Consider the group $\mathbb{Z}/6\mathbb{Z}$.

if $m, 6$ relatively prime, subgroup generates entire group.

Subgroups: $\{[0]_6\}$, $\langle [1]_6 \rangle = \mathbb{Z}/6\mathbb{Z}$.

$$\langle [2]_6 \rangle = \{[0]_6, [2]_6, [4]_6\}$$
$$\langle [3]_6 \rangle = \{[0]_6, [3]_6\}$$

$\mathbb{Z}/6\mathbb{Z} \dashrightarrow G/N$ where $G = \mathbb{Z}$, $N = 6\mathbb{Z}$

$\{[0]_6, [2]_6, [4]_6\}$ $\qquad 6\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$

everything div. by 6 is div. by 2.

$$2\mathbb{Z}/6\mathbb{Z} = \{2m + 6\mathbb{Z} : m \in \mathbb{Z}\}$$
$$= \{[0]_6, [2]_6, [4]_6\}$$

$\{[0]_6, [3]_6\}$ $\qquad 6\mathbb{Z} \subseteq 3\mathbb{Z} \subseteq \mathbb{Z}$

$3\mathbb{Z}/6\mathbb{Z}$

If we want subgroups of G/N, we should look at subgroups H of G so that $N \subseteq H \subseteq G$ and then form H/N.

$$\langle [0]_6 \rangle = 6\mathbb{Z}/6\mathbb{Z}$$
$$\langle [3]_6 \rangle = 3\mathbb{Z}/6\mathbb{Z}$$
$$\langle [2]_6 \rangle = 2\mathbb{Z}/6\mathbb{Z}$$
$$\langle [1]_6 \rangle = \mathbb{Z}/6\mathbb{Z}$$

Proposition: Let $N \trianglelefteq G$, and $N \leq H \leq G$, then H/N is a subgroup of G/N.

proof: HW

Theorem: (3rd Isomorphism Thm) Let K, N be normal subgroups of G with $N \leq K \leq G$.
Then $K/N \trianglelefteq G/N$ and

$$(G/N)/(K/N) \cong G/K.$$

proof: Let $\varphi: G/N \to G/K$
$$gN \to gK$$

By def:
$g_2^{-1}g_1 \in N$
$g_2^{-1}g_1 = n$

Suppose $g_1N = g_2N$. Note $g_1 = g_2 n$ for some $n \in N$
Observe that
$$\begin{aligned}\varphi(g_1N) &= g_1K \\ &= g_2nK \\ &= g_2K \quad \text{since } N \leq K \\ &= \varphi(g_2N)\end{aligned}$$

Thus φ is well-defined.

Let $g_1N, g_2N \in G/N$. We have

$$\begin{aligned}\varphi(g_1N\, g_2N) &= \varphi(g_1g_2N) \\ &= g_1g_2K \\ &= g_1Kg_2K \\ &= \varphi(g_1N)\varphi(g_2N)\end{aligned}$$

Let $gK \in G/K$. Then $\varphi(gN) = gK$, φ is surjective.

It only remains to show $\ker\varphi = K/N$.

Let $kN \in K/N$. The $\varphi(kN) = kK = K$, so $kN \in \ker\varphi$. Hence $K/N \in \ker\varphi$.

Let $gN \in \ker\varphi$, i.e., $\varphi(gN) = K$.
So $gK = K$, so $g \in K$ since $g = k$, for some $k \in K$.
Hence $\ker\varphi \subseteq K/N$ and we have equality.

So K/N is normal b/c it is the ker of a homom. 1^{st} iso. thm. gives

$$(G/N)/(K/N) \cong G/K.$$

Corollary: Let $N \trianglelefteq G$ and K any subgroup of G w/ $N \leq K$. Then $K \trianglelefteq G$ iff $K/N \trianglelefteq G/N$

Proof: If $K \trianglelefteq G$, 3rd Iso Thm gave $K/N \trianglelefteq G/N$.

Assume $K/N \trianglelefteq G/N$. W.T.S. $gKg^{-1} = K \ \forall g \in G$.
Let $g \in G$, $k \in K$. We have

$$g k g^{-1} N = gN\, kN\, g^{-1}N$$

$gNg^{-1} = N$
$gng^{-1} \in N$

$$= gN\, kN\, (gN)^{-1} \in K/N$$

b/c $K/N \trianglelefteq G/N$

if $gkg^{-1}N \in K/N$, then $gkg^{-1} \in K$

Thus $gkg^{-1} \in K$. Hence $gKg^{-1} \subseteq K$. elm of K (so by def $k \in gKg^{-1}$ where $k \in K$ So $K \subseteq gKg^{-1}$

Let $k \in K$. Then $k = g(g^{-1}kg)g^{-1} \in g^{-1}Kg \subseteq K$

$g^{-1}Kg$ in K

$g^{-1}Kg \subseteq K$
so $g^{-1}kg \in K$

$\in gKg^{-1}$ equal to

Thus $K \subseteq gKg^{-1}$, so $gKg^{-1} = K \ \forall g \in G$; i.e., $K \trianglelefteq G$. □

Theorem: Let T be a subgroup of G/N. Then $T = H/N$ for some subgroup $H \leq G$ w/ $N \subseteq H \subseteq G$.

proof: Define $H = \{g \in G : gN \in T\}$.
We show H is a subgroup.
Since $T \leq G/N$, it contains eN. Thus $e \in H$.
Let $g_1, g_2 \in H$; i.e., $g_1N, g_2N \in T$.

Since $T \leq G/N$, $g_2^{-1}N \in T$.

and $g_1 N g_2^{-1} N \in T$.
So $g_1 g_2^{-1} N \in T$
So $g_1 g_2^{-1} \in H$. So H is a subgroup.

Let $n \in N$. We have $nN = N \in T$ b/c T is a subgroup ($N = e_{G/N}$)
Thus, $n \in H$. Hence $N \leq H$.

subgroup so has to contain identity of G/N, i.e. $e_{G/N} = N$.

We have $H/N = \{hN : h \in H\} = T$

In Groups: List all the subgroups

$$(\mathbb{Z}/12\mathbb{Z})/H \quad \text{where} \quad H = \langle [6]_{12} \rangle$$

$$\langle [6]_{12} \rangle = 6\mathbb{Z}/12\mathbb{Z}$$

$$(\mathbb{Z}/12\mathbb{Z})/(6\mathbb{Z}/12\mathbb{Z})$$

Theorem: Let T be a subgroup of G/N. Then $T = H/N$ for some subgroup $H \leq G$ w/ $N \leq H \leq G$.

$$(\mathbb{Z}/12\mathbb{Z})/(6\mathbb{Z}/12\mathbb{Z})$$

$$6\mathbb{Z}/12\mathbb{Z} \subseteq H \subseteq \mathbb{Z}/12\mathbb{Z}$$

$$\{[0]_{12}, [6]_{12}\}$$

$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}/3\mathbb{Z}$

$$(\mathbb{Z}/12\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}) \leq (\mathbb{Z}/12\mathbb{Z})/(6\mathbb{Z}/12\mathbb{Z})$$

$(\mathbb{Z}/2\mathbb{Z})$

$$H = \langle [6]_{12} \rangle \subseteq N \subseteq \mathbb{Z}/12\mathbb{Z}$$
$$= \{ [0]_{12}, [6]_{12} \}$$

$$\langle [2]_{12} \rangle$$
$$\langle [3]_{12} \rangle$$

$$\langle [2]_{12} \rangle / [6]_{12} \quad \text{and} \quad \langle [3]_{12} \rangle / [6]_{12}$$

<u>Example</u>: Compute all the homomorphic images of S_3.

Let $\varphi : S_3 \longrightarrow G$ for G some group.

$$\varphi(S_3) = \{ e_G \}$$

$$\varphi : S_3 \twoheadrightarrow \text{im}(\varphi) \leq G$$

Map always surjective onto image

$$S_3 / \ker \varphi \cong \text{im}\,\varphi.$$

No matter what φ is, $\ker \varphi \trianglelefteq S_3$. only possible kernels.

$$\ker \varphi \overset{\text{" "}}{=} \{ e_G \}, \ S_3, \ \langle (123) \rangle.$$

no other normal subgroups.

$$S_3 / \langle (123) \rangle = \{ \langle (123) \rangle, (12)\langle (123) \rangle \} \cong \mathbb{Z}/2\mathbb{Z}.$$