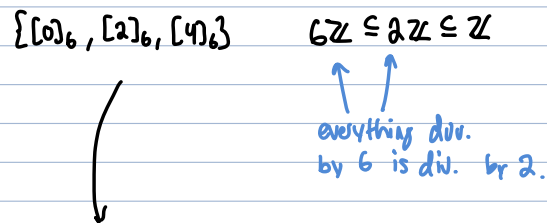


Consider the group $\mathbb{Z}/6\mathbb{Z}$.
 Subgroups: $\{[0]_6\}$, $\langle [0]_6 \rangle = \mathbb{Z}/6\mathbb{Z}$.
 if $m, 6$ relatively prime,
 subgroup generates entire group.

$$\begin{aligned}\langle [2]_6 \rangle &= \{[0]_6, [2]_6, [4]_6\} \\ \langle [3]_6 \rangle &= \{[0]_6, [3]_6\}\end{aligned}$$

$\mathbb{Z}/6\mathbb{Z} \dots \rightarrow G/N$ where $G = \mathbb{Z}$, $N = 6\mathbb{Z}$

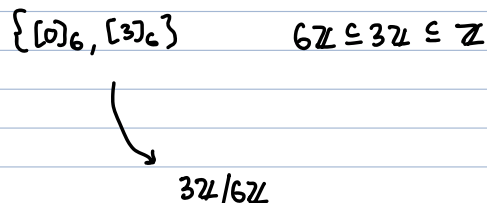
$$\{[0]_6, [2]_6, [4]_6\} \quad 6\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$$



 everything div.
 by 6 is div. by 2.

$$\begin{aligned}2\mathbb{Z}/6\mathbb{Z} &= \{2m + 6\mathbb{Z} : m \in \mathbb{Z}\} \\ &= \{[0]_6, [2]_6, [4]_6\}\end{aligned}$$

$$\{[0]_6, [3]_6\} \quad 6\mathbb{Z} \subseteq 3\mathbb{Z} \subseteq \mathbb{Z}$$



 $3\mathbb{Z}/6\mathbb{Z}$

If we want subgroups of G/N , we should look at subgroups H of G so that $N \subseteq H \subseteq G$ and then form H/N .

$$\begin{aligned}\langle [0]_6 \rangle &= 6\mathbb{Z}/6\mathbb{Z} \\ \langle [3]_6 \rangle &= 3\mathbb{Z}/6\mathbb{Z} \\ \langle [2]_6 \rangle &= 2\mathbb{Z}/6\mathbb{Z} \\ \langle [1]_6 \rangle &= \mathbb{Z}/6\mathbb{Z}\end{aligned}$$

Proposition: Let $N \trianglelefteq G$, and $N \leq H \leq G$, then H/N is a subgroup of G/N .

proof: HW

Theorem: (3rd Isomorphism Thm) Let K, N be normal subgroups of G with $N \leq K \leq G$.

Then $K/N \trianglelefteq G/N$ and

$$(G/N)/(K/N) \cong G/K.$$

proof: Let $\varphi: G/N \rightarrow G/K$
 $gN \rightarrow gK$

By def:

$$g_2^{-1}g_1 \in N \\ g_2^{-1}g_1 = n$$

Suppose $g_1N = g_2N$. Note $g_1 = g_2n$ for some $n \in N$
 Observe that $\varphi(g_1N) = g_1K$
 $= g_2nK$
 $= g_2K$ since $N \leq K$
 $= \varphi(g_2N)$

Thus φ is well-defined.

Let $g_1N, g_2N \in G/N$. We have

$$\begin{aligned} \varphi(g_1N g_2N) &= \varphi(g_1 g_2 N) \\ &= g_1 g_2 K \\ &= g_1 K g_2 K \\ &= \varphi(g_1N) \varphi(g_2N) \end{aligned}$$

Let $gK \in G/K$. Then $\varphi(gN) = gK$, φ is surjective.

It only remains to show $\ker \varphi = K/N$.

Let $kN \in K/N$. The $\varphi(kN) = kK = K$,
 so $kN \in \ker \varphi$. Hence $K/N \subseteq \ker \varphi$.

Let $gN \in \ker \varphi$, i.e., $\varphi(gN) = K$.
 So $gK = K$, so $g \in K$ since $g = k$, for some $k \in K$.
 Hence $\ker \varphi \subseteq K/N$ and we have equality.

So K/N is normal b/c it is the ker of a homom. 1st iso. thm. gives

$$(G/N)/(K/N) \cong G/K.$$

Corollary: Let $N \trianglelefteq G$ and K any subgroup of G w/ $N \leq K$.
Then $K \trianglelefteq G$ iff $K/N \trianglelefteq G/N$

Proof: If $K \trianglelefteq G$, 3rd Iso Thm gives $K/N \trianglelefteq G/N$.

Assume $K/N \trianglelefteq G/N$. W.T.S. $gKg^{-1} = K \quad \forall g \in G$.

Let $g \in G, k \in K$. We have

$$\begin{aligned} gkg^{-1}N &= gNkNg^{-1}N \\ &= gNkN(gN)^{-1} \in K/N \end{aligned}$$

$gNg^{-1} = N$
 $gng^{-1} \in N$

b/c $K/N \trianglelefteq G/N$

if $gkg^{-1}N \in K/N$, then $gkg^{-1} \in K$

Thus $gkg^{-1} \in K$. Hence $gKg^{-1} \subseteq K$. elm of K (so by def $k \in gKg^{-1}$)

Let $k \in K$. Then $k = g(g^{-1}kg)g^{-1} \in g^{-1}Kg \subseteq K$ where $k \in K$
so $K \subseteq gkg^{-1}$

$g^{-1}Kg \subseteq K$
so $g^{-1}kg \in K$

$g^{-1}Kg \subseteq K$
 \downarrow
 $\in gKg^{-1}$ equal to

Thus $K \subseteq gKg^{-1}$, so $gKg^{-1} = K \quad \forall g \in G$ i.e., $K \trianglelefteq G$. □

Theorem: Let T be a subgroup of G/N . Then $T = H/N$ for some subgroup $H \leq G$ w/ $N \leq H \leq G$.

proof: Define $H = \{g \in G : gN \in T\}$.

We show H is a subgroup.

Since $T \leq G/N$, it contains eN . Thus $e \in H$.

Let $g_1, g_2 \in H$; i.e., $g_1N, g_2N \in T$.

Since $T \leq G/N$, $g_2^{-1}N \in T$.

and $g_1 N g_2^{-1} N \in T$.

So $g_1 g_2^{-1} N \in T$

So $g_1 g_2^{-1} \in H$. So H is a subgroup.

Let $n \in N$. We have $nN = N \in T$

b/c T is a subgroup ($N = eG/N$)

Thus, $n \in H$. Hence $N \subseteq H$.

subgroup so has to contain identity of G/N , i.e. $e_{G/N} = N$.

We have $H/N = \{hN : h \in H\}$

$= T$

In Groups: List all the subgroups

$(\mathbb{Z}/12\mathbb{Z})/H$ where $H = \langle [6] \rangle$

$\langle [6]_{12} \rangle = 6\mathbb{Z}/12\mathbb{Z}$

$(\mathbb{Z}/12\mathbb{Z})/(6\mathbb{Z}/12\mathbb{Z})$

Theorem: Let T be a subgroup of G/N . Then $T = H/N$ for some subgroup $H \subseteq G$ w/ $N \subseteq H \subseteq G$.

$(\mathbb{Z}/12\mathbb{Z})/(6\mathbb{Z}/12\mathbb{Z})$

$6\mathbb{Z}/12\mathbb{Z} \subseteq H \subseteq \mathbb{Z}/12\mathbb{Z}$

$\{[0]_{12}, [6]_{12}\}$

$\mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/3\mathbb{Z}$

$(\mathbb{Z}/12\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}) \subseteq (\mathbb{Z}/12\mathbb{Z})/(6\mathbb{Z}/12\mathbb{Z})$
 $(\mathbb{Z}/2\mathbb{Z})$

$$H = \langle [6]_{12} \rangle \subseteq N \subseteq \mathbb{Z}/12\mathbb{Z}$$

$$= \{[0]_{12}, [6]_{12}\}$$

$$\downarrow$$

$$\langle [2]_{12} \rangle$$

$$\langle [3]_{12} \rangle$$

$$\langle [2]_{12} \rangle / [6]_{12} \quad \text{and} \quad \langle [3]_{12} \rangle / [6]_{12}$$

Example: Compute all the homomorphic images of S_3 .

Let $\varphi: S_3 \rightarrow G$ for G some group.

$$\varphi(S_3) = \{e_G\}$$

$$\varphi: S_3 \rightarrow \text{im}(\varphi) \subseteq G$$

Map always surjective onto image

$$S_3 / \ker \varphi \cong \text{im} \varphi.$$

No matter what φ is, $\ker \varphi \triangleq S_3$.

$$\ker \varphi = \{e_G\}, S_3, \langle (123) \rangle.$$

only possible
kernels.

no other normal subgroups.

$$S_3 / \langle (123) \rangle = \{ \langle (123) \rangle, (12) \langle (123) \rangle \} \cong \mathbb{Z}/2\mathbb{Z}.$$