Problem 1.

(1) Assume that $\sum_{n=1}^{\infty} a_n$ is convergent and $(b_n)_n$ is a bounded sequence. Does $\sum_{n=1}^{\infty} a_n b_n$ converge? Prove or provide a counterexample.

(2) Assume that $\sum_{n=1}^{\infty} |a_n|$ is convergent and $(b_n)_n$ is a bounded sequence. Does $\sum_{n=1}^{\infty} a_n b_n$ converge? Prove or provide a counter example.

Proof.

Problem 2. Show that the least upper bound property of the real numbers implies the Cauchy completeness property, that is, show that the property that every bounded set of real numbers has a least upper bound implies that every Cauchy sequence of real numbers converges in $\mathbf{R}$.

Proof.

Problem 3. Let $(x_n)_n$ be a sequence in $\mathbf{R}$ with $|x_n - x_{n+1}| < \frac{1}{n}$ for all $n \in \mathbf{N}$

(1) If $(x_n)_n$ is bounded, must $(x_n)_n$ converge?

(2) If the subsequence $(x_{2n})_n$ converges, must (x_n) converge?

Proof.

Problem 4. Suppose that $(a_n)_n$ and $(b_n)_n$ are Cauchy sequences in $\mathbf{R}$. Prove, using the definition of Cauchy sequences, that $(|a_n - b_n|)_n$ converges in $\mathbf{R}$.

Proof.

Problem 5. Let $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + s_n}$ for $n \in \mathbf{N}$.

(1) Show that $s_n \leqslant 2$ for all n.

(2) Show that $(s_n)_n$ converges and then compute the limit of the sequence.

Proof.

Problem 6. Consider the sequence $(a_n)_n$ given by:

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$

(1) Prove that $(a_n)_n$ is increasing.

(2) Prove that $(a_n)_n$ converges.

Proof.

Problem 7. Prove that the sequence $(a_n)_n$, where:

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}},$$

converges and compute its limit.

Proof.

Problem 8.

(1) Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \frac{1}{4-x_n}$ converges.

(2) Explicitly compute $\text{limit}_{n\to\infty} x_n$.

Proof.

Problem 9.

(1) Argue from the definition of Cauchy sequences that if $(a_n)_n$ and $(b_n)_n$ are Cauchy sequences, then so is $(a_n b_n)_n$.

(2) Give an example of a sequence (a_n) with $\text{limit}_{n\to\infty} |a_{n+1} - a_n| = 0$ but which is *not* Cauchy.

Proof.

Problem 10. Let $(x_n)_n$ be a sequence of real numbers satisfying:

$$|x_{n+1} - x_n| \leqslant C\,|x_n - x_{n-1}|\,,$$

for all $n \geqslant 1$, where $0 < C < 1$ is a constant. Prove that $(x_n)_n$ converges.

Proof.

Problem 11.

(1) Exhibit, with proof, a sequence of real numbers which has $[0, 1]$ as its set of limit points.

(2) Does there exist a sequence with $(0, 1)$ as its set of limit points? Give an example with proof or prove that no such sequence exists.

Proof.

Problem 12. Show that the sequence $(x_n)_n$ is Cauchy, where:

$$x_n = \int_1^n \frac{\cos t}{t^2}\,dt.$$

Proof.

Problem 13. Prove that every convergent sequence of real numbers has a maximum or minimum value.

Proof.

Problem 14. Suppose that for a function $f : \mathbf{R} \to \mathbf{R}$, there is a number $k \in (0, 1)$ such that for all $x, y \in \mathbf{R}$:

$$|f(x) - f(y)| \leqslant k|x - y|.$$

Fix a number x_0, and define a sequence by:

$$x_n = f(x_{n-1})$$

for each $n \geqslant 1$. Prove that $(x_n)_n$ is a Cauchy sequence.

Proof.

Problem 15. Let $(x_n)_n$ be a sequence such that $(x_{2n})_n$, $(x_{2n+1})_n$, and $(x_{3n})_n$ are convergent. Show that $(x_n)_n$ is convergent.

Proof.

Problem 16. Prove that the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges by showing that the sequence of partial sums is Cauchy.

Proof.

Problem 17. Suppose that $\sum_{n=1}^{\infty} x_n$ is convergent series of positive terms. Show that $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}$ are also convergent.

Proof.

Problem 18.

(1) Let $(f_n)_n$ and $(g_n)_n$ be sequences of bounded functions on a subset A of $\mathbf{R}$. Suppose that $(f_n)_n$ converges uniformly to a bounded function f and $(g_n)_n$ converges uniformly to a bounded function g. Show that $(f_n g_n)_n \to fg$ uniformly on A.

(2) Show that (a) may be false if g is unbounded. *Hint:* Consider $f_n(x) = \frac{1}{n}$ and $g_n(x) = x + \frac{1}{n}$. Prove that the convergence $(f_n g_n)_n \to fg$ in this case is not uniform on $\mathbf{R}$.

Proof.

Problem 19. Suppose that $f : [0, \infty) \to \mathbf{R}$ is a continuous, increasing, and bounded function. Prove that f is uniformly continuous on $[0, \infty)$.

Proof.

Problem 20. Let $(f_n)_n$ be a sequence of functions defined on $A \subseteq \mathbf{R}$.

(1) Prove if each f_n is uniformly continuous on A and $(f_n)_n$ converges uniformly on A to a function f, then f is uniformly continuous on A.

(2) Give a counter example to show that (a) is false if we assume pointwise convergence instead of uniform convergence.

Proof.

Problem 21.

(1) Let $f : \mathbf{R} \to \mathbf{R}$ be uniformly continuous. Show that if $(x_n)_n$ is a Cauchy sequence of real numbers, then $(f(x_n))_n$ is a Cauchy sequence.

(2) Suppose that f_n is a sequence of continuous functions that converge uniformly on a subset $A \subseteq \mathbf{R}$ to a function f. Show that f is continuous on A.

Proof.

Problem 22. Consider the function:

$$g(x) = \begin{cases} e^x, & x \in \mathbf{Q}, \\ 1, & x \notin \mathbf{Q}. \end{cases}$$

Find, with proof, the set $C = \{x \in \mathbf{R} \mid g \text{ is continuous at } x\}$.

Proof.

Problem 23. Define $f : (-1, 0) \cup (0, 1) \to \mathbf{R}$ by:

$$f(x) = \begin{cases} 4, & x \in (-1, 0), \\ 5, & x \in (0, 1). \end{cases}$$

(1) Show that f is continuous on $(-1, 0) \cup (0, 1)$.

(2) Show that f is not uniformly continuous on $(-1, 0) \cup (0, 1)$.

Proof.

Problem 24. Let $(f_n)_n$ be a sequence of functions $f_n : \mathbf{R} \to \mathbf{R}$ and let $f : \mathbf{R} \to \mathbf{R}$ be a function. Suppose f_n is bounded for each $n \in \mathbf{N}$.

(1) Prove that if $(f_n)_n \to f$ uniformly on $\mathbf{R}$, then f is bounded.

(2) If each f_n is continuous and $(f_n)_n \to f$ pointwise on $\mathbf{R}$, does f have to be bounded? Give a proof or a counterexample.

Proof.

Problem 25. Show that the sequence of functions:

$$f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$$

converges pointwise to $f(x) = 0$ on $[0, 1]$, but does not converge uniformly.

Proof.

Problem 26.

(1) Let $A \subseteq \mathbf{R}$. Define what it means for $f : A \to \mathbf{R}$ to be uniformly continuous.

(2) Use the definition to show that $f(x) = \frac{1}{x}$ is uniformly continuous.

(3) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Proof.

Problem 27.

(1) Let $(f_n)_n$ be a sequence of functions defined on $A \subseteq \mathbf{R}$ that converges uniformly on A to a function f. Prove that if each f_n is continuous at $c \in A$, then f is continuous at c.

(2) Give an example to show that the result is false if we only assume that $(f_n)_n$ converges pointwise to f on A.

Proof.

Problem 28. Define $f_n : [0, \infty) \to \mathbf{R}$ by:

$$f_n(x) = \frac{\sin(nx)}{1 + nx}.$$

(1) Show that f_n converges pointwise on $[0, \infty)$ and find the pointwise limit f.

(2) Show that $(f_n)_n \to f$ uniformly on $[a, \infty)$ for every $a > 0$.

(3) Show that f_n does not converge uniformly to f on $[0, \infty)$.

Proof.

Problem 29. Let $f_n : \mathbf{R} \to \mathbf{R}$ be a sequence of continuous functions that converges uniformly on $\mathbf{R}$ to a function f. Let $(x_n)_n$ be a sequence of real numbers that converges to $x_0 \in \mathbf{R}$. Prove that $(f_n(x_n))_n \to f(x_0)$.

Proof.

Problem 30. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that:

$$\operatorname*{limit}_{x \to +\infty} f(x) = \operatorname*{limit}_{x \to -\infty} f(x) = +\infty.$$

Prove that f attains an absolute minimum value of $\mathbf{R}$. In other words, prove that there exists a real number c such that $f(c) \leqslant f(x)$ for all $x \in \mathbf{R}$.

Proof.

Problem 31. Suppose that $\mathtt{f} : [0, \infty)$ is a continuous, increasing, and bounded function. Prove that $\mathtt{f}$ is uniformly continuous on $[0, \infty)$.

Proof.

Problem 32. A zero of a continuous function is called *isolated* if there exists an open set containing that zero but no other zeros of f.

(1) Give an example of a continuous function $f : (0, 1) \to \mathbf{R}$ with infinitely many isolated zeros.

(2) If $f : [0, 1] \to \mathbf{R}$ is continuous and all of its zeros are isolated, show that f has only finitely many zeros on $[0, 1]$.

Proof.

Problem 33.

(1) Give a definition for a function $f : [a, b] \to \mathbf{R}$ to be uniformly continuous.

(2) Using your definition (and not a theorem), prove that the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, 2]$.

Proof.

Problem 34. Suppose that $f(x)$ is continuous and unbounded on $[a, b)$. Prove that $\text{limit}_{x \to b^-} f(x)$ does not exist.

Proof.

Problem 35. For each $n \in \mathbf{N}$, the function:

$$f_n(x) = \frac{nx}{e^{nx}}$$

is continuous on $[0, 2]$. Find the pointwise limit function $f(x) = \text{limit}_{n\to\infty} f_n(x)$ and show that $(f_n)_n$ does not converge uniformly to f.

Proof.

Problem 36. Let f be continuous on $[0, 1]$ with $f(x) > 0$ for all $x \in [0, 1]$. Let $S = \sup_{x \in [0,1]} f(x)$. Show that for every $\epsilon > 0$, there is some open interval I on which $f(x) > S - \epsilon$.

Proof.

Problem 37. Show that if $f_n(x)$ is a uniformly continuous function on $[0, 1]$ for each $n \in \mathbf{N}$ and $(f_n)_n \to f$ uniformly on $[0, 1]$, then $f(x)$ is also uniformly continuous on $[0, 1]$.

Proof.

Problem 38. Define a sequence of functions by:

$$f_n(x) = \frac{nx^n}{1 + nx^n}$$

for $n \in \mathbf{N}$.

(1) Find the pointwise limit $f(x)$ for each $x \in [0, \infty)$.

(2) Prove that $(f_n)_n$ does not converge uniformly on $[0, \infty)$.

(3) Prove that $(f_n)_n$ converges uniformly on $[1, 2]$.

Proof.

Problem 39. Consider the sequence of functions $f_n : \mathbf{R} \to \mathbf{R}$ given by:

$$f_n(x) = \frac{nx}{\sqrt{1 + n^2x^2}}.$$

Find the pointwise limit $f(x) = \text{limit}_{n\to\infty} f_n(x)$. Does $(f_n)_n$ converge to f uniformly on $\mathbf{R}$? Justify your answer.

Proof.

Problem 40. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function with a continuous derivative. Suppose there exist four distinct points w, x, y, z in $\mathbf{R}$ with $f(w) = f(x)$ and $f(y) = y$ and $f(z) = z$. Prove that there is a point u where $f'(u) = \frac{1}{2}$.

Proof.

Problem 41. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Suppose that f is differentiable, that $f(0) = 1$, and that $|f'(x)| \leqslant 1$ for all $x \in \mathbf{R}$. Prove that $|f(x)| \leqslant |x| + 1$ for all $x \in \mathbf{R}$.

Proof.

Problem 42. Prove that there does not exist a differentiable function $f : \mathbf{R} \to \mathbf{R}$ such that $f'(0) = 0$ and $f'(x) \geqslant 1$ for all $x \neq 0$. [*Hint:* Use the Mean Value Theorem].

Proof.

Problem 43. A function $f : \mathbf{R} \to \mathbf{R}$ is *Lipschitz continuous* on a set $A \subseteq \mathbf{R}$ if there exists a constant $M \geqslant 0$ such that $|f(x) - f(y)| \leqslant M|x - y|$ for all $x, y \in A$.

(1) Assume that f is a differentiable function on $\mathbf{R}$ and that f' is continuous on $[a, b]$. Prove that f is Lipschitz on $[a, b]$.

(2) Prove that a Lipschitz function $f : \mathbf{R} \to \mathbf{R}$ is uniformly continuous on $\mathbf{R}$.

Proof.

Problem 44. A function $f : \mathbf{R} \to \mathbf{R}$ is *Lipschitz continuous* on a set $A \subseteq \mathbf{R}$ if there exists a constant $M \geqslant 0$ such that $|f(x) - f(y)| \leqslant M|x - y|$ for all $x, y \in A$.

(1) Show that $f(x) = \sqrt{x}$ is Lipschitz continuous on $[1, \infty)$ but not $[0, \infty)$.

(2) Prove that a Lipschitz function $f : \mathbf{R} \to \mathbf{R}$ is uniformly continuous on $\mathbf{R}$.

Proof.

Problem 45. Show that the function:

$$f(x) = \begin{cases} x^2, & x \in \mathbf{Q} \\ 0, & x \notin \mathbf{Q} \end{cases}$$

is differentiable only at $x = 0$.

Proof.

Problem 46.

(1) State the Mean Value Theorem.

(2) Use the Mean Value Theorem to prove that $|\tan(x)| \geqslant |x|$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Proof.

Problem 47. Suppose that $f : \mathbf{R} \to \mathbf{R}$ satisfies:

$$|f(x) - f(y)| \leqslant (x - y)^2$$

for all $x, y \in \mathbf{R}$. Show that f is a constant function on $\mathbf{R}$. (*Hint:* Is f differentiable)

Proof.

Problem 48.

(1) Suppose that f is a real valued function on $(0, \infty)$ whose derivative exists and is bounded on $(0, \infty)$. Prove that f is uniformly continuous on $(0, \infty)$.

(2) Give an example of a differentiable real valued function f on $(0, \infty)$ that is uniformly continuous on $(0, \infty)$ yet f' is unbounded on $(0, \infty)$.

Proof.

Problem 49. Suppose that f is differentiable on $\mathbf{R}$ and that $f'(x) \leqslant 4$ for all $x \in \mathbf{R}$. Prove that there is at most one point $x > 0$ such $f(x) = x^2$.

Proof.

Problem 50. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is differentiable and that $|f'(x)| < 1$ for all $x \in \mathbf{R}$.

(1) Prove that f has at most one fixed point.

(2) Show that the following function satisfies $|f'(x)| < 1$ for all $x \in \mathbf{R}$ but has no fixed points:

$$f(x) = \ln(1 + e^x).$$

Proof.

Problem 51.

(1) Prove that $\ln x \leqslant x - 1$ for all $x > 0$.

(2) Prove that $\ln x \geqslant x-1-\frac{1}{2}(x-1)^2$ for all $x \geqslant 1$, and that $\ln x \leqslant x-1-\frac{1}{2}(x-1)^2$, for all $0 < x \leqslant 1$.

Proof.

Problem 52. Let f be a function that is continuous on $[0, 1]$ and differentiable;e on $(0, 1)$. Show that if $f(0) = 0$, $|f'(x)| \leqslant |f(x)|$ for all $x \in (0, 1)$, then $f(x) = 0$ for all $x \in [0, 1]$.

Proof.

Problem 53. Prove that for all real numbers x and y:

$$|\cos^2(x) - \cos^2(y)| \leqslant |x - y|.$$

Proof.

Problem 54. Suppose that f is continuous on $[0, 1]$. Show that there is some $c \in [0, 1]$ with:

$$\int_0^1 x^2 f(x) dx = \frac{1}{3} f(c).$$

Proof.

Problem 55.

(1) State the Mean Value Theorem

(2) Show that if $f : \mathbf{R} \to \mathbf{R}$ is differentiable, $f(0) = 0$, and for all x, $|f'(x)| < |x|^3$, then $|f(x)| \leqslant x^4$ for all x.

Proof.

Problem 56. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable everyone on (a, b) except perhaps at one number $c \in (a, b)$, and let $\text{limit}_{x \to c} f'(x)$ exist. Show that f is differentiable at c and $f'(c) = \text{limit}_{x \to c} f'(x)$.

Proof.

Problem 57. Let $f : [a, b] \to \mathbf{R}$ be continuous and twice differentiable on (a, b). Assume that the line segment from $A = (a, f(a))$ and $B = (b, f(b))$ intersects the graph of f in a third point different from A and B. Show that $f''(c) = 0$ for some $c \in (a, b)$.

Proof.

Problem 58. Prove that if f is a function which is differentiable on all of $\mathbf{R}$ and $f'(x) > 0$ for all x, then f is injective.

Proof.

Problem 59. Suppose f and g are continuous on $[a, b]$ and f' and g' are continuous on (a, b) with $f(a) = g(a)$ and $f(b) = g(b)$. Prove there is a number $c \in (a, b)$ such that the line tangent to the graph of f at the point $(c, f(c))$ is parallel to the line tangent to the graph of g at $(c, g(c))$.

Proof.

Problem 60. Find, with proof, the maximum number of real roots of the function $f(x) = x^{16} + ax + b$ where a and b are real numbers.

Proof.

Problem 61. A function $f : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous if there is a constant $M \geqslant 0$ such that

$$|f(x) - f(y)| \leqslant M|x - y|$$

for all $x, y \in \mathbf{R}$.

(1) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is differentiable and $f' : \mathbf{R} \to \mathbf{R}$ is bounded. Prove that f is Lipschitz continuous.

(2) Give an example, with proof, of a function $f : \mathbf{R} \to \mathbf{R}$ that is differentiable but not Lipschitz continuous.

(3) Give an example, with proof, of a function $f : \mathbf{R} \to \mathbf{R}$ that is Lipschitz continuous but not differentiable.

Proof.

Problem 62. Let $a > 0$. For each $n \in \mathbf{N}$, consider the function $f_n : \mathbf{R} \to \mathbf{R}$ given by $f_n(x) = \frac{\sin(x/n)}{\sqrt{1+n^2}}$

(1) Show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[-a, a]$.

(2) Show that the series $\sum_{n=1}^{\infty} f_n(x)$ is continuously differentiable on $(-a, a)$.

Proof.

Problem 63. Suppose f is continuous on $[0, 1]$ and $|f(x)| < 1$ for all x on $[0, 1]$. Prove that F is uniformly continuous on $[0, 1]$, where

$$F(x) = \sum_{k=1}^{\infty} (f(x))^k.$$

Proof.

Problem 64. Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

(1) Find the domain of $f(x)$ precisely.

(2) Prove that f is uniformly continuous on this domain.

Proof.

Problem 65. Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin x^n}{n^2 x^n}.$$

(1) Prove that f is continuous on $[1, \infty)$.

(2) Prove that, in fact, f is continuous on $(0, \infty)$.

Proof.

Problem 66. Consider the function $f(x) = \sum_{k=1}^{\infty}(1 - \cos(x/k))$.

(1) Prove that the series for f converges uniformly on every interval of the form $[-M, M]$ in $\mathbf{R}$.

(2) Prove that f is differentiable on $\mathbf{R}$.

You may use without proof the following inequalities in this problem:

$$|\sin t| \leqslant |t|, \ \ |1 - \cos t| \leqslant \frac{t^2}{2}, \ \ t \in \mathbf{R}.$$

Proof.

Problem 67. Show that the following series converges uniformly on (r, ∞) for any real number $r > 1$.

$$\sum_{n=1}^{\infty} \frac{n \ln(1 + nx)}{x^n}.$$

Proof.

Problem 68. Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2 + x^4}{n^4 x^2}.$$

(1) Prove that the series converges uniformly on $[-R, R]$ for any $R > 0$.

(2) Prove that f is continuous on $\mathbf{R}$.

Proof.

Problem 69. Consider the function

$$f(x) = \sum_{k=0}^{\infty} e^{-kx} \cos kx.$$

(1) Prove that the series converges uniformly on $[a, \infty)$ for any $a > 0$.

(2) Prove that f is a continuous function on $(0, \infty)$.

Proof.

Problem 70. Consider the series

$$\sum_{n=1}^{\infty} e^{-nx^2} \sin(nx).$$

(1) Prove that this series converges uniformly on $[a, \infty)$ for each $a > 0$.

(2) Does this series converge uniformly on $[0, \infty)$? Justify your answer.

Proof.

Problem 71. Let $P = \{2, 3, 5, 7, 11, 13, ...\}$ be the set of prime numbers.

(1) Find the radius of convergence R of the power series

$$f(x) = \sum_{p \in P} x^p = x^2 + x^3 + x^5 + x^7 + ...$$

(2) Show that $0 \leqslant f(x) \leqslant \frac{x^2}{1-x}$ for $0 \leqslant x < R$.

Proof.

Problem 72. Consider

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sin(2^k x).$$

(1) Show that f is continuous on $\mathbf{R}$.

(2) Show that f is not differentiable at $x = 0$. (*Hint:* Consider the sequence $\left(\frac{\pi}{2^n}\right)_n$).

Proof.

Problem 73. Suppose that $(a_k)_k$ is a sequence with $|a_k| \leqslant 1$ for all $k \in \mathbf{N}$.

(1) Prove that the series $\sum_{k=1}^{\infty} a_k x^k$ and $\sum_{k=1}^{\infty} k a_k x^{k-1}$ converge uniformly and absolutely on any closed interval contained in $(-1, 1)$.

(2) Prove that

$$\frac{d}{dx}\left(\sum_{k=1}^{\infty} a_k x^k\right) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

for all $x \in (-1, 1)$.

Proof.

Problem 74. Let $a > 0$ and define $f(x) = \sum_{n=1}^{\infty} \frac{1}{n}(ax)^n$.

(1) Find the interval of convergence.

(2) Let $0 < c < R$ where R is the radius of convergence. Show the convergence is uniform on $[-c, c]$.

Proof.

Problem 75. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{n^x}{3^n - 7}.$$

Show f is continuous on $[0, \infty)$.

Proof.

Problem 76. Show that $f(x) = \sum_{n=1}^{\infty} \arctan\left(\frac{x}{n^2}\right)$ is a continuous function on all of $\mathbf{R}$.

Proof.

Problem 77. Prove that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2 + x^4}{n^4 + x^2}$$

converges to a continuous function $f : \mathbf{R} \to \mathbf{R}$.

Proof.

Problem 78. Prove that

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \right)^2$$

is continuous on $\mathbf{R}$.

Proof.

Problem 79. Let

$$f_n(x) = \frac{x}{(x + \cos(x/n))^n}$$

for each $n \in \mathbf{N}$. Prove that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is continuous on $[1, 2]$.

Proof.

Problem 80. Let $(f_n)_n$ be a sequence of increasing functions on $[a, b]$ with $\sum_{n=1}^{\infty} f_n(x)$ absolutely convergent when $x = a$ and when $x = b$. Show that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for every $x \in [a, b]$ and that also the series converges uniformly on $[a, b]$.

Proof.

Problem 81.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Let $f : [a, b] \to \mathbf{R}$ be a bounded function, and assume that the lower integral of f on $[a, b]$ is positive. Show that there exists an interval $[c, d] \subseteq [a, b]$ with $c < d$ with $f(x) > 0$ for $x \in [c, d]$.

Proof.

Problem 82.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Prove that if f is continuous on $[0, 1]$, then

$$\operatorname*{limit}_{n\to\infty} \int_0^1 f(x^n)dx = f(0).$$

Proof.

Problem 83.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Suppose that $f : [0, 1] \to \mathbf{R}$ is continuous and monotonically increasing, with $f(0) = 0$, $f(1/2) = 1$, and $f(1) = 2$. Prove that

$$\int_0^1 f(x)dx > \frac{1}{2}.$$

Proof.

Problem 84. Let $f : [0, 1] \to \mathbf{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)x^n \, dx = 0.$$

Proof.

Problem 85.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Let a_n be a positive sequence of real numbers converging to 0 and let $B = \{b_1, b_2, b_3, ...\}$ be a countably infinite subset of $[0, 1]$. Consider the function f on $[0, 1]$ defined by

$$f(x) = \begin{cases} a_n, & x = b_n, \\ 0, & x \notin B \end{cases}.$$

Use your definition from (a) to prove that f is Riemann integrable on $[0, 1]$.

Proof.

Problem 86.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Let f be bounded on $[a, b]$ and assume that there exists a partition P with $L(f, P) = U(f, P)$. Use the definition of Riemann integrability to characterize f.

Proof.

Problem 87.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function with the property that f is Riemann integrable on $[a, c]$ for all $a < c < b$. Use the definition of Riemann integrability to show that f is Riemann integrable on $[a, b]$.

Proof.

Problem 88.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Use your definition from (a) to prove that if $f : [a, b] \to \mathbf{R}$ is continuous and

$$\int_a^b |f(x)|\, dx = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

Proof.

Problem 89.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Use your definition from (a) to prove that

$$f(x) = \begin{cases} 1, & x = \frac{1}{n} \text{ for some } n \in \mathbf{N} \\ 0, & \text{otherwise} \end{cases}$$

is integrable on $[0, 1]$ and compute the value of the integral $\int_0^1 f(x)dx$.

Proof.

Problem 90.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Use the definition of the Riemann integral to prove that $f(x) = \frac{1}{1+x}$ is Riemann integrable on $[0, b]$, for any $b > 0$.

Proof.

Problem 91.

(1) State the definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Let

$$g_n(x) = \begin{cases} n, & 0 \leqslant x \leqslant \frac{1}{n} \\ 0, \frac{1}{n} < x \leqslant 1 \end{cases}$$

and let f be any continuous function on $[0, 1]$. Use the definition of the Riemann integral to compute

$$\operatorname*{limit}_{n\to\infty} \int_0^1 f(x) g_n(x) dx$$

in terms of f.

Proof.

Problem 92.

(1) State the definition for the real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on the interval $[a, b]$.

(2) Let $f : [a, b] \to \mathbf{R}$ be increasing on the interval $[a, b]$. Use the definition to prove that f is Riemann integrable on $[a, b]$.

Proof.

Problem 93.

(1) State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable.

(2) Let $f : [a, b] \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} 0, & x \in [a, b] \cap \mathbf{Q} \\ x, & x \in [a, b] \setminus \mathbf{Q} \end{cases}.$$

Use your definition to decide with proof if f is Riemann integrable.

Proof.

Problem 94.

(1) State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable.

(2) Use this definition to prove that the function f defined on $[0, \frac{\pi}{2}]$ by

$$f(x) = \begin{cases} \cos^2 x, & x \in \mathbf{Q} \\ 0, & \text{otherwise} \end{cases}$$

is not Riemann integrable.

Proof.

Problem 95. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous, with

$$\int_0^1 f(xt)dt = 0$$

for all $x \in \mathbf{R}$. Show that $f(x) = 0$.

Proof.

Problem 96.

(1) State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on $[a, b]$.

(2) Let $f : [a, b] \to \mathbf{R}$ be Riemann integrable. Prove that $|f(x)|$ is also Riemann integrable and that

$$\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx.$$

Proof.

Problem 97.

(1) State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on $[a, b]$.

(2) Let

$$f(x) = \begin{cases} 1, & 1 \leqslant x < 2 \\ 10, & x = 2 \\ 2, & 2 < x \leqslant 3. \end{cases}$$

Use your definition to prove that f is integrable on $[1, 3]$.

Proof.

Problem 98.

(1) State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on $[a, b]$.

(2) Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Use your definition to prove that f is integrable on $[a, b]$.

Proof.

Problem 99.

(1) State a definition for a real valued function $f : [a, b] \to \mathbf{R}$ to be Riemann integrable on $[a, b]$.

(2) Let f, g be Riemann integrable functions and suppose that the set E is finite where

$$E = \{x \in (a, b) \mid f(x) \neq g(x)\}.$$

Use your definition of Riemann integrability to show that $\int_a^b f(x)dx = \int_a^b g(x)dx$. (*Hint:* Consider the function $f - g$).

Proof.

Problem 100. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous, periodic function. Prove that the set $f(\mathbf{R})$ is compact. (Recall that a function $f : \mathbf{R} \to \mathbf{R}$ is *periodic* if there exists a nonzero constant P such that $f(x) = f(x + P)$ for all $x \in \mathbf{R}$).

Proof.