Math 397

Homework 4

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Exercise 1. Let X be a metric space. Show that X is second countable if and only if X is separable. Conclude that if X is a separable metric space, then every open set is the union of countably many open balls.

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be a countable base for X. Let $x \in X$ and $\epsilon > 0$. Then $x \in U(x,\epsilon) \subseteq X$. We can find $U_n \in \{U_n\}_{n=1}^{\infty}$ with $x \in U_n \subseteq U(x,\epsilon)$. So for any $a_n \in U_n$, we have $a_n \in U(x,\epsilon)$, giving $d(x,a_n) < \epsilon$. Thus $\{a_n\}_{n=1}^{\infty}$ is dense; i.e., X is separable.

Let $\{a_n\}_{n=1}^{\infty}$ be a countable dense subset. Claim: $\mathcal{B} = \{U(a_n, \frac{1}{m}) \mid n, m \geqslant 1\}$ is a base. Let $U \in \tau_X$ and $x \in U$. Since U is open, there exists $\epsilon > 0$ such that $U(x,\epsilon) \subseteq U$. Moreover, we can find $m \geqslant 1$ with $\epsilon > \frac{1}{m}$. Since $\{a_n\}_{n=1}^{\infty}$ is dense, we can find $a_j \in \{a_n\}_{n=1}^{\infty}$ such that $d(x,a_j) < \frac{1}{2m}$. Let $y \in U(a_j, \frac{1}{2m})$. Then:

$$\begin{aligned} d(x,y) &\leqslant d(x,a_j) + d(a_j,y) \\ &< \frac{1}{2m} + \frac{1}{2m} \\ &= \frac{1}{m} \\ &< \epsilon. \end{aligned}$$

So $y \in U(x,\epsilon)$. Thus $x \in U(a_j, \frac{1}{2m}) \subseteq U(x,\epsilon) \subseteq U$, establishing $\mathcal{B}$ as a base.x $\square$

Exercise 2. Let (X,d) be a metric space, $(x_n)_n$ a sequence in X, and $x \in X$. Show the following are equivalent:

(1) $(x_n)_n \to x$ in X;

(2) $(d(x_n,x))_n \to 0$ in $\mathbf{R}$;

(3) $(\forall V \in \mathcal{N}_x)(\exists N \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies x_n \in V)$.

Proof. (1) $\Leftrightarrow$ (2) Let $\epsilon > 0$. Find N large so for $n \geqslant N$ we have $d(x_n,x) < \epsilon$. This is equivalent to $|d(x_n,x) - 0| < \epsilon$, whence $(d(x_n,x))_n \to 0$. The other direction is identical.

(1) $\Rightarrow$ (3) Let $V \in \mathcal{N}_x$. Then there exists $\epsilon > 0$ so $U(x,\epsilon) \subseteq V$. Since $(x_n)_n \to x$, find N large so for $n \geqslant N$ we have $d(x_n,x) < \epsilon$. Thus $x_n \in U(x,\epsilon) \subseteq V$.

(3) $\Rightarrow$ (1) Let $\epsilon > 0$. Find N large so $n \geqslant N$ implies $x_n \in U(x,\epsilon) \in \mathcal{N}_x$. Then $d(x_n,x) < \epsilon$, giving $(x_n)_n \to x$. $\square$

Exercise 4. Let $\{(X_k,d_k)\}_{k\geqslant 1}$ be a family of metric spaces. Assume that for every $k \geqslant 1$ we have $d_k(x,y) \leqslant 1$ for all $x,y \in X_k$. Let:

$$X := \prod_{k\geqslant 1} X_k$$
$$d(f,g) := \sum_{k=1}^{\infty} 2^{-k} d_k(f(k),g(k)).$$

Show that a sequence $(f_n)_n$ converges to f in (X,d) if and only if $(f_n(k))_n \xrightarrow{d_k} f(k)$ for every $k \geqslant 1$.

Proof. Let $(f_n)_n \xrightarrow{d} f$. Fix $k \geqslant 1$. We have:

$$0 \leqslant 2^{-k} d_k(f_n(k), f(k)) \leqslant d(f_n, f).$$

Since $(d(f_n, f))_n \to 0$, multiplying 2^{-k} on all sides and applying the squeeze theorem yields $(d_k(f_n(k), f(k)))_n \to 0$. Whence $(f_n(k))_n \xrightarrow{d_k} f(k)$ for every $k \geqslant 1$.

Now suppose $(f_n(k))_n \xrightarrow{d_k} f(k)$ for every $k \geqslant 1$. Then $(d_k(f_n(k), f(k)))_n \xrightarrow{d_k} 0$ for every $k \geqslant 1$. Find K large so that:

$$\sum_{k>K} 2^{-k} < \frac{\epsilon}{2}.$$

Find $N_1, N_2, ..., N_K$ sufficiently large so that for $n \geqslant N_i$ we have $d_i(f_n(i), f(i)) < \frac{\epsilon}{2}$. For $n \geqslant \max_{i=1}^{K} N_i$ observe that:

$$\begin{aligned}
d(f_n, f) &= \sum_{k=1}^{\infty} 2^{-k} d_k(f_n(k), f(k)) \\
&= \sum_{k=1}^{K} 2^{-k} d_k(f_n(k), f(k)) + \sum_{k>K} 2^{-k} d_k(f_n(k), f(k)) \\
&\leqslant \sum_{k=1}^{K} 2^{-k} d_k(f_n(k), f(k)) + \sum_{k>K} 2^{-k} \\
&< \sum_{k=1} 2^{-k} \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Since $(d(f_n, f))_n \to 0$, we have $(f_n)_n \xrightarrow{d} f$. □

Exercise 5. Let V be a normed space. Show that the vector operations:

$$\begin{aligned}
a &: V \times V \to V; \quad a(v, w) = v + w; \\
\mu &: F \times V \to V; \quad \mu(\alpha, w) = \alpha w
\end{aligned}$$

are continuous.

Proof. Let $((v_n, w_n))_n$ be a sequence in $V \times V$ converging to (v_0, w_0). Then $(v_n)_n \to v_0$ and $(w_n)_n \to w_0$. Observe that:

$$\begin{aligned}
(a(v_n, w_n))_n &= (v_n + w_n)_n \\
&= (v_n)_n + (w_n)_n \\
&\xrightarrow{n \to \infty} v_0 + w_0 \\
&= a(v_0, w_0).
\end{aligned}$$

Thus a is continuous at (v_0, w_0). Since this point was arbitrary, a is continuous.

Now let $((\alpha_n, v_n))_n$ be a sequence in $F \times V$ converging to (α_0, v_0). Then $(\alpha_n)_n \to \alpha_0$ and $(v_n)_n \to v_0$. Observe that:

$$\begin{aligned}(\mu(\alpha_n, v_n))_n &= (a_n v_n)_n \\ &= (a_n)_n (v_n)_n \\ &\xrightarrow{n\to\infty} \alpha_0 v_0 \\ &= \mu(\alpha_0, v_0).\end{aligned}$$

Thus μ is continuous at (α_0, v_0). Since this point was arbitrary, μ is continuous. □

Exercise 7. Consider the two metrics on $(0, \infty)$:

$$\begin{aligned}d(s,t) &:= |s - t| \\ \rho(s,t) &:= \left|\frac{1}{s} - \frac{1}{t}\right|\end{aligned}$$

Show that d and ρ are topologically equivalent. Are they uniformly equivalent?

Proof. Let $x, x_0 \in X$. We will first show that $\mathrm{id} : (X, d) \to (X, \rho)$ is continuous. Note that:

$$\begin{aligned}\rho(\mathrm{id}(x), \mathrm{id}(x_0)) &= \left|\frac{1}{x} - \frac{1}{x_0}\right| \\ &= \frac{1}{x \cdot x_0} |x - x_0| \\ &= \frac{1}{x \cdot x_0} d(x, x_0).\end{aligned}$$

Thus id is Lipschitz. We will now show that $\mathrm{id}^{-1} : (X, \rho) \to (X, d)$ is continuous. Let $(x_n)_n$ be a sequence in (X, ρ) such that $(x_n)_n \xrightarrow{\rho} x_0$. Then $(\rho(x_n, x_0))_n \to 0$, which is equivalent to $\left(d\left(\frac{1}{x_n}, \frac{1}{x_0}\right)\right)_n \to 0$. Since $\left(\frac{1}{x_n}\right)_n$ is a sequence of non-zero numbers converging to a non-zero limit $\frac{1}{x_0}$, the sequence of reciprocals $(x_n)_n$ will converge to x_0; i.e, $(\mathrm{id}^{-1}(x_n))_n \xrightarrow{d} \mathrm{id}^{-1}(x_0)$. This establishes id^{-1} as continuous, giving that d and ρ are topologically equivalent.

Let $\epsilon_0 = 1$. Consider the sequences $\left(\frac{1}{n}\right)_n$ and $\left(\frac{1}{n+1}\right)_n$ in (X, d). We have $\left(d\left(\frac{1}{n}, \frac{1}{n+1}\right)\right)_n \to 0$ and $\rho\left(\frac{1}{n}, \frac{1}{n+1}\right) \geqslant \epsilon_0$. Whence $\mathrm{id} : (X, d) \to (X, \rho)$ is not uniformly continuous; i.e., d and ρ are not uniformly equivalent. □

Exercise 9. Suppose $T : V \to W$ is a bijective linear map between normed spaces with with $\|T\|_{\mathrm{op}} \leqslant 1$ and $\|T^{-1}\| \leqslant 1$. Show that T is an isometry.

Proof. Since $\|T\|_{\mathrm{op}} \leqslant 1$, we have $\sup_{v \in B_V} \|T(v)\|_W \leqslant \sup_{v \in B_V} \|v\|_V$. So $\|T(v)\|_W \leqslant \|v\|_V$ for all $v \in V$. Since $\|T^{-1}\|_{\mathrm{op}} \leqslant 1$, we have $\sup_{w \in B_W} \|T^{-1}(w)\|_V \leqslant \sup_{w \in B_W} \|w\|_W$. So $\|T^{-1}(w)\|_V \leqslant \|w\|_W$ for all $w \in W$. Since T is a bijection, given $x \in X$ take $w = T(v)$. Then $\|v\|_V \leqslant \|T(v)\|_W$ for all $v \in V$. By antisymmetry, we have $\|T(v)\|_W = \|v\|_V$. Thus T is an isometry. □