**Recall:** A group $G$ acts on a set $X$ by the following properties:

- $e_G . x = x$
- $g_1 g_2 . x = g_1 . (g_2 . x)$.

The orbit of an element $x \in X$ is:

$$\mathrm{Orb}(x) = \{g.x : g \in G\}.$$

Likewise, $x \sim y$ iff $y \in \mathrm{Orb}(x)$ is an equivalence relation on $X$.

**Definition:** Let $G$ act on $X$. The stabilizer of $x \in X$ is:

$$\mathrm{Stab}_G(x) = \{g \in G : g.x = x\}.$$

**Definition:** Let $G$ act on $X$. The fixed points of $g \in G$ are:

$$\mathrm{Fix}_X(g) = \{x \in X : g.x = x\}.$$

**Example:** Let $G$ act on $X$. We will show that $\mathrm{Stab}_G(x) \leq G$. Let $g \in \mathrm{Stab}_G(x)$. We have that $g.x = x$, so:

$$g^{-1}.(g.x) = g^{-1}.x$$
$$\Rightarrow (g^{-1}g).x = g^{-1}.x$$
$$\Rightarrow e_G.x = g^{-1}.x$$
$$\Rightarrow x = g^{-1}.x.$$

Hence if $g \in \mathrm{Stab}_G(x)$, then $g^{-1} \in \mathrm{Stab}_G(x)$. Let $g, h \in \mathrm{Stab}_G(x)$. Then:

$$\begin{aligned}(gh).x &= g.(h.x) \\ &= g.x \\ &= x\end{aligned}$$

Thus $\mathrm{Stab}_G(x)$ is a subgroup of $G$.

**Theorem:** (Orbit-Stabilizer Theorem) Let $G$ be a finite group that acts on $X$. Then:

$$|\mathrm{Orb}(x)| = |G| / |\mathrm{Stab}_G(x)|.$$

Proof. We will prove the equality by showing there exists a bijection. Define $\varphi: G/\mathrm{Stab}_G(x) \longrightarrow \mathrm{Orb}(x)$ by $g\,\mathrm{Stab}_G(x) \longmapsto g.x$. We must first show our mapping is well-defined. Suppose $g_1\mathrm{Stab}_G(x) = g_2\mathrm{Stab}_G(x)$. We can write $g_1 = g_2 h$, where $h \in \mathrm{Stab}_G(x)$. Observe that:

$$\begin{aligned}\varphi(g_1\mathrm{Stab}_G(x)) &= g_1.x \\ &= (g_2 h).x \\ &= g_2.(h.x) \\ &= g_2.x \qquad \text{Since } h \in \mathrm{Stab}_G(x). \\ &= \varphi(g_2\mathrm{Stab}_G(x)).\end{aligned}$$

$g_2^{-1}g_1 \in H$
$g_2^{-1}g_1 = h$
$g_1 = g_2 h$
$g_1.x = g_2.x$
$g_1.x =$

Thus $\varphi$ is well-defined. It remains to show that $\varphi$ is a bijection. Let $g_1\mathrm{Stab}(x), g_2\mathrm{Stab}(x) \in G/\mathrm{Stab}_G(x)$ with $\varphi(g_1\mathrm{Stab}(x)) = \varphi(g_2\mathrm{Stab}(x))$. So $g_1.x = g_2.x$, which is equivalent to $g_2^{-1}.(g_1.x) = x$, i.e., $(g_2^{-1}g_1).x = x$. Thus $g_2^{-1}g_1 \in \mathrm{Stab}_G(x)$, so $g_1\mathrm{Stab}_G(x) = g_2\mathrm{Stab}_G(x)$. Let $g.x \in \mathrm{Orb}(x)$. We have $\varphi(g\,\mathrm{Stab}(x)) = g.x$. Thus $\varphi$ is surjective. Hence $\varphi$ is bijective and we have that $|G/\mathrm{Stab}_G(x)| = |\mathrm{Orb}(x)|$. From Lagrange's Theorem, $|G|/|\mathrm{Stab}(x)| = |\mathrm{Orb}(x)|$. $\square$

**Lemma:** (Burnside's Lemma) Let $G$ be a finite group acting on a finite set $X$. We have:

$$\#\text{orbits} = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}_X(G)|.$$

Proof. Note that:

$$\begin{aligned}\#\text{orbits} = \sum_{x\in X}\frac{1}{|\mathrm{Orb}(x)|} &= \frac{1}{|G|}\sum_{x\in X}|\mathrm{Stab}_G(x)| \\ &= \frac{1}{|G|}\sum_{x\in X}\Big(\sum_{\substack{g\in G\\ g.x=x}} 1\Big) \\ &= \frac{1}{|G|}\sum_{g\in G}\Big(\sum_{\substack{x\in X\\ g.x=x}} 1\Big) \\ &= \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}_X(g)|.\end{aligned}$$

$\square$

<u>Example</u>: Let $G$ be finite and acts on itself by conjugation. Note then that:

$$\mathrm{Orb}(h) = \{g.h : g \in G\}$$
$$= \{ghg^{-1} : g \in G\}$$

$$\mathrm{Stab}(h) = \{g \in G,\ g.h = h\} \qquad (1)$$
$$= \{g \in G,\ ghg^{-1} = h\}$$

From (1), this is called the <u>centralizer</u> of $h$, denoted $C_G(h)$. If $G$ acts on $X$, $|X| = \sum_{\substack{\text{dist.} \\ \text{orb.}}} |\mathrm{Orb}(x)|$. From our conjugation action, we will get $|G|$ on the left hand side. The center of $h$, denoted $Z_G(h)$, is defined as:

$$Z_G(h) = \{h \in G : gh = hg\ \forall g \in G\}.$$

If $x \in Z_G(g)$, then $gxg^{-1} = x\ \forall g \in G$. Hence $\mathrm{Orb}(x) = \{x\}$. We have that:

$$|G| = \sum_{\substack{\text{dist.} \\ \text{orb}}} |\mathrm{Orb}(x)|$$
$$= \sum_{x \in Z(g)} \{x\} + \sum_{\substack{\text{dist.} \\ \text{orb.} \\ \text{not in } Z(g)}} |\mathrm{Orb}(x)|$$
$$= |Z(G)| + \sum_{\substack{\text{dist.} \\ \text{orb.} \\ \text{not in } Z(g)}} |\mathrm{Orb}(x)|.$$

Let $x_1, \dots, x_n$ be representatives for the distinct orbits not overlapping $Z(G)$. Then:

$$|G| = |Z(G)| + \sum_{i=1}^{n} |\mathrm{Orb}(x_i)|$$
$$= |Z(G)| + \sum_{i=1}^{n} \frac{|G|}{|\mathrm{Stab}_G(x_i)|}$$
$$= |Z(G)| + \sum_{i=1}^{n} \frac{|G|}{|C_G(x_i)|}. \qquad (2)$$

Equation (2) is defined as <u>The Class Equation</u>.

<u>Theorem</u>: (Cauchy's Theorem) Let $G$ be a finite group and assume $p \mid |G|$ for $p$ prime. Then $G$ has an element of order $p$; i.e., a subgroup of order $p$.

Proof. If $|G| = p$, then $G$ is cyclic (Take $G = \langle g \rangle$. Then $|g| = p$). We will proceed with the following cases:

<u>Case 1</u>: $G$ is abelian.

We use induction on $|G|$. If $|G| = 2, 3$, we are done ($2, 3$ prime). Assume the result is true for any abelian group with size less than $|G|$.

Let $g \in G$, $g \neq e_G$. Then $g^{|G|/p}$ is an element of order $p$. So if $G$ is cyclic, we are done. If $G$ is not cyclic, we have a proper non trivial subgroup $H$. We have $|H| < |G|$, so if $p \mid |H|$, then induction gives an element of order $p$ in $H$, so also in $G$ aswell.

Assume $p \nmid |H|$. Since $G$ is abelian, $H \trianglelefteq G$. So $G/H$ is an abelian group.

We have $|G| = |G/H| \cdot |H|$. Since $p$ prime, $p$ divides $|G|$, hence $p \mid |G/H|$ or $p \mid |H|$. Since $p \nmid |H|$, we get $p \mid |G/H|$. We have by induction an element of order $p$ in $G/H$, say $|gH| = p$.

In particular, $(gH)^p = g^p H = H$, so $g^p \in H$ and $g^k \notin H$ for $1 \leq k < p$.

Let $|H| = n$; i.e., $p \nmid n$. Since $g^p \in H$ and $|H| = n$, we have $(g^p)^n = e$.

So $g^{pn} =$