Abstract

We will survey many results related to sequences and series of functions. The culmination of this document will be proving the Weierstrass M-test and Cauchy-Hadamard theorem.

Sequences of Functions

Definition 1. Let Ω be a set, (X, d) a metric space, and $(f_n)_n$ a sequence of functions in X^Ω.

(1) $(f_n)_n$ converges *pointwise* to $f \in X^\Omega$ if:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N_{x,\epsilon} \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies d(f_n(x), f(x)) < \epsilon).$$

(2) $(f_n)_n$ converges *uniformly* to $f \in X^\Omega$ if:

$$\begin{aligned}(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) &: (\forall n \in \mathbf{N})(\forall x \in \Omega)\big(n \geqslant N \implies d(f_n(x), f(x)) < \epsilon\big) \\ &\equiv (\forall n \in \mathbf{N})\big(n \geqslant N \implies \sup_{x\in\Omega} d(f_n(x), f(x)) < \epsilon\big).\end{aligned}$$

Example 1. Let $(f_n)_n$ be a sequence in $\mathbf{R}^{[0,1]}$ defined by $f_n(x) = x^n$ for all $n \in \mathbf{N}$. If $x \in [0, 1)$, then $(f_n(x))_n \to 0$. If $x = 1$, then $(f_n(x))_n \to 1$. Thus $(f_n)_n \to \mathbf{1}_1$ pointwise.

Example 2. Let $(f_n)_n$ be a sequence in $\mathbf{R}^\mathbf{R}$ defined by $f_n(x) = \frac{nx}{1+n^2x^2}$. If $x = 0$, then $(f_n(x))_n \to 0$. If $x \neq 0$, observe that:

$$\begin{aligned}|f_n(x)| &= \left|\frac{nx}{1+n^2x^2}\right| \\ &= \frac{n|x|}{1+n^2x^2} \\ &\leqslant \frac{|x|}{nx^2} \\ &= \frac{1}{n|x|}.\end{aligned}$$

Since x is fixed, $(f_n(x))_n \to 0$. Thus $(f_n)_n \to 0_{\mathbf{R}^\mathbf{R}}$ pointwise.

Example 3. Let $(h_n)_n$ be a sequence in $\mathbf{R}^{[0,\infty)}$ defined by $h_n(x) = x^{\frac{1}{n}}$. If $x > 0$, then $(h_n(x))_n \to 1$. If $x = 0$, then $(h_n(x))_n \to 0$. Thus $(h_n)_n \to \mathbf{1}_{(0,\infty)}$ pointwise.

Definition 2. Let Ω be a set and (X, d) a metric space.

(1) The set of all bounded functions from Ω to X is denoted $\mathrm{Bd}(\Omega, X)$.

(2) The *uniform metric* between two bounded functions $f, g \in \mathrm{Bd}(\Omega, X)$ is defined by $D_u(f, g) = \sup_{x\in\Omega} d(f(x), g(x))$.

Proposition 1. *Let Ω be a set, (X, d) a metric space, and $(f_n)_n$ a sequence in X^Ω. The following are equivalent:*

(1) The sequence $(f_n)_n$ converges uniformly to f in X^Ω;

(2) The sequence $(D_u(f_n, f))_n$ converges to 0 in $\mathbf{R}$.

Proof. ($\Rightarrow$) Let $\epsilon > 0$. By assumption, find $N \in \mathbf{N}$ sufficiently large so that, for all $x \in X$, $n \geqslant N$ implies $d(f_n(x), f(x)) < \frac{\epsilon}{2}$. It follows then that:

$$\begin{aligned} |D_u(f_n, f)| &= \left|\sup_{x\in X} d(f_n(x), f(x))\right| \\ &\leqslant \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

Thus $(D_u(f_n, f))_n \to 0$.

($\Leftarrow$) Let $\epsilon > 0$. Find $N \in \mathbf{N}$ large so that $n \geqslant N$ implies $D_u(f_n, f) < \epsilon$. If $n \geqslant N$ and $z \in X$, then:

$$\begin{aligned} d(f_n(z), f(z)) &\leqslant \sup_{x\in X} d(f_n(x), f(x)) \\ &< \epsilon. \end{aligned}$$

Since z was arbitrary, $(f_n)_n \to f$ uniformly. □

Example 4. Let $(f_n)_n$ be a sequence in $\mathbf{R}^{\mathbf{R}}$ defined by $f_n = \mathbf{1}_{[n,n+1]}$. Claim: $(f_n)_n \to 0_{\mathbf{R}^{\mathbf{R}}}$. Let $x \in \mathbf{R}$ and $\epsilon > 0$. Find N large so $N > x$. If $n \geqslant N$, then $|f_n(x) - 0_{\mathbf{R}^{\mathbf{R}}}(x)| = |f_n(x)| = |\mathbf{1}_{[n,n+1]}(x)| = 0$. Thus $(f_n)_n \to 0_{\mathbf{R}^{\mathbf{R}}}$ pointwise.

Note that:

$$\begin{aligned} D_u(f_n, f) &= \sup_{x\in\mathbf{R}} |f_n(x) - 0_{\mathbf{R}^{\mathbf{R}}}(x)| \\ &= \sup_{x\in\mathbf{R}} |f_n(x)| \\ &= 1. \end{aligned}$$

Thus $(f_n)_n$ does *not* converge uniformly to $0_{\mathbf{R}^{\mathbf{R}}}$.

Proposition 2. *Let (X, d) and (Y, ρ) be metric spaces. Suppose $(f_n)_n$ is a sequence in $C(X, Y)$ which converges uniformly to $f : X \to Y$. Then $f \in C(X, Y)$.*

Proof. Let $\epsilon > 0$. Since $(f_n)_n$ converges uniformly to f, choose $N \in \mathbf{N}$ large so that $D_u(f_N, f) < \frac{\epsilon}{3}$. Let $c \in X$. Since f_N is continuous, there exists $\delta > 0$ such that, for all $x \in X$, $d(x,c) < \delta$ implies $\rho(f_N(x), f_N(c)) < \frac{\epsilon}{3}$. For $c \in X$ and $d(x,c) < \delta$:

$$\begin{aligned}\rho(f(x), f(c)) &\leqslant \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(c)) + \rho(f_N(c), f(c)) \\ &\leqslant 2D_u(f_N, f) + \rho(f_N(x), f_N(c)) \\ &< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Thus f is continuous at c. Since c was arbitrary, $f \in C(X,Y)$. □

Theorem 3. *Let $J \subseteq \mathbf{R}$ be a bounded interval and let $(f_n)_n$ be a sequence of functions in $\mathbf{R}^J$. Suppose there exists $x_0 \in J$ such that $(f_n(x_0))_n$ converges, and that the sequence $(f'_n)_n$ of derivatives exists on J and converges uniformly on J to a function g.*

Then the sequence $(f_n)_n$ converges uniformly on J to a function f that has a derivative at every point of J and $f' = g$.

Lemma 1. *Let (X, d) be a compact metric space. Suppose $(f_n)_n$ is a monotonically decreasing sequence in $C(X, \mathbf{R})$ which converges pointwise to 0. Then $(f_n)_n$ converges uniformly to 0.*

Proof. Let $\epsilon > 0$. Since $(f_n)_n$ converges pointwise to 0, for each $x \in X$ there exists $N_x \in \mathbf{N}$ such that $n \geqslant N_x$ implies $f_n(x) < \frac{\epsilon}{2}$. Because f_{N_x} is continuous at x, there exists $\delta_x > 0$ such that, for every $z \in X$, $d(x,z) < \delta_x$ implies $|f_{N_x}(x) - f_{N_x}(z)| < \frac{\epsilon}{2}$. The collection $\{U(x, \delta_x)\}_{x \in X}$ covers X, so by compactness there is a finite set $F \subseteq X$ with $X = \bigcup_{x \in F} U(x, \delta_x)$. Set $N = \max_{x \in F} N_x$. Let $z \in X$ be arbitrary and locate $x \in F$ such that $z \in U(x, \delta_x)$. Notice that our choice of N does not depend on z. For $n \geqslant N$:

$$\begin{aligned}f_n(z) &\leqslant f_{N_x}(z) \\ &= f_{N_x}(z) - f_{N_x}(x) + f_{N_x}(x) \\ &\leqslant |f_{N_x}(z) - f_{N_x}(x)| + f_{N_x}(x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Thus for $n \geqslant N$, we have $\|f_n\|_u \leqslant \epsilon$. □

Theorem 4 (Dini's Theorem)**.** *Let (X, d) be compact. Suppose $(f_n)_n$ is a monotone sequence in $C(X, \mathbf{R})$ which converges pointwise to f. Then $(f_n)_n$ converges uniformly to f.*

Proof. If $(f_n)_n$ is decreasing, apply Lemma 1 to $f_n - f$. If $(f_n)_n$ is increasing, apply Lemma 1 to $f - f_n$. □

Series of Functions

Definition 3. Let Ω be any set and $(V, \|\cdot\|)$ a normed vector space.

(1) If $(f_n)_n$ is a sequence of functions in V^Ω, the sequence of *partial sums* $(s_n)_n$ is defined by:

$$\begin{aligned} s_1(x) &:= f_1(x), \\ s_2(x) &:= s_1(x) + f_2(x), \\ &\vdots \\ s_{n+1}(x) &:= s_n(x) + f_{n+1}(x) \\ &= \sum_{k=1}^{n+1} f_k(x). \end{aligned}$$

(2) If the sequence of functions $(s_n)_n$ converges to $f \in V^\Omega$, we say that the infinite series of functions $\sum f_n$ *converges* to f and write $\sum f_n = f$.

(3) If the series $\sum \|f_n\|$ converges, we say that $\sum f_n$ is *absolutely convergent*.

(4) If the sequence $(s_n)_n$ of partial sums converges uniformly to $f \in X^\Omega$, we say that $\sum f_n$ is *uniformly convergent*.

Proposition 5. *Let $(V, \|\cdot\|)$ be a normed space. The following are equivalent:*

(1) V is a Banach space;

(2) If $(v_k)_k$ is a sequence in V with $\sum_{k=1}^\infty \|v_k\|$ convergent, then $\sum_{k=1}^\infty v_k$ converges.

Proof. ($\Rightarrow$) Let $\epsilon > 0$. Let $s_n = \sum_{k=1}^n v_k$ and $t_n = \sum_{k=1}^n \|v_k\|$. Since $(t_n)_n$ converges, it is Cauchy. Find N sufficiently large so that $p, q > N$ implies $|t_p - t_q| < \epsilon$. For $p > q > N$:

$$\begin{aligned} \left\|s_p - s_q\right\| &= \left\| \sum_{k=q+1}^{p} v_k \right\| \\ &\leqslant \sum_{k=q+1}^{p} \|v_k\| \\ &= |t_p - t_k| \\ &< \epsilon. \end{aligned}$$

Thus $(s_n)_n$ is $\|\cdot\|$-Cauchy. Since V is complete, $\sum_{k=1}^\infty v_k$ converges.

($\Leftarrow$) Since $\sum_{k=1}^\infty v_k$ converges, the sequence $(v_n)_n$ also converges. In particular, the sequence $(v_n)_n$ is Cauchy, so find $N_1 \in \mathbf{N}$ sufficiently large such that

$p, q \geqslant N_1$ implies $\|v_p - v_q\| < 2^{-1}$. Again, find $N_2 > N_1$ large so that $p, q \geqslant N_2$ implies $\|v_p - v_q\| < 2^{-2}$. Inductively, find $N_k > N_{k-1}$ such that $p, q \geqslant N_k$ implies $\|v_p - v_q\| < 2^{-k}$. Now consider the sequence $(v_{n_{k+1}} - v_{n_k})_k$. Note that:

$$\begin{aligned}\sum_{k=1}^{\infty} \|v_{n_{k+1}} - v_{n_k}\| &\leqslant \sum_{k=1}^{\infty} 2^{-k} \\ &= 1.\end{aligned}$$

By our hypothesis $\sum_{k=1}^{\infty}(v_{n_{k+1}} - v_{n_k})$ converges, so the sequence of partial sums:

$$\begin{aligned}w_m &= \sum_{k=1}^{m} v_{n_{k+1}} - v_{n_k} \\ &= v_{n_m} - v_{n_1}\end{aligned}$$

also converges. Let $\operatorname{limit}_{m\to\infty} w_m := w$ and observe that:

$$\begin{aligned}\operatorname*{limit}_{m\to\infty} v_{n_m} &= \operatorname*{limit}_{m\to\infty}(w_m + v_{n_1}) \\ &= w + v_{n_1}.\end{aligned}$$

Since $(v_n)_n$ is a $\|\cdot\|$-Cauchy sequence which admits a convergent subsequence, $(v_n)_n$ converges. Thus V is a Banach space. $\square$

Example 5. Note that Proposition 5 does not only apply to series in $\mathbf{R}$ or $\mathbf{C}$. Given a metric space (X, d), it can be proven that the set of all continuous and bounded functions:

$$C_b(X) := C(X) \cap \mathrm{Bd}(X)$$

is a $\|\cdot\|_u$-Banach space, where $\|\cdot\|_u$ is the *uniform norm* defined as:

$$\|f\|_u := \sup_{x\in X} |f(x)|.$$

Thus, given a sequence of functions $(f_n)_n$ in $C_b(X)$, if $\sum \|f_n\|_u$ converges, then $\sum f_n$ converges.

Proposition 6. *Let $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ be normed vector spaces. Suppose $(f_n)_n$ is a sequence in $C(V, W)$ and $\sum f_n$ converges uniformly to $f : V \to W$. Then $f \in C(V, W)$.*

Proof. The proof follows similarly to Proposition 2. As an exercise, one could verify that if $f_1, ..., f_N$ are continuous, then $\sum_{k=1}^{N} f_k$ is also continuous. $\square$

Theorem 7. *Something about interchange of series and derivative*

Lemma 2 (Cauchy's Criterion)**.** *Let Ω be a set and $(V, \|\cdot\|)$ a Banach space. Let $(f_n)_n$ be a sequence of functions in V^Ω. The infinite series $\sum f_n$ is uniformly convergent if and only if:*

$$(\forall \epsilon > 0)(\exists M > 0) : (\forall x \in \Omega)(\forall m, n \in \mathbf{N}) \left(m > n \geqslant M \implies \left\| \sum_{k=1}^{m} f_k(x) - \sum_{k=1}^{n} f_k(x) \right\| < \epsilon \right).$$

Proof. ($\Rightarrow$) Let $\epsilon > 0$. Since $\left(\sum_{k=1}^{n} f_k\right)_n \to f$ uniformly, there exists $N \in \mathbf{N}$ large so that $n \geqslant N$ implies $\left\|\sum_{k=1}^{n} f_k - f\right\|_u < \frac{\epsilon}{2}$. For $m > n \geqslant N$, observe that:

$$\begin{aligned} \left\| \sum_{k=1}^{m} f_k - \sum_{k=1}^{n} f_k \right\|_u &\leqslant \left\| \sum_{k=1}^{m} f_k - f \right\|_u + \left\| f - \sum_{k=1}^{n} f_k \right\|_u \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

($\Leftarrow$) Let $\epsilon > 0$. Let $\sum f_n$ be a series which satisfies the above hypothesis. Then there exists $M > 0$ such that... finish tomorrow I think I figured it out

Let $\sum f_n$ be a series which satisfies the above hypothesis. Observe that:

$$\begin{aligned} \left\| \sum_{k=1}^{m} f_k(x) - \sum_{k=1}^{n} f_k(x) \right\| &\leqslant \left\| \sum_{k=1}^{m} f_k - \sum_{k=1}^{n} f_k \right\|_u \\ &< \epsilon. \end{aligned}$$

Thus $\left(\sum_{k=1}^{n} f_k(x)\right)_n$ is $\|\cdot\|$-Cauchy. $\square$

Theorem 8 (Weierstrass M-test)**.** *Let Ω be a set and $(V, \|\cdot\|)$ a Banach space. Let $(f_n)_n$ be a sequence of functions on V^Ω and let $(M_n)_n$ be a sequence of positive real numbers such that $\sup_{x\in\Omega} \|f_n(x)\| \leqslant M_n$ for all $n \in \mathbf{N}$. If $\sum_{n=1}^{\infty} M_n$ is convergent, then $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.*

Proof. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ is convergent, it is Cauchy. Find $N \in \mathbf{N}$ sufficiently large so $m > n \geqslant N$ implies $\left|\sum_{k=1}^{m} M_k - \sum_{k=1}^{n} M_k\right| < \epsilon$. It immediately follows that:

$$\begin{aligned} \sup_{x\in\Omega} \left\| \sum_{k=1}^{m} f_k - \sum_{k=1}^{n} f_k \right\| &\leqslant \left| \sum_{k=1}^{m} M_k - \sum_{k=1}^{n} M_k \right| \\ &< \epsilon \end{aligned}$$

By Lemma 2, $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. $\square$