Math 397

Homework 1

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Exercise 1. Let V be a vector space, and suppose $\{W_i\}_{i\in I}$ is a family of subspaces of V.

(1) Show that $\bigcap_{i\in I} W_i$ is the largest subspace of V contained in every W_i.

(2) Show that:

$$\sum_{i\in I} W_i = \left\{ \sum_{i\in F} w_i \mid w_i \in W_i,\ F \subseteq I \text{ finite} \right\}$$

is the smallest subspace containing each W_i.

(3) We say that V is the *internal direct sum* of the family $\{W_i\}_{i\in I}$ and we write $V = \bigoplus_{i\in I} W_i$ if:

(i) $V = \sum_{i\in I} W_i$;

(ii) For each $j \in I$, $W_j \cap \sum_{i\neq j} W_i n\{0\}$.

If $V = \bigoplus_{i\in I} W_i$, show that every $v \in V$ has a unique expression $v = \sum_{i\in F} w_i$ where $F \subseteq I$ is finite and $0 \neq w_i$ for each $w_i \in W_i$.

Proof. (1) Let U be a subspace of V with $U \subseteq W_i$ for each $i \in I$. Then clearly $U \subseteq \bigcap_{i\in I} W_i$.

(2) Let $W = \sum_{i\in I} W_i$ and let U be a subspace of V with $U \supseteq W_i$ for each $i \in I$. If $x \in W$, then $x = \sum_{i\in I} w_i$. But since W_i is a subspace, it is closed under addition. Whence $x \in W_i$ for each $i \in I$. By inclusion then, $x \in U$. Hence $W \subseteq U$.

(3) By the definition of internal direct sums $V = \sum_{i\in I} W_i$, whence each $v \in V$ can be written as $v = \sum_{i\in F} w_i$. It remains to show that this expression is unique. Suppose $v = \sum_{i\in F} w_i = \sum_{i\in F} u_i$ with $w_i, u_i \in W_i$. For each j we have:

$$w_j - u_j = \sum_{\substack{i\in F\\ i\neq j}} (w_i - u_i)$$

But notice that $w_j - u_j \in W_j$ and $\sum_{i\in F, i\neq j}(w_i - u_i) \in \sum_{i\neq j} W_i$. So $w_j - u_j \in W_j \cap \sum_{i\neq j} W_i$. By the definition of internal direct sums this gives $w_j - u_j = 0$, which simplifies to $w_j = u_j$. □

Exercise 3. Let V be a vector space with subspaces $W_i \subseteq V$ for $i = 1, 2$. If $W_1 \cup W_2 \subseteq V$ is a subspace, show that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. Suppose towards contradiction $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Then there exists $w_1 \in W_1 \setminus W_2$ and $w_2 \in W_2 \setminus W_1$. Let $v = w_1 + w_2$. Then $v \in W_1 \cup W_2$. But this means $w_2 = v - w_1 \in W_2$. Whence $w_1 \in W_2$, which is a contradiction. □

Exercise 4. Let V be a vector space over F and suppose $W \subset V$ is a subspace.

(1) Show that the quotient space $V/W = \{[v]_w \mid v \in V\}$ is a vector space with operations:

$$[u]_W + [v]_W = [u+v]_W\,; \quad \alpha[v]_W = [\alpha v]_W.$$

(2) Suppose $\|\cdot\|$ is a norm on V. Show that:

$$\|[v]_W\|_{V/W} := \inf_{w \in W} \|v - w\|$$

is a seminorm.

Proof. (1) Since V is an abelian group and $W \subseteq V$ is normal, V/W is an abelian group. It only remains to show that $\alpha[v]_W = [\alpha v]_W$ satisfies the vector space axioms. We have that:

$$\begin{aligned} \alpha\left([u]_W + [v]_W\right) &= \alpha[u+v]_W \\ &= [\alpha(u+v)]_W \\ &= [\alpha u + \alpha v]_W \\ &= [\alpha u]_W + [\alpha v]_W, \end{aligned}$$

$$\begin{aligned} \alpha\left(\beta[v]_W\right) &= \alpha[\beta v]_W \\ &= [\alpha(\beta v)]_W \\ &= [(\alpha\beta) v]_W \\ &= (\alpha\beta)[v]_W, \end{aligned}$$

$$\begin{aligned} 1_F[v]_W &= [1_F v]_W \\ &= [v]_W. \end{aligned}$$

Whence V/W is a vector space.

(2) We must first show that $\|\cdot\|_{V/W} : V/W \to F$ is well-defined. Let $[v_1]_W = [v_2]_W$. Then $v_2 - v_1 \in W$. Observe that:

$$\begin{aligned} \|[v_1]_W\|_{V/W} &= \inf_{w \in W} \|v_1 - w\| \\ &= \inf_{w-(v_2-v_1) \in W} \left\|v_1 - \left(w - (v_2 - v_1)\right)\right\| \\ &= \inf_{w-(v_2-v_1) \in W} \|v_1 - w + v_2 - v_1\| \\ &= \inf_{w \in W} \|v_2 - w\| \\ &= \|[v_2]_W\|_{V/W}. \end{aligned}$$

We also have that:

$$\begin{aligned} \|\alpha[v]_W\|_{V/W} &= \|[\alpha v]_W\|_{V/W} \\ &= \inf_{w \in W} \|\alpha v - w\| \\ &= \inf_{w' \in W} \|\alpha v - \alpha w'\| \\ &= \inf_{w' \in W} \|\alpha(v - w')\| \\ &= |\alpha| \inf_{w' \in W} \|v - w'\| \\ &= |\alpha| \, \|[v]_W\|_{V/W}. \end{aligned}$$

Whence $\|\cdot\|_{V/W}$ is homogenous. Finally, we can see that:

$$\begin{aligned}
\|[u]_W + [v]_W\|_{V/W} &= \|[u+v]_W\|_{V/W} \\
&= \inf_{w \in W} \|u + v - w\| \\
&= \inf_{w, w' \in W} \|u + v - (w + w')\| \\
&= \inf_{w, w' \in W} \|u - w + v - w'\| \\
&\leqslant \inf_{w, w' \in W} (\|u - w\| + \|v - w'\|) \\
&= \inf_{w \in W} \|u - w\| + \inf_{w' \in W} \|v - w'\| \\
&= \|[u]_W\|_{V/W} + \|[v]_W\|_{V/W}.
\end{aligned}$$

Thus $\|\cdot\|_{V/W}$ is a seminorm. □

Exercise 5. Show that the quantity:

$$\|f\|_1 := \int_0^1 |f(t)|\,dt$$

defines a norm on $C([0,1])$ with $\|f\|_1 \leqslant \|f\|_u$. Are $\|\cdot\|_1$ and $\|\cdot\|_u$ equivalent norms?

Proof. $\|\cdot\|_1$ is homogenous because:

$$\begin{aligned}
\|\alpha f\|_1 &= \int_0^1 |(\alpha f)(t)|\,dt \\
&= \int_0^1 |\alpha f(t)|\,dt \\
&= |\alpha| \int_0^1 |f(t)|\,dt \\
&= |\alpha|\,\|f\|_1.
\end{aligned}$$

Note that $|f(t) + g(t)| \leqslant |f(t)| + |g(t)|$. Integrating both sides gives:

$$\begin{aligned}
\int_0^1 |f(t) + g(t)|\,dt &= \int_0^1 |(f+g)(t)|\,dt \\
&= \|f + g\|_1 \\
&\leqslant \int_0^1 (|f(t)| + |g(t)|)\,dt \\
&= \int_0^1 |f(t)|\,dt + \int_0^1 |g(t)|\,dt \\
&= \|f\|_1 + \|g\|_1.
\end{aligned}$$

Whence our norm satisfies the triangle inequality. Now suppose $\|\cdot\|_1 = 0$. Then $\int_0^1 |f(t)|\,dt = 0$. Suppose $f \geqslant 0$ on $[0,1]$. Since f is continuous, it is continuous at $f(c)$ for some $c \in [0,1]$. If $f(c) > 0$, then there exists $\delta > 0$ such that $f(t) \geqslant \frac{f(c)}{2} > 0$ for all $t \in V_\delta(c)$. This gives:

$$0 = \int_0^1 f(t)dt \geqslant \int_{c-\delta}^{c+\delta} f(t)dt \geqslant \int_{c-\delta}^{c+\delta} \frac{f(c)}{2} = f(c) > 0.$$

This is a contradiction. Since $c \in [0,1]$ was arbitrary, it must be that $f = 0$, satisfying positive-definiteness. Moreover, note that $|f(t)| \leqslant \sup_{t\in[0,1]} |f(t)|$. We have that $\int_0^1 |f(t)|\,dt \leqslant \int_0^1 \sup_{t\in[0,1]} |f(t)|\,dt$, which is equivalent to $\|f\|_1 \leqslant \int_0^1 \|f\|_u\,dt = \|f\|_u$.

Suppose that $\|\cdot\|_1$ and $\|\cdot\|_u$ are equivalent. Then $\|f\|_u \leqslant c\,\|f\|_1$. Consider $g(t) = t^N$, where $N > c$. Then:

$$\begin{aligned} 1 &= \sup_{t\in[0,1]} |t^N| \\ &\leqslant \int_0^1 |t^N|dt \\ &= \frac{c}{N} \\ &< 1, \end{aligned}$$

This is a contradiction, hence $\|\cdot\|_1$ and $\|\cdot\|_u$ are not equivalent. □

Exercise 6. Show that all the p-norms $\|\cdot\|_p$ $(1 \leqslant p \leqslant \infty)$ on F^n are equivalent and if $1 \leqslant p \leqslant q \leqslant \infty$, then $\ell_p \subseteq \ell_q$.

Proof. Let $x \in F^n$. We have that:

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^n \left(\max_{i=1}^n |x_i|\right)^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n \|x\|_\infty^p\right)^{\frac{1}{p}} = n^p \|x\|_\infty.$$

$$\|x\|_\infty = \left(\left(\max_{i=1}^n |x_i|\right)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = \|x\|_p.$$

$$\|x\|_\infty = \max_{i=1}^n |x_i| \leqslant \sum_{i=1}^n |x_i| = \|x_i\|_1.$$

$$\|x\|_1 = \sum_{i=1}^n |x_i| \leqslant \sum_{i=1}^n \max_{i=1}^n |x_i| = n \max_{i=1}^n |x_i| = n\,\|x\|_\infty.$$

From this, and since equivalent norms form an equivalence relation, all norms on F^n are equivalent.

Suppose $p = 1$ and $q = \infty$. Let $(x_n)_n \in \ell_1$. Then clearly $\sup_{i=1}^\infty |x_i| \leqslant \sum_{i=1}^\infty |x_i| < \infty$. Whence $\ell_1 \subseteq \ell_\infty$. Now suppose $p, q < \infty$ with $p \leqslant q$. Let $(x_n)_n \in \ell_p$. Then $\sum_{n=1}^\infty |x_n|^p < \infty$. In particular, $(x_n)_n \longrightarrow 0$, which implies that $(|x_n|)_n \longrightarrow 0$. From this, there exists $K \in \mathbf{N}$ large such that for all $n \geqslant K$, we have $0 \leqslant x_n < 1$. It then follows that the tail $\sum_{n\geqslant K} |x_n|^p$ converges. Whence:

$$\sum_{n\geqslant K} |x_n|^q \leqslant \sum_{n\geqslant K} |x_n|^p < \infty.$$

Thus $\sum_{n=1}^\infty |x_n|^q < \infty$, establishing that $(x_n)_n \in \ell_q$.

□

Exercise 7. Let $M_{m,n}(\mathbf{C})$ denote the linear space of all $m \times n$ matrices with coefficients from $\mathbf{C}$. For $A \in M_{m,n}(\mathbf{C})$, set:

$$\|A\|_{op} := \sup_{\xi \in B_{\ell_2^n}} \|A\xi\|_{\ell_2^m}.$$

Show that $\|\cdot\|_{op}$ is a norm on $M_{m,n}(\mathbf{C})$.

Proof. Observe that:

$$\begin{aligned}
\|\alpha A\|_{op} &= \sup_{\xi \in B_{\ell_2^n}} \|(\alpha A)\xi\|_{\ell_2^n} \\
&= \sup_{\xi \in B_{\ell_2^n}} \|\alpha(A\xi)\|_{\ell_2^n} \\
&= |\alpha| \sup_{\xi \in B_{\ell_2^n}} \|A\xi\|_{\ell_2^n} \\
&= \alpha \|A\|_{op}.
\end{aligned}$$

Thus $\|\cdot\|_{op}$ is homogenous. We also have:

$$\begin{aligned}
\|A + B\|_{op} &= \sup_{\xi \in B_{\ell_2^n}} \|(A + B)\xi\|_{\ell_2^n} \\
&= \sup_{\xi \in B_{\ell_2^n}} \|A\xi + B\xi\|_{\ell_2^n} \\
&\leqslant \sup_{\xi \in B_{\ell_2^n}} \left(\|A\xi\|_{\ell_2^n} + \|B\xi\|_{\ell_2^n}\right) \\
&= \sup_{\xi \in B_{\ell_2^n}} \|A\xi\|_{\ell_2^n} + \sup_{\xi \in B_{\ell_2^n}} \|B\xi\|_{\ell_2^n} \\
&= \|A\|_{op} + \|B\|_{op}.
\end{aligned}$$

Hence $\|\cdot\|_{op}$ satisfies the triangle inequality. Now suppose $\|A\|_{op} = 0$. Then $\sup_{\xi \in B_{\ell_2^n}} \|A\xi\|_{\ell_2^n} = 0$. Since $\|\cdot\|_{\ell_2^n}$ is positive definite, the set $\{\|A\xi\|_{\ell_2^n} \mid \xi \in B_{\ell_2^n}\}$ must only contain positive real numbers. Whence if the supremum of this set equals 0, it must be that $\|A\xi\|_{\ell_2^n} = 0$ for all $\xi \in B_{\ell_2^n}$. Again, by the positive-definiteness of $\|\cdot\|_{\ell_2^n}$, we have that $A\xi = 0$ for all $\xi \in B_{\ell_2^n}$. Whence $A = 0$. This gives that $\|\cdot\|_{op}$ is a norm. $\square$

Exercise 10. Let p be a semi-norm on a vector space V.

(1) Show that $N_p = \{w \in V \mid p(w) = 0\}$ is a subspace of V.

(2) We form the quotient vector space V/N_p. Show that

$$\left\|[v]_{N_p}\right\|_p := p(v)$$

defines a norm on V/N_p.

(3) If $(E, \|\cdot\|)$ is a normed space and $T : V \to E$ is a linear map, show that $p(v) := \|T(v)\|$ is a semi-norm on V. In this case what is N_p?

Proof. (1) Let $w_1, w_2 \in N_p$ and $\alpha \in F$. Then:

$$p(w_1 + \alpha w_2) \leqslant p(w_1) + |\alpha| p(w_2) = 0.$$

Since $w_1 + \alpha w_2 \in N_p$, N_p is a subspace.

(2) We must first show that $\|\cdot\|_p$ is well-defined. Let $[v_1]_{N_p} = [v_2]_{N_p}$. Then $v_1 = v_2 + w$ for some $w \in N_p$. Then:

$$\begin{aligned}
\left\|[v_1]_{N_p}\right\|_p &= p(v_1) \\
&= p(v_1 + w) \\
&\leqslant p(v_2) + p(w) \\
&= p(v_2) \\
&= \left\|[v_2]_{N_p}\right\|_p
\end{aligned}$$

So $\left\|[v_1]_{N_p}\right\|_p \leqslant \left\|[v_2]_{N_p}\right\|_p$. But note that we also have $v_2 = v_1 + w$ for some $w \in N_p$. This will give $\left\|[v_2]_{N_p}\right\|_p \leqslant \left\|[v_1]_{N_p}\right\|_p$, whence by antisymmetry $\left\|[v_1]_{N_p}\right\|_p = \left\|[v_2]_{N_p}\right\|_p$. Now let $\alpha \in F$. Observe that:

$$\begin{aligned}
\left\|\alpha[v]_{N_p}\right\|_p &= \left\|[\alpha v]_{N_p}\right\|_p \\
&= p(\alpha v) \\
&= |\alpha| p(v) \\
&= |\alpha| \left\|[v]_{N_p}\right\|_p .
\end{aligned}$$

Thus $\|\cdot\|_p$ satisfies homogeneity. The triangle inequality is also satisfied because:

$$\begin{aligned}
\left\|[v]_{N_p} + [w]_{N_p}\right\|_p &= \left\|[v + w]_{N_p}\right\|_p \\
&= p(v + w) \\
&\leqslant p(v) + p(w) \\
&= \left\|[v]_{N_p}\right\|_p + \left\|[w]_{N_p}\right\|_p .
\end{aligned}$$

Suppose $\left\|[v]_{N_p}\right\|_p = 0$. Then $p(v) = 0$. But this means $v \in N_p$, whence $[v]_{N_p} = [0]_{N_p}$. Hence $\|\cdot\|_p$ is a norm.

(3) We have that:

$$\begin{aligned}
p(\alpha v) &= \|T(\alpha v)\| \\
&= \|\alpha T(v)\| \\
&= |\alpha| \, \|T(v)\| \\
&= |\alpha| p(v).
\end{aligned}$$

Thus p satisfies homogeneity. We also get:

$$\begin{aligned} p(v+w) &= \|T(v+w)\| \\ &= \|T(v)+T(w)\| \\ &\leqslant \|T(v)\| + \|T(w)\| \\ &= p(v)+p(w). \end{aligned}$$

Thus p is a semi-norm. Observe that:

$$\begin{aligned} N_p &= \{v \in V \mid p(v) = 0\} \\ &= \{v \in V \mid \|T(v)\| = 0\} \\ &= \{v \in V \mid T(v) = 0\} \\ &= \ker(T). \end{aligned}$$

$\square$