Math 397

Homework 5

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Exercise 1. Show that $C_0(\mathbf{R})$ is a Banach space.

Proof. Note that $C_b(\mathbf{R}) \supseteq C_0(\mathbf{R})$ is a Banach space. Let $(f_n)_n$ be a sequence in $C_0(\mathbf{R})$ converging to $f \in C_b(\mathbf{R})$. Let $\epsilon > 0$ and find N large so that $\|f - f_N\| < \frac{\epsilon}{2}$. Since $f_N \in C_0(\mathbf{R})$, find M large so $|x| > M$ implies $|f_N(x)| < \frac{\epsilon}{2}$. For $|x| > M$ we have:

$$\begin{aligned} |f(x)| &= |f(x) - f_N(x) + f_N(x)| \\ &\leqslant |f(x) - f_N(X)| + |f_N(x)| \\ &\leqslant \|f - f_N\| + |f_N(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $\operatorname{limit}_{|x|\to\infty} = 0$. Since $f \in C_0(\mathbf{R})$, we have that $C_0(\mathbf{R}) \subseteq C_b(\mathbf{R})$ is closed; i.e., it is complete. $\square$

Exercise 2. Show that ℓ_2 is a Hilbert space.

Proof. Let $(f_n)_n$ be $\|\cdot\|_{\ell_2}$-Cauchy. Let $\epsilon > 0$. Find N_1 large so $n, m \geqslant N_1$ implies $\|f_n - f_m\|_{\ell_2} < \epsilon$. Then:

$$\begin{aligned} |f_n(k) - f_m(k)| &\leqslant \|f_n - f_m\|_{\ell_2} \\ &< \epsilon. \end{aligned}$$

So $(f_n(k))_n$ is Cauchy in $\mathbf{C}$. Since $\mathbf{C}$ is complete, define $f(k) := \operatorname{limit}_{n\to\infty} f_n(k)$. Claim: $f \in \ell_2$ and $\operatorname{limit}_{n\to\infty} f_n = f$.

We will first show that $f \in \ell_2$. Note that since $(f_n)_n$ is $\|\cdot\|_{\ell_2}$-Cauchy, it is bounded. For $K > 1$, observe that:

$$\begin{aligned} \sum_{j=1}^{K} |f(j)|^2 &= \sum_{j=1}^{K} \Big|\operatorname*{limit}_{n\to\infty} f_n(j)\Big|^2 \\ &= \operatorname*{limit}_{n\to\infty} \sum_{j=1}^{K} |f_n(j)|^2 \\ &\leqslant \sup_{n\geqslant 1} \|f_n\|_{\ell_2}^2 \\ &< \infty. \end{aligned}$$

Since $\left(\sum_{j=1}^{K} |f(j)|^2\right)_K$ is increasing and bounded above, the Monotone Convergence Theorem says its limit exists. This means:

$$\begin{aligned}\operatorname*{limit}_{K\to\infty}\sum_{j=1}^{K}|f(j)|^2 &= \sum_{j=1}^{\infty}|f(j)|^2\\ &= \|f\|_{\ell_2}^2\\ &< \infty.\end{aligned}$$

Thus $f \in \ell_2$.

We will now show that f is the limit of our Cauchy sequence. With the same epsilon as before, find N_2 large so that $n, m \geqslant N_2$ implies $\|f_n - f_m\|_2 < \frac{\epsilon^2}{2}$. Then:

$$\begin{aligned}\sum_{j=1}^{K}|f_n(j) - f_m(j)| &\leqslant \|f_n - f_m\|_{\ell_2}^2\\ &< \frac{\epsilon^2}{4}.\end{aligned}$$

Taking the limit as $m \to \infty$ and considering all $n \geqslant N_2$ gives $\sum_{j=1}^{K}|f_n(j) - f(j)| \leqslant \frac{\epsilon^2}{4}$. Taking the limit as $K \to \infty$ gives:

$$\begin{aligned}\sum_{j=1}^{\infty}|f_n(j) - f(j)| &= \|f_n - f\|_{\ell_2}^2\\ &\leqslant \frac{\epsilon^2}{4}\\ &< \epsilon^2.\end{aligned}$$

Square-rooting both sides establishes $(f_n)_n \to f$. Thus ℓ_2 is a Banach space.

Define $\langle\cdot,\cdot\rangle : \ell_2 \times \ell_2 \to \mathbf{C}$ by $\langle f, g\rangle = \sum_{k=1}^{\infty} f(j)\overline{g(j)}$. We must first verify that this series exists. Note that:

$$\begin{aligned}\sum_{j=1}^{K}|f(j)\overline{g(j)}| &= \sum_{j=1}^{K}|f(j)||g(j)|\\ &\leqslant \left(\sum_{j=1}^{K}|f(j)|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{K}|g(j)|^2\right)^{\frac{1}{2}}\\ &\leqslant \|f\|_{\ell_2}\|g\|_{\ell_2}\\ &< \infty.\end{aligned}$$

Since $\left(\sum_{j=1}^{K}|f(j)\overline{g(j)}|\right)_K$ is increasing and bounded above, its limit exists by the Monotone Convergence Theorem. So $\sum_{j=1}^{\infty}|f(j)\overline{g(j)}|$ converges. In particular, since $(\mathbf{C}, |\cdot|)$ is a Banach space, $\sum_{j=1}^{\infty} f(j)\overline{g(j)}$ will converge.

Let $f, g_1, g_2 \in \ell_2$ and $\alpha \in \mathbf{C}$. Observe that:

$$\begin{aligned}\langle f, g_1 + \alpha g_2\rangle &= \sum_{j=1}^{\infty} f(j)\overline{(g_1 + \alpha g_2)(j)}\\ &= \sum_{j=1}^{\infty} f(j)\overline{g_1(j)} + \overline{\alpha}\sum_{j=1}^{\infty} f(j)\overline{g_2(j)}\\ &= \langle f, g_1\rangle + \overline{a}\langle f, g_2\rangle.\end{aligned}$$

Now let $f_1, f_2, g \in \ell_2$ and $\alpha \in \mathbf{C}$. Observe that:

$$\begin{aligned}
\langle f_1 + \alpha f_2, g \rangle &= \sum_{j=1}^{\infty} (f_1 + \alpha f_2)(j)\overline{g(j)} \\
&= \sum_{j=1}^{\infty} f_1(j)\overline{g(j)} + \alpha \sum_{j=1}^{\infty} f_2(j)\overline{g(j)} \\
&= \langle f_1, g \rangle + \alpha \langle f_2, g \rangle.
\end{aligned}$$

Thus $\langle \cdot, \cdot \rangle$ is a sesquilinear form. Moreover, we can see:

$$\begin{aligned}
\langle f, g \rangle &= \sum_{j=1}^{\infty} f(j)\overline{g(j)} \\
&= \overline{\sum_{j=1}^{\infty} g(j)\overline{f(j)}} \\
&= \overline{\langle g, f \rangle}.
\end{aligned}$$

Whence $\langle \cdot, \cdot \rangle$ is Hermitian. Finally, if $f \neq 0$, we have:

$$\begin{aligned}
\langle f, f \rangle &= \sum_{j=1}^{\infty} f(j)\overline{f(j)} \\
&= \sum_{j=1}^{\infty} |f(j)|^2 \\
&> 0.
\end{aligned}$$

Thus $\langle \cdot, \cdot \rangle$ is positive definite, establishing it as an inner-product. Thus ℓ_2 is a Hilbert space. $\square$

Exercise 3. Suppose (X, d) is a complete metric space and $(x_n)_n$ is a *contractive* sequence in X, that is, there exists a $\theta \in (0, 1)$ with $d(x_{n+1}, x_n) \leqslant \theta d(x_n, x_{n-1})$. Show that $(x_n)_n$ is convergent.

Proof. Note that $d(x_{n+1}, x_n) \leqslant \theta^{n-1} d(x_2, x_1)$. Without loss of generality, for $n > m$ we have:

$$\begin{aligned}
d(x_n, x_m) &\leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \\
&\vdots \\
&\leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\
&\leqslant \theta^{n-2} d(x_2, x_1) + \theta^{n-3} d(x_2, x_1) + ... + \theta^{m-1} d(x_2, x_1) \\
&= (\theta^{n-m-1} + \theta^{n-m-2} + ... + 1)\theta^{m-1} d(x_2, x_1) \\
&= \left(\frac{1 - \theta^{n-m}}{1 - \theta}\right) \theta^{m-1} d(x_2, x_1) \\
&\leqslant \left(\frac{1}{1 - \theta}\right) \theta^{m-1} d(x_2, x_1) \\
&\to 0.
\end{aligned}$$

Thus $(x_n)_n$ is Cauchy. Since (X, d) is complete, $(x_n)_n$ converges. □

Exercise 4. Let (X, d) be a complete metric space and suppose $f : X \to X$ is a contractive map; i.e., for all $x, y \in X$ there is a $\theta \in (0, 1)$ with:

$$d(f(x), f(y)) \leqslant \theta d(x, y).$$

Prove that f has a unique fixed point.

Proof. Define $(x_n)_n$ in X by $x_n = f(x_{n-1})$. We can see:

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\
&\leqslant \theta d(x_n, x_{n-1}).
\end{aligned}$$

Thus $(x_n)_n$ is contractive. By Exercise 3, $(x_n)_n$ is convergent. Define $x := \operatorname{limit}_{n\to\infty} x_n$. Observe that:

$$\begin{aligned}
x &= \operatorname*{limit}_{n\to\infty} x_n \\
&= \operatorname*{limit}_{n\to\infty} f(x_{n-1}) \\
&= f\left(\operatorname*{limit}_{n\to\infty} x_{n-1}\right) \\
&= f(x).
\end{aligned}$$

So f admits a fixed point. Suppose $x' \in X$ is also a fixed point. Then:

$$\begin{aligned}
d(x, x') &= d(f(x), f(x')) \\
&\leqslant \theta d(x, x').
\end{aligned}$$

Note that this only holds if $d(x, x') = 0$ Thus $x = x'$, establishing that f admits a unique fixed point. □

Exercise 6. Let $T : V \to W$ be a continuous linear map between normed spaces which is bounded below, that is, there is a $C > 0$ with $\|Tv\| \geqslant C\|v\|$ for all $v \in V$. If V is complete, show that $\operatorname{im}(T) \subseteq W$ is a closed subspace, and that $V \cong \operatorname{im}(T)$ are uniformly isomorphic.

Proof. Let $(T(v_n))_n$ be a sequence in $\operatorname{im}(T)$ converging to $w \in W$. Given ϵ, find N large so that $n \geqslant M$ implies $\|T(v_n) - w\| < \frac{C\epsilon}{2}$. For $n, m \geqslant N$, observe that:

$$\begin{aligned} \|v_n - v_m\| &\leqslant \frac{1}{C}\|T(v_n - v_m)\| \\ &= \frac{1}{C}\|T(v_n) - T(v_m)\| \\ &\leqslant \frac{1}{C}\|T(v_n) - w\| + \frac{1}{C}\|w - T(v_m)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $(v_n)_n$ is Cauchy. Since V is complete, let $v_0 := \operatorname{limit}_{n\to\infty} v_n$. Since T is continuous, we can see $(T(v_n))_n \to T(v_0)$. It must be the case that $T(v_0) = w$; i.e., $w \in \operatorname{im}(T)$. Thus $\operatorname{im}(T)$ is a closed subspace.

Since T is continuous, there exists some $\alpha > 0$ such that $\|Tv\| \leqslant \alpha\|v\|$. Clearly if $v = 0$, then $Tv = 0$, implying that T is injective. Whence $V \cong \operatorname{im}(T)$ as vector spaces. Since T is continuous, it is uniformly continuous, so it remains to show that $T^{-1} : \operatorname{im}(T) \to V$ (which exists) is also continuous. Let $w \in \operatorname{im}(T)$, then there exists $v \in V$ with $T(v) = w$. Observe that:

$$\begin{aligned} \left\|T^{-1}w\right\| &= \left\|T^{-1}(T(v))\right\| \\ &= \|v\| \\ &\leqslant \frac{1}{C}\|Tv\| \\ &= \frac{1}{C}\|w\|. \end{aligned}$$

Thus T is uniformism. □

Exercise 7. Let (X, d) and (Y, ρ) be metric spaces with completions $(\widetilde{X}, \iota_X)$ and $(\widetilde{Y}, \iota_Y)$ respectively. If $f : X \to Y$ is an isometry, show that there is a unique isometry $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ that extends f, that is, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle \iota_x}\uparrow & & \uparrow{\scriptstyle \iota_y} \\ X & \xrightarrow{f} & Y \end{array}$$

Proof. Define $\varphi : \iota(X) \to \widetilde{Y}$ by $\varphi(\iota(x)) = \iota_Y(f(x))$. Since f and ι_Y are isometries, note that their composition $\iota_Y \circ f$ is also an isometry. This gives:

$$\begin{aligned} \rho(\varphi(\iota_X(x_1)), \varphi(\iota_X(x_2))) &= \rho(\iota_Y(f(x_1)), \iota_Y(f(x_2))) \\ &= d(x_1, x_2). \end{aligned}$$

Since φ is an isometry, the unique uniformly continuous extension $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is also an isometry. □

Exercise 8. Let V be a normed space, W a Banach space, and $U \subseteq V$ a dense linear subspace. Moreover, let $T_0 : U \to W$ be a bounded linear map. Show that there is a unique bounded linear map $T : V \to W$ that extends T_0, that is, $T|_U = T_0$.

Proof. Clearly V is a metric space, $U \subseteq V$ is a dense subset, and W is a complete metric space. So there exists a uniformly continuous map $T : V \to W$ with $T(v) = T_0(v)$ for all $v \in U$. Hence we only need to show T is linear and bounded. Let $v, v' \in V$ and $\alpha \in F$. Let $(x_n)_n$ and $(y_n)_n$ be sequences in U with $(x_n)_n \to v$ and $(y_n)_n \to v'$. Observe that:

$$\begin{aligned} T(v + \alpha v') &= \operatorname*{limit}_{n\to\infty} T_0(x_n + \alpha y_n) \\ &= \operatorname*{limit}_{n\to\infty} T_0(x_n) + \alpha \operatorname*{limit}_{n\to\infty} T_0(y_n) \\ &= T(v) + \alpha T(v'). \end{aligned}$$

Thus T is linear. To show T is bounded, it suffices to show $\|T\|_{\text{op}} = \|T_0\|_{\text{op}}$, since T_0 is bounded. Note that the composition $V \xrightarrow{T} W \xrightarrow{\|\cdot\|_W} F$ will be continuous and bounded, which means:

$$\begin{aligned} \|T\|_{\text{op}} &= \sup_{v\in B_V} \|T(v)\|_W \\ &= \sup_{v\in B_U} \|T(v)\|_W \\ &= \sup_{v\in B_U} \|T_0(v)\|_W \\ &= \|T_0\|_{\text{op}} . \end{aligned}$$

Thus $T \in B(V, W)$. $\square$

Exercise 9. Let X be a metric space. Show that the following are equivalent:

(1) Every meager set has empty interior.

(2) The complement of a meager set is dense.

Moreover, show that these equivalent statements hold true if the metric space is complete.

Proof. If $A \subseteq X$ is meager with $A^0 = \emptyset$, then $\overline{A^c} = (A^o)^c = \emptyset^c = X$. The converse is identical.

Now suppose X is a complete metric space. If $A \subseteq X$ is meager, then $A = \bigcup_{n\geqslant 1} A_n$ We will show that the complement of A is dense. Clearly $\overline{\bigcap_{n\geqslant 1} A_n^c} \subseteq X$, so it remains to show the other direction of inclusion. Define $B_n = \overline{A_n}$. Clearly $A_n \subseteq B_n$, which implies that $A_n^c \supseteq B_n^c$ for each n. Whence $\bigcap_{n\geqslant 1} A_n^c \supseteq \bigcap_{n\geqslant 1} B_n^c$. Furthermore, $\overline{\bigcap_{n\geqslant 1} A_n^c} \supseteq \overline{\bigcap_{n\geqslant 1} B_n^c}$. Note that each B_n^c is open and dense, so by Baire's theorem we have $\overline{\bigcap_{n\geqslant 1} B_n^c} = X$. Thus $\overline{\left(\bigcup_{n\geqslant 1} A_n\right)^c} = X$. Since (1) and (2) are equivalent, we've established that both statements hold true if X is a complete metric space. $\square$

Exercise 10. Let V be a normed space with linear basis B.

(1) If $W \subseteq V$ is a proper subspace, show that $W^o = \emptyset$.

(2) If V is a Banach space, show that B is uncountable. You may use the fact that finite-dimensional subspaces are always closed.

Proof. Suppose towards contradiction that $W^o \neq \emptyset$. Then we can find some $v_0 \in W^o$. In particular, there exists $\delta > 0$ such that $U(v_0, \delta) \subseteq W$. Let $v \in V$. We can see that $\frac{\delta}{2}\frac{v}{\|v\|} + v_0 \in U(v_0, \delta) \subseteq W$, so for some $w \in W$ we have:

$$\frac{\delta}{2}\frac{v}{\|v\|} + v_0 = w.$$

Solving for v yields $v = \|v\| \frac{2}{\delta}(w - v_0)$. But $\|v\| \frac{2}{\delta}(w - v_0) \in W$, so $v \in W$, which contradicts $W \subseteq V$ being a proper subspace. Thus $W^o = \emptyset$.

Suppose towards contradiction B is countable, that is, $B = \{e_n \mid n \geqslant 1\}$. Then:

$$\begin{aligned} V &= \operatorname{span}\{e_n \mid n \geqslant 1\} \\ &= \bigcup_{n \geqslant 1} \operatorname{span}\{e_1, ..., e_n\}. \end{aligned}$$

Note that $\operatorname{span}\{e_1, ..., e_n\}$ is a finite *and* proper subspace of V. This means $\overline{\operatorname{span}\{e_1, e_2, ..., e_n\}}^o = \emptyset$. But this contradicts Baire's theorem, since we've written V —a complete normed space—as the countable union of nowhere dense sets. It must be the case that B is uncountable. $\square$