

Contents

Contents	i
1 Introduction	1
1.1 Basic Properties of Vector Spaces	1
2 Bases and Dimension	6
2.1 Basic Definitions	6
2.2 Every Vector Space Admits a Basis	6
2.3 Cardinality and Dimension	7
2.4 Direct Sums and Quotient Spaces	11
2.5 Dual Spaces	14
3 Linear Transformations and Matrices	17
3.1 Choosing Coordinates	17
3.2 Row Operations	23
3.3 Column-space and Null-space	25
3.4 The Transpose of a Matrix	27
4 Generalized Eigenvectors and Jordan Canonical Form	29
4.1 Diagonalization	29
4.2 Eigenvalues and Eigenvectors	31
4.3 Characteristic Polynomials	39
4.4 Jordan Canonical Form	46
4.5 Diagonalization, II	49
5 Tensor Products and the Trace	51
5.1 Complexification	51
5.2 Free Vector Spaces	54
5.3 Extension of Scalars	55
5.4 Tensor Products of Vector Spaces	60
5.5 The Trace	62
6 Tensor Algebras, Exterior Algebras, and the Determinant	66
6.1 Tensor Algebras	66
6.2 Exterior Product	66

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1

Introduction

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out"

-Emil Artin

1.1 Basic Properties of Vector Spaces

Definition 1.1.1. Let F be any field. Let V be a nonempty set with binary operations:

$$\begin{aligned} V \times V &\rightarrow V \\ (v, w) &\mapsto v + w \end{aligned}$$

called vector addition and

$$\begin{aligned} F \times V &\rightarrow V \\ (c, v) &\mapsto cv \end{aligned}$$

called scalar multiplication. Then V is an F -vector space if the following properties are satisfied:

- (1) V is an abelian group, that is:
 - (i) there exists a $0_v \in V$ such that $0_v + v = v = v + 0_v$,
 - (ii) for every $v \in V$ there exists a $-v \in V$ such that $v + (-v) = 0_v = (-v) + v$,
 - (iii) for every $u, v, w \in V$, $(u + v) + w = u + (v + w)$, and
 - (iv) $v + w = w + v$ for all $v, w \in V$.
- (2) $c(v + w) = cv + cw$ for all $c \in F, v, w \in V$,
- (3) $(c + d)v = cv + dv$ for all $c, d \in F, v \in V$,
- (4) $(cd)v = c(dv)$ for all $c, d \in F, v \in V$,
- (5) there exists a $1_F \in F$ such that $1_F v = v$.

Example 1.1.1.

- (1) Let F be any field. Define $F^n = \{(a_1, \dots, a_n) \mid a_i \in F\}$ as affine n -space. Then F^n is an F -vector space.
- (2) Let $n \in \mathbf{Z}_{\geq 0}$. Define $P_n(F) = \{a_0 + a_1x + \dots + a_nx^n \mid a_i \in F\}$. This is an F -vector space with polynomial addition and scalar multiplication. Define $F[x] = \bigcup_{n \geq 0} P_n(F)$. This is also an F -vector space, but either via polynomial addition or polynomial multiplication.

- (3) Let $m, n \in \mathbf{Z}_{\geq 0}$. Set $V = \text{Mat}_{n,m}(F) = \{\text{all } m \times n \text{ matrices with entries in } F\}$. This is an F -vector space with matrix addition and scalar multiplication. If $m = n$ then write $\text{Mat}_n(F)$ for $\text{Mat}_{n,n}(F)$.

Lemma 1.1.1. *Let V be an F -vector space.*

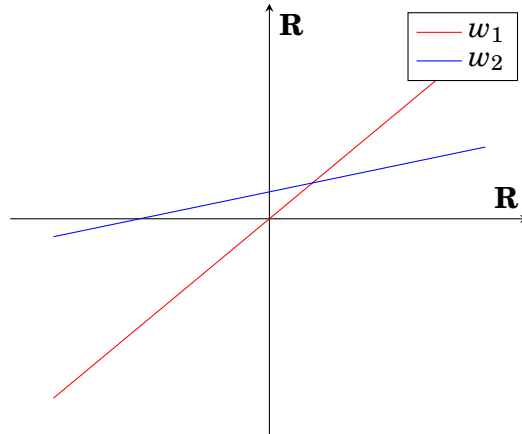
1. *The element $0_v \in V$ is unique,*
2. *$0v = 0_v$ for all $v \in V$,*
3. *$(-1_F)v = -v$ for all $v \in V$.*

Proof. (1) Let $0, 0'$ satisfy the following properties: $0 + v = v$ and $0' + v = v$ for all $v \in V$. Observe that $0 = 0' + 0 = 0 + 0' = 0'$. (2) Note that $0_F v = (0_F + 0_F)v = 0_F v + 0_F v$. Subtracting both sides by $0_F v$ yields $0 = 0_F v$. (3) Observe that $(-1_F)v + v = (-1_F)v + 1_F v = (-1_F + 1_F)v = 0_F v = 0$. Hence $(-1_F)v = -v$. \square

Definition 1.1.2. Let V be an F -vector space. We say $W \subseteq V$ is an F -subspace (or just subspace if F is obvious by context) if W is an F -vector space under the same addition and scalar multiplication.

Example 1.1.2.

- (1) Consider the plane $V = \mathbf{R}^2$. Let w_1, w_2 be subsets of \mathbf{R}^2 as follows:



Note that w_2 is not a subspace, as it does not contain $0_{\mathbf{R}^2}$. On the other hand w_1 is a subspace; note that every element of w_1 is of the form (x, ax) , hence $(x_1, ax_1) + (x_2, ax_2) = (x_1 + x_2, a(x_1 + x_2))$. The other axioms follow similarly.

- (2) Let $V = \mathbf{C}$ and $W = \{a + 0i \mid a \in \mathbf{R}\}$. If $F = \mathbf{R}$, then clearly W is an \mathbf{R} -subspace. If $F = \mathbf{C}$, then W is not a \mathbf{C} -subspace; given $2 \in W$ and $i \in \mathbf{C}$, $2i \notin W$.
- (3) $\text{Mat}_2(\mathbf{R})$ is not a subspace of $\text{Mat}_4(\mathbf{R})$, as $\text{Mat}_2(\mathbf{R}) \not\subseteq \text{Mat}_4(\mathbf{R})$.
- (4) Let $m, n \in \mathbf{Z}_{\geq 0}$. If $m \leq n$, then $P_m(F)$ is a subspace of $P_n(F)$.

Lemma 1.1.2. *Let V be an F -vector space and $W \subseteq V$. Then W is an F -subspace of V if:*

- (1) W is nonempty,
- (2) W is closed under addition, and
- (3) W is closed under scalar multiplication.

Proof. Let $x, y \in W$ and $\alpha \in F$, then by assumption $x + \alpha y \in W$. Take $\alpha = -1$, then $x - y \in W$ which implies W is an abelian subgroup of V . Then by (3) it must be the case that W is an F -subspace of V . \square

Definition 1.1.3. Let V, W be F -vector spaces. Let $T : V \rightarrow W$. We say T is a linear transformation (or linear map) if for every $v_1, v_2 \in V$ and $c \in F$ we have

$$T(v_1 + cv_2) = T(v_1) + cT(v_2).$$

The collection of all linear maps from V to W is denoted $\text{Hom}_F(V, W)$ (some textbooks write this as $\mathcal{L}(V, W)$).

Example 1.1.3.

- (1) Let V be an F -vector space. Define $\text{id}_V : V \rightarrow V$ by $\text{id}_V(v) = v$. This is a linear map; i.e., $\text{id}_V \in \text{Hom}_F(V, V)$ because $\text{id}_V(v_1 + cv_2) = v_1 + cv_2 = \text{id}_V(v_1) + c \text{id}_V(v_2)$.
- (2) Let $V = \mathbf{C}$. Define $T : V \rightarrow V$ by $z \mapsto \bar{z}$. Observe that:

$$\begin{aligned} T(z_1 + cz_2) &= \overline{z_1 + cz_2} = \bar{z}_1 + \bar{c} \bar{z}_2 \\ T(z_1) + cT(z_2) &= \bar{z}_1 + c \bar{z}_2. \end{aligned}$$

Note that these two are only equal if $c = \bar{c}$. Hence $T \in \text{Hom}_F(\mathbf{C}, \mathbf{C})$ if $F = \mathbf{R}$ but not if $F = \mathbf{C}$.

- (3) Let $A \in \text{Mat}_{m,n}(F)$. Define $T_A : F^n \rightarrow F^m$ by $x \mapsto Ax$. Then $T_A \in \text{Hom}_F(F^n, F^m)$.
- (4) Recall that $C^\infty(\mathbf{R})$ is the set of all smooth functions $f : \mathbf{R} \rightarrow \mathbf{R}$ (another way of saying "smooth" is "infinitely differentiable"). Let $V = C^\infty(\mathbf{R})$. This is an \mathbf{R} -vector space under pointwise addition and scalar multiplication. If $a \in \mathbf{R}$ then:

- $E_a : V \rightarrow \mathbf{R}$ defined by $f \mapsto f(a)$ is an element of $\text{Hom}_{\mathbf{R}}(V, \mathbf{R})$,
- $D : V \rightarrow V$ defined by $f \mapsto f'$ is an element of $\text{Hom}_{\mathbf{R}}(V, V)$,
- $I_a : V \rightarrow V$ defined by $f \mapsto \int_a^x f(t)dt$ is an element of $\text{Hom}_{\mathbf{R}}(V, V)$, and
- $\tilde{E}_a : V \rightarrow V$ defined by $f \mapsto f(a)$ (where $f(a)$ is the constant function) is an element of $\text{Hom}_{\mathbf{R}}(V, V)$.

From this, we can express the fundamental theorem of calculus as follows:

$$\begin{aligned} D \circ I_a &= \text{id}_V \\ I_a \circ D &= \text{id}_V - \tilde{E}_a. \end{aligned}$$

Proposition 1.1.3. $\text{Hom}_F(V, W)$ is an F -vector space.

Proof. **do this** □

Lemma 1.1.4. Let $T \in \text{Hom}_F(V, W)$. Then $T(0_v) = 0_w$.

Proof. **do this** □

Definition 1.1.4. Let $T \in \text{Hom}_F(V, W)$ be invertible; i.e., there exists a linear transformation $T^{-1} : W \rightarrow V$ such that $T \circ T^{-1} = \text{id}_w$ and $T^{-1} \circ T = \text{id}_v$. If this is the case we say T is an isomorphism and say V and W are isomorphic, written as $V \cong W$.

Proposition 1.1.5. Let $T \in \text{Hom}_F(V, W)$ be an isomorphism. Then $T^{-1} \in \text{Hom}_F(W, V)$.

Proof. **do this** □

Example 1.1.4.

- (1) Let $V = \mathbf{R}^2$ and $W = \mathbf{C}$. Define $T : \mathbf{R}^2 \rightarrow \mathbf{C}$ by $(x, y) \mapsto x + iy$. This is an isomorphism: note that $T \in \text{Hom}_{\mathbf{R}}(\mathbf{R}^2, \mathbf{C})$ because

$$\begin{aligned} T((x_1, y_1) + r(x_2, y_2)) &= \dots \text{fill this out} \\ &= T((x_1, y_1)) + rT((x_2, y_2)). \end{aligned}$$

Defining $T^{-1} : \mathbf{C} \rightarrow \mathbf{R}^2$ by $x + iy \mapsto (x, y)$ (and showing it's linear) clearly satisfies $(T \circ T^{-1})(x + iy) = x + iy$ and $(T^{-1} \circ T)((x, y)) = (x, y)$, hence $\mathbf{R}^2 \cong \mathbf{C}$ as \mathbf{R} -vector spaces.

- (2) Set $V = P_n(F)$ and $W = F^{n+1}$. Define $T : P_n(F) \rightarrow F^{n+1}$ by

$$a_0 + a_1x + \dots + a_nx^n \mapsto (a_0, a_1, \dots, a_n).$$

This is an isomorphism; $P_n(F) \cong F^{n+1}$.

Definition 1.1.5. Let $T \in \text{Hom}_F(V, W)$. Define the kernel of T as:

- (1) The kernel of T is defined as $\ker(T) = \{v \in V \mid T(v) = 0_w\}$.
 (2) The image of T is defined as $\text{im}(T) = \{w \in W \mid T(v) = w \text{ for some } v \in V\}$.

Lemma 1.1.6. Let $T \in \text{Hom}_F(V, W)$. Then:

- (1) $\ker(T)$ is a subspace of V ,
 (2) $\text{im}(T)$ is a subspace of W .

Proof. Let $v_1, v_2 \in \ker(T)$ and $\alpha \in F$. Observe that $T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2) = 0_w + \alpha 0_w = 0_w$, hence $v_1 + \alpha v_2 \in \ker(T)$ establishing $\ker(T)$ as a subspace of V .

Let $w_1, w_2 \in \text{im}(T)$ and $\alpha \in F$. Then there exists $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Observe that $w_1 + \alpha w_2 = T(v_1) + \alpha T(v_2) = T(v_1 + \alpha v_2)$, hence $w_1 + \alpha w_2 \in \text{im}(T)$ establishing $\text{im}(T)$ as a subspace of W . □

Lemma 1.1.7. *Let $T \in \text{Hom}_F(V, W)$. T is injective if and only if $\ker(T) = \{0_v\}$*

Proof. Let T be injective. Let $v \in \ker(T)$. Then $T(v) = 0_w = T(0_v)$, and since T is injective $v = 0_v$.

Conversely, assume $\ker(T) = \{0_v\}$. Let $v_1, v_2 \in V$ with $T(v_1) = T(v_2)$. Subtracting both sides by $T(v_2)$ gives $T(v_1) - T(v_2) = 0_w$, and since T is a linear transformation yields $T(v_1 - v_2) = 0_w$. Since the kernel is trivial, it must be the case that $v_1 = v_2$. \square

Example 1.1.5. Let $m > n$. Define $T : F^m \rightarrow F^n$ by

$$(a_0, a_1, \dots, a_{n-1}, a_n, a_{n+1}, \dots, a_m) \mapsto (a_0, a_1, \dots, a_n)$$

Then $\text{im}(T) = F^n$ and $\ker(T) = \{(0, \dots, 0, a_{n+1}, a_{n+2}, \dots, a_m) \in F^m\} \cong F^{m-n}$.

2

Bases and Dimension

2.1 Basic Definitions

Unless otherwise stated assume V to be an F -vector space.

Definition 2.1.1. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a subset of V where I is an indexing set (possibly infinite). We say $v \in V$ is an F -linear combination of \mathcal{B} (or just *linear combination*) if there is a set $\{a_i\}_{i \in I}$ with $a_i = 0$ for all but finitely many i such that $v = \sum_{i \in I} a_i v_i$. The collection of F -linear combinations is denoted $\text{span}_F(\mathcal{B})$.

Example 2.1.1. Let $V = P_2(F)$.

- (1) Set $\mathcal{B} = \{1, x, x^2\}$. We have $\text{span}_F(\mathcal{B}) = P_2(F)$.
- (2) Set $\mathcal{C} = \{1, (x-1), (x-1)^2\}$. We have $\text{span}_F(\mathcal{C}) = P_2(F)$.

Definition 2.1.2. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a subset of V . We say \mathcal{B} is *F -linearly independent* (or just *linearly independent*) if whenever $\sum_{i \in I} a_i v_i = 0$ then $a_i = 0$ for all $i \in I$.

Definition 2.1.3. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a subset of V . We say \mathcal{B} is an *F -basis* (or just *basis*) of V if:

- $\text{span}_F(\mathcal{B}) = V$, and
- \mathcal{B} is linearly independent.

Example 2.1.2. Let $V = F^n$. Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ with

$$\begin{aligned} e_1 &= (1, 0, 0, \dots, 0) \\ e_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \dots, 1). \end{aligned}$$

We have that \mathcal{E}_n is a basis of F^n and is referred to as the *standard basis*.

2.2 Every Vector Space Admits a Basis

Definition 2.2.1. A *relation* from A to B is a subset $R \subseteq A \times B$. Typically, when one says "a relation on A " that means a relation from A to A ; i.e., $R \subseteq A \times A$.

Definition 2.2.2. Let A be a set. An *ordering* of A is a relation R on A that is

- (1) reflexive: $(a, a) \in R$ for all $a \in A$,
- (2) transitive: $(a, b), (b, c) \in R$ implies $(a, c) \in R$, and
- (3) antisymmetric: $(a, b), (b, a) \in R$ implies $a = b$.

If this is the case, we write $(a, b) \in R$ as $a \leq_R b$. If A is an ordered set we write it as the ordered pair (A, \leq_R) (or just A if the ordering is obvious by context).

Definition 2.2.3. An ordered set (X, \leq_R) is total if for all $a, b \in X$ we have that $a \leq_R b$ or $b \leq_R a$.

Definition 2.2.4. Let (X, \leq) be an ordered set and $A \subseteq X$ nonempty.

- (1) A is called a chain if (A, \leq) is a total ordering.
- (2) A is called bounded above if there exists an element $u \in X$ with $a \leq u$ for all $a \in A$. Such a u is called an upperbound for A .
- (3) A maximal element of A is an element $m \in A$ such that if $a \geq m$, then $a = m$.

Lemma 2.2.1 (Zorn's Lemma). *Let X be an ordered set with the property that every chain has an upperbound. Then X contains a maximal element.*

Theorem 2.2.2. *Let \mathcal{A} and C be subsets of V with $\mathcal{A} \subseteq C$. Assume \mathcal{A} is linearly independent and $\text{span}_F(C) = V$. Then there exists a basis \mathcal{B} of V with $\mathcal{A} \subseteq \mathcal{B} \subseteq C$ ¹.*

Proof. Let $X = \{\mathcal{B}' \subseteq V \mid \mathcal{A} \subseteq \mathcal{B}' \subseteq C, \mathcal{B}' \text{ is linearly independent}\}$. We have $\mathcal{A} \in X$, so $X \neq \emptyset$. X is ordered with respect to inclusion, and has an upperbound of C . By **Zorn's Lemma** we have a maximal element in X , call it \mathcal{B} .

Claim: $\text{span}_F(\mathcal{B}) = V$. Suppose towards contradiction it's not, then there exists a $v \in C$ with $v \notin \text{span}_F(\mathcal{B})$. But then $\mathcal{B} \cup \{v\}$ is still linearly independent, and $\mathcal{B} \cup \{v\} \subseteq C$. This gives $\mathcal{B} \subseteq \mathcal{B} \cup \{v\}$, which is a contradiction because \mathcal{B} is maximal in X . Thus $\text{span}_F(\mathcal{B}) = V$. \square

2.3 Cardinality and Dimension

Lemma 2.3.1. *A homogenous system of m linear equations in n unknowns with $m < n$ has a nonzero solution.*

Proof. **do this** \square

Corollary 2.3.2. *Let $\mathcal{B} \subseteq V$ with $\text{span}_F(\mathcal{B}) = V$ and $|\mathcal{B}| = m$. Any set with more than m elements cannot be linearly independent.*

¹Given any linearly-independent set \mathcal{A} , we can constructing a basis \mathcal{B} by adding elements. Given any spanning set C , we can construct a basis \mathcal{B} by removing elements.

Proof. Let $C = \{w_1, \dots, w_n\}$ with $n > m$. We will show C cannot be linearly independent. Write $\mathcal{B} = \{v_1, \dots, v_m\}$ with $\text{span}_F(\mathcal{B}) = V$. For each i , write

$$w_i = \sum_{j=1}^m a_{ji} v_j \text{ for some } a_{ji} \in F.$$

Consider the equations

$$\sum_{i=1}^n a_{ji} x_i = 0.$$

By Lemma 2.3.1 there exists nonzero solutions $(x_1, \dots, x_n) = (c_1, \dots, c_n) \neq (0, \dots, 0)$. We have

$$\begin{aligned} 0 &= \sum_{j=1}^m \left(\sum_{i=1}^n a_{ji} c_i \right) v_j \\ &= \sum_{i=1}^n c_i \left(\sum_{j=1}^m a_{ji} v_j \right) \\ &= \sum_{i=1}^n c_i w_i. \end{aligned}$$

Thus $C = \{w_1, \dots, w_n\}$ is not linearly independent. \square

Corollary 2.3.3. *If \mathcal{B} and C are both finite bases of V , then $|\mathcal{B}| = |C|$.*

Proof. Let $|\mathcal{B}| = m$ and $|C| = n$. Because $\text{span}_F(\mathcal{B}) = V$ and C is linearly independent, it must be the case that $n \leq m$. But since $\text{span}_F(C) = V$ and \mathcal{B} is also linearly independent, it must be the case that $m \leq n$. By antisymmetry, $n = m$. \square

Definition 2.3.1. Let \mathcal{B} be a basis of V . The dimension of V , written $\dim_F(V)$, is the cardinality of \mathcal{B} ; i.e., $\dim_F(V) = |\mathcal{B}|$.

Theorem 2.3.4. *Let V be a finite dimensional vector space with $\dim_F(V) = n$. Let $C \subseteq V$ with $|C| = m$.*

- (1) *If $m > n$, then C is not linearly independent.*
- (2) *If $m < n$, then $\text{span}_F(C) \neq V$.*
- (3) *If $m = n$, then the following are equivalent:*
 - C is a basis;
 - C is linearly independent;
 - $\text{span}_F(C) = V$.

Corollary 2.3.5. *Let $W \subseteq V$ be a subspace. We have $\dim_F(W) \leq \dim_F(V)$. If $\dim_F(V) < \infty$, then $V = W$ if and only if $\dim_F(V) = \dim_F(W)$.*

Example 2.3.1. Let $V = \mathbf{C}$.

- (1) If $F = \mathbf{C}$, then $\mathcal{B} = \{1\}$ is a basis and $\dim_{\mathbf{C}}(\mathbf{C}) = 1$.
- (2) If $F = \mathbf{R}$, then $\mathcal{B} = \{1, i\}$ is a basis and $\dim_{\mathbf{R}}(\mathbf{C}) = 2$.
- (3) If $F = \mathbf{Q}$, then $|\mathcal{B}| = \mathfrak{c}$ and $\dim_{\mathbf{Q}}(\mathbf{C}) = \mathfrak{c}$ (the *continuum*).

Example 2.3.2. Let $V = F[x]$ and let $f(x) \in F[x]$. We can use this polynomial to split $F[x]$ into equivalence classes analogous to how one creates the field \mathbf{F}_p . Define $g(x) \sim h(x)$ if $f(x) \mid (g(x) - h(x))$. This is an equivalence relation. We let $[g(x)]$ denote the equivalence class containing $g(x) \in F[x]$. Let $F[x]/(f(x)) = \{[g(x)] \mid g(x) \in F[x]\}$ denote the collection of equivalence classes. Define $[g(x)] + [h(x)] = [g(x) + h(x)]$ and $\alpha[g(x)] = [\alpha g(x)]$, this makes $F[x]/(f(x))$ into a vector space.

Set $n = \deg(f(x))$. Let $\mathcal{B} = \{[1], [x], \dots, [x^{n-1}]\}$. We will show this is a basis for $F[x]/(f(x))$. Suppose there exists $a_0, \dots, a_{n-1} \in F$ with $a_0[1] + a_1[x] + \dots + a_{n-1}[x^{n-1}] = [0]$. So $[a_0 + a_1x + \dots + a_{n-1}x^{n-1}] = [0]$, hence $f(x) \mid (a_0 + a_1x + \dots + a_{n-1}x^{n-1})$. But $\deg(f(x)) = n$, so we must have $a_0 = a_1 = \dots = 0$ (linear independence).

Let $[g(x)] \in F[x]/(f(x))$. The Euclidean algorithm of polynomials gives $g(x) = f(x)q(x) + r(x)$ for some $q(x), r(x) \in F[x]/(f(x))$ with $r(x) = 0$ or $\deg(r(x)) \leq \deg(f(x))$. Observe that $[g(x)] = [f(x)q(x) + r(x)] = [f(x)q(x)] + [r(x)] = [r(x)]$. Since $[r(x)]$ can be written as a linear combination of basis elements from \mathcal{B} , we have $[g(x)] \in \text{span}_F(\mathcal{B})$. Note that any element of $\text{span}_F(\mathcal{B})$ is clearly contained in $F[x]/(f(x))$, establishing $\text{span}_F(\mathcal{B}) = F[x]/(f(x))$.

Lemma 2.3.6. Let V be an F -vector space and $C = \{v_i\}_{i \in I}$ be a subset of V . Then C is a basis if and only if each $v \in V$ can be written uniquely as a linear combination of elements of C .

Proof. Suppose C is a basis. Let $v \in V$ and suppose

$$v = \sum_{i \in I} a_i v_i = \sum_{i \in I} b_i v_i,$$

for some $a_i, b_i \in F$. Observe that:

$$0_v = \sum_{i \in I} (a_i - b_i) v_i.$$

Since C is a basis, it is linearly independent, so $a_i - b_i = 0$ for all i . Thus $a_i = b_i$ for all i establishing that the expansion is unique.

Conversely, suppose every vector $v \in V$ is a unique linear combination of C . Certainly we have $\text{span}_F(C) = V$. Suppose $0_v = \sum_{i \in I} a_i v_i$ for some $a_i \in F$. We also have that $0_v = \sum_{i \in I} 0 \cdot v_i$. Uniqueness gives $a_i = 0$ for all $i \in I$; i.e., C is linearly independent. \square

Proposition 2.3.7. Let V, W be F -vector spaces.

- (1) Let $T \in \text{Hom}_F(V, W)$. We have that T is determined by what it does to a basis (where it maps it).
- (2) Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a basis of V and $C = \{w_i\}_{i \in I}$ be a subset of W . If $|\mathcal{B}| = |C|$, there is a $T \in \text{Hom}_F(V, W)$ such that $T(v_i) = w_i$ for all $i \in I$.

Proof. (1) Let $v \in V$. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a basis of V and write $v = \sum_{i \in I} a_i v_i$. We have $T(v) = T(\sum_{i \in I} a_i v_i) = \sum_{i \in I} a_i T(v_i)$.

(2) Define $T : V \rightarrow W$ by $v \mapsto \sum_{i \in I} a_i w_i$. If $v = \sum_{i \in I} a_i v_i$ this map is linear (show this). \square

Corollary 2.3.8. *Let $T \in \text{Hom}_F(V, W)$ with $\mathcal{B} = \{v_i\}_{i \in I}$ a basis of V and $C = \{w_i = T(v_i)\}_{i \in I}$ a subset of W . We have C is a basis of W if and only if T is an isomorphism.*

Proof. Suppose C is a basis of W . Using the result from Proposition 2.3.7, define $S \in \text{Hom}_F(W, V)$ with $S(w_i) = v_i$. Check $T \circ S = \text{id}_W$ and $S \circ T = \text{id}_V$. Thus T is an isomorphism.

Conversely, let T be an isomorphism. Let $w \in W$. As T is surjective, there exists a $v \in V$ such that $T(v) = w$. Using \mathcal{B} as a basis of V , write $v = \sum_{i \in I} a_i v_i$. So observe that:

$$w = T(v) = T\left(\sum_{i \in I} a_i v_i\right) = \sum_{i \in I} a_i T(v_i) \in \text{span}_F(C),$$

hence $W = \text{span}_F(C)$ (note the other direction is trivial—you never need to show that). Now suppose there exists a collection of elements $a_i \in F$ with $\sum_{i \in I} a_i T(v_i) = 0_W$. Since T is linear, this is equivalent to $T(\sum_{i \in I} a_i v_i) = 0_W$, and since T is injective it must be the case that $\sum_{i \in I} a_i v_i = 0_V$. Since \mathcal{B} is a basis we get $a_i = 0$ for all $i \in I$, establishing that C is linearly independent. \square

Theorem 2.3.9 (Rank-Nullity Theorem). *Let V be an F -vector space with $\dim_F(V) < \infty$. Then:*

$$\dim_F(V) = \dim_F(\ker(T)) + \dim_F(\text{im}(T)).$$

Proof. Let $\dim_F(\ker(T)) = k$ and $\dim_F(V) = n$. Let $\mathcal{A} = \{v_1, \dots, v_k\}$ be a basis of $\ker(T)$. Extend this to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V . We'd like to show that $C = \{T(v_{k+1}), \dots, T(v_n)\}$ is a basis of $\text{im}(T)$.

Let $w \in \text{im}(T)$. So there exists a $v \in V$ with $T(v) = w$. Write $v = \sum_{i=1}^n a_i v_i$. We have:

$$\begin{aligned} w &= T(v) \\ &= T\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i T(v_i) \\ &= \sum_{i=k+1}^n a_i T(v_i) \in \text{span}_F(C). \quad \text{b/c } v_1, \dots, v_k \in \ker(T) \end{aligned}$$

Thus $\text{span}_F(C) = \text{im}(T)$. Now suppose we have $\sum_{i=k+1}^n a_i T(v_i) = 0_W$. Since T is linear we have $T(\sum_{i=k+1}^n a_i v_i) = 0_W$, which gives $\sum_{i=k+1}^n a_i v_i \in \ker(T)$. Thus we can write it in terms of the basis \mathcal{A} of $\ker(T)$: there exists a_1, \dots, a_k such that

$$\sum_{i=k+1}^n a_i v_i = \sum_{i=1}^k a_i v_i,$$

which is equivalent to $\sum_{i=1}^k a_i v_i + \sum_{i=k+1}^n a_i v_i = 0_V$. However, \mathcal{B} is a basis of V so $a_1 = \dots = a_n = 0$. \square

Corollary 2.3.10. *Let V, W be F -vector spaces with $\dim_F(V) = n$. Let $V_1 \subseteq V$ be a subspace with $\dim_F(V_1) = k$ and $W_1 \subseteq W$ a subspace with $\dim_F(W_1) = n - k$. Then there exists a $T \in \text{Hom}_F(V, W)$ such that $\ker(T) = V_1$ and $\text{im}(T) = W_1$.*

Proof. **do it** □

Corollary 2.3.11. *Let $T \in \text{Hom}_F(V, W)$ with $\dim_F(V) = \dim_F(W) < \infty$. The following are equivalent:*

- (1) T is an isomorphism.
- (2) T is injective.
- (3) T is surjective.

Proof. **do it** □

Corollary 2.3.12. *Let $A = \text{Mat}_n(F)$. The following are equivalent:*

- (1) A is invertible.
- (2) There exists an element $B \in \text{Mat}_n(F)$ such that $BA = 1_n$.
- (3) There exists an element $B \in \text{Mat}_n(F)$ such that $AB = 1_n$.

Proof. **do it** □

Corollary 2.3.13. *Let $\dim_F(V) = m$ and $\dim_F(W) = n$.*

- (1) *If $m < n$ and $T \in \text{Hom}_F(V, W)$, then T is not surjective.*
- (2) *If $m > n$ and $T \in \text{Hom}_F(V, W)$, then T is not injective.*
- (3) *If $m = n$ then $V \cong W$.*

Example 2.3.3. **This follows shortly after corollary 2.2.30 (write it down later)**

2.4 Direct Sums and Quotient Spaces

Definition 2.4.1. Let V be an F -vector space and V_1, \dots, V_k be subspaces. The sum of V_1, \dots, V_k is

$$V_1 + \dots + V_k = \{v_1 + \dots + v_k \mid v_i \in V_i\}.$$

Proposition 2.4.1. *Let V be an F -vector space and V_1, \dots, V_k be subspaces. Then $V_1 + \dots + V_k$ is also a subspace of V .*

Proof. **do this** □

Definition 2.4.2. Let V_1, \dots, V_k be subspaces of V . We say V_1, \dots, V_k are independent if whenever $v_1 + \dots + v_k = 0_V$ then $v_i = 0_V$.

Definition 2.4.3. Let V_1, \dots, V_k be subspaces of V . We say V is the direct sum of V_1, \dots, V_k and write $V = V_1 \oplus \dots \oplus V_k$ if:

- (1) $V = V_1 + \dots + V_k$, and
- (2) V_1, \dots, V_k are independent.

Example 2.4.1.

- (1) Let $V = F^2$ with $V_1 = \{(x, 0) \mid x \in F\}$ and $V_2 = \{(0, y) \mid y \in F\}$. Then

$$\begin{aligned} V_1 + V_2 &= \{(x, 0) + (0, y) \mid x, y \in F\} \\ &= \{(x, y) \mid x, y \in F\} \\ &= V \end{aligned}$$

If $(x, 0) + (y, 0) = (0, 0)$, then $x = y = 0$ which means V_1 and V_2 are independent. Hence $F^2 = V_1 \oplus V_2$.

- (2) Let $V = F[x]$ and $V_1 = F$, $V_2 = Fx = \{\alpha x \mid \alpha \in F\}$, and $V_3 = P_1(F)$. Note that $P_1(F) = V_1 \oplus V_2$. But V_1, V_3 are not independent because $1_F \in V_1$ and $-1_F \in V_3$ and $(-1_F) + 1_F = 0$.

- (3) Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V and $\text{span}_F(v_i) = V_i$. Then $V = V_1 \oplus \dots \oplus V_n$.

Lemma 2.4.2. Let V be an F -vector space with V_1, \dots, V_k as subspaces. We have $V = V_1 \oplus \dots \oplus V_k$ if and only if every $v \in V$ can be written uniquely in the form $v = v_1 + \dots + v_k$ for all $v_i \in V_i$.

Proof. Suppose $V = V_1 \oplus \dots \oplus V_k$. Let $v \in V$. Suppose $v = v_1 + \dots + v_k = \tilde{v}_1 + \dots + \tilde{v}_k$ for $v_i, \tilde{v}_i \in V_i$. Then $0_V = (v_1 - \tilde{v}_1) + \dots + (v_k - \tilde{v}_k)$. Since V_1, \dots, V_k are independent and $v_i - \tilde{v}_i \in V$, this gives $v_i - \tilde{v}_i = 0_V$ for all i . Thus the expansion for V is unique.

Conversely, suppose every $v \in V$ can be written uniquely in the form $v = v_1 + \dots + v_k$ with $v_i \in V_i$. Then $V = V_1 + \dots + V_k$ by definition of sums of subspaces. If $0_V = v_1 + \dots + v_k$ for some $v_i \in V_i$, and $0_v = 0_v + \dots + 0_v$, then (by uniqueness) it must be the case that $v_i = 0_V$ for all i . \square

Note 1. It suffices to show that $\dim_F(V) = \dim_F(V_1) + \dots + \dim_F(V_k)$ and $V_1 \cap \dots \cap V_k = \{0_V\}$.

Exercise 2.4.1. Let V_1, \dots, V_k be subspaces of V . For each $1 \leq i \leq k$, let \mathcal{B}_i be a basis of V_i . Let $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}_i$. Show that:

- (1) \mathcal{B} spans V if and only if $V = V_1 + \dots + V_k$.
- (2) \mathcal{B} is linearly independent if and only if V_1, \dots, V_k are independent.
- (3) \mathcal{B} is a basis if and only if $V = V_1 \oplus \dots \oplus V_k$.

Proof. **do this shit**

\square

Lemma 2.4.3. *Let $U \subseteq V$ be a subspace. Then U has a complement.*

Proof. **do this shit** □

Definition 2.4.4. Let $W \subseteq V$ be a subspace. Define $v_1 \sim v_2$ if $v_1 - v_2 \in W$ for some $v_1, v_2 \in V$. This forms an equivalence relation. Denote the equivalence class containing v as $[v]_W = v + W = \{\tilde{v} \in V \mid v - \tilde{v} \in W\} = \{v + w \mid w \in W\}$. The set containing all equivalence classes over W is denoted $V/W = \{v + W \mid v \in V\}$.

Proposition 2.4.4. *Let $v_1 + W, v_2 + W \in V/W$ and $\alpha \in F$. With addition and scalar multiplication defined as follows:*

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ \alpha(v_1 + W) &= \alpha v_1 + W,\end{aligned}$$

its operations are well-defined and V/W forms an F -vector space.

Proof. Let $v_1 + W = \tilde{v}_1 + W$ and $v_2 + W = \tilde{v}_2 + W$. Then $v_1 = \tilde{v}_1 + w_1$ and $v_2 = \tilde{v}_2 + w_2$ for some $w_1, w_2 \in W$. Observe that:

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2 + W) \\ &= (\tilde{v}_1 + w_1 + \tilde{v}_2 + w_2 + W) \\ &= (\tilde{v}_1 + \tilde{v}_2) + W \\ &= (\tilde{v}_1 + W) + (\tilde{v}_2 + W).\end{aligned}$$

$$\begin{aligned}c(v_1 + W) &= cv_1 + W \\ &= c(\tilde{v}_1 + w_1) + W \\ &= c\tilde{v}_1 + W \\ &= c(\tilde{v}_1 + W).\end{aligned}$$

Hence addition and scalar multiplication are well-defined. **show the vector space axioms here.** □

Example 2.4.2. Let $V = \mathbf{R}^2$ and $W = \{(x, 0) \mid x \in \mathbf{R}\}$. Let $(x_0, y_0) \in V$. We have that $(x_0, y_0) \sim (x, y)$ if $(x_0, y_0) - (x, y) = (x_0 - x, y_0 - y) \in W$. So $(x_0, y_0) + W = \{(x, y_0) \mid x \in \mathbf{R}\}$. Then V/W is a vector space only when $y = 0$.

Define $\tau : \mathbf{R} \rightarrow V/W$ by $y \mapsto (0, y) + W$. This is an isomorphism. Let $y_1, y_2, c \in \mathbf{R}$. Observe that:

$$\begin{aligned}\tau(y_1 + cy_2) &= (0, y_1 + cy_2) + W \\ &= ((0, y_1) + (0, cy_2)) + W \\ &= ((0, y_1) + c(0, y_2)) + W \\ &= ((0, y_1) + W) + c((0, y_2) + W) \\ &= \tau(y_1) + c\tau(y_2).\end{aligned}$$

Hence $\tau \in \text{Hom}_F(\mathbf{R}, V/W)$. Let $(x, y) + W \in V/W$. Then $(x, y) + W = (0, y) + W$. So τ is surjective because $\tau(y) = (0, y) + W$. Now let $y \in \ker(\tau)$. Then $\tau(y) = (0, y) + W = (0, 0) + W$. This implies $y = 0$, meaning the kernel is trivial and so τ is injective.

Alternatively, it is routine to show that $\tau^{-1} \in \text{Hom}_F(V/W, \mathbf{R})$ with $\tau^{-1} \circ \tau = \text{id}_{\mathbf{R}}$ and $\tau \circ \tau^{-1} = \text{id}_{V/W}$.

Definition 2.4.5. Let $W \subseteq V$ be a subspace. The canonical projection map $\pi_W : V \rightarrow V/W$ is defined by $v \mapsto v + W$. Note that $\pi_W \in \text{Hom}_F(V, V/W)$.

Note 2. To define a map $T : V/W \rightarrow V'$, you always have to check it is well-defined.

Theorem 2.4.5 (First Isomorphism Theorem). Let $T \in \text{Hom}_F(V, W)$. Define $\bar{T} : V/\ker(T) \rightarrow W$ by $v + \ker(T) \mapsto T(v)$. Then \bar{T} is a linear map. Moreover, $V/\ker(T) \cong \text{im}(T)$.

Proof. finish this □

2.5 Dual Spaces

Note that when one refers to something as “canonical”, it means the object in question does not depend on a basis.

Definition 2.5.1. Let V be an F -vector space. The dual space, denoted V^\vee , is defined to be $V^\vee = \text{Hom}_F(V, F)$.

Theorem 2.5.1. We have V is isomorphic to a subspace of V^\vee . If $\dim_F(V) < \infty$, then $V \cong V^\vee$.

Proof. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be a basis (hence this theorem is not canonical). For each $i \in I$, define:

$$v_i^\vee(v_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

We get $\{v_i^\vee\}_{i \in I}$ are elements of V^\vee . We obtain $T \in \text{Hom}_F(V, V^\vee)$ by $T(v_i) = v_i^\vee$. To show that V is isomorphic to a subspace of V^\vee , it is enough to show T is injective, then by the first isomorphism theorem $V \cong \text{im}(T)$ (a subspace of V^\vee).

Let $v \in \ker(T)$, then $T(v) = 0_{V^\vee}$. Write $v = \sum_{i \in I} a_i v_i$. So:

$$\begin{aligned} 0_{V^\vee} &= T(v) \\ &= T\left(\sum_{i \in I} a_i v_i\right) \\ &= \sum_{i \in I} a_i T(v_i) \\ &= \sum_{i \in I} a_i v_i^\vee. \end{aligned}$$

Towards contradiction, pick some j with $a_j \neq 0$. Note that $0_{V^\vee} = \sum_{i \in I} a_i v_i^\vee(v_j) = a_j$ (every term except for $a_j v_j^\vee(v_j)$ equals 0). This is a contradiction, hence T is injective.

Now assume $\dim_F(V) = n$ and write $\mathcal{B} = \{v_1, \dots, v_n\}$. Let $v^\vee \in V^\vee$. Define $a_i = v^\vee(v_i)$. Set $v = \sum_{i=1}^n a_i v_i$ and define $S : V^\vee \rightarrow V$ by $S(v^\vee) = v = \sum_{i=1}^n v^\vee(v_i) v_i$. We'd like to show that $S \in \text{Hom}_F(V^\vee, V)$ and is the inverse of T . Let $v^\vee, w^\vee \in V^\vee$ and $c \in F$. Set $a_i = v^\vee(v_i)$ and $b_i = w^\vee(v_i)$. Then:

$$\begin{aligned} S(v^\vee + cw^\vee) &= \sum_{i=1}^n [(v^\vee + cw^\vee)(v_i)] v_i \\ &= \sum_{i=1}^n [v^\vee(v_i) + cw^\vee(v_i)] v_i \\ &= \sum_{i=1}^n v^\vee(v_i) v_i + c \sum_{i=1}^n w^\vee(v_i) v_i \\ &= S(v^\vee) + cS(w^\vee). \end{aligned}$$

Hence S is linear. Now observe that:

$$\begin{aligned} (S \circ T)(v_j) &= S(T(v_j)) \\ &= S(v_j^\vee) \\ &= \sum_{i=1}^n v_j^\vee(v_i) v_i \\ &= v_j \end{aligned}$$

Let $v^\vee \in V^\vee$. Note that $(T \circ S)(v^\vee)$ is a function, so it will require an input. Observe that

$$\begin{aligned} (T \circ S)(v^\vee)(v_j) &= T(S(v^\vee))(v_j) \\ &= T\left(\sum_{i=1}^n v^\vee(v_i) v_i\right)(v_j) \\ &= \left[\sum_{i=1}^n v^\vee(v_i) T(v_i)\right](v_j) \\ &= \sum_{i=1}^n v^\vee(v_i) (v_i^\vee(v_j)) \\ &= v^\vee(v_j). \end{aligned}$$

□

Definition 2.5.2. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V . The dual basis for V^\vee is $\mathcal{B}^\vee = \{v_1^\vee, \dots, v_n^\vee\}$.

Proposition 2.5.2. *There is a canonical injective linear map from V to $(V^\vee)^\vee$. If $\dim_F(V) < \infty$, this is an isomorphism.*

Proof. Let $v \in V$. Define $\hat{v} : V^\vee \rightarrow F$ by $\varphi \mapsto \varphi(v)$ ². We can easily verify that \hat{v} is linear. Therefore, we have $\hat{v} \in \text{Hom}_F(V^\vee, F) = (V^\vee)^\vee$. We have a map:

$$\Phi : V \rightarrow (V^\vee)^\vee \text{ defined by } v \mapsto \hat{v}.$$

We want to verify that Φ is an injective linear map. Let $v_1, v_2 \in V$ and $c \in F$. Let $\varphi \in V^\vee$, then:

$$\begin{aligned} \Phi(v_1 + cv_2)(\varphi) &= \widehat{v_1 + cv_2}(\varphi) \\ &= \varphi(v_1 + cv_2) \\ &= \varphi(v_1) + c\varphi(v_2) \\ &= \hat{v}_1(\varphi) + c\hat{v}_2(\varphi) \\ &= \Phi(v_1)(\varphi) + c\Phi(v_2)(\varphi). \end{aligned}$$

We will now show that Φ is injective. Let $v \in V$ and assume $v \neq 0_V$. We will form a basis \mathcal{B} of V that contains v (**why is this still canonical?**). Let $v^\vee \in V^\vee$, then $v^\vee(v) = 1$ and $v^\vee(w) = 0$ for all $w \in \mathcal{B}$, $w \neq v$. Now assume $v \in \ker(\Phi)$. Then $\Phi(v)(\varphi) = \varphi(v) = 0$ for all $\varphi \in V^\vee$. But picking $\varphi = v^\vee$ gives:

$$\begin{aligned} 0 &= \Phi(v)(v^\vee) \\ &= v^\vee(v) \\ &= 1. \end{aligned}$$

This is a contradiction, hence Φ is injective. □

Definition 2.5.3. Let $T \in \text{Hom}_F(V, W)$. We get an induced map $T^\vee : W^\vee \rightarrow V^\vee$ with $T^\vee(\varphi) = \varphi \circ T$. The following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & \searrow T^\vee(\varphi) & \downarrow \varphi \\ & & F. \end{array}$$

²This can be notated as eval_v , but \hat{v} appears more often in literature

3

Linear Transformations and Matrices

3.1 Choosing Coordinates

Example 3.1.1 (Choosing Coordinates). Let V be an F -vector space with $\dim_F(V) < \infty$. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V . This basis fixes an isomorphism $V \cong F^n$. We can see this as follows: let $v \in V$, then $v = \sum_{i=1}^n a_i v_i$. Define $T_{\mathcal{B}} : V \rightarrow F^n$ by:

$$T_{\mathcal{B}}(v) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n.$$

This is an isomorphism. Given $v \in V$, we define $[v]_{\mathcal{B}} := T_{\mathcal{B}}(v)$. We refer to this as *choosing coordinates* on V .

Example 3.1.2.

(1) Let $V = \mathbf{Q}^2$ and $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$. This forms a basis of V . Let $v \in V$ with $v = \begin{pmatrix} a \\ b \end{pmatrix}$. We have:

$$v = \frac{a+b}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a-b}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ hence } [v]_{\mathcal{B}} = \begin{pmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{pmatrix}.$$

Had we considered the standard basis $\mathcal{E}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, then $[v]_{\mathcal{E}_2} = \begin{pmatrix} a \\ b \end{pmatrix}$.

(2) Let $V = P_2(\mathbf{R})$. Let $C = \{1, (x-1), (x-1)^2\}$. This forms a basis of V . Let $f(x) = a + bx + cx^2 \in P_2(\mathbf{R})$. Written in terms of C , we have $f(x) = (a + b + c) + (b + 2c)(x-1) + c(x-1)^2$. Thus:

$$[f(x)]_C = \begin{pmatrix} a + b + c \\ b + 2c \\ c \end{pmatrix}$$

Example 3.1.3 (Linear Transformations as Matrices). Recall that given a matrix $A \in \text{Mat}_{m,n}(F)$, we obtain a linear map $T_A \in \text{Hom}_F(F^n, F^m)$ by $T_A(v) = Av$. We aim to show that this process "works in reverse" —given a linear transformation $T \in \text{Hom}_F(F^n, F^m)$, we want to find a matrix A so that $T = T_A$.

Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ be the standard basis of F^n and $\mathcal{F}_m = \{f_1, \dots, f_m\}$ be the standard basis of F^m . We have that $T(e_j) \in F^m$ for each j . Hence $T(e_j) = v$ for some $v \in F^m$. Representing v in

terms of \mathcal{F}_n , let $a_{1j}, a_{2j}, \dots, a_{mj}$ be the unique vectors such that $T(e_j) = \sum_{i=1}^m a_{ij}f_i$. We obtain a matrix $(a_{ij}) := A \in \text{Mat}_{m,n}(F)$, and furthermore:

$$T_A(e_j) = Ae_j = \sum_{i=1}^m a_{ij}f_i = a_{1j}f_1 + \dots + a_{mj}f_m.$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Example 3.1.4 (Linear Transformations as Matrices). Recall that given a matrix $A \in \text{Mat}_{m,n}(F)$, we obtain a linear map $T_A \in \text{Hom}_F(F^n, F^m)$ by $T_A(v) = Av$. This process "works in reverse"—given a linear transformation $T \in \text{Hom}_F(F^n, F^m)$, there is a matrix A so that $T = T_A$.

Let $\mathcal{E}_n = \{e_1, \dots, e_n\}$ be the standard basis of F^n and $\mathcal{F}_m = \{f_1, \dots, f_m\}$ be the standard basis of F^m . We have that $T(e_j) \in F^m$ for each j , meaning we have elements $a_{ij} \in F$ with $T(e_j) = \sum_{i=1}^m a_{ij}f_i$. Define $A = (a_{ij}) \in \text{Mat}_{m,n}(F)$. Observe that:

$$T_A(e_j) = Ae_j = \sum_{i=1}^m a_{ij}f_i = a_{1j}f_1 + \dots + a_{mj}f_m.$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Working "in reverse", let $T \in \text{Hom}_F(V, W)$ with $\mathcal{B} = \{v_1, \dots, v_n\}$ a basis for V and $\mathcal{C} = \{w_1, \dots, w_m\}$ a basis for W . Define:

$$P = T_{\mathcal{B}} : V \rightarrow F^n \text{ by } v \mapsto [v]_{\mathcal{B}}$$

$$Q = T_{\mathcal{C}} : W \rightarrow F^m \text{ by } w \mapsto [w]_{\mathcal{C}}$$

From the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ P \downarrow & & \downarrow Q \\ F^n & \xrightarrow{Q \circ T \circ P^{-1}} & F^m \end{array}$$

we have that $Q \circ T \circ P^{-1}$ corresponds to a matrix $A \in \text{Mat}_{m,n}(F)$. Write $[T]_{\mathcal{B}}^C = A$, this is the unique matrix that satisfies $[T]_{\mathcal{B}}^C[v]_{\mathcal{B}} = [T(v)]_C$. Given $T(v_j) = \sum_{i=1}^m a_{ij}w_i$, observe that:

$$[T]_{\mathcal{B}}^C v_j = [T(v_j)]_C = \left[\sum_{i=1}^m a_{ij}w_i \right]_C = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

So $[T]_{\mathcal{B}}^C v_j$ corresponds to the j^{th} column of the matrix $[T]_{\mathcal{B}}^C$. Thus we have:

$$[T]_{\mathcal{B}}^C = \left([T(v_1)]_C \mid \dots \mid [T(v_n)]_C \right)$$

Example 3.1.5.

- (1) Let $V = P_3(\mathbf{R})$ with $\mathcal{B} = \{1, x, x^2, x^3\}$. Define $T \in \text{Hom}_{\mathbf{R}}(V, V)$ by $T(f(x)) = f'(x)$. Following Example 3.1.4 gives:

$$\begin{aligned} T(1) &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x^2) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x^3) &= 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3 \end{aligned}$$

$$[T(1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x^2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x^3)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- (2) Let $V = P_3(\mathbf{R})$ with $\mathcal{B} = \{1, x, x^2, x^3\}$ with $C = \{1, (1-x), (1-x)^2, (1-x^3)\}$. Then

$$T(1) = 0$$

$$T(x) = 1$$

$$T(x^2) = 2 + 2(x-1)$$

$$T(x^3) = -9 - 6(x-1) + 3(x-1)^2$$

$$\begin{aligned}
[T(1)]_C &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
[T(x)]_C &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
[T(x^2)]_C &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \\
[T(x^3)]_C &= \begin{pmatrix} -9 \\ -6 \\ 3 \\ 0 \end{pmatrix}
\end{aligned}$$

$$[T]_{\mathcal{B}}^C = \begin{pmatrix} 0 & 1 & 2 & -9 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 3.1.1.

- (1) Let \mathcal{A} be a basis of U , \mathcal{B} a basis of V and \mathcal{C} a basis of W . Let $S \in \text{Hom}_F(U, V)$ and $T \in \text{Hom}_F(V, W)$. Show

$$[T \circ S]_{\mathcal{A}}^C = [T]_{\mathcal{B}}^C [S]_{\mathcal{A}}^{\mathcal{B}}.$$

- (2) Given $A \in \text{Mat}_{m,k}(F)$ and $B \in \text{Mat}_{n,m}(F)$, we have corresponding linear maps T_A and T_B . Show that you can recover the definition of matrix multiplication by using part (1).

Note 3. Instead of $[T]_{\mathcal{B}}^{\mathcal{B}}$ we will write $[T]_{\mathcal{B}}$.

Example 3.1.6 (Change of Basis). Let V be an F -vector space and $\mathcal{B}, \mathcal{B}'$ bases of V . Given v expressed in terms of \mathcal{B} , we'd like to express it in terms of \mathcal{B}' (or vice versa).

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ and $\mathcal{B}' = \{v'_1, \dots, v'_n\}$. Define:

$$\begin{aligned}
T : V &\rightarrow F^n \text{ by } v \mapsto [v]_{\mathcal{B}} \\
S : V &\rightarrow F^n \text{ by } v \mapsto [v]_{\mathcal{B}'}.
\end{aligned}$$

We obtain a diagram similar to Example 3.1.4:

$$\begin{array}{ccc}
V & \xrightarrow{\text{id}_V} & V \\
T \downarrow & & \downarrow S \\
F^n & \xrightarrow{S \circ \text{id}_V \circ T^{-1}} & F^n
\end{array}$$

Hence the change of basis matrix is $[\text{id}_V]_{\mathcal{B}}^{\mathcal{B}'}$

Exercise 3.1.2. Let $\mathcal{B} = \{v_1, \dots, v_n\}$. Show that $[\text{id}_V]_{\mathcal{B}}^{\mathcal{B}'} = ([v_1]_{\mathcal{B}'} \mid \dots \mid [v_n]_{\mathcal{B}'})$.

Example 3.1.7.

(1) Let $V = \mathbf{Q}^2$ with $\mathcal{B} = \{e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ and $\mathcal{B}' = \{v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$. Observe that:

$$\begin{aligned} e_1 &= \frac{1}{2}v_1 + \frac{1}{2}v_2 \\ e_2 &= -\frac{1}{2}v_1 + \frac{1}{2}v_2 \end{aligned}$$

$$\begin{aligned} [e_1]_{\mathcal{B}} &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ [e_2]_{\mathcal{B}} &= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

$$[\text{id}_V]_{\mathcal{E}_2}^{\mathcal{B}'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Consider $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbf{Q}^2$. We can express v in terms of \mathcal{B}' by doing the following calculation:

$$\begin{aligned} [\text{id}_V]_{\mathcal{E}_2}^{\mathcal{B}'} [v_2]_{\mathcal{E}_2} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix} \\ &= [v]_{\mathcal{B}'}. \end{aligned}$$

(2) Let $V = P_2(\mathbf{R})$ with $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1, (x-2), (x-2)^2\}$. Then:

$$\begin{aligned} 1 &= 1 \cdot 1 + 0 \cdot (x-2) + 0 \cdot (x-2)^2 \\ x &= 2 \cdot 1 + 1 \cdot (x-2) + 0 \cdot (x-2)^2 \\ x^2 &= 4 \cdot 1 + 4 \cdot (x-2) + 1 \cdot (x-2)^2 \end{aligned}$$

$$\begin{aligned} [1]_{\mathcal{B}'} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ [x]_{\mathcal{B}'} &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\ [x^2]_{\mathcal{B}'} &= \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \end{aligned}$$

$$[\text{id}_V]_{\mathcal{B}}^{\mathcal{B}'} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 3.1.8 (Similar Matrices). Let $A, B \in \text{Mat}_n(F)$. Let \mathcal{E}_n be the standard basis for F^n and $T_A \in \text{Hom}_F(F^n, F^n)$ such that $A = [T_A]_{\mathcal{E}_n}$. We can relate A in terms of an arbitrary basis \mathcal{B} as follows:

$$\begin{array}{ccc} F^n & \xrightarrow{T_A} & F^n \\ T_{\mathcal{B}} \downarrow & & \downarrow T_{\mathcal{B}} \\ F^n & \xrightarrow{[T_A]_{\mathcal{B}}} & F^n. \end{array}$$

But by extending our diagram using our change of basis algorithm, we obtain the following:

$$\begin{array}{ccccccc} F^n & \xrightarrow{\text{id}_{F^n}} & F^n & \xrightarrow{T_A} & F^n & \xrightarrow{\text{id}_{F^n}} & F^n \\ T_{\mathcal{B}} \downarrow & & T_{\mathcal{E}_n} \downarrow & & \downarrow T_{\mathcal{E}_n} & & \downarrow T_{\mathcal{B}} \\ F^n & \xrightarrow{[\text{id}_{F^n}]_{\mathcal{B}}^{\mathcal{E}_n}} & F^n & \xrightarrow{[T_A]_{\mathcal{E}_n}} & F^n & \xrightarrow{[\text{id}_{F^n}]_{\mathcal{E}_n}^{\mathcal{B}}} & F^n \end{array}$$

So $[T_A]_{\mathcal{B}} = [\text{id}_{F^n}]_{\mathcal{B}}^{\mathcal{E}_n} [T_A]_{\mathcal{E}_n} [\text{id}_{F^n}]_{\mathcal{E}_n}^{\mathcal{B}}$. Assigning $P^{-1} = [\text{id}_{F^n}]_{\mathcal{B}}^{\mathcal{E}_n}$ and $P = [\text{id}_{F^n}]_{\mathcal{E}_n}^{\mathcal{B}}$ yields the familiar equation $[T_A]_{\mathcal{B}} = P^{-1}AP$; i.e., $A = P[T_A]_{\mathcal{B}}P^{-1}$. In particular, the matrix $A = [T_A]_{\mathcal{E}_n}$ is similar to $[T_A]_{\mathcal{B}}$ for any basis \mathcal{B} .

Example 3.1.9. Let $A = \begin{pmatrix} 1 & 3 & -5 \\ -2 & -1 & 6 \\ 3 & 2 & 1 \end{pmatrix}$. Let $\mathcal{E}_3 = \{e_1, e_2, e_3\}$ be the standard basis of F^3 . We have:

$$T_A(e_1) = e_1 - 2e_2 + 3e_3$$

$$T_A(e_2) = 3e_1 - e_2 + 2e_3$$

$$T_A(e_3) = 3e_1 + 2e_2 + e_3.$$

Now consider $\mathcal{B} = \{v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}\}$. One can check this is indeed a basis. Observe that:

$$e_1 = -2v_1 + -3v_2 + v_3$$

$$e_2 = 3v_1 + 3v_2 - v_3$$

$$e_3 = -2v_1 - 2v_2 + v_3.$$

So the change of basis matrix from \mathcal{E}_3 to \mathcal{B} is given by $P = [\text{id}_{F^3}]_{\mathcal{E}_3}^{\mathcal{B}} = \begin{pmatrix} -2 & 3 & -2 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{pmatrix}$. We have $P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Thus A is similar to the matrix $B = P^{-1}AP = \begin{pmatrix} -29 & 32 & -25 \\ -38 & 45 & -31 \\ -20 & 27 & -15 \end{pmatrix}$.

3.2 Row Operations

Definition 3.2.1. Let $A = (a_{ij}) \in \text{Mat}_{m,n}(F)$. We say a_{kl} is a pivot of A if $a_{kl} \neq 0$ and $a_{ij} = 0$ if $i > k$ or $j < l$.

Example 3.2.1. Let $A = \begin{pmatrix} 2 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then 2, 1, and 5 are pivots.

Definition 3.2.2. Let $A \in \text{Mat}_{m,n}(F)$. We say A is in row echelon form if all its nonzero rows have a pivot and all its zero rows are located below nonzero rows. We say it is reduced row echelon form if it is in row echelon form and all of its pivots are 1 and the only nonzero elements in the columns containing pivots.

Example 3.2.2. From the previous example, expressing $A = \begin{pmatrix} 2 & 1 & 4 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in reduced row echelon form yields $A' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Example 3.2.3. Let $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$. Then $T_A : F^4 \rightarrow F^4$. Let $\mathcal{B}_4 = \{e_1, e_2, e_3, e_4\}$ and $\mathcal{F}_3 = \{f_1, f_2, f_3\}$. So $A = [T_A]_{\mathcal{B}_3}^{\mathcal{F}_3}$. We have the following set of equations:

$$\begin{aligned} T_A(e_1) &= 3f_1 + f_2 + f_3 \\ T_A(e_2) &= 4f_1 + 2f_2 + f_3 \\ T_A(e_3) &= 5f_1 + 3f_2 + 2f_3 \\ T_A(e_4) &= 6f_1 + 4f_2 + 3f_3. \end{aligned}$$

We are going to perform row operations of A by making substitutions to its basis elements. Consider the operation $R_1 \leftrightarrow R_3$.

$$\mathcal{F}_3^{(2)} = \{f_1^{(2)} = f_3, f_2^{(2)} = f_2, f_3^{(2)} = f_1\}.$$

$$\begin{aligned} T_A(e_1) &= f_1^{(2)} + f_2^{(2)} + 3f_3^{(2)} \\ T_A(e_2) &= f_1^{(2)} + 2f_2^{(2)} + 4f_3^{(2)} \\ T_A(e_3) &= 2f_1^{(2)} + 3f_2^{(2)} + 5f_3^{(2)} \\ T_A(e_4) &= 3f_1^{(2)} + 4f_2^{(2)} + 6f_3^{(2)}. \end{aligned}$$

So $[T_A]_{\mathcal{B}_3}^{\mathcal{F}_3^{(2)}} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{pmatrix}$. Now consider the row operation $-R_1 + R_2 \leftrightarrow R_2$.

$$\mathcal{F}_3^{(3)} = \{f_1^{(3)} = f_1^{(2)} + f_2^{(2)}, f_2^{(3)} = f_2^{(2)}, f_3^{(3)} = f_3^{(2)}\}.$$

$$\begin{aligned} T_A(e_1) &= f_1^{(2)} + f_2^{(2)} + 3f_3^{(2)} \\ &= f_1^{(3)} + 3f_3^{(3)}. \end{aligned}$$

$$\begin{aligned} T_A(e_2) &= f_1^{(2)} + 2f_2^{(2)} + 4f_3^{(2)} \\ &= f_1^{(2)} + f_2^{(2)} + f_2^{(2)} + 4f_3^{(2)} \\ &= f_1^{(3)} + f_2^{(3)} + 4f_3^{(3)}. \end{aligned}$$

$$T_A(e_3) = \dots$$

$$T_A(e_4) = \dots$$

So $[T_A]_{\mathcal{B}_3}^{\mathcal{F}_3^{(3)}} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 \end{pmatrix}$. Now consider the row operation $-3R_1 + R_3 \leftrightarrow R_3$.

$$\mathcal{F}_3^{(4)} = \{f_1^{(4)} = f_1^{(3)} + 3f_3^{(3)}, f_2^{(4)} = f_2^{(3)}, f_3^{(4)} = f_3^{(3)}\}.$$

$$\begin{aligned} T_A(e_1) &= f_1^{(3)} + 3f_3^{(3)} \\ &= f_1^{(4)} \end{aligned}$$

$$T_A(e_2) = \dots$$

$$T_A(e_3) = \dots$$

$$T_A(e_4) = \dots$$

The rest of the steps to convert A to reduced row echelon form follow similarly.

Theorem 3.2.1. *Let $A \in \text{Mat}_{m,n}(F)$. The matrix A can be put in row echelon form through a series of row operations of the form:*

$$(1) R_i \leftrightarrow R_j$$

$$(2) R_i \leftrightarrow cR_i$$

$$(3) cR_i + R_j \leftrightarrow R_j.$$

Example 3.2.4. Instead of directly changing the basis of a matrix, we can use linear maps to perform row operations. Let $C = \{w_1, \dots, w_n\}$ be a basis of W .

(1) Define $T_{i,j} : W \rightarrow W$ by

$$T_{i,j}(w_k) = w_k \text{ if } k \neq i, j,$$

$$T_{i,j}(w_i) = w_j,$$

$$T_{i,j}(w_j) = w_i.$$

Then $E_{i,j} = [T_{i,j}]_C^C$ corresponds to the identity matrix except the i^{th} and j^{th} rows are switched.

(2) Let $c \in F$, $c \neq 0$. Define $T_i^{(c)} : W \rightarrow W$ by:

$$\begin{aligned} T_i^{(c)}(w_j) &= w_j \text{ if } j \neq i, \\ T_i^{(c)}(w_i) &= cw_i \end{aligned}$$

Then $E_i^{(c)} = \left[T_i^{(c)} \right]_C^C$ corresponds to the identity matrix with the i^{th} row multiplied by c .

(3) Define $T_{i,j}^{(c)} : W \rightarrow W$ by:

$$\begin{aligned} T_{i,j}^{(c)}(w_k) &= w_k \text{ if } k \neq j, \\ T_{i,j}^{(c)}(w_j) &= w_j + cw_i \end{aligned}$$

Then $E_{i,j}^{(c)} = \left[T_{i,j}^{(c)} \right]_C^C$ corresponds to the identity matrix with the **what does this mean?**

Now let $T_A : F^4 \rightarrow F^3$ with $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$ and \mathcal{E}_4 and \mathcal{F}_3 their respective standard bases. Performing the row operation $R_1 \leftrightarrow R_3$ using the above method yields:

$$\begin{aligned} (T_{1,3} \circ T_A)(e_1) &= T_{1,3}(3f_1 + f_2 + f_3) \\ &= 3T_{1,3}(f_1) + T_{1,3}(f_2) + T_{1,3}(f_3) \\ &= 3f_3 + f_2 + f_1 \end{aligned}$$

$$\left[T_{1,3} \circ T_A \right]_{\mathcal{E}_4}^{\mathcal{F}_3} = \left[T_{1,3} \right]_{\mathcal{F}_3}^{\mathcal{F}_3} \left[T_A \right]_{\mathcal{E}_4}^{\mathcal{F}_3}$$

$$= E_{1,3}A$$

$$= \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{pmatrix}.$$

The rest of the row operations follow similarly. The reduced-row echelon form of A can then be expressed as:

$$\left[T_{1,3}^{(-1)} \circ T_{2,3}^{(-1)} \circ T_{(3)}^{(\frac{1}{2})} \circ T_{3,2}^{(-1)} \circ T_{3,1}^{(-3)} \circ T_{1,2}^{(-1)} \circ T_{1,3} \circ T_A \right]_{\mathcal{E}_4}^{\mathcal{F}_3}.$$

3.3 Column-space and Null-space

Definition 3.3.1. Let $A \in \text{Mat}_{m,n}(F)$.

(1) The column-space of A is the F -span of the column vectors, denoted as $CS(A)$.

- (2) The null-space of A is the F -span of vectors $v \in F^n$ such that $Av = 0_V$, denoted as $NS(A)$.
- (3) The rank of A is $\text{rank } A = \dim_F CS(A)$.

Example 3.3.1. Let $T_A \in \text{Hom}_F(F^n, F^m)$ where $\mathcal{E}_n = \{e_1, \dots, e_n\}$ is the standard basis of F^n and $\mathcal{F}_m = \{f_1, \dots, f_m\}$ is the standard basis of F^m . Since

$$[T_A]_{\mathcal{E}_n}^{\mathcal{F}_m} = A = \left(T_A(e_1) \mid \dots \mid T_A(e_n) \right),$$

we have that $CS(A) = \text{im}(T_A)$, so $\text{rank } A = \dim_F \text{im}(T_A)$. Recall from an introductory linear algebra course that the column space is calculated by:

- (a) Put A into row echelon form,
- (b) Look at which columns have pivots,
- (c) The same columns in A are then a basis of $CS(A)$.

Why does this work? There exists an isomorphism $E : F^n \rightarrow F^m$ so that $[E \circ T_A]_{\mathcal{E}_n}^{\mathcal{F}_m} = [E]_{\mathcal{E}_n}^{\mathcal{F}_m} A$ is in row echelon form. The column space of $[E \circ T_A]_{\mathcal{E}_n}^{\mathcal{F}_m}$ has as its basis the columns containing pivots (denoted e_{i_1}, \dots, e_{i_k}):

$$\underbrace{[E \circ T_A(e_{i_1})]_{\mathcal{F}_m}, \dots, [E \circ T_A(e_{i_k})]_{\mathcal{F}_m}}_{\text{this is a basis of } CS([E \circ T_A]_{\mathcal{E}_n}^{\mathcal{F}_m})}$$

Since E is an isomorphism, there is an inverse $E^{-1} : F^m \rightarrow F^n$ with:

$$\begin{aligned} E^{-1}(w_1) &= [E \circ T_A(e_{i_1})]_{\mathcal{F}_m} \\ &\vdots \\ E^{-1}(w_k) &= [E \circ T_A(e_{i_k})]_{\mathcal{F}_m}. \end{aligned}$$

These are linearly independent since E^{-1} is an isomorphism. If there is a vector $v \in CS(A)$ with $v \notin \text{span}_F([E \circ T_A(e_{i_1})]_{\mathcal{F}_m}, \dots, [E \circ T_A(e_{i_k})]_{\mathcal{F}_m})$, then $E(v)$ cannot be in $\text{span}_F(w_1, \dots, w_k)$. So the columns

$[E \circ T_A(e_{i_1})]_{\mathcal{F}_m}, \dots, [E \circ T_A(e_{i_k})]_{\mathcal{F}_m}$ give a basis for the column space of A .

Example 3.3.2. Let $A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$. Rewritten in row echelon form is $A' = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{pmatrix}$. Thus:

$$\begin{aligned} CS(A) &= \text{span}_F \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \right) \\ CS(A) &= \text{span}_F \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \right). \end{aligned}$$

Example 3.3.3. We have $v \in NS(A)$ if and only if $Av = 0_{F^m} = T_A(v)$. Note that $T_A(v) = 0_{F^m}$ if and only if $v \in \ker(T_A)$, hence $NS(A) = \ker(T_A)$. In an introductory algebra class, the null space of a matrix A is calculated by:

- (1) Putting A into reduced row echelon form,

(2) Solving the equation $A'x = 0_{F^n}$.

This works because given a map $T_A : F^n \rightarrow F^m$, row operations change the basis of the codomain, not the domain. So $NS(A) = NS(A')$.

Example 3.3.4. Let $A = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -1 & 1 \end{pmatrix}$. The reduce row echelon form of A is $A' = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Solving the equation:

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives $x_2 = 0$ and $x_1 = -\frac{1}{2}x_3$. Hence $NS(A) = \text{span}_F \left(\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right)$.

3.4 The Transpose of a Matrix

Definition 3.4.1. Let $A \in \text{Mat}_{m,n}(F)$ with $\mathcal{E}_n = \{e_1, \dots, e_n\}$ and $\mathcal{F}_m = \{f_1, \dots, f_m\}$ as standard bases. Then $A = [T_A]_{\mathcal{E}_n}^{\mathcal{F}_m}$, and furthermore $T_A \in \text{Hom}_F(F^n, F^m)$ induces a dual map $T_A^\vee \in \text{Hom}_F(F^{m^\vee}, F^{n^\vee})$. The transpose of A is defined as:

$$A^t = [T_A^\vee]_{\mathcal{F}_m^\vee}^{\mathcal{E}_n^\vee}.$$

Lemma 3.4.1. Let $A = (a_{ij}) \in \text{Mat}_{m,n}(F)$. Then $A^t = (b_{ij}) \in \text{Mat}_{n,m}(F)$ with $b_{ij} = a_{ji}$.

Proof. We use the same setup as Definition 3.4.1. We have:

$$\begin{aligned} T_A(e_i) &= \sum_{k=1}^m a_{ki} f_k \\ T_A^\vee(f_j^\vee) &= \sum_{k=1}^n b_{kj} e_k^\vee. \end{aligned}$$

Applying f_j^\vee to $T_A(e_i)$ yields¹:

$$\begin{aligned} (f_j^\vee \circ T_A)(e_i) &= f_j^\vee \left(\sum_{k=1}^m a_{ki} f_k \right) \\ &= \sum_{k=1}^m a_{ki} f_j^\vee(f_k) \\ &= a_{ji}. \end{aligned}$$

¹I was really confused about this. In short, given a $T \in \text{Hom}_F(V, V)$ and basis \mathcal{B} we have a matrix representation $[T]_{\mathcal{B}}$. It is natural to wonder what, $[T^\vee]_{\mathcal{B}^\vee}$ looks like, and it turns out to be the "transpose" we were familiar with from 214. Basically, applying f_j^\vee to $T_A(e_i)$ gives us coefficients (by definition of dual basis elements) which correspond to a particular column vector of $[T_A]_{\mathcal{B}}$. Likewise, since we have that fancy property from Definition 2.5.3, naturally we should evaluate $T_A^\vee(f_j^\vee)$ at e_i , which gives us coefficients which correspond to column vectors of $[T_A^\vee]_{\mathcal{B}^\vee}$. The rest is self-explanatory.

Evaluating the $T_A^\vee(f_j^\vee)$ at e_i gives:

$$\begin{aligned} T_A^\vee(f_j^\vee)(e_i) &= \sum_{k=1}^n b_{kj} e_k^\vee(e_i) \\ &= b_{ij}. \end{aligned}$$

By Definition 2.5.3, we have $(f_j^\vee \circ T_A)(e_i) = T_A^\vee(f_j^\vee)(e_i)$. Hence $a_{ji} = b_{ij}$ □

Exercise 3.4.1. Let $A_1, A_2 \in \text{Mat}_{m,n}(F)$ and $c \in F$. Show that:

$$\begin{aligned} (A_1 + A_2)^t &= A_1^t + A_2^t \\ (cA_1)^t &= cA_1^t. \end{aligned}$$

Lemma 3.4.2. Let $A \in \text{Mat}_{m,n}(F)$ and $B \in \text{Mat}_{p,m}(F)$. Then $(BA)^t = A^t B^t$.

Proof. Let $\mathcal{E}_m, \mathcal{E}_n$, and \mathcal{E}_p be standard bases with $[T_A]_{\mathcal{E}_n}^{\mathcal{E}_m} = A$ and $[T_B]_{\mathcal{E}_m}^{\mathcal{E}_p} = B$. Then $BA = [T_B \circ T_A]_{\mathcal{E}_n}^{\mathcal{E}_p}$. Thus:

$$\begin{aligned} (BA)^t &= [(T_B \circ T_A)^\vee]_{\mathcal{E}_p^\vee}^{\mathcal{E}_n^\vee} \\ &= [T_A^\vee \circ T_B^\vee]_{\mathcal{E}_p^\vee}^{\mathcal{E}_n^\vee} \\ &= [T_A^\vee]_{\mathcal{E}_m^\vee}^{\mathcal{E}_n^\vee} [T_B^\vee]_{\mathcal{E}_p^\vee}^{\mathcal{E}_m^\vee} \\ &= A^t B^t. \end{aligned}$$

□

Lemma 3.4.3. Let $A \in \text{GL}_n(F)$. Then $(A^{-1})^t = (A^t)^{-1}$.

Proof. Let $A = [T_A]_{\mathcal{E}_n}^{\mathcal{E}_n}$. Then $A^{-1} = [T_A^{-1}]_{\mathcal{E}_n}^{\mathcal{E}_n}$. We have:

$$\begin{aligned} 1_n &= [\text{id}_{F^n}]_{\mathcal{E}_n}^{\mathcal{E}_n^\vee} \\ &= [(T_A^{-1} \circ T_A)^\vee]_{\mathcal{E}_n}^{\mathcal{E}_n^\vee} \\ &= [T_A^\vee \circ (T_A^{-1})^\vee]_{\mathcal{E}_n}^{\mathcal{E}_n^\vee} \\ &= [T_A^\vee]_{\mathcal{E}_n}^{\mathcal{E}_n^\vee} [(T_A^{-1})^\vee]_{\mathcal{E}_n}^{\mathcal{E}_n^\vee} \\ &= A^t (A^{-1})^t. \end{aligned}$$

By the uniqueness of inverses, we must have that $(A^{-1})^t = (A^t)^{-1}$. Showing left invertibility follows identically. □

4

Generalized Eigenvectors and Jordan Canonical Form

4.1 Diagonalization

Recall. We say $A \sim B$ if and only if $A = PBP^{-1}$ for some $P \in \text{GL}_n(F)$. In particular, this means $A = [T]_{\mathcal{A}}$ and $B = [T]_{\mathcal{B}}$ for some bases \mathcal{A} and \mathcal{B} (Example 3.1.8).

Definition 4.1.1. We say A is *diagonalizable* if $A \sim D$ for some diagonal matrix D . In terms of linear transformations, $A = [T]_{\mathcal{A}}$ is diagonalizable if there is a basis \mathcal{B} such that $[T]_{\mathcal{B}} = D$.

Example 4.1.1. If $A \sim B$ then A is diagonalizable if and only if B is diagonalizable. If A and B are diagonalizable, they must be similar to the same diagonal matrix up to reordering the diagonals.

Example 4.1.2. Let $V = F^2$ and $T \in \text{Hom}_F(V, V)$. Let $T(e_1) = 3e_1$ and $T(e_2) = -2e_2$. We have that:

$$[T]_{\mathcal{E}_2} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

It follows that $V = V_1 \oplus V_2$, where $V_1 = \text{span}_F(e_1)$ and $V_2 = \text{span}_F(e_2)$. In this case, we have that $T(V_1) \subseteq V_1$ and $T(V_2) \subseteq V_2$, allowing us to write T as a diagonal matrix.

Example 4.1.3. Let $V = F^2$ and $T \in \text{Hom}_F(V, V)$. Consider $T(e_1) = 3e_1$ and $T(e_2) = e_1 + 3e_2$. Then:

$$[T]_{\mathcal{E}_2} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Then $V = V_1 \oplus V_2$ with $V_1 = \text{span}_F(e_1)$ and $V_2 = \text{span}_F(e_2)$. But while we have $T(V_1) \subseteq V_1$, we do not have $T(V_2) \subseteq V_2$.

Suppose towards contradiction we have $W_1, W_2 \neq \{0\}$ with $T(W_1) \subseteq W_1$ and $T(W_2) \subseteq W_2$. Write $W_i = \text{span}_F(w_i)$. In particular, this means we can write $T(w_1) = \alpha w_1$ and $T(w_2) = \beta w_2$. For $\mathcal{B} = \{w_1, w_2\}$, we have:

$$[T]_{\mathcal{B}} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Write $w_1 = ae_1 + be_2$ and $w_2 = ce_1 + de_2$. Then:

$$\begin{aligned} \alpha w_1 &= T(w_1) \\ &= aT(e_1) + bT(e_2) \\ &= a(3e_1) + b(e_1 + 3e_2) \\ &= (3a + b)e_1 + (3b)e_2. \end{aligned}$$

Thus, $\alpha(ae_1 + be_2) = (3a + b)e_1 + (3b)e_2$, meaning $\alpha a = 3a + b$ and $\alpha b = 3b$. Either $b = 0$ or $\alpha = 3$. It must be the case that $\alpha = 3$, hence $T(w_1) = 3w_1$. A similar argument for w_2 gives:

$$\begin{aligned} \beta w_2 &= T(w_2) \\ &= \dots \\ &= (3c + d)e_1 + (3d)e_2. \end{aligned}$$

This implies $\beta c = c + d$ and $\beta d = 3d$. If $\beta = 3$, then this contradicts the first equation. If $w_2 = ce_1$, this contradicts w_1, w_2 being a basis.

Example 4.1.4. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Let $F = \mathbf{Q}$. Let $P \in GL_2(\mathbf{Q})$, where $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have:

$$P^{-1}AP = \frac{1}{ad - bc} \begin{pmatrix} ad - 2ab + 2cd - 4bc & -3bd - 3b^2 + 2d^2 \\ 3ac + 3a^2 - 2c^2 & -bc + 3ab - 2cd + 4ad \end{pmatrix}.$$

We must have that $3a^2 + 4ac - 2c^2 = 0$. If $c = 0$, then $a = 0$, which contradicts P being invertible. So $c \neq 0$, meaning we can divide by c^2 and set $x = \frac{a}{c}$. Then the roots of $3x^2 + 3x - 2 = 0$ are:

$$x = \frac{-3 \pm \sqrt{33}}{6},$$

which gives:

$$a = \frac{-3 \pm \sqrt{33}}{6}c.$$

Since $c \neq 0$, $a \notin \mathbf{Q}$. Thus we cannot diagonalize A over \mathbf{Q} . But if we were to take $F = \mathbf{Q}(\sqrt{33})$, then we have that:

$$\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{4} \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{4} \end{pmatrix} \right\},$$

$$[T]_{\mathcal{B}} = \begin{pmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{pmatrix}.$$

Definition 4.1.2. Let V be an F -vector space and $T \in \text{Hom}_F(V, V)$. A subspace $W \subseteq V$ is said to be T -invariant or T -stable if $T(W) \subseteq W$.

Theorem 4.1.1. Let $\dim_F(V) = n$ and $W \subseteq V$ a k -dimensional subspace. Let $\mathcal{B}_W = \{v_1, \dots, v_k\}$ be a basis of W and extend to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V . Let $T \in \text{Hom}_F(V, V)$. We have W is T -stable if and only if $[T]_{\mathcal{B}}$ is block upper-triangular of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where $A = [T|_W]_{\mathcal{B}_W}$.

Example 4.1.5. Let $V = \mathbb{Q}^4$ with basis $\mathcal{E}_4 = \{e_1, e_2, \dots, e_4\}$ and define T by:

$$\begin{aligned} T(e_1) &= 2e_1 + 3e_3 \\ T(e_2) &= e_1 + e_4 \\ T(e_3) &= e_1 - e_3 \\ T(e_4) &= 2e_1 - 2e_2 + 5e_3 - 4e_4. \end{aligned}$$

Set $W = \text{span}_{\mathbb{Q}}(e_1, e_3)$, then W is T -stable. Since $\mathcal{B}_W = \{e_1, e_3\}$ and $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$, we have:

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 3 & -1 & 0 & 5 \\ \boxed{0} & \boxed{0} & 0 & -2 \\ \boxed{0} & \boxed{0} & 1 & -4 \end{pmatrix}$$

Example 4.1.6. A special case is when $\dim_F W = 1$. If $W = \text{span}_F(w_1)$ and W is T -stable, then $T(w_1) \in W$; i.e., $T(w_1) = \lambda w_1$ for some $\lambda \in F$. Equivalently, this can be written as $(T - \lambda \text{id}_V)(w_1) = 0_V$, meaning $w_1 \in \ker(T - \lambda \text{id}_V)$.

4.2 Eigenvalues and Eigenvectors

Definition 4.2.1. Let $T \in \text{Hom}_F(V, V)$ and $\lambda \in F$. If $\ker(T - \lambda \text{id}_V) \neq \{0_V\}$, we say λ is an eigenvalue of T . Any nonzero vector in $\ker(T - \lambda \text{id}_V)$ is called a λ -eigenvector. The set $E_{\lambda}^1 = \ker(T - \lambda \text{id}_V)$ is called the eigenspace associated with λ .

Exercise 4.2.1. Show that E_{λ}^1 is a subspace.

Exercise 4.2.2. Let $T \in \text{Hom}_F(V, V)$. If $\lambda_1, \lambda_2 \in F$ with $\lambda_1 \neq \lambda_2$, then $E_{\lambda_1}^1 \cap E_{\lambda_2}^1 = \{0_V\}$.

Example 4.2.1. Let $A = \begin{pmatrix} 12 & 35 \\ -6 & 17 \end{pmatrix} \in \text{Mat}_2(\mathbf{Q})$ and $T_A \in \text{Hom}_{\mathbf{Q}}(\mathbf{Q}^2, \mathbf{Q}^2)$. We have:

$$\begin{pmatrix} -12 & 35 \\ -6 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{2}{5} \end{pmatrix} = 2 \begin{pmatrix} 1 \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} -12 & 35 \\ -6 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3}{7} \end{pmatrix} = 3 \begin{pmatrix} 1 \\ \frac{3}{7} \end{pmatrix}$$

So T_A has eigenvalues of 2 and 3. Then

$$E_2^1 = \text{span}_{\mathbf{Q}} \left(v_1 = \begin{pmatrix} 1 \\ 2/5 \end{pmatrix} \right)$$

$$E_3^1 = \text{span}_{\mathbf{Q}} \left(v_2 = \begin{pmatrix} 1 \\ 3/7 \end{pmatrix} \right)$$

gives:

$$[T_A]_{\{v_1, v_2\}} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Example 4.2.2 ($F[x]$ -Modules). Let $T \in \text{Hom}_F(V, V)$. Note that V is by definition an F -module, but we are able to view V as an $F[x]$ -module given some linear transformation T . The action $F[x] \times V \rightarrow V$ is defined by $(f(x), v) \mapsto f(T)(v)$.

Write $T^m = \underbrace{T \circ T \circ \dots \circ T}_{m\text{-times}}$. Write $f(x) \in F[x]$ as $f(x) = a_m x^m + \dots + a_1 x + a_0$. Then

$$f(T) = a_m T^m + \dots + a_1 T + a_0 \text{id}_V \in \text{Hom}_F(V, V).$$

For example, let $g(x) = 2x^2 + 3 \in \mathbf{R}[x]$. Then $g(T) = 2T^2 + 3 \text{id}_V$ and $g(T)(v) = 2T(T(v)) + 3v$. If $f(x) = g(x)h(x)$ for some $g(x), h(x) \in F[x]$, then $f(T) = g(T) \circ h(T)$. Instead of writing $f(T)(v) = g(T)(h(T)(v))$, we will abuse notation and write $g(T)h(T)(v)$. Normally function composition does not commute, but these do **for some reason**.

Theorem 4.2.1. Let $\dim_F(V) = n$ and $T \in \text{Hom}_F(V, V)$. There is a unique monic polynomial $m_T(x) \in F[x]$ of lowest degree so that $m_T(T)(v) = 0_V$ for all $v \in V$. Moreover, $\deg_{m_T}(T) \leq n^2$.

Proof. Recall that $\text{Hom}_F(V, V)$ is an F -vector space. We have $\text{Hom}_F(V, V) \cong \text{Mat}_n(F)$, hence $\dim_F(\text{Hom}_F(V, V)) = n^2$.

Given $T \in \text{Hom}_F(V, V)$, consider the set $\{\text{id}_V, T, T^2, \dots, T^{n^2}\} \subseteq \text{Hom}_F(V, V)$. This has $n^2 + 1$ elements, so it must be linearly dependent (meaning a linear combination of some subset can equal 0). Let m be the smallest integer so that

$$a_m T^m + \dots + a_1 T + a_0 \text{id}_V \stackrel{!}{=} 0_{\text{Hom}_F(V, V)}.$$

We obtain a set $\{\text{id}_V, T, T^2, \dots, T^m\}$. Since m is minimal, $a_m \neq 0$. Define:

$$m_T(x) = x^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0 \in F[x], \text{ where } b_i = \frac{a_i}{a_m}.$$

This gives $m_T(T) = 0_{\text{Hom}_F(V,V)}$; i.e., $m_T(T)(v) = 0_V$ for all $v \in V$. It remains to that $m_T(x)$ is unique. Suppose there exists an $f(x) \in F[x]$ which satisfies $f(T)(v) = 0_V$ for all $v \in V$. Write:

$$f(x) = m_T(x)q(x) + r(x)$$

for some $q(x), r(x) \in F[x]$ with $r(x) = 0$ or $\deg(r(x)) < \deg(m_T(x))$. We have for all $v \in V$:

$$\begin{aligned} 0_V &= f(T)(v) \\ &= q(T)m_T(T)(v) + r(T)(v) \\ &= q(T)(0_V) + r(T)(v) \\ &= r(T)(v) \end{aligned}$$

It must be the case that $r(x) = 0$, otherwise we have a polynomial of lower degree than $m_T(x)$ which kills all vectors. So $f(x) = m_T(x)q(x)$; i.e., $m_T(x) \mid f(x)$. But if $m_T(x)$ and $f(x)$ are both monic and of minimal degree, it must be the case that they are the same degree. This gives $m_T(x) = f(x)$. \square

Definition 4.2.2. The unique monic polynomial $m_T(x)$ is called the minimal polynomial of T .

Corollary 4.2.2. If $f(x) \in F[x]$ satisfies $f(T)(v) = 0_V$ for all $v \in V$, then $m_T(x) \mid f(x)$.

Proof. **wat** \square

Example 4.2.3. Let $F = \mathbf{Q}$ and $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. We can see that:

$$A - a_0 1_2 \neq 0_2 \text{ for any } a_0 \in F.$$

But $A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ gives $A^2 - 5A - 2 \cdot 1_2 = 0_2$. Hence $m_A(x) = x^2 - 5x - 2$. Note the relationship between this example and Example 4.1.4.

Example 4.2.4. Let $V = \mathbf{Q}^3$, $\mathcal{E}_3 = \{e_1, e_2, e_3\}$, and

$$[T_A]_{\mathcal{E}_3} = A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $W = \text{span}_{\mathbf{Q}}(e_1)$ Then $T(W) = T(ae_1) = ae_1 \in W$. Hence $T(W) \subseteq W$, meaning W is T -stable. This gives 1 as an eigenvalue. On a completely unrelated note, $m_{T_A}(x) = (x-1)^2(x+1)$.

¹This seems kind of out of nowhere, so think of it like this: Let $I_T = \{p \in F[x] \mid p(T)(v) = 0_V \text{ for all } v \in V\}$. $F[x]$ is a P.I.D., so every ideal is generated by a single element. The minimal polynomial $m_T(x)$ is the generator of this ideal.

Theorem 4.2.3. *Let V be an F -vector space and $T \in \text{Hom}_F(V, V)$. We have λ is an eigenvalue if and only if λ is the root of $m_T(x)$. In particular, if $(x - \lambda) \mid m_T(x)$, then $E_\lambda^1 \neq \{0_V\}$ (i.e., there is a nonzero $v \in V$ such that $T(v) = \lambda v$).*

Proof. Let λ be an eigenvalue with eigenvector v and write $m_T(x) = x^m + \dots + a_1x + a_0$. We have:

$$\begin{aligned} 0_V &= m_T(T)(v) \\ &= (T^m + a_{m-1}T^{m-1} + \dots + a_1T + a_0 \text{id}_V)(v) \\ &= T^m(v) + a_{m-1}T^{m-1}(v) + \dots + a_1T(v) + a_0v \\ &= \lambda^m v + a_{m-1}\lambda^{m-1}v + \dots + a_1\lambda v + a_0v \\ &= (\lambda^m + a_{m-1}\lambda^{m-1} + \dots + a_1\lambda + a_0)v \\ &= m_T(\lambda) \cdot v. \end{aligned}$$

Since $v \neq 0$ and $m_T(\lambda) \in F$, it must be the case that $m_T(\lambda) = 0$. Hence λ is a root.

Now suppose $m_T(\lambda) = 0$. This gives $m_T(x) = (x - \lambda)f(x)$ for some $f(x) \in F[x]$. Since $\deg f(x) < \deg m_T(x)$, this gives a nonzero vector $v \in V$ so that $f(T)(v) \neq 0$ (since $m_T(x)$ is the smallest polynomial that satisfies $m_T(T)(v) = 0_V$, it must be the case that there is a nonzero $v \in V$ that satisfies $f(T)(v) \neq 0$). Set $w = f(T)(v)$, then:

$$\begin{aligned} 0_V &= (T - \lambda \text{id}_V)f(T) \\ &= (T - \lambda \text{id}_V)w, \end{aligned}$$

which simplifies to $T(w) = \lambda w$. Thus λ is an eigenvalue. \square

Corollary 4.2.4. *Let $\lambda_1, \dots, \lambda_n \in F$ be distinct eigenvalues of T . For each i , let v_i be an eigenvector with eigenvalue λ_i . The set $\{v_1, \dots, v_m\}$ is linearly independent.*

Proof. We have $m_T(x) = (x - \lambda_1)(x - \lambda_2)\dots(x - \lambda_m)f(x)$ for some $f(x) \in F[x]$. Suppose $a_1v_1 + \dots + a_mv_m = 0_V$ for $a_i \in F$. Define $g_1(x) = (x - \lambda_2)\dots(x - \lambda_m)f(x)$. Note that $g_1(T)(v_i) = 0_V$ for $2 \leq i \leq m$. Then:

$$\begin{aligned} 0_V &= g_1(T)(0_V) \\ &= \sum_{j=1}^m a_j g_1(T)(v_j) \\ &= a_1 g_1(T)(v_1) \\ &= a_1 g_1(\lambda_1)v_1 \end{aligned}$$

But $g_1(\lambda_1) \neq 0$ and $v \neq 0$, so it must be that case that $a_1 = 0$. Inductively, it follows for $2, \dots, m$. \square

Corollary 4.2.5. *If $\deg(m_T(x)) = \dim_F(V)$ and $m_T(x)$ has distinct roots, all of which are in F , then we can find a basis \mathcal{B} so that $[T]_{\mathcal{B}}$ is diagonal.*

Example 4.2.5. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. These matrices are not similar, however $m_A(x) = m_B(x) = (x - 1)(x - 2)$. The minimal polynomial is not enough information on the similarity of matrices.

Example 4.2.6. Let:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

We have that $m_A(x) = (x-1)^2(x+1)$. Note that $Ae_1 = e_1$, so $E_1^1 \supseteq \text{span}_F(e_1)$ (or, more simply, $e_1 \in E_1^1$). Note that $Ae_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. So $e_2 \notin E_1^1$ (another way of saying this is $(A - 1_3)e_2 \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$). But now consider:

$$(A - 1_3)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 4 \end{pmatrix}.$$

We have $(A - 1_3)^2 e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Thus $e_1, e_2 \in \ker(A - \text{id}_{F^3})^2$.

Definition 4.2.3. Let $T \in \text{Hom}_F(V, V)$. For $k \geq 1$, the k^{th} *generalized eigenspace* of T associated to λ is $E_\lambda^k = \ker(T - \lambda \text{id}_V)^k = \{v \in V \mid (T - \lambda \text{id}_V)^k v = 0_V\}$. Elements of E_λ^k are called *generalized eigenvectors*. Set $E_\lambda^\infty = \bigcup_{k \geq 1} E_\lambda^k$.

Example 4.2.7. Continuing Example 4.2.6, let $\alpha e_1 + \beta e_2 \in \text{span}_F(e_1, e_2)$. Then:

$$(A - 1_3)^2(\alpha e_1 + \beta e_2) = \alpha(A - 1_3)^2 e_1 + \beta(A - 1_3)^2 e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So $\text{span}_F(e_1, e_2) \subseteq E_1^2$. We also have -1 as an eigenvalue with eigenvector $v_3 = \begin{pmatrix} \frac{1}{2} \\ -2 \\ 1 \end{pmatrix}$. Check that $v_3 \notin E_1^2$. So $\dim_F(E_1^2) \leq 2$; i.e., $E_1^2 = \text{span}_F(e_1, e_2)$. **why does $v_3 \notin E_1^2$ imply the dimension which implies containment in the other direction.**

Lemma 4.2.6. Let V be a finite dimensional F -vector space, $\dim_F(V) = n$, and $T \in \text{Hom}_F(V, V)$. There exists m with $1 \leq m \leq n$ such that $\ker(T^j) = \ker(T^{m+1})$. Moreover, for such an m , $\ker(T^m) = \ker(T^{m+j})$ for all $j \geq 0$.

Proof. We have $\ker(T^1) \subseteq \ker(T^2) \subseteq \dots$ If these containments are always strict, then the dimension increases indefinitely, which contradicts $\dim_F(V) = n$. Hence we have an m with $1 \leq m \leq n$ and $\ker(T^m) = \ker(T^{m+1})$.

Let m be the smallest value where $\ker(T^m) = \ker(T^{m+1})$. We use induction on j . Base case of $j = 1$ is what defines m . Assume $\ker(T^m) = \ker(T^{m+j})$ for all $1 \leq j \leq N$. Let $v \in \ker(T^{m+N+1})$. This gives:

$$\begin{aligned} 0_V &= T^{m+N+1}(v) \\ &= T^{m+1}(T^N(v)). \end{aligned}$$

So $T^N(v) \in \ker(T^{m+1})$. However $\ker(T^{m+1}) = \ker(T^m)$, so $T^N(v) \in \ker(T^m)$. Hence:

$$\begin{aligned} 0_V &= T^m(T^N(v)) \\ &= T^{m+N}(v), \end{aligned}$$

so $v \in \ker(T^{m+N})$. Induction hypothesis gives $\ker(T^{m+N}) = \ker(T^m)$, giving $v \in \ker(T^m)$. Thus $\ker(T^{m+N+1}) \subseteq \ker(T^m)$. The other direction of containment is trivial. \square

Example 4.2.8. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V and $T \in \text{Hom}_F(V, V)$, $\lambda \in F$ such that:

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

In other words, $[T]_{\mathcal{B}}$ contains λ along the diagonal and 1 along the super-diagonal. Let $A = [T]_{\mathcal{B}}$. Consider:

$$(A - \lambda 1_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We get:

$$\begin{aligned} (A - \lambda 1_n)v_1 &= 0_V \\ (A - \lambda 1_n)v_2 &= v_1 \\ &\vdots \\ (A - \lambda 1_n)v_n &= v_{n-1}. \end{aligned}$$

This gives $E_\lambda^1 = \text{span}_F(v_1)$ (by the first equation). Now observe:

$$(A - \lambda 1_n)^2 v_1 = 0_V$$

$$\begin{aligned} (A - \lambda 1_n)^2 v_2 &= (A - \lambda 1_n)(A - \lambda 1_n)v_2 \\ &= (A - \lambda 1_n)v_1 \\ &= 0_V \end{aligned}$$

$$(A - \lambda 1_n)^2 v_3 = v_1$$

$$\vdots$$

$$(A - \lambda 1_n)^2 v_n = v_{n-2}.$$

So $E_\lambda^2 = \text{span}_F(v_1, v_2)$. In general, we have that $E_\lambda^k = \text{span}_F(v_1, \dots, v_k)$. Moreover, Lemma 4.2.6 gives $E_\lambda^1 \subseteq E_\lambda^2 \subseteq \dots \subseteq E_\lambda^k$.

Corollary 4.2.7. *If $\dim_F(V) = n$ and $T \in \text{Hom}_F(V, V)$, there exists an m with $1 \leq m \leq n$ so that for any $\lambda \in F$, $E_\lambda^\infty = E_\lambda^m$.*

Theorem 4.2.8. *Let $T \in \text{Hom}_F(V, V)$, and $\lambda \in F$ with $(x - \lambda)^k \mid m_T(x)$. We have:*

$$\dim_F(E_\lambda^k) \geq k.$$

Proof. Write $m_T(x) = (x - \lambda)^k f(x)$ where $f(x) \in F[x]$, $f(\lambda) \neq 0$. Define $g_k(x) = (x - \lambda)^k$. We have that $(x - \lambda)^{k-1} f(x) = g_{k-1}(x) f(x)$ is *not* the minimal polynomial. So there is a $v \in V$ with $v \neq 0_V$ such that:

$$g_{k-1}(T) f(T)(v) \neq 0_V.$$

Set $v_k = f(T)(v)$. Observe that:

$$\begin{aligned} (T - \lambda \text{id}_V)^k(v_k) &= (T - \lambda \text{id}_V)^k f(T)(v) \\ &= m_T(T)(v) \\ &= 0_V. \end{aligned}$$

So $v_k \in E_\lambda^k$. Moreover, by our construction:

$$\begin{aligned} (T - \lambda \text{id}_V)^{k-1}(v_k) &= g_{k-1}(T)(v_k) \\ &= g_{k-1}(T) f(T)(v) \\ &\neq 0_V. \end{aligned}$$

Hence $v_k \in E_\lambda^k \setminus E_\lambda^{k-1}$. Now set $v_{k-1} = (T - \lambda \text{id}_V)v_k = (T - \lambda \text{id}_V)f(T)(v)$. Note:

$$\begin{aligned} (T - \lambda \text{id}_V)^{k-1}(v_{k-1}) &= (T - \lambda \text{id}_V)^{k-1}(T - \lambda \text{id}_V)f(T)(v) \\ &= (T - \lambda \text{id}_V)^k(v_k) \\ &= (T - \lambda \text{id}_V)^k f(T)(v) \\ &= m_T(T)(v) \\ &= 0_V. \end{aligned}$$

So $v_{k-1} \in E_\lambda^{k-1}$. Again, by our construction:

$$\begin{aligned} (T - \lambda \text{id}_V)^{k-2}(v_{k-1}) &= (T - \lambda \text{id}_V)^{k-2}(T - \lambda \text{id}_V)(v_k) \\ &= (T - \lambda \text{id}_V)^{k-1}(v_k) \\ &\neq 0_V. \end{aligned}$$

So $v_{k-1} \in E_\lambda^{k-1} \setminus E_\lambda^{k-2}$. Setting $v_{k-2} = (T - \lambda \text{id}_V)^2 v_k$ gives a similar result. By this construction, we obtain a set $\{v_k, v_{k-1}, \dots, v_2, v_1\}$. Claim: this set is linearly independent. Suppose towards contradiction it's not, that is, $a_1 v_1 + \dots + a_k v_k = 0_V$ does not imply $a_1 = \dots = a_k = 0$. This gives $v_k = \frac{-1}{a_k}(a_1 v_1 + \dots + a_{k-1} v_{k-1}) \in E_\lambda^{k-1}$, which is a contradiction. It follows that $a_1 = \dots = a_k = 0$, hence $\{v_k, v_{k-1}, \dots, v_2, v_1\}$ is linearly independent (linear independent set \subseteq a basis, so that's why the theorem is established). \square

Example 4.2.9. Let $T_A \in \text{Hom}_F(F^3, F^3)$ be defined by:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

We have that $m_T(x) = (x - 2)^3$. Now observe:

$$(A - 2 \cdot 1_3)^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note $(A - 2 \cdot 1_3)^2 e_3 = 4e_3 \neq 0_F^3$, but $(A - 2 \cdot 1_3)^3 e_3 = 0_{F^3}$. Set $v_3 = e_3$, we have $v_3 \in E_2^3$. Now observe:

$$\begin{aligned} v_2 &= (A - 2 \cdot 1_3)(v_3) \\ &= \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}. \end{aligned}$$

Similarly:

$$\begin{aligned} v_1 &= (A - 2 \cdot 1_3)(v_2) \\ &= \dots \\ &= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence:

$$\begin{aligned} E_2^3 &= \text{span}_F \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right) \\ E_2^2 &= \text{span}_F \left(\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right) \\ E_2^1 &= \text{span}_F \left(\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right). \end{aligned}$$

Setting $\mathcal{B} = \{v_1, v_2, v_3\}$, we have:

$$[T_A]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

4.3 Characteristic Polynomials

Definition 4.3.1. Let $A \in \text{Mat}_n(F)$. The characteristic polynomial is $c_A(x) = \det(x1_n - A)$.

Definition 4.3.2. Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in F[x]$. The companion matrix of $f(x)$ is given by:

$$C(f(x)) = \begin{pmatrix} -a_0 & 0 & 0 & \dots & 0 \\ -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \dots & 1 \end{pmatrix}$$

The companion matrix shows that any polynomial $f(x) \in F[x]$ can be realized as the characteristic polynomial of a matrix.

Lemma 4.3.1. If $A = C(f(x))$, then $c_A(x) = f(x)$.

Lemma 4.3.2. Let $A, B \in \text{Mat}_n(F)$ be similar matrices. Then $c_A(x) = c_B(x)$.

Proof. Let $A = PBP^{-1}$ for some $P \in GL_n(F)$. We have:

$$\begin{aligned}
 c_A(x) &= \det(x1_n - A) \\
 &= \det(x1_n - PBP^{-1}) \\
 &= \det(P(x1_n)P^{-1} - PBP^{-1}) \\
 &= \det(P(x1_n - B)P^{-1}) \\
 &= \det(P) \det(x1_n - B) \det(P^{-1}) \\
 &= \det(x1_n - B) \\
 &= c_B(x).
 \end{aligned}$$

□

Definition 4.3.3. For $T \in \text{Hom}_F(V, V)$, let \mathcal{B} be a basis of V and set $c_T(x) = c_{[T]_{\mathcal{B}}}(x)$.

Theorem 4.3.3. Let $v \in V$, $v \neq 0_V$. Let $\dim_F(V) = n$. Then there is a unique monic polynomial $m_{T,v}(x) \in F[x]$ so that $m_{T,v}(T)(v) = 0_V$. Moreover, if $f(x) \in F[x]$ with $f(T)(v) = 0_V$, then $m_{T,v}(x) \mid f(x)$.

Proof. Consider the set $\{v, T(v), T^2(v), \dots, T^n(v)\}$. Since this set contains $n + 1$ elements and the dimension of V is n , the set must be linearly dependent. Write:

$$a_m T^m(v) + \dots + a_1 T(v) + a_0 = 0_V$$

for some $m \leq n$ of minimal order and $a_i \neq 0$ for all i . Set:

$$p(x) = x^m + \frac{a_{m-1}}{a_m} x^{m-1} + \dots + \frac{a_1}{a_m} x + \frac{a_0}{a_m} \in F[x].$$

By construction $p(T)(v) = 0_V$. Set $I_v = \{g(x) \in F[x] \mid g(T)(v) = 0_V\}$. We have that $p(x)$ is a monic nonzero polynomial in I_v of minimal degree. Set $m_{T,v}(x) = p(x)$.

Let $f(x) \in I_v$. We'd like to show that $m_{T,v}(x) \mid f(x)$. Write:

$$f(x) = q(x)m_{T,v}(x) + r(x),$$

with $q(x), r(x) \in F[x]$ and $\deg(r(x)) = 0$ or $\deg(r) < \deg(m_{T,v}(x))$. Observe that:

$$\begin{aligned}
 r(T)(v) &= f(T)(v) - q(T)m_{T,v}(T)(v) \\
 &= 0_V - q(T)0_V \\
 &= 0_V.
 \end{aligned}$$

So $r(x) \in I_v$. But $m_{T,v}(x)$ had minimal degree, so it must be the case that $r(x) = 0$. Thus $f(x) = q(x)m_{T,v}(x)$, implying $m_{T,v}(x) \mid f(x)$ ². Now suppose $h(x) \in I_v$ with $\deg(h(x)) = \deg(m_{T,v}(x))$. Since both polynomials are monic and of equal degree, if $m_{T,v}(x) \mid h(x)$ then $m_{T,v}(x) = h(x)$. □

Definition 4.3.4. We refer to $m_{T,v}(x)$ as the T -annihilator of v .

²The proof of $F[x]$ being a P.I.D. follows identically. Instead of considering I_v we would consider an arbitrary polynomial in $F[x]$.

Example 4.3.1. Let $V = F^n$ and $\mathcal{E}_n = \{e_1, \dots, e_n\}$. Define $T \in \text{Hom}_F(V, V)$ by:

$$\begin{aligned} T(e_1) &= 0_V \\ T(e_j) &= e_{j-1} \text{ for } 2 \leq j \leq n. \end{aligned}$$

Consider $f(x) = x$. Then $f(T)(e_1) = T(e_1) = 0_V$. Hence $m_{T,e_1}(x) \mid x$. So either $m_{T,e_1}(x) = 1$ or $m_{T,e_1}(x) = x$. But $\text{id}_V(e_1) = e_1 \neq 0_V$, hence it must be the case that $m_{T,e_1}(x) = x$.

Now consider $g(x) = x^2$. Then $g(T)(e_2) = T^2(e_2) = T(T(e_2)) = T(e_1) = 0_V$. Hence $m_{T,e_2}(x) \mid x^2$. So $m_{T,e_2}(x) = 1$ or x or x^2 . If $m_{T,e_2}(x) = 1$, then $\text{id}_V(e_2) = e_2 \neq 0_V$. If $m_{T,e_2}(x) = x$, then $T(e_2) = e_1 \neq 0$. So $m_{T,e_2}(x) = x^2$. It follows for $i \leq j \leq n$, $m_{T,e_j}(x) = x^j$.

Example 4.3.2. Let $V = \mathbf{Q}^2$. Define $T \in \text{Hom}_{\mathbf{Q}}(\mathbf{Q}^2, \mathbf{Q}^2)$ by:

$$\begin{aligned} T(e_1) &= e_1 + 3e_2 \\ T(e_2) &= 2e_1 + 4e_2. \end{aligned}$$

We are trying to find $m_{T,e_1}(x)$. Since V is two-dimensional, $\deg(m_{T,e_1}(x)) = 1$ or 2 . Write $m_{T,e_1}(x) = x + a$. Then:

$$\begin{aligned} m_{T,e_1}(T)(e_1) &= T(e_1) + ae_1 \\ &= e_1 + 3e_2 + ae_1 \\ &\neq 0_V. \end{aligned}$$

So it must be that $\deg(m_{T,e_1}(x)) = 2$. Note that:

$$\begin{aligned} T^2(e_1) &= T(e_1 + 3e_2) \\ &= T(e_1) + 3T(e_2) \\ &= 7e_1 + 15e_2. \end{aligned}$$

Now let:

$$T^2(e_1) + bT(e_1) + ce_1 = 0_V,$$

for some $b, c \in \mathbf{Q}$. This will yield a system of equations, and solving for it gives:

$$\begin{aligned} b &= -5 \\ c &= -2. \end{aligned}$$

Hence $m_{T,e_1}(x) = x^2 - 5x - 2$.

Exercise 4.3.1.

1. Show $m_{T,e_2}(x) = x^2 - 5x - 2$.
2. Calculate $m_{T,e_1}(x)$ and $m_{T,e_2}(x)$ of $F = \mathbf{F}_3$.

Theorem 4.3.4. Let $\dim_F(V) = n$ and $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V . Let $T \in \text{Hom}_F(V, V)$. We have:

$$m_T(x) = \text{lcm}_{1 \leq i \leq n} m_{T,v_i}(x).$$

Proof. Let $f(x) = \text{lcm}_{1 \leq i \leq n} m_{T, v_i}(x)$. Note that $m_T(T)(v_i) = 0_V$, so $m_{T, v_i}(x) \mid m_T(x)$ for each i . Hence $f(x) \mid m_T(x)$.

Now let $v \in V$. Write $v = \sum_{i=1}^n a_i v_i$. We have:

$$\begin{aligned} f(T)(v) &= f(T)\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i f(T)(v_i) \\ &= 0_V, \end{aligned}$$

because $m_{T, v_i}(x) \mid f(x)$ for all i . Hence $m_T(x) \mid f(x)$. **i dont quite get this number theory stuff** \square

Lemma 4.3.5. *Let $T \in \text{Hom}_F(V, V)$. Let $v_1, \dots, v_k \in V$, and set $p_i(x) = m_{T, v_i}(x)$. Suppose $p_i(x)$ are pairwise relatively prime. Set $v = v_1 + \dots + v_k$. Then:*

$$m_{T, v}(x) = p_1(x) \dots p_k(x).$$

Proof. We prove this for $k \geq 2$; i.e., $m_{T, v_1+v_2}(x) = m_{T, v_1}(x)m_{T, v_2}(x)$. Since $p_1(x)$ and $p_2(x)$ are relatively prime, there exists $q_1(x), q_2(x) \in F[x]$ so that $1 = p_1(x)q_1(x) + p_2(x)q_2(x)$. In particular, $\text{id}_V = p_1(T)q_1(T) + p_2(T)q_2(T)$. Set $v = v_1 + v_2$. We have:

$$\begin{aligned} v &= \text{id}_V(v) \\ &= (p_1(T)q_1(T) + p_2(T)q_2(T))(v) \\ &= p_1(T)q_1(T)(v) + p_2(T)q_2(T)(v) \\ &= p_1(T)q_1(T)(v_1 + v_2) + p_2(T)q_2(T)(v_1 + v_2) \\ &= p_1(T)q_1(T)(v_2) + p_2(T)q_2(T)(v_2). \end{aligned}$$

Write $w_1 = p_1(T)q_1(T)(v_2)$ and $w_2 = p_2(T)q_2(T)(v_1)$. This means $v = w_1 + w_2$. Note:

$$\begin{aligned} p_1(T)(w_1) &= p_1(T)p_2(T)q_2(T)(v_1) \\ &= q_2(T)p_2(T) \underbrace{p_1(T)(v_1)}_{=0_V} \\ &= 0_V. \end{aligned}$$

Hence $w_1 \in \ker(p_1(T))$. It follows similarly that $w_1 \in \ker(p_2(T))$. Let $r(x) \in F[x]$ with $r(T)(v) = 0_V$. We have $v = w_1 + w_2$ and $w_2 \in \ker(p_2(T))$, so:

$$\begin{aligned} p_2(T)(v) &= p_2(T)(w_1 + w_2) \\ &= p_2(T)(w_1). \end{aligned}$$

Thus:

$$\begin{aligned} 0_V &= p_2(T)q_2(T)(0_V) \\ &= p_2(T)q_2(T)r(T)(v) \\ &= r(T)p_2(T)q_2(T)(v) \\ &= r(T)p_2(T)q_2(T)(w_1). \end{aligned}$$

We also know $r(T)q_1(T)p_1(T)(w_1) = 0_V$ because $w_1 \in \ker(p_1(T))$. Hence:

$$\begin{aligned} 0_V &= r(T)p_2(T)q_2(T)(w_1) + r(T)p_1(T)q_1(T)(w_1) \\ &= r(T) \underbrace{(p_2(T)q_2(T) + p_1(T)q_1(T))}_{\text{id}_V}(w_1) \\ &= r(T)(w_1). \end{aligned}$$

This gives:

$$\begin{aligned} 0_V &= r(T)(w_1) \\ &= r(T)p_2(T)q_2(T)(v_1). \end{aligned}$$

So $r(T)p_2(T)q_2(T)(v_1) = 0_V$. Thus $p_1(x) \mid r(x)p_2(x)q_2(x)$. Now note that:

$$\gcd(p_1(x), p_2(x)q_2(x)) = 1,$$

which means $p_1(x) \mid r(x)$. A similar argument shows $p_2(x) \mid r(x)$. And since $\gcd(p_1(x), p_2(x)) = 1$, this gives $\text{lcm}(p_1(x), p_2(x)) = p_1(x)p_2(x)$. So $p_1(x)p_2(x) \mid r(x)$. Since $r(x)$ was arbitrary, take $r(x) = m_{T,v}(x)$. Then $p_1(x)p_2(x) \mid m_{T,v}(x)$. Finally, since $p_1(x)p_2(x)(v) = 0_V$, $m_{T,v}(x) \mid p_1(x)p_2(x)$, establishing the lemma. \square

Exercise 4.3.2. Show inductively that $m_{T,v} = p_1(x)p_2(x)\dots p_k(x)$ ³.

Theorem 4.3.6. Let $T \in \text{Hom}_F(V, V)$. There exists $v \in V$ such that $m_{T,v}(x) = m_T(x)$. In particular, $\deg(m_T(x)) \leq n$.

Proof. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis. We know:

$$m_T(x) = \text{lcm}_{1 \leq i \leq n} m_{T,v_i}(x).$$

Factor $m_T(x) = p_1(x)^{e_1} \dots p_k(x)^{e_k}$, with each $p_i(x)$ relatively prime and $e_1 \geq 1$. For $1 \leq j \leq k$, there exists $i_j \in \{1, \dots, n\}$ and $q_{i_j}(x) \in F[x]$ with:

$$m_{T,v_{i_j}}(x) = p_j(x)^{e_j} q_{i_j}(x).$$

Set $w_j = q_{i_j}(T)(v_{i_j})$. This gives:

$$m_{T,w_j}(x) = p_j(x)^{e_j}.$$

Now set $w = w_1 + \dots + w_k$. The previous result gives $m_{T,w}(x) = p_1(x)^{e_1} \dots p_k(x)^{e_k} = m_T(x)$????. \square

Lemma 4.3.7. Let $W \subseteq V$ be a T -invariant subspace. Then there is an induced map $\bar{T} \in \text{Hom}_F(V/W, V/W)$ defined by $\bar{T}(v + W) = T(v) + W$.

Lemma 4.3.8. Let $v \in V$. Then $m_{\bar{T}, [v]}(x) \mid m_{T,v}(x)$. Similarly, $m_{\bar{T}}(x) \mid m_T(x)$.

³Pairwise coprime is a stronger statement than just coprime. It means that $\gcd(p_i, p_j) = 1$ for all $1 \leq i, j \leq k$

Proof. We have:

$$\begin{aligned} m_{T,v}(\bar{T})([v]) &= m_{T,v}(\bar{T})(v + W) \\ &= m_{T,v}(T)(v) + W \\ &= 0_V + W \\ &= 0_{V/W}. \end{aligned}$$

Then by definition of (in this case, $m_{\bar{T},[v]}(x)$) annihilator polynomials, $m_{\bar{T},[v]}(x) \mid m_{T,v}(x)$. \square

Definition 4.3.5. Let $T \in \text{Hom}_F(V, V)$ and $\mathcal{A} = \{v_1, \dots, v_k\}$ a set of vectors in V . The T -span of \mathcal{A} is the subspace:

$$W = \left\{ \sum_{i=1}^k p_i(T)(v_i) \mid v_i \in \mathcal{A}, p_i(x) \in F[x] \right\}.$$

We say the subset W is T -generated by \mathcal{A} .

Exercise 4.3.3. Show W is a T -invariant subspace of V . Moreover, show it is the smallest T -invariant subspace with respect to inclusion of V that contains \mathcal{A} .

Example 4.3.3. Let $V = \mathbf{Q}^4$. Define $T \in \text{Hom}_{\mathbf{Q}}(\mathbf{Q}^4, \mathbf{Q}^4)$ by:

$$\begin{aligned} T(e_1) &= 2e_1 + 3e_3 \\ T(e_2) &= e_1 + e_2 \\ T(e_3) &= e_1 - e_3 \\ T(e_4) &= 2e_1 - 2e_2 + 5e_3 - 4e_4. \end{aligned}$$

Let $\mathcal{A} = \{e_1\}$. Our goal is to find $T\text{-span}_{\mathbf{Q}}(\mathcal{A})$. Set $p(x) = 1$, then $p(T)(e_1) = \text{id}_V(e_1) = e_1$. Hence $e_1 \in T\text{-span}_{\mathbf{Q}}(\mathcal{A})$. Now set $q(x) = \frac{1}{3}(x - 2)$. Then:

$$\begin{aligned} q(T)(e_1) &= \frac{1}{3}(T - 2\text{id}_V)(e_1) \\ &= \frac{1}{3}(T(e_1) - 2e_1) \\ &= e_3. \end{aligned}$$

Hence $e_3 \in T\text{-span}_{\mathbf{Q}}(\mathcal{A})$. So $\text{span}_{\mathbf{Q}}(e_1, e_3) \subseteq T\text{-span}_{\mathbf{Q}}(\mathcal{A})$ (basically $\alpha p(x) + \beta q(x) \in \text{span}_T(F(\mathcal{A}))$, so plugging in a linear combination of e_1 and e_3 will give you back a linear combination of e_1 and e_3). Note that $\text{span}_F(e_1, e_3)$ is T -invariant. By Exercise 4.3.3, since $T\text{-span}_{\mathbf{Q}}(\mathcal{A})$ is the smallest T -invariant subspace by inclusion, it must be the case that $T\text{-span}_{\mathbf{Q}}(\mathcal{A}) \subseteq \text{span}_F(e_1, e_3)$. Hence $T\text{-span}_{\mathbf{Q}}(\mathcal{A}) = \text{span}_F(e_1, e_3)$.

Lemma 4.3.9. Let $T \in \text{Hom}_F(V, V)$, $w \in V$, and W the subspace of V that is T -generated by $\{w\}$. Then $\dim_F(W) = \deg(m_{T,w}(x))$.

Proof. Let $\deg(m_{T,w}(x)) = k$. Consider the set $\{w, T(w), \dots, T^{k-1}(w)\}$. This is a basis of the T -span of $\{w\}$. \square

Theorem 4.3.10. *Let $\dim_F(V) = n$.*

(1) *We have $m_T(x) \mid c_T(x)$.*

(2) *Every irreducible factor of $c_T(x)$ is a factor of $m_T(x)$.*

Proof. (1) Let $\deg(m_T(x)) = k \leq n$. Let $v \in V$ with $m_T(x) = m_{T,v}(x)$. Let W_1 be the T -span of $\{v\}$. By Lemma 4.3.9, $\dim_F(W_1) = k$. Write:

$$\begin{aligned} v &= v_k \\ T(v) &= v_{k-1} \\ T^2(v) &= v_{k-2} \\ &\vdots \\ T^i(v) &= v_{k-i}. \end{aligned}$$

We have $\mathcal{B}_1 = \{v_1, \dots, v_k\}$ is a basis of W_1 (see proof of previous lemma). Since:

$$\begin{aligned} 0_V &= m_T(T)(v) \\ &= T^k(v) + a_{k-1}T^{k-1}(v) + \dots + a_1T(v) + a_0v, \end{aligned}$$

we have that $T^k(v) = -a_{k-1}T^{k-1}(v) - \dots - a_1T(v) - a_0v$. Thus:

$$\begin{aligned} T(v_1) &= T(T^{k-1}(v)) = T^k(v) = -a_{k-1}T^{k-1}(v) - \dots - a_1T(v) - a_0v. \\ T(v_2) &= T(T^{k-2}(v)) = T^{k-1}(v) \\ T(v_3) &= T(T^{k-3}(v)) = T^{k-2}(v) \\ &\vdots \end{aligned}$$

So $[T|_{W_1}]_{\mathcal{B}_1} = C(m_T(x))$. We proceed with cases:

Case 1: $k = n$. Then $W_1 = V$, and $[T]_{\mathcal{B}_1} = C(m_T(x))$, which has characteristic polynomial $m_T(x)$, meaning $m_T(x) = c_T(x)$.

Case 2: $k < n$. Expand \mathcal{B}_1 to a full basis of V as follows: Let $\mathcal{B}_2 = \{v_{k+1}, \dots, v_n\}$ and write:

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2.$$

Since W_1 is T -invariant, by Theorem 4.1.1 $[T]_{\mathcal{B}}$ will be block diagonal. So we have:

$$[T]_{\mathcal{B}} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad A = [T|_{W_1}]_{\mathcal{B}_1} = C(m_T(x)).$$

Hence:

$$\begin{aligned} c_T(x) &= \det(x1_n - [T]_{\mathcal{B}}) \\ &= \det \begin{pmatrix} x1_k - A & -B \\ 0 & x1_{n-k} - D \end{pmatrix} \\ &= \det(x1_k - A) \det(x1_{n-k} - D) \\ &= c_A(x) \det(x1_{n-k} - D) \\ &= m_T(x) \det(x1_{n-k} - D). \end{aligned}$$

Thus $m_T(x) \mid c_T(x)$.

For (2), we induct on $\dim_F(V) = n$. If $n = 1$, then both the characteristic polynomial and minimal polynomial are monic and of degree 1, hence they are equal. If $\deg(m_T(x)) = n$, then $m_T(x) \mid c_T(x)$. Since both are degree n and monic, they must be equal. Now suppose $\deg m_T(x) = k < n$ **The rest of this proof is hard.** \square

Example 4.3.4. Consider:

$$A = \begin{pmatrix} 1 & 2 & & & & \\ & 3 & 4 & & & \\ & & 3 & 7 & & \\ & & -1 & 2 & & \\ & & & & -5 & 6 \\ & & & & 2 & -3 \end{pmatrix} \in \text{Mat}_9(\mathbf{Q}).$$

We can verify that $c_A(x) = (x^2 - 5x - 2)(x^2 - x - 1)(x^2 + 8x + 3)$. Since every irreducible factor of $c_T(x)$ is a factor of $m_T(x)$, we have that $m_T(x) = (x^2 - 5x - 2)(x^2 - x - 1)(x^2 + 8x + 3)$.

Theorem 4.3.11 (Cayley-Hamilton).

(1) Let $T \in \text{Hom}_F(V, V)$ and $\dim_F(V) < \infty$. Then $c_T(T) = 0_{\text{Hom}_F(V, V)}$.

(2) Let $A \in \text{Mat}_n(F)$. Then $c_A(A) = 0_n$.

Proof. Write $c_T(x) = f(x)m_T(x)$. Then for any $v \in V$:

$$\begin{aligned} c_T(T)(v) &= f(T)m_T(T)(v) \\ &= f(T)(0_V) \\ &= 0_V. \end{aligned} \quad \square$$

4.4 Jordan Canonical Form

For this section V is always finite-dimensional.

Definition 4.4.1. Let $T \in \text{Hom}_F(V, V)$. A Jordan basis for V with respect to T is a basis \mathcal{B} so

that:

$$[T]_{\mathcal{B}} = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix} \lambda_n & 1 \\ & \lambda_n \end{matrix}} \end{pmatrix}.$$

for some $\lambda_1, \dots, \lambda_n \in F$. More generally, if $V = V_1 \oplus \dots \oplus V_k$ is a decomposition into T -invariant subspaces, and each V_i has a Jordan basis \mathcal{B}_i , we say $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}_i$ is a Jordan basis for V .

Definition 4.4.2. A matrix of the form:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

is called a Jordan block associated to λ_i . We say a matrix J is in Jordan canonical form if it is a block diagonal matrix with each block representing a Jordan block.

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}.$$

Lemma 4.4.1. Let $T \in \text{Hom}_F(V, V)$. We have that $\ker(T - \lambda \text{id}_V)^j$ and $\text{im}(T - \lambda \text{id}_V)^j$ are T -invariant subspaces for all $j \geq 0$.

Proof. Note that $T \circ (T - \lambda \text{id}_V)^j = (T - \lambda \text{id}_V)^j \circ T$. Let $v \in \ker(T - \lambda \text{id}_V)^j$. We have:

$$\begin{aligned} (T - \lambda \text{id}_V)^j(T(v)) &= T((T - \lambda \text{id}_V)^j(v)) \\ &= T(0_V) \\ &= 0_V. \end{aligned}$$

So $T(v) \in \ker(T - \lambda \text{id}_V)^j$. Now let $w \in \text{im}(T - \lambda \text{id}_V)^j$. We can write $w = (T - \lambda \text{id}_V)^j(v)$ for some $v \in V$. Then:

$$\begin{aligned} T(w) &= T((T - \lambda \text{id}_V)^j(v)) \\ &= (T - \lambda \text{id}_V)(T(v)). \end{aligned}$$

Thus $T(w) \in \text{im}(T - \lambda \text{id}_V)^j$. □

Lemma 4.4.2. *Suppose $m_T(x) = (x - \lambda)^m p(x)$ with $p(\lambda) \neq 0$. Then $E_\lambda^\infty = E_\lambda^m$.*

Proof. Let $E_\lambda^\infty = E_\lambda^e$. Let $v \in E_\lambda^e \setminus E_\lambda^{e-1}$. Since $(T - \lambda \text{id}_V)^e(v) = 0_V$, we know that $m_{T,v}(x) \mid (x - \lambda)^e$. Note that $m_{T,v}(x) \nmid (x - \lambda)^{e-1}$, otherwise $(T - \lambda \text{id}_V)^{e-1}(v) = q(T)m_{T,v}(T)(v) = 0_V$, implying $v \in E_\lambda^{e-1}$ which is a contradiction. Now since $m_{T,v}$ is by definition the monic polynomial of minimal degree which kills v , it must be the case that $m_{T,v}(x) = (x - \lambda)^e$. So we have:

$$(x - \lambda)^e \mid (x - \lambda)^m p(x).$$

Note that $p(x)$ does not have irreducible factors of the form $(x - \lambda)$, hence $(x - \lambda)^e$ and $p(x)$ are coprime, implying:

$$(x - \lambda)^e \mid (x - \lambda)^m.$$

But this is a contradiction if $e > m$, so it must be the case that $e \leq m$. Now if we were to assume that $e < m$, then:

$$\begin{aligned} (T - \lambda)^m(v) &= g(T)(T - \lambda)^e(v) \\ &= g(T)(0_V) \\ &= 0_V. \end{aligned}$$

So $m_T(x) \mid (x - \lambda)^m$, a contradiction. Thus $e = m$. □

Lemma 4.4.3. *Let $\dim_F(V) = n$. Let $m_T(x) = (x - \lambda)^m p(x)$ with $p(\lambda) \neq 0$. We have $V = E_\lambda^m \oplus \text{im}(T - \lambda)^m$.*

Proof. Recall that $E_\lambda^m = \ker(T - \lambda)^m$. The dimensions are correct by the rank-nullity theorem. It only remains to show that $E_\lambda^m \cap \text{im}(T - \lambda)^m = \{0_V\}$.

Let $v \in E_\lambda^m \cap \text{im}(T - \lambda)^m$. Since $v \in \text{im}(T - \lambda)^m$, let $v = (T - \lambda)^m(w)$ for some $w \in V$. Applying $(T - \lambda)^m$ to both sides gives:

$$(T - \lambda)^m(v) = (T - \lambda)^{2m}(w).$$

Since $v \in E_\lambda^m$ by our assumption, we have that $0_V = (T - \lambda)^{2m}(w)$. But Lemma 4.4.2 gives that $E_\lambda^\infty = E_\lambda^m$, hence $(T - \lambda)^{2m}(w) = (T - \lambda)^m(w) = 0_V$. Thus $v = 0_V$. □

Theorem 4.4.4. *Assume $m_T(x) = (x - \lambda_1)^{m_1} \dots (x - \lambda_k)^{m_k}$ with each $\lambda_i \in F$ distinct and $m_j \geq 1$. We have:*

$$V = E_{\lambda_1}^{m_1} \oplus \dots \oplus E_{\lambda_k}^{m_k}.$$

Proof. We use induction on k . If $k = 1$, then $m_T(x) = (x - \lambda_1)^{m_1}$. Since $m_T(T)(v) = 0_V$ for all $v \in V$, we have $V = E_{\lambda_1}^{m_1}$. Now assume the result is true for any vector space W and $S \in \text{Hom}_F(W, W)$ where $m_S(x)$ splits completely over F and has less than k distinct roots. Write:

$$V = E_{\lambda_1}^{m_1} \oplus \text{im}(T - \lambda_1)^{m_1}.$$

Set $W = \text{im}(T - \lambda_1)^{m_1}$. Lemma 4.4.1 gives that W is T -invariant, so $T|_W \in \text{Hom}_F(W, W)$. We want to show that $m_{T|_W}(x) = (x - \lambda_2)^{m_2} \dots (x - \lambda_k)^{m_k}$. Set $p(x) = (x - \lambda_2)^{m_2} \dots (x - \lambda_k)^{m_k}$. Suppose $p(T)(w) \neq 0$. We have:

$$\begin{aligned} 0_V &= m_T(T)(w) \\ &= (T - \lambda_1)^{m_1} p(T)(w). \end{aligned}$$

So $p(T)(w) \in E_{\lambda_1}^{m_1}$. But also $p(T)(w) = p(T|_W)(w) \in W$. So Lemma 4.4.3 gives that $p(T)(w) = p(T|_W)(w) = 0_V$. Thus:

$$m_{T|_W}(x) \mid p(x).$$

Suppose $m_{T|_W}(x)$ is a proper divisor of $p(x)$, that is, $p(x) = m_{T|_W}(x)h(x)$ where $\deg(h(x)) > 1$. Consider $f(x) = (x - \lambda_1)^{m_1} m_{T|_W}(x)$. Let $v \in V$ and write $v = v_1 + w$ for some $v_1 \in E_{\lambda_1}^{m_1}$ and $w \in W$. Then:

$$\begin{aligned} f(T)(v) &= f(T)(v_1) + f(T)(w) \\ &= m_{T|_W}(T)(T - \lambda_1)^{m_1}(v_1) + (T - \lambda_1)^{m_1} m_{T|_W}(T)(w) \\ &= 0_V. \end{aligned}$$

Thus $m_T(x) \mid f(x)$. But note that:

$$\begin{aligned} m_T(x) &= (x - \lambda_1)^{m_1} p(x) \\ &= (x - \lambda_1)^{m_1} m_{T|_W}(x) h(x) \\ f(x) &= (x - \lambda_1)^{m_1} m_{T|_W}(x). \end{aligned}$$

This contradicts $m_T(x) \mid f(x)$, so our original assumption that $m_{T|_W}(x)$ is a proper divisor of $p(x)$ is false, it must be the case that $m_{T|_W}(x) = p(x)$. Since $V = E_{\lambda_1}^{m_1} \oplus W$, applying our induction hypothesis to W and $T|_W$ gives:

$$V = E_{\lambda_1}^{m_1} \oplus (E_{\lambda_1}^{m_2} \oplus \dots \oplus E_{\lambda_k}^{m_k}).$$

□

4.5 Diagonalization, II

The following theorem relates all that was discussed in this chapter.

Theorem 4.5.1. *If $c_T(x)$ does not split into a product of linear factors over F , T is not diagonalizable. If $c_T(x)$ does split into linear factors, the following are equivalent:*

- (1) T is diagonalizable;
- (2) for every eigenvalue λ , $E_\lambda^\infty = E_\lambda^1$;
- (3) $m_T(x)$ splits into a product of (distinct) linear factors;
- (4) for every eigenvalue λ , if $c_T(x) = (x - \lambda)^{e_\lambda} p(x)$ with $p(\lambda) \neq 0$, then $e_\lambda = \dim_F(E_\lambda^1)$;
- (5) if we set $d_\lambda = \dim_F(E_\lambda^1)$, then $\sum_\lambda d_\lambda = \dim_F(V)$;
- (6) if $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues of T , then

$$V = E_{\lambda_1}^1 \oplus \dots \oplus E_{\lambda_m}^1$$

5

Tensor Products and the Trace

5.1 Complexification

Recall that if V is a \mathbf{C} -vector space, then V is also an \mathbf{R} -vector space by restricting the scalars of \mathbf{C} . A natural question to ask is if V is an \mathbf{R} -vector space, can we "extend" V to be a \mathbf{C} -vector space?

Example 5.1.1 (Complexification of \mathbf{R}). Let $V = \mathbf{R}$. We cannot make \mathbf{R} into a \mathbf{C} -vector space. However, we do have $\mathbf{R} \hookrightarrow \mathbf{C}$ by $x \mapsto x + 0i$, with \mathbf{C} as a \mathbf{C} -vector space. But note that $z \in \mathbf{C}$ can be written as $z = x + yi$. There is an isomorphism $\mathbf{R} \oplus \mathbf{R} \cong \mathbf{C}$ as \mathbf{R} -vector spaces by:

$$x + yi \mapsto (x, y)$$

If we take $z = x + yi \in \mathbf{C}$ to be a vector, and $a + bi \in \mathbf{C}$ to be a scalar, we have:

$$(a + bi)(x + yi) = (ax - by) + (ay + bx)i,$$

meaning in $\mathbf{R} \oplus \mathbf{R}$ we define:

$$(a + bi)(x, y) = (ax - by, ay + bx)$$

With scalar multiplication defined as above, then $\mathbf{R} \oplus \mathbf{R}$ is a \mathbf{C} -vector space. Furthermore, we have $\mathbf{R} \oplus \mathbf{R} \cong \mathbf{C}$ as \mathbf{C} -vector spaces!

Definition 5.1.1. Let V be a real vector space. The complexification of V is denoted $V_{\mathbf{C}} = V \oplus V$, where complex scalar multiplication is defined by:

$$(a + bi)(v_1, v_2) = (av_1 - bv_2, av_2 + bv_1).$$

Upon investigation one can see:

$$i(v_1, v_2) = (-v_2, v_1).$$

Exercise 5.1.1. Prove that $V_{\mathbf{C}}$ is a \mathbf{C} -vector space.

Proposition 5.1.1. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be an \mathbf{R} -basis of V . The set $\mathcal{B}_{\mathbf{C}} = \{(v_j, 0_V)\}_{j \in I}$ is a \mathbf{C} -basis of $V_{\mathbf{C}}$.

Proof. Let $(w_1, w_2) \in V_{\mathbf{C}}$. We can write:

$$\begin{aligned} w_1 &= \sum_{j \in I} a_j v_j \\ w_2 &= \sum_{j \in I} b_j v_j \end{aligned}$$

for some $a_j, b_j \in \mathbf{R}$. We have:

$$\begin{aligned}
 (w_1, w_2) &= \left(\sum_{j \in I} a_j v_j, \sum_{j \in I} b_j v_j \right) \\
 &= \left(\sum_{j \in I} a_j v_j, 0_V \right) + \left(0_V, \sum_{j \in I} b_j v_j \right) \\
 &= \sum_{j \in I} a_j (v_j, 0_V) + \sum_{j \in I} b_j (0_V, v_j) \\
 &= \sum_{j \in I} a_j (v_j, 0_V) + \sum_{j \in I} i b_j (v_j, 0_V) \\
 &\in \text{span}_{\mathbf{C}} \{ (v_j, 0_V) \}_{j \in I}.
 \end{aligned}$$

Now suppose we have $(0_V, 0_V) = \sum_{j \in I} (a_j + i b_j) (v_j, 0_V)$. Then:

$$\begin{aligned}
 (0_V, 0_V) &= \sum_{j \in I} (a_j + i b_j) (v_j, 0_V) \\
 &= \sum_{j \in I} a_j (v_j, 0_V) + \sum_{j \in I} i b_j (v_j, 0_V) \\
 &= \left(\sum_{j \in I} a_j v_j, 0_V \right) + i \left(\sum_{j \in I} b_j v_j, 0_V \right) \\
 &= \left(\sum_{j \in I} a_j v_j, 0_V \right) + \left(\sum_{j \in I} 0_V, b_j v_j \right) \\
 &= \left(\sum_{j \in I} a_j v_j, \sum_{j \in I} b_j v_j \right),
 \end{aligned}$$

meaning:

$$\begin{aligned}
 \sum_{j \in I} a_j v_j &= 0_V \\
 \sum_{j \in I} b_j v_j &= 0_V.
 \end{aligned}$$

So $a_j = 0$ for all j and $b_j = 0$ for all j . Thus $\{(v_j, 0_V)\}_{j \in I}$ are linearly independent. \square

Proposition 5.1.2. *Let V, W be \mathbf{R} -vector spaces, and let $T \in \text{Hom}_{\mathbf{R}}(V, W)$. There is a unique $T_{\mathbf{C}} \in \text{Hom}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}})$ that makes the following diagram commute:*

$$\begin{array}{ccc}
 V & \xrightarrow{T} & W \\
 \iota_V \downarrow & & \downarrow \iota_W \\
 V_{\mathbf{C}} & \xrightarrow{T_{\mathbf{C}}} & W_{\mathbf{C}}
 \end{array}$$

Proof. Define

$$T_{\mathbf{C}}(v_1, v_2) = (T(v_1), T(v_2)).$$

Let $v \in V$. We have $\iota_V(v) = (v, 0_V)$, meaning:

$$\begin{aligned} T_{\mathbf{C}}(\iota_V(v)) &= T_{\mathbf{C}}((v, 0_V)) \\ &= (T(v), T(0_V)) \\ &= (T(v), 0_W), \end{aligned}$$

and:

$$\iota_W(T(v)) = (T(v), 0_W).$$

Hence the diagram commutes. We claim that $T_{\mathbf{C}}$ is \mathbf{C} -linear. Let $x + iy \in \mathbf{C}$ and $(v_1, v_2), (v'_1, v'_2) \in V_{\mathbf{C}}$. Then:

$$\begin{aligned} T_{\mathbf{C}}((v_1, v_2) + (x + iy)(v'_1, v'_2)) &= T_{\mathbf{C}}((v_1, v_2) + (xv'_1 - yv'_2, xv'_2 + yv'_1)) \\ &= T_{\mathbf{C}}((v_1 + xv'_1 - yv'_2, v_2 + xv'_2 + yv'_1)) \\ &= (T(v_1 + xv'_1 - yv'_2), T(v_2 + xv'_2 + yv'_1)) \\ &= (T(v_1), T(v_2)) + x(T(v'_1), T(v'_2)) + y(-T(v'_2), T(v'_1)) \\ &= (T(v_1), T(v_2)) + (x + iy)(T(v'_1), T(v'_2)) \\ &= T_{\mathbf{C}}(v_1, v_2) + (x + iy)T_{\mathbf{C}}(v'_1, v'_2). \end{aligned}$$

Hence $T_{\mathbf{C}}$ is linear. Now suppose there is an $S \in \text{Hom}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}})$ making the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \iota_V \downarrow & & \downarrow \iota_W \\ V_{\mathbf{C}} & \xrightarrow{S} & W_{\mathbf{C}} \end{array}$$

Let $v_1, v_2 \in V_{\mathbf{C}}$. Then:

$$\begin{aligned} S((v_1, v_2)) &= S((v_1, 0_V) + (0_V, v_2)) \\ &= S((v_1, 0_V) + i(v_2, 0_V)) \\ &= S((v_1, 0_V)) + iS((v_2, 0_V)) \\ &= S(\iota_V(v_1)) + iS(\iota_V(v_2)) \\ &= \iota_W(T(v_1)) + i\iota_W(T(v_2)) \\ &= (T(v_1), 0_W) + i(T(v_2), 0_W) \\ &= (T(v_1), 0_W) + (0_W, T(v_2)) \\ &= (T(v_1), T(v_2)) \\ &= T_{\mathbf{C}}((v_1, v_2)). \end{aligned}$$

Thus $T_{\mathbf{C}}$ is unique. □

5.2 Free Vector Spaces

We showed in Section 2.2 that every vector space has a basis. In this section we show that given a set X , we can build a vector space that "has" X as a basis. We will give a few basic definitions before investigating the properties of these spaces.

Definition 5.2.1. Let $f : \Omega \rightarrow F$ be a map whose domain is an arbitrary set Ω . The support of f , denoted $\text{supp}(f)$ is the set of points in Ω where f is nonzero:

$$\text{supp}(f) = \{x \in \Omega \mid f(x) \neq 0\}.$$

Definition 5.2.2. Let F be a field. The set of all functions from Ω to F is denoted:

$$\mathcal{F}(\Omega, F) = \{f \mid f : \Omega \rightarrow F\}.$$

Exercise 5.2.1. Show that $\mathcal{F}(\Omega, F)$ is an F -vector space.

Example 5.2.1. Fix $t \in \Gamma$. Recall that $\delta_t : \Gamma \rightarrow F$ is defined by:

$$\delta_t(s) = \begin{cases} 1, & s = t \\ 0, & s \neq t \end{cases}.$$

We have that $\delta_t \in \mathcal{F}(\Gamma, F)$, and furthermore $\text{supp}(\delta_t) = \{t\}$. If $f \in \mathcal{F}(\Gamma, F)$ has finite support, then $\text{supp}(f) = \{t_1, \dots, t_n\}$ for some $t_i \in \Gamma$. If:

$$\begin{aligned} f(t_1) &= \alpha_1 \neq 0 \\ f(t_2) &= \alpha_2 \neq 0 \\ &\vdots \\ f(t_n) &= \alpha_n \neq 0, \end{aligned}$$

then we can write $f = \sum_{j=1}^n \alpha_j \delta_{t_j}$.

Theorem 5.2.1 (Existence of Free Vector Spaces). *Let F be a field and Γ a set. There is an F -vector space $\mathbb{F}(\Gamma)$ that has a basis isomorphic to Γ as sets. Moreover, $\mathbb{F}(\Gamma)$ has the following universal property: if W is any F -vector space and $t : \Gamma \rightarrow W$ is a map of sets, there is a unique $T \in \text{Hom}_F(\mathbb{F}(\Gamma), W)$ such that $T(x) = t(x)$ for every $x \in \Gamma$; i.e., the following diagram commutes:*

$$\begin{array}{ccc} \Gamma & \xhookrightarrow{t} & \mathbb{F}(\Gamma) \\ & \searrow t & \downarrow T \\ & & W \end{array}$$

Proof. If Γ is the empty set, take $\mathbb{F}(\Gamma) = \{0\}$. Let $\Gamma \neq \emptyset$. Define:

$$\mathbb{F}(\Gamma) = \{f : \Gamma \rightarrow F \mid \text{supp}(f) < \infty\}.$$

Since $\mathbb{F}(\Gamma) \subseteq \mathcal{F}(\Gamma, F)$, this space inherits a natural vector space structure. In particular, if f, g are finitely supported functions and $c \in F$, then $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ will be finitely supported. Moreover, the zero element of this set is $f(x) = 0_{\mathbb{F}(\Gamma)}$. The rest of the vector space axioms are left as an exercise.

We obtain an inclusion $\iota : \Gamma \hookrightarrow \mathbb{F}(\Gamma)$ by $x \mapsto \delta_x$. Let $\mathcal{X} = \{\delta_x \mid x \in \Gamma\}$. This is a subset of $\mathbb{F}(\Gamma)$ and furthermore we have a bijection $\Gamma \xrightarrow{\sim} \mathcal{X}$.

Let $f \in \mathbb{F}(\Gamma)$. We can write $f = \sum_{x \in \Gamma} f(x)\delta_x \in \text{span}_F(\mathcal{X})$. Hence $\text{span}_F(\mathcal{X}) = \mathbb{F}(\Gamma)$. Note that:

$$\begin{aligned} f(y) &= f(y)\delta_y(y) \\ &= f(y)\delta(y)(y) + \sum_{x \neq y} f(x)\delta_x(y) \\ &= \sum_{x \in \Gamma} f(x)\delta_x(y). \end{aligned}$$

Note that $f(y)$ is just a scalar in F , hence an arbitrary element of $\mathbb{F}(\Gamma)$ looks like $\sum_{i=1}^n a_i \delta_{x_i}$. Suppose then that $\sum_{i=1}^n a_i \delta_{x_i} = 0_{\mathbb{F}(\Gamma)}$. We have that $\sum_{i=1}^n a_i \delta_{x_i}(y) = 0_F$ for all $y \in \Gamma$. Thus:

$$\begin{aligned} 0_F &= \sum_{i=1}^n a_i \delta_{x_i}(x_j) \\ &= a_j. \end{aligned}$$

This establishes \mathcal{X} as a basis for $\mathbb{F}(\Gamma)$.

Now suppose we have $t : \Gamma \rightarrow W$. Define $T : \mathbb{F}(\Gamma) \rightarrow W$ by:

$$T\left(\sum_{i=1}^n a_i \delta_{x_i}\right) = \sum_{i=1}^n a_i t(\delta_{x_i}).$$

Because \mathcal{X} is a basis, this gives a well-defined linear map. It is unique because any linear map that agrees with t on \mathcal{X} must agree with T on $\mathbb{F}(\Gamma)$, establishing the proof. \square

Example 5.2.2. If $\Gamma = \mathbf{R}$, we can form $\mathbb{F}_{\mathbf{R}}(\mathbf{R})$. An example of an element of $\mathbb{F}_{\mathbf{R}}(\mathbf{R})$ is $2 \cdot \pi + 3 \cdot 2$, where $\pi, 2$ are basis elements and $2, 3$ are scalars. Note that, from this construction, we cannot simplify this expression.

Exercise 5.2.2. Show that if $\Gamma = \{x_1, \dots, x_n\}$, then $\mathbb{F}(\Gamma) \cong F^n$.

5.3 Extension of Scalars

Let V be an F -vector space and K/F an extension of fields. We can naturally consider K as an F -vector space. As we did with complexification, we want to define a way to "multiply" vectors in V by scalars in K . The way we define "multiplication" should be obvious: Let $a, a_1, a_2 \in K$, $c \in F$, and $v, v_1, v_2 \in V$. We want multiplication to satisfy:

- (1) $(a_1 + a_2) \star v$;
- (2) $a \star (v_1 + v_2) = a \star v_1 + a \star v_2$;

$$(3) (ac) \star v = a \star (cv).$$

We will construct a vector space that satisfies exactly this by constructing the *tensor product* of V with K .

Definition 5.3.1. Let V be an F -vector space and K/F be an extension of fields. Let $K \times V$ be the Cartesian product of K and V and define:

$$\mathcal{A}_1 = \{(a_1 + a_2, v) - (a_1, v) - (a_2, v) \mid a_1, a_2 \in K, v \in V\},$$

$$\mathcal{A}_2 = \{(a, v_1 + v_2) - (a, v_1) - (a, v_2) \mid a \in K, v_1, v_2 \in V\},$$

$$\mathcal{A}_3 = \{(ca, v) - (a, cv) \mid c \in F, a \in K, v \in V\},$$

$$\mathcal{A}_4 = \{a_1(a_2, v) - (a_1a_2, v) \mid a_1, a_2 \in K, v \in V\}.$$

Define $\text{Rel}_K(K \times V) = \text{span}_F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$. The *tensor product* of K and V over F is:

$$K \otimes_F V = \mathbb{F}(K \times V) / \text{Rel}_K(K \times V).$$

For any arbitrary element $\sum_{\text{finite}} c_i \delta_{a_i, v_i} \in \mathbb{F}(K \times V)$, we denote the equivalence class $\sum_{\text{finite}} c_i \delta_{a_i, v_i} + \text{Rel}_K(K \times V)$ as:

$$\begin{aligned} \sum_{\text{finite}} c_i (a_i \otimes v_i) &= \sum_{\text{finite}} c_i a_i (1 \otimes v_i) \\ &= \sum_{\text{finite}} b_i \otimes v_i \end{aligned}$$

for some $b_i \in K$. An element of the form $a \otimes v$ is referred to as a *pure tensor*. Both arbitrary elements of $K \otimes_F V$ and pure tensors admit the following properties:

$$(1) (a_1 + a_2) \otimes v = a_1 \otimes v + a_2 \otimes v \text{ for all } a_1, a_2 \in K, v \in V;$$

$$(2) a \otimes (v_1 + v_2) = a \otimes v_1 + a \otimes v_2 \text{ for all } a \in K, v_1, v_2 \in V;$$

$$(3) ca \otimes v = a \otimes cv \text{ for all } c \in F, a \in K, \text{ and } v \in V;$$

$$(4) a_1(a_2 \otimes v) = (a_1a_2) \otimes v \text{ for all } a_1, a_2 \in K, v \in V.$$

Note 4.

- (1) An **arbitrary element of $K \otimes_F V$ is a finite sum**. It is a common mistake when working with tensor products to check things for pure tensors and not with arbitrary elements.
- (2) Since $K \otimes_F V$ is a quotient space, there must be care in checking things are well-defined when working with tensor products.

Exercise 5.3.1. Show that $K \otimes_F V$ is a K -vector space. (Hint: $0 \otimes 0_V$ is the additive identity in $K \otimes_F V$).

Proposition 5.3.1. Let K/F be a field extension and V an F -vector space with basis $\mathcal{B} = \{v_i\}_{i \in I}$. We have $\text{span}_K \{1 \otimes v_i\}_{i \in I} = K \otimes_F V$.

Proof. Let $a \otimes v \in K \otimes_F V$. Write $v = \sum_{i=1}^n c_i v_i$ for some $c_i \in F$. We have:

$$\begin{aligned} a \otimes v &= a \otimes \left(\sum_{i=1}^n c_i v_i \right) \\ &= \sum_{i=1}^n a \otimes c_i v_i \\ &= \sum_{i=1}^n a c_i \otimes v_i \\ &= \sum_{i=1}^n a c_i (1 \otimes v_i). \end{aligned}$$

From this, it follows that every pure tensor $a \otimes v$ is also in the span of $\{1 \otimes v_i\}_{i \in I}$. This gives that all finite sums of the form $\sum_{j \in I} a_j \otimes \tilde{v}_j$ are also in the span of $\{1 \otimes v_i\}_{i \in I}$. Hence $\text{span}_F \{1 \otimes v_i\}_{i \in I} = K \otimes_F V$. \square

Theorem 5.3.2. *Let K/F be an extension of fields, V an F -vectorspace, and $\iota_V : V \rightarrow K \otimes_F V$ defined by $\iota_V(v) = 1 \otimes v$. Let W be any K -vector space and $S \in \text{Hom}_F(V, W)$. There is a unique $T \in \text{Hom}_K(K \otimes_F V, W)$ so that $S = T \circ \iota_V$; i.e., the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\iota_V} & K \otimes_F V \\ & \searrow S & \downarrow T \\ & & W \end{array}$$

Conversely, if $T \in \text{Hom}_K(K \otimes_F V, W)$, then $T \circ \iota_V \in \text{Hom}_F(V, W)$.

Proof. Let $S \in \text{Hom}_F(V, W)$. Recall we constructed $K \otimes_F V$ as a quotient of $\mathbb{F}(K \times V)$. Define:

$$t : K \times V \rightarrow W \text{ by } (a, v) \mapsto aS(v)$$

as a map of sets. Theorem 5.2.1 tells us that t extends to a map $T \in \text{Hom}_K(\mathbb{F}(K \times V), W)$ such that $T((a, v)) = t((a, v))$. Since T is linear:

$$\begin{aligned} T\left(\sum_{i \in I} c_i(a_i, v_i)\right) &= \sum_{i \in I} T(c_i(a_i, v_i)) \\ &= \sum_{i \in I} c_i T((a_i, v_i)) \\ &= \sum_{i \in I} c_i t((a_i, v_i)) \\ &= \sum_{i \in I} c_i a_i S(v_i). \end{aligned}$$

We must check that T is the zero map when restricted to $\text{Rel}_K(K \times V)$. We have:

$$\begin{aligned} T((a+b, v) - (a, v) - (b, v)) &= T((a+b, v)) - T((a, v)) - T((b, v)) \\ &= (a+b)S(v) - aS(v) - bS(v) \\ &= 0_W. \end{aligned}$$

The rest of the relations are left as an exercise. Thus we have $T \in \text{Hom}_K(K \otimes_F V, W)$ defined by $T(\sum_{i \in I} c_i(a_i \otimes v_i)) = \sum_{i \in I} c_i a_i S(v_i)$. To see that the diagram commutes, observe that:

$$T(\iota_V(v)) = T(1 \otimes v) = 1 \cdot S(v) = S(v).$$

From Proposition 5.3.1, we saw that $K \otimes_F V$ is spanned by elements of the form $1 \otimes v$. Hence any linear map on $K \otimes_F V$ is determined by the image of these elements. Since $T(1 \otimes v) = S(v)$, we get T is uniquely determined by S .

The converse statement that for any $T \in \text{Hom}_K(K \otimes_F V, W)$ one has $T \circ \iota_V \in \text{Hom}_F(V, W)$ is left as an exercise. \square

Exercise 5.3.2. Complete the proof that T vanishes on all the relations.

Exercise 5.3.3. Given $T \in \text{Hom}_K(K \otimes_F V, W)$, show $T \circ \iota_V \in \text{Hom}_F(V, W)$.

Proposition 5.3.3. Let K/F be an extension of fields. Then $K \otimes_F F \cong K$ as K -vector spaces.

Proof. There is a natural inclusion map $i : F \rightarrow K$. Let $\iota : F \rightarrow K \otimes_F F$. By the universal property we obtain a unique K -linear map $T : K \otimes_F F \rightarrow K$ so that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\iota} & K \otimes_F F \\ & \searrow i & \downarrow T \\ & & K \end{array}$$

We see that $T(1 \otimes x) = i(x) = x$. Since T is K -linear this completely determines T because, for $\sum_{i \in I} a_i \otimes x_i \in K \otimes_F F$, we have:

$$\begin{aligned} T\left(\sum_{i \in I} a_i \otimes x_i\right) &= \sum_{i \in I} T(a_i \otimes x_i) \\ &= \sum_{i \in I} T(a_i(1 \otimes x_i)) \\ &= \sum_{i \in I} a_i T(1 \otimes x_i) \\ &= \sum_{i \in I} a_i x_i. \end{aligned}$$

If we show T has an inverse map, then we obtain an isomorphism. Let $S : K \rightarrow K \otimes_F F$ defined by $y \mapsto y \otimes 1$. Let $a \in K$, and $y_1, y_2 \in K$. Then:

$$S(y_1 + ay_2) = \dots$$

Hence $S \in \text{Hom}_K(K, K \otimes_F F)$. Since S, T are linear, it is enough to check that they are inverses with pure tensors. Observe that:

$$\begin{aligned} T(S(y)) &= T(y \otimes 1) = yT(1 \otimes 1) = y \\ S(T(a \otimes x)) &= S(aT(1 \otimes x)) = S(ax) = ax \otimes 1 = a \otimes x. \end{aligned}$$

Thus $T^{-1} = S$, and so $K \otimes_F F \cong K$ as K -vector spaces. \square

Example 5.3.1. From the previous section, we can now see that $\mathbf{R}_C = \mathbf{C} \otimes_{\mathbf{R}} \mathbf{R} \cong \mathbf{C}$.

Example 5.3.2. It is not always obvious that an element of $K \otimes_F F$ is nonzero. Take for example $\mathbf{Z}/2\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}$. We have that $1 \otimes 2 = 2 \otimes 1 = 0_{\mathbf{Z}/2\mathbf{Z}} \otimes 1 = 0_{\mathbf{Z}/2\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}}$.

Exercise 5.3.4. Show that $V_C \cong \mathbf{C} \otimes_{\mathbf{R}} V$.

Proposition 5.3.4. Let K/F be an extension of fields and V an F -vector space with $\dim_F V = n$. Then $K \otimes_F V \cong K^n$ as K -vector spaces.

Proof. We want a K -linear map $K \otimes_F V \rightarrow K^n$. Take $\mathcal{B} = \{v_1, \dots, v_n\}$ to be a basis for V and $\mathcal{E}_n = \{e_1, \dots, e_n\}$ the standard basis for K^n . Define a map $t : V \rightarrow K^n$ by $t(v_i) = e_i$. Since this map is defined on a basis, it extends to an F -linear map. So $t \in \text{Hom}_F(V, K^n)$. The universal property gives $T \in \text{Hom}_K(K \otimes_F V, K^n)$ so that $T(1 \otimes v_i) = e_i$. We will show that T has an inverse. Define $S \in \text{Hom}_K(K^n, K \otimes_F V)$ by $S(e_i) = 1 \otimes v_i$. These are clearly inverse maps, so $K \otimes_F V \cong K^n$. Moreover, since S is invertible and the e_i are a basis, $\{S(e_i)\}_{i=1}^n$ gives a basis of $K \otimes_F V$; i.e., $\{1 \otimes v_i\}_{i=1}^n$ is a basis. \square

Proposition 5.3.5. Let K/F be an extension of fields and V an F -vector space. Let $\mathcal{B} = \{v_i\}_{i \in I}$ be an F -basis of V . We have $\mathcal{B}_K = \{1 \otimes v_i\}_{i \in I}$ is a basis of $K \otimes_F V$.

Proof. We saw in Proposition 5.3.1 that \mathcal{B}_K spans $K \otimes_F V$. Suppose there exists a linear dependence $\sum_{i \in I} c_i(1 \otimes v_i) = 0_{K \otimes_F V}$. Given $(b, v) \in K \times V$, write $(b, \sum_{i \in I} a_i v_i)$ for some $a_i \in F$. Fix $i_0 \in I$ and define:

$$t_{i_0} : V \rightarrow K$$

by $t_{i_0}(v) = t_{i_0}(\sum_{i \in I} a_i v_i) = a_{i_0}$. One can check that $t_{i_0} \in \text{Hom}_F(V, K)$. The universal property extends this to $T_{i_0} \in \text{Hom}_K(K \otimes_F V, K)$ so that $T_{i_0}(1 \otimes v) = t_{i_0}(v) = a_{i_0}$. Recall that $\sum_{i \in I} c_i(1 \otimes v_i) = 0_{K \otimes_F V}$. Observe that:

$$\begin{aligned} 0_K &= T_{i_0}(0_{K \otimes_F V}) \\ &= T_{i_0}\left(\sum_{i \in I} c_i(1 \otimes v_i)\right) \\ &= \sum_{i \in I} c_i T_{i_0}(1 \otimes v_i) \\ &= \sum_{i \in I} c_i t_{i_0}(v_i) \\ &= c_{i_0}. \end{aligned}$$

Since $i_0 \in I$ was arbitrary, we have that $c_i = 0$ for all $i \in I$; i.e., \mathcal{B}_K is linearly independent. Thus \mathcal{B}_K is a basis of $K \otimes_F V$. \square

Theorem 5.3.6. Let K/F be an extension of fields and V, W both F -vector spaces. Given $T \in \text{Hom}_F(V, W)$, there is a unique map $T_K \in \text{Hom}_K(K \otimes_F V, K \otimes_F W)$ so that the following diagram commutes:

$$\begin{array}{ccc}
V & \xrightarrow{T} & W \\
\downarrow \iota_V & & \downarrow \iota_W \\
K \otimes_F V & \xrightarrow{T_K} & K \otimes_F W
\end{array}$$

Proof. Define $t : V \rightarrow K \otimes_F W$ by $t(v) = 1 \otimes T(v)$. It can be shown that $t \in \text{Hom}_F(V, K \otimes_F W)$. The universal property extends this to a unique map $T_K \in \text{Hom}_K(K \otimes_F V, K \otimes_F W)$ so that $t = T_K \circ \iota_V$. Let $v \in V$. We have that $\iota_W(T(v)) = 1 \otimes T(v) = t(v) = (T_K \circ \iota_V)(v)$, meaning the diagram commutes. \square

Exercise 5.3.5. Let V be an \mathbf{R} -vector space. We have $\mathbf{C} \otimes_{\mathbf{R}} V \cong V_{\mathbf{C}}$.

5.4 Tensor Products of Vector Spaces

Definition 5.4.1. Let U, V, W be F -vector spaces. If $t : V \times W \rightarrow U$ satisfies:

- (1) $t(v_1 + v_2, w) = t(v_1, w) + t(v_2, w)$;
- (2) $t(v, w_1 + w_2) = t(v, w_1) + t(v, w_2)$;
- (3) $ct(v, w) = t(cv, w) = t(v, cw)$;

we call t a *bilinear map*. The collection of bilinear maps is denoted $\text{Hom}_F(V, W; U)$. If $t \in \text{Hom}_F(V, V; F)$, then we say t is a *bilinear form*.

Example 5.4.1.

- 1. The standard dot product $\mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$ is a bilinear form.
- 2. The standard cross-product in $\mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is a bilinear map.

Definition 5.4.2. Let V, W be F -vector spaces and consider the Cartesian product $V \times W$. For $v_1, v_2, v \in V, w_1, w_2, w \in W$, and $c \in F$, let:

$$\begin{aligned}
\mathcal{A}_1 &= (v_1 + v_2, w) - (v_1, w) - (v_2, w) \\
\mathcal{A}_2 &= (v, w_1 + w_2) - (v, w_1) - (v, w_2) \\
\mathcal{A}_3 &= (cv, w) - (v, cw) \\
\mathcal{A}_4 &= c(v, w) - (cv, w)
\end{aligned}$$

and define $\text{Rel}_F(V \times W) = \text{span}_F\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$. The *tensor product* of V and W over F is:

$$V \otimes_F W = \mathbb{F}(V \times W) / \text{Rel}_F(V \times W).$$

For any arbitrary element $\sum_{\text{finite}} c_i \delta_{v_i, w_i} \in \mathbb{F}(V \times W)$, we denote the equivalent class $\sum_{\text{finite}} c_i \delta_{v_i, w_i} + \text{Rel}_F(V \times W)$ as:

$$\begin{aligned}
\sum_{\text{finite}} c_i (v_i \times w_i) &= \sum_{\text{finite}} c_i v_i (1 \otimes w_i) \\
&= \sum_{\text{finite}} v_i \otimes w_i,
\end{aligned}$$

where v_i is reassigned as $c_i v_i$. An element of the form $v \otimes w$ is referred to as a pure tensor. Both arbitrary elements of $V \otimes_F W$ and pure tensors admit the following properties:

- (1) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$ for all $v_1, v_2 \in V, w \in W$;
- (2) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$ for all $v \in V, w_1, w_2 \in W$;
- (3) $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$ for all $c \in F, v \in V, w \in W$.

Exercise 5.4.1. Define $\iota : V \times W \rightarrow V \otimes_F W$ by $(v, w) \mapsto v \otimes w$. Show that $\iota \in \text{Hom}_F(V, W; V \otimes_F W)$.

Theorem 5.4.1. Let U, V, W be F -vector spaces.

- (1) If $T \in \text{Hom}_F(V \otimes_F W, U)$, then $T \circ \iota \in \text{Hom}_F(V, W; U)$.
- (2) If $t \in \text{Hom}_F(V, W; U)$, then there exists a unique $T \in \text{Hom}_F(V \otimes_F W, U)$ so that $t = T \circ \iota$; i.e., the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\iota} & V \otimes_F W \\ & \searrow t & \downarrow T \\ & & U \end{array}$$

Proof. Let $T \in \text{Hom}_F(V \otimes_F W, U)$ and set $t = T \circ \iota$. Let $v_1, v_2 \in V, w \in W$, and $c \in F$. We have:

$$\begin{aligned} t(v_1 + cv_2, w) &= T(\iota(v_1 + cv_2, w)) \\ &= T((v_1 + cv_2) \otimes w) \\ &= T(v_1 \otimes w + c(v_2 \otimes w)) \\ &= T(v_1 \otimes w) + cT(v_2 \otimes w) \\ &= t(v_1, w) + ct(v_2, w). \end{aligned}$$

The same argument is true for the second variable. Thus $t \in \text{Hom}_F(V, W; U)$.

Now let $t \in \text{Hom}_F(V, W; U)$. This says that $t : V \times W \rightarrow U$. By the universal property of free vector spaces, t extends to a unique F -linear map $T : \mathbb{F}(V \times W) \rightarrow U$ that satisfies $t(v, w) = T(v, w)$. Taking the canonical projection $\pi : \mathbb{F}(V \times W) \rightarrow \mathbb{F}(V \times W)/\text{Rel}_F(V \times W)$, it remains to show that T vanishes on $\text{Rel}_F(V \times W)$. T does indeed vanish, and by diagram chasing we can show that $T(v, w) = t(v, w)$, establishing the proof. \square

Exercise 5.4.2. Show that $\text{Hom}_F(V, W; U) \cong \text{Hom}_F(V \otimes_F W, U)$ as F -vector spaces.

Corollary 5.4.2. Let U, V, W be F -vector spaces. We have:

- (1) $V \otimes_F W \cong W \otimes_F V$;
- (2) $(U \otimes_F V) \otimes_F W \cong U \otimes_F (V \otimes_F W)$.

Proof. (1) is left an exercise. (2) Fix $w \in W$. Define: $t : U \times V \rightarrow U \otimes_F (V \otimes_F W)$

\square

buncha stuff to rewrite

5.5 The Trace

Given $A \in \text{Mat}_n(F)$ with $A = (a_{ij})$, then $\text{tr}(A) = \sum_{i=1}^n a_{ii}$. But in Chapter 3 we showed that $A = [T]_{\mathcal{B}}$ for some basis \mathcal{B} . We should be able to define the trace on linear maps so that:

$$\text{tr}(T) = \text{tr}([T]_{\mathcal{B}_1}) = \text{tr}([T]_{\mathcal{B}_2})$$

for different bases \mathcal{B}_1 and \mathcal{B}_2 . Normally trace is defined as $\text{tr}(\cdot) : \text{Mat}_n(F) \rightarrow F$. We want to "convert this" somehow to $\text{tr}(\cdot) : \text{Hom}_F(V, V) \rightarrow F$. Working with $\text{Hom}_F(V, V)$ is rather tedious, so this will be our first obstacle to overcome.

Lemma 5.5.1. *Let V be a finite dimensional F -vector space. Then $V \otimes_F V^\vee \cong \text{Hom}_F(V, V)$.*

Proof. Define:

$$t : V \times V^\vee \rightarrow \text{Hom}_F(V, V)$$

by:

$$(v, \varphi) \mapsto (v' \mapsto \varphi(v') \cdot v).$$

Let $v_1, v_2 \in V$ and $c \in F$. We have:

$$\begin{aligned} t(v, \varphi)(v_1 + cv_2) &= \varphi(v_1 + cv_2)v \\ &= (\varphi(v_1) + c\varphi(v_2))v \\ &= \varphi(v_1)v + c\varphi(v_2)v \\ &= t(v, \varphi)(v_1) + ct(v, \varphi)(v_2). \end{aligned}$$

So for each $(v, \varphi) \in V \times V^\vee$, we have that $t(v, \varphi) \in \text{Hom}_F(V, V)$. We'd like to now show that:

$$\begin{aligned} t(v_1 + cv_2, \varphi) &= t(v_1, \varphi) + ct(v_2, \varphi) \\ t(v, \varphi_1 + c\varphi_2) &= t(v, \varphi_1) + ct(v, \varphi_2). \end{aligned}$$

Since these are both functions, we need to show that they are equal for all $v' \in V$; i.e.:

$$t(v_1 + cv_2, \varphi)(v') = t(v_1, \varphi)(v') + ct(v_2, \varphi)(v').$$

Observe that:

$$\begin{aligned} t(v_1 + cv_2, \varphi)(v') &= \varphi(v')(v_1 + cv_2) \\ &= t(v_1, \varphi)(v') + ct(v_2, \varphi)(v'). \end{aligned}$$

Showing t is linear in the second variable is left as an exercise. This gives a well-defined linear map $\mathcal{T} : V \otimes_F V^\vee \rightarrow \text{Hom}_F(V, V)$ defined by $v \otimes \varphi \mapsto (v' \mapsto \varphi(v')v)$. Since $\dim_F(V \otimes_F V^\vee) = \dim_F(\text{Hom}_F(V, V)) = n^2$, it is enough to show that \mathcal{T} is injective. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis

of V and $\mathcal{B}^\vee = \{v_1^\vee, \dots, v_n^\vee\}$ its dual basis. Suppose $\mathcal{T}(\sum_{i,j} a_{ij}(v_i \otimes v_j^\vee)) = 0_{\text{Hom}_F(V,V)}$. Take $v_m \in \mathcal{B}$. We have:

$$\begin{aligned} 0_V &= \mathcal{T}\left(\sum_{i,j} a_{ij}(v_i \otimes v_j^\vee)\right)(v_m) \\ &= \sum_{i,j} a_{ij}(\mathcal{T}(v_i \otimes v_j^\vee)(v_m)) \\ &= \sum_{i,j} a_{ij}v_j^\vee(v_m)v_i \\ &= \sum_i a_{im}v_i. \end{aligned}$$

Thus $a_{im} = 0$ for all i . Since m was arbitrary, this gives that $a_{ij} = 0$ for all i, j . Thus \mathcal{T} is injective, establishing $V \otimes_F V^\vee \cong \text{Hom}_F(V, V)$. \square

Proposition 5.5.2. *Let $C : \text{Hom}_F(V, V) \times \text{Hom}_F(V, V) \rightarrow \text{Hom}_F(V, V)$ be defined by the usual function composition. There exists a unique $\Psi : (V \otimes_F V^\vee) \times (V \otimes_F V^\vee) \rightarrow V \otimes_F V^\vee$ making the following diagram commute:*

$$\begin{array}{ccc} (V \otimes_F V^\vee) \times (V \otimes_F V^\vee) & \xrightarrow{\Psi} & V \otimes_F V^\vee \\ \downarrow \mathcal{T} \times \mathcal{T} & & \downarrow \mathcal{T} \\ \text{Hom}_F(V, V) \times \text{Hom}_F(V, V) & \xrightarrow{C} & \text{Hom}_F(V, V) \end{array}$$

Proof. Since we cannot define \mathcal{T}^{-1} , we will define $\Phi : (V \otimes_F V^\vee) \times (V \otimes_F V^\vee) \rightarrow V \otimes_F V^\vee$ by $(v \otimes \varphi, w \otimes \psi) \mapsto \varphi(w)v \otimes \psi$ and show that it makes the diagram commute. Given $x \in V$, from the following diagram chase:

$$\begin{array}{ccc} (v \otimes \varphi, w \otimes \psi) & & \\ \downarrow \mathcal{T} \times \mathcal{T} & & \\ (\mathcal{T}(v \otimes \varphi), \mathcal{T}(w \otimes \psi)) & \xrightarrow{C} & \mathcal{T}(v \otimes \varphi) \circ \mathcal{T}(w \otimes \psi), \end{array}$$

we can see that:

$$\begin{aligned} \mathcal{T}(v \otimes \varphi)(\mathcal{T}(w \otimes \psi)(x)) &= \mathcal{T}(v \otimes \varphi)(\psi(x)(w)) \\ &= \psi(x)\mathcal{T}(v \otimes \varphi)(w) \\ &= \psi(x)\varphi(w)v. \end{aligned}$$

Another diagram chase:

$$\begin{array}{ccc} (v \otimes \varphi, w \otimes \psi) & \xrightarrow{\Psi} & \varphi(w)v \otimes \psi \\ & & \downarrow \mathcal{T} \\ & & \mathcal{T}(\varphi(w)v \otimes \psi) \end{array}$$

gives:

$$\begin{aligned}\mathcal{T}(\varphi(w)v \otimes \psi)(x) &= \varphi(w)\mathcal{T}(v \otimes \psi)(x) \\ &= \varphi(w)\psi(x)v.\end{aligned}$$

Hence our diagram commutes. Thus $\Phi = \Psi$ by uniqueness. \square

Definition 5.5.1. Let $T \in \text{Hom}_F(V, V)$, $\mathcal{B} = \{v_1, \dots, v_n\}$ a basis of V , and $A = [T]_{\mathcal{B}}$. The trace of A is:

$$\begin{aligned}\text{Tr}(A) &= \sum_{i=1}^n v_i^\vee(T(v_i)) \\ &= \sum_{i=1}^n v_i^\vee\left(\sum_{j=1}^n a_{ij}v_j\right) \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}v_i^\vee(v_j)\right) \\ &= \sum_{i=1}^n a_{ii}.\end{aligned}$$

Definition 5.5.2. Let $\mathcal{T} : V \otimes_F V^\vee \rightarrow \text{Hom}_F(V, V)$ be defined by $v \otimes \varphi \mapsto \varphi(v')v$. Given $s : V \times V^\vee \rightarrow F$ defined by $(v, \varphi) \mapsto \varphi(v)$ and the induced map $S : V \otimes_F V^\vee \rightarrow F$ defined by $v \otimes \varphi \mapsto \varphi(v)$, the trace is $\text{tr} = S \circ \mathcal{T}^{-1}$.

$$\begin{array}{ccc}\text{Hom}_F(V, V) & \xrightarrow{\text{tr}} & F \\ \mathcal{T} \uparrow & \nearrow S & \\ V \otimes_F V & & \end{array}$$

Proposition 5.5.3. $\text{tr} = \text{Tr}$.

Proof. Define $\mathcal{T}(v_i \otimes v_j^\vee) = \mathcal{T}_{ij} \in \text{Hom}_F(V, V)$. Since \mathcal{T} is an isomorphism, $\{\mathcal{T}_{ij}\}_{i,j}$ is a basis of $\text{Hom}_F(V, V)$. Observe that:

$$\begin{aligned}\text{Tr}(\mathcal{T}_{kl}) &= \text{Tr}(\mathcal{T}(v_k \otimes v_l^\vee)) \\ &= \sum_{i=1}^n v_i^\vee(\mathcal{T}(v_k \otimes v_l^\vee)(v_i)) \\ &= \sum_{i=1}^n v_i^\vee(v_l^\vee(v_i)v_k) \\ &= \sum_{i=1}^n v_l^\vee(v_i)v_i^\vee(v_k) \\ &= v_l^\vee v_k\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(\mathcal{T}_{kl}) &= (S \circ \mathcal{T}^{-1})(\mathcal{T}_{kl}) \\
&= S(v_k \otimes v_l^\vee) \\
&= v_l^\vee(v_k)
\end{aligned}$$

Thus both of our definitions of trace are consistent. □

Corollary 5.5.4. *Let $A, B \in \mathrm{Mat}_n(F)$. We have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.*

Proof. Define:

$$\begin{aligned}
t_1 : \mathrm{Hom}_F(V, V) \times \mathrm{Hom}_F(V, V) &\rightarrow F \text{ by } (S, T) \mapsto \mathrm{Tr}(S \circ T) \\
t_2 : \mathrm{Hom}_F(V, V) \times \mathrm{Hom}_F(V, V) &\rightarrow F \text{ by } (S, T) \mapsto \mathrm{Tr}(T \circ S).
\end{aligned}$$

One can check that these are both bilinear. This induces:

$$\begin{aligned}
\tilde{t}_1 : \mathrm{Hom}_F(V, V) \otimes_F \mathrm{Hom}_F(V, V) &\rightarrow F \text{ by } S \otimes T \mapsto \mathrm{Tr}(S \circ T) \\
\tilde{t}_2 : \mathrm{Hom}_F(V, V) \otimes_F \mathrm{Hom}_F(V, V) &\rightarrow F \text{ by } S \otimes T \mapsto \mathrm{Tr}(T \circ S).
\end{aligned}$$

Let □

6

Tensor Algebras, Exterior Algebras, and the Determinant

6.1 Tensor Algebras

6.2 Exterior Product

Definition 6.2.1. Let $\{V_i\}_{i=1}^n, W$ be F -vector spaces. If $t : V_1 \times \dots \times V_n \rightarrow W$ is linear in each component:

$$t(v_1, \dots, v_{j-1}, v_j + c\tilde{v}_j, \dots, v_n) = t(v_1, \dots, v_n) + ct(v_1, \dots, \tilde{v}_j, \dots, v_n).$$

The collection of multilinear maps is denoted $\text{Hom}_F(V_1, \dots, V_n; W)$.

Theorem 6.2.1. Let V_1, \dots, V_n, W be F -vector spaces. Define $\iota : V_1 \times \dots \times V_n \rightarrow V_1 \otimes_F \dots \otimes_F V_n$ by $\iota(v_1, \dots, v_n) = v_1 \otimes \dots \otimes v_n$.

(1) Given $T \in \text{Hom}_F(V_1 \otimes_F \dots \otimes_F V_n, W)$, then $T \circ \iota \in \text{Hom}_F(V_1, \dots, V_n; W)$.