$$g(x) = \sum_{n=0}^{k} a_n x^n = a_0 + \sum_{n=1}^{k} a_n x^n \ldots$$

Isomorphism Theorems:

Recall: $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$
$a \longmapsto [a]_n$.

This is clearly a surjective homomorphism.
$\operatorname{Ker} \pi = n\mathbb{Z}$.

We have $\mathbb{Z}/\operatorname{Ker}\pi \cong \mathbb{Z}/n\mathbb{Z}$

Consider $\varphi : G \twoheadrightarrow H$ a homomorphism.
Do we have something like $G/\ker\varphi \cong H$.

Theorem (1st Isomorphism Thm): Let $\varphi : G \longrightarrow H$ be a homomorphism. Then $G/\operatorname{Ker}\varphi \cong \operatorname{im}\varphi$.

Proof: Let $K = \ker \varphi$.
Define $\Phi : G/K \longrightarrow \operatorname{im}\varphi$
$gK \longmapsto \varphi(g)$

Suppose $g_1 K = g_2 K$. So we have $g_1 = g_2 k$ for some $k \in K$.
We have
$$\Phi(g_1K) = \varphi(g_1) = \varphi(g_2 k) = \varphi(g_2)\varphi(k)$$
$$= \varphi(g_2) e_H \quad \text{b/c } k \in K = \ker\varphi$$
$$= \varphi(g_2)$$
$$= \Phi(g_2 K)$$

Let $g_1K, g_2K \in G/K$.

Observe
$$\Phi(g_1K\, g_2K) = \Phi(g_1 g_2 K)$$
$$= \varphi(g_1 g_2)$$
$$= \varphi(g_1)\varphi(g_2)$$
$$= \Phi(g_1K)\Phi(g_2K).$$
So Φ is a homom.

Let $h \in \operatorname{im}(\varphi)$. There exists $g \in G$ w/ $\varphi(g) = h$ by definition of $\operatorname{im}(\varphi)$. This gives us $\Phi(gK) = \varphi(g) = h$.
Thus Φ is surjective.

Let $gK \in \operatorname{Ker}\Phi$. This gives that $\Phi(gK) = e_H$ and $\Phi(gK) = g$. So $\varphi(g) = e_H$; i.e., $g \in \operatorname{Ker}\varphi = K$. Thus $gK = K = e_{G/K}$. Hence $\operatorname{Ker}\Phi = \{e_{G/K}\}$, so Φ is injective. $\square$

Example: Let $G = \mathbb{R} \times \mathbb{R}$.
Define $\varphi: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$
$(a,b) \longmapsto a$

Let $(a,b), (c,d) \in \mathbb{R} \times \mathbb{R}$. We have $\varphi((a,b)+(c,d)) = \ldots = \varphi((a,b)) + \varphi((c,d))$.
Let $a \in \mathbb{R}$, we have $\varphi((a,1)) = a$, so φ is surj.
Note that $(a,b) \in \operatorname{Ker}\varphi$ iff $\varphi(a,b) = 0$; i.e., $a = 0$.
Thus $\operatorname{Ker}\varphi = \{(0,b) : b \in \mathbb{R}\} = \{0\} \times \mathbb{R}$
1st iso. Thm. gives $(\mathbb{R} \times \mathbb{R})/(\{0\} \times \mathbb{R}) \cong \mathbb{R}$

Example: $G = 2\mathbb{Z}$ and $K = 6\mathbb{Z} \trianglelefteq 2\mathbb{Z}$. What is $2\mathbb{Z}/6\mathbb{Z}$?

$$2\mathbb{Z}/6\mathbb{Z} = \{0 + 6\mathbb{Z}, 2 + 6\mathbb{Z}, 4 + 6\mathbb{Z}\}$$

$a + 6\mathbb{Z}$ where $a \in 2\mathbb{Z}$

Let $a \in 2\mathbb{Z}$. Write $a = 2m$, for some $m \in \mathbb{Z}$.
Write $m = 3q + r$ w/ $0 \le r < 2$.

Define $\varphi: 2\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$
$a = 2(3q+r) \longmapsto [r]_3$.

We have $0 \longmapsto [0]_3$
$2 \longmapsto [1]_3$
$4 \longmapsto [2]_3$

$0 = 0(0) = 2(3q+r)$
$2(3 \cdot 0 + 0) = 0$

$2 = 2(1) = 2(3q + r)$
$= 2(3 \cdot 0 + 1) = 2$

$4 = 2(2) = 2(3q + r)$
$= 2(3(0) + 2) = 4$

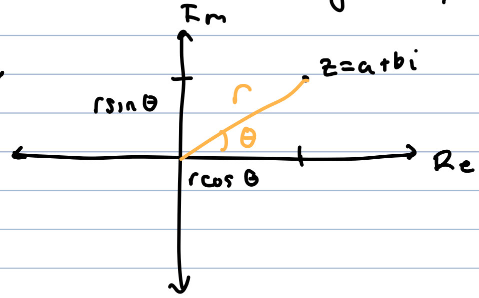
Let $a = 2(3q_1 + r_1)$, $b = 2(3q_2 + r_2)$
$\varphi(a+b) = \ldots = \varphi(a) + \varphi(b)$.

We have $a \in \operatorname{Ker}\varphi$ iff $a = 2(3q + 0)$
iff $a = 6q$
iff $a \in 6\mathbb{Z}$.

Thus $\operatorname{Ker}\varphi = 6\mathbb{Z}$.
Thus $2\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$.

<u>Example</u>: Let $S^1 \subseteq \mathbb{C}$ be given by $\{z \in \mathbb{C} : |z| = 1\}$

For $z \in \mathbb{C}$,



$$\begin{aligned} z &= r\cos\theta + i r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= r(\cos 2\pi t + i \sin 2\pi t) \\ &= r e^{2\pi i t} \end{aligned}$$

So $S^1 = \{e^{2\pi i t} : 0 \le t < 1\} \subseteq \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

This is a group under mult.

Define a map $\varphi : \mathbb{R} \to S^1$
$t \longmapsto e^{2\pi i t}$.

This is certainly surjective.

Let $t_1, t_2 \in \mathbb{R}$. Then $\varphi(t_1 + t_2) = e^{2\pi i (t_1 + t_2)} = e^{2\pi i t_1} e^{2\pi i t_2}$
$= \varphi(t_1)\varphi(t_2)$

What is the $\ker \varphi$? If $t \in \ker \varphi$ then $e^{2\pi i t} = 1$
$\cos(2\pi t) + i \sin(2\pi t) = 1$
$\cos(2\pi t) = 1$ $\Longrightarrow t \in \mathbb{Z}$.
$\sin(2\pi t) = 0$

Thus $\mathbb{R}/\mathbb{Z} \cong S^1$.

<u>In groups</u>: Let $GL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_2(\mathbb{R}) : ad - bc \neq 0 \right\}$
and

$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) : ab - bc = 1 \right\}$.

Show $SL(2,\mathbb{R}) \trianglelefteq GL(2,\mathbb{R})$ and determine a familiar group $GL(2,\mathbb{R}) / SL(2,\mathbb{R})$ is isomorphic to.

Define $\varphi : GL(2,\mathbb{R}) \longrightarrow \mathbb{R}^\times$
$g \longmapsto \det(g)$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto ad - bc$

Let $g_1, g_2 \in GL(2,\mathbb{R})$. Then $\varphi(g_1 g_2) = \det(g_1 g_2) = \det(g_1)\det(g_2)$
$= \varphi(g_1)\varphi(g_2)$

Let $a \in \mathbb{R}^\times$. Then $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in GL(2,\mathbb{R})$ and $\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = a \cdot 1 - 0 = a$.
Thus φ is surj.

$$\ker \varphi = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) : \varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 1 \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) : ad - bc = 1 \right\} = SL(2,\mathbb{R}).$$

Since $\ker \varphi = SL(2,\mathbb{R})$, it is a normal subgroup.
1st iso. thm. gives $GL(2,\mathbb{R}) / SL(2,\mathbb{R}) \cong \mathbb{R}^\times$.

$[-1]_4 = [3]_4$

$[-2]_4 = [2]_4$

$[-3]_4 = [1]_4$

$[-4]_4 = [0]_4$

$[-5]_4 = [$