Abstract

We will survey many of the important results related to sequences and series of functions. The culmination of this document will be proving the Weierstrass M-test and Cauchy-Hadamard theorem.

Sequences of Functions

Definition 1. Let Ω be a set, (X, d) a metric space, and $(f_n)_n$ a sequence of functions in X^Ω.

(1) $(f_n)_n$ converges *pointwise* to $f \in X^\Omega$ if:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N_{x,\epsilon} \in \mathbf{N}) : (\forall n \in \mathbf{N})(n \geqslant N \implies d(f_n(x), f(x)) < \epsilon).$$

(2) $(f_n)_n$ converges *uniformly* to $f \in X^\Omega$ if:

$$\begin{aligned}(\forall \epsilon > 0)(\exists N_\epsilon \in \mathbf{N}) : (\forall n \in \mathbf{N})(\forall x \in \Omega)\big(n \geqslant N &\implies d(f_n(x), f(x)) < \epsilon\big)\\ \equiv (\forall n \in \mathbf{N})\big(n \geqslant N &\implies \sup_{x\in\Omega} d(f_n(x), f(x)) < \epsilon\big)\\ \equiv (\forall n \in \mathbf{N})\big(n \geqslant N &\implies D_u(f_n, f) < \epsilon\big).\end{aligned}$$

Example 1. Let $(f_n)_n$ be a sequence in $\mathbf{R}^{[0,1]}$ defined by $f_n(x) = x^n$ for all $n \in \mathbf{N}$. If $x \in [0, 1)$, then $(f_n(x))_n \to 0$. If $x = 1$, then $(f_n(x))_n \to 1$. Thus $(f_n)_n \to \mathbf{1}_1$ pointwise.

Example 2. Let $(f_n)_n$ be a sequence in $\mathbf{R}^\mathbf{R}$ defined by $f_n(x) = \frac{nx}{1+n^2x^2}$. If $x = 0$, then $(f_n(x))_n \to 0$. If $x \neq 0$, observe that:

$$\begin{aligned}|f_n(x)| &= \left|\frac{nx}{1+n^2x^2}\right|\\ &= \frac{n|x|}{1+n^2x^2}\\ &\leqslant \frac{|x|}{nx^2}\\ &= \frac{1}{n|x|}.\end{aligned}$$

Since x is fixed, $(f_n(x))_n \to 0$. Thus $(f_n)_n \to 0_{\mathbf{R}^\mathbf{R}}$ pointwise.

Example 3. Let $(h_n)_n$ be a sequence in $\mathbf{R}^{[0,\infty)}$ defined by $h_n(x) = x^{\frac{1}{n}}$. If $x > 0$, then $(h_n(x))_n \to 1$. If $x = 0$, then $(h_n(x))_n \to 0$. Thus $(h_n)_n \to \mathbf{1}_{(0,\infty)}$ pointwise.

Definition 2. Let Ω be a set and (X, d) a metric space.

(1) The set of all bounded functions from Ω to X is denoted $\mathrm{Bd}(\Omega, X)$.

(2) The *uniform metric* between two bounded functions $f, g \in \mathrm{Bd}(\Omega, X)$ is defined by $D_u(f, g) = \sup_{x\in\Omega} d(f(x), g(x))$.

Proposition 1. *Let Ω be a set, (X, d) a metric space, and $(f_n)_n$ a sequence in X^Ω. The following are equivalent:*

(1) The sequence $(f_n)_n$ converges uniformly to f;

(2) The sequence $(f_n)_n$ converges to f in $(\mathrm{Bd}(\Omega, X), D_u)$;

(3) The sequence $(D_u(f_n, f))_n$ converges to 0.

Example 4. Let $(f_n)_n$ be a sequence in $\mathbf{R}^\mathbf{R}$ defined by $f_n = \mathbf{1}_{[n,n+1]}$. Claim: $(f_n)_n \to 0_{\mathbf{R}^\mathbf{R}}$. Let $x \in \mathbf{R}$ and $\epsilon > 0$. Find N large so $N > x$. If $n \geqslant N$, then $|f_n(x) - 0_{\mathbf{R}^\mathbf{R}}(x)| = |f_n(x)| = |\mathbf{1}_{[n,n+1]}(x)| = 0$. Thus $(f_n)_n \to 0_{\mathbf{R}^\mathbf{R}}$ pointwise.

Note that each f_n is bounded, and furthermore:

$$\begin{aligned} D_u(f_n, f) &= \sup_{x\in\mathbf{R}} |f_n(x) - 0_{\mathbf{R}^\mathbf{R}}(x)| \\ &= \sup_{x\in\mathbf{R}} |f_n(x)| \\ &= 1. \end{aligned}$$

Thus $(f_n)_n$ does *not* converge uniformly to $0_{\mathbf{R}^\mathbf{R}}$.

Theorem 2. *Let (X, d) be a metric space. Suppose $(f_n)_n$ is a sequence in $C(\Omega, X)$ which converges uniformly to $f : \Omega \to X$. Then $f \in C(\Omega, X)$.*

Proof. □

Theorem 3. *Interchange of derivative and limit*

Theorem 4. *Let (X, d) be compact. Suppose $(f_n)_n$ is a monotonically decreasing sequence in $C(X, \mathbf{R})$ which converges pointwise to 0. Then $(f_n)_n$ converges uniformly to 0.*

Proof. Let $\epsilon > 0$. Since $(f_n)_n$ converges pointwise to 0, for each $x \in X$ there exists $N_x \in \mathbf{N}$ such that $n \geqslant N_x$ implies $f_n(x) < \frac{\epsilon}{2}$. Because f_{N_x} is continuous at x, there exists $\delta_x > 0$ such that, for every $z \in X$, $d(x, z) < \delta_x$ implies $|f_{N_x}(x) - f_{N_x}(z)| < \frac{\epsilon}{2}$. The collection $\{U(x, \delta_x)\}_{x\in X}$ covers X, so by compactness there is a finite set $F \subseteq X$ with $X = \bigcup_{x\in F} U(x, \delta_x)$. Set $N = \max_{x\in F} N_x$. Let $z \in X$ be arbitrary and locate $x \in F$ such that $z \in U(x, \delta_x)$. Notice that our choice of N does not depend on z. For $n \geqslant N$:

$$\begin{aligned} f_n(z) &\leqslant f_{N_x}(z) \\ &= f_{N_x}(z) - f_{N_x}(x) + f_{N_x}(x) \\ &\leqslant |f_{N_x}(z) - f_{N_x}(x)| + f_{N_x}(x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus for $n \geqslant N$, we have $\|f_n\|_u \leqslant \epsilon$. □

Theorem 5 (Dini's Theorem)**.** *Let (X, d) be compact. Suppose $(f_n)_n$ is a monotone sequence in $C(X, \mathbf{R})$ which converges pointwise to f. Then $(f_n)_n$ converges uniformly to f.*

Proof. If $(f_n)_n$ is decreasing, apply Theorem 4 to $f_n - f$. If $(f_n)_n$ is increasing, apply Theorem 4 to $f - f_n$. $\square$