<u>Definition</u>: Let G be a group and X a set. We say G <u>acts</u> on X if there is a map $\Psi: G \times X \longrightarrow X$, which we denote $\Psi(g,x) = g.x$, satisfying:

$$\Psi(e_G, x) = e_G . x = x$$
$$\Psi(g_1, \Psi(g_2, x)) = g_1 . (g . x) = g_1 g_2 . x = \Psi(g_1 g_2, x)$$

<u>Example</u>: Let $G = S_n$ and $X = \{1, 2, \dots, n\}$. Let $\sigma \in S_n$. We know $\sigma(j) \in X$ for any $j \in X$. We have a map $S_n \times X \longrightarrow X$ defined by $(\sigma, j) \mapsto \sigma(j)$. Observe that $e_{S_n}(j) = j$ and $(\sigma\tau)(j) = \sigma(\tau(j))$ for all $\sigma, \tau \in S_n$ and $j \in X$.

<u>Example</u>: The group D_3 acts on the set X = equilateral triangle.

<u>Example</u>: Let $G = GL_n(\mathbb{R})$. This is a group under matrix multiplication. Let $X = \mathbb{R}^n$. Define $\Psi: G \times X \longrightarrow X$ by $\Psi(M, x) = Mx$. For $n = 2$, we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \in \mathbb{R}^2 .$$

Observe that $I_n x = x$ and $M_1(M_2 x) = (M_1 M_2) x$.

<u>Example</u>: Let $H \trianglelefteq G$. We have that H acts on G by conjugation; i.e., $\Psi: H \times G \longrightarrow H$ defined by $\Psi(h, g) = h.g = hgh^{-1}$. Observe that:

$$e_G . g = e_G g e_G^{-1} = g \quad \forall g \in G \qquad e_G \in H \text{ since } H \trianglelefteq G.$$
$$\begin{aligned} (h_1 h_2) \cdot g &= (h_1 h_2) g (h_1 h_2)^{-1} \\ &= h_1 h_2 g h_2^{-1} h_1^{-1} \\ &= h_1 (h_2 . g) h_1^{-1} \\ &= h_1 . (h_2 . g). \end{aligned}$$

<u>Example</u>: Let G be a group and $X = \{ f : G \longrightarrow \mathbb{C} \}$ be the set of functions from G to $\mathbb{C}$. G acts on X via $(g.f)(h) = f(g^{-1}h)$.

$$(e_G \cdot f)(h) = f(e_G^{-1} h) = f(h)$$
$$\begin{aligned} ((g_1 g_2) . f)(h) &= f((g_1 g_2)^{-1} h) \\ &= f(g_2^{-1} g_1^{-1} h) \\ &= f(g_2^{-1}(g_1^{-1} h)) \\ &= (g_2 \cdot f)(g_1^{-1} h) \\ &= g_1 . (g_2 . f)(h). \end{aligned}$$

Definition: Let G act on a set X. Let $x \in X$. The orbit of x is:

$$\mathrm{Orb}(x) = \{g.x : g \in G\}.$$

Example: Let $H \leq G$ and set $X = G/H = \{gH : g \in G\}$. Since we are not requiring H to be normal, this is just a set. Define $g_1.(gH) = g_1 gH$. Observe that:

$$\begin{aligned} e_G . gH &= e_G gH = gH \\ (g_1 g_2).gH &= g_1 g_2 g H \\ &= g_1.(g_2.gH). \end{aligned}$$

Consider $G = S_3$ and $H = \langle (12) \rangle$. Then $X = G/H = \{H, (123)H, (13)H\}$
Let $x = (123)H$. We have that:

$$\begin{aligned} \mathrm{Orb}(x) &= \mathrm{Orb}((123)H) \\ &= \{e_G \cdot (123)H, (12).(123)H, (13).(123)H \ldots\} \\ &= \{(123)H, (23)H, (12)H, (13)H, (132)H, H\} \\ &= \{H, (132)H, (13)H\} \\ &= G/H. \end{aligned}$$

Example: G acts on itself by $g.x = gxg^{-1}$; i.e., $\Psi : G \times G \longrightarrow G$ is defined by $\Psi(g,x) = g.x = gxg^{-1}$ $\forall g, x \in G$. Observe that:

$$\begin{aligned} e_G . x &= e_G \cdot x . e_G^{-1} = x \\ (g_1 g_2).x &= \ldots \\ &= g_1.(g_2.x). \end{aligned}$$

Let $g = D_3$. Then:

$$\begin{aligned} \mathrm{Orb}(r) &= \{g.r : g \in D_3\} \\ &= \{id.r, r.r, r^2.r, \ldots\} \\ &= \{r, r^2\}. \end{aligned}$$

$$\begin{aligned} \sigma.r &= \sigma r \sigma^{-1} \\ &= \sigma r r r^{-1} \sigma \\ &= \sigma r \sigma \\ &= \sigma \sigma r^2 = r^2 \end{aligned}$$

Lemma: Let G act on X. There is an equivalence relation on X given by $x \sim y$ iff $y \in \mathrm{Orb}(x)$. Under this equivalence relation, the equivalence classes are the orbits.

Proof. See draft. Just showing symmetry, reflexivity, and transitivity.