**Recall**: (Lagrange's Theorem) Let $|G| < \infty$ and $H \leq G$.
We have $|H| \mid |G|$.

**Corollary**: Let $G$ be a finite group.
(1) For any $g \in G$, $|g| \mid |G|$.
(2) If $|G| = p = $ prime, then $G$ is cyclic.

proof: (1) Recall that $|g| = |\langle g \rangle|$. prop (iii)
From Lagrange's Theorem, $|\langle g \rangle| \mid |G|$.
Hence $|g| \mid |G|$.
(2) Let $g \in G$, $g \neq e_G$.
Know $\{e_G\} \subset |\langle g \rangle|$ and $|\langle g \rangle| \mid |G| \Leftrightarrow |\langle g \rangle| \mid p$. assuming $|G| = p$.
Hence $|\langle g \rangle| = 1, p$.
However $|\langle g \rangle| \neq 1$ since $g \neq e_G$.
So $|\langle g \rangle| = p$, thus $\langle g \rangle = G$.

**Exercise**: Let $G = (\mathbb{Z}/8\mathbb{Z})^\times$. Calculate all left cosets of $H = \langle [5]_8 \rangle$
$\langle [5]_8 \rangle = \{ [5]_8^k : k \in \mathbb{Z} \} = \{ [1]_8, [5]_8 \}$
$(\mathbb{Z}/8\mathbb{Z})^\times = \{ [1]_8, [3]_8, [5]_8, [7]_8 \}$.
$gH = \{ [1]_8 H = H = \{[1]_8, [5]_8\},\ [3]_8 H = \{[3]_8, [7]_8\} = [7]_8 H \}$

**Note**: If we have a group $G$ and a subgroup $H$, we don't always have $gH = Hg$.

**Definition**: Given a subgroup $H \leq G$, we say $H$ is a <u>normal subgroup</u> and write $H \trianglelefteq G$ if $gH = Hg$ for every $g \in G$.
Equality of sets
<u>Warning</u>: $gH = Hg$ does <u>NOT</u> mean $gh = hg$ $\forall h \in H$.

**Example**: Let $G = D_3$. Let $H = \langle r \rangle = \{e, r, r^2\}$. Claim: $H \trianglelefteq D_3$.
Note that $|H| = 3$ and $|D_3| = 6$. There are two distinct cosets.
$H = \{e, r, r^2\}$
$\sigma H = \{\sigma, \sigma r, \sigma r^2\}$
$He = \{e, r, r^2\} = eH$
$H\sigma = \{\sigma, r\sigma, r^2\sigma\}$.
Note $eH = He$ and $\sigma H = H\sigma$. Thus $H \trianglelefteq D_3$.
However, $\sigma r \neq r\sigma$ and ...

$(123)(123) = (132)$
$(123)(132) = (1)(2)(3) = e$

**Exercise:** (1) Let $G = S_3$ and $H = \langle (123) \rangle$. Compute the left and right cosets of $H$.

Is $H$ normal?

$H = \{e, (123), (132)\}$

$S_3 = \{e, (123), (132), (23), (12), (13)\}$

$\longrightarrow$ Two unique cosets

$eH = \{e, (123), (132)\} = He$

$(12)H = \{(12), (13), (23)\} = H(12)$

Recall Prop 18.
$G = \bigsqcup_{g \in G} gH$

Hence $H \trianglelefteq G$.

$gK = Kg$, $gN = Ng$

$gk = k_1 g$ $\quad$ $gn = n_1 g$

$g = k_1 g k^{-1}$

$k_1 g k^{-1} n = n_1 g$

let $h \in N$, $h \in K$

(2) Let $G$ be a group. Prove $H \trianglelefteq G$ iff for every $g \in G$, $gHg^{-1} = H$,

where $gHg^{-1} = \{ghg^{-1} : h \in H\}$

proof. ($\Rightarrow$) Assume $ghg^{-1} \in gHg^{-1}$.

Since $H$ is normal, $gh = h_1 g$ for some $h_1 \in H$.

Thus $ghg^{-1} = h_1 g g^{-1} = h_1 \in H$.

So $ghg^{-1} \in H$, i.e., $gHg^{-1} \subseteq H$.

Let $h \in H$. Since $H \trianglelefteq G$, $hg = gh_1$ for some $h_1 \in H$.

Thus $h = gh_1 g^{-1} \in gHg^{-1}$, i.e., $H \subseteq gHg^{-1}$

Thus $gHg^{-1} = H$.

($\Leftarrow$) Assume $gHg^{-1} = H$ $\forall g \in G$. WTS $gH = Hg$ $\forall g \in G$.

Let $gh \in gH$. We have $ghg^{-1} \in gHg^{-1} = H$, so

$ghg^{-1} = h_1$ for some $h_1 \in H$. Thus $gh = h_1 g$, so $gH \subseteq Hg$.

Let $hg \in Hg$. We have __________

**Theorem:** Let $H \trianglelefteq G$. If $aH = cH$ and $bH = dH$, then $abH = cdH$ for any $a, b, c, d \in G$.

Proof: Let $abh \in abH$. Since $aH = cH$, $\exists h_1 \in H$ s.t. $ae_G = ch_1$ for some $h_1 \in H$.

Since $bH = dH$, $bh = dh_2$ for some $h_2 \in H$.

Note now that $abh = ch_1 dh_2$.

Since $H \trianglelefteq G$, we have $h_1 d = dh_3$ for some $h_3 \in H$.

Thus $abh = ch_1 dh_2 = cdh_3 h_2 \in H$.

Hence $abH \subseteq cdH$.

Show $cdH \subseteq abH$ as exercise. $\square$

**Exercise:** Let $\varphi : G \longrightarrow H$ be a homomomorphism. Show $\operatorname{Ker}\varphi \trianglelefteq G$.

Proof: Let $K = \ker \varphi$.

We can show $gKg^{-1} = K$ $\forall g \in G$.

Let $gkg^{-1} \in gKg^{-1}$. We have $\varphi(gkg^{-1}) = \dots = e_H$.

Thus $gkg^{-1} \in K$, and $gKg^{-1} \subseteq K$.

Let $k \in K$. W.T.S. $k = g k_1 g^{-1}$ for some $k_1 \in K$.

Note