Apr 4.

Lemma: Let $H, K \leq G$, $|G| < \infty$. If $H \cap K = \{e_G\}$, then $|HK| = |H||K|$.

Proposition: Let $H, K \leq G$. Then $HK \leq G$ iff $HK = KH$.

Proposition: Let $H, K \leq G$. If $K \trianglelefteq G$ then $HK \trianglelefteq G$.
Proof. See draft.

Proposition: If H, K are both normal subgroups of G and $H \cap K = \{e_G\}$, then $HK \cong H \times K$.
Proof. Homework #7.

Exercise: Classify all groups of order 115.
We have that $115 = 5 \cdot 23$. Hence from Sylow's Theorem, $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 23$. This implies that $n_5 = 1$; i.e., there is a unique subgroup $P \in Syl_5(G)$. Again, from Sylow's Theorem, this implies $P \trianglelefteq G$, where $|G| = 115$.
We also have that $n_{23} \equiv 1 \pmod{23}$ and $n_{23} \mid 5$, which implies $n_{23} = 1$. So we have a unique subgroup $Q \in Syl_{23}(G)$, which gives us that $Q \trianglelefteq G$.
Note $P \cap Q \leq P$ and $P \cap Q \leq Q$, so $|P \cap Q| \mid 5$ and $|P \cap Q| \mid 23$.
We must have that $|P \cap Q| = 1$, so $P \cap Q = \{e_G\}$. Thus $PQ \cong P \times Q$.
We have $PQ \leq G$ and $|PQ| = |P||Q| = 115$, so $G = PQ$. Thus $G \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/23\mathbb{Z}$ and $G \cong \mathbb{Z}/115\mathbb{Z}$.

Recall that if $H, K \trianglelefteq G$, $H \cap K = \{e_G\}$, and $|G| = |H| \cdot |K|$, then $G \cong H \times K$.

Similarly, if $H, K \leq G$ and $H \trianglelefteq G$, then HK is a subgroup. To add to this, if $H \cap K = \{e_G\}$ and $|G| = |H||K|$, then $|G| = |HK|$.

Example: We aim to motivate the semi-direct product. (b/c HK is a subgroup).
Let $h_1 k_1, h_2 k_2 \in HK$. We have $h_1 k_1 h_2 k_2 = h_3 k_3$ for some $h_3 \in H$, $k_3 \in K$.
Observe that:

$$h_1 k_1 h_2 k_2 = h_1 \overbrace{k_1 h_2 k_1^{-1}}^{\in H} k_1 k_2$$
$$= \underbrace{(h_1 k_1 h_2 k_1^{-1})}_{h_3} \underbrace{k_1 k_2}_{k_3}.$$

Since $H \trianglelefteq G$, we have an action of G on H given by $(g, h) \mapsto ghg^{-1}$. In particular, K acts on H via conjugation: $(k, h) \mapsto k.h := khk^{-1}$. So:

$$h_1 k_1 h_2 k_2 = (h_1 k_1 h_2 k_1^{-1}) k_1 k_2$$
$$= h_1 (k_1 . h_2) k_1 k_2.$$

Let the <u>automorphisms</u> of H be $\mathrm{Aut}(H) = \{\psi : H \to H \mid \psi \text{ is an iso.}\}$.
Fix $k \in K$. Define $\varphi(k) : H \to H$ by $h \mapsto khk^{-1}$. For example, $\varphi(k)(x) = kxk^{-1}$.
Claim: $\varphi(k) \in \mathrm{Aut}(H)$.

Proof. $\varphi(k)(h_1h_2) = kh_1h_2k^{-1}$
$= kh_1k^{-1}kh_2k^{-1}$
$= \varphi(k)(h_1)\,\varphi(k)(h_2)$.

Thus $\varphi(k)$ is a homomorphism.
Suppose $\varphi(k)(h_1) = \varphi(k)(h_2)$. Then $kh_1k^{-1} = kh_2k^{-1} \Rightarrow h_1 = h_2$. Thus $\varphi(k)$ is inj.
Let $h \in H$. Since $H \trianglelefteq G$, $khk^{-1} \in H$. Then $\varphi(k)(k^{-1}hk) = k(k^{-1}hk)k^{-1} = h$.
So $\varphi(k)$ is surj.
Thus $\varphi(k) \in \mathrm{Aut}(H)$. $\square$

Define $\varphi : K \to \mathrm{Aut}(H)$ by $k \mapsto [\varphi(k) : h \mapsto khk^{-1}]$. From this new definition, observe that:

$$h_1k_1h_2k_2 = h_1\,\varphi(k_1)(h_2)\,k_1k_2.$$

We would like to remove G as a starting point. Consider the groups H, K and a homomorphism $\varphi : K \to \mathrm{Aut}(H)$. We use this to construct a new group

$$H \rtimes_\varphi K := G.$$

This group operation has the following properties:

i) H and K inject into G by:
$H \hookrightarrow H \rtimes_\varphi K;\ h \mapsto (h, e_K)$,
$K \hookrightarrow H \rtimes_\varphi K;\ k \mapsto (e_H, k)$.

ii) As sets, $H \rtimes_\varphi K = H \times K$.

iii) $(h_1, k_1) * (h_2, k_2) = (h_1\,\varphi(k_1)(h_2),\ k_1k_2)$.

<u>Example</u>: We can show $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.