Math 374

Homework 5

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(b) I wrote the following code which compares the Jacobi and Gauss-Seidel methods. I've only run the code for parts (i), (iii), and (iv). I made the code stop when 5 decimal digits are stable, just to be safe.

```
In[1]:= Clear["Global`*"]
        CompareJacobiGauss[A_, b_, x0_, it_ : 20] :=
          Module[{JD, JL, JU, GL, GU, x, y, xLastit, yLastit, n, table},
           (*Jacobi Method*)
           JD = DiagonalMatrix[Diagonal[A]];
           JL = LowerTriangularize[A, -1];
           JU = UpperTriangularize[A, 1];
           x = Table[0, {it + 1}, {Length[b]}];
           x[[1]] = x0;
           For[n = 1, n <= it, n++,
            x[[n + 1]] = N[Inverse[JD] . (b - (JL + JU) . x[[n]]), 100000];
            If[Max[Abs[x[[n + 1]] - x[[n]]]] < 0.00001, Break[];];
              xLastit = n + 1;
            ];
           (*Gauss-Seidel Method*)
           GL = LowerTriangularize[A];
           GU = UpperTriangularize[A, 1];
           y = Table[0, {it + 1}, {Length[b]}];
           y[[1]] = x0;
           For[n = 1, n <= it, n++,
            y[[n + 1]] = N[Inverse[GL] . (b - GU . y[[n]]), 100000];
            If[Max[Abs[y[[n + 1]] - y[[n]]]] < 0.00001, Break[];];
              yLastit = n + 1;
            ];
           table = Table[{
              n - 1,
              NumberForm[x[[n]], {Infinity, 8}, ExponentFunction -> (Null &),
               NumberPadding -> {"", "0"}],
              NumberForm[y[[n]], {Infinity, 8}, ExponentFunction -> (Null &),
               NumberPadding -> {"", "0"}]
              }, {n, 1, Max[xLastit, yLastit]}];
           TableForm[
            Prepend[table, {"n", "Jacobi Method", "Gauss-Seidel Method"}]]
           ];

        A1 = {{9, 0, -2}, {4, 12, 3}, {1, 6, 11}};
        b1 = {2, -5, 0};
        x1 = {0, 0, 0};

        A2 = {{0, 1, 4}, {9, -2, 6}, {12, 11, 3}};
        b2 = {2, -5, 0};
        x2 = {0, 0, 0};
```

```
A3 = {{4, 8, -10, 22}, {7, -5, 8, 18}, {4, 1, -9, 9}, {-2, 5, 12,
    3}};
b3 = {-1, 4, 0, 9};
x3 = {0, 0, 0, 0};

A4 =
  {{40, 0, 7, 0, 6, 0, 2, 0},
   {0, 41, 0, 8, 0, 6, 0, 1},
   {18, 0, 42, 0, 9, 0, 5, 0},
   {0, 17, 0, 43, 0, 10, 0, 5},
   {3, 0, 16, 0, 44, 0, 11, 0},
   {0, 3, 0, 15, 0, 45, 0, 12},
   {1, 0, 4, 0, 14, 0, 46, 0},
   {0, 2, 0, 4, 0, 13, 0, 47}};
b4 = {1, 4, 1, 4, 2, 1, 3, 6};
x4 = {0, 0, 0, 0, 0, 0, 0, 0};

CompareJacobiGauss[A1, b1, x1]
(*CompareJacobiGauss[A2,b2,x2]*)
CompareJacobiGauss[A3, b3, x3]
CompareJacobiGauss[A4, b4, x4]
```

Out[1]=

n	Jacobi Method	Gauss-Seidel Method
0	{0, 0, 0}	{0, 0, 0}
1	{0.22222222, -0.41666667, 0}	{0.22222222, -0.49074074, 0.24747475}
2	{0.22222222, -0.49074074, 0.20707071}	{0.27721661, -0.57094089, 0.28622079}
3	{0.26823793, -0.54250842, 0.24747475}	{0.28582684, -0.58349748, 0.29228709}
4	{0.27721661, -0.56794800, 0.27152842}	{0.28717491, -0.58546341, 0.29323687}
5	{0.28256187, -0.57695431, 0.28458831}	{0.28738597, -0.58577121, 0.29338557}
6	{0.28546407, -0.58200103, 0.28901491}	{0.28741902, -0.58581940, 0.29340885}
7	{0.28644776, -0.58407508, 0.29150383}	{0.28742419, -0.58582694, 0.29341250}
8	{0.28700085, -0.58502521, 0.29254570}	{0, 0, 0}
9	{0.28723238, -0.58547004, 0.29301367}	{0, 0, 0}
10	{0.28733637, -0.58566421, 0.29323526}	{0, 0, 0}
11	{0.28738561, -0.58575427, 0.29333172}	{0, 0, 0}
12	{0.28740705, -0.58579480, 0.29337637}	{0, 0, 0}
13	{0.28741697, -0.58581311, 0.29339652}	{0, 0, 0}

Out[2]=

n	Jacobi Method	Gauss-Seidel Method
0	{0, 0, 0, 0}	{0, 0, 0, 0}
1	{-0.25000000, -0.80000000, 0, 3.00000000}	{-0.25000000, -1.15000000, -0.23888889, 5.70555556}
2	{-15.15000000, 9.65000000, 2.80000000, 4.16666667}	{-29.92777778, -22.54111111, -10.10024691, 61.01765432}
3	{-35.46666667, -2.53000000, -1.49444444, -34.38333333}	{-316.01549383, -239.71853086, -106.06906859, 616.13016324}
4	{190.18222222, -176.62444444, -50.42740741, -10.45000000}	{-3174.70150754, -2397.02403265, -1061.18428819, 6126.30953547}
5	{284.40537037, 147.15125926, 54.45049383, 625.87185185}	{-31553.86510024, -23819.39167376, -10544.22958395, 60842.99439190}
6	{-3600.72146914, 2737.62697531, 768.62437860, -270.45049383}	{-313358.50976778, -236538.70119838, -104709.53230471, 604166.62470430}
7	{-2066.46528807, -4785.63282881, -1566.59037174, -10034.69011934}	{-3111613.11423869, -2348794.56268622, -1039749.71081136, 5999250.70489668}
8	{60845.33538464, -41525.28042771, -11484.85611724, 12867.77267627}	{-30897664.27858773, -23323027.78969294, -10324492.06221930, 59571241.37930694}
9	{-16434.57915716, 113130.88138548, 35296.22391081, 155714.78210489}	{-306807002.41235056, -231592522.51133667, -102519928.86077516, 591529254.35376139}
10	{-994452.75457085, 594037.96301488, 160980.62263343, -340689.75072379}	{-3046525676.32495217, -2299662518.15863231, -1018000214.91939873, 5873754608.72534731}
11	{1088169.00953467, -2361148.76279135, -716664.53464252, -2296951.26527244}	{-30251325849.22064239, -22835139942.16868700, -10108516873.39145898, 58325083500.69988597}
12	{15563867.89797480, -7892251.99706026, -2075670.45690051, 7527355.08291209}	{-300388971553.24064628, -226747886570.24364966, -100375335697.43414012, 579155172707.64887904}
13	{-30805125.35414725, 45566819.82460742, 13567712.81567197, 31832350.08801901}	{-2982796015995.41688570, -2251556337762.74229962, -996704871930.61888123, 5750882706666.43476714}
14	{-232292283.34413947, 93177624.52613744, 23204163.24446550, -150752298.20646509}	{-29618504390966.70382308, -22357456198444.01040059, -9897058822479.21253208, 57104992693348.73158057}
15	{700792798.94444687, -830790809.83392478, -243640243.63428958, -402974213.08418406}	{-294105194472728.28422215, -222004592681731.70427228, -98275604036945.13965953, 567039940968850.87627715}
16	{3268839182.29513792, -859421639.19570033, -183820836.86819931, 2826407526.55666421}	{-2920399500057579.51012857, -2204456479051861.18303751, -975855001173613.48189527, 5630581136409172.55922457}
17	{-14285950210.09075076, 14457328611.02806532, 4183733647.66609213, 4346878870.66238975}	{-28998920795080760.66439828, -21889765023917826.08798159, -9690024219617590.63472818, 55910424721612901.90928317}
18	{-42363156891.78404395, 2342407475.72329946, -396062488.15260333, -50354449079.43831124}	{-287952866470079285.16191469, -217360522811692698.16882625, -96219796244165969.07032628, 555178145349432186.12140574}
19	{271274498764.83260460, -24121813631 6.31974733, -68922251311.81751972, -30561867098.45111509}	{-2859308244408906550.25589478, -2158341892905178851.63371406, -955441284710657264.17384905, 5512796130745322778.91432314}
20	{478220913394.32682832, 159485974616.63360056, 63202561650.77229281, 858568898287.35806084}	{-28392298145065560741.44597178, -21431857387945674657.41095568, -9487320532388890290.32954828, 54740845679421311765.70580473}

Out[3]=

n	Jacobi Method	Gauss-Seidel Method
0	{0, 0, 0, 0, 0, 0, 0, 0}	{0, 0, 0, 0, 0, 0, 0, 0}
1	{0.02500000, 0.09756098, 0.02380952, 0.09302326, 0.04545455, 0.02222222, 0.06521739, 0.12765957}	{0.02500000, 0.09756098, 0.01309524, 0.05445264, 0.03898810, -0.00243272, 0.05166925, 0.11954666}
2	{0.01075428, 0.07304442, -0.00440900, 0.03444054, 0.01878764, -0.04933215, 0.04876953, 0.10944460}	{0.01427666, 0.08437631, 0.00318526, 0.04633016, 0.03040555, -0.03072536, 0.05537619, 0.12862460}
3	{0.02051495, 0.09539083, 0.00936873, 0.06289171, 0.03413219, -0.02331281, 0.05964902, 0.13526525}	{0.01711294, 0.08988015, 0.00336753, 0.04967832, 0.02921915, -0.03462912, 0.05565976, 0.12918521}
4	{0.01525819, 0.08540190, 0.00060229, 0.04500367, 0.02573674, -0.04117180, 0.05356868, 0.12469613}	{0.01724482, 0.08978446, 0.00353148, 0.05055882, 0.02907965, -0.03506574, 0.05568509, 0.12923511}
5	{0.01835566, 0.09176354, 0.00537806, 0.05433500, 0.03080303, -0.03172476, 0.05700040, 0.13158330}	{0.01723579, 0.08967534, 0.00356223, 0.05069770, 0.02906275, -0.03511806, 0.05568776, 0.12924241}
6	{0.01658837, 0.08839232, 0.00255640, 0.04882211, 0.02799727, -0.03709589, 0.05497586, 0.12790542}	{0.01723281, 0.08965572, 0.00356681, 0.05071678, 0.02906062, -0.03512506, 0.05568807, 0.12924356}
7	{0.01760425, 0.09034373, 0.00415606, 0.05183168, 0.02964995, -0.03405275, 0.05611357, 0.13000369}	{0.01723231, 0.08965299, 0.00356744, 0.05071935, 0.02906035, -0.03512604, 0.05568811, 0.12924373}
8	{0.01701952, 0.08925998, 0.00323110, 0.05010850, 0.02871457, -0.03574557, 0.05544939, 0.12882280}	{0, 0, 0, 0, 0, 0, 0, 0}
9	{0.01735490, 0.08987275, 0.00376121, 0.05106795, 0.02925683, -0.03478402, 0.05582722, 0.12948380}	{0, 0, 0, 0, 0, 0, 0, 0}
10	{0.01716190, 0.08952870, 0.00345629, 0.05052522, 0.02894674, -0.03532096, 0.05560880, 0.12911011}	{0, 0, 0, 0, 0, 0, 0, 0}
11	{0.01727270, 0.08972229, 0.00363145, 0.05082956, 0.02912538, -0.03501746, 0.05573388, 0.12931945}	{0, 0, 0, 0, 0, 0, 0, 0}
12	{0.01720899, 0.08961339, 0.00353080, 0.05065810, 0.02902286, -0.03518764, 0.05566187, 0.12920137}	{0, 0, 0, 0, 0, 0, 0, 0}
13	{0.01724559, 0.08967463, 0.00358864, 0.05075446, 0.02908181, -0.03509174, 0.05570321, 0.12926766}	{0, 0, 0, 0, 0, 0, 0, 0}
14	{0.01722456, 0.08964017, 0.00355541, 0.05070024, 0.02904795, -0.03514562, 0.05567945, 0.12923033}	{0, 0, 0, 0, 0, 0, 0, 0}
15	{0.01723664, 0.08965955, 0.00357451, 0.05073074, 0.02906741, -0.03511529, 0.05569310, 0.12925131}	{0, 0, 0, 0, 0, 0, 0, 0}
16	{0.01722970, 0.08964865, 0.00356353, 0.05071358, 0.02905622, -0.03513234, 0.05568525, 0.12923951}	{0, 0, 0, 0, 0, 0, 0, 0}

I didn't include the matrix from (ii) because it was crashing my Mathematica. Let $\mathbf{x}, \mathbf{b} \in \mathbb{C}^n$ and $A \in \mathrm{Mat}_n(\mathbb{C})$. Recall that the iterative formula for the Jacobi method is:

$$\mathbf{x}^{(k+1)} = D^{-1}\left(\mathbf{b} - (L+U)\mathbf{x}^{(k)}\right),$$

where $A = D + L + U$, that is, it is decomposed into a diagonal, strictly lower triangular, and strictly upper triangular matrix. Similarly, The iterative formula for the Gauss-Seidel method is:

$$\mathbf{x}^{(k+1)} = L^{-1}(\mathbf{b} - U\mathbf{x}^{(k)}),$$

where $A = L + U$, that is, it is decomposed into a lower triangular and strictly upper triangular matrix. For part (ii), we can clearly see that $\det(D^{-1}) = 0$ and $\det(L^{-1}) = 0$, implying the matrix cannot be inverted. Mathematica doesn't like this.

Recall $A \in \mathrm{Mat}_n(\mathbf{C})$ is *invertible* if there exists a $B \in \mathrm{Mat}_n(\mathbf{C})$ such that $AB = BA = I_n$. For $A \in \mathrm{Mat}_{n,m}(\mathbf{C})$, a *pseudoinverse* of A is a matrix $A^+ \in \mathrm{Mat}_{m,n}(\mathbf{C})$ satisfying the following properties:

(1) $AA^+A = A$;

(2) $A^+AA^+ = A$;

(3) $(AA^+)^* = AA^+$;

(4) $(A^+A)^* = A^+A$,

where $\square^*$ denotes the conjugate transpose. Replacing each instance of `Inverse[]` with `PseudoInverse[]` in the above code, we have:

In[4]:= CompareJacobiGauss[A2, b2, x2]

Out[4]=

n	Jacobi Method	Gauss-Seidel Method
0	{0, 0, 0}	{0, 0, 0}
1	{0, 2.50000000, 0}	{-0.45111363, 0.46998867, 0.08116270}
2	{0, 2.50000000, -9.16666667}	{-0.49504995, 0.51576334, 0.08906756}
3	{0, -25.00000000, -9.16666667}	{-0.49932914, 0.52022157, 0.08983746}
4	{0, -25.00000000, 91.66666667}	{-0.49974591, 0.52065578, 0.08991244}
5	{0, 277.50000000, 91.66666667}	{-0.49978650, 0.52069807, 0.08991975}
6	{0, 277.50000000, -1017.50000000}	{-0.49979046, 0.52070219, 0.08992046}
7	{0, -3050.00000000, -1017.50000000}	{0, 0, 0}
8	{0, -3050.00000000, 11183.33333333}	{0, 0, 0}
9	{0, 33552.50000000, 11183.33333333}	{0, 0, 0}
10	{0, 33552.50000000, -123025.83333333}	{0, 0, 0}
11	{0, -369075.00000000, -123025.83333333}	{0, 0, 0}
12	{0, -369075.00000000, 1353275.00000000}	{0, 0, 0}
13	{0, 4059827.50000000, 1353275.00000000}	{0, 0, 0}
14	{0, 4059827.50000000, -14886034.16666667}	{0, 0, 0}
15	{0, -44658100.00000000, -14886034.16666667}	{0, 0, 0}
16	{0, -44658100.00000000, 163746366.66666667}	{0, 0, 0}
17	{0, 491239102.50000000, 163746366.66666667}	{0, 0, 0}
18	{0, 491239102.50000000, -1801210042.50000000}	{0, 0, 0}
19	{0, -5403630125.00000000, -1801210042.50000000}	{0, 0, 0}
20	{0, -5403630125.00000000, 19813310458.33333333}	{0, 0, 0}

Neither the Jacobi method nor the Gauss-Seidel method converged to a correct solution, but it's still cool nonetheless!

(c) I wrote the code below to do parts (i), (ii), and (iii). From the table at the very end, the rate at which the absolute errors change is approximately linear.

```
In[5]:= Clear["Global`*"]
EulersMethod[f_, leftEndpoint_, rightEndpoint_, deltaX_,
   initialCondition_] :=
  Module[{a, b, h, x, y, yVals, n, points, plot},
   a = leftEndpoint;
   b = rightEndpoint;
   h = deltaX;
   x = Table[x, {x, a, b, h}];
   y = Table[0, {Length[x]}];
   y[[1]] = initialCondition;
   For[n = 1, n < Length[x], n++,
    y[[n + 1]] = N[y[[n]] + h*f[x[[n]], y[[n]]], 10];];
   points = Transpose[{x, y}];
   plot = ListLinePlot[points, PlotRange -> All];
   {MatrixForm[points], plot, points}
   ];

f[x_, y_] := x*y^2 + x^2*y - 2;
p1 = EulersMethod[f, 0, 2, 1, 1];
p2 = EulersMethod[f, 0, 2, 1/2, 1];
p3 = EulersMethod[f, 0, 2, 1/4, 1];
p4 = EulersMethod[f, 0, 2, 1/8, 1];
p5 = EulersMethod[f, 0, 2, 1/16, 1];
```

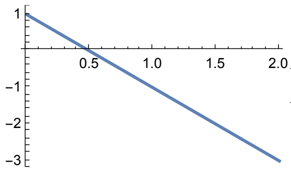
```
{p1[[1]], p1[[2]]}
{p2[[1]], p2[[2]]}
{p3[[1]], p3[[2]]}
{p4[[1]], p4[[2]]}
{p5[[1]], p5[[2]]}
ListLinePlot[{p1[[3]], p2[[3]], p3[[3]], p4[[3]], p5[[3]]},
 PlotLabel -> "y'[x] = x*y^2+x^2*y-2", PlotRange -> All,
 PlotLegends -> {"h=1", "h=1/2", "h=1/4", "h=1/8", "h=1/16"}]

g[x_, y_] := -8*y + Sin[x];
q1 = EulersMethod[g, 0, 2, 1, 0];
q2 = EulersMethod[g, 0, 2, 1/2, 0];
q3 = EulersMethod[g, 0, 2, 1/4, 0];
q4 = EulersMethod[g, 0, 2, 1/8, 0];
q5 = EulersMethod[g, 0, 2, 1/16, 0];
{q1[[1]], q1[[2]]}
{q2[[1]], q2[[2]]}
{q3[[1]], q3[[2]]}
{q4[[1]], q4[[2]]}
{q5[[1]], q5[[2]]}
ListLinePlot[{q1[[3]], q2[[3]], q3[[3]], q4[[3]], q5[[3]]},
 PlotLabel -> "y'[x] = -8*y +Sin[x]", PlotRange -> All,
 PlotLegends -> {"h=1", "h=1/2", "h=1/4", "h=1/8", "h=1/16"}]

h[x_, y_] := y *Cos[x];
r1 = EulersMethod[h, 0, 2, 1, 2];
r2 = EulersMethod[h, 0, 2, 1/2, 2];
r3 = EulersMethod[h, 0, 2, 1/4, 2];
r4 = EulersMethod[h, 0, 2, 1/8, 2];
r5 = EulersMethod[h, 0, 2, 1/16, 2];
{r1[[1]], r1[[2]]}
{r2[[1]], r2[[2]]}
{r3[[1]], r3[[2]]}
{r4[[1]], r4[[2]]}
{r5[[1]], r5[[2]]}
ListLinePlot[{r1[[3]], r2[[3]], r3[[3]], r4[[3]], r5[[3]]},
 PlotLabel -> "y'[x] = y*Cos[x]", PlotRange -> All,
 PlotLegends -> {"h=1", "h=1/2", "h=1/4", "h=1/8", "h=1/16"}]

y[x_] := 2 Exp[Sin[x]];
delta = {1, 1/2, 1/4, 1/8, 1/16};
approx = {r1[[3, -1, 2]], r2[[3, -1, 2]], r3[[3, -1, 2]],
   r4[[3, -1, 2]], r5[[3, -1, 2]]};
errors = {Abs[approx[[1]] - y[2]], Abs[approx[[2]] - y[2]],
   Abs[approx[[3]] - y[2]], Abs[approx[[4]] - y[2]],
   Abs[approx[[5]] - y[2]]};
table = Table[{
    delta[[n]],
    approx[[n]],
    errors[[n]]
    }, {n, 1, Length[delta]}];
TableForm[
 Prepend[table, {"deltaX", "y_{approx}[2]", "|y_{approx}[2] - y_{exact}[2]|"
}]]
```

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & -1.000000000 \\ 2 & -3.000000000 \end{pmatrix}, \right.$$



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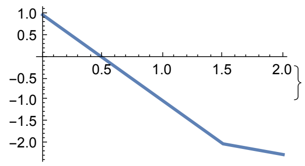
$$\left\{ \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \\ 1 & -1.000000000 \\ \frac{3}{2} & -2.000000000 \\ 2 & -2.25000000 \end{pmatrix}, \right.$$



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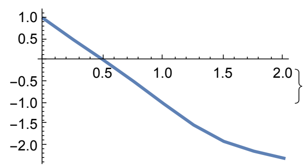
$$\left\{ \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0.5000000000 \\ \frac{1}{2} & 0.0234375000 \\ \frac{3}{4} & -0.4750289917 \\ 1 & -0.9995200919 \\ \frac{5}{4} & -1.499640011 \\ \frac{3}{2} & -1.882649340 \\ \frac{7}{4} & -2.11250139 \\ 2 & -2.27747059 \end{pmatrix}, \right.$$



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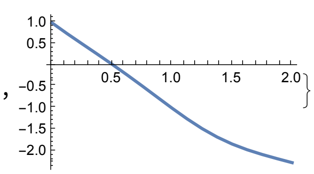
$$\left\{ \begin{pmatrix} 0 & 1 \\ \frac{1}{8} & 0.7500000000 \\ \frac{1}{4} & 0.5102539063 \\ \frac{3}{8} & 0.2723764852 \\ \frac{1}{2} & 0.0306419601 \\ \frac{5}{8} & -0.2183417956 \\ \frac{3}{4} & -0.4752785520 \\ \frac{7}{8} & -0.737519416 \\ 1 & -0.998609450 \\ \frac{9}{8} & -1.248783027 \\ \frac{5}{4} & -1.477045476 \\ \frac{11}{8} & -1.674646024 \\ \frac{3}{2} & -1.838398973 \\ \frac{13}{8} & -1.97175291 \\ \frac{7}{4} & -2.08287410 \\ \frac{15}{8} & -2.18120710 \\ 2 & -2.2746678 \end{pmatrix}, \right.$$



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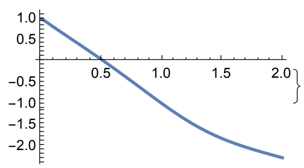
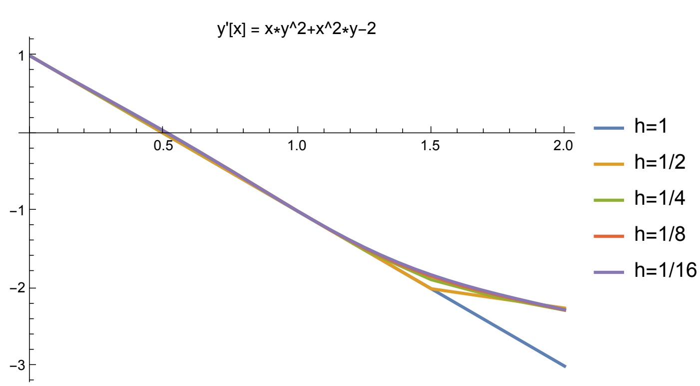
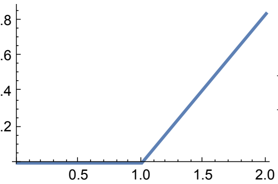
0	1
$\frac{1}{16}$	0.8750000000
$\frac{1}{8}$	0.7532043457
$\frac{3}{16}$	0.6333720592
$\frac{1}{4}$	0.5144648416
$\frac{5}{16}$	0.3956100022
$\frac{3}{8}$	0.2760813968
$\frac{7}{16}$	0.1552943216
$\frac{1}{2}$	0.0328115214
$\frac{9}{16}$	-0.091642155
$\frac{5}{8}$	-0.218159162
$\frac{11}{16}$	-0.346626196
$\frac{3}{4}$	-0.476703202
$\frac{13}{16}$	-0.607810146
$\frac{7}{8}$	-0.739127971
$\frac{15}{16}$	-0.869620063
1	-0.998078838
$\frac{17}{16}$	-1.123198680
$\frac{9}{8}$	-1.243671322
$\frac{19}{16}$	-1.358294032
$\frac{5}{4}$	-1.466076135
$\frac{21}{16}$	-1.566327380
$\frac{11}{8}$	-1.658713660
$\frac{23}{16}$	-1.74327150
$\frac{3}{2}$	-1.82038094
$\frac{25}{16}$	-1.89070450
$\frac{13}{8}$	-1.95510511
$\frac{27}{16}$	-2.01455767
$\frac{7}{4}$	-2.07006679
$\frac{29}{16}$	-2.1225993
$\frac{15}{8}$	-2.1730357
$\frac{31}{16}$	-2.2221401
2	-2.2705471



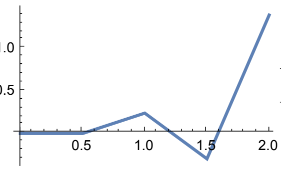


$$\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 0.8414709848 \end{pmatrix},\right.$$



$$\}$$

$$\left\{\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \\ 1 & 0.2397127693 \\ \frac{3}{2} & -0.298402816 \\ 2 & 1.393955940 \end{pmatrix},\right.$$



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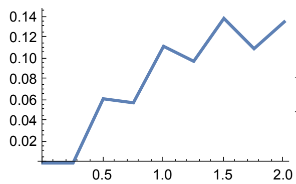
$$\left\{\begin{pmatrix} 0 & 0 \\ \frac{1}{4} & 0 \\ \frac{1}{2} & 0.06185098981 \\ \frac{3}{4} & 0.0580053948 \\ 1 & 0.1124042952 \\ \frac{5}{4} & 0.097963451 \\ \frac{3}{2} & 0.139282704 \\ \frac{7}{4} & 0.11009104 \\ 2 & 0.13590544 \end{pmatrix},\right.$$



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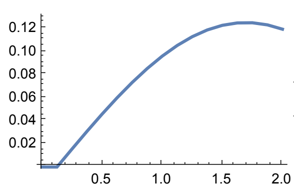
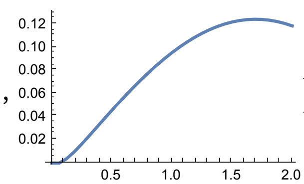
$$\left\{\begin{pmatrix} 0 & 0 \\ \frac{1}{8} & 0 \\ \frac{1}{4} & 0.01558434167 \\ \frac{3}{8} & 0.03092549491 \\ \frac{1}{2} & 0.04578406614 \\ \frac{5}{8} & 0.0599281923 \\ \frac{3}{4} & 0.0731371591 \\ \frac{7}{8} & 0.0852048450 \\ 1 & 0.0959429378 \\ \frac{9}{8} & 0.105183873 \\ \frac{5}{4} & 0.112783449 \\ \frac{11}{8} & 0.118623077 \\ \frac{3}{2} & 0.12261163 \\ \frac{13}{8} & 0.12468687 \\ \frac{7}{4} & 0.12481642 \\ \frac{15}{8} & 0.1229982 \\ 2 & 0.1192607 \end{pmatrix},\right.$$



$$\}$$

$$\left\{\begin{pmatrix} 0 & 0 \\ \frac{1}{16} & 0 \\ \frac{1}{8} & 0.003903707365 \\ \frac{3}{16} & 0.009744024519 \\ \frac{1}{4} & 0.01652221831 \\ \frac{5}{16} & 0.02372385661 \\ \frac{3}{8} & 0.03107683546 \\ \frac{7}{16} & 0.03843045080 \\ \frac{1}{2} & 0.04569499148 \\ \frac{9}{16} & 0.0528115919 \\ \frac{5}{8} & 0.0597372130 \\ \frac{11}{16} & 0.0664371861 \\ \frac{3}{4} & 0.0728815355 \\ \frac{13}{16} & 0.0790431903 \\ \frac{7}{8} & 0.0848971361 \\ \frac{15}{16} & 0.090420037 \\ 1 & 0.095590088 \\ \frac{17}{16} & 0.100386980 \\ \frac{9}{8} & 0.104791924 \\ \frac{19}{16} & 0.108787686 \\ \frac{5}{4} & 0.11235865 \\ \frac{21}{16} & 0.11549086 \\ \frac{11}{8} & 0.11817209 \\ \frac{23}{16} & 0.12039186 \\ \frac{3}{2} & 0.12214151 \\ \frac{25}{16} & 0.12341419 \\ \frac{13}{8} & 0.1242049 \\ \frac{27}{16} & 0.1245107 \\ \frac{7}{4} & 0.1243302 \\ \frac{29}{16} & 0.1236642 \\ \frac{15}{8} & 0.1225153 \\ \frac{31}{16} & 0.1208880 \\ 2 & 0.118789 \end{pmatrix}, \right.$$



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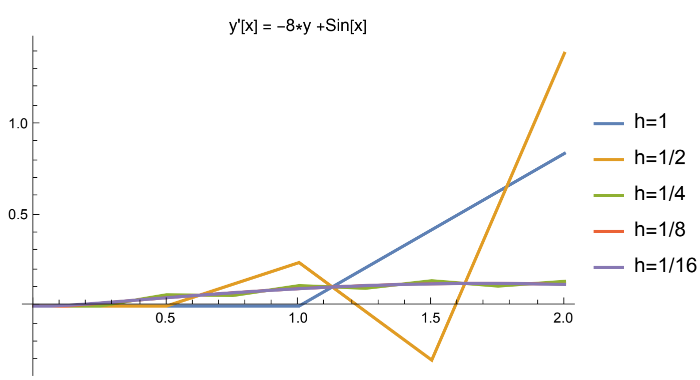
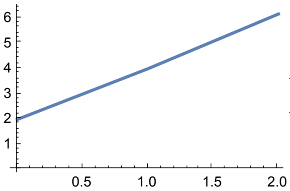
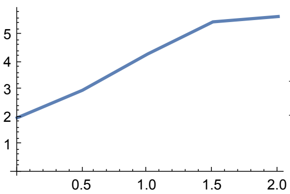


$$\left\{\begin{pmatrix} 0 & 2 \\ 1 & 4.000000000 \\ 2 & 6.161209223 \end{pmatrix},\right.$$



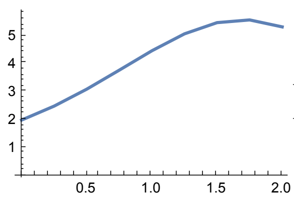
$$\left.\right\}$$

$$\left\{\begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 3.000000000 \\ 1 & 4.316373843 \\ \frac{3}{2} & 5.482447213 \\ 2 & 5.676353700 \end{pmatrix},\right.$$



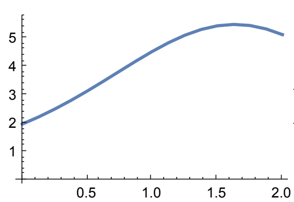
$$\left.\right\}$$

$$\left\{\begin{pmatrix} 0 & 2 \\ \frac{1}{4} & 2.500000000 \\ \frac{1}{2} & 3.105570264 \\ \frac{3}{4} & 3.786918841 \\ 1 & 4.479630431 \\ \frac{5}{4} & 5.084719094 \\ \frac{3}{2} & 5.485550503 \\ \frac{7}{4} & 5.582558626 \\ 2 & 5.333791363 \end{pmatrix},\right.$$



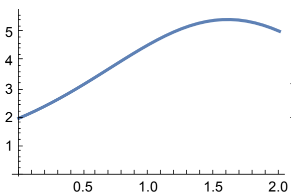
$$\left.\right\}$$

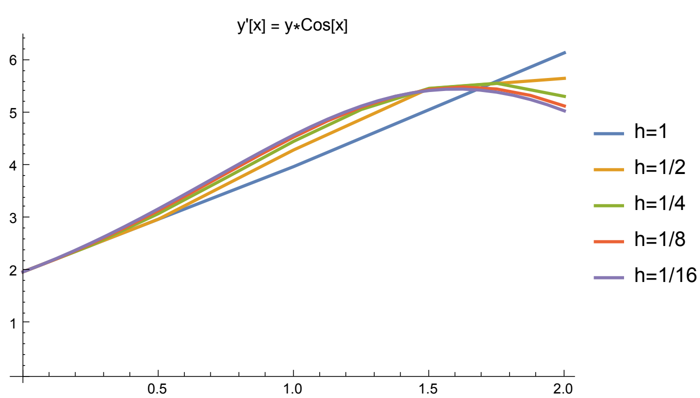
$$\left\{\begin{pmatrix} 0 & 2 \\ \frac{1}{8} & 2.250000000 \\ \frac{1}{4} & 2.529055594 \\ \frac{3}{8} & 2.835359766 \\ \frac{1}{2} & 3.165150251 \\ \frac{5}{8} & 3.512360334 \\ \frac{3}{4} & 3.868409670 \\ \frac{7}{8} & 4.222218707 \\ 1 & 4.560522323 \\ \frac{9}{8} & 4.868529914 \\ \frac{5}{4} & 5.130929385 \\ \frac{11}{8} & 5.333166482 \\ \frac{3}{2} & 5.462860897 \\ \frac{13}{8} & 5.511164333 \\ \frac{7}{4} & 5.473841947 \\ \frac{15}{8} & 5.351880605 \\ 2 & 5.151497159 \end{pmatrix},\right.$$



$$\left.\right\}$$

0	2
$\frac{1}{16}$	2.125000000
$\frac{1}{8}$	2.257553185
$\frac{3}{16}$	2.397549373
$\frac{1}{4}$	2.544769890
$\frac{5}{16}$	2.698873587
$\frac{3}{8}$	2.859383687
$\frac{7}{16}$	3.025676082
$\frac{1}{2}$	3.196969757
$\frac{9}{16}$	3.372320064
$\frac{5}{8}$	3.550615574
$\frac{11}{16}$	3.730579216
$\frac{3}{4}$	3.910774341
$\frac{13}{16}$	4.089616219
$\frac{7}{8}$	4.265389346
$\frac{15}{16}$	4.436270669
1	4.600358637
$\frac{17}{16}$	4.755707661
$\frac{9}{8}$	4.900367272
$\frac{19}{16}$	5.032424978
$\frac{5}{4}$	5.150051505
$\frac{21}{16}$	5.251546906
$\frac{11}{8}$	5.335385806
$\frac{23}{16}$	5.400259999
$\frac{3}{2}$	5.445116565
$\frac{25}{16}$	5.469189834
$\frac{13}{8}$	5.472025688
$\frac{27}{16}$	5.453497021
$\frac{7}{4}$	5.413809557
$\frac{29}{16}$	5.353497670
$\frac{15}{8}$	5.273410307
$\frac{31}{16}$	5.174687615
2	5.058729263





deltaX	$y_{approx}[2]$	$\lvert y_{approx}[2] - y_{exact}[2] \rvert$
1	6.161209223	1.196053767
$\frac{1}{2}$	5.676353700	0.711198244
$\frac{1}{4}$	5.333791363	0.368635907
$\frac{1}{8}$	5.151497159	0.186341703
$\frac{1}{16}$	5.058729263	0.093573807