Consider the group $\mathbb{Z}/6\mathbb{Z}$. Recall that if $\gcd(m,6)=1$ for some $0 \le m \le 6$, then $\langle [m]_6 \rangle = \mathbb{Z}/6\mathbb{Z}$. Thus the subgroups of $\mathbb{Z}/6\mathbb{Z}$ are:

$$\langle [0]_6 \rangle = \{[0]_6\}$$
$$\langle [2]_6 \rangle = \{[0]_6, [2]_6, [4]_6\}$$
$$\langle [3]_6 \rangle = \{[0]_6, [3]_6\}.$$

Note that $\mathbb{Z}/6\mathbb{Z} = G/N$, where $G = \mathbb{Z}$ and $N = 6\mathbb{Z}$. Observe that:

$$6\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \longrightarrow 2\mathbb{Z}/6\mathbb{Z} = \{2m + 6\mathbb{Z} : m \in \mathbb{Z}\} = \{[0]_6, [2]_6, [4]_6\}.$$

$$6\mathbb{Z} \subseteq 3\mathbb{Z} \subseteq \mathbb{Z} \longrightarrow 3\mathbb{Z}/6\mathbb{Z} = \{3m + 6\mathbb{Z} : m \in \mathbb{Z}\} = \{[0]_6, [3]_6\}.$$

If we want subgroups of G/N, we should look at subgroups H of G so that $N \subseteq H \subseteq G$, where we can then form $H/N \le G/N$.

<u>Proposition</u>: Let $N \trianglelefteq G$ and $N \le H \le G$. Then H/N is a subgroup of G/N.
Proof. Verify this!

<u>Theorem</u>: (3^{rd} Isomorphism Theorem) Let K, N be normal subgroups of G with $N \subseteq K \subseteq G$. Then $K/N \trianglelefteq G/N$ and $(G/N)/(K/N) \cong G/K$.
Proof. Let $\varphi : G/N \longrightarrow G/K$ defined by $gN \longmapsto gK$. We must show φ is well-defined. Let $g_1 N = g_2 N$. Then $g_2^{-1} g_1 = n$ for some $n \in N$; i.e., $g_1 = g_2 n$. Observe that:

$$\begin{aligned} \varphi(g_1 N) &= g_1 K \\ &= g_2 n K \\ &= g_2 K \quad \text{Since } N \subseteq K. \\ &= \varphi(g_2 N) \end{aligned}$$

Thus φ is well-defined. Let $g_1 N, g_2 N \in G/N$. We have:

$$\begin{aligned} \varphi(g_1 N g_2 N) &= \varphi(g_1 g_2 N) \\ &= g_1 g_2 K \\ &= g_1 K \, g_2 K \\ &= \varphi(g_1 N)\varphi(g_2 N). \end{aligned}$$

Thus φ is a homomorphism. Let $gK \in G/K$. Then $\varphi(gN) = gK$, hence φ is surjective.

It only remains to show that $\operatorname{Ker}\varphi = K/N$. Let $kN \in K/N$. Then $\varphi(kN) = kK = K$; i.e., $kN \in \operatorname{Ker}\varphi$. Hence $kN \subseteq \operatorname{Ker}\varphi$.
Let $gN \in \operatorname{Ker}\varphi$; i.e., $\varphi(gN) = K$. So $gK = K$, which means $g \in K$ since $g = k$ for some $k \in K$. Hence $\operatorname{Ker}\varphi \subseteq K/N$ and we have equality. Likewise, K/N is normal because it is the Kernel of a homomorphism. Thus the first isomorphism theorem gives $(G/N)/(K/N) \cong G/K$. $\square$

<u>Corollary</u>: Let $N \trianglelefteq G$ and K any subgroup of G w/ $N \leq K$. Then $K \trianglelefteq G$ iff $K/N \trianglelefteq G/N$.
Proof. If $K \trianglelefteq G$, then the third isomorphism theorem gives $K/N \trianglelefteq G/N$.
Assume $K/N \trianglelefteq G/N$. WTS $gKg^{-1} = K$ $\forall g \in G$. Let $g \in G$, $k \in K$. Observe that:

$$\begin{aligned} gkg^{-1}N &= gN\,kN\,\boxed{g^{-1}N} \\ &= \underbrace{gN\,kN\,\boxed{(gN)^{-1}}}_{\text{know this is } \in K/N,\text{ but not } gkg^{-1}N.} \in K/N \quad \text{b/c } K/N \trianglelefteq G/N \end{aligned}$$

Thus $gkg^{-1} \in K$. Hence $gKg^{-1} \subseteq K$.
Let $k \in K$. Then $k = g(g^{-1}kg)g^{-1} \in g^{-1}Kg \subseteq K$. Thus $K \subseteq gKg^{-1}$, so $gKg^{-1} = K$ for all $g \in G$; i.e., $K \trianglelefteq G$. $\square$

<u>Theorem</u>: Let T be a subgroup of G/N. Then $T = H/N$ for some subgroup $H \leq G$ with $N \leq H \leq G$.
Proof. Define $H = \{g \in G : gN \in T\}$. We will show that H is a subgroup. Since $T \leq G/N$, we have that T contains eN. Thus $e \in H$. Let $g_1, g_2 \in H$; i.e., $g_1N \in T$ and $g_2N \in T$. But since $T \leq G/N$, we have that $g_2^{-1}N \in T$, and similarly $g_1Ng_2^{-1}N \in T$. So $g_1g_2^{-1}N \in T$; i.e., $g_1g_2^{-1} \in H$. Thus H is a subgroup.

Let $n \in N$. We have $nN = N \in T$ b/c T is a subgroup. Thus $n \in H$, hence $N \leq H$.
We have $H/N = \{hN : h \in H\} = T$.

<u>Exercise</u>: List all the subgroups $(\mathbb{Z}/12\mathbb{Z})/H$ where $H = \langle [6]_{12} \rangle$.
We have that $\langle [6]_{12} \rangle \subseteq N \subseteq \mathbb{Z}/12\mathbb{Z}$. So $N = \langle [2]_{12} \rangle$ or $\langle [3]_{12} \rangle$. Hence $\langle [2]_{12} \rangle / \langle [6]_{12} \rangle$ and $\langle [3]_{12} \rangle / \langle [6]_{12} \rangle$ are normal to $(\mathbb{Z}/12\mathbb{Z})/\langle [6]_{12} \rangle$.

<u>Example</u>: What are all of the homomorphic images of S_3? Let $\varphi : S_3 \to G$ for some group G. Recall that $\varphi : S_3 \twoheadrightarrow \operatorname{im}(\varphi) \leq G$, so $S_3/\operatorname{Ker}\varphi \cong \operatorname{im}\varphi$.
Thus, no matter what φ is, $\operatorname{Ker}\varphi \trianglelefteq S_3$. Thus $\operatorname{Ker}\varphi$ can equal $\{e_G\}$, S_3, or $\langle (123) \rangle$.
Hence $S_3/\langle (123) \rangle = \{\langle (123) \rangle, (12)\langle (123) \rangle\} \cong \mathbb{Z}/2\mathbb{Z}$.