Problem 1.

(1) Assume that $\sum_{n=1}^{\infty} a_n$ is convergent and $(b_n)_n$ is a bounded sequence. Does $\sum_{n=1}^{\infty} a_n b_n$ converge? Prove or provide a counterexample.

(2) Assume that $\sum_{n=1}^{\infty} |a_n|$ is convergent and $(b_n)_n$ is a bounded sequence. Does $\sum_{n=1}^{\infty} a_n b_n$ converge? Prove or provide a counter example.

Proof.

Problem 2. Show that the least upper bound property of the real numbers implies the Cauchy completeness property, that is, show that the property that every bounded set of real numbers has a least upper bound implies that every Cauchy sequence of real numbers converges in **R**.

Proof.

Problem 3. Let $(x_n)_n$ be a sequence in $\mathbf{R}$ with $|x_n - x_{n+1}| < \frac{1}{n}$ for all $n \in \mathbf{N}$

(1) If $(x_n)_n$ is bounded, must $(x_n)_n$ converge?

(2) If the subsequence $(x_{2n})_n$ converges, must (x_n) converge?

Proof.

Problem 4. Suppose that $(a_n)_n$ and $(b_n)_n$ are Cauchy sequences in $\mathbf{R}$. Prove, using the definition of Cauchy sequences, that $(|a_n - b_n|)_n$ converges in $\mathbf{R}$.

Proof.

Problem 5. Let $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + s_n}$ for $n \in \mathbf{N}$.

(1) Show that $s_n \leqslant 2$ for all n.

(2) Show that $(s_n)_n$ converges and then compute the limit of the sequence.

Proof.

Problem 6. Consider the sequence $(a_n)_n$ given by:

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}.$$

(1) Prove that $(a_n)_n$ is increasing.

(2) Prove that $(a_n)_n$ converges.

Proof.

Problem 7. Prove that the sequence $(a_n)_n$, where:

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}},$$

converges and compute its limit.

Proof.

Problem 8.

(1) Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \frac{1}{4-x_n}$ converges.

(2) Explicitly compute $\text{limit}_{n\to\infty} x_n$.

Proof.

Problem 9.

(1) Argue from the definition of Cauchy sequences that if $(a_n)_n$ and $(b_n)_n$ are Cauchy sequences, then so is $(a_n b_n)_n$.

(2) Give an example of a sequence (a_n) with $\text{limit}_{n\to\infty} |a_{n+1} - a_n| = 0$ but which is *not* Cauchy.

Proof.

Problem 10. Let $(x_n)_n$ be a sequence of real numbers satisfying:

$$|x_{n+1} - x_n| \leqslant C\,|x_n - x_{n-1}|\,,$$

for all $n \geqslant 1$, where $0 < C < 1$ is a constant. Prove that $(x_n)_n$ converges.

Proof.

Problem 11.

(1) Exhibit, with proof, a sequence of real numbers which has $[0, 1]$ as its set of limit points.

(2) Does there exist a sequence with $(0, 1)$ as its set of limit points? Give an example with proof or prove that no such sequence exists.

Proof.

Problem 12. Show that the sequence $(x_n)_n$ is Cauchy, where:

$$x_n = \int_1^n \frac{\cos t}{t^2} \, dt.$$

Proof.

Problem 13. Prove that every convergent sequence of real numbers has a maximum or minimum value.

Proof.

Problem 14. Suppose that for a function $f : \mathbf{R} \to \mathbf{R}$, there is a number $k \in (0, 1)$ such that for all $x, y \in \mathbf{R}$:

$$|f(x) - f(y)| \leqslant k|x - y|.$$

Fix a number x_0, and define a sequence by:

$$x_n = f(x_{n-1})$$

for each $n \geqslant 1$. Prove that $(x_n)_n$ is a Cauchy sequence.

Proof.

Problem 15. Let $(x_n)_n$ be a sequence such that $(x_{2n})_n$, $(x_{2n+1})_n$, and $(x_{3n})_n$ are convergent. Show that $(x_n)_n$ is convergent.

Proof.

Problem 16. Prove that the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges by showing that the sequence of partial sums is Cauchy.

Proof.

Problem 17. Suppose that $\sum_{n=1}^{\infty} x_n$ is convergent series of positive terms. Show that $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}$ are also convergent.

Proof.

Problem 18.

(1) Let $(f_n)_n$ and $(g_n)_n$ be sequences of bounded functions on a subset A of $\mathbf{R}$. Suppose that $(f_n)_n$ converges uniformly to a bounded function f and $(g_n)_n$ converges uniformly to a bounded function g. Show that $(f_n g_n)_n \to fg$ uniformly on A.

(2) Show that (a) may be false if g is unbounded. *Hint:* Consider $f_n(x) = \frac{1}{n}$ and $g_n(x) = x + \frac{1}{n}$. Prove that the convergence $(f_n g_n)_n \to fg$ in this case is not uniform on $\mathbf{R}$.

Proof.

Problem 19. Suppose that $f : [0, \infty) \to \mathbf{R}$ is a continuous, increasing, and bounded function. Prove that f is uniformly continuous on $[0, \infty)$.

Proof.

Problem 20. Let $(f_n)_n$ be a sequence of functions defined on $A \subseteq \mathbf{R}$.

(1) Prove if each f_n is uniformly continuous on A and $(f_n)_n$ converges uniformly on A to a function f, then f is uniformly continuous on A.

(2) Give a counter example to show that (a) is false if we assume pointwise convergence instead of uniform convergence.

Proof.

Problem 21.

(1) Let $f : \mathbf{R} \to \mathbf{R}$ be uniformly continuous. Show that if $(x_n)_n$ is a Cauchy sequence of real numbers, then $(f(x_n))_n$ is a Cauchy sequence.

(2) Suppose that f_n is a sequence of continuous functions that converge uniformly on a subset $A \subseteq \mathbf{R}$ to a function f. Show that f is continuous on A.

Proof.

Problem 22. Consider the function:

$$g(x) = \begin{cases} e^x, & x \in \mathbf{Q}, \\ 1, & x \notin \mathbf{Q}. \end{cases}$$

Find, with proof, the set $C = \{x \in \mathbf{R} \mid g \text{ is continuous at } x\}$.

Proof.

Problem 23. Define $f : (-1, 0) \cup (0, 1) \to \mathbf{R}$ by:

$$f(x) = \begin{cases} 4, & x \in (-1, 0), \\ 5, & x \in (0, 1). \end{cases}$$

(1) Show that f is continuous on $(-1, 0) \cup (0, 1)$.

(2) Show that f is not uniformly continuous on $(-1, 0) \cup (0, 1)$.

Proof.

Problem 24. Let $(f_n)_n$ be a sequence of functions $f_n : \mathbf{R} \to \mathbf{R}$ and let $f : \mathbf{R} \to \mathbf{R}$ be a function. Suppose f_n is bounded for each $n \in \mathbf{N}$.

(1) Prove that if $(f_n)_n \to f$ uniformly on $\mathbf{R}$, then f is bounded.

(2) If each f_n is continuous and $(f_n)_n \to f$ pointwise on $\mathbf{R}$, does f have to be bounded? Give a proof or a counterexample.

Proof.

Problem 25. Show that the sequence of functions:

$$f_n(x) = \frac{n^2 x}{1 + n^4 x^2}$$

converges pointwise to $f(x) = 0$ on $[0, 1]$, but does not converge uniformly.

Proof.

Problem 26.

(1) Let $A \subseteq \mathbf{R}$. Define what it means for $f : A \to \mathbf{R}$ to be uniformly continuous.

(2) Use the definition to show that $f(x) = \frac{1}{x}$ is uniformly continuous.

(3) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Proof.

Problem 27.

(1) Let $(f_n)_n$ be a sequence of functions defined on $A \subseteq \mathbf{R}$ that converges uniformly on A to a function f. Prove that if each f_n is continuous at $c \in A$, then f is continuous at c.

(2) Give an example to show that the result is false if we only assume that $(f_n)_n$ converges pointwise to f on A.

Proof.

Problem 28. Define $f_n : [0, \infty) \to \mathbf{R}$ by:

$$f_n(x) = \frac{\sin(nx)}{1 + nx}.$$

(1) Show that f_n converges pointwise on $[0, \infty)$ and find the pointwise limit f.

(2) Show that $(f_n)_n \to f$ uniformly on $[a, \infty)$ for every $a > 0$.

(3) Show that f_n does not converge uniformly to f on $[0, \infty)$.

Proof.

Problem 29. Let $f_n : \mathbf{R} \to \mathbf{R}$ be a sequence of continuous functions that converges uniformly on $\mathbf{R}$ to a function f. Let $(x_n)_n$ be a sequence of real numbers that converges to $x_0 \in \mathbf{R}$. Prove that $(f_n(x_n))_n \to f(x_0)$.

Proof.

Problem 30. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that:

$$\operatorname*{limit}_{x \to +\infty} f(x) = \operatorname*{limit}_{x \to -\infty} f(x) = +\infty.$$

Prove that f attains an absolute minimum value of $\mathbf{R}$. In other words, prove that there exists a real number c such that $f(c) \leqslant f(x)$ for all $x \in \mathbf{R}$.

Proof.