

Recall: A group G acts on a set X by the following properties:

- $e_G \cdot x = x$
- $g_1 g_2 \cdot x = g_1 \cdot (g_2 \cdot x)$.

The orbit of an element $x \in X$ is:

$$\text{Orb}(x) = \{g \cdot x : g \in G\}.$$

Likewise, $x \sim y$ iff $y \in \text{Orb}(x)$ is an equivalence relation on X .

Definition: Let G act on X . The stabilizer of $x \in X$ is:

$$\text{Stab}_G(x) = \{g \in G : g \cdot x = x\}.$$

Definition: Let G act on X . The fixed points of $g \in G$ are:

$$\text{Fix}_X(g) = \{x \in X : g \cdot x = x\}.$$

Example: Let G act on X . We will show that $\text{Stab}_G(x) \leq G$. Let $g \in \text{Stab}_G(x)$. We have that $g \cdot x = x$, so:

$$g^{-1} \cdot (g \cdot x) = g^{-1} \cdot x$$

$$\Rightarrow (g^{-1}g) \cdot x = g^{-1} \cdot x$$

$$\Rightarrow e_G \cdot x = g^{-1} \cdot x$$

$$\Rightarrow x = g^{-1} \cdot x.$$

Hence if $g \in \text{Stab}_G(x)$, then $g^{-1} \in \text{Stab}_G(x)$. Let $g, h \in \text{Stab}_G(x)$. Then:

$$\begin{aligned} (gh) \cdot x &= g \cdot (h \cdot x) \\ &= g \cdot x \\ &= x \end{aligned}$$

Thus $\text{Stab}_G(x)$ is a subgroup of G .

Theorem: (Orbit-Stabilizer Theorem) Let G be a finite group that acts on X .
Then:

$$|\text{Orb}(x)| = |G| / |\text{Stab}_G(x)|.$$

Proof. We will prove the equality by showing there exists a bijection.
Define $\varphi: G/\text{Stab}_G(x) \rightarrow \text{Orb}(x)$ by $g\text{Stab}_G(x) \mapsto g.x$. We must first show our mapping is well-defined. Suppose $g_1\text{Stab}_G(x) = g_2\text{Stab}_G(x)$. We can write $g_1 = g_2h$, where $h \in \text{Stab}_G(x)$. Observe that:

$$\begin{aligned} \varphi(g_1\text{Stab}_G(x)) &= g_1.x \\ &= (g_2h).x \\ &= g_2.(h.x) \\ &= g_2.x && \text{Since } h \in \text{Stab}_G(x). \\ &= \varphi(g_2\text{Stab}_G(x)). \end{aligned}$$

Handwritten notes on the left:
 $g_2^{-1}g_1 \in H$
 $g_2^{-1}g_1 = h$
 $g_1 = g_2h$
 $\rightarrow g_1.x = g_2.x$
 $g_1.x =$

Thus φ is well-defined. It remains to show that φ is a bijection. Let $g_1\text{Stab}(x), g_2\text{Stab}(x) \in G/\text{Stab}_G(x)$ with $\varphi(g_1\text{Stab}(x)) = \varphi(g_2\text{Stab}(x))$. So $g_1.x = g_2.x$, which is equivalent to $g_2^{-1}(g_1.x) = x$, i.e., $(g_2^{-1}g_1).x = x$. Thus $g_2^{-1}g_1 \in \text{Stab}_G(x)$, so $g_1\text{Stab}_G(x) = g_2\text{Stab}_G(x)$. Let $g.x \in \text{Orb}(x)$. We have $\varphi(g\text{Stab}(x)) = g.x$. Thus φ is surjective. Hence φ is bijective and we have that $|G/\text{Stab}_G(x)| = |\text{Orb}(x)|$. From Lagrange's Theorem, $|G|/|\text{Stab}_G(x)| = |\text{Orb}(x)|$. \square

Lemma: (Burnside's Lemma) Let G be a finite group acting on a finite set X . We have:

$$\# \text{orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}_X(g)|.$$

Proof. Note that:

$$\begin{aligned} \# \text{orbits} &= \frac{1}{|G|} \sum_{x \in X} \frac{1}{|\text{Orb}(x)|} = \frac{1}{|G|} \sum_{x \in X} |\text{Stab}_G(x)| \\ &= \frac{1}{|G|} \sum_{x \in X} \left(\sum_{\substack{g \in G \\ g.x = x}} 1 \right) \\ &= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{\substack{x \in X \\ g.x = x}} 1 \right) \\ &= \frac{1}{|G|} \sum_{g \in G} |\text{Fix}_X(g)|. \end{aligned}$$

\square

Example: Let G be finite and acts on itself by conjugation. Note then that:

$$\begin{aligned}\text{Orb}(h) &= \{g.h : g \in G\} \\ &= \{ghg^{-1} : g \in G\}\end{aligned}$$

$$\begin{aligned}\text{Stab}(h) &= \{g \in G, g.h = h\} \\ &= \{g \in G, ghg^{-1} = h\}\end{aligned} \quad (1)$$

From (1), this is called the centralizer of h , denoted $C_G(h)$. If G acts on X , $|X| = \sum_{\substack{\text{dist.} \\ \text{orb.}}} |\text{Orb}(x)|$. From our conjugation action, we will get

$|G|$ on the left hand side. The center of h , denoted $Z_G(h)$, is defined as:

$$Z_G(h) = \{h \in G : gh = hg \forall g \in G\}.$$

If $x \in Z_G(g)$, then $gxg^{-1} = x \forall g \in G$. Hence $\text{Orb}(x) = \{x\}$. We have that:

$$\begin{aligned}|G| &= \sum_{\substack{\text{dist.} \\ \text{orb.}}} |\text{Orb}(x)| \\ &= \sum_{x \in Z(G)} |\{x\}| + \sum_{\substack{\text{dist.} \\ \text{orb.} \\ \text{not in } Z(G)}} |\text{Orb}(x)| \\ &= |Z(G)| + \sum_{\substack{\text{dist.} \\ \text{orb.} \\ \text{not in } Z(G)}} |\text{Orb}(x)|.\end{aligned}$$

Let x_1, \dots, x_n be representatives for the distinct orbits not overlapping $Z(G)$. Then:

$$\begin{aligned}|G| &= |Z(G)| + \sum_{i=1}^n |\text{Orb}(x_i)| \\ &= |Z(G)| + \sum_{i=1}^n \frac{|G|}{|\text{Stab}_G(x_i)|} \\ &= |Z(G)| + \sum_{i=1}^n \frac{|G|}{|C_G(x_i)|}.\end{aligned} \quad (2)$$

Equation (2) is defined as The Class Equation.

Theorem: Let G be a finite group and assume $p \mid |G|$ for p prime.
Then G has an element of order p i.e., a subgroup of order p .