Math 395-Real Analysis II

Problem Set 6-Compactness

Problem 1. Show that a discrete metric space is compact if and only if it is finite.

Problem 2. Let X be a metric space and suppose $Y \subseteq X$. If $K \subseteq Y$, show that K is compact in Y (with the relative topology) if and only if K is compact in X.

Problem 3. Let X be a metric space. Let $(x_n)_n$ is a sequence in X which converges to a point $x_0 \in X$. Show that $\{x_0, x_1, x_2, \dots\}$ is compact.

Problem 4. Let (X, d) be a metric space. If $C, K \subseteq X$ we define

$$\operatorname{dist}(C, K) := \inf_{x \in C,\ y \in K} d(x, y).$$

(i) If K is compact and C is closed, show that

$$K \cap C = \emptyset \iff \operatorname{dist}(C, K) > 0.$$

Can we remove the requirement that K is compact and only require it to be closed?

(ii) If both K and C are compact show that there is $x \in C$ and $y \in K$ with $\operatorname{dist}(C, K) = d(x, y)$.

Problem 5. Let V be a finite-dimensional normed space. Show that the unit ball

$$B := \big\{v \in V \mid \|v\| \leq 1\big\}$$

is compact.

Problem 6. Prove that the unit ball in $\mathrm{C}([0, 1])$ is not compact.

Problem 7. Let V be a normed space, and let $K, L \subseteq V$ be compact. Show that

$$K + L := \big\{x + y \mid x \in K, y \in L\big\}$$

is also compact.

Problem 8. Let $(f_n : [0, 1] \to \mathbb{R})_{n \geq 1}$ be a sequence of differentiable functions with

$$\sup_{n \geq 1} \|f_n\|_{\mathrm{u}} < \infty, \quad \text{and} \quad \sup_{n \geq 1} \|f_n'\|_{\mathrm{u}} < \infty.$$

Show that there is a subsequence $(f_{n_k})_k$ that converges uniformly to a continuous function $f : [0, 1] \to \mathbb{R}$.

Problem 9. Let $(X_n, d_n)_{n \geq 1}$ be a sequence of compact metric spaces. Show that the product $\prod_{n \geq 1} X_n$ with the product metric is also compact.

Problem 10. Let (X, d) be a compact metric space, and let $\mathcal{V}$ be an open cover of X. Show that there is a number $L(\mathcal{V})$ (the Lebesgue number of $\mathcal{V}$) satisfying: given any nonempty $E \subseteq X$ with $\operatorname{diam}(S) < L(\mathcal{V})$, there exists $V \in \mathcal{V}$ satisfying $E \subseteq V$.