.3. Let V be a vector space and $S \subseteq V$ a subset.

(1) S is *spanning* for V if $\operatorname{span}(S) = V$.

(2) S is *independent* if, given $n \in \mathbf{N}$, $\alpha_1, ..., \alpha_n \in F$, $v_1, ..., v_n \in S$, then $\sum_{j=1}^{n} \alpha_j v_j = 0$ implies $\alpha_j = 0$ for all j.

———————//———————

Our goal is to show that every vector space admits a basis. As such, recall the following definitions from a standard course in Real Analysis.

Definition 1.1.4. An *ordering* on a set X is a relation $R \subseteq X \times X$ on X that is reflexive, transitive, and antisymmetric. We write xRy as $x \leqslant_R y$. The pair $(X, \leqslant_R)$ is called an *ordered set*. An ordering $\leqslant$ on X is called *total* (or *linear*) if for all $x, y \in X$, $x \leqslant y$ or $y \leqslant x$.

Note that if $(X, \leqslant)$ is an ordered set and $Y \subseteq X$ is a subset, then $(Y, \leqslant)$ is an ordered set as well.

Definition 1.1.5. Let $(X, \leqslant)$ be an ordered set and $Y \subseteq X$. An *upper bound* for Y is an element $u \in X$ with $u \geqslant y$ for all $y \in Y$. An element $m \in X$ is called *maximal* if $x \in X$, $x \geqslant m$ implies $x = m$.

Lemma 1.1.1 (Zorn's Lemma)**.** *Let* $(X, \leqslant_X)$ *be an ordered set. Suppose every subset* $Y \subseteq X$ *for which* $(Y, \leqslant_X)$ *is totally ordered has an upper bound in* X*. Then* X *admits a maximal element.*

The proof of Zorn's Lemma is outside the interest of this text.

Theorem 1.1.2. *Every vector space admits a basis. Moreover, every independent set is contained in a basis.*

Proof. Let $S \subseteq V$ be linearly independent. Define:

$$\mathfrak{T}(S) = \{T \subseteq V \mid S \subseteq T, T \text{ linearly independent}\}.$$

Let $\mathfrak{C} \subseteq \mathfrak{T}(S)$ be a totally ordered subset. Set $R = \bigcup_{T \in \mathfrak{C}} T$. Clearly $R \supseteq S$. Assume $\sum_{j=1}^{n} \alpha_j v_j = 0$, where $\alpha_j \in F$ and $v_j \in R$. Since $\mathfrak{C}$ is totally ordered, there exists $T_0 \in \mathfrak{C}$ with $v_j \in T_0$ for all $j = 1, ..., n$. Since T_0 is independent, $\alpha_j = 0$ for all $j = 1, ..., n$. Thus R is independent as well. Whence R is an upper bound for $\mathfrak{C}$. By Zorn's Lemma, $\mathfrak{T}(S)$ admits a maximal element, call it B.

Claim: B is a basis for V. Suppose towards contradiction it's not, then there exists $v_0 \in V \setminus \operatorname{span}(B)$. Consider $B \cup \{v_0\}$ and let $\alpha_0 v_0 + \sum_{j=1}^{n} \alpha_j v_j = 0_V$. If $\alpha_0 \neq 0$, then $\sum_{j=1}^{n} \alpha_j v_j = -\alpha_0 v_0$, giving $v_0 \in \operatorname{span}(B)$ which is a contradiction. If $\alpha_0 = 0$, then $\sum_{j=1}^{n} \alpha_j v_j = 0_V$. Since B is independent, $\alpha_j = 0$ for all $j = 1, ..., n$. Thus $B \cup \{v_0\}$ is independent, contradicting the maximality of B. Whence B is a basis for V. ☐

Theorem 1.1.3. *If* B_1 *and* B_2 *are bases for* V*, then* $\operatorname{card}(B_1) = \operatorname{card}(B_2)$.

Definition 1.1.6. If V is a vector space, its *dimension* is the cardinality of any of its bases.

Corollary 1.1.4. *If* B *is a basis for* V*, then every* $v \in V$ *can be written* $v = \sum_{j=1}^{n} \alpha_k \beta_k$*,* $\alpha_k \in F$*,* $b_k \in B$ *in a unique way.*

Theorem 1.1.5. *Let* V *be a linear space and* $B \subseteq V$ *a subset. The following are equivalent:*

(1) B *is a basis for* V*;*

(2) B *is a maximal element in* $\mathfrak{T} = \{T \subseteq V \mid T \text{ independent}\}$*;*

(3) B *is a minimal element in* $\mathfrak{S} = \{S \subseteq V \mid S \text{ spans } V\}$*;*

Definition 1.1.7. Let $\{V_i\}_{i \in I}$ be a family of vector spaces over a field F.

(1) The *product* of $\{V_i\}_{i \in I}$ is denoted:

$$\prod_{i \in I} V_i := \{(v_i)_{i \in I} \mid v_i \in V_i\}.$$

(2) The *co-product* (or *sum*) is denoted

$$\bigoplus_{i\in I} V_i := \big\{(v_i)_{i\in I} \mid v_i \in V_i,\ \operatorname{supp}\big((v_i)_{i\in I}\big) < \infty\big\}.$$

Exercise 1.1.4.

(1) Show that $\prod_{i\in I} V_i$ equipped with pointwise operations:

$$\begin{aligned}(v_i)_{i\in I} + (w_i)_{i\in I} &= (v_i + w_i)_{i\in I}\\ \alpha(v_i)_{i\in I} &= (\alpha v_i)_{i\in I}\end{aligned}$$

is a linear space.

(2) Show that $\bigoplus_{i\in I} V_i$ is a subspace of $\prod_{i\in I} V_i$.

Proposition 1.1.6. *Let V be a vector space over F and $W \subseteq V$. The (additive, abelian) quotient group V/W can be made into a vector space by defining multiplication by scalars as $\alpha(v+W) = \alpha v + W$ for all $\alpha \in F$, $v + W \in V/W$.*

Example 1.1.3.

(1) The set $F^n = \{(x_1, ..., x_n) \mid x_j \in F\}$ with component-wise operations is a vector space.

(2) The set $M_{n,m}(F) = \{(a_{ij}) \mid a_{ij} \in F\}$ with linear operations is a vector space.

(3) Let Ω be a nonempty set. Then $\mathcal{F}(\Omega, F) = \{f \mid f : \Omega \to F\}$ with pointwise operations is a vector space.

(4) The set $\ell_\infty(\Omega, F) = \{f \in \mathcal{F}(\Omega, F) \mid \|f\|_u < \infty\}$ with pointwise operations is a vector space.

Exercise 1.1.5. Show $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a subspace.

(5) Let $f : [a, b] \to \mathbf{R}$ be any function. Let $\mathcal{P} = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$. The *variation of f on $\mathcal{P}$* is defined as:

$$\operatorname{Var}(f; \mathcal{P}) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|.$$

We say f is a *bounded variation* if:

$$\operatorname{Var}(f) := \sup_{\mathcal{P}} \operatorname{Var}(f; \mathcal{P}) < \infty.$$

The set of all functions of bounded variation is defined:

$$\operatorname{BV}([a, b]) = \{f : [a, b] \to \mathbf{R} \mid \operatorname{Var}(f) < \infty\}.$$

This is a vector space by defining addition and scalar multiplication componentwise.

Exercise 1.1.6. Show that BV([a,b]) $\subseteq \ell_\infty([a,b],\mathbf{R})$ is a subspace.

(6) Let $K \subseteq V$ be a convex subset of a vector space V, that is, for all $v, w \in K$ and $t \in [0,1]$, then $(1-t)v + tw \in K$. A function $f : K \to F$ is said to be *affine* if $x, y \in K$ and $t \in [0,1]$ implies $f((1-t)x+ty) = (1-t)f(x)+tf(y)$. The set
$\mathrm{Aff}(K,F) = \{f \in \mathcal{F}(\Omega,F) \mid f \text{ affine}\}$ with pointwise operations is a vector space.

Exercise 1.1.7. Show $\mathrm{Aff}(\Omega,F) \subseteq \mathcal{F}(\Omega,F)$ is a subspace.

(7) The set $C([a,b],F) = \{f : [a,b] \to F \mid f \text{ continuous}\}$ with pointwise operations is a vector space.

Exercise 1.1.8. Explain why $C([a,b],F) \subseteq \ell_\infty([a,b],F)$ is a subspace.

(8) Consider the following sequence spaces:

- $s = \{(a_k)_k \mid a_k \in F\} = \mathcal{F}(\mathbf{N},F)$;
- $\ell_\infty = \ell_\infty(\mathbf{N},F) = \{(a_k)_k \mid \sup_{k\geqslant 1}|a_k| < \infty\}$;
- $c = \{(a_k)_k \mid (a_k)_k \text{ converges}\}$;
- $c_0 = \{(a_k)_k \mid (a_k)_k \to 0\}$;
- $c_{00} = \{(a_k)_k \mid \mathrm{supp}\big((a_k)_k\big) < \infty\}$;
- $\ell_1 = \{(a_k)_k \mid \sum_{k=1}^\infty |a_k| < \infty\}$.

These are all vector spaces with pointwise operations. In fact, $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subspaces.

Exercise 1.1.9. Show that $\ell_1 \subseteq c_0$ is a subspace.

(9) Consider the following continuous function spaces on $\mathbf{R}$:

- $C(\mathbf{R}) = \{f : \mathbf{R} \to F \mid f \text{ continuous}\}$;
- $C_b(\mathbf{R}) = C(\mathbf{R}) \cap \ell_\infty(\mathbf{R})$;
- $C_0(\mathbf{R}) = \{f \in C(\mathbf{R}) \mid \mathrm{limit}_{x\to\pm\infty} f(x) = 0\}$;
- Recall that a function is *compactly supported* if for all $\epsilon > 0$, there exists $\alpha > 0$ such that $|x| \geqslant \alpha$ implies $f(x) = 0$. The set of compactly supported functions is denoted $C_c(\mathbf{R}) = \{f \in C(\mathbf{R}) \mid f \text{ compactly supported}\}$.

These are all vector spaces with pointwise operations, and $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq C_b(\mathbf{R}) \subseteq C(\mathbf{R})$ are all subspace inclusion.

Definition 1.1.8. If V and W are linear spaces over a common field F, a map $T : V \to W$ is called *linear* if $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha \in F$.

Example 1.1.4. Let $A \in M_{m,n}(F)$. Then $T_A : F^n \to F^m$ defined by $T_A(v) = Av$ is linear. Let $\{e_1, ..., e_n\}$ be a basis for F^n. If $T : F^n \to F^m$ is linear, set:

$$[T] = \Big(T(e_1) \Big| \, T(e_2) \Big| \quad ... \quad \Big| \, T(e_n) \Big).$$

This gives $T(v) = [T]v$ for all $v \in F^n$. In fact, we also have $[T_A] = A$ and $T_{[T]} = T$.

Example 1.1.5. The *canonical projection* is linear:

$$\pi_j : \prod_{i \in I} V_i \to V_j \; \text{ defined by } \; \pi_j\big((v_i)_i\big) = v_i.$$

We also have that the *coordinate exclusions* are linear:

$$\iota_j : V_j \hookrightarrow \bigoplus_{i \in I} V_i \; \text{ defined by } \; \iota_j(v) = (v_i)_i \,, \text{ where } \; v_i = \begin{cases} 0_v, & i \neq j \\ v_j, & \text{otherwise.} \end{cases}$$

The *evaluation map* is linear as well. For $s \in S$, consider:

$$e_s : \mathcal{F}(S, F) \to F \; \text{ defined by } \; e_s(f) = f(s).$$

Proposition 1.1.7. *Let V be a vector space with basis B. Let W be a vector space and suppose $\varphi : B \to W$ is a map. Then there exists a unique linear map $T_\varphi : V \to W$ with $T_\varphi(b) = \varphi(b)$ for all $b \in B$. We have the following diagram.*

$$\begin{tikzcd} B \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & V \arrow[d, dotted, "T_\varphi"] \\ & W \end{tikzcd}$$

Proof. Define $T_\varphi : V \to W$ by:

$$\begin{aligned} T_\varphi(v) &= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\ &= \sum_{j=1}^{n} \alpha_j \varphi(b_j). \end{aligned}$$

Let $v_1, v_2 \in V$ and $c \in F$. We have that:

$$\begin{aligned}
T_\varphi(v_1 + cv_2) &= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j + c\sum_{j=1}^{n} \beta_j b_j\right) \\
&= T_\varphi\left(\sum_{j=1}^{n} (\alpha_j + c\beta_j) b_j\right) \\
&= \sum_{j=1}^{n} (\alpha_j + c\beta_j)\varphi(b_j) \\
&= \sum_{j=1}^{n} \alpha_j \varphi(b_j) + c\sum_{j=1}^{n} \beta_j \varphi(b_j) \\
&= T_\varphi(v_1) + cT_\varphi(v_2).
\end{aligned}$$

Thus T_φ is linear. Chasing the above diagram makes it clear that $T_\varphi(b) = \varphi(b)$. It remains to show that T_φ is unique. Let T be another linear transformation satisfying $T(b) = \varphi(b)$ for all $b \in B$. Then:

$$\begin{aligned}
T(v) &= T\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\
&= \sum_{j=1}^{n} \alpha_j \varphi(b_j) \\
&= T_\varphi\left(\sum_{j=1}^{n} \alpha_j b_j\right) \\
&= T_\varphi(v).
\end{aligned}$$

Thus T_φ is unique. $\square$

Proposition 1.1.8. *Let $T : V \to W$ be linear.*

(1) $\ker(T) = \{v \in V \mid T(v) = 0_W\}$ *is a linear subspace of V.*

(2) $\mathrm{im}(T) = \{T(v) \mid v \in V\}$ *is a linear subspace of W.*

(3) $\ker(T) = \{0_V\}$ *if and only if T is injective.*

(4) $\mathrm{im}(T) = W$ *if and only if T is surjective.*

Proof. (1) Let $v_1, v_2 \in \ker(T)$ and $\alpha \in F$. Observe that:

$$\begin{aligned}
T(v_1 + cv_2) &= T(v_1) + cT(v_2) \\
&= 0.
\end{aligned}$$

Thus $v_1 + cv_2 \in \ker(T)$, giving $\ker(T)$ as a linear subspace of V.

(2) Let $w_1, w_2 \in \operatorname{im}(T)$. Then there exists $v_1.v_2 \in V$ with $T(v_1) = w_1$ and $T(v_2) = w_2$. We have:

$$\begin{aligned} w_1 + cw_2 &= T(v_1) + cT(v_2) \\ &= T(v_1 + cv_2). \end{aligned}$$

Whence $w_1 + cw_2 \in \operatorname{im}(T)$, giving $\operatorname{im}(T)$ as a linear subspace of W.

(3) Let $\ker(T) = \{0\}$. Suppose $T(v_1) = T(v_2)$. Then $T(v_1) - T(v_2) = T(v_1 - v_2) = 0_W$. It must be that $v_1 - v_2 = 0_W$, giving $v_1 = v_2$. Thus T is injective. Conversely, suppose T is injective and let $v \in \ker(T)$. Then $T(v) = 0_W = T(0_V)$. Hence $v = 0_V$, establishing $\ker(T) = \{0\}$.

(4) This is by definition of surjectivity. □

Proposition 1.1.9. *If $T : V \to W$ is linear and bijective, then the inverse map $T^{-1} : W \to V$ is linear.*

Proof. We have that:

$$T(T^{-1}(w_1) + \alpha T^{-1}(w_2)) = w_1 + \alpha w_2 = T \circ T^{-1}(w_1 + \alpha w_2).$$

Applying T^{-1} to both sides gives the desired result. □

Proposition 1.1.10 (Vector Spaces are Injective)**.** *Let U, V, W be vector spaces and $0 \to U \xrightarrow{j} V$ be exact (that is, j is injective). Let $\varphi : U \to W$ be linear. There exists a linear map $\Psi : V \to W$ such that $\varphi = \Psi \circ j$; i.e., the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "j"] \arrow[d, "\varphi"'] & V \arrow[dl, dotted, "\Psi"] \\ & W & \end{tikzcd}$$

Proof. Let $\{u_i\}_{i\in I}$ be a basis for U. We must first show that $\{j(u_i)\}_{i\in I}$ is linearly independent. Notice that:

$$\begin{aligned} 0_V &= \sum_{i\in I} \alpha_i j(u_i) \\ &= j\left(\sum_{i\in I} \alpha_i u_i\right). \end{aligned}$$

By the injectivity of j, we have that $\sum_{i\in I} \alpha_i u_i = 0_U$. Thus $\alpha_i = 0$ for all $i \in I$, giving $\{j(u_i)\}_{i\in I}$ as linearly independent.

Since $\{j(u_i)\}_{i\in I}$ is linearly independent in V, we can extend it to a basis $B = \{v_i\}_{i\in J}$ where $I \subseteq J$ and $v_i = j(u_i)$ whenever $i \in I$. Now define $\psi : B \to W$

by:

$$\psi(v_i) = \begin{cases} \varphi(u_i), & i \in I \\ w, & i \in J \setminus I \ , \end{cases}$$

where $w \in W$ is arbitrary. Since this is a map of basis elements, there exists a unique linear map $\Psi : V \to W$ with $\Psi(v_i) = \psi(v_i)$ for all $v_i \in B$. We can finally see that:

$$\begin{aligned} \varphi(u_i) &= \psi(v_i) \\ &= \Psi(v_i) \\ &= \Psi(j(u_i)). \end{aligned}$$

This establishes that $\varphi = \Psi \circ j$. □

Proposition 1.1.11 (Vector Spaces are Projective)**.** *Let U, V, W be vector spaces and $V \xrightarrow{\pi} U \to 0$ be exact (that is, π is onto). Let $\varphi : W \to U$ be linear. There exists a linear map $\Psi : V \to W$ such that $\varphi = \pi \circ \Psi$; i.e., the following diagram commutes:*

$$\begin{tikzcd} & W \arrow[dl, dotted, "\Psi"'] \arrow[d, "\varphi"] & \\ V \arrow[r, "\pi"] & U \arrow[r] & 0 \end{tikzcd}$$

Proof. Let $B = \{w_i\}_{i \in I}$ be a basis for W. Define $\psi : B \to V$ by $\psi(w_i) = \pi^{-1}(\varphi(w_i))$. Since this is a map of basis elements, it extends to a unique (dependent on π^{-1}) linear map $\Psi : W \to V$ with $\Psi(w_i) = \psi(w_i)$ for all $w_i \in B$. Moreover, we have that:

$$\begin{aligned} (\pi \circ \Psi)(w_i) &= (\pi \circ \psi)(w_i) \\ &= (\pi \circ (\pi^{-1} \circ \varphi))(w_i) \\ &= \varphi(w_i). \end{aligned}$$

□

Definition 1.1.9. Let V and W be vector spaces over F. A *linear isomorphism* between V and W is a bijective linear map $T : V \to W$. If such a T exists, we say V and W are *linearly isomorphic*, and write $V \cong W$.

——————//——————

Finite dimensional vector spaces are boring. This is illustrated through the following theorem.

Theorem 1.1.12. *Let V and W be finite-dimensional vector spaces over F. Then $V \cong W$ if and only if* $\dim(V) = \dim(W)$.

Proof. Suppose $V \cong W$. Then there is an isomorphism taking basis of V to a basis of W. Therefore they have the same dimension.

Conversely, if $\dim(V) = \dim(W) = n$, then they are each isomorphic to F^n, giving that they are isomorphic to eachother. $\square$

——————//——————

Example 1.1.6. Let V be a vector space, $W \subseteq V$ a subspace. The *natural projection*:

$$\pi : V \to V/W \text{ defined by } \pi(v) = v + W$$

is a linear surjective map.

Theorem 1.1.13 (First Isomorphism Theorem for Vector Spaces)**.** *Let $T : V \to V'$ be a linear map and $W \subseteq V$ a subspace.*

(1) If T "kills" W (that is, $W \subseteq \ker(T)$), then there exists a linear map $\widetilde{T} : V/W \to V'$ with $\widetilde{T} \circ \pi = T$; i.e., the following diagram commutes.

$$\begin{tikzcd} V \arrow[rr, "T"] \arrow[dr, "\pi"'] & & V' \\ & V/W \arrow[ur, dotted, "\widetilde{T}"'] & \end{tikzcd}$$

(2) If $\ker(T) = W$, then $\widetilde{T}$ is injective.

(3) If $\ker(T) = W$ and $\operatorname{im}(T) = V'$, then $V/W \cong V'$.

Proof. (1) As stipulated, define $\widetilde{T}(v + W) = T(v)$. We must show that $\widetilde{T}$ is well-defined: suppose $v_1 + W = v_2 + W$ for some $v_1, v_2 \in V$. Then $v_1 = v_2 + w$ for some $w \in W$. This gives:

$$\begin{aligned} \widetilde{T}(v_1 + W) &= \widetilde{T}(v_2 + w + W) \\ &= \widetilde{T}(v_2 + W). \end{aligned}$$

Whence $\widetilde{T}$ is well-defined. Now given $v_1 + W, v_2 + W \in V/W$ and $\alpha \in F$, observe that:

$$\begin{aligned} \widetilde{T}\big((v_1 + W) + c(v_2 + W)\big) &= \widetilde{T}\big((v_1 + cv_2) + W\big) \\ &= T(v_1 + cv_2) \\ &= T(v_1) + cT(v_2) \\ &= \widetilde{T}(v_1 + W) + c\widetilde{T}(v_2 + W). \end{aligned}$$

Thus $\widetilde{T}$ is linear.

(2) If $\ker(T) = W$, then:

$$\begin{aligned}
\ker(\widetilde{T}) &= \{v + W \mid \widetilde{T}(v + W) = 0_{V'}\} \\
&= \{v + W \mid T(v) = 0_{V'}\} \\
&= \{v + W \mid v \in \ker(T)\} \\
&= \{v + W \mid v \in W\} \\
&= \{0\}.
\end{aligned}$$

Thus $\widetilde{T}$ is injective.

(3) It remains to show that $\mathrm{im}(T) = V'$ implies $\widetilde{T}$ is surjective. Observe that:

$$\begin{aligned}
\mathrm{im}(\widetilde{T}) &= \{\widetilde{T}(v + W) \mid v + W \in V/W\} \\
&= \{\widetilde{T}(\pi(v)) \mid v \in V\} \\
&= \{T(v) \mid v \in V\} \\
&= \mathrm{im}(T) \\
&= V'.
\end{aligned}$$

Thus $\widetilde{T}$ is surjective, which establishes it as a bijection. This gives $V/W \cong V'$. $\square$

Definition 1.1.10. Let S be a nonempty set. The *free vector space* of S is:

$$\mathbf{F}(S) = \{f : S \to F \mid \mathrm{supp}(f) < \infty\}.$$

Exercise 1.1.10. Show $\mathbf{F}(S) \subseteq \mathcal{F}(S, F)$ is a subspace.

Proposition 1.1.14. *The set $\{\delta_s \mid s \in S\}$ is a basis for $\mathbf{F}(S)$, where $\delta_s : S \to F$ is defined by:*

$$\delta_s(t) = \begin{cases} 1, & t = 0 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. If $f \in \mathbf{F}(S)$ with $\mathrm{supp}(f) = \{s_1, ..., s_n\}$, then $f = \sum_{k=1}^{n} f(s_k)\delta_{s_k}$. If $\sum_{k=1}^{n} \alpha_k \delta_{s_k} = 0$, then for $j = 1, ..., n$ we have $0 = \left(\sum_{k=1}^{n} \alpha_k \delta_{s_k}\right)(s_j) = \alpha_j$. $\square$

Theorem 1.1.15. *Given any vector space V and a map (of sets) $\varphi : S \to V$, there exists a unique linear map $T_\varphi : \mathbf{F}(S) \to V$ with $T_\varphi \circ \iota = \varphi$, where $\iota : S \to \mathbf{F}(S)$ is defined by $\iota(s) = \delta_s$ for all $s \in S$. The following diagram commutes:*

$$\begin{tikzcd}
S \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathbf{F}(S) \arrow[d, dotted, "T_\varphi"] \\
& V
\end{tikzcd}$$

Proof. By the previous proposition, we have that $B = \{\delta_s \mid s \in S\}$ is a basis for $\mathbf{F}(S)$. Define $T : B \to V$ by $T(\delta_s) = \varphi(s)$. Since this is a map of basis elements, there exists a unique linear map $T_\varphi : \mathbf{F}(S) \to V$ with $T_\varphi(\delta_s) = T(\delta_s)$ for all $\delta_s \in B$. The diagram commutes because:

$$\begin{aligned}\varphi(s) &= T(\delta_s)\\ &= T_\varphi(\delta_s)\\ &= T_\varphi(\iota(s)).\end{aligned}$$

Moreover, if T' satisfies $\varphi = T' \circ \iota$, then:

$$\begin{aligned}T'(\delta_s) &= T'(\iota(s))\\ &= \varphi(s)\\ &= T_\varphi(\iota(s))\\ &= T_\varphi(\delta_s).\end{aligned}$$

Thus T_φ is unique. □

Definition 1.1.11. Let V and W be vector spaces. The set of linear transformations between V and W is $\mathcal{L}(V, W) = \{T \mid T : V \to W \text{ linear}\}$. The set of linear functionals is $V' := \mathcal{L}(V, F)$.

Exercise 1.1.11. Show $\mathcal{L}(V, W)$ is a vector space.

Exercise 1.1.12. Show $M_{m,n}(F) \cong \mathcal{L}(F^m, F^n)$ by $a \mapsto T_a : (v \mapsto av)$.

§ 1.2. Algebras

Definition 1.2.1. An *algebra* over F is a linear space A over F equipped with a multiplication operation:

$$A \times A \to A \;\text{ defined by }\; (a, b) \mapsto ab$$

satisfying:

(1) $(ab)c = a(bc)$ for all $a, b, c \in A$;

(2) $(\alpha a)b = \alpha(ab) = a(\alpha b)$ for all $a, b \in A$, $\alpha \in F$;

(3) $a(b + c) = ab + ac$ for all $a, b, c \in A$;

(4) $(a + b)c = ac + bc$ for all $a, b, c \in A$.

If $ab = ba$ for all $a, b \in A$ we say that A is *commutative*. If there exists $1_A \in A$ with $1_A a = a 1_A = a$ for all $a \in A$ we say A is *unital*.

Example 1.2.1.

(1) $M_n(F)$ is a noncommutative unital algebra over F under the usual matrix multiplication.

(2) If V is a vector space over F, $\mathcal{L}(V)$ is a unital algebra over F. It is noncommutative provided $\dim(V) > 1$.

(3) $\mathcal{F}(S, F)$ is a unital commutative algebra over F.

Definition 1.2.2. Let B be a (unital) algebra over F.

(1) A (unital) *subalgebra* of B is a subspace $A \subseteq B$ ($1_B \in A$) satisfying the property that if $a, a' \in A$, then $aa' \in A$.

(2) An *ideal* of B is a subspace $I \subseteq B$ with $b \in B$, $a \in I$ implying $ba, ab \in I$.

Example 1.2.2.

(1) $\ell_\infty(\Omega, F) \subseteq \mathcal{F}(\Omega, F)$ is a unital subalgebra.

(2) $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq s$ are all subalgebras. In particular, $c_0 \subseteq \ell_\infty$ and $c_{00} \subseteq s$ are ideals.

(3) $C\big([a,b]\big) \subseteq \ell_\infty\big([a,b]\big)$ is a unital subalgebra.

(4) $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq C_b(\mathbf{R}) \subseteq \ell_\infty(\mathbf{R})$ are all subalgebras. In fact, $C_b(\mathbf{R}) \subseteq C(\mathbf{R})$ and $C_b(\mathbf{R}) \subseteq \ell_\infty(\mathbf{R})$ are unital, whereas $C_0(\mathbf{R}) \subseteq C_b(\mathbf{R})$ and $C_c(\mathbf{R}) \subseteq C(\mathbf{R})$ are ideals.

(5) The set $T_n(F) = \{(a_{ij}) \in M_n(F) \mid a_{ij} = 0, i > j\}$ is a unital subalgebra of $M_n(F)$.

Example 1.2.3 (Group Algebra)**.** Let Γ denote a group (not necessarily abelian). Take the free vector space $\mathbf{F}(\Gamma)$ and define multiplication as *convolution*: given $f, g \in \mathbf{F}(\Gamma)$ let:

$$(f * g)(r) = \sum_{\substack{\big\{(s,t)\,\mid \\ s\in\mathrm{supp}(f),\\ t\in\mathrm{supp}(g),\\ st=r\big\}}} f(s)g(t).$$

Since $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are finite, this is a finite sum. We often suppress this notation and write $(f * g)(r) = \sum_{st=r} f(s)g(t)$.

We can also make substitutions:

$$\begin{aligned}(f * g)(r) &= \sum_{st=r} f(s)g(t)\\ &= \sum_{t\in\Gamma} f(rt^{-1})g(t)\\ &= \sum_{s\in\Gamma} f(s)g(s^{-1}r).\end{aligned}$$

It is clear that:

$$\begin{aligned}(f+g)*h &= f*h+g*h\\ g*(g+h) &= f*g+f*h\\ \alpha(f*g) &= (\alpha f)*g = f*(\alpha g)\end{aligned}$$

for $f,g,h \in \mathbf{F}(\Gamma)$, $\alpha \in F$. Associativity can be similarly shown using the above definition. Rather, we will prove associativity by first show that $\delta_s * \delta_t = \delta_{st}$. Given:

$$(\delta_s * \delta_t)(r) = \sum_{q\in\Gamma} \delta_s(rq^{-1})\delta_t(q),$$

notice that:

$$\delta_s(rt^{-1}) = \begin{cases} 1, & s = rt^{-1} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & r = st \\ 0, & \text{otherwise} \end{cases} = \delta_{st}(r).$$

Since $\{\delta_t \mid t \in \Gamma\}$ is a basis for $\mathbf{F}(\Gamma)$, every $f \in \mathbf{F}(\Gamma)$ looks like:

$$f = \sum_{t\in J} \alpha_t\delta_t,\ J \subseteq T \text{ finite.}$$

Using distributivity we get:

$$\begin{aligned}\delta_r * (\delta_s * \delta_t) &= \delta_r * \delta_{st}\\ &= \delta_{rst}\\ &= \delta_{rs} * \delta_t\\ &= (\delta_r * \delta_s) * \delta_t.\end{aligned}$$

Whence convolution is associative.

Exercise 1.2.1. Let $\{A_i\}_{i\in I}$ be a family of algebras over F.

(1) $\prod_{i\in I} A_i$ is an algebra under $(a_i)_i(b_i)_i = (a_ib_i)_i$.

(2) $\bigoplus_{i\in I} A_i \subseteq \prod_{i\in I} A_i$ is an ideal.

Exercise 1.2.2. Let A be an algebra over F and $I \subseteq A$ an ideal. Then A/I is an algebra under $(a+I)(b+I) = ab + I$.

§ 1.3. Normed Vector Spaces

To each vector v in a vector space V, we want to assign a "length", denoted $\|v\|$.

Definition 1.3.1. A *norm* on a vector space V is a map:

$$\|\cdot\| : V \to [0,\infty),\ v \mapsto \|v\|$$

satisfying:

(1) $\|\alpha v\| = |\alpha| \|v\|$ for all $\alpha \in F$, $v \in V$ (homogeneity);

(2) $\|v + w\| \leqslant \|v\| + \|w\|$ (triangle inequality);

(3) If $\|v\| = 0$, then $v = 0_V$ (positive-definite).

If $\|\cdot\|$ satisfies (1) and (2), it is called a *seminorm*. The pair $(V, \|\cdot\|)$ is called a *normed space*.

Definition 1.3.2. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space V are called *equivalent* if there exists $c_1 \geqslant 0$ and $c_2 \geqslant 0$ with $\|v\| \leqslant c_1 \|v\|'$ and $\|v\|' \leqslant c_2 \|v\|$ for all $v \in V$.

Exercise 1.3.1. If p is a seminorm on V, show that $|p(v) - p(w)| \leqslant p(v - w)$.

Definition 1.3.3. Let $(V, \|v\|)$ be a normed space.

(1) The *closed unit ball* is denoted $B_V = \{v \in V \mid \|v\| \leqslant 1\}$.

(2) The *open unit ball* is denoted $U_V = \{v \in V \mid \|v\| < 1\}$.

(3) The *unit sphere* is denoted $S_V = \{v \in V \mid \|v\| = 1\}$.

Example 1.3.1. Let $V = F^n$ and $x = (x_1, ..., x_n)$. We define:

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|;$$

$$\|x\|_\infty = \max_{j=1}^{n} |x_j|;$$

$$\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}.$$

For $p \geqslant 1$:

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}.$$

Exercise 1.3.2. Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms

——————//——————

We aim to show that $\|\cdot\|_p$ is a norm for $p \in [0, \infty]$.

Lemma 1.3.1. *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [0, \infty) \to \mathbf{R}$ be defined by $f(t) = \frac{1}{p}t^p - t + \frac{1}{q}$. Then $f(t) \geqslant 0$ for $t \geqslant 0$.*

Proof. Note that $f'(t) = t^{p-1} - 1$. Since:

$$\begin{aligned} &f'(1) = 0 \\ &f'(t) > 0 \text{ for } t > 1 \\ &f'(t) < 0 \text{ for } 0 \leqslant t < 1, \end{aligned}$$

we can see that $f(t) \geqslant 0$ for all $t \geqslant 0$. □

Lemma 1.3.2 (Young's Inequality)**.** *Let $p, q \in [0, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $x, y \geqslant 0$, then $xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q$.*

Proof. By Lemma 1.3.1, $t \leqslant \frac{1}{p}t^p + \frac{1}{q}$. Multiplying both sides by y^q gives:

$$ty^q \leqslant \frac{1}{p}t^p y^q + \frac{1}{q}y^q.$$

Let $t = xy^{1-q}$. Then:

$$xy^{1-q}y^q \leqslant \frac{1}{p}x^p y^{p-pq} y^q + \frac{1}{q}y^q.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have that $p - pq = -q$. Whence:

$$xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q.$$

□

Lemma 1.3.3 (Hölders Inequality)**.** *Let $p, q \in [0, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for $x, y \in F^n$:*

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leqslant \|x\|_p \|y\|_q.$$

Proof. We proceed by cases.

Case 1: $p = 1$. Then:

$$\begin{aligned} \left| \sum_{j=1}^{n} x_j y_j \right| &\leqslant \sum_{j=1}^{n} |x_j||y_j| \\ &\leqslant \sum_{j=1}^{n} |x_j| \|y\|_\infty \\ &= \|x\|_1 \|y\|_\infty. \end{aligned}$$

Case 2: $p = \infty$. This follows similarly to Case 1.

Case 3: $1 < p < \infty$. Suppose $\|x\|_p = \|y\|_q = 1$. Then:

$$\begin{aligned}
\left|\sum_{j=1}^n x_j y_j\right| &\leqslant \sum_{j=1}^n |x_j||y_j| \\
&\leqslant \sum_{j=1}^n \left(\frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q\right) \\
&= \frac{1}{p}\left(\sum_{j=1}^n |x_j|^p\right) + \frac{1}{q}\left(\sum_{j=1}^n |y_j|^q\right) \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

Whence the inequality holds. Now suppose $\|x\|_p = 0$ or $\|y\|_q = 0$. Then $x = 0_{F^n}$ or $y = 0_{F^n}$, whence the inequality holds. Suppose $\|x\|_p \neq 0$ and $\|y\|_p \neq 0$. Set:

$$\begin{aligned}
x' &= \frac{x}{\|x\|_p} \\
y' &= \frac{y}{\|y\|_p}.
\end{aligned}$$

Then $\|x'\|_p = 1 = \|y'\|_p$. Observe that:

$$\begin{aligned}
1 &\geqslant \left|\sum_{j=1}^n x'_j y'_j\right| \\
&= \left|\sum_{j=1}^n \frac{x}{\|x\|_p}\frac{y}{\|y\|_p}\right|.
\end{aligned}$$

Multiplying both sides by $\|x\|_p\|y\|_q$ gives the desired result. $\square$

Lemma 1.3.4 (Minkowski's Inequality). *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $x, y \in F^n$:*

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p.$$

Proof. The only nontrivial case is for $1 < p < \infty$. Observe that:

$$\begin{aligned}
(\|x+y\|_p)^p &= \sum_{j=1}^{n} |x_j + y_j|^p \\
&= \sum_{j=1}^{n} |x_j + y_j||x_j + y_j|^{p-1} \\
&\leqslant \sum_{j=1}^{n} |x_j||x_j + y_j|^{p-1} + |y_j||x_j + y_j|^{p-1} \\
&\leqslant \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{p-1\left(\frac{p}{p-1}\right)}\right)^{1-\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{p-1\left(\frac{p}{p-1}\right)}\right)^{1-\frac{1}{p}} \\
&= \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{p}\right)^{1-\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j + y_j|^{p}\right)^{1-\frac{1}{p}} \\
&= (\|x\|_p + \|y\|_p)\frac{\|x+y\|_p^p}{\|x+y\|_p}.
\end{aligned}$$

Multiplying boths sides by $\frac{\|x+y\|_p}{\|x+y\|_p^p}$ gives the desired inequality. □

Theorem 1.3.5. *Let $V = F^n$. Then $(F^n, \|\cdot\|_p)$ is a normed space.*

Proof. Let $x = (x_1, ..., x_n) \in F^n$ and $\alpha \in F$. Observe that:

$$\begin{aligned}
\|\alpha x\|_p &= \left(\sum_{j=1}^{n} |\alpha x_j|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{j=1}^{n} |\alpha|^p |x_j|^p\right)^{\frac{1}{p}} \\
&= |\alpha|\|x\|_p.
\end{aligned}$$

This satisfies homogeneity. Moreover, Minkowski's Inequality satisifes the triangle inequality. It remains to show that $\|\cdot\|_p$ is positive-definite. If $\|x\|_p = 0$, then $x_j = 0$ for all $1 \leqslant j \leqslant n$. Thus $x = 0_{F^n}$. □

Corollary 1.3.6. *Let $p \in [1, \infty]$. Then $\ell_p = \left\{(a_k)_k \mid \sum_{k=1}^{\infty} |a_k|^p < \infty\right\}$ with norm $\|(a_k)_k\|_p = (\sum_{k=1}^{\infty} |a_k|^p)^{\frac{1}{p}}$ is a normed space.*

Proof. Homogeneity and positive-definiteness are trivial to prove. Let $(x_k)_k, (y_k)_k \in \ell_p$. It is clear that:

$$\begin{aligned}\left(\sum_{k=1}^{n}|x_k+y_k|^p\right)^{\frac{1}{p}} &\leqslant \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}|y_k|^p\right)^{\frac{1}{p}} \\ &\leqslant \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}|y_k|^p\right)^{\frac{1}{p}} \\ &= \|(x_k)_k\|_p + \|(y_k)_k\|_p .\end{aligned}$$

We have that $\sum_{k=1}^{n}|x_k+y_k|^p$ is increasing and bounded above by $(\|(x_k)_k\|_p + \|(y_k)_k\|_p)^p$. By the Monotone Convergence Theorem $\text{limit}_{n\to\infty}\sum_{k=1}^{n}|x_k+y_k|^p = \sum_{k=1}^{\infty}|x_k+y_k|^p$ exists. Whence $(\sum_{k=1}^{\infty}|x_k+y_k|^p)^{\frac{1}{p}} = \|(x_k)_k+(y_k)_k\|_p \leqslant \|(x_k)_k\|_p + \|(y_k)_k\|_p$ □

Theorem 1.3.7. *All p-norms on ℓ_p^n are equivalent for $1 \leqslant p \leqslant \infty$.*

Proof. Let $x \in \ell_p^n$. We have that:

$$\|x\|_p = \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n}\left(\max_{i=1}^{n}|x_i|\right)^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n}\|x\|_\infty^p\right)^{\frac{1}{p}} = n^p\|x\|_\infty .$$

$$\|x\|_\infty = \left(\left(\max_{i=1}^{n}|x_i|\right)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} = \|x\|_p .$$

$$\|x\|_\infty = \max_{i=1}^{n}|x_i| \leqslant \sum_{i=1}^{n}|x_i| = \|x_i\|_1 .$$

$$\|x\|_1 = \sum_{i=1}^{n}|x_i| \leqslant \sum_{i=1}^{n}\max_{i=1}^{n}|x_i| = n\max_{i=1}^{n}|x_i| = n\|x\|_\infty .$$

By transitivity, all norms on ℓ_p^n are equivalent. □

——————//——————

Example 1.3.2.

(1) $\left(\ell_\infty(\Omega,F), \|\cdot\|_u\right)$ where $\|f\|_u = \sup_{x\in\Omega}|f(x)|$ is a normed space. This includes its subspaces, such as $C\big([a,b],F\big) \subseteq \ell_\infty\big([a,b],F\big)$ and $C_c(\mathbf{R}) \subseteq C_0(\mathbf{R}) \subseteq \ell_\infty(\mathbf{R})$, all with $\|\cdot\|_u$.

(2) Take $\Omega = \mathbf{N}$ in the previous example. Then $\big(\ell_\infty, \|\cdot\|_\infty\big)$ is a normed space. This includes its subspaces $c_{00} \subseteq c_0 \subseteq \ell_\infty$ with $\|\cdot\|_\infty$.

(3) $\big(\ell_1, \|\cdot\|_1\big)$ is a normed space.

(4) $\big(C\big([a,b]\big), \|\cdot\|_1\big)$ with $\|f\|_1 = \int_a^b |f(t)|dt$ is a normed space.

(5) $\big(\mathrm{BV}\big([a,b]\big), \|\cdot\|_{\mathrm{BV}}\big)$ where $\|f\|_{\mathrm{BV}} = |f(a)| + \mathrm{Var}(f)$ is a normed space.

(6) Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Then $\big(B(V,W), \|\cdot\|_{\mathrm{op}}\big)$ is a normed space, where $B(V,W) = \{T \in \mathcal{L}(V,W) \mid \|T\|_{\mathrm{op}} < \infty\}$ is the set of bounded linear maps and $\|T\|_{\mathrm{op}} = \sup_{v \in B_V} \|T(v)\|_W$. Intuitively, $\|T\|_{\mathrm{op}}$ measures the radius of the smallest ball which contains B_V.

Exercise 1.3.3. Show that $V^* := B(V,F)$ is a subspace of V'.

(7) Let S be a nonempty set. Both $\big(\mathbf{F}(S), \|\cdot\|_1\big)$ and $\big(\mathbf{F}(S), \|\cdot\|_p\big)$ are normed spaces, where $\|f\|_1 = \sum_{s \in S} |f(s)|$ and $\|f\|_p = \big(\sum_{s \in S} |f(s)|^p\big)^{\frac{1}{p}}$. Note that since $f(s) \neq 0$ for finitely many $s \in S$, both $\|\cdot\|_1$ and $\|\cdot\|_p$ are well-defined.

Exercise 1.3.4. Show that $\|f\|_\infty := \sup_{s \in S} |f(s)|$ is a norm on $\mathbf{F}(S)$.

§ 1.4. Inner Product Spaces

Definition 1.4.1. Let V be a vector space over F and $\varphi : V \times V \to F$ a map.

(1) The map φ is said to be a *bilinear form* if is is linear in the first and second variable seperately; i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have:

  (i) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$

  (ii) $\varphi(v, cv_1 + v_2) = c\varphi(v, v_1) + \varphi(v, v_2)$.

(2) The map φ is said to be a *sesquilinear form* if it is linear in the first variable and conjugate linear in the second variable; i.e., for all $v_1, v_2, v \in V$ and $c \in F$ we have:

  (i) $\varphi(cv_1 + v_2, v) = c\varphi(v_1, v) + \varphi(v_2, v)$

  (ii) $\varphi(v, cv_1 + v_2) = \bar{c}\varphi(v, v_1) + \varphi(v, v_2)$.

If we wish to keep track of a bilinear form on V we write (V, φ).

Definition 1.4.2. Let V be a vector space over F.

(1) A bilinear form φ on V is said to be *symmetric* if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

(2) A sesquilinear form φ on V is said to be *Hermitian* if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.

Definition 1.4.3. Let (V, φ) be a vector space over F such that if φ is symmetric, then $F = \mathbf{R}$ or if φ is Hermitian, then $F = \mathbf{C}$. We say φ is *positive-definite* if for all nonzero $v \in V$ we have $\varphi(v, v) \neq 0$.

Definition 1.4.4. Let (V, φ) be a vector space over $\mathbf{R}$ with φ a positive-definite symmetric bilinear form or over $\mathbf{C}$ with φ a positive-definite Hermitian sesquilinear form. Then we say φ is an *inner product* on V and write φ as $\langle \cdot, \cdot \rangle$. We say $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space.*

———————//———————

Definition 1.4.5. If V is an inner product space we define $\|v\|_2 = \langle v, v \rangle^{\frac{1}{2}}$.

Definition 1.4.6. Let V be an inner product space. Two vectors $v, w \in V$ are *orthogonal* if $\langle u, v \rangle = 0$. We denote this as $u \perp v$.

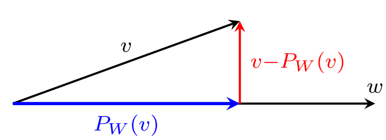
Theorem 1.4.1 (Pythagorean Theorem)**.** *Let $v_1, ..., v_n$ be mutually orthogonal. Then $\sum_{j=1}^{n} \|v_j\|_2^2 = \left\|\sum_{j=1}^{n} v_j\right\|_2^2$.*

Proof. Because $v_i \perp v_j$ for $1 \leqslant i, j \leqslant n$, we have $\langle v_i, v_j \rangle = 0$. Observe that:

$$\begin{aligned}
\left\| \sum_{j=1}^{n} v_j \right\|_2^2 &= \left\langle \sum_{j=1}^{n} v_j, \sum_{j=1}^{n} v_j \right\rangle \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \langle v_j, v_i \rangle \right) \\
&= \sum_{j=1}^{n} \langle v_j, v_j \rangle \\
&= \sum_{j=1}^{n} \|v_j\|_2^2 .
\end{aligned}$$

□

Definition 1.4.7. Let V be an inner product space and $w \in V$ nonzero. The *projection* of a vector $v \in V$ onto w is a map $P_W : V \to V$ defined by $P_W(v) = \frac{\langle v,w \rangle}{\langle w,w \rangle} w$.



Proposition 1.4.2. **** Let V be an inner product space and $w \in V$ a nonzero vector. Then $P_w(v) \perp v - P_w(v)$.*

Proof. □

Corollary 1.4.3. **** Let V be an inner product space and $w \in W$ a nonzero vector. Then $\|v\|_2^2 = \|P_w(v)\|_2^2 + \|v - P_w(v)\|_2^2$.*

Proof. □

Lemma 1.4.4 (Cauchy-Schwartz Inequality)**.** *Let V be an inner product space and $v, w \in V$. Then $|\langle v, w \rangle| \leqslant \|v\|_2 \|w\|_2$.*

Proof. The previous corollary gives $\|v\|_2 \geqslant \|P_w(v)\|_2$. We have that:

$$\begin{aligned}\|v\|_2 &\geqslant \left\|\frac{\langle v,w\rangle}{\langle w,w\rangle}w\right\|_2\\ &= \frac{|\langle v,w\rangle|}{\|w\|_2^2}\|w\|_2\\ &= \frac{\langle v,w\rangle}{\|w\|_2}.\end{aligned}$$

Multiplying both sides by $\|w\|_2$ gives the desired result. □

Theorem 1.4.5. *Let V be an inner product space. Then $(V, \|\cdot\|_2)$ is a normed space.*

Proof. Let $v, w \in V$ and $\alpha \in F$. We have that:

$$\begin{aligned}\|\alpha v\|_2 &= \langle \alpha v, \alpha v\rangle^{\frac{1}{2}}\\ &= (\alpha\overline{\alpha}\langle v,v\rangle)^{\frac{1}{2}}\\ &= \left(|\alpha|^2\langle v,v\rangle\right)^{\frac{1}{2}}\\ &= |\alpha|\,\|v\|_2\,.\end{aligned}$$

Thus $\|\cdot\|_2$ satisfies homogeneity. It follows from the Cauchy-Schwartz Inequality that:

$$\begin{aligned}\|v+w\|_2^2 &= \langle v+w, v+w\rangle\\ &= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle\\ &= \|v\|_2^2 + \langle v,w\rangle + \overline{\langle v,w\rangle} + \|w\|_2^2\\ &= \|v\|_2^2 + 2\Re\left(\langle v,w\rangle\right) + \|w\|_2^2\\ &\leqslant \|v\|_2^2 + 2|\langle v,w\rangle| + \|w\|_2^2\\ &\leqslant \|v\|_2^2 + 2\,\|v\|_2\,\|w\|_2 + \|w\|_2^2\\ &= (\|v\|_2 + \|w\|_2)^2\,,\end{aligned}$$

where we used the fact that $2\Re(\langle v,w\rangle) = 2|\langle v,w\rangle|$. Squaring both sides proves that $\|\cdot\|_2$ satisfies the triangle inequality. It remains to show positive-definiteness. Suppose $\|v\|_2 = 0$. Then $\langle v,v\rangle = 0$, but since the inner-product is by definition positive-definite, we get that $v = 0_V$. □

——————//——————

Example 1.4.1.

(1) $\ell_2^n = F^n$ is an inner product space where $\langle(x_1, ..., x_n), (y_1, ..., y_n)\rangle := \sum_{j=1}^n x_j\overline{y_j}$.

(2) ℓ_2 is an inner product space where $\langle (a_k)_k, (b_k)_k \rangle := \sum_{k=1}^{\infty} a_k \overline{b_k}$. Note that:

$$\begin{aligned}\sum_{k=1}^{n} \left|a_k \overline{b_k}\right| &= \sum_{k=1}^{n} |a_k||b_k| \\ &\leqslant \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |b_k|^2\right)^{\frac{1}{2}} \\ &= \|(a_k)_k\|_2 \, \|(b_k)_k\|_2 \\ &< \infty \qquad \text{(Because } (a_k)_k, (b_k)_k \in \ell_2\text{)}.\end{aligned}$$

Since $\sum_{k=1}^{n} |a_k \overline{b_k}|$ is increasing and bounded above, the Monotone Convergence Theorem says $\sum_{k=1}^{\infty} |a_k \overline{b_k}|$ exists and is finite. Whence $\langle (a_k)_k, (b_k)_k \rangle$ converges.

(3) Recall that $\mathrm{Tr} : M_n(\mathbf{C}) \to \mathbf{C}$ is defined by $\mathrm{Tr}(a_{ij}) = \sum_{i=1}^{n} a_{ii}$. Then $M_n(\mathbf{C})$ is an inner product space where $\langle a_{ij}, b_{ij} \rangle := \mathrm{Tr}(b_{ij}^* a_{ij})$.

(4) $C([0,1])$ is an inner product space where $\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}dx$.

§ 1.5. Normed Algebras

Definition 1.5.1. A *normed algebra* is an algebra A equipped with a norm $\|\cdot\|_A$ such that $\|ab\|_A \leqslant \|a\|_A \|b\|_A$. If A is unital, we require $\|1\|_A = 1$.

Example 1.5.1.

(1) $\ell_\infty(\Omega)$ equipped with $\|\cdot\|_u$ is a normed algebra.

(2) $C_c(\mathbf{R})$, $C_0(\mathbf{R})$, and $C([0,1])$ are all normed algebras when equipped with $\|\cdot\|_u$.

(3) $M_n(F)$ equipped with $\|\cdot\|_{\mathrm{op}}$ is a normed algebra.

(4) If V is a normed space, then $B(V,V)$ with $\|\cdot\|_{\mathrm{op}}$ is a normed algebra: for $T, S \in B(V,V)$ and $v \in B_V$, we have that

$$\begin{aligned}\|(T \circ S)(v)\| &\leqslant \|T\|_{\mathrm{op}} \|S(v)\| \\ &\leqslant \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}.\end{aligned}$$

Taking the supremum over all $v \in B_V$ gives $\|T \circ S\|_{\mathrm{op}} \leqslant \|T\|_{\mathrm{op}} \|S\|_{\mathrm{op}}$.

(5) Let S be a group. Equip the algebra $\mathbf{F}(S)$ with $\|\cdot\|_1$. We get a normed algebra.

Exercise 1.5.1. For $a, b \in \ell_1(\mathbf{Z})$, define $a * b : \mathbf{Z} \to F$ by $(a*b)(n) = \sum_{k \in \mathbf{Z}} a(n-k)b(k)$. Show that $\ell_1(\mathbf{Z})$ with this multiplication is a normed algbera.

Chapter 2

Metric Spaces

§ 2.1. Basic Definitions and Examples

Definition 2.1.1. A *metric* on a nonempty set X is a map

$$d : X \times X \to [0, \infty)$$

satisfying for all $x, y, z \in X$:

(1) $d(x, y) = d(y, x)$ (symmetry);

(2) $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangle inequality);

(3) $d(x, x) = 0$;

(4) if $d(x, y) = 0$ then $x = y$ (positivity).

If d satisfies all but (iv), then d is called a *semi-metric*. The pair (X, d) is called a *metric space*.

Definition 2.1.2. Two metrics d, ρ on X are called *equivalent* if there exists constants c, c' with $d(x, y) \leqslant c\rho(x, y)$ and $\rho(x, y) \leqslant c'd(x, y)$.

Definition 2.1.3. A family of metrics $\{d_k\}_{k=1}^{\infty}$ on X is *uniformly bounded* if, for all $x, y \in X$ and $k \in \mathbf{N}$, we have $d_k(x, y) \leqslant C$.

Example 2.1.1.

(1) The *discrete metric* on $X \neq \emptyset$ is:

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

(2) The *hamming distance* between two bit strings of equal length: given $X = \{0, 1\}^n$, then $d_H : X \times X \to [0, \infty)$ is defined by $d_H\big((x_j)_{n\geqslant 1}, (y_j)_{n\geqslant 1}\big) = |\{j \mid x_j \neq y_j\}|$.

(3) If $(V, \|\cdot\|)$ is any normed space, then $d(v, w) = \|v - w\|$ is a metric on V.

Exercise 2.1.1. If $\|\cdot\|$ and $\|\cdot\|'$ are norms on a linear space V, show they are equivalent if and only if their induced metrics are equivalent.

(4) If (X, d) is a metric and $Y \subseteq X$ is a subset, then (Y, d) is a metric space.

(5) Let (X, ρ) be a metric space. Fix $p \in X$. Then:

$$d(x,y) := \begin{cases} 0, & x = y \\ \rho(x,p) + \rho(p,y), & x \neq y \end{cases}$$

is a metric.

(6) It is often beneficial to work with metrics that are bounded. Let ρ be a (semi)-metric of X. Set:

$$d(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}.$$

Defining $d(x,y)$ as above gives $0 \leqslant d(x,y) \leqslant 1$. Although d and ρ are not equivalent metrics, they are topologically equivalent.

Clearly d is symmetric and $d(x,x) = 0$. Moreover, if $d(x,y) = 0$, then $\rho(x,y) = 0$, giving $x = y$ if ρ is a metric. For the triangle inequality, consider the function $g : [0,\infty) \to [0,1)$ given by $g(t) = \frac{t}{1+t}$. We have that $g'(t) = \frac{(1+t)-t}{(1+t)^2} = \frac{1}{(1+t)^2} > 0$, whence g is strictly increasing. Since we know $\rho(x,z) \leqslant \rho(x,y) + \rho(y,z)$, observe that:

$$\begin{aligned} d(x,z) &= \frac{\rho(x,z)}{1+\rho(x,z)} \\ &\leqslant \frac{\rho(x,y)+\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\ &= \frac{\rho(x,y)}{1+\rho(x,y)+\rho(y,z)} + \frac{\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\ &\leqslant \frac{\rho(x,y)}{1+\rho(x,y)} + \frac{\rho(y,z)}{1+\rho(y,z)} \\ &= d(x,y) + d(y,z). \end{aligned}$$

(7) If $d_1, ..., d_n$ are metrics on X and $c_1, ..., c_n > 0$, then:

$$d(x,y) = \sum_{i=1}^{n} c_i d_i(x,y)$$

is a metric on X.

(8) Let $\{\rho_k\}_{k=1}^{\infty}$ be a family of semi-metrics on X. Assume that the family is *separating*: if $x, y \in X$ and $x \neq y$, then there exists k such that $\rho_k(x, y) \neq 0$. Let $d_k(x, y) = \frac{\rho_k(x,y)}{1+\rho_k(x,y)}$. Then:

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x, y)$$

is a metric on X. Since $0 \leqslant d_k(x, y) \leqslant 1$, by comparison $d(x, y)$ will converge.

(9) Let $(X_k, \rho_k)_{k \geqslant 1}$ be a sequence of metric spaces. For each k let d_k be as above. Let $X = \prod_{k=1}^{\infty} X_k$. Then the map $D : X \times X \to [0, \infty)$ defined by

$$D(f, g) = \sum_{k=1}^{\infty} 2^{-k} d_k(f(k), g(k))$$

is a metric on X. Note that we need not make d_k from ρ_k if all the d_k are uniformly bounded.

(10) Let $X = \{0, 2\}$ with the discrete metric. The *abstract Cantor set* is $\Delta := \prod_{k \in \mathbf{N}} X$. Then the map $D : \Delta \times \Delta \to [0, \infty)$ defined by

$$D(f, g) = \sum_{k=1}^{\infty} 3^{-k} |f(k) - g(k)|$$

is a metric on Δ.

(11) Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbf{R}^3$ (or $\mathbf{R}^n$). The unit sphere $S^2 = \{x \in \mathbf{R}^3 \mid \|x\|_2 = 1\}$ paired with $d(x, y) := \arccos(\langle x, y \rangle)$ is a metric space.

Exercise 2.1.2. Show that (S^2, d) defined as above is indeed a metric space.

Definition 2.1.4.

(1) Let (X, d) be a metric space with $E \subseteq X$. The *diameter* of E is $\operatorname{diam}(E) = \sup_{x,y \in E} d(x, y)$. We say E is *bounded* (metrically) if $\operatorname{diam}(E) < \infty$.

(2) If Ω is any set and (Y, d) is a metric space, $f : \Omega \to Y$ is *bounded* if $\operatorname{diam}(f(\Omega)) < \infty$. The set of bounded functions is $\operatorname{Bd}(\Omega, Y) := \{f : \Omega \to Y \mid f \text{ bounded}\}$.

Exercise 2.1.3. If $(V, \|\cdot\|)$ is a normed space and $E \subseteq V$, the following are equivalent:

(1) E is bounded;

(2) $\sup_{v \in E} \|v\| < \infty$;

(3) there exists $r > 0$ such that $E \subseteq B(0, r)$.

Example 2.1.2. The set $\mathrm{Bd}(\Omega, Y)$ is a metric space with:

$$D_u(f,g) := \sup_{x\in\Omega} d(f(x), g(x)).$$

Clearly $D_u(f,g) = D_u(g,f)$ and $D_u(f,f) = 0$. If $D_u(f,g) = 0$, then $f(x) = g(x)$ for all $x \in \Omega$, giving $f = g$. Moreover, for every $x \in \Omega$:

$$\begin{aligned} d(f(x), h(x)) &\leqslant d(f(x), g(x)) + d(g(x), h(x)) \\ &\leqslant D_u(f,g) + D_u(g,h). \end{aligned}$$

Whence $D_u(f,h) \leqslant D_u(f,g) + D_u(g,h)$. Note that if we take the normed space $(\ell_\infty(\Omega), \|\cdot\|_u)$, the induced metric is:

$$\begin{aligned} d(f,g) &= \|f - g\|_u \\ &= \sup_{x\in\Omega} |(f-g)(x)| \\ &= \sup_{x\in\Omega} |f(x) - g(x)| \\ &= D_u(f,g). \end{aligned}$$

So as metric spaces, $\ell_\infty(\Omega) \cong \mathrm{Bd}(\Omega, F)$. Now consider the subset $E = \{f \in \mathrm{Bd}(\Omega, F) \mid f(x) \in \{0,1\}\}$. We get:

$$\begin{aligned} D_u(f,g) &= \sup_{x\in\Omega} |f(x) - g(x)| \\ &= \begin{cases} 0, & f = g \\ 1, & f \neq g\,. \end{cases} \end{aligned}$$

So (E, D_u) is discrete.

§ 2.2. Topology of Metric Spaces

Unless otherwise stated, let (X, d) be a metric space.

Definition 2.2.1. Let X be a set. A collection T of subsets of X is called a *topology on* X if they satisfy:

(1) $\emptyset, X \in T$;

(2) arbitrary unions of elements in T are in T;

(3) finite intersections of elements in T are in T.

Definition 2.2.2. Let (X, d) be a metric space.

(1) Let $x_0 \in X$ and $\delta > 0$.

(i) The *open ball* centered at x_0 of radius δ is $U(x_0, \delta) = \{x \mid d(x, x_0) < \delta\}$.

(ii) The *closed ball* centered at x_0 of radius δ is $B(x_0, \delta) = \{x \mid d(x, x_0) \leqslant \delta\}$.

(iii) The *sphere* centered at x_0 of radius δ is $S(x_0, \delta) = \{x \mid d(x, x_0) = \delta\}$.

(2) A subset $U \subseteq X$ is *open in* X if:

$$(\forall x \in U)(\exists \delta > 0) : U(x, \delta) \subseteq U.$$

The collection of open sets is denoted $\tau_X := \{U \subseteq X \mid U \text{ is open}\}$.

(3) A subset $D \subseteq X$ is *closed in* X if $D^c \subseteq X$ is open in X.

(4) If $x \in U \in \tau_X$, then U is called an *open neighborhood of* x. If $x \in U \in \tau_X$ and $U \subseteq N \subseteq X$, then N is called a *neighborhood of* x. The collection of neighborhoods of x is denoted $\mathcal{N}_x = \{N \mid N \text{ is a neighborhood of } x\}$.

(5) Let $A \subseteq X$.

(i) The *interior of* A is:

$$A^o := \bigcup \{V \in \tau_X \mid V \subseteq A\}.$$

(ii) The *closure of* A is:

$$\overline{A} := \bigcap \{C \mid C \supseteq A, C \text{ closed}\}.$$

(iii) The *boundary of* A is $\partial A := \overline{A} \setminus A^o$.

Exercise 2.2.1. Show that $\overline{A^c} = (A^o)^c$ and $\overline{A}^c = (A^c)^o$.

Proposition 2.2.1. *Let (X, d) be a metric space. The open sets τ_X form a topology.*

Proof. Both $\emptyset$ and X are open by assumption. Let $\{V_i\}_{i \in I}$ be a family of open sets of X. Let $x \in \bigcup_{i \in I} V_i$. Then $x \in V_i$ for some i. Since V_i is open, there exists $\delta > 0$ with $B(x, \delta) \subseteq V_i \subseteq \bigcup_{i \in I} V_i$. Whence the arbitrary union of open sets is open.

Now let $\{V_k\}_{k=1}^n$ be a family of open sets. Let $x \in \bigcap_{k=1}^n V_k$. Then $x \in V_k$ for all k. Since V_k is open, there exists $\delta_k > 0$ with $B(x, \delta_k) \subseteq V_k$. Pick $\delta = \min\{\delta_1, \delta_2, ..., \delta_k\}$. Then $B(x, \delta) \subseteq \bigcap_{k=1}^n V_k$. Whence $\bigcap_{k=1}^n V_k$ is open. $\square$

Note that an arbitrary intersection of open sets is not necessarily open. Consider the sequence of intervals $(I_n)_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^\infty I_n = \{0\}$, which is closed.

Exercise 2.2.2. Let (X, d) be a metric space and consider a collection $\mathcal{C}$ of subsets of X with $\emptyset, X \in \mathcal{C}$. Show that:

(1) if $\{C_i\}_{i \in I}$ is a family of closed sets, then $\bigcap_{i \in I} C_i$ is closed;

(2) if $\{C_i\}_{i=1}^n$ is a family of closed sets then $\bigcup_{i=1}^n C_i$ is closed.

Proposition 2.2.2. *Let (X, d) be a metric space and $x \in X$.*

(1) $N \in \mathcal{N}_x$ if and only if there exists $\delta > 0$ such that $U(x, \delta) \subseteq N$.

(2) If $N \in \mathcal{N}_x$ and $N \subseteq M$, then $M \in \mathcal{N}_x$.

(3) If $N_1, N_2 \in \mathcal{N}_x$, then $N_1 \cap N_2 \in \mathcal{N}_x$.

Proof. (1) Let $N \in \mathcal{N}_x$. Then there is an open set $x \in U$ with $U \subseteq N \subseteq X$. Since U is open, there exists $\delta > 0$ such that $U(x, \delta) \subseteq U \subseteq N$. Conversely, suppose there exists $\delta > 0$ such that $U(x, \delta) \subseteq N$. Clearly $U(x, \delta) \subseteq N \subseteq X$, whence $N \in \mathcal{N}_x$.

(2) If $N \in \mathcal{N}_x$, then there is an open set U with $x \in U$ and $U \subseteq N \subseteq X$. So $U \subseteq N \subseteq M \subseteq X$. Whence $M \in \mathcal{N}_x$.

(3) If $N_1, N_2 \in \mathcal{N}_x$, then there are open sets U_1, U_2 with $x \in U_1$, $x \in U_2$ and $U_1 \subseteq N_1 \subseteq X$, $U_2 \subseteq N_2 \subseteq X$. Whence $U_1 \cap U_2 \subseteq N_1 \cap N_2 \subseteq X$. □

Proposition 2.2.3. *Let $U \subseteq \mathbf{R}$ be open. Then:*

$$U = \bigsqcup_{j \in J} I_j,$$

where J is countable and I_j are open intervals.

Proof. For each $x \in U$, define:

$$I_x := \bigcup \{I \mid x \in I \subseteq U,\ I \text{ open interval}\}.$$

Clearly $x \in I_x \subseteq U$. If $s, t \in I_x$ with $s < t$, then there exists open intervals I, I' with $x \in I \subseteq U$, $x \in I' \subseteq U$, and $s \in I$, $t \in I'$. Since $I \cap I' \neq \emptyset$, $I \cup I'$ is an open interval. Moreover, since $s, t \in I \cup I'$, we know $[s, t] \subseteq I \cup I' \subseteq I_x$. This shows I_x is an interval —in particular, since I_x is the union of open intervals, it must be open.

Now suppose $x, y \in U$ and $I_x \cap I_y \neq \emptyset$. Then there exists $z \in I_x \cap I_y$, but $z \in I_x$ implies $I_x \subseteq I_z$ and $z \in I_y$ implies $I_y \subseteq I_z$. But we also have $x \in I_x \subseteq I_z$ which gives $I_z \subseteq I_x$, and similarly $y \in I_y \subseteq I_z$ gives $I_z \subseteq I_y$. Together, we have $I_x = I_y$, which means for any $x, y \in U$, then $I_x \cap I_y = \emptyset$ or $I_x = I_y$. Thus there exists $J \subseteq U$ with $U = \bigsqcup_{j \in J} I_j$.

It remains to show that J is countable. Define $J \to \mathbf{Q}$ by $x \mapsto q_x$, where $q_x \in \mathbf{Q} \cap I_x$. This map is injective, establishing the proposition. □

Proposition 2.2.4. *Let $A \subseteq X$.*

(1) $x \in A^o$ if and only if there exists $\delta > 0$ such that $U(x, \delta) \subseteq A$.

(2) $x \in \overline{A}$ if and only if for all $\delta > 0$, $U(x, \delta) \cap A \neq \emptyset$.

(3) $x \in \partial A$ if and only if for all $\delta > 0$, $U(x, \delta) \cap A \neq \emptyset$ and $U(x, \delta) \cap A^c \neq \emptyset$.

Proof. (1) If $x \in A^o$, then by definition $x \in \bigcup\{V \in \tau_X \mid V \subseteq A\}$. So there exists some $V \in \tau_X$ such that $x \in V \subseteq A$. Since V is open, there exists $\delta > 0$ with $U(x, \delta) \subseteq V \subseteq A$. Conversely, if $U(x, \delta) \subseteq A$ for some $\delta > 0$, then clearly $x \in A^o$, as $U(x, \delta) \in \tau_X$.

(2) We prove the converse of this statement. Note that $x \notin \overline{A}$ if and only if $x \in (\overline{A})^c = (A^c)^o$. By (1) there exists $\delta > 0$ with $U(x, \delta) \subseteq (A^c)^o \subseteq A^c$. This is true if and only if $U(x, \delta) \cap A = \emptyset$, establishing the proposition.

(3) Note that:

$$\begin{aligned}\partial A &= \overline{A} \setminus A^o \\ &= \overline{A} \cap (A^o)^c \\ &= \overline{A} \cap \overline{A^c}.\end{aligned}$$

So $x \in \partial A$ if and only if $x \in \overline{A} \cap \overline{A^c}$. Applying (2) to $x \in \overline{A}$ and $x \in \overline{A^c}$ establishes (3). □

Exercise 2.2.3. Show that open balls are open, closed balls are closed, and spheres are closed.

Proposition 2.2.5. **** For any normed space:*

(1) $\overline{U(x, \delta)} = B(x, \delta)$

(2) $B(x, \delta)^o = U(x, \delta)$

(3) $\partial U(x, \delta) = \partial B(x, \delta) = S(x, \delta)$.

Proof. □

Proposition 2.2.6. **** Let (X, d) be a metric space with $\{A_i\}_{i \in I}$ a family of subsets. Let $K \subseteq I$ be finite.*

(1) $\displaystyle\bigcup_{i \in I} A_i^o \subseteq \left(\bigcup_{i \in I} A_i\right)^o$ *(Inclusion may be strict).*

(2) $\displaystyle\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$ *(Inclusion may be strict).*

(3) $\displaystyle\bigcap_{i \in K} A_i^o = \left(\bigcap_{i \in K} A_i\right)^o$

(4) $\displaystyle\overline{\bigcup_{i \in K} A_i} = \bigcup_{i \in K} \overline{A_i}$.

Proof. □

Proposition 2.2.7. *Let $S \subseteq X$.*

(1) $\partial S = \partial S^c$.

(2) ∂S *is closed.*

(3) $\overline{S} = S \cup \partial S$.

(4) $S \setminus \partial S = S^o$.

Proof. (1) This follows from the characterization of ∂S. (2) We have $\partial S = \overline{S} \setminus S^o = \overline{S} \cap (S^o)^c$, which is closed. (3) Clearly $S \cup \partial S \subseteq \overline{S}$. Let $x \in \overline{S}$. If $x \in S$ we are done. Otherwise $x \in \overline{S} \setminus S \subseteq \overline{S} \setminus S^o = \partial S$. (4) Observe that:

$$\begin{aligned} S \setminus \partial S &= S \cap (\partial S)^c \\ &= S \cap (\overline{S} \setminus S^o)^c \\ &= S \cap (\overline{S} \cap (S^o)^c)^c \\ &= S \cap (\overline{S}^c \cup S^o) \\ &= S \cap \overline{S}^c \cup S \cap S^o \\ &= S^o. \end{aligned}$$

□

Definition 2.2.3. Let (X, d) be a metric space.

(1) A subset $A \subseteq X$ is *d-dense* if $\overline{A} = X$.

(2) A subset $N \subseteq X$ is *nowhere dense* if $(\overline{N})^o = \emptyset$.

(3) The space (X, d) is *separable* if there exists a countable dense subset $D \subseteq X$.

Exercise 2.2.4. If $N \subseteq X$ is closed, then N is nowhere dense if and only if N^c is dense.

Proposition 2.2.8. **** Let $A \subseteq X$. The following are equivalent:*

(1) A is dense;

(2) $(\forall U \in \tau_X), U \cap A \neq \emptyset$;

(3) $(\forall x \in X)(\forall \epsilon > 0), U(x, \epsilon) \cap A \neq \emptyset$;

(4) $(\forall x \in X)(\forall \epsilon > 0)(\exists a \in A) : d(a, x) < \epsilon$.

Proof. □

Definition 2.2.4. Let (X, d) be a metric space.

(1) A *base* for τ_X is a family of open subsets $\mathcal{B} \subseteq \tau_X$ such that:

$$(\forall U \in \tau_X)(\forall x \in U)(\exists B \in \mathcal{B}) : x \in B \subseteq U.$$

Equivalently, for all $U \in \tau_X$, we can write $U = \bigcup_{i \in I} B_i$, where $\{B_i\}_{i \in I} \subseteq \tau_X$.

(2) X is *second countable* if it has a countable base.

Note that this definition can be generalized to any topological space. Clearly $\mathcal{B} = \{U(x, \epsilon) \mid x \in X, \epsilon > 0\}$ forms a base for any metric space.

Example 2.2.1. The set $\mathcal{B} = \{U(q, \frac{1}{n}) \mid n \geqslant 1, q \in \mathbf{Q}^d\}$ is a base for $\mathbf{R}^d$.

Proposition 2.2.9. *Let (X, d) be a metric space. X is separable if and only if X is second countable.*

Proof. Let $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ be a countable base. Choose any $a_n \in U_n$. Then $\{a_n\}_{n=1}^{\infty}$ is dense. Indeed, given any $x \in X$ and $\epsilon > 0$, there exists U_m with $x \in U_m \subseteq U(x, \epsilon)$ (since $U_m \in \mathcal{B}$). Whence $d(a_m, x) < \epsilon$.

Let $\{a_n\}_{n=1}^{\infty}$ be dense. Consider:

$$\mathcal{B} = \left\{U(a_n, \tfrac{1}{m}) \mid n \geqslant 1, m \geqslant 1\right\}.$$

Clearly $\mathcal{B}$ is countable —it remains to show that it is a base for X. Given $x \in V \in \tau_X$, find $\epsilon > 0$ such that $U(x, \epsilon) \subseteq V$. Then there exists $m \geqslant 1$ with $\epsilon > \frac{1}{m}$. Since $\{a_n\}_{n=1}^{\infty}$ is dense, there exists $a_j \in \{a_n\}_{n=1}^{\infty}$ such that $d(a_j, x) < \frac{1}{2m}$. Let $y \in U(a_j, \frac{1}{2m})$. Observe that:

$$\begin{aligned} d(x, y) &\leqslant d(x, a_j) + d(a_j, y) \\ &< \frac{1}{2m} + \frac{1}{2m} \\ &= \frac{1}{m} \\ &< \epsilon. \end{aligned}$$

So $y \in U(x, \epsilon)$. Thus $x \in U(a_j, \frac{1}{2m}) \subseteq U(x, \epsilon) \subseteq V$, establishing $\mathcal{B}$ as a base. □

Example 2.2.2.

(1) The space $(\mathbf{R}^d, \|\cdot\|_p)$ is separable for any $1 \leqslant p \leqslant \infty$. Indeed, if $(r_1, ..., r_d) \in \mathbf{R}^d$ and $\epsilon > 0$, find $q_j \in \mathbf{Q}$, $j = 1, ..., d$ with:

$$|r_j - q_j| < \frac{\epsilon}{d}.$$

Then:

$$\|r - q\|_1 = \sum_{j=1}^{d} |r_j - q_j| < \epsilon.$$

So $\mathbf{Q}^d$ is $\|\cdot\|_1$-dense in $\mathbf{R}^d$. For $1 \leqslant p \leqslant \infty$, let $C > 0$ be such that $\|\cdot\|_p \leqslant C \|\cdot\|_1$. So given $\epsilon > 0$, find $q \in \mathbf{Q}^d$ with $\|r - q\|_1 \leqslant \frac{\epsilon}{C}$. Then $\|r - q\|_p < \epsilon$.

(2) Similarly, $\mathbf{C}^d_{\mathbf{Q}} \subseteq \mathbf{C}^d$ is $\|\cdot\|_p$-dense, where:

$$\mathbf{C}_{\mathbf{Q}} = \{a + bi \mid a, b \in \mathbf{Q}\}.$$

(3) Recall that $c_{00} = \big\{(z_k)_k \mid \operatorname{supp}\big((z_k)_k\big) < \infty\big\}$. The space $(c_{00}, \|\cdot\|_u)$ is separable.

Note that $c_{00} = \mathbf{C}\text{-}\operatorname{span}\{e_k \mid k \in \mathbf{N}\}$. This space is not countable —clearly $\mathbf{C}\text{-}\operatorname{span}\{e_1\} = \{\alpha e_1 \mid \alpha \in \mathbf{C}\}$ is not countable, so it must be that c_{00} is also not countable.

Instead, consider:

$$\begin{aligned}
\mathbf{C}_{\mathbf{Q}}\text{-}\operatorname{span}\{e_k \mid k \in \mathbf{N}\} &= \left\{\sum_{k=1}^{\infty} t_k e_k \mid t_k \in \mathbf{C}_{\mathbf{Q}},\right\} \\
&= \bigcup_{k=1}^{\infty} \{C_k \mid C_k = \mathbf{C}_{\mathbf{Q}}\text{-}\operatorname{span}\{e_1, ..., e_k\}\}
\end{aligned}$$

Note that C_k is in bijection with $\mathbf{Q}^{2k}$, whence $\mathbf{C}_{\mathbf{Q}}\text{-}\operatorname{span}\{e_k \mid k \in \mathbf{N}\}$ is countable.

Given $z \in c_{00}$, let $z = \sum_{k=1}^{N} z_k e_k$ and $\epsilon > 0$. Find $t_k \in \mathbf{C}_{\mathbf{Q}}$ with $|z_k - t_k| < \epsilon$. Then:

$$\begin{aligned}
\|z - t\|_u &= \left\|\sum_{k=1}^{N} z_k e_k - \sum_{k=1}^{K} t_k e_k\right\|_u \\
&= \left\|\sum_{k=1}^{N} (z_k - t_k) e_k\right\|_u \\
&= \sup_{k=1}^{N} |z_k - t_k| \\
&< \epsilon.
\end{aligned}$$

Thus $\mathbf{C}_{\mathbf{Q}}\text{-}\operatorname{span}\{e_k \mid k \in \mathbf{N}\}$ is dense in c_{00}, whence the space $(c_{00}, \|\cdot\|_u)$ is separable.

Proposition 2.2.10. *If (X, d) is a separable metric space and $Y \subseteq X$, then (Y, d) is separable.*

Proof. Let $A = \{a_k\}_{k=1}^{\infty}$ be dense in X. Let:

$$N = \{(m, n) \mid U(a_m, \tfrac{1}{n}) \cap Y \neq \emptyset\}.$$

For each $(m, n) \in N$, choose $b_{(m,n)} \in Y \cap U(a_m, \frac{1}{n})$. Claim: the set

$$\{b_{(m,n)} \mid (m, n) \in N\}$$

is dense in Y. Let $y \in Y$ and $\epsilon > 0$. Then there exists $n \geqslant 1$ with $\frac{\epsilon}{2} > \frac{1}{n}$. Since A is dense, $U(y, \frac{1}{n}) \cap A \neq \emptyset$ (this is because for all $U \in \tau_X$, we have $U \cap A \neq \emptyset$). So $d(a_m, y) < \frac{1}{n}$. Whence:

$$\begin{aligned} d(b_{(m,n)}, y) &\leqslant d(b_{(m,n)}, a_m) + d(a_m, y) \\ &< \frac{1}{n} + \frac{1}{n} \\ &< \epsilon. \end{aligned}$$

□

Example 2.2.3.

(1) The space $(\ell_\infty, \|\cdot\|_u)$ is not separable. If it were, consider:

$$E = \{(x_k)_k \mid x_k \in \{0,1\}\} \subseteq \ell_\infty.$$

This set is uncountable, and by the previous proposition $(E, \|\cdot\|_u)$ is also separable. Let $a, b \in E$. We have:

$$\begin{aligned} \|(a_k)_k - (b_k)_k\|_u &= \|(a_k - b_k)_k\|_u \\ &= \sup_{k \geqslant 1} |a_k - b_k| \\ &= \begin{cases} 0, & (a_k)_k = (b_k)_k \\ 1, & (a_k)_k \neq (b_k)_k \,. \end{cases} \end{aligned}$$

So $(E, \|\cdot\|_u)$ is discrete. Note that a metric space is discrete if and only if its singletons are open. Whence if $(E, \|\cdot\|_u)$ is separable, then $A = \overline{A} = E$, which contradicts the countability of A. It must be that $(\ell_\infty, \|\cdot\|_u)$ is not separable.

(2) The space $(\ell_p, \|\cdot\|_p)$ is separable for $1 \leqslant p < \infty$. Let $a = (a_k)_k \in \ell_p$ and $\epsilon > 0$. Find N large so that $\sum_{k>N} |a_k|^p < \frac{\epsilon^p}{2}$. Find $b_k \in \mathbf{C}_{\mathbf{Q}}$ with $|a_k - b_k| < \frac{\epsilon}{(2N)^{\frac{1}{p}}}$. Let $b = (b_1, b_2, ..., b_{N-1}, b_N, 0, 0, ...)$. We have:

$$\begin{aligned} \|a - b\|_p^p &= \sum_{k=1}^{\infty} |a_k - b_k|^p \\ &= \sum_{k=1}^{N} |a_k - b_k|^p + \sum_{k>N} |a_k|^p \\ &< N \cdot \frac{\epsilon^p}{2N} + \frac{\epsilon^p}{2} \\ &= \epsilon. \end{aligned}$$

Whence the set $\mathbf{C}_{\mathbf{Q}}$- span$\{e_k \mid k \in \mathbf{N}\}$ is $\|\cdot\|_u$-dense, giving $(\ell_p, \|\cdot\|_p)$ as separable.

(3) We will eventually show that the set of polynomial functions:

$$P\big([0,1]\big) = \left\{ \sum_{k=0}^{n} a_k x^k \mid a_k \in F, n \geqslant 0 \right\}$$

is $\|\cdot\|_u$-dense in $C\big([0,1]\big)$ (note that this set is not countable). With this fact, we can show that $C\big([0,1]\big)$ is separable. Indeed, given $f \in C\big([0,1]\big)$ and $\epsilon > 0$, find $p \in P\big([0,1]\big)$ with $\|f - p\|_u < \frac{\epsilon}{2}$. Now let $p(x) = \sum_{k=0}^{n} a_k x^k$. Find $b_k \in \mathbf{C_Q}$ with $|a_k - b_k| < \frac{\epsilon}{2(n+1)}$ and define $q(x) = \sum_{k=0}^{n} b_k x^k$. Observe that:

$$\begin{aligned} \|f - q\|_u &= \|f - p + p - q\|_u \\ &\leqslant \|f - p\|_u + \|p - q\|_u \\ &= \|f - p\|_u + \sum_{k=0}^{n} |a_k - b_k| \\ &< \frac{\epsilon}{2} + (n+1) \cdot \frac{\epsilon}{2(n+1)} \\ &= \epsilon. \end{aligned}$$

Thus the set $\mathbf{C_Q}\text{-}\operatorname{span}\{x^k \mid k \in \mathbf{N}\}$ is $\|\cdot\|_u$-dense in $C\big([0,1]\big)$. In particular, since it is countable, $C\big([0,1]\big)$ is separable.

————//————

We have seen that subsets of metric spaces are metric spaces in their own right. Then what are their open sets?

Proposition 2.2.11. *Let (X, d) be a metric space and let $Y \subseteq X$ be any subset. Then $V \subseteq Y$ is open in Y if and only if there exists an open set $U \subseteq X$ with $U \cap Y = V$. That is, $\tau_Y = \{U \cap Y \mid U \in \tau_X\}$.*

Proof. Let $V \subseteq Y$ be open. Then for every $x \in V$, there exists $\delta_x > 0$ with $U_y(x, \delta_x) \subseteq V$, where:

$$\begin{aligned} U_y(x, \delta_x) &= \{y \in Y \mid d(y, x) < \delta_x\} \\ &= U(x, \delta_x) \cap Y. \end{aligned}$$

Set $U = \bigcup_{x \in V} U(x, \delta_x)$. Then U is indeed open in X. Also:

$$\begin{aligned} U \cap Y &= \left(\bigcup_{x \in V} U(x, \delta_x) \right) \cap Y \\ &= \bigcup_{x \in V} \left(U(x, \delta_x) \cap Y \right) \\ &= \bigcup_{x \in V} U_y(x, \delta_x) \\ &= V. \end{aligned}$$

Conversely, suppose $V = U \cap Y$ for some open $U \in \tau_X$. Let $x \in V$. Since $x \in U$ and U is open, there exists $\delta > 0$ such that $U(x, \delta) \subseteq U$. So $U_y(x, \delta) = U(x, \delta) \cap Y \subseteq U \cap Y = V$. Thus V is open in Y. $\square$

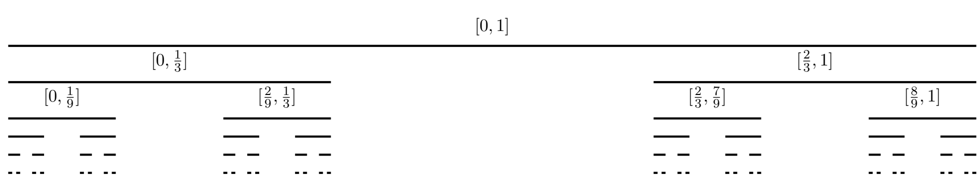
Example 2.2.4.

(1) $[0, \frac{1}{2})$ is not open in $\mathbf{R}$, but it is open in $[0, 1]$.

(2) ℓ_∞ is not a discrete metric space, but $\{0, 1\}^{\mathbf{N}} \subseteq \ell_\infty$ is.

§ 2.3. The Cantor Set

Given the interval $[0, 1]$, start by deleting the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, leaving two line segments $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments $\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.



We are interested in studying the topological properties of the points which are not deleted at any step of this infinite process. The set of these points have a special name and are defined below.

Definition 2.3.1. Let $C_0 := [0, 1]$ and $C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3} \right)$ for $n \geqslant 1$. The *Cantor set* is $\mathfrak{C} = \bigcap_{n=0}^{\infty} C_n$.

Proposition 2.3.1. *The Cantor set is closed.*

Proof. Since the Cantor set is defined as the intersection of closed sets, it must be closed. □

Proposition 2.3.2. *The Cantor set is nowhere dense.*

Proof. Suppose towards contradiction its not, that is, $\overline{\mathfrak{C}}^o \neq \emptyset$. Then there is some $x \in \overline{\mathfrak{C}}^o$. We can find an $\epsilon > 0$ with $(x-\epsilon, x+\epsilon) \subseteq \mathfrak{C}$, in particular $(x-\epsilon, x+\epsilon) \subseteq C_n$ for all $n \geqslant 1$. Find m large so that $\epsilon > \frac{1}{3^m}$ and consider $(x-\epsilon, x+\epsilon) \subseteq C_m$. We have that $C_m = \bigsqcup_{j=1}^{2^m} C_{m,j}$ where length$(C_{m,j}) = \frac{1}{3^m}$. Since each $C_{m,j}$ is disjoint, it must be the case that $(x-\epsilon, x+\epsilon) \subseteq C_{m,j}$ for some $1 \leqslant j \leqslant 2^m$. But the length of $(x-\epsilon, x+\epsilon)$ is 2ϵ, which is impossible. It must be that $\mathfrak{C}$ is nowhere dense. □

Proposition 2.3.3. *The total length of the Cantor set is* 0.

Proof. The total length of the removed intervals is:

$$\begin{aligned}\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots &= \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \\ &= \frac{1}{2} \cdot \frac{2/3}{1 - 2/3} \\ &= 1.\end{aligned}$$

Thus length$(\mathfrak{C}) = 0$. □

Lemma 2.3.4. *****

Proof. □

Lemma 2.3.5. *****

Proof. □

Lemma 2.3.6. *****

Proof. □

Proposition 2.3.7. ***** card$(\mathfrak{C}) = \mathfrak{c}$.

Proof. □

§ 2.4. Convergent Sequences

Definition 2.4.1. Let (X,d) be a metric space.

(1) A *sequence* in X is a map $x_\bullet : \mathbf{N} \to X$ defined by $n \mapsto x_n$. We denote a sequence as $(x_n)_{n\geqslant 1}$, $(x_n)_{n=1}^\infty$, or $(x_n)_n$.

(2) A *natural sequence* is a sequence $(x_n)_n$ in $\mathbf{N}$ with $n_1 < n_2 < n_3 < ...$

(3) A *subsequence* of a sequence $(x_n)_n$ is a sequence $(x_{n_k})_k$, where $(n_k)_k$ is a natural sequence. This is equivalent to the composition of maps $\mathbf{N} \xrightarrow{k\mapsto n_k} \mathbf{N} \xrightarrow{n_k \mapsto x_{n_k}} X$.

Definition 2.4.2. A sequence $(x_n)_n$ *converges* to $x \in X$ if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies d(x_n, x) < \epsilon).$$

We write $(x_n)_n \xrightarrow{d} x$ or $\operatorname{limit} x_n = x$.

Exercise 2.4.1. Show that a sequence can have at most one limit.

Proposition 2.4.1. *Let $(x_n)_n$ be a sequence in X and $x \in X$. The following are equivalent:*

(1) $(x_n)_n \to x$*;*

(2) $(d(x_n, x))_n \to 0$ *in* $\mathbf{R}$*;*

(3) $(\forall V \in \mathcal{N}_x)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies x_n \in V)$.

Proof. Exercise. □

Exercise 2.4.2. Let (X,d) be a metric space and $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$. Then $(x_n)_n \xrightarrow{d} x$ if and only if $(x_n)_n \xrightarrow{\rho} x$.

Proposition 2.4.2. *Convergent sequences are bounded.*

Proof. Suppose that $(x_n)_n \to x$ and let $\epsilon = 1$. Find N large so that for $n \geqslant N$ we have $d(x_n, x) < 1$. Then for all $m, n \geqslant N$, we have $d(x_m, x_n) \leqslant d(x_m, x) + d(x, x_n) < 2$. Set $C = \max_{1\leqslant n,m\leqslant N} d(x_m, x_n)$. Now if $n \geqslant N$ and $m \leqslant N$, we have:

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_N) + d(x_N, x_m) \\ &\leqslant 1 + C. \end{aligned}$$

Let $K = \max\{2, 1+C, C\}$. Then $\operatorname{diam}(\{x_n\}_{n\geqslant 1}) = \sup_{m,n\geqslant 1} d(x_n, x_m) \leqslant K$. □

Definition 2.4.3. Let $(v_k)_k$ be a sequence in $(V, \|\cdot\|)$.

(1) A *sequence of partial sums* $(s_n)_n$ is defined as $s_n = \sum_{k=1}^n v_k$.

(2) If $(s_n)_n \to s$ in V we say the *series* $\sum_{k=1}^{\infty} v_k$ converges and write $\sum_{k=1}^{\infty} v_k = s$.

(3) The series $\sum v_k$ converges *absolutely* if $\sum \|v_k\|$ converges.

——————//——————

Definition 2.4.4. Let $(f_n : \Omega \to X)_n$ be a sequence of functions.

(1) $(f_n)_n$ converges *pointwise* to $f : \Omega \to X$ if:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N_{x,\epsilon} \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies |f_n(x) - f(x)| < \epsilon).$$

(2) $(f_n)_n$ converges *uniformly* to $f : \Omega \to X$ if:

$$(\forall \epsilon > 0)(\exists N_{\epsilon} \in \mathbf{N}) : (\forall n \in \mathbf{N})(\forall x \in \Omega)(n \geqslant N \implies |f_n(x) - f(x)| < \epsilon).$$

Example 2.4.1. Let $(f_n)_n$ be a sequence of functions in $\big(\mathrm{Bd}(\Omega, Y), D_u\big)$ converging to $f \in \mathrm{Bd}(\Omega, Y)$. Proposition 2.4.1 says this is equivalent to $(D_u(f_n, f))_n \to 0$ in $\mathbf{R}$. Therefore, we have $(\sup_{x\in\Omega} d(f_n(x), f(x)))_n \to 0$ in $\mathbf{R}$. Notice that this is precisely the definition of uniform convergence. Whence convergence in $c_{00} \subseteq c_0 \subseteq c \subseteq \ell_\infty \subseteq \ell_\infty(\Omega) = \mathrm{Bd}(\Omega, F)$ is uniform.

Proposition 2.4.3. *Let $\{d_k\}_k$ be a separating family of semi-metrics which are uniformly bounded. Define:*

$$d(x, y) := \sum_{k=1}^{\infty} 2^{-k} d_k(x, y).$$

Then $(x_n)_n \xrightarrow{d} x$ if and only if $(d_k(x_n, x))_n \to 0$ for all k.

Proof. ($\Rightarrow$) Let $k \geqslant 1$ be arbitrary. We have:

$$0 \leqslant 2^{-k} d_k(x_n, x) \leqslant d(x_n, x).$$

Multiplying all sides of the above equation by 2^k gives $0 \leqslant d_k(x_n, x) \leqslant 2^k d(x_n, x)$. Since $\mathrm{limit}_{n\to\infty} 2^k d(x_n, x) = 0$, by the Squeeze Theorem $(d_k(x_n, x))_n \to 0$.

($\Leftarrow$) Let $\epsilon > 0$. For all x, y, k, there exists $C > 0$ with $d_k(x, y) \leqslant C$. Since $\sum_{k=1}^{\infty} 2^{-k}$ converges (in particular, it converges to 1), find K large so:

$$\sum_{k>K} 2^{-k} < \frac{\epsilon}{2C}.$$

Note that $(d_k(x_n, x))_n \to 0$ for $k = 1, 2, ..., K$. So there exists $N_1, N_2, ..., N_K \in \mathbf{N}$ with $n \geqslant N_j$ implying $d_j(x_n, x) < \frac{\epsilon}{2}$ for $1 \leqslant j \leqslant K$. Let $N = \max_{j=1}^{K} N_j$. For

$n \geqslant N$, we have:

$$\begin{aligned}
d(x_n, x) &= \sum_{k=1}^{K} 2^{-k} d_k(x_n, x) + \sum_{k>K} 2^{-k} d_k(x_n, x) \\
&= \sum_{k=1}^{K} 2^{-k} d_k(x_n, x) + \sum_{k>K} 2^{-k} C \\
&< \sum_{k=1}^{K} 2^{-k} \frac{\epsilon}{2} + C \cdot \frac{\epsilon}{2C} \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

□

Thus $(x_n)_n \xrightarrow{d} x$.

Example 2.4.2. Consider the space $C(\mathbf{R})$. How does one define a distance between two functions? Given $f, g \in C(\mathbf{R})$, note that the uniform metric:

$$d(f, g) := \sup_{x \in \Omega} |f(x) - g(x)|$$

does not guarantee $d(f, g) < \infty$. We can fix this as follows: define $\rho_k(f, g) := \sup_{x \in [-k,k]} |f(x) - g(x)|$. Note that the family of metrics $\{\rho_k\}_k$ is separating, but not uniformly bounded. Defining $d_k(f, g) := \frac{\rho_k(f,g)}{1+\rho_k(f,g)}$ gives $\{d_k\}_k$ as a family of uniformly bounded semi-metrics. We can now define what is called the *Fréchet metric*:

$$d_F(f, g) := \sum_{k=1}^{\infty} 2^{-k} d_k(f, g).$$

By the comparison test, $d_F(f, g) < \infty$. In $(C(\mathbf{R}), d_F)$, observe that:

$$\begin{aligned}
(f_n)_n \xrightarrow{d_F} f &\iff \forall k, (d_k(f_n, f))_n \to 0 \\
&\iff \forall k, (\rho_k(f_n, f))_n \to 0 \\
&\iff \forall k, \left(\sup_{x \in [-k,k]} |f_n(x) - f(x)|\right)_n \to 0 \\
&\iff \forall k, (f_n)_n \to f \text{ uniformly on } [-k, k].
\end{aligned}$$

We've obtained a new type of convergence called *compact convergence*.

——————//——————

Proposition 2.4.4. *Let (X, d) be a metric space and $A \subseteq X$. We have $x \in \overline{A}$ if and only if there exists a sequence $(a_n)_n$ in A with $(a_n)_n \to x$.*

Proof. ($\Rightarrow$) If $x \in \overline{A}$, then for each $n \geqslant 1$ we have $U(x, \frac{1}{n}) \cap A \neq \emptyset$. For each n choose $a_n \in U(x, \frac{1}{n}) \cap A$. Then $d(x, a_n) < \frac{1}{n}$, so $(a_n)_n \to x$. ($\Leftarrow$) Given $\epsilon > 0$, find N large so $d(x, a_N) < \epsilon$. Then $a_N \in U(x, \epsilon) \cap A$. Since $U(x, \epsilon) \cap A \neq \emptyset$, we have $x \in \overline{A}$. □

Proposition 2.4.5. *Let (X, d) be a metric space and $A \subseteq X$. The following are equivalent:*

(1) A is closed;

(2) If $(a_n)_n$ is a sequence in A which converges to $x \in X$, then $x \in A$.

Proof. Let $(a_n)_n$ be a sequence in A which converges to $x \in X$. By Proposition 2.4.5 $x \in \overline{A}$. Since A is closed, $A = \overline{A}$. Thus $x \in A$.

We will show A is closed by proving $A = \overline{A}$ Clearly $A \subseteq \overline{A}$. Let $x \in \overline{A}$. Then there exists a sequence $(a_n)_n$ in A with $(a_n)_n \to x$. Thus $x \in A$, giving $\overline{A} \subseteq A$. □

Exercise 2.4.3. Show $x \in \overline{A}$ if and only if $x \in A$ or there exists a sequence $(a_n)_n$ in $A \setminus \{a\}$ with $(a_n)_n \to x$.

Proposition 2.4.6. *****

(1) The space $c_0 \subseteq \ell_\infty$ is closed.

(2) $\overline{c_{00}} = c_0$.

Proof. (1) Let $(z_n)_n$ be a sequence in c_0 converging to $f \in \ell_\infty$. Let $\epsilon > 0$. Find N large so that:

$$\|z_N - f\|_u < \frac{\epsilon}{2}.$$

Since $z_N \in c_0$, we know $\text{limit}_{k\to\infty} z_N(k) = 0$. Find K large so that for $k \geqslant K$:

$$|z_N(k)| < \frac{\epsilon}{2}.$$

Together, for $k \geqslant K$ we have:

$$\begin{aligned}|f(k)| &= |f(k) - z_N(k) + z_N(k)| \\ &\leqslant |f(k) - z_N(k)| + |z_N(k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Thus $\text{limit}_{k\to\infty} f(k) = 0$; i.e., $f \in c_0$. By Proposition 2.4.5, c_0 is closed in ℓ_∞. $\square$

Definition 2.4.5. Let (X, d) be a metric space and $A \subseteq X$. The *distance* of an element $x \in X$ to A is defined as the map $\mathrm{dist}_A : X \to [0, \infty)$ given by $\mathrm{dist}_A(x) = \inf_{a \in A} d(x, a)$.

Proposition 2.4.7. *Let (X, d) be a metric space and $A \subseteq X$.*

(1) $\overline{A} = \{x \mid \mathrm{dist}_A(x) = 0\}$.

(2) $\mathrm{dist}_A = \mathrm{dist}_{\overline{A}}$.

(3) $|\mathrm{dist}_A(x) - \mathrm{dist}_A(y)| \leqslant d(x, y)$.

Proof. (1) Let $D := \{x \mid \mathrm{dist}_A(x) = 0\}$. By Proposition 2.4.4, $x \in \overline{A}$ if and only if there exists a sequence $(a_n)_n$ in A which converges to x. Equivalently, the seqence $(d(x, a_n))_n$ converges to 0 in $\mathbf{R}$. Since $\inf_{a \in A} d(x, a) \leqslant d(x, a_n)$, it must be the case that $\mathrm{dist}_A(x) = 0$. Thus $\overline{A} \subseteq D$. Now suppose $x \in D$. Then $\inf_{a \in A} d(x, a) = 0$ implies the existance of a sequence $(a_n)_n$ in A with $(d(x, a_n))_n \to 0$. Thus $(a_n)_n \to x$, establishing that $x \in \overline{A}$ by Proposition 2.4.4. Thus $D = \overline{A}$.

(2) Let $x \in X$. Since $A \subseteq \overline{A}$, it is clear that:

$$\begin{aligned}
\mathrm{dist}_{\overline{A}}(x) &= \inf_{a \in \overline{A}} d(x, a) \\
&\leqslant \inf_{a \in A} d(x, a) \\
&= \mathrm{dist}_A(x).
\end{aligned}$$

Let $\epsilon > 0$ and $y \in \overline{A}$. From part (1), we have that $\mathrm{dist}_A(y) = 0$. In particular, $\mathrm{dist}_A(y) < \epsilon$. So there exists $a \in A$ such that $d(a, y) < \epsilon$. Observe that:

$$\begin{aligned}
\mathrm{dist}_A(x) &= \inf_{a \in A} d(x, a) \\
&\leqslant d(x, a) \\
&\leqslant d(x, y) + d(y, a) \\
&< d(x, y) + \epsilon.
\end{aligned}$$

It follows that $\mathrm{dist}_A(x) \leqslant d(x, y)$ for all $y \in \overline{A}$. Finally:

$$\begin{aligned}
\mathrm{dist}_A(x) &\leqslant \inf_{y \in \overline{A}} d(x, y) \\
&= \mathrm{dist}_{\overline{A}}(x).
\end{aligned}$$

Thus $\mathrm{dist}_A = \mathrm{dist}_{\overline{A}}$.

(3) □

§ 2.5. Continuity

Definition 2.5.1. Let (X, d) and (Y, ρ) be metric spaces. A map $f : X \to Y$ is *continuous at* $x_0 \in X$ if any of the equivalent definitions are satisfied:

(1) $(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X)\Big(d(x, x_0) < \delta \implies \rho\big(f(x), f(x_0)\big) < \epsilon\Big)$

(2) $(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X)\Big(x \in U_X(x_0, \delta) \implies f(x) \in U_Y\big(f(x_0), \epsilon\big)\Big)$

(3) $(\forall \epsilon > 0)(\exists \delta > 0) : f\big(U_X(x_0, \delta)\big) \subseteq U_Y\big(f(x_0), \epsilon\big)$

Proposition 2.5.1. *Let $f : (X, d) \to (Y, \rho)$ be a map between metric spaces and $x_0 \in X$. The following are equivalent:*

(1) f is continuous at x_0;

(2) $(\forall V \in \mathcal{N}_{f(x_0)})(\exists U \in \mathcal{N}_{x_0}) : f(U) \subseteq V$;

(3) $(\forall (x_n)_n \in X^{\mathbf{N}})\big((x_n)_n \to x_0 \implies (f(x_n))_n \to f(x_0)\big)$

Proof. (1)⇒(2) follows from Definition 2.5.1.

(1)⇒(3) Let $(x_n)_n \to x_0$. Let $\epsilon > 0$. Since f is continuous, find $\epsilon > 0$ so that $x \in U(x, \delta)$ implies $f(x) \in U(f(x_0), \epsilon)$. Whence $d(f(x_n), f(x_0)) < \epsilon$, establishing $(f(x_n))_n \to f(x_0)$.

(3)⇒(1) We prove the contrapositive of this statement. If f is not continuous, choose $\epsilon_0 > 0$ so that $d(x_n, x_0) < \frac{1}{n}$ and $d(f(x_n), f(x_0)) \geqslant \epsilon_0$. Whence $(x_n)_n \to x_0$ and $(f(x_n))_n \not\to f(x_0)$. □

Proposition 2.5.2. *Let $f : (X, d) \to (Y, \rho)$ be a map of metric spaces. The following are equivalent:*

(1) f is continuous;

(2) $(\forall V \in \tau_Y), f^{-1}(V) \in \tau_X$;

Proof. Let $V \subseteq Y$ be open. If $f^{-1}(V) = \emptyset$, we're done. If not, let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since V is open, find $\epsilon > 0$ so that $U(f(x), \epsilon) \subseteq V$. Since f is continuous, find $\delta > 0$ so that $f(U(x, \delta)) \subseteq U(f(x), \epsilon) \subseteq V$. Whence $U(x, \delta) \subseteq f^{-1}(V)$.

Let $x \in X$ and $\epsilon > 0$. Since $U(f(x), \epsilon) \in \tau_Y$, we have $f^{-1}(U(f(x)), \epsilon) \in \tau_X$. Note that $x \in f^{-1}(U(f(x)), \epsilon)$. Since this set is open, find $\delta > 0$ so that $U(x, \delta) \subseteq f^{-1}(U(f(x)), \epsilon)$. Thus $f(U(x, \delta)) \subseteq U(f(x), \epsilon)$; i.e., f is continuous. □

Proposition 2.5.3. *** *Let $(X, d) \xrightarrow{f} (Y, \rho) \xrightarrow{g} (Z, \gamma)$ be maps of metric spaces. If f is continuous at $x \in X$ and g is continuous at $y = f(x)$, then $g \circ f$ is continuous at x.*

Proof. Exercise. □

Proposition 2.5.4. *Let (X, d) be a metric space with $A \subseteq X$ dense. Let $f : X \to F$ be a continuous and bounded function. Then* $\sup_{x\in A} f(x) = \sup_{x\in X} f(x)$.

Proof. Since $A \subseteq X$ we have $\sup_{x\in A} f(x) \leqslant \sup_{x\in X} f(x)$. Conversely, let $\epsilon > 0$. Find $x' \in X$ so that $\sup_{x\in X} f(x) - \frac{\epsilon}{2} < f(x')$. Since f is continuous, find $\delta > 0$ so that for all $x \in X$, $d(x, x') < \delta$ implies $|f(x) - f(x')| < \frac{\epsilon}{2}$. Since A is dense, find $a \in A$ so that $d(x', a) < \delta$. This implies $|f(x') - f(a)| < \frac{\epsilon}{2}$, or equivalently $f(x') - \frac{\epsilon}{2} < f(a)$. This gives:

$$\begin{aligned} \sup_{x\in X} f(x) - \frac{\epsilon}{2} &< f(x') \\ &< f(a) + \frac{\epsilon}{2}. \end{aligned}$$

Upon simplifying, we have:

$$\sup_{x\in X} f(x) - \epsilon < f(a).$$

Hence for $\epsilon > 0$ we have:

$$\sup_{x\in X} f(x) < \sup_{x\in A} f(x) + \epsilon.$$

Taking $\epsilon \to 0$ gives $\sup_{x\in X} f(x) \leqslant \sup_{x\in A} f(x)$. ☐

Definition 2.5.2. Let $f : (X, d) \to (Y, \rho)$ be a map of metric spaces.

(1) We say f is *uniformly continuous* if:

$$(\forall\epsilon > 0)(\exists\delta > 0) : (\forall x, y \in X)\big(d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon\big).$$

(2) We say f is *Lipschitz* if:

$$(\exists C > 0) : (\forall x, y \in X)\big(\rho(f(x), f(y)) \leqslant Cd(x, y)\big).$$

If $C < 1$, we say f is *contractive.*

(3) We say f is an *isometry* if:

$$(\forall x, x' \in X)\big(\rho(f(x), f(x')) = d(x, x')\big)$$

Exercise 2.5.1. *** Show that Lipschitz implies uniform continuity. Show that uniform continuity implies continuity. Show that the converse direction fails in general.

Example 2.5.1. Let $(V, \|\cdot\|)$ be a normed space. Then $V \xrightarrow{\|\cdot\|} [0, \infty)$ is continuous. Indeed, we have $|\|v\| - \|w\|| \leqslant \|v - w\|$, so $\|\cdot\|$ is Lipschitz.

Example 2.5.2. Let (X,d) be a metric space and equip $X \times X$ with the product metric D_1. Claim: $d : X \times X \to [0,\infty)$ is continuous. Indeed, given $(x,y),(x',y') \in X \times X$, then we have:

$$d(x,y) \leqslant d(x,x') + d(x',y') + d(y',y).$$

So we have $d(x,y) - d(x',y') \leqslant d(x,x') + d(y,y')$. But this is equivalent to $|d(x,y) - d(x',y')| \leqslant D_1((x,y),(x',y'))$. So d is Lipschitz.

Example 2.5.3. If (X,d) is a metric space and $A \subseteq X$, then $\operatorname{dist}_A : X \to [0,\infty)$ is continuous. We've shown that $|\operatorname{dist}_A(x) - \operatorname{dist}_A(y)| \leqslant d(x,y)$, so dist_A is Lipschitz.

Definition 2.5.3. Let X be a topological space. We say X is *normal* (or T4) if, given any disjoint closed sets $E, F \subseteq X$, there are neighbourhoods U

Definition 2.5.4. Let X be a topological space. We say X is *normal* (or T4) if, for any $A, B \subseteq X$ closed satisfying $A \cap B = \emptyset$, then there exists $U, V \in \tau_X$ with $A \subseteq U$, $B \subseteq V$ satisfying $U \cap V = \emptyset$.

Proposition 2.5.5. *Metric spaces are normal.*

Proof. Let $A, B \subseteq (X,d)$ with $A \cap B = \emptyset$. Define $f : (X,d) \to \mathbf{R}$ by:

$$f(x) = \frac{\operatorname{dist}_A(x)}{\operatorname{dist}_A(x) + \operatorname{dist}_B(x)}.$$

Then f is continuous. Moreover, define:

$$U := f^{-1}\left((-1/2, 1/2)\right)$$
$$V := f^{-1}\left((-1/2, 3/2)\right)$$

Then U and V open with $U \cap V = \emptyset$. $\square$

Proposition 2.5.6. *Let V and W be normed spaces and $T : V \to W$ linear. The following are equivalent:*

(1) T is continuous at 0_V;

(2) T is continuous;

(3) T is uniformly continuous;

(4) T is Lipschitz;

(5) there exists $C \geqslant 0$ such that $\|Tv\| \leqslant C\|v\|$ for all $v \in V$;

(6) T is bounded.

Proof. We will show (6)⇔(5)⇒(4)⇒(3)⇒(2)⇒(1)⇒(5). Let T be bounded. Given $v \in V$, $v \neq 0$, we have:

$$\begin{aligned}\|T\|_{\text{op}} &= \sup_{v \in B_V} \|Tv\| \\ &\geqslant \left\| T\frac{v}{\|v\|} \right\| \text{ for all } v \in V \\ &= \frac{1}{\|v\|} \|Tv\| \text{ for all } v \in V.\end{aligned}$$

Thus $\|Tv\| \leqslant \|T\|_{\text{op}} \|v\|$ for all $v \in V$. The converse is clear by inspection.

Suppose there exists $C \geqslant 0$ satisfying (5). We have that $\|Tv - Tv'\| \leqslant C \|v - v'\|$. Thus T is Lipschitz.

Let T be Lipschitz. Let $\epsilon > 0$ and find $\delta = \frac{\epsilon}{c}$. Then $\|v - v'\| < \delta$ implies:

$$\begin{aligned}\left\|Tv - Tv'\right\| &\leqslant C \left\|v - v'\right\| \\ &< c\frac{\epsilon}{c} \\ &= \epsilon.\end{aligned}$$

Thus T is uniformly continuous.

Suppose that T be uniformly continuous. Fix $x \in V$. Given $\epsilon > 0$, we can find $\delta > 0$ so that $\|v - x\| < \delta$ implies $\|Tv - Tx\| < \epsilon$. Thus T is continuous at $x \in V$. Since x was arbitrary, T is continuous. Moreover, T will be continuous at 0_V, establishing (1).

We will now show (1) implies (5). Let $\epsilon = 1$. We can find a $\delta > 0$ such that $T\big(U(0,\delta)\big) \subseteq U(0,1)$. If $v \in V$, $v \neq 0$, then $\frac{\delta v}{2\|v\|} \in U(0,\delta)$. Since T is continuous at 0, we have $T\frac{\delta v}{2\|v\|} \in U(0,1)$[1]. This gives $\left\|T\frac{\delta v}{2\|v\|}\right\| < 1$, which is equivalent to $\|Tv\| < \frac{2}{\delta}\|v\|$. This establishes (5). □

Corollary 2.5.7. *Let V be a normed space with* $\dim(V) = n$. *If $T : \ell_p^n \to V$ is linear, then T is continuous.*

Proof. Let $\mathcal{B} = \{e_1, ..., e_n\}$ be a basis for ℓ_p^n. Let $v \in V$. Then $v = \sum_{j=1}^{n} \alpha_j e_j$ where

[1]Recall that $T(0) = 0$.

each $\alpha_j \in \ell_p$. Then:

$$\begin{aligned}\|Tv\| &= \left\| T\left(\sum_{j=1}^{n} \alpha_j e_J\right)\right\| \\ &= \left\|\sum_{j=1}^{n} \alpha_j T e_j\right\| \\ &\leqslant \sum_{j=1}^{n} |\alpha_j| \, \|Te_j\| \,.\end{aligned}$$

Let $c = \max_{j=1}^{n} \|Te_j\|$. We have:

$$\begin{aligned}\sum_{j=1}^{n} |\alpha_j| \, \|Te_j\| &\leqslant c \sum_{j=1}^{n} |\alpha_j| \\ &= c \left\|\sum_{j=1}^{n} \alpha_j e_j\right\|_1 \,.\end{aligned}$$

We showed that all norms are equivalent in Theorem 1.3.7. So there exists $c' > 0$ such that $\|\cdot\|_1 \leqslant \|\cdot\|_p$. Thus:

$$\begin{aligned}\|Tv\| &\leqslant c \cdot c' \left\|\sum_{j=1}^{n} \alpha_j e_j\right\|_p \\ &= c \cdot c' \, \|v\|_p \,.\end{aligned}$$

By Proposition 2.5.6, we have that T is continuous. □

Proposition 2.5.8. *Let (X, d) be a metric space with $A \subseteq X$ dense. If $f, g : X \to (Y, \rho)$ are continuous with $f(a) = g(a)$ for all $a \in A$, then $f = g$.*

Proof. If $x \in X$, we can find a sequence $(a_n)_n$ in A with $(a_n)_n \xrightarrow{d} x$. Then:

$$\begin{array}{c}(f(a_n))_n \to f(x) \\ \shortparallel \\ (g(a_n))_n \to g(x).\end{array}$$

Thus $f(x) = g(x)$. □

Definition 2.5.5. Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$.

(1) f is a *homeomorphism* if f is bijective with f and f^{-1} continuous. If such an f exists, we say $X \cong Y$ are *homeomorphic*.

(2) f is a *uniformism* if f is bijective with f and f^{-1} uniformly continuous. If such an f exists, we say $X \cong Y$ are *uniformly isomorphic.*

(3) f is an *metric isomorphism* if f is bijective with f and f^{-1} Lipschitz. If such an f exists, we say $X \cong Y$ are *metrically isomorphic.*

(4) f is an *isometric isomorphism* if f and f^{-1} are isometries. We say $X \cong Y$ are *isometrically isomorphic.*

Example 2.5.4. $(0, 1) \cong \mathbf{R}$ are homeomorphic, but not uniformly isomorphic.

Example 2.5.5. *** Let $a = (a_k)_k \in \ell_1$. Define $\varphi_a : c_0 \to F$ by $\varphi_a(z) = \sum_{k \geqslant 1} a_k z_k$. This series converges since:

Definition 2.5.6. Let X be a set with two metrics d_1 and d_2.

(1) d_1 and d_2 are *metrically equivalent* if $\mathrm{id} : (X, d_1) \to (X, d_2)$ and $\mathrm{id}^{-1} : (X, d_2) \to (X, d_1)$ are Lipschitz.

(2) d_1 and d_2 are *uniformly equivalent* if $\mathrm{id} : (X, d_1) \to (X, d_2)$ and $\mathrm{id}^{-1} : (X, d_2) \to (X, d_1)$ are uniformisms.

(3) d_1 and d_2 are *topologically equivalent* if $\mathrm{id} : (X, d_1) \to (X, d_2)$ and $\mathrm{id}^{-1} : (X, d_2) \to (X, d_1)$ are homeomorphisms.

Example 2.5.6. ***

§ 2.6. Completeness

Definition 2.6.1. A sequence $(x_n)_n$ in a metric space (X, d) is d-*Cauchy* if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) : (\forall p, q \in \mathbf{N})(p, q \geqslant N \implies d(x_p, x_q) < \epsilon).$$

Proposition 2.6.1. *Let (x_n) be a sequence in (X, d).*

(1) If $(x_n)_n$ converges, then $(x_n)_n$ is Cauchy.

(2) If $(x_n)_n$ is Cauchy, then $(x_n)_n$ is bounded.

Proof. (1) Let $x \in X$ and suppose $(x_n)_n \to x$. Let $\epsilon > 0$. Find N large so that for $p \geqslant N$ we have $d(x_p, x) < \frac{\epsilon}{2}$. Then $p, q \geqslant N$ implies:

$$\begin{aligned} d(x_p, x_q) &\leqslant d(x_p, x) + d(x, x_q) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

(2) Let $\epsilon = 1$. Find N large so $p, q \geqslant N$ implies $d(x_p, x_q) < 1$. Let $C = \max_{1 \leqslant p,q \leqslant N} d(x_p, x_q)$. Without loss of generality, if $p \geqslant N$ and $q \leqslant N$, then

$d(x_p, x_q) \leqslant d(x_p, x_N) + d(x_N, x_q) < 1 + C$. Set $K = \max\{1, 1 + C\}$. Then $\operatorname{diam}(\{x_n\}_{n\geqslant 1}) = \sup_{p,q\geqslant 1} d(x_p, x_q) < K$. □

Proposition 2.6.2. *Let $(x_n)_n$ be a Cauchy sequence in X and suppose there exists a subsequence $(x_{n_k})_k$ converging to $x \in X$. Then $(x_n)_n$ converges to x.*

Proof. Let $\epsilon > 0$. Since $(x_n)_n$ is Cauchy, there exists N large so $n, n_k \geqslant N$ implies $d(x_n, x_{n_k}) < \frac{\epsilon}{2}$. This gives:

$$\begin{aligned} d(x_n, x) &= d(x_n, \operatorname*{limit}_{k\to\infty} x_{n_k}) \\ &= \operatorname*{limit}_{k\to\infty} d(x_n, x_{n_k}) \\ &\leqslant \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

□

Definition 2.6.2. A metric space is said to be *complete* if every Cauchy sequence converges. A complete normed space is called a *Banach space*. A complete inner product space is called a *Hilbert space*.

Lemma 2.6.3. *Let $f : (X, d) \to (Y, \rho)$ be uniformly continuous. If $(x_n)_n$ is d-Cauchy then $(f(x_n))_n$ is ρ-Cauchy.*

Proof. Let $\epsilon > 0$. Find $\delta > 0$ so that $d(x, x') < \delta$ implies $\rho(f(x), f(x')) < \epsilon$. Pick N sufficiently large so that $p, q \geqslant N$ implies $d(x_p, x_q) < \delta$. This gives $\rho(f(x_p), f(x_q)) < \epsilon$, whence $(f(x_n))_n$ is ρ-Cauchy. □

Corollary 2.6.4. *If $f : (X, d) \to (Y, \rho)$ is a uniformism, then (X, d) is complete if and only if (Y, ρ) is complete.*

Proof. Let (X, d) be complete. If $(y_n)_n$ is ρ-Cauchy, then $(f^{-1}(y_n))_n$ is d-Cauchy in X. So we can find some $x \in X$ such that $(f^{-1}(y_n))_n \to x$. Then $(f(f^{-1}(y_n)))_n = (y_n)_n \to f(x)$. The converse follows similarly. □

Corollary 2.6.5. *If d_1 and d_2 are uniformly equivalent metrics on a set X, then (X, d_1) is complete if and only if (X, d_2) is complete.*

Proof. Since the map $\mathrm{id} : (X, d_1) \to (X, d_2)$ is a uniformism, the previous corollary gives that (X, d_1) is complete if and only if (X, d_2) is complete. □

Proposition 2.6.6. *ℓ_p^d is a Banach space for $1 \leqslant p \leqslant \infty$.*

Proof. We only need to show this for ℓ_∞^d since all the p-norms are equivalent. Let $(x_n)_n$ be $\|\cdot\|_\infty$-Cauchy in ℓ_∞^d. Let $\epsilon > 0$. Find N large so for $n, m \geqslant N$ we have

$\|x_n - x_m\|_\infty < \epsilon$. Observe that:

$$\begin{aligned}|x_n(k) - x_m(k)| &\leqslant \max_{1\leqslant k\leqslant d} |x_n(k) - x_m(k)| \\ &= \|x_n - x_m\|_\infty \\ &< \epsilon,\end{aligned}$$

where $x_n(k)$ is the k^{th} entry of the d-tuple x_n. So for each $k = 1, ..., d$, we know $(x_n(k))_n$ is Cauchy in F. Set $\operatorname{limit}_{n\to\infty} x_n(k) = x(k)$ for $k = 1, .., d$. This gives:

$$\|x - x_n\|_\infty = \max_{1\leqslant k\leqslant d} |x(k) - x_n(k)| \xrightarrow{n\to\infty} 0.$$

Whence $(x_n)_n \to x$ in ℓ_∞^d. Now set $y = (y_1, ..., y_n)$. We have:

$$\begin{aligned}\|x_n - y\|_p &\leqslant c' \|x_n - y\|_1 \\ &= c' \sum_{j=1}^{d} |x_n(j) - y_j| \\ &\leqslant c' \max_{1\leqslant j\leqslant d} |x_n(j) - y_j| \xrightarrow{n\to\infty} 0.\end{aligned}$$

Thus ℓ_p^d is complete. □

Proposition 2.6.7. *ℓ_p is a Banach space for $1 \leqslant p \leqslant \infty$.*

Proof. Suppose that $(f_n)_n$ is $\|\cdot\|_{\ell_p}$-Cauchy. Observe that

$$\begin{aligned}|f_n(k) - f_m(k)|^p &\leqslant \sum_{j=1}^{\infty} |f_n(j) - f_m(j)|^p \\ &= \|f_n - f_m\|_{\ell_p}^p\end{aligned}$$

So $(f_n(k))_n$ is Cauchy in F. Since this space is complete, define $\operatorname{limit}_{n\to\infty} f_n(k) := f(k)$.

Our goal is to find some function $f : \mathbf{N} \to F$ satisfying $f \in \ell_p$ and $\|f_n - f\|_{\ell_p} \to 0$. The $f(k)$ we've just obtained will lead us to the most suitable candidate.

Since $(f_n)_n$ is $\|\cdot\|_{\ell_p}$-Cauchy, it is bounded by some constant. Fix $K \geqslant 1$ and observe that:

$$\begin{aligned}\sum_{k=1}^{K} |f(k)|^p &= \sum_{k=1}^{K} \left|\operatorname*{limit}_{n\to\infty} f_n(k)\right|^p \\ &= \operatorname*{limit}_{n\to\infty} \sum_{k=1}^{K} |f_n(k)|^p \\ &\leqslant \sup_{n\geqslant 1} \|f_n\|_{\ell_p}^p \\ &:= C.\end{aligned}$$

Since the sequence $\left(\sum_{k=1}^{K}|f(k)|^p\right)_{K=1}^{\infty}$ is increasing and bounded above C, the Monotone Convergence Theorem says that it's limit exists. We obtain:

$$\begin{aligned}\operatorname*{limit}_{K\to\infty}\sum_{k=1}^{K}|f(k)|^p &= \sum_{k=1}^{\infty}|f(k)|^p\\ &= \|f\|_{\ell_p}^p\\ &< \infty.\end{aligned}$$

Thus $f \in \ell_p$.

It remains to show that $(f_n)_n$ converges to f. Given $\epsilon > 0$, find N large so that $n, m \geqslant N$ implies $\|f_n - f_m\|_p < \epsilon$. For every $n, m \geqslant N$ we have:

$$\begin{aligned}\sum_{k=1}^{K}|f_m(k) - f_n(k)|^p &\leqslant \|f_m - f_n\|_{\ell_p}^p\\ &< \epsilon^p.\end{aligned}$$

Taking the limit as $m \to \infty$ and considering all $n \geqslant N$ gives:

$$\sum_{k=1}^{K}|f(k) - f_n(k)|^p < \epsilon^p.$$

Finally, taking the limit as $K \to \infty$ and simplifying gives $\|f - f_n\|_{\ell_p} < \epsilon$. Thus ℓ_p is complete. □

Proposition 2.6.8. *Let (Y, d) be a complete metric space. The set of bounded functions $\mathrm{Bd}(\Omega, Y)$ with $\|\cdot\|_u$ is complete.*

Proof. Let $(f_n)_n$ be D_u-Cauchy. Fix $x, x' \in \Omega$ and let $\epsilon > 0$. Find N large so that $n, m \geqslant N$ implies:

$$d(f_n(x), f_m(x)) \leqslant D_u(f_n, f_m) < \epsilon.$$

Thus $(f_n(x))_n$ is Cauchy in Y. Since Y is complete, define $\operatorname{limit}_{n\to\infty} f_n(x) := f(x)$. Now find N_1 large so $n \geqslant N_1$ implies $d(f(x), f_n(x)) < \epsilon$. Find N_2 large so $n \geqslant N_2$ implies $d(f(x'), f_n(x')) < \epsilon$. Observe that:

$$\begin{aligned}d(f(x), f(x')) &\leqslant d(f(x), f_{N_1}(x)) + d(f_{N_1}(x), f_{N_2}(x')) + d(f_{N_2}(x'), f(x'))\\ &< 2\epsilon + d(f_{N_1}(x), f_{N_2}(x'))\\ &\leqslant 2\epsilon + d(f_{N_1}(x), f_{N_2}(x)) + d(f_{N_2}(x), f_{N_2}(x'))\\ &\leqslant 2\epsilon + \sup_{n,m\geqslant 1} d(f_n(x), f_m(x)) + \sup_{x,x'\in\Omega} d(f_{N_2}(x), f_{N_2}(x'))\\ &= 2\epsilon + \operatorname{diam}(\{f_n(x)\}_{n\geqslant 1}) + \operatorname{diam}(f_{N_2}(\Omega)).\end{aligned}$$

Note that the sequence $(f_n(x))_n$ is bounded because it is Cauchy —so there exists $C_1 \geqslant 0$ such that $\operatorname{diam}(\{f_n(x)\}_{n\geqslant 1}) < C_1$. Moreover, since $f_{N_2} \in \mathrm{Bd}(\Omega, Y)$, there exists $C_2 \geqslant 0$ such that $\operatorname{diam}(f_{N_2}(\Omega)) < C_2$. This gives:

$$d(f(x), f(x')) < 2\epsilon + C_1 + C_2.$$

Since this inequality is independent of any n, and since $x, x' \in \Omega$ was arbitrary, we have that $\operatorname{diam}(f(\Omega)) < \infty$; i.e., $f \in \mathrm{Bd}(\Omega, Y)$. With the same ϵ as above, find N_3 large so that for all $n, m \geqslant N_3$ then $D_u(f_n, f_m) < \frac{\epsilon}{2}$. We know:

$$d(f_n(x), f_m(x)) \leqslant D_u(f_n, f_m) < \frac{\epsilon}{2}.$$

Taking $m \to \infty$ gives $d(f_n(x), f(x)) \leqslant \frac{\epsilon}{2}$ for all $n \geqslant N$. But note that N does not depend on our fixed $x \in \Omega$. It follows that $D_u(f_n, f) \leqslant \frac{\epsilon}{2} < \epsilon$. Thus $(f_n)_n \to f$, establishing $\mathrm{Bd}(\Omega, Y)$ as complete. □

Corollary 2.6.9. *$\ell_\infty(\Omega)$ is complete.*

Proposition 2.6.10. *Let (X, d) be a complete metric space and $Y \subseteq X$. Y is complete if and only if Y is closed.*

Proof. Let $(y_n)_n$ be a sequence in Y converging to $x \in X$. Then $(y_n)_n$ is sequence in X. Since X is complete, $(y_n)_n$ is Cauchy. Since Y is complete, $(y_n)_n$ must converge to some $y \in Y$. Since sequences can have at most one limit, it must be that $y = x$. Thus $x \in Y$; i.e., Y is closed.

Conversely, if $(y_n)_n$ is Cauchy in Y, then it is Cauchy in X. Since X is complete, there exists $x \in X$ with $(y_n)_n \to x$. Since Y is closed, $x \in Y$. Thus Y is complete. □

Proposiiton 2.6.10 and Proposition 2.4.5 are extremely useful tools for showing a space is complete.

Corollary 2.6.11. *Let (X, d) and (Y, ρ) be metric spaces.*

(1) $C_b(X, Y) := C(X, Y) \cap \mathrm{Bd}(X, Y)$ is D_u-complete.

(2) $C_b(X)$ is a $\|\cdot\|_u$-Banach space.

(3) $C_0(\mathbf{R})$ is a $\|\cdot\|_u$-Banach space.

Proof. (1) Let $(f_n)_n$ be a sequence in $C_b(X, Y)$ converging to $f \in \mathrm{Bd}(X, Y)$. Let $x \in X$ and $\epsilon > 0$. Find N large so that $D_u(f_N, f) < \frac{\epsilon}{3}$. Find $\delta > 0$ so that for all $x' \in X$, $d(x, x') < \delta$ implies $\rho(f_N(x), f_N(x')) < \frac{\epsilon}{3}$. For $d(x, x') < \delta$:

$$\begin{aligned}
\rho(f(x), f(x')) &\leqslant \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(x')) + \rho(f_N(x'), f(x')) \\
&\leqslant 2D_u(f_n, f) + \rho(f_N(x), f_N(x')) \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Thus $f \in C_b(X,Y)$ because it is bounded and continuous. Since $C_b(X,Y) \subseteq \mathrm{Bd}(X,Y)$ is closed, it is complete.

(2) As we've just shown, the space $C_b(X)$ is complete. It only remains to show it is a vector space.

(3) Let $(f_n)_n$ be a sequence in $C_0(\mathbf{R})$ converging to $f \in C_b(\mathbf{R})$. Let $\epsilon > 0$ and find N large so that $\|f - f_N\|_u < \frac{\epsilon}{2}$. Since $f_N \in C_0(\mathbf{R})$, we know that $\operatorname{limit}_{x\to\pm\infty} f_N(x) = 0$. So there exists $M > 0$ with $|x| \geqslant M$ implying $|f_N(x)| < \frac{\epsilon}{2}$. For $|x| \geqslant M$ observe that:

$$\begin{aligned}
|f(x)| &\leqslant |f(x) - f_N(x)| + |f_N(x)| \\
&\leqslant \|f - f_N\|_u + |f_N(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus $\operatorname{limit}_{x\to\infty} f(x) = 0$; i.e., $f \in C_0(\mathbf{R})$. Since $C_0(\mathbf{R}) \subseteq C_b(\mathbf{R})$ is closed, it is complete. □

Proposition 2.6.12. *Let V be a normed space and W a Banach space. $B(V,W)$ with $\|\cdot\|_{\mathrm{op}}$ is a Banach space.*

Proof. Let $(T_n)_n$ be $\|\cdot\|_{\mathrm{op}}$-Cauchy. Let $v \in V$ and $\epsilon > 0$. Find N_1 large so that $n, m \geqslant N_1$ implies $\|T_n - T_m\|_{\mathrm{op}} < \frac{\epsilon}{\|v\|}$. We can see:

$$\begin{aligned}
\|T_n v - T_m v\|_W &= \|(T_n - T_m)v\|_W \\
&\leqslant \|T_n - T_m\|_{\mathrm{op}} \|v\| \\
&< \frac{\epsilon}{\|v\|} \cdot \|v\| \\
&= \epsilon.
\end{aligned}$$

So $(T_n v)_n$ is $\|\cdot\|_W$-Cauchy. Since W is a Banach space, define $\operatorname{limit}_{n\to\infty} T_n v = Tv$. We must show that T is linear, bounded, and $\|T_n - T\|_{\mathrm{op}} \to 0$. Given $v_1, v_2 \in V$ and $c \in F$ we can see:

$$\begin{aligned}
T(v_1 + cv_2) &= \operatorname*{limit}_{n\to\infty} T_n(v_1 + cv_2) \\
&= \operatorname*{limit}_{n\to\infty} T_n v_1 + cT_n v_2 \\
&= \operatorname*{limit}_{n\to\infty} T v_1 + c \operatorname*{limit}_{n\to\infty} T v_2 \\
&= Tv_1 + cTv_2.
\end{aligned}$$

Thus T is linear. Now since $(T_n)_n$ is $\|\cdot\|_{\mathrm{op}}$-Cauchy, it is bounded, so there exists

$C > 0$ with $\|T_n\|_{\text{op}} \leqslant C$ for all $n \geqslant 1$. Using the fact norms are continuous, we have:

$$\begin{aligned}\|Tv\|_W &= \left\|\operatorname*{limit}_{n\to\infty} T_n v\right\|_W \\ &= \operatorname*{limit}_{n\to\infty} \|T_n v\|_W \\ &\leqslant \limsup_{n\to\infty} \|T_n\|_{\text{op}} \|v\| \\ &\leqslant C\|v\|.\end{aligned}$$

Thus $T \in B(V, W)$. With the same epsilon as before, find N_2 so that $n, m \geqslant N_2$ implies $\|T_n - T_m\|_{\text{op}} < \frac{\epsilon}{2}$. We can show:

$$\|T_n v - T_m v\|_W \leqslant \|T_n - T_m\|_{\text{op}} < \frac{\epsilon}{2}.$$

Taking $m \to \infty$ gives:

$$\|T_n v - Tv\|_W \leqslant \frac{\epsilon}{2}.$$

Taking the supremum over all $v \in B_V$ gives:

$$\|T_n - T\|_{\text{op}} \leqslant \frac{\epsilon}{2} < \epsilon.$$

Thus $B(V, W)$ is complete. □

Proposition 2.6.13. *Let $(V, \|\cdot\|)$ be a normed space. The following are equivalent:*

(1) V is a Banach space;

(2) If $(v_k)_k$ is a sequence in V with $\sum_{k=1}^{\infty} \|v_k\|$ convergent, then $\sum_{k=1}^{\infty} v_k$ converges.

Proof. Suppose V is a Banach space. Let $s_n = \sum_{k=1}^{n} v_k$ and $t_n = \sum_{k=1}^{n} \|v_k\|$. For $p > q > 1$:

$$\begin{aligned}\|s_p - s_q\| &= \left\|\sum_{k=q+1}^{p} v_k\right\| \\ &\leqslant \sum_{k=q+1}^{p} \|v_k\| \\ &= |t_p - t_k|.\end{aligned}$$

Since $(t_n)_n$ is convergent, it is Cauchy. So $(s_n)_n$ is Cauchy, implying it is convergent. Thus $\sum_{k=1}^{\infty} v_k$ converges.

Now let $(v_n)_n$ be Cauchy. Find $n_1 \in \mathbf{N}$ such that $p, q \geqslant n_1$ implies $\|v_p - v_q\| < 2^{-1}$. Find $n_2 > n_1$ such that $p, q \geqslant n_2$ implies $\|v_p - v_q\| < 2^{-2}$. Inductively, find

$n_k > n_{k-1}$ such that $p, q \geqslant n_k$ implies $\|v_p - v_q\| < 2^{-k}$. Consider the sequence $(v_{n_{k+1}} - v_{n_k})_k$. Then:

$$\sum_{k=1}^{\infty} \left\| v_{n_{k+1}} - v_{n_k} \right\| \leqslant \sum_{k=1}^{\infty} 2^{-k} = 1.$$

By our hypothesis, $\sum_{k=1}^{\infty}(v_{n_{k+1}} - v_{n_k})$ converges. So the sequence of partial sums:

$$\begin{aligned} w_m &= \sum_{k=1}^{m} v_{n_{k+1}} - v_{n_k} \\ &= v_{n_m} - v_{n_1} \end{aligned}$$

also converges to some $w \in V$ as $m \to \infty$. However, notice:

$$\begin{aligned} (v_{n_m})_m &= (v_{n_m} - v_{n_1})_m + v_{n_1} \\ &\xrightarrow{m\to\infty} w + v_{n_1}. \end{aligned}$$

Since $(v_n)_n$ is a Cauchy sequence which admits a convergent subsequence, $(v_n)_n$ converges. Thus V is a Banach space. □

Example 2.6.1. *** Let $\mathcal{H}$ be a Hilbert space. Suppose $(e_n)_n$ is an orthonormal sequence in $\mathcal{H}$ and $(t_k)_k \in \ell_2$. We will show $\sum_{k=1}^{\infty} t_k e_k$ converges in $\mathcal{H}$, but not absolutely in general.

Let $s_n = \sum_{k=1}^{n} t_k e_k$. For $n > m > 1$:

$$\begin{aligned} \|s_n - s_m\|^2 &= \left\| \sum_{k=m+1}^{n} t_k e_k \right\|^2 \\ &= \sum_{k=m+1}^{n} |t_k|^2. \end{aligned}$$

Since $\left(\sum_{k=1}^{n} |t_k|^2\right)_n$ is Cauchy, then $(s_n)_n$ is Cauchy, whence $\sum_{k=1}^{\infty} t_k e_k$ converges. But notice $\|s_n\|^2 = \sum_{k=1}^{n} |t_k|^2$. As $n \to \infty$, we see $\|\sum_{k=1}^{\infty} t_k e_k\|^2 = \sum_{k=1}^{\infty} |t_k|^2$.

————//————

Recall that if $f : (X, d) \to (Y, \rho)$ is a uniformly continuous map between metric spaces and $(x_n)_n$ is Cauchy, then $(f(x_n))_n$ is Cauchy. Complete spaces have the unique property that, given a map from a dense subset $A \to Y$, we can define an extension of such function.

Theorem 2.6.14. *Let (X, d) and (Y, ρ) be metric spaces with Y complete. Suppose $A \subseteq X$ is dense and $f : A \to Y$ is uniformly continuous. There exists a unique uniformly continuous map $\widetilde{f} : X \to Y$ with $\widetilde{f}(x) = f(x)$ for all $x \in A$.*

Proof. Let $x \in X$. We know there exists a sequence $(a_n)_n$ in A with $(a_n)_n \to x$. Since $(a_n)_n$ is convergent, it is Cauchy, so $(f(a_n))_n$ is also Cauchy. By the completeness of Y, $(f(a_n))_n$ converges. Define $\widetilde{f}(x) := \text{limit}_{n\to\infty} f(a_n)$.

We must show this extension is well-defined. Suppose $(b_n)_n$ is another sequence in A with $(b_n)_n \to x$. Then the mixed sequence $(a_1, b_1, a_2, b_2, ...)$ will converge to x. The same reasoning as above tells us $(f(a_1), f(b_1), f(a_2), f(b_2), ...)$ converges in Y. The two subsequences $(f(a_n))_n$ and $(f(b_n))_n$ must then converge to the same limit.

We will now show that $\widetilde{f}$ is uniformly continuous. Let $\epsilon > 0$. Find $\delta > 0$ so that for all $a, b \in A$, $d(a, b) < \delta$ implies $\rho(f(a), f(b)) < \frac{\epsilon}{2}$. Now let $x, x' \in X$ with $d(x, x') < \frac{\delta}{4}$. Find sequences $(a_n)_n$ and $(b_n)_n$ in A with $(a_n)_n \to x$ and $(b_n)_n \to x'$. Find N large so that $n \geqslant N$ implies $d(a_n, x) < \frac{\delta}{4}$ and $d(b_n, x) < \frac{\delta}{4}$. The triangle inequality gives $d(a_n, b_n) < \frac{3\delta}{4} < \delta$. So $\rho(f(a_n), f(b_n)) < \frac{\epsilon}{2}$ for all $n \geqslant N$. Observe that:

$$\begin{aligned}\rho(\widetilde{f}(x), \widetilde{f}(x')) &= \operatorname*{limit}_{n\to\infty} \rho(f(a_n), f(b_n)) \\ &\leqslant \frac{\epsilon}{2} \\ &< \epsilon.\end{aligned}$$

Thus $\widetilde{f}$ is uniformly continuous. It remains to show that $\widetilde{f}$ is unique. Suppose $g : X \to Y$ is also a continuous extension of f. Then $g(x) = f(x) = \widetilde{f}(x)$ for all $x \in A$. Since g and $\widetilde{f}$ agree on elements of a dense set, we must have $g = \widetilde{f}$. □

Proposition 2.6.15. *Let (X, d) and (Y, ρ) be metric spaces with $A \subseteq X$ dense, Y complete, and $f : A \to Y$ an isometry. Then the continuous extension $\widetilde{f} : X \to Y$ is an isometry.*

Proof. Let $x, x' \in X$. Let $(a_n)_n$ and $(b_n)_n$ be sequences in A with $(a_n)_n \to x$ and $(b_n)_n \to x'$. We have:

$$\begin{aligned}\rho(\widetilde{f}(x), \widetilde{f}(x')) &= \operatorname*{limit}_{n\to\infty} \rho(f(a_n), f(b_n)) \\ &= \operatorname*{limit}_{n\to\infty} d(a_n, b_n) \\ &= d(x, x').\end{aligned}$$ □

Corollary 2.6.16. *Let V be a normed space, W a Banach space, and $U \subseteq V$ a dense linear subspace. Let $T_0 \in B(U, W)$. There exists a unique $T \in B(V, W)$ with $\|T\|_{\text{op}} = \|T_0\|_{\text{op}}$. Moreover, if T_0 is isometric, then so if T.*

Proof. We only need to show T is linear and $\|T\|_{\text{op}} = \|T_0\|_{\text{op}}$. Let $v, v' \in V$ and $\alpha \in F$. Let $(x_n)_n$ and $(y_n)_n$ be sequences in U with $(x_n)_n \to v$ and $(y_n)_n \to v'$.

Observe that:

$$\begin{aligned} T(v+\alpha v') &= \operatorname*{limit}_{n\to\infty} T_0(x_n+\alpha y_n) \\ &= \operatorname*{limit}_{n\to\infty} T_0(x_n) + \alpha \operatorname*{limit}_{n\to\infty} T_0(y_n) \\ &= T(v) + \alpha T(v'). \end{aligned}$$

Note that the composition $V \xrightarrow{T} W \xrightarrow{\|\cdot\|_W} F$ will be continuous and bounded, so by Proposition **??** we have:

$$\begin{aligned} \|T\|_{\mathrm{op}} &= \sup_{v\in B_V} \|T(v)\| \\ &= \sup_{v\in B_U} \|T(v)\| \\ &= \sup_{v\in B_U} \|T_0(v)\| \\ &= \|T_0\|_{\mathrm{op}}. \end{aligned}$$

Thus $T \in B(V,W)$. $\square$

Example 2.6.2. *** (Something about Hilbert spaces...)

Proposition 2.6.17. *Let $T : V \to W$ be a continuous linear map between normed spaces which is bounded below, that is, there is a $C > 0$ with $\|Tv\| \geqslant C\|v\|$ for all $v \in V$. If V is complete, then* $\operatorname{im}(T) \subseteq W$ *is a closed subspace, and $V \cong \operatorname{im}(T)$ are uniformly isomorphic.*

Proof. Let $(T(v_n))_n$ be a sequence in $\operatorname{im}(T)$ converging to $w \in W$. Given ϵ, find N large so that $n \geqslant M$ implies $\|T(v_n) - w\| < \frac{C\epsilon}{2}$. For $n, m \geqslant N$, observe that:

$$\begin{aligned} \|v_n - v_m\| &\leqslant \frac{1}{C}\|T(v_n - v_m)\| \\ &= \frac{1}{C}\|T(v_n) - T(v_m)\| \\ &\leqslant \frac{1}{C}\|T(v_n) - w\| + \frac{1}{C}\|w - T(v_m)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $(v_n)_n$ is Cauchy. Since V is complete, let $v_0 := \operatorname{limit}_{n\to\infty} v_n$. Since T is continuous, we can see $(T(v_n))_n \to T(v_0)$. It must be the case that $T(v_0) = w$; i.e., $w \in \operatorname{im}(T)$. Thus $\operatorname{im}(T)$ is a closed subspace.

Since T is continuous, there exists some $\alpha > 0$ such that $\|Tv\| \leqslant \alpha\|v\|$. Clearly if $v = 0$, then $Tv = 0$, implying that T is injective. Whence $V \cong \operatorname{im}(T)$ as vector spaces. Since T is continuous, it is uniformly continuous, so it remains to show that $T^{-1} : \mathrm{im}(T) \to V$ (which exists) is also continuous. Let $w \in \mathrm{im}(T)$, then there exists $v \in V$ with $T(v) = w$. Observe that:

$$\begin{aligned} \left\|T^{-1}w\right\| &= \left\|T^{-1}(T(v))\right\| \\ &= \|v\| \\ &\leqslant \frac{1}{C}\|Tv\| \\ &= \frac{1}{C}\|w\|. \end{aligned}$$

Thus T is a uniformism. □

——————//——————

Definition 2.6.3. Let (X, d) be a metric space. A *completion* of (X, d) is a pair $\big((Z, \rho), \iota\big)$ where:

(1) (Z, ρ) is a complete metric space;

(2) $\iota : X \hookrightarrow Z$ is an isometry;

(3) $\overline{\iota(X)}^{\rho} = Z$.

Example 2.6.3. $\big(([0,1], \|\cdot\|), \iota(t) = t\big)$ is a completion of $(0, 1)$.

Lemma 2.6.18. *Let $f : (X, d) \to (Y, \rho)$ be an isometry between metric spaces. If X is complete, then $f(X) \subseteq Y$ is closed. In particular, if $f(X)$ is dense, then f is onto.*

Proof. If $(f(x_n))_n \to y$ in Y, then $(f(x_n))_n$ is ρ-Cauchy. Since f is an isometry, $d(x_n, x_m) = \rho(f(x_n), f(x_m))$, so $(x_n)_n$ is d-Cauchy. Let $x \in X$ such that $(x_n)_n \to x$. Since f is continuous, we have $(f(x_n))_n \to f(x)$. It must be that $y = f(x) \in f(X)$.

Since we've just shown that $f(X)$ is closed, if it were also dense then $f(X) = \overline{f(X)}^{\rho} = Y$, whence f is surjective. □

Theorem 2.6.19. *Let (X, d) be a metric space. If $\big((Z, \rho), \iota\big)$ and $\big((Z', \rho'), \jmath\big)$ are completions of X, there exists a unique isometric isomorphism $\varphi : Z \to Z'$ with $\varphi \circ \iota = \jmath$; that is, the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\jmath"'] & Z \arrow[d, dotted, "\varphi"] \\ & Z' \end{tikzcd}$$

Proof. Let $z \in Z$. Since $\iota(X)$ is dense in Z, there exists a sequence $(\iota(x_n))_n$ in $\iota(X)$ such that $(\iota(x_n))_n \to z$. It is clear that $(\iota(x_n))_n$ is ρ-Cauchy, and furthermore we have:

$$\begin{aligned} \rho'(\jmath(x_n), \jmath(x_m)) &= d(x_n, x_m) \\ &= \rho(\iota(x_n), \iota(x_m)). \end{aligned}$$

So if $(\iota(x_n))_n$ is ρ-Cauchy, then $(\jmath(x_n))_n$ is ρ'-Cauchy. Since $\jmath(X)$ is complete, define $\varphi(z) := \text{limit}_{n\to\infty}\, \jmath(x_n)$.

We will show that $\varphi : Z \to Z'$ is well-defined, that is, it does not depend on our particular choice of sequence. Let (y_n) be another sequence in X with $(y_n)_n \to z$. We can see that:

$$\begin{aligned} d(x_n, y_n) &= \rho(\iota(x_n), \iota(y_n)) \\ &\leqslant \rho(\iota(x_n), z) + \rho(z, \iota(y_n)) \\ &\to 0, \end{aligned}$$

which gives:

$$\begin{aligned} \rho'(\jmath(y_n), \varphi(z)) &\leqslant \rho'(\jmath(y_n), \jmath(x_n)) + \rho'(\jmath(x_n), \varphi(z)) \\ &= d(y_n, x_n) + \rho'(\jmath(x_n), \varphi(z)) \\ &\to 0. \end{aligned}$$

Thus $(\jmath(y_n))_n \to \varphi(z)$, so φ is well-defined.

We will now show that φ is an isometric isomorphism. Given $z_1, x_2 \in Z$, let $(\iota(x_n))_n \to z_1$ and $(\iota(y_n))_n \to z_2$, corresponding to $(\jmath(x_n))_n \to \varphi(z_1)$ and $(\jmath(y_n))_n \to \varphi(z_2)$. Observe that:

$$\begin{aligned} \rho'(\varphi(z_1), \varphi(z_2)) &= \underset{n\to\infty}{\text{limit}}\, \rho'(\jmath(x_n), \jmath(y_n)) \\ &= \underset{n\to\infty}{\text{limit}}\, d(x_n, y_n) \\ &= \underset{n\to\infty}{\text{limit}}\, \rho(\iota(x_n), \iota(y_n)) \\ &= \rho(z_1, z_2). \end{aligned}$$

Since φ is an isometry, it is injective. Clearly $\varphi \circ \iota = \jmath$ holds, so we have $\jmath(X) = \varphi(\iota(X)) \subseteq \varphi(Z)$. Then $Z' = \overline{\jmath(X)}^{\rho} \subseteq \overline{\varphi(Z)}$. Since Z is complete, by the previous lemma $\varphi(Z)$ is closed, whence $Z' \subseteq \overline{\varphi(Z)} = \varphi(Z)$. Thus φ is onto.

It remains to show that φ is unique. Let $\psi : Z \to Z'$ be another isometric isomorphism satisfying $\psi \circ \iota = \jmath$. Given $z \in Z$, we can find a sequence $(\iota(x_n))_n$ in $\iota(X)$ such that $(\iota(x_n))_n \to z$. Observe that:

$$\begin{aligned} \varphi(z) &= \underset{n\to\infty}{\text{limit}}\, \varphi(\iota(x_n)) \\ &= \underset{n\to\infty}{\text{limit}}\, \jmath(x_n) \\ &= \underset{n\to\infty}{\text{limit}}\, \psi(\iota(x_n)) \\ &= \psi(z). \end{aligned}$$

Since $z \in Z$ was arbitrary, this proves $\varphi = \psi$. $\square$

Lemma 2.6.20. *If (X, d) is a metric space and $i : (X, d) \to (Y, \rho)$ is an isometry into a complete metric space (Y, ρ), then $\left((\overline{i(X)}^{\rho}, \rho), i\right)$ is a completion of X.*

Proof. The space $\left(\overline{i(X)}^{\rho}, \rho\right)$ is complete by Proposition 2.6.10. Clearly $i : X \to \overline{i(X)}^{\rho}$ is an isometry because $\overline{i(X)}^{\rho} \subseteq Y$. $\square$

Theorem 2.6.21. *Every metric space admits a unique completion up to isometric isomorphism.*

Proof. Uniqueness was shown in Theorem 2.6.19. Given (X, d), consider the Banach space $(C_b(X), \|\cdot\|_u)$. By the previous lemma we only need to construct an isometry $X \overset{i}{\hookrightarrow} C_b(X)$.

Fix any $x_0 \in X$. Define $f_x : X \to F$ by $f_x(t) = d(t, x) - d(t, x_0)$. Clearly f_x is continuous, and it is bounded because $|f_x(t)| = |d(t, x) - d(t, x_0)| \leqslant d(x, x_0)$. So $f_x \in C_b(X)$. Now define $i : X \to C_b(X)$ by $x \mapsto f_x$. Observe that:

$$\begin{aligned}
\|f_x - f_y\|_u &= \sup_{t \in X} |f_x(t) - f_y(t)| \\
&= \sup_{t \in X} |d(t, x) - d(t, x_0) - d(t, y) + d(t, x_0)| \\
&= \sup_{t \in X} |d(t, x) - d(t, y)| \\
&= d(x, y).
\end{aligned}$$

Thus i is an isometry, making $\left((\overline{i(X)}^{\|\cdot\|_u}, \|\cdot\|_u), i\right)$ a completion of X. $\square$

Theorem 2.6.22. *Let (X, d) be a metric space with completion $\big((Z, \rho), i\big)$. If $f : X \to Y$ is a uniformly continuous function into a complete metric space Y, then there exists a unique uniformly continuous function $\widetilde{f} : Z \to Y$ such that $\widetilde{f} \circ i = f$; i.e., the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[d, dotted, "\widetilde{f}"] \\
& Y
\end{tikzcd}$$

Proof. Define $g : i(X) \to Z$ by $g = f \circ i^{-1}$. Since g is uniformly continuous and $i(X) \subseteq Z$ is dense, Theorem 2.6.14 says there exists a unique uniformly continuous map $\widetilde{f} : Z \to Y$. Clearly $\widetilde{f} \circ i = f$. $\square$

Theorem 2.6.23. **** Let $(V, \|\cdot\|)$ be a normed space. The completion of V is a Banach space.*

Proof. Let $\big((W,\rho),i\big)$ be the completion of V. We must first verify that W is a vector space. In doing so, we have to define what the vector space operations of W are. Let $w \in W$ and $\alpha \in F$. We know there exists a sequence $(i(v_n))_n$ in $i(V)$ converging to w. Since this sequence is convergent, it is ρ-Cauchy. From the following:

$$\begin{aligned}\rho(i(\alpha v_n), i(\alpha v_m)) &= \|\alpha v_n - \alpha v_m\| \\ &= |\alpha|\,\|v_n - v_m\| \\ &= |\alpha|\rho(i(v_n), i(v_m)),\end{aligned}$$

we can see $(i(\alpha v))_n$ is also ρ-Cauchy, whence it is convergent. Define $s : F \times W \to W$ by $s(\alpha, w) = \operatorname{limit}_{n\to\infty} i(\alpha v_n) := \alpha w$. We will first show that this action is well-defined. Let $(i(u_n))_n$ be another sequence in $i(V)$ converging to w. As above, we have:

$$\begin{aligned}\rho(i(\alpha v_n), i(\alpha u_n)) &= |\alpha|\rho(i(v_n), i(u_n)) \\ &\leqslant \alpha\big(\rho(i(v_n), w) + \rho(w, i(u_n))\big) \\ &\to 0.\end{aligned}$$

This gives:

$$\begin{aligned}\rho(i(\alpha u_n), \alpha w) &\leqslant \rho(i(\alpha u_n), i(\alpha v_n)) + \rho(i(\alpha v_n), \alpha w) \\ &\to 0.\end{aligned}$$

Thus $(i(\alpha u_n))_n \to \alpha w$, meaning scalar multiplication is well-defined. Now let $w_1, w_2 \in W$. Let $(i(v_n))_n$ and $(i(u_n))_n$ be sequences in $i(V)$ converging respectively to w_1 and w_2. Since these sequences are convergent, they are ρ-Cauchy. From the fact that:

$$\begin{aligned}\rho(i(v_n + u_n), i(v_m + u_m)) &= \|v_n + u_n - v_m - u_m\| \\ &\leqslant \|v_n - v_m\| + \|u_n - u_m\| \\ &= \rho(i(v_n), i(v_m)) + \rho(i(u_n), i(u_m)),\end{aligned}$$

we can see $(i(v_n+u_n))_n$ is also ρ-Cauchy, hence it is convergent. Define $a : W \times W \to W$ by $a(w_1, w_2) = \operatorname{limit}_{n\to\infty} i(v_n+u_n) := w_1+w_2$. We will show this binary operation is well-defined. Let $(i(x_n))_n$ and $(i(y_n))_n$ be sequences in $i(V)$ also converging to w_1 and w_2 respectively. Note that:

$$\begin{aligned}\rho(i(v_n + u_n), i(x_n + y_n)) &= \|v_n + u_n - x_n - y_n\| \\ &\leqslant \|v_n - x_n\| + \|u_n - y_n\| \\ &= \rho(i(v_n), i(x_n)) + \rho(i(u_n), i(y_n)) \\ &\leqslant \rho(i(v_n), w_1) + \rho(w_1, (x_n)) + \rho(i(u_n), w_2) + \rho(w_2, i(y_n)) \\ &\to 0.\end{aligned}$$

This gives:

$$\begin{aligned}\rho(i(x_n+y_n), w_1+w_2) &\leqslant \rho(i(x_n+y_n), i(v_n+u_n)) + \rho(i(v_n+u_n), w_1+w_2)\\ &\to 0.\end{aligned}$$

Thus $(i(x_n+y_n))_n \to w_1+w_2$, meaning vector addition is well-defined.

Before showing W paired with the above operations is a vector space, we need to verify that the isometry $i : V \to W$ is linear. If $v, v' \in V$ and $\alpha \in F$, by taking $v_n = v$ and $u_n = v'$ for all $n \geqslant 1$, we can see that:

$$\begin{aligned}i(v+\alpha v') &= \operatorname*{limit}_{n\to\infty} i(v+\alpha v')\\ &= i(v) + \alpha i(v').\end{aligned}$$

With this fact, we can show that W is a vector space —this is left as an exercise. Moreover, we have that $i(V) \subseteq W$ is a ρ-dense linear subspace[2].

It remains to show that W is a normed space. Exactly as before, if $w \in W$, we can find a sequence $(i(v_n))_n$ in $i(V)$ converging to w. Since this sequence is Cauchy, observe that:

$$\begin{aligned}\Big|\|v_n\| - \|v_m\|\Big| &\leqslant \|v_n - v_m\|\\ &= \rho(i(v_n), i(v_m)).\end{aligned}$$

So $\big(\|v_n\|\big)_n$ is Cauchy, hence it is convergent. Define $\|w\|_W := \text{limit}_{n\to\infty} \|v_n\|$. Showing this definition is well-defined, and that is satisfies the properties of a norm are left as an exercise. □

——————//——————

There is a softer proof of Theorem 2.6.23, but it requires heavier machinery. If V is a vector space over F, recall that

$$V' = \{\varphi \mid \varphi : V \to F \text{ linear}\}$$

is the linear space of all linear functionals on V. By Zorn's Lemma, $V' \neq \emptyset$. If V is a normed space, then

$$V^* = \{\varphi \in V' \mid \varphi \text{ continuous}\}$$

is called the *continuous dual space*. This is in fact a Banach space with norm $\|\varphi\|_{\text{op}} = \sup_{v\in B_V} |\varphi(v)|$. However, is $V^* \neq \emptyset$?

[2]If $T : V \to W$ is injective, then $V \cong T(V)$

Theorem 2.6.24 (Hahn-Banach)**.** *Let V be a normed space. For any nonzero $v_0 \in V$, there is a $\varphi_{v_0} \in V^*$ with $\varphi_{v_0}(v_0) = \|v_0\|$. Such a φ_{v_0} is called a norming functional.*

Corollary 2.6.25. *Let V be a normed space and $v \in V$. Then $\|v\| = \sup_{\varphi \in B_{V^*}} |\varphi(v)|$.*

Proof. If $\varphi \in B_{V^*}$, then $|\varphi(v)| \leqslant \|\varphi\|_{\text{op}} \|v\| \leqslant \|v\|$. Choose $\varphi_v \in V^*$. Since $\varphi_v(v) = \|v\|$, we have $\sup_{v \in B_{V^*}} |\varphi(v)| \geqslant |\varphi_v(V)| = \|v\|$. □

Corollary 2.6.26. *There is a linear isometry $V \hookrightarrow (V^*)^*$.*

Proof. Let $v \in V$. Define $\widehat{v} : V^* \to F$ by $\varphi \mapsto \varphi(v)$. We can easily verify that $\widehat{v} \in (V^*)'$. By the above corollary:

$$\begin{aligned} \|\widehat{v}\| &= \sup_{\varphi \in B_{V^*}} |\widehat{v}(\varphi)| \\ &= \sup_{\varphi \in B_{V^*}} |\varphi(v)| \\ &= \|v\| . \end{aligned}$$

Whence $\widehat{v} \in (V^*)^*$. Define $i_V : V \to (V^*)^*$ by $v \mapsto \widehat{v}$. Note that this is an isometry since $\|i_V(v)\| = \|\widehat{v}\| = \|v\|$. Given $v_1, v_2 \in V$, $c \in F$, and $\varphi \in V^*$ we can see:

$$\begin{aligned} i_V(v_1 + cv_2)(\varphi) &= \widehat{v_1 + cv}_2(\varphi) \\ &= \varphi(v_1 + cv_2) \\ &= \varphi(v_1) + c\varphi(v_2) \\ &= \widehat{v_1}(\varphi) + c\widehat{v_2}(\varphi) \\ &= i_V(v_1)(\varphi) + ci_V(v_2)(\varphi). \end{aligned}$$

Thus i_V is linear. □

Using the tools above, we can now demonstrate in a cleaner way that the completion of any normed space forms a Banach space.

Alternative proof of Theorem 2.6.23. If V is a normed space, then $(V^*)^* = B(V^*, F)$ is complete by Proposition 2.6.12. Then $\overline{i_V(V)}^{\|\cdot\|_{\text{op}}} \subseteq (V^*)^*$ is complete by Proposition 2.6.10. Clearly $i_V : V \to \overline{i_V(V)}^{\|\cdot\|_{\text{op}}}$ is an isometry because $\overline{i_V(V)}^{\|\cdot\|_{\text{op}}} \subseteq (V^*)^*$. Thus the completion of V is $\left((\overline{i_V(V)}^{\|\cdot\|_{\text{op}}}, \|\cdot\|_{\text{op}}), i_V \right)$ □

Lemma 2.6.27. *Let $T : V \to W$ be continuous and linear. There is an induced continous and linear map $T^* : W^* \to V^*$ with $T^*(\psi) = \psi \circ T$; i.e., the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "T^*(\psi)"'] & W \arrow[d, "\psi"] \\ & F \end{tikzcd}$$

Proof. Let $\psi_1\,\psi_2 \in W^*$ and $\alpha \in F$. We have:

$$\begin{aligned} T^*(\psi_1 + \alpha\psi_2) &= (\psi_1 + \alpha\psi_2) \circ T \\ &= \psi_1 \circ T + \alpha(\psi_2 \circ T) \\ &= T^*(\psi_1) + \alpha T^*(\psi_2). \end{aligned}$$

Thus T^* is linear. Moreover:

$$\begin{aligned} \|T^*\|_{\mathrm{op}} &= \sup_{\|\psi\|_{\mathrm{op}} \leqslant 1} \|T^*(\psi)\|_{\mathrm{op}} \\ &= \sup_{\|\psi\|_{\mathrm{op}} \leqslant 1} \|\psi \circ T\|_{\mathrm{op}} \\ &\leqslant \sup_{\|\psi\|_{\mathrm{op}} \leqslant 1} \left(\|\psi\|_{\mathrm{op}} \|T\|_{\mathrm{op}}\right) \\ &= \|T\|_{\mathrm{op}} \cdot \sup_{\|\psi\|_{\mathrm{op}} \leqslant 1} \|\psi\|_{\mathrm{op}} \\ &\leqslant \|T\|_{\mathrm{op}}. \end{aligned}$$

Thus T^* is continuous. □

Theorem 2.6.28. *Let V and W be normed spaces with completions $\widetilde{V} = \overline{i_V(V)}$ and $\widetilde{W} = \overline{i_W(W)}$. If $T : V \to W$ is a continuous and linear map, there exists a unique continuous and linear map $\widetilde{T} : \widetilde{V} \to \widetilde{W}$ such that $\widetilde{T} \circ i_V = i_W \circ T$; i.e., the following diagram commutes:*

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\widetilde{T}"] & \widetilde{W} \\ V \arrow[u, "i_V"] \arrow[r, "T"'] & W \arrow[u, "i_W"'] \end{tikzcd}$$

Proof. Since T is continuous, we can induce the map $T^* : W^* \to V^*$ where $T^*(\psi) = \psi \circ T$. Since T^* is continuous, we can induce the map $T^{**} : V^{**} \to W^{**}$ where $T^{**}(\xi) = \xi \circ T^*$. If $v \in V$ and $\psi \in V^*$, note that:

$$\begin{aligned} (\widehat{v} \circ T^*)(\psi) &= (T^* \circ \psi)(v) \\ &= (\psi \circ T)(v) \\ &= \widehat{T(v)}(\psi). \end{aligned}$$

This allows us to show:

$$\begin{aligned} T^{**} \circ i_V(v) &= \widehat{v} \circ T^* \\ &= \widehat{T(v)} \\ &= i_W(T(v)). \end{aligned}$$

Whence $T^{**}(i_V(V)) \subseteq i_W(W)$. Since T^{**} is continuous:

$$\begin{aligned} T^{**}\left(\widetilde{V}\right) &= T^{**}\left(\overline{i_V(V)}\right) \\ &\subseteq \overline{T^{**}(i_V(V))} \\ &\subseteq \overline{i_W(W)} \\ &= \widetilde{W}. \end{aligned}$$

Thus $T^{**}|_{\widetilde{V}} : \widetilde{V} \to \widetilde{W}$ is the desired linear extension. We obtain the following diagram:

$$\begin{array}{ccc} V^{**} & \xrightarrow{T^{**}} & W^{**} \\ \cup\!| & & \cup\!| \\ \widetilde{V} & \xrightarrow{T^{**}|_{\widetilde{V}}} & \widetilde{W} \\ \cup\!| & & \cup\!| \\ i_V(V) & \xrightarrow{T^*|_{i_V(V)}} & i_W(W) \\ {\scriptstyle i_V}\uparrow & & \uparrow{\scriptstyle i_W} \\ V & \xrightarrow[T]{} & W \end{array}$$

$\square$

Exercise 2.6.1. Show that $\left\|\widetilde{T}\right\|_{\text{op}} = \|T\|_{\text{op}}$.

Definition 2.6.4. If V is a normed space and $i_V : V \to (V^*)^*$ is surjective, then V is called *reflexive*.

Example 2.6.4. Hilbert spaces are reflexive by the Riesz representation theorem.

——————//——————

Recall that $A \subseteq X$ is *nowhere dense* if $\overline{A}^o = \emptyset$. For example, if $f : \mathbf{R} \to \mathbf{R}$ is any map, the set $\{(x, y) \in \mathbf{R}^2 \mid y = f(x)\}$ is nowhere dense.

Proposition 2.6.29. *For a metric space (X, d) and $A \subseteq X$, the following are equivalent:*

(1) A is nowhere dense;

(2) There exists a closed subset $F \subseteq X$ such that $F^o = \emptyset$ and $F \supseteq A$;

(3) There exists an open and dense subset $U \subseteq X$ such that $U \subseteq A^c$.

Proof. (1) $\Rightarrow$ (2) Take $F = \overline{A}$.

(2) $\Rightarrow$ (1) Let F be such a set. Then $\overline{A} \subseteq \overline{F}$. So $\overline{A}^o \subseteq \overline{F}^o = \emptyset$.

(2) $\Rightarrow$ (3) Let F be such a set. Let $U = F^c$. Then $\overline{U} = \overline{F^c} = (F^o)^c = \emptyset^c = X$. We also have $U = F^c \subseteq A^c$.

(3) $\Rightarrow$ (2) Let U be such a set. Take $F = U^c$. Then $F^o = (U^c)^o = (\overline{U})^c = X^c = \emptyset$. Also $F = U^c \supseteq (A^c)^c = A$. $\square$

Definition 2.6.5. A point $x \in X$ is *isolated* if there exists an $\epsilon > 0$ such that $U(x, \epsilon) = \{x\}$.

Proposition 2.6.30. *Let (X, d) be a metric space.*

(1) If $A \subseteq X$ is nowhere dense and $B \subseteq A$, then B is nowhere dense.

(2) If $A \subseteq X$ is nowhere dense, then $\overline{A}$ is nowhere dense.

(3) If $A_1, A_2, ..., A_n$ are nowhere dense, then $\bigcup_{k=1}^{n} A_k$ is nowhere dense.

(4) If X has no isolated points, then every finite subset is nowhere dense.

Proof. (1) If $B \subseteq A$ then $\overline{B} \subseteq \overline{A}$, so $(\overline{B})^o \subseteq (\overline{A})^o = \emptyset$.

(2) Note that $\overline{A} = \overline{\overline{A}}$. So $(\overline{\overline{A}})^o = (\overline{A})^o = \emptyset$.

(3) We show this for $n = 2$. Let A_1 and A_2 be nowhere dense. By Proposition 2.6.29, $A_2^c \supseteq U_1$, where U_1 is open and dense. Similarly, $A_2^c \supseteq U_2$, where U_2 is open and dense. Then:

$$\begin{aligned}(A_1 \cup A_2)^c &= A_1^c \cap A_2^c \\ &\supseteq U_1 \cap U_2.\end{aligned}$$

Clearly $U_1 \cap U_2$ is open by Proposition 2.2.1. Claim: $U_1 \cap U_2$ is dense. Let $x \in X$ and $\epsilon > 0$. We'd like to show, by Proposition 2.2.8, that $(U_1 \cap U_2) \cap U(x, \epsilon) \neq \emptyset$. Since U_1 is dense, we know $U_1 \cap U(x, \epsilon) \neq \emptyset$. Let $z \in U_1 \cap U(x, \epsilon)$. Since $U_1 \cap U(x, \epsilon)$ is open, by Definition 2.2.2 there exists a $\delta > 0$ such that $U(z, \delta) \subseteq U_1 \cap U(x, \epsilon)$. Now since U_2 is dense, $U(z, \delta) \cap U_2 \neq \emptyset$. Therefore $U(x, \epsilon) \cap (U_1 \cap U_2) \neq \emptyset$. Thus $U_1 \cap U_2$ is dense.

(4) Since X has no isolated points, $\{x\}$ is closed but not open. Then $(\overline{\{x\}})^o = \{x\}^o = \emptyset$. So $\{x\}$ is nowhere dense. By (3), any finite set $\{x_1, x_2, ..., x_n\} = \{x_1\} \cup \{x_2\} \cup ... \cup \{x_n\}$ is nowhere dense. $\square$

Note that $\mathbf{Q}$ is not nowhere dense, but it is the *countable* union of nowhere dense sets.

Definition 2.6.6. Let (X, d) be a metric space.

(1) A set $A \subseteq X$ is *meager* if $A = \bigcup_{k=1}^{\infty} A_k$, where each A_k is nowhere dense.

(2) A set $B \subseteq X$ is *residual* if B^c is meager.

Example 2.6.5. A meager set is "topologically small", whereas a residual set is "topologically big". Consider $\mathbf{Q} \subseteq \mathbf{R}$ meager. Then $\mathbf{R} \setminus \mathbf{Q} \subseteq \mathbf{R}$ is residual.

Note that this definition only applies to *subsets* of a specified metric space. For instance, $\mathbf{Z} \subseteq \mathbf{R}$ is meager, but $\mathbf{Z} \subseteq \mathbf{Z}$ is not meager. Indeed, $\mathbf{Z}$ with the discrete topology means that every subset of $\mathbf{Z}$ is open, and every subset of $\mathbf{Z}$ is closed. So given $A \subseteq \mathbf{Z}$ nowhere dense, we have $A = (\overline{A})^c = \emptyset$. So if $A \subseteq \mathbf{Z}$ is meager, then $A \neq \emptyset$.

Proposition 2.6.31. *Let (X, d) be a metric space.*

(1) If $A \subseteq X$ is meager and $B \subseteq A$, then B is meager.

(2) If $A_1, A_2, ..., A_k$ are meager, then $\bigcup_{k=1}^{\infty} A_k$ is meager.

(3) If X has no isolated points, every countable set is meager.

Proof. (1) If A is meager, then $A = \bigcup_{k=1}^{\infty} A_k$, where each A_k is nowhere dense. Then:

$$\begin{aligned} B &= B \cap A \\ &= B \cap \bigcup_{k=1}^{\infty} A_k \\ &= \bigcup_{k=1}^{\infty} (B \cap A_k) \end{aligned}$$

Since $B \cap A_k$ is nowhere dense for all k, B is meager.

(2) If A_k is meager, then $A_k = \bigcup_{j=1}^{\infty} A_{kj}$, where A_{kj} is nowhere dense for each j. We have:

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} A_{kj}.$$

Since the countable union of countable sets is countable, we can see $\bigcup_{k=1}^{\infty} A_k$ is meager.

(3) We can write $\{x_k \mid k \geqslant 1\}$ as $\bigcup_{k=1}^{\infty} \{x_k\}$. Since singletons are nowhere dense, every countable set is meager. $\square$

Proposition 2.6.32 (Cantor's Intersection Theorem)**.** *Let (X, d) be a complete metric space and $F_1 \supseteq F_2 \supseteq F_3 \supseteq ...$ be a nested sequence of closed nonempty sets with $(\mathrm{diam}(F_n))_n \to 0$. Then $\bigcap_{n \geqslant 1} F_n = \{x\}$ for some $x \in X$.*

Proof. Let $x_n \in F_n$ for all $n \geqslant 1$. Note that $(x_n)_n$ is Cauchy —given $\epsilon > 0$, let N be large so that $n \geqslant N$ implies $\text{diam}(F_n) < \epsilon$. For $m, n \geqslant N$, we have $x_m, x_n \in F_N$ (because $F_N \supseteq F_{N+1} \supseteq ...$), which gives:

$$\begin{aligned} d(x_n, x_m) &\leqslant \sup_{x_n, x_m \in F_N} d(x_n, x_m) \\ &= \text{diam}(F_N) \\ &< \epsilon. \end{aligned}$$

Since X is complete, $(x_n)_n \to x_0$ for some $x_0 \in X$.

Claim: $\bigcap_{n \geqslant 1} F_n = \{x_0\}$. To see this, fix $m \in \mathbf{N}$ and consider the sequence $(x_{m+k})_k$ in F_m. Since $(x_n)_n$ converges to x_0, we know the tail $(x_{m+k})_k \xrightarrow{k \to \infty} x_0$. Since F_m is closed, we know $x_0 \in F_m$. Since m was arbitrary, $x_0 \in \bigcap_{n \geqslant 1} F_n$. Now suppose $x_0, x \in \bigcap_{n \geqslant 1} F_n$ with $d(x_0, x) > 0$. We can find N large so that $\text{diam}(F_n) < d(x_0, x)$ But $x_0, x \in \bigcap_{n \geqslant 1} F_n$ implies $x_0, x \in F_n$, giving $d(x_0, x) < d(x_0, x)$, which is a contradiction. Thus $\bigcap_{n \geqslant 1} F_n = \{x_0\}$. □

Example 2.6.6. Note that $\bigcap_{n \geqslant 1}(0, \frac{1}{n}) = \emptyset$ becaue $(0, \frac{1}{n})$ is not closed, and $\bigcap_{n \geqslant 1}[n, \infty) = \emptyset$ because $(\text{diam}([n, \infty)))_n \nrightarrow 0$.

Theorem 2.6.33 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space.*

(1) If $\{V_k\}_{k=1}^{\infty}$ is a countable family of open and dense subsets, then $\bigcap_{k=1}^{\infty} V_k$ is also dense (but not necessarily open).

(2) X is not meager.

Proof. (1) Let U_0 be any open ball. Since V_1 is open and dense, $U_0 \cap V_1$ is open and nonempty. So there exists an open ball $U(x, \delta) \subseteq U_0 \cap V_1$. Notice we also have $U_1 := U(x, \frac{\delta}{2}) \subseteq U_0 \cap V_1$. Define $B_1 = \overline{U_1} \subseteq U_0 \cap V_1$. Since δ was arbitrary, we arbitrarily shrink out open ball to ensure $\text{diam}(B_1) < 1$. Now $U_1 \cap V_2$ is open and nonempty. So there exists some open ball contained in $U_1 \cap V_2$. Exactly as we just did, find $B_2 := \overline{U_2} \subseteq U_1 \cap V_2$ such that $\text{diam}(B_2) < \frac{1}{2}$. Inductively, having constructed $U_1, B_1, U_2, B_2, ...$, we note that $U_{n-1} \cap V_n$ is open and nonempty, so it contains an open ball. Find $B_n := \overline{U_n} \subseteq U_{n-1} \cap V_n$ such that $\text{diam}(B_n) < \frac{1}{n}$. Observe that:

$$B_1 \supseteq U_1 \supseteq B_2 \supseteq U_2 \supseteq B_3 \supseteq U_3 \supseteq ...$$

So $\{B_n\}_{n \geqslant 1}$ is a nested sequence of closed sets with $(\text{diam}(B_n))_n \to 0$. By Cantor's Intersection Theorem, $\cap_{n \geqslant 1} B_n = \{x\}$. Claim: $x \in U_0 \cap \left(\bigcap_{k=1}^{\infty} V_k\right)$. Note that $B_n \subseteq U_{n-1} \cap V_n \subseteq V_n$. So $x \in \bigcap_{n \geqslant 1} B_n \subseteq \bigcap_{n \geqslant 1} V_n$. Also, $x \in B_1 = \overline{U_1} \subseteq U_0 \cap V_1 \subseteq U_0$. This means $x \in U_0 \cap \left(\bigcap_{k=1}^{\infty} V_k\right)$. Since U_0 was an arbitrary open set, and since it's intersection with $\bigcap_{k=1}^{\infty} V_k$ is nonempty, we can conclude that $\bigcap_{k=1}^{\infty} V_k$ is dense.

(2) Suppose towards contadiction that X is meager, that is, $X = \bigcup_{k=1}^{\infty} A_k$, where each A_k is nowhere dense. Fixing k, by Proposition 2.6.29 we can find an open and dense set V_k with $V_k \subseteq A_k^c$. But then:

$$\begin{aligned}\emptyset &= X^c \\ &= \left(\bigcup_{k=1}^{\infty} A_k\right)^c \\ &= \bigcap_{k=1}^{\infty} A_k^c \\ &\supseteq \bigcap_{k=1}^{\infty} V_k.\end{aligned}$$

This is a contradiction. Thus, if X is complete, then X is not meager. □

Exercise 2.6.2. Show that $\mathbf{R} \setminus \mathbf{Q}$ is not meager.

Skipped all of the "application" stuff

Example 2.6.7. Let $X = \big(C([0,1]), \|\cdot\|_u\big)$. Recall that $f(x) = |x|$ is continuous everywhere, but not differentiable at $x = 0$. Does there exist a function $f : X \to \mathbf{R}$ which is continuous *everywhere*, but differentiable *nowhere*. The answer is yes, and in fact, the set:

$$\{f \in X \mid f \text{ differentiable nowhere}\}$$

is *residual*. This is surprising, as the complement of this set—the set containing every continuous function which is differentiable at one or more points, is "topologically small."

§ 2.7. Compactness

Compactness is a generalization of "closed and bounded in $\mathbf{R}^n$ for metric spaces. It is the algebraic analogue of "finite-dimensional".

Definition 2.7.1. Let (X, d) be a metric space and $K \subseteq X$ a subset.

(1) A *cover* for K is a family $\mathcal{U} = \{U_i\}_{i \in I}$ with $U_i \subseteq X$ for all i and $K \subseteq K \subseteq \bigcup_{i \in I} U_i$.
(2) The family $\mathcal{U}$ is an *open cover* for K if each U_i is open.

(3) The family $\mathcal{U}$ is a *finite cover* for K if I is finite.

(4) A *subcover* of $\mathcal{U}$ is a subfamily $\{U_i\}_{i\in J}$, where $J \subseteq I$ and $K \subseteq \bigcup_{i\in J} U_i$.

(5) K is called *compact* if every open cover $\mathcal{U}$ of K admits a finite subcover. That is, if $K \subseteq \bigcup_{i\in I} U_i$ with each $U_i \subseteq X$ open, there exists a finite subset $F \subseteq I$ with $K \subseteq \bigcup_{i\in F} U_i$.

Since K is not necessarily proper, a metric space X is compact if it can be written as the union of finitely many open sets.

Example 2.7.1. The set $(0, 1]$ is not compact in $\mathbf{R}$. The family $\left\{\left(\frac{1}{n}, 2\right)\right\}_{n\geqslant 1}$ is an open cover of $(0, 1]$, but admits no finite subcover.

Exercise 2.7.1. Let (X, d) be a metric space with $Y \subseteq X$ and $K \subseteq Y$. $K \subseteq X$ is compact if and only if $K \subseteq Y$ is compact.

Proposition 2.7.1. *Let (X, d) be a metric space and $K \subseteq X$.*

(1) If K is compact, then K is closed and bounded,

(2) If X is compact and K is closed, then K is compact.

Proof. (1) We will first show that K is bounded. Pick $x_0 \in K$, Note that $\{U(x_0, n)\}_{n=1}^{\infty}$ is a cover for K, whence $K \subseteq \bigcup_{n\geqslant 1} U(x_0, n)$. By the compactness of K, $K \subseteq \bigcup_{n=1}^{N} U(x_0, n)$. For $x, x' \in K$:

$$\begin{aligned} d(x, x') &\leqslant d(x, x_0) + d(x_0, x') \\ &\leqslant 2N. \end{aligned}$$

Thus $\operatorname{diam}(K) \leqslant 2N$, so K is bounded. We will now show that K^c is open. Let $x_0 \in K^c$ be arbitrary. For each $x \in K$, we can find a $\delta_x > 0$ such that $U(x_0, \delta_x) \cap U(x, \delta_x) = \emptyset$. Note that $K \subseteq \bigcup_{x\in K} U(x, \delta_x)$. Since K is compact, there exists $x_1, ..., x_n \in K$ with $K \subseteq \bigcup_{j=1}^{n} U(x_j, \delta_{x_j})$. Let $\delta = \min_{j=1}^{n} \delta_{x_j}$. Claim: $U(x_0, \delta) \subseteq K^c$. If not, we can find some $z \in U(x_0, \delta) \cap K$, which implies $z \in U(x_0, \delta) \cap U(x_j, \delta_{x_j})$ for some j. But this means $\emptyset \neq U(x_0, \delta) \cap U(x_j, \delta_{x_j}) \subseteq U(x_0, \delta_{x_j}) \cap U(x_j, \delta_{x_j}) = \emptyset$. Thus K is closed.

(2) Let $K \subseteq \bigcup_{i\in I} U_i$, where $U_i \subseteq X$ is open. Then $X = K^c \cup \left(\bigcup_{i\in I} U_i\right)$. Since $K^c \cup \left(\bigcup_{i\in I} U_i\right)$ is an open cover of X, there exists a finite subset $F \subseteq I$ with $X = K^c \cup \left(\bigcup_{i\in F} U_i\right)$. It follows that $K \subseteq \bigcup_{i\in F} U_i$. □

Definition 2.7.2. Let (X, d) be a metric space. A non-empty family of subsets $\{A_i\}_{i\in I}$ is said to have the *finite intersection property* if for any finite subset $F \subseteq I$, we have $\bigcap_{i\in F} A_i \neq \emptyset$.

Proposition 2.7.2. *Let (X, d) be a metric space. The following are equivalent:*

(1) X is compact;

(2) For any family of closed subsets $\{C_i\}_{i\in I}$ satisfying the finite intersection property, we have $\bigcap_{i\in I} C_i \neq \emptyset$.

Proof. Suppose that X is compact and let $\{C_i\}_{i\in I}$ be a family of closed sets satisfying the finite intersection property. Assume towards contradiction that $\bigcap_{i\in I} C_i = \emptyset$. Then $\bigcup_{i\in I} C_i^c = X$. Since X is compact, there exists some finite subset $F \subseteq I$ with $\bigcup_{i\in F} C_i^c = X$. But this implies $\bigcap_{i\in F} C_i = \emptyset$, contradicting our family of closed subsets satisfying the finite intersection property.

We will assume the contrapositive of (2). Let $X = \bigcup_{i\in I} V_i$, where each V_i is open. Then $\emptyset = \bigcap_{i\in I} V_i^c$. Since $\{V_i^c\}_{i\in I}$ is a family of closed subsets, it must be the case that $\bigcap_{i\in F} V_i^c = \emptyset$ for some finite subset $F \subseteq I$. Whence $\bigcup_{i\in F} V_i = X$. □

Proposition 2.7.3. **** Every compact metric space is separable.*

Proof. Fix $n \geqslant 1$. We have an open cover $X = \bigcup_{x\in X} U(x, \frac{1}{n})$, which admits a subcover $X = \bigcup_{j=1}^{J_n} U\left(x_{n,j}, \frac{1}{n}\right)$. Let $S = \{x_{n,j} \mid n \geqslant 1, 1 \leqslant j \leqslant J_n\}$.

Let $x \in X$ and $\epsilon > 0$. Let m be such that $\epsilon > \frac{1}{m}$. Find $j \in \{1, ..., J_m\}$ with $x \in U(x_{m,j}, \frac{1}{m})$. Thus $d(x, x_{m,j}) < \frac{1}{m}$. Whence $S \cap U(x, \epsilon) \neq \emptyset$. So S is dense and certainly countable. □

Definition 2.7.3. Let (X, d) be a metric space and let $K \subseteq X$, K is called *sequentially compact* if every sequence $(x_n)_n$ in K admits a convergent subsequence in K.

Proposition 2.7.4. *Let (X, d) be a metric space. If $K \subseteq X$ is compact, then K is sequentially compact.*

Proof. Let $(x_k)_k$ be a sequence in K. Consider $C_n = \overline{\{x_k \mid k > n\}}$. Note that:

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq ...$$

In particular, $\bigcap_{j=1}^{n} C_j \neq \emptyset$ for any $n > 1$. So $\{C_n\}_{n\geqslant 1}$ is a family of closed sets which satisfies the finite intersection property. By Proposition 2.7.2 we have $\bigcap_{n\geqslant 1} C_n \neq \emptyset$. Let $x \in \bigcap_{n\geqslant 1} C_n$.

Clearly $x \in C_1$. This means $U(x, 1) \cap \{x_k \mid k > 1\} \neq \emptyset$. Let $x_{k_1} \in U(x, 1) \cap C_1$. Since $x_{k_1} \in C_1$, we know that $k_1 > 1$. Moreover, $d(x, x_{k_1}) < 1$.

Similarly, $x \in C_{k_1}$. This means $U(x, \frac{1}{2}) \cap \{x_k \mid k > k_1\} \neq \emptyset$. Let $x_{k_2} \in U(x, \frac{1}{2}) \cap \{x_k \mid k > k_1\}$. Since $x_{k_2} \in \{x_k \mid k > k_1\}$, we know $k_2 > k_1$. Moreover, $d(x, x_{k_2}) < \frac{1}{2}$.

Inductively, $x \in C_{k_{j-1}}$. This means $U(x, \frac{1}{j}) \cap \{x_k \mid k > k_{j-1}\} \neq \emptyset$. Let $x_{k_j} \in U(x, \frac{1}{j}) \cap \{x_k \mid k > k_{j-1}\}$. Since $x_{k_j} \in \{x_k \mid k > k_{j-1}\}$, we know $k_j > k_{j-1}$. Moreover, $d(x, x_{k_j}) < \frac{1}{j}$. We've obtained a sequence $(x_{k_j})_j$ which converges to x. Thus (X, d) is sequentially compact. □

Proposition 2.7.5. *If (X, d) is sequentially compact, then (X, d) is complete.*

Proof. Let $(x_n)_n$ be Cauchy. We know there exists a subsequence $(x_{n_k})_k$ which converges to some $x \in X$. Let $\epsilon > 0$. Find N large so $p, q \geqslant N$ implies $d(x_p, x_q) < \frac{\epsilon}{2}$. Find K large so $k \geqslant K$ implies $d(x_{n_k}, x) < \frac{\epsilon}{2}$. For $n, k \geqslant \max\{N, K\}$:

$$\begin{aligned} d(x_n, x) &\leqslant d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $(x_n)_n \to x$. Whence (X, d) is complete. □

Definition 2.7.4. Let (X, d) be a metric space. We say $Y \subseteq X$ is *totally bounded* if for all $\epsilon > 0$, there exists a finite subset $F \subseteq Y$ such that $Y \subseteq \bigcup_{z \in F} U(z, \epsilon)$.

Proposition 2.7.6. *Let (X, d) be a metric space. If $K \subseteq X$ is sequentially compact, then K is totally bounded.*

Proof. Suppose towards contradiction that K is not totally bounded. Then there exists some $\epsilon_0 > 0$ such that, for any finite subset $F \subseteq K$, we have $K \nsubseteq \bigcup_{z \in F} U(z, \epsilon_0)$. Let $x_1 \in K$ be arbitrary. Then $K \nsubseteq U(x_1, \epsilon_0)$, since K cannot be covered by finitely many open balls. Find $x_2 \in K \setminus U(x_1, \epsilon_0)$. Then $K \nsubseteq U(x_1, \epsilon_0) \cup U(x_2, \epsilon_0)$. Find $x_3 \in K \setminus (U(x_1, \epsilon_0) \cup U(x_2, \epsilon_0))$. Then $K \nsubseteq U(x_1, \epsilon_0) \cup U(x_2, \epsilon_0) \cup U(x_3, \epsilon_0)$. Inductively, we obtain a sequence $(x_n)_n$, where $x_n \in K \setminus \bigcup_{j=1}^{n-1} U(x_j, \epsilon_0)$. Since $x_n \notin \bigcup_{j=1}^{n-1} U(x_j, \epsilon_0)$, we have that $d(x_n, x_j) \geqslant \epsilon_0$ for all $j \neq n$. Now since X is sequentially compact, $(x_n)_n$ admits a convergent subsequence, call it $(x_{n_k})_k$. Since $(x_{n_k})_k$ converges, it is Cauchy. But this is a contradiction, because $d(x_{n_l}, x_{n_k}) \geqslant \epsilon_0$ for all $l \neq k$. It must be the case that K is totally bounded. □

Proposition 2.7.7. *If (X, d) is a metric space and $K \subseteq X$ is totally bounded, then K is bounded.*

Proof. Let $\epsilon = 1$. We have that $K \subseteq \bigcup_{j=1}^{n} U(x_j, 1)$. Define $C = \max_{1 \leqslant i,j \leqslant n} d(x_i, x_j)$. If $x, y \in K$, then $x \in U(x_i, 1)$ and $y \in U(x_j, 1)$ for some i, j. This gives:

$$\begin{aligned} d(x, y) &\leqslant d(x, x_i) + d(x_i, x_j) + d(x_j, y) \\ &\leqslant 2 + C. \end{aligned}$$

Thus $\operatorname{diam}(K) \leqslant 2 + C$. □

Corollary 2.7.8. *If $K \subseteq \mathbf{R}$ is compact and nonempty, then* $\sup(K) \in K$.

Proof. Define $u := \sup(K)$. Let $(x_n)_n$ be a sequence in K converging to u. Note that $u \in \mathbf{R}$ since K is bounded. Since K is closed, $u \in K$. □

Theorem 2.7.9. *Let (X, d) be a metric space. The following are equivalent:*

(1) X is compact;

(2) X is sequentially compact;

(3) X is complete and totally bounded.

Proof. Propositions 2.7.4, 2.7.5, and 2.7.6 established (1)⇒(2) and (2)⇒(3). It remains to prove (3)⇒(1). Suppose towards contradiction $\mathcal{V}$ is an open cover for X which fails to admit a finite subcover. Let $\epsilon = 1$. By total boundedness:

$$X = U(x_1, 1) \cup U(x_2, 1) \cup ... \cup U(x_{m_1}, 1).$$

There must be at least one $U(x_j, 1)$ that can't be covered by finitely many members of $\mathcal{V}$. Label one of these balls at $B_1(x_1)$. Let $\epsilon = \frac{1}{2}$. By total boundedness,

$$X = U(x_1, {}^1\!/_2) \cup U(x_2, {}^1\!/_2) \cup ... \cup U(x_{m_2}, {}^1\!/_2).$$

Note that:

$$\begin{aligned} \bigcup_{j=1}^{m_2} \Big(B_1(x_1) \cap U(x_j, {}^1\!/_2) \Big) &= B_1(x_1) \cap \left(\bigcup_{j=1}^{m_2} U(x_j, {}^1\!/_2) \right) \\ &= B_1(x_1) \cap X \\ &= B_1(x_1). \end{aligned}$$

There must be some j such that $B_{x_1}(1) \cap U(x_j, {}^1\!/_2)$ cannot be covered by finitely many mebers of $\mathcal{V}$. Label $U(x_j, {}^1\!/_2)$ as $B_{1/2}(x_2)$. We continue this process inductively, along which we obtain a sequence $(x_n)_n$ such that

$$F_n := B_1(x_1) \cap B_{1/2}(x_2) \cap B_{1/3}(x_3) \cap ... \cap B_{1/n}(x_n),$$

cannot be covered by finitely many members of $\mathcal{V}$. Define $C_n = \overline{F_n}$. Note that $C_1 \supseteq C_2 \supseteq C_3 \supseteq ...$ and $\operatorname{diam}(C_n) \leqslant \frac{2}{n}$. Since X is complete, by Cantor's Intersection Theorem $\bigcap_{n \geqslant 1} C_n = \{x\}$. Locate this x in $V \in \mathcal{V}$. Since V is open, we can find some $\epsilon > 0$ with $U(x, \epsilon) \subseteq V$. Find N large so that $\frac{2}{N} < \epsilon$. We can see that $C_N \subseteq U(x, \epsilon)$ since $d(z, x) \leqslant \frac{2}{N} < \epsilon$ for all $z \in C_N$. But this means $F_N \subseteq C_N \subseteq U(x, \epsilon) \subseteq V$. This is a contradiction, as we've asserted F_n cannot be covered by finitely many members of $\mathcal{V}$. Thus X is compact. □

—————//—————

We're interested in studying compact subsets of $\mathbf{R}^d$.
(Heine-Borel stuff here...)

—————//—————

Compactness and Continuity

Proposition 2.7.10. *Let $f : (X, d) \to (Y, d)$ be continuous and $K \subseteq X$ compact. Then $f(K) \subseteq Y$ is compact.*

Proof. Let $f(K) \subseteq \bigcup_{i \in I} V_i$, where each $V_i \subseteq Y$ is open. Then:

$$\begin{aligned} K &\subseteq f^{-1}(f(K)) \\ &\subseteq f^{-1}\left(\bigcup_{i \in I} V_i\right) \\ &= \bigcup_{i \in I} f^{-1}(V_i). \end{aligned}$$

Since f is continuous, $f^{-1}(V_i) \subseteq X$ is open. So there exists a finite subset $F \subseteq I$ with $K \subseteq \bigcup_{i \in F} f^{-1}(V_i)$. Whence:

$$\begin{aligned} f(K) &\subseteq f\left(\bigcup_{i \in F} f^{-1}(V_i)\right) \\ &= \bigcup_{i \in F} f(f^{-1}(V_i)) \\ &= \bigcup_{i \in F} V_i. \end{aligned}$$ □

Corollary 2.7.11. *If d_1 and d_2 are topologically equivalent metrics on X, then $K \subseteq X$ is d_1-compact if and only if $K \subseteq X$ is d_2-compact.*

Proof. By the definition of topological equivalence $id : (X, d_1) \to (X, d_2)$ is continuous. Let $K \subseteq X$ be d_1-compact. Then $K = \mathrm{id}(K) \subseteq (X, d_2)$. The previous proposition says K is d_2-compact. The converse direction is identical. □

Theorem 2.7.12 (Extreme Value Theorem)**.** *Let $f : (X, d) \to \mathbf{R}$ be continuous and $K \subseteq X$ compact. Then there exists $x_m, x_M \in K$ with $f(x_m) = \inf_{x \in K} f(x)$ and $f(x_M) = \sup_{x \in K} f(x)$.*

Proof. If f is continuous, then $f(K) \subseteq \mathbf{R}$ is compact. Then Proposition 2.7.8 gives $\inf(f(K)), \sup(f(K)) \in f(K)$. Hence there exists x_m, x_M with $f(x_m) = \inf(K)$ and $f(x_M) = \sup(K)$. □

Proposition 2.7.13. *Let V be a finite dimensional vector space over F. All norms on V are equivalent.*

Proof. Let $\{v_1, ..., v_n\}$ be a basis for V. Define $\|\cdot\|_1 : V \to [0, \infty)$ by:

$$\left\| \sum_{i=1}^{n} t_j v_j \right\|_1 = \sum_{i=1}^{n} |t_j|.$$

Then $\varphi : \ell_1^n \to (V, \|\cdot\|_1)$ given by $(t_1, ..., t_n) \mapsto \sum_{i=1}^n t_i v_i$ is a linear isomorphism and isometry. Indeed:

$$\begin{aligned}\left\|\varphi((t_1, ..., t_n)) - \varphi((t_1', ..., t_n'))\right\|_1 &= \left\|\sum_{i=1}^n t_i v_i - \sum_{i=1}^n t_i' v_i\right\|_1 \\ &= \left\|\sum_{i=1}^n (t_i - t_i') v_i\right\|_1 \\ &= \sum_{i=1}^n |t_i - t_i'| \\ &= \left\|(t_1, ..., t_n) - (t_1', ..., t_n')\right\|_{\ell_2^n}.\end{aligned}$$

By the Heine-Borel Theorem $B_{\ell_1^n} = \{t \in \ell_1^n \mid \|t\|_{\ell_1^n} \leqslant 1\}$ is compact as it is a closed and bounded subset of ℓ_1^n. Moreover, $B_1 = \{v \in V \mid \|V\| \leqslant 1\}$ is compact by Proposition 2.7.10 since $B_1 = \varphi(B_{\ell_1^n})$. Now $S_1 = \{v \in V \mid \|v\|_1 = 1\} \subseteq B_1$ is closed, hence compact. Let $\|\cdot\|$ be any norm on V. We can see:

$$\begin{aligned}\left\|\sum_{i=1}^n t_i v_i\right\| &= \sum_{i=1}^n |t_i| \, \|v_i\| \\ &\leqslant c \sum_{i=1}^n |t_i| \\ &= c \left\|\sum_{i=1}^n t_i v_i\right\|_1,\end{aligned}$$

where $c = \max_{i=1}^n \|v_j\|$.

Now consider $g : (V, \|\cdot\|_1) \to \mathbf{R}$ given by $g(v) = \|v\|$. We have:

$$\begin{aligned}|g(v) - g(v')| &= | \, \|v\| - \|v'\| \, | \\ &\leqslant \|v - v'\| \\ &\leqslant c \|v - v'\|_1.\end{aligned}$$

Thus g is continuous. Since $S_1 \subseteq (V, \|\cdot\|_1)$ is compact, the Extreme Value Theorem says there exists $w \in S_1$ with $g(w) = \inf_{v \in S_1} g(v)$. If $g(w) = \|w\| = 0$, then $w = 0$. This is a contradiction since $w \in S_1$, so it must be the case that $g(w) > 0$. Define $c' := g(w)$, then $c' \leqslant g(v) = \|v\|$ for every $v \in V$. Using the fact that $\frac{v}{\|v\|_1} \in S_1$ for any nonzero $v \in V$, we can see $c' \leqslant \left\|\frac{v}{\|v\|_1}\right\|$ implies $\|v\|_1 \leqslant \frac{1}{c'} \|v\|$. By transitivity, all norms on V are equivalent. ☐

Corollary 2.7.14. *Let V be a normed space and $W \subseteq V$ a finite dimensional subspace. Then $W \subseteq V$ is closed.*

Proof. We know that all norms on V are equivalent, so there exists a uniformism and linear isomorphism $\varphi : W \to \ell_1^n$. Let $(w_n)_n$ be a sequence in W converging to some $v \in V$. Since $(w_n)_n$ is convergent, it is $\|\cdot\|$-Cauchy. Since φ is uniformly continuous, $(\varphi(w_n))_n$ is $\|\cdot\|_{\ell_1^n}$-Cauchy, whence it converges to some $x \in \ell_1^n$. Since φ^{-1} is uniformly continuous, we have that $(w_n)_n \to \varphi^{-1}(x) \in W$. Thus $v = \varphi^{-1}(x) \in W$; i.e., W is closed. $\square$

Lemma 2.7.15 (Riesz). **** Let V be a normed space and W a closed, proper subspace of V. For each $t \in (0,1)$, there is a $v_t \in S_V$ with $\operatorname{dist}_W(v_t) \geqslant t$.*

Proof. Recall that $\operatorname{dist}_W : V \to [0,\infty)$ is given by $v \mapsto \inf_{w\in W} \|v-w\|$. Let $v_0 \in V \setminus W$ and define $\delta := \operatorname{dist}_W(v_0)$. Clearly $\delta > 0$ —if not the closedness of W implies $v_0 \in W$, which is a contradiction. Since $t \in (0,1)$, we have $\delta < \frac{\delta}{t}$. Choose $w_t \in W$ with $\delta < \|v_0 - w_t\| < \frac{\delta}{t}$. Define $v_t := \frac{w_t - v_0}{\|w_t - v_0\|}$. $\square$

Theorem 2.7.16. *Let V be a normed space and $B_V = \{v \in V \mid \|v\| \leqslant 1\}$. The following are equivalent:*

(1) B_V is compact;

(2) $\dim(V) < \infty$.

Proof. Let B_V be compact and suppose towards contradiction $\dim(V) = \infty$. Choose $v_1 \in S_V$. Since V is infinite dimensional, $\operatorname{span}\{v_1\} \neq V$. By Riesz' Lemma, there exists a $v_2 \in S_V$ so that $\operatorname{dist}_{\operatorname{span}\{v_1\}}(v_2) \geqslant \frac{1}{2}$. In particular, $\|v_2 - v_1\| \geqslant \frac{1}{2}$. Again, since V is infinite dimensional, $\operatorname{span}\{v_1, v_2\} \neq V$. By Riesz' Lemma, there exists a $v_3 \in S_V$ so that $\operatorname{dist}_{\operatorname{span}\{v_1,v_2\}}(v_3) \geqslant \frac{1}{2}$. Then $\|v_3 - v_2\| \geqslant \frac{1}{2}$ and $\|v_3 - v_1\| \geqslant \frac{1}{2}$. Inductively, we obtain a sequence $(v_n)_n$ in S_V with $\|v_n - v_j\| \geqslant \frac{1}{2}$ for all $1 \leqslant j \leqslant n-1$. Since $(v_n)_n$ is not Cauchy, by Proposition 2.7.5 $(v_n)_n$ does not admit a convergent subsequence. So S_V is not compact by Theorem 2.7.9. But this contradicts Proposition 2.7.1, as we've assumed $B_V \supseteq S_V$ is compact.

Suppose V is finite dimensional. By Proposition 2.5.7 there exists a continuous linear isomorphism $f : \mathbf{C}^n \to V$. Note that $B_V = \{v \in V \mid \|v\| \leqslant 1\}$ is closed and bounded. Since f is continuous, by definition $f^{-1}(B_V)$ is closed. Since f^{-1} is continuous, Proposition 2.5.6 says f^{-1} is Lipschitz. So $f^{-1}(B_V)$ is bounded . By the Heine-Borel Theorem, $f^{-1}(B_V)$ is compact. Thus $f(f^{-1}(B_V)) = B_V$ is also compact by Proposition 2.7.10. $\square$

Proposition 2.7.17. *Let (X,d) be compact and $f : (X,d) \to (Y,\rho)$ continuous. Then f is uniformly continuous*

Proof. Let $\epsilon > 0$. Since f is continuous, for every $x \in X$ there exists a $\delta_x > 0$ such that, for any $z \in X$, we have $d(x,z) < \delta_x$ implies $\rho(f(x), f(z)) < \frac{\epsilon}{2}$. Note that $X = \bigcup_{x\in X} U(x, \frac{\delta_x}{2})$. Since X is compact, there exists $x_1, ..., x_n \in X$ with

$X = \bigcup_{i=1}^n U\big(x_i, \frac{\delta_{x_i}}{2}\big)$. Let $x, x' \in X$ be arbitrary. Define $\delta = \min_{i=1}^n \delta_{x_i}$. Locate $U\big(x_i, \frac{\delta_{x_i}}{2}\big) \ni x$. If $d(x, x') < \delta$, then

$$\begin{aligned} d(x', x_i) &\leqslant d(x', x) + d(x, x_i) \\ &< \delta + \frac{\delta_{x_i}}{2} \\ &\leqslant \delta_{x_i}, \end{aligned}$$

which gives:

$$\begin{aligned} \rho(f(x), f(x')) &\leqslant \rho(f(x), f(x_i)) + \rho(f(x_i), f(x')) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

Alternative proof of Proposition 2.7.17. Suppose towards contradiction f is not uniformly continuous. Then there exists an $\epsilon_0 > 0$ and sequences $(u_n)_n$, $(v_n)_n$ in X such that $d(u_n, v_n) < \frac{1}{n}$ and $\rho(f(u_n), f(v_n)) \geqslant \epsilon_0$. Since X is compact there exists a subsequence $(u_{n_k})_k$ convering to $u \in X$. Then:

$$\begin{aligned} d(v_{n_k}, u) &\leqslant d(v_{n_k}, u_{n_k}) + d(u_{n_k}, u) \\ &< \frac{1}{n_k} + d(u_{n_k}, u). \end{aligned}$$

Taking the limit as $k \to \infty$ gives $(d(v_{n_k}, u))_k \to 0$; i.e., $(v_{n_k})_k$ converges to u. Since f is continuous, both $(f(u_{n_k}))_k$ and $(f(v_{n_k}))_k$ converge to $f(u)$. But this is a contradiction, as $\rho(f(u_n), f(v_n)) \to 0 \not\geqslant \epsilon_0$. □

Lemma 2.7.18. *Let (X, d) be compact. Suppose $(f_n)_n$ is a monotonically decreasing seqence of continuous real-valued functions on X which converges pointwise to 0. Then $(f_n)_n$ converges uniformly to 0.*

Proof. Let $\epsilon > 0$. Since $(f_n)_n$ converges pointwise to 0, for each $x \in X$ there exists $N_x \in \mathbf{N}$ such that $n \geqslant N_x$ implies $f_n(x) < \frac{\epsilon}{2}$. Because f_{N_x} is continuous at x, there exists $\delta_x > 0$ such that, for every $z \in X$, $d(x, z) < \delta_x$ implies $|f_{N_x}(x) - f_{N_x}(z)| < \frac{\epsilon}{2}$. The collection $\{U(x, \delta_x)\}_{x \in X}$ covers X, so by compactness there is a finite set $F \subseteq X$ with $X = \bigcup_{x \in F} U(x, \delta_x)$. Set $N = \max_{x \in F} N_x$. Let $z \in X$ be arbitrary and locate $x \in F$ such that $z \in U(x, \delta_x)$. Notice that our choice of N does not depend on z.

For $n \geqslant N$:

$$\begin{aligned} f_n(z) &\leqslant f_{N_x}(z) \\ &= f_{N_x}(z) - f_{N_x}(x) + f_{N_x}(x) \\ &\leqslant |f_{N_x}(z) - f_{N_x}(x)| + f_{N_x}(x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus for $n \geqslant N$, we have $\|f_n\|_u \leqslant \epsilon$. □

Alternative proof of Lemma 2.7.18. Let $\epsilon > 0$. For $n \geqslant 1$, define:

$$\begin{aligned} U_n &:= \{x \mid f_n(x) < \frac{\epsilon}{2}\} \\ &= f_n^{-1}\big((-\infty, \frac{\epsilon}{2})\big) \end{aligned}$$

Note $U_1 \subseteq U_2 \subseteq U_3 \subseteq ...$, and in particular $\bigcup_{n\geqslant 1} U_n = X$. Since X is compact, $X = U_N$ for some N. That is, $f_N < \frac{\epsilon}{2}$ for all x, so $\|f_n\|_u < \epsilon$. □

Theorem 2.7.19 (Dini's Theorem)**.** *Let (X, d) be compact. Suppose $(f_n)_n$ is a monotone sequence of continuous real-valued functions on X which converges pointwise to f. Then $(f_n)_n$ converges uniformly to f.*

Proof. If $(f_n)_n$ is decreasing, apply Lemma 2.7.18 to $f_n - f$. If $(f_n)_n$ is increasing, apply Lemma 2.7.18 to $f - f_n$. □

——————//——————

The goal of this subsection is to develop the structure of $C(X)$. If (X, d) is compact, by the Extreme Value Theorem $C(X) = C_b(X)$; i.e., $C(X)$ is a Banach algebra. We know that compact subsets of $C(X)$ must be closed and bounded. Conversely however, we'd like to know which closed and bounded subsets of $C(X)$ are compact. This leads us to the following definition.

Definition 2.7.5. Let (X, d) be a metric space and $\mathcal{F} \subseteq C(X)$.

(1) $\mathcal{F}$ is *pointwise equicontinuous* if:

$$(\forall x_0 \in X)(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X)(\forall f \in \mathcal{F})(d(x, x_0) < \delta \implies |f(x) - f(x_0)| < \epsilon)$$

(2) $\mathcal{F}$ is *uniformly equicontinuous* if:

$$(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x, x' \in X)(\forall f \in \mathcal{F})(d(x, x') < \delta \implies |f(x) - f(x')| < \epsilon)$$

Proposition 2.7.20. *Let $\mathcal{F} \subseteq C(X)$ be finite. Then $\mathcal{F}$ is pointwise equicontinuous.*

Proof. Suppose $\mathcal{F} = \{f_1, f_2, ..., f_n\}$. Let $x_0 \in X$ and $\epsilon > 0$. For each $k \in \{1, 2, ..., n\}$, there exists $\delta_k > 0$ such that, for any $x \in X$, $d(x, x_0) < \delta_k$ implies $|f_k(x) - f_k(x_0)| < \epsilon$. Set $\delta = \min_{k=1}^n \delta_k$. If $x \in X$ and $d(x, x_0)$, then by our choice of δ we have $|f(x) - f(x_0)| < \epsilon$ for any $f \in \mathcal{F}$. Hence $\mathcal{F}$ is pointwise equicontinuous. $\square$

Theorem 2.7.21 (Arzela-Ascoli)**.** *Let (X, d) be compact and $\mathcal{F} \subseteq C(X)$. The following are equivalent:*

(1) $\mathcal{F}$ is compact;

(2) $\mathcal{F}$ is closed, bounded, and uniformly equicontinuous.

Proof. Suppose $\mathcal{F}$ is compact. Then $\mathcal{F}$ is closed and (totally) bounded. We only need to show that $\mathcal{F}$ is uniformly equicontinuous. Let $\epsilon > 0$. Since $\mathcal{F}$ is totally bounded, there exists $f_1, ..., f_n \in \mathcal{F}$ with $\mathcal{F} \subseteq \bigcup_{j=1}^n U(f_j, \frac{\epsilon}{3})$. Now since X is compact, each $f_1, ..., f_n$ are uniformly continuous. So there exists $\delta_j > 0$ such that, for all $x, y \in X$, $d(x, y) < \delta_j$ implies $|f_j(x) - f_j(y)| < \frac{\epsilon}{3}$. Let $\delta = \min_{j=1}^n \delta_j$. Given $f \in F$, locate f_j such that $f \in U(f_j, \frac{\epsilon}{3})$. Then given any $x, y \in X$ satsifying $d(x, y) < \delta$, we have:

$$\begin{aligned} |f(x) - f(y)| &\leqslant |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) + f(y)| \\ &\leqslant 2\left\|f - f_j\right\|_u + \frac{\epsilon}{3} \\ &< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Thus $\mathcal{F}$ is uniformly equicontinuous.

Now suppose $\mathcal{F}$ is closed, bounded, and uniformly equicontinuous. Since X is compact, $C(X)$ is complete. Since $\mathcal{F}$ is closed, it is complete. We only need to show $\mathcal{F}$ is totally bounded. Let $\epsilon > 0$. By the uniform equicontinuity of $\mathcal{F}$, there exists $\delta > 0$ such that, for any $f \in \mathcal{F}$ and $x, y \in X$ satisfying $d(x, y) < \delta$, we have $|f(x) - f(y)| < \frac{\epsilon}{4}$. Note that since X is compact, it is totally bounded. So we can find $x_1, ..., x_n \in X$ with $X = \bigcup_{i=1}^n U(x_i, \delta)$. Note that any $f \in \mathcal{F}$ is uniformly continuous, hence bounded. So it must be the case that $C_{\mathcal{F}} = \big\{\big(f(x_1), ..., f(x_n)\big) \mid f \in \mathcal{F}\big\} \subseteq \mathbf{R}^n$ is bounded, hence totally bounded. As a result, there exists $f_1, ..., f_m \in \mathcal{F}$ such that $C_{\mathcal{F}} \subseteq \bigcup_{i=1}^m U\big((f_i(x_1), ..., f_i(x_n)), \frac{\epsilon}{4}\big)$. This means, given any $f \in \mathcal{F}$, there exists $i = 1, ..., m$ such that $(f(x_1), ..., f(x_n)) \in \bigcup_{i=1}^m U\big((f_i(x_1), ..., f_i(x_n)), \frac{\epsilon}{4}\big)$. Claim: $\mathcal{F} \subseteq \bigcup_{i=1}^m U(f_i, \epsilon)$. Let $f \in \mathcal{F}$ and $x \in X$ be arbitrary. Locate x_j so that $x \in U(x_j, \delta)$. Locate i so that $(f(x_1), ..., f(x_n)) \in U\big((f_i(x_1), ..., f_i(x_n)), \frac{\epsilon}{4}\big)$. Then $\sum_{j=1}^n |f(x_j) - f_i(x_j)| < \frac{\epsilon}{4}$. We finally have:

$$\begin{aligned} |f(x) - f_i(x)| &\leqslant |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)| \\ &< \frac{3\epsilon}{4}. \end{aligned}$$

Thus $\|f - f_i\| < \epsilon$, so our claim holds. Since $\mathcal{F}$ is both complete and totally bounded, it is compact. $\square$

Just as we were interested in compact subsets of $C(X)$, we'd like to know which subalgebras of $C(X)$ are dense. It turns out that the crucial property that a subalgebra must satisfy is that it separates points: a set A of functions defined on X is said to *separate* points if, for every two different points x and y in X, there exists a function f in A with $f(x) \neq f(y)$. We restate this in one of the most vital results in mathematical analysis.

Theorem 2.7.22 (Stone-Weierstrass)**.** *Let (X, d) be a compact metric space. Suppose $A \subseteq C(X)$ is a separating unital subalgebra. Then A is $\|\cdot\|_u$-dense in $C(X)$.*

The Stone-Weierstrass Theorem for continuous complex-valued functions requires $A \subseteq C(X, \mathbf{C})$ to be separating, unital, *and* self-adjoint.

Example 2.7.2.

(1) The polynomials $\left\{\sum_{k=0}^{n} a_k x^k \mid a_k \in F,\, n \geqslant 0\right\}$ are dense in $C([a, b])$.

(2) By Theorem 2.7.16 and Proposition 2.7.1, the set $S_{\mathbf{C}} = \{z \in \mathbf{C} \mid |z| = 1\}$ is compact. Consider the set of all trigonometric polynomials $\mathcal{T} = \{z \mapsto \sum_{k=-n}^{n} a_k z^k \mid a_k \in \mathbf{C}\}$. Note that $\overline{z} = z^{-1}$ for $z \in S_{\mathbf{C}}$. Then $\mathcal{T} \subseteq C(S_{\mathbf{C}})$ is dense.

§ 2.8. Connectedness

Definition 2.8.1. Let (X, d) be a metric space and $Y \subseteq X$.

(1) A *splitting for Y in X* is a pair of open subsets $U, V \subseteq X$ with $Y \subseteq U \cup V$ and $U \cap V \cap Y = \emptyset$.

(2) Such a splitting for Y is *trivial* if $U \cap Y = \emptyset$ or $V \cap Y = \emptyset$.

(3) Y is *connected in X* if every splitting is trivial. Otherwise, Y is said to be *disconnected*.

Lemma 2.8.1.

Proposition 2.8.2. $[a, b] \subseteq \mathbf{R}$ *is connected.*

Proof. Let $[a, b] \subseteq U \cup V$ be a splitting. Suppose towards contradiction that $V \cap [a, b] \neq \emptyset$; i.e., our splitting is not trivial. Set $c = \inf(V \cap [a, b])$. Since U is open, there exists $\epsilon > 0$ such that $[a, a + \epsilon) \subseteq U$.

Claim: $V \cap [a, b] \subseteq [a + \epsilon, b]$. If not, then there exists so $x \in V \cap [a, b]$ with $x \notin [a+\epsilon, b]$. It must be the case that $x \in [a, a+\epsilon) \subseteq U$. But this gives $x \in U \cap V \cap [a, b]$, which contradicts U, V being a splitting. From this, it follows that $c \geqslant a + \epsilon > a$.

Claim: $[a, c) \subseteq U$. If not, there must be some $x \in [a, c)$ with $x \notin U$. Since U, V splits $[a, b]$, it must be that $x \in V$. But $x < c$, and $x \in V \cap [a, b]$, contradicting our definition of c.

Claim: $c \in V$. Suppose towards contradiction $c \in U$. We proceed by cases. Case 1: $c < b$. Since U is open, there exists some $\delta > 0$ with $(c - \delta, c + \delta) \subseteq U$. Then $[a, c + \delta) \subseteq U \cap [a, b]$. Consider $x \in (c, c + \delta)$. If $x \in V$, then $x \in U \cap V \cap [a, b]$, contradicting the fact that U, V is a splitting. If $x \in U$, then this contradicts c being the least upper bound of $V \cap [a, b]$. Case 2: $c = b$. Then $[a, b) \subseteq U$ and $\{b\} \subseteq U$. But this means $[a, b] \subseteq U$. In particular, $U \cap [a, b] = [a, b]$. Thus $U \cap V \cap [a, b] \neq \emptyset$, contradicting again that U, V is a splitting.

Since V is open, there exists $\gamma > 0$ such that $(c - \gamma, c + \gamma) \subseteq V$. This clearly contradicts c being the least upper bound, as we can find an element less than c still contained in $V \cap [a, b]$. $\square$

Chapter 3

Measure and Integration

Chapter A

Sequences of Functions and Series

Only Real Valued Functions

§ A.1. Sequences of Functions

Definition A.1.1. Let Ω be a set, (X, d) a metric space, and $(f_n)_n$ a sequence of functions in X^Ω.

(1) $(f_n)_n$ converges *pointwise* to $f \in X^\Omega$ if:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N_{x,\epsilon} \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies d(f_n(x), f(x)) < \epsilon).$$

(2) $(f_n)_n$ converges *uniformly* to $f \in X^\Omega$ if:

$$(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) : (\forall n \in \mathbf{N})(\forall x \in \Omega)(n \geqslant N \implies d(f_n(x), f(x)) < \epsilon).$$

Theorem A.1.1. *Let (X, d) and (Y, ρ) be metric spaces. If $(f_n)_n$ is a sequence of functions in $C(X, Y)$ which converges uniformly to $f : X \to Y$, then f is continuous.*

Proof. Let $\epsilon > 0$. Since $(f_n)_n$ converges uniformly to f, pick N large so that $n \geqslant N$ implies $\rho(f_n(x), f(x)) < \frac{\epsilon}{3}$ for all $x \in X$. Let $c \in X$ be arbitrary. Since $f_N \in C(X, Y)$, there exists $\delta > 0$ such that $d(x, c) < \delta$ implies $\rho(f(x), f(c)) < \frac{\epsilon}{3}$. If $d(x, c) < \delta$, then:

$$\begin{aligned}
\rho(f(x), f(c)) &\leqslant \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(c)) + \rho(f_N(c), f(c)) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Thus $f \in C(X, Y)$. □

§ A.2. Series of Functions

For the remainder of this section assume $\Omega \subseteq \mathbf{R}$.

Definition A.2.1.

(1) If $(f_n : \Omega \to \mathbf{R})_n$ is a sequence of functions, the *partial sums* $(s_n)_n$ of the infinite series $\sum f_n$ is defined for $x \in \Omega$ by:

$$\begin{aligned} s_1(x) &:= f_1(x), \\ s_2(x) &:= s_1(x) + f_2(x), \\ &\vdots \\ s_{n+1}(x) &:= s_n(x) + f_{n+1}(x). \end{aligned}$$

If the sequence $(s_n)_n$ of functions converges to a function $f : \Omega \to \mathbf{R}$, we say that the infinite series of functions $\sum f_n$ *converges* to f.

(2) If the series $\sum |f_n(x)|$ converges for each $x \in \Omega$, we say that $\sum f_n$ is *absolutely convergent* on Ω.

(3) If the sequence $(s_n)_n$ of partial sums is uniformly convergent on Ω to f, we say that $\sum f_n$ is *uniformly convergent* on Ω.

Theorem A.2.1. *If $f_n : \Omega \to \mathbf{R}$ is continuous for each $n \in \mathbf{N}$ and if $\sum f_n$ converges to f uniformly on Ω, then f is continuous on Ω.*