Definition. A zero of a continuous function is called *isolated* if there exists an open set containing that zero but no others zeros of f.

Proposition. *If $f : [0,1] \to \mathbf{R}$ is continuous and all of its zeros are isolated, show that f has only finitely many zeros on $[0,1]$.*

Proof. This statement is equivalent to the following: if f has infinitely many zeros on $[0,1]$, then f being continuous implies there exists a zero which is not isolated.

Let $\{x_i\}_{i\in I} \subseteq [0,1]$ be an infinite family of zeros of f. Since this set is bounded, by the Bolzano-Weierstrass theorem there exists a convergent sequence $(x_n)_n$, where $x_n \in \{x_i\}_{i\in I}$ for each $n \geqslant 1$. Define $x_0 := \text{limit}_{n\to\infty} x_n$. Since f is continuous, $(x_n)_n \to x_0$ implies $(f(x_n))_n \to f(x_0)$. But since $f(x_n) = 0$ for each n, it must be the case that $f(x_0) = 0$... $\square$

This is where I'm starting to have trouble. What would the negation of the definition of an isolated zero be? If c is not isolated, does that mean for every open set containing c, there is another zero contained in the open set?

I'm also having trouble using the negation of this definition in my proof. If $(x_n)_n \to x_0$, then *for every* neighborhood of x_0, call it V, there is some $N \in \mathbf{N}$ such that $x_N \in V$. Hence $x_N, x_0 \in V$. But V isn't necessarily open!