Math 397

Homework 6

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Exercise 3. Let X be a metric space. Let $(x_n)_n$ be a sequence in X which converges to a point $x_0 \in X$. Show that $\{x_0, x_1, x_2, ...\}$ is compact.

Proof. Let $\{V_i\}_{i \in I}$ be an open cover of $\{x_0, x_1, x_2, ...\}$. Since $x_0 \in \bigcup_{i \in I} V_i$, there exists some $V \in \{V_i\}_{i \in I}$ such that $x_0 \in V$. Because V is open, there exists $\epsilon > 0$ such that $U(x_0, \epsilon) \subseteq V$. Since $(x_n) \to x_0$, find N large so that $n \geqslant N$ implies $x_n \in U(x_0, \epsilon) \subseteq V$. For the remaining $x_1, x_2, ..., x_{N-1} \in \bigcup_{i \in I} V_i$, there exists $V_1, V_2, ..., V_{N-1} \in \{V_i\}_{i \in I}$ with $x_j \in V_j$ for each $1 \leqslant j \leqslant N-1$. So the set $\{x_0, x_1, x_2, ...\}$ is covered by the finite union $V \cup \bigcup_{i=1}^{N-1} V_i$. Thus $\{x_0, x_1, x_2, ...\}$ is compact. $\square$

Exercise 4. Let (X, d) be a metric space. If $C, K \subseteq X$ we define:

$$\operatorname{dist}(C, K) := \inf_{x \in C, y \in K} d(x, y).$$

If K is compact and C is closed, show that $K \cap C = \emptyset$ if and only if $\operatorname{dist}(C, K) > 0$. Can we remove the requirement that K is compact and only require it to be closed?

Proof. $(\Rightarrow)$ Since $\operatorname{dist}(C, K) = \inf_{x \in C, y \in K} d(x, y)$, we'd like to show:

$$(\forall \epsilon > 0)(\forall x \in C)(\forall y \in K) : d(x, y) > \epsilon.$$

Because $K \cap C = \emptyset$, for all $k \in K$ we have $k \notin C$. This means for each $k \in K$ we can find $\epsilon_k > 0$ such that $U(k, \epsilon_k) \cap C = \emptyset$. We obtain a family of open sets $\{U(k, \frac{\epsilon_k}{2})\}_{k \in K}$ which covers K. Since K is compact, we have that $K \subseteq \bigcup_{i=1}^{n} U(k_i, \frac{\epsilon_{k_i}}{2})$. Define $\epsilon = \min_{i=1}^{n} \frac{\epsilon_{k_i}}{2}$. Let $k \in K$ and $c \in C$ be arbitrary. Then $k \in U(k_i, \frac{\epsilon_{k_i}}{2})$ for some i. This gives:

$$d(k_i, c) \leqslant d(k_i, k) + d(k, c).$$

Solving for $d(k, c)$ yields:

$$d(k, c) \geqslant d(k_i, c) - d(k_i, k).$$

But note that $d(k_i, c) \geqslant \epsilon_{k_i}$ because $U(k_i, \epsilon_{k_i}) \cap C = \emptyset$. So:

$$d(k, c) \geqslant \epsilon_{k_i} - d(k_i, k).$$

Moreover, $d(k_i, k) > \frac{\epsilon_{k_i}}{2}$. We can finally see that:

$$\begin{aligned} d(k, c) &> \epsilon_{k_i} - \frac{\epsilon_{k_i}}{2} \\ &= \frac{\epsilon_{k_i}}{2} \\ &\geqslant \epsilon. \end{aligned}$$

Since we've shown ϵ is a lower bound for $\{d(x, y) \mid x \in C, y \in K\}$, it must be the case that $\inf_{x \in C, y \in K} d(x, y) \geqslant \epsilon$. Since ϵ is bounded away from zero, we have that $\operatorname{dist}(K, C) > 0$.

($\Leftarrow$) Define $A := \{d(x, y) \mid x \in C, y \in K\}$. Suppose $\operatorname{dist}(C, K) = 0$. Then $\inf(A) = 0$. Since A is both bounded below and a subset of $\mathbf{R}$, we have that $\inf(A) \in A$. So there exists $x \in C$ and $y \in K$ such that $d(x, y) = 0$. Thus $x = y$; i.e., $K \cap C \neq \emptyset$.

If we remove the requirement that K is compact, in the forward direction it could be the case that $\inf_{k \in K} \frac{\epsilon_{k_i}}{2} = 0$. This gives $\inf_{x \in C, y \in K} d(x, y) \geqslant 0$, which does not satisfy the given proposition. $\square$

Exercise 5. Let V be a finite-dimensional normed space. Show that the closed unit ball is compact.

Proof. We know there exists a homeomorphism and linear isomorphism $\varphi : \ell_p^n \to V$. Note that B_V is closed and bounded. Since φ is a continuous linear operator, $\varphi^{-1}(B_V)$ is closed by definition. Since φ^{-1} is a continuous linear operator, it is Lipschitz. So $\varphi^{-1}(B_V)$ is bounded. By the Heine-Borel Theorem, $\varphi^{-1}(B_V)$ is compact, whence $\varphi(\varphi^{-1}(B_V)) = B_V$ is also compact. $\square$

Exercise 6. Prove that the unit ball in $C([0, 1])$ is not compact.

Proof. Claim: A normed space V is finite-dimensional if and only if it's closed unit ball is compact. We've proven the forward direction in Exercise 5. For the converse direction, suppose towards contradiction $\dim(V) = \infty$. Choose $v_1 \in S_V$. Since V is infinite dimensional, $\operatorname{span}\{v_1\} \neq V$. By Riesz' Lemma, there exists a $v_2 \in S_V$ so that $\operatorname{dist}_{\operatorname{span}\{v_1\}}(v_2) \geqslant \frac{1}{2}$. In particular, $\|v_2 - v_1\| \geqslant \frac{1}{2}$. Again, since V is infinite dimensional, $\operatorname{span}\{v_1, v_2\} \neq V$. By Riesz' Lemma, there exists a $v_3 \in S_V$ so that $\operatorname{dist}_{\operatorname{span}\{v_1, v_2\}}(v_3) \geqslant \frac{1}{2}$. Then $\|v_3 - v_2\| \geqslant \frac{1}{2}$ and $\|v_3 - v_1\| \geqslant \frac{1}{2}$. Inductively, we obtain a sequence $(v_n)_n$ in S_V with $\|v_n - v_j\| \geqslant \frac{1}{2}$ for all $1 \leqslant j \leqslant n - 1$. Since $(v_n)_n$ is not Cauchy, $(v_n)_n$ does not admit a convergent subsequence. So S_V is not compact. But this is a contradiction, as any closed subset of a compact set must be compact.

Since $\dim(C([0, 1])) = \infty$, its unit ball is not compact. $\square$