

Monotone Convergence Theorem

Definitions

- (1) A sequence $(x_n)_n$ is monotone if it is either increasing, decreasing, strictly increasing, or strictly decreasing.

Theorems/Propositions/Lemmas

- (1) A convergent sequence is bounded.

Proof. Suppose $(x_n)_n \rightarrow x$. Since $(x_n)_n$ is convergent, we know:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni n \geq N \implies |x_n - x| < \epsilon.$$

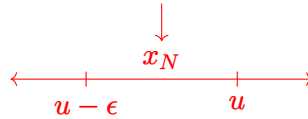
Pick $\epsilon = 1$. Then there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $x_n \in V_1(x)$. Define:

$$c = \max\{|x_1|, |x_2|, \dots, |x_{N_1}|, |x - 1|, |x + 1|\}.$$

If $n \leq N_1$, then $|x_n| \leq c$. If $n \geq N_1$, then $|x_n| \leq c$. □

- (2) (Monotone Convergence Theorem) Let $(x_n)_n$ be a monotone sequence. $(x_n)_n$ is convergent if and only if $(x_n)_n$ is bounded. Moreover, If $(x_n)_n$ is increasing and bounded above, then $\lim x_n = \sup\{x_n \mid n \in \mathbf{N}\}$ or if $(x_n)_n$ is decreasing and bounded below, then $\lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}$

Proof. (\implies) This direction was showed in (1). (\impliedby) Suppose $(x_n)_n$ is bounded above and increasing. Let $u = \sup\{x_n \mid n \in \mathbf{N}\}$. Supremum property says given $\epsilon > 0$, there exists $N \in \mathbf{N}$ with $u - \epsilon < x_N$.



But for $n \geq N$:

$$u - \epsilon < x_N \leq x_n \leq u < u + \epsilon.$$

Hence $|x_n - u| < \epsilon$, establishing that $(x_n)_n \rightarrow u$. Now let $y_n = -x_n$. Then y_n is increasing and bounded above. We get:

$$\begin{aligned} \lim y_n = \sup\{y_n \mid n \in \mathbf{N}\} &\implies -\lim x_n = \sup\{-x_n \mid n \in \mathbf{N}\} \\ &\implies -\lim x_n = -\inf\{x_n \mid n \in \mathbf{N}\} \\ &\implies \lim x_n = \inf\{x_n \mid n \in \mathbf{N}\}. \end{aligned}$$

□

(3) If $(x_n)_n$ is increasing and unbounded, then $(x_n)_n$ diverges properly to $+\infty$.

Proof. Pick M large. Since $(x_n)_n$ is unbounded, there exists $N \in \mathbf{N}$ with $x_N > M$. Hence if $n \geq N$, then $x_n \geq x_N > M$, establishing $(x_n)_n \rightarrow +\infty$. \square

Examples

(1) Let $x_1 = 8$ and inductively set $x_{n+1} = \frac{1}{2}x_n + 2$. Show that $(x_n)_n$ converges and find its limit.

Solution. Note that $(x_n)_n = (8, 6, 5, \frac{9}{2}, \dots)$. We will show this sequence is bounded below by 4 and decreasing. Clearly $x_1 = 8 \geq 4$. Now assume $x_n \geq 4$. Then:

$$\begin{aligned} x_{n+1} &= \frac{1}{2}x_n + 2 \\ &\geq \frac{1}{2}(4) + 2 \\ &= 4. \end{aligned}$$

Moreover,

$$\begin{aligned} x_{n+1} \leq x_n &\iff \frac{1}{2}x_n + 2 \leq x_n \\ &\iff 4 \leq x_n. \end{aligned}$$

Thus $(x_n)_n$ is bounded below by 4 and decreasing. By MCT $(x_n)_n \rightarrow L$. Observe that:

$$\begin{aligned} x_{n+1} = \frac{1}{2}x_n + 2 &\xrightarrow{n \rightarrow \infty} L = \frac{1}{2}L + 2 \\ &\iff L = 4. \end{aligned}$$

(2) Let $x_n = \sum_{k=1}^n \frac{1}{k^2}$. Show that $(x_n)_n$ converges.

Solution. Clearly $x_n \leq x_{n+1}$. We have:

$$\begin{aligned} x_n &= \sum_{k=1}^n \frac{1}{k^2} \\ &= 1 + \sum_{k=2}^n \frac{1}{k^2} \\ &\leq 1 + \sum_{k=2}^n \frac{1}{k(k-1)} \quad \text{Since } k^2 \geq k(k-1) \\ &= 1 + \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k} \right) \quad \text{Partial fractions} \\ &= 1 + \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n} \right) \right] \\ &= 1 + 1 - \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &\leq 2. \end{aligned}$$

Since $(x_n)_n$ is increasing and bounded above, by MCT $(x_n)_n \rightarrow L$.

(3) Given $a > 0$, construct a sequence $(x_n)_n$ which converges to \sqrt{a} .

Solution. Let $x_1 = 1$ and inductively set $x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$. Observe that:

$$\begin{aligned} 2x_{n+1} = x_n + \frac{a}{x_n} &\implies 2x_{n+1}x_n = x_n^2 + a \\ &\implies x_n^2 - 2x_{n+1}x_n + a = 0. \end{aligned}$$

By assumption $(x_n)_n$ converges, hence this polynomial has a real root. So:

$$\begin{aligned} \Delta \geq 0 &\implies 4x_{n+1}^2 - 4a \geq 0 \\ &\implies x_{n+1}^2 \geq a. \end{aligned}$$

Whence $(x_n)_n$ bounded below. It remains to show that $(x_n)_n$ is decreasing. Observe that:

$$\begin{aligned} x_n \geq x_{n+1} &\iff x_n \geq \frac{1}{2} \left(x_n + \frac{a}{x_n} \right) \\ &\iff 2x_n \geq x_n + \frac{a}{x_n} \\ &\iff x_n \geq \frac{a}{x_n} \quad \text{Since } x_n + x_n \geq x_n + \frac{a}{x_n} \\ &\iff x_n^2 \geq a \\ &\iff x_{n+1}^2 \geq a. \quad \text{Since } a \text{ is a lowerbound} \end{aligned}$$

Hence by MCT, $(x_n)_n \rightarrow L$. This gives:

$$\begin{aligned} x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right) &\xrightarrow{n \rightarrow \infty} L = \frac{1}{2} \left(L + \frac{a}{L} \right) \\ &\implies L^2 = a \\ &\implies L = \sqrt{a}. \end{aligned}$$

(4) Let $h_n = \sum_{k=1}^n \frac{1}{k}$. Show that $(h_n)_n \rightarrow +\infty$.

Solution. Clearly $(h_n)_n$ is increasing. Observe that:

$$\begin{aligned} h_2 &= 1 + \frac{1}{2} \geq 1 + \frac{1}{2} \\ h_{2^2} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + 2 \left(\frac{1}{2} \right) \\ h_{2^3} &= \dots = 1 + 3 \left(\frac{1}{2} \right) \end{aligned}$$

Inductively, $h_{2^n} \geq 1 + \frac{n}{2}$. Since $(1 + \frac{1}{n})_n \rightarrow +\infty$, $(h_n)_n \rightarrow +\infty$.

Subsequences

Definitions

- (1) A natural sequence is a strictly increasing sequence of natural numbers $(n_k)_k$ with $n_k \in \mathbf{N}$.
- (2) Let $(x_n)_n$ be a sequence. A subsequence of $(x_n)_n$ is a sequence $(x_{n_k})_k$ where $(n_k)_k$ is a natural sequence. Formally, a subsequence is a composition of maps $\mathbf{N} \xrightarrow{k \mapsto n_k} \mathbf{N} \xrightarrow{n_k \mapsto x_{n_k}} X$
- (3) If $(x_n)_n$ is a sequence of real numbers, a peak of a sequence is a term x_m with $x_m \geq x_n$ for all $n \geq m$.

Theorems/Propositions/Lemmas

- (1) Given a natural sequence $(n_k)_k$, $n_k \geq k$ for all k .

Proof. Clearly $n_1 \geq 1$. Now assume $n_k \geq k$. Then $n_{k+1} \geq n_k + 1 \geq k + 1$. □

- (2) Suppose $(x_n)_n \rightarrow x$. For any subsequence $(x_{n_k})_k$, we have $(x_{n_k})_k \rightarrow x$.

Proof. Since $(x_n)_n \rightarrow x$, $(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni n \geq N \implies |x_n - x| < \epsilon$. Consider $K = N$. Then $k \geq K$ implies $k \geq N$. But by (1) $n_k \geq k \geq N$. Hence $|x_{n_k} - x| < \epsilon$, establishing $(x_{n_k})_k \rightarrow x$. □

- (3) Let $(x_n)_n$ be a sequence. Then $(x_n)_n \not\rightarrow x$ if and only if there exists $\epsilon_0 > 0$ and a subsequence $(x_{n_k})_k$ such that $d(x_{n_k}, x) \geq \epsilon_0$.

Proof. (\Leftarrow) If $(x_n)_n \rightarrow x$, then any subsequence $(x_{n_k})_k$ converges to x . (\Rightarrow) Since $(x_n)_n \not\rightarrow x$:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(n \geq N \wedge d(x_n - x) \geq \epsilon_0).$$

Note that:

$$\begin{aligned} N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geq 1 \wedge d(x_{n_1}, x) \geq \epsilon_0) \\ N = n_1 + 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 > n_1 \wedge d(x_{n_2}, x) \geq \epsilon_0) \\ N = n_2 + 1 &\implies (\exists n_3 \in \mathbf{N})(n_3 > n_2 \wedge d(x_{n_3}, x) \geq \epsilon_0) \\ &\vdots \\ N = n_k + 1 &\implies (\exists n_{k+1} \in \mathbf{N})(n_{k+1} > n_k \wedge d(x_{n_{k+1}}, x) \geq \epsilon_0) \end{aligned}$$

Hence $(x_{n_k})_k$ is a subsequence satisfying $d(x_{n_k}, x) \geq \epsilon_0$. □

- (4) Let $(x_n)_n$ be a real sequence. There is a subsequence that is monotone.

Proof. We proceed with cases. Case 1: there are infinitely many peaks. Let $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ be an enumeration of peaks. Then $(x_{n_k})_k$ is decreasing by definition. Case 2: there are finitely many peaks. Let $x_{m_1}, x_{m_2}, \dots, x_{m_r}$ be the peaks of our sequence. Then $m_1 < m_2 < \dots < m_r$ by definition. Let $n_1 = m_r + 1$. Since x_{n_1} is not a peak, there exists $n_2 > n_1$ such that $x_{n_3} > x_{n_2}$. Inductively, we obtain a sequence $(x_{n_k})_k$ with $x_{n_k} < x_{n_{k+1}}$. \square

- (5) (Bolzano-Weierstrass Theorem) If $(x_n)_n$ is a real sequence that is bounded, it admits a convergent subsequence.

Proof. Since $(x_n)_n$ is a bounded real sequence it admits a monotone subsequence $(x_{n_k})_k$ which is bounded. By the monotone convergence theorem $(x_{n_k})_k$ converges. \square

- (6) If $(x_n)_n$ is an unbounded sequence of real numbers, show that there is a subsequence $(x_{n_k})_k$ such that $\left(\frac{1}{x_{n_k}}\right)_k \xrightarrow{k \rightarrow \infty} 0$.

Proof. Since $(x_n)_n$ is an unbounded real sequence:

$$(\exists \epsilon_0 > 0)(\forall N \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(n \geq N \wedge |x_n - 0| \geq \epsilon_0).$$

We can construct a subsequence as follows:

$$\begin{aligned} N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geq 1 \wedge |x_{n_1}| \geq \epsilon_0) \\ N = n_1 + 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 \geq n_1 \wedge |x_{n_2}| \geq \epsilon_0) \\ &\vdots \end{aligned}$$

Inductively, we obtain a sequence $(x_{n_k})_k$ which properly diverges to $+\infty$. Given $\epsilon > 0$, let K be arbitrarily big so that $\epsilon > \frac{1}{n_K}$. Then for $k \geq K$, we have $\left|\frac{1}{x_{n_k}}\right| < \epsilon$. \square

- (7) Suppose that every subsequence of a sequence $(x_n)_n$ has a subsequence that converges to 0. Show that $(x_n)_n \rightarrow 0$.

Proof. Suppose towards contradiction that $(x_n)_n \not\rightarrow 0$. Then there exists a subsequence $(x_{n_k})_k \not\rightarrow 0$. By definition:

$$(\exists \epsilon_0 > 0)(\forall K \in \mathbf{N}) \ni (\exists k \in \mathbf{N})(k \geq K \wedge d(x_{n_k}, 0) \geq \epsilon_0).$$

We will construct a subsequence of $(x_{n_k})_k$ as follows:

$$\begin{aligned} K = 1 &\implies (\exists k_1 \in \mathbf{N})(k_1 \geq 1 \wedge d(x_{n_{k_1}}, 0) \geq \epsilon_0) \\ K = k_1 + 1 &\implies (\exists k_2 \in \mathbf{N})(k_2 \geq k_1 \wedge d(x_{n_{k_2}}, 0) \geq \epsilon_0) \\ &\vdots \end{aligned}$$

Inductively, we obtain a sequence $(x_{n_{k_j}})_j \not\rightarrow 0$. But this contradicts our claim that every subsequence has a subsequence which converges to 0. Hence it must be that $(x_n)_n \rightarrow 0$. \square

- (8) If $(x_n)_n$ is a bounded sequence and $s := \sup_n x_n$ is such that $s \notin \{x_n \mid n \geq 1\}$, show that there is a subsequence $(x_{n_k})_k$ that converges to s .

Examples

Limit Inferior & Limit Superior

Definitions

- (1) Let $X = (x_n)_n$ be a fixed bounded sequence who's limit may not exist. Then $\overline{X} = \{t \in \mathbf{R} \mid t = \lim_{k \rightarrow \infty} x_{n_k}, x_{n_k} \text{ some subsequence}\}$ is the set containing all subsequential limits (or limit points) of X .
- (2) Let $(x_n)_n$ be a bounded sequence.
 - (i) $l = \lim_{m \rightarrow \infty} l_m = \lim_{m \rightarrow \infty} (\inf_{n \geq m} x_n) := \liminf x_n$
 - (ii) $u = \lim_{m \rightarrow \infty} u_m = \lim_{m \rightarrow \infty} (\sup_{n \geq m} x_n) := \limsup x_n$.

Theorems/Propositions/Lemmas

- (1) Let $X = (x_n)_n$ be a bounded sequence with $l = \liminf x_n$ and $u = \limsup x_n$. If $x \in X$, then $x \in [l, u]$.

Proof. Note that:

$$\begin{aligned} \inf_{n \geq n_k} x_n \leq x_{n_k} &\implies \lim_{k \rightarrow \infty} (\inf_{n \geq n_k} x_n) \leq \lim_{k \rightarrow \infty} x_{n_k} \\ &\implies l \leq x. \end{aligned}$$

$$\begin{aligned} \sup_{n \geq n_k} x_n \geq x_{n_k} &\implies \lim_{k \rightarrow \infty} (\sup_{n \geq n_k} x_n) \geq \lim_{k \rightarrow \infty} x_{n_k} \\ &\implies u \geq x. \end{aligned}$$

□

- (2) Let $(x_n)_n = X$ be a bounded sequence. Let $l = \liminf x_n$ and $u = \limsup x_n$. Then $l, u \in \overline{X}$.

Proof. Let $u_m = \sup_{n \geq m} x_n$. By the supremum property:

$$\begin{aligned} N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geq 1 \wedge u_1 - 1 < x_{n_1} \leq u_1) \\ N = n_1 + 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 > n_1 \wedge u_2 - \frac{1}{2} < x_{n_2} \leq u_2) \\ N = n_2 + 1 &\implies (\exists n_3 \in \mathbf{N})(n_3 > n_2 \wedge u_3 - \frac{1}{3} < x_{n_3} \leq u_3) \\ &\vdots \end{aligned}$$

Inductively:

$$\begin{aligned} u_k - \frac{1}{k} < x_{n_k} \leq u_k &\implies \lim_{k \rightarrow \infty} u_k < \lim_{k \rightarrow \infty} x_{n_k} \leq \lim_{k \rightarrow \infty} u_k \\ &\implies u < \lim_{k \rightarrow \infty} x_{n_k} \leq u. \end{aligned}$$

By the squeeze theorem, $(x_{n_k})_k \rightarrow u$. Now let $l_m = \inf_{n \geq m} x_n$. By the infimum property:

$$\begin{aligned} N = 1 &\implies (\exists n_1 \in \mathbf{N})(n_1 \geq 1 \wedge l_1 \leq x_{n_1} < l_1 + 1) \\ N = n_1 + 1 &\implies (\exists n_2 \in \mathbf{N})(n_2 > n_1 \wedge l_2 \leq x_{n_2} < l_2 + \frac{1}{2}) \\ N = n_2 + 1 &\implies (\exists n_3 \in \mathbf{N})(n_3 > n_2 \wedge l_3 \leq x_{n_3} < l_3 + \frac{1}{3}) \\ &\vdots \end{aligned}$$

Inductively:

$$\begin{aligned} l_k \leq x_{n_k} < l_k + \frac{1}{k} &\implies \lim_{k \rightarrow \infty} l_k \leq \lim_{k \rightarrow \infty} x_{n_k} < \lim_{k \rightarrow \infty} l_k + \frac{1}{k} \\ &\implies l \leq \lim_{k \rightarrow \infty} x_{n_k} < l. \end{aligned}$$

By the squeeze theorem, $(x_{n_k})_k \rightarrow l$. Hence $l, u \in \overline{X}$. □

(3) * Let $(x_n)_n$ be bounded.

(i) $\liminf x_n \leq \limsup x_n$.

(ii) $(x_n)_n \rightarrow x$ if and only if $\liminf x_n = \limsup x_n = x$.

Proof. (i) Note that $l_m \leq u_m$ for all $m \geq 1$. Taking the limit $m \rightarrow \infty$ gives $l \leq u$.

(ii) (\Rightarrow) If $(x_n)_n \rightarrow x$, then every subsequence $(x_{n_k})_k \rightarrow x$. But we showed in (2) that there exists subsequences which converge to l and u . Whence $x = l = u$. (\Leftarrow) If $l = u = x$, then $\overline{X} = [x, x] = \{x\}$. Hence every subsequence $(x_{n_k})_k \rightarrow x$. Thus $(x_n)_n \rightarrow x$. □

Examples

Cauchy Sequences

Definitions

- (1) A sequence $(x_n)_n$ is Cauchy if:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (m, n \in \mathbf{N})(m, n \geq N \implies d(x_n, x_m) < \epsilon)$$

- (2) A sequence $(x_n)_n$ is contractive if there exists $0 < \rho < 1$ with $|x_{n+1} - x_n| \leq \rho|x_n - x_{n-1}|$ for all $n \geq 2$. We say ρ is the constant of contraction.

Theorems/Propositions/Lemmas

- (1) Cauchy sequences are bounded.

Proof. Pick $\epsilon = 1$. Then $(\exists N \in \mathbf{N}) \ni (\forall m, n \in \mathbf{N})(m, n \geq N \implies |x_n - x_m| < 1)$. Let $c = \max\{|x_1|, \dots, |x_N|\}$. But consider that:

$$n \geq N \implies |x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| < 1 + |x_N|.$$

So $|x_n| \leq c'$, where $c' = \max\{c, 1 + |x_N|\}$. □

- (2) If $(x_n)_n$ is Cauchy and there exists a subsequence $(x_{n_k})_k \rightarrow x$, then $(x_n)_n \rightarrow x$.

Proof. Since $(x_n)_n$ is Cauchy, given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $n, n_k \geq N$ implies $d(x_n, x_{n_k}) < \frac{\epsilon}{2}$. Since $(x_{n_k})_k \rightarrow x$, given $\epsilon > 0$ there exists $K \in \mathbf{N}$ such that $k \geq K$ implies $d(x_{n_k}, x) < \frac{\epsilon}{2}$. Let $J = \max\{K, N\}$. For $n, n_k, k \geq J$:

$$|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \epsilon.$$

□

- (3) Let $(x_n)_n$ be a sequence. $(x_n)_n$ is Cauchy if and only if $(x_n)_n$ converges.

Proof. (\implies) Suppose $(x_n)_n \rightarrow x$. Let $\epsilon > 0$. There exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - x| < \frac{\epsilon}{2}$. For $m, n \geq N$, we have $|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \epsilon$. (\impliedby) If $(x_n)_n$ is Cauchy then $(x_n)_n$ is bounded. Bolzano-Weierstrass theorem says there exists a convergent subsequence. By (2) $(x_n)_n$ converges. □

- (4) Contractive Sequences are Cauchy.

Proof. Let $(x_n)_n$ be a contractive sequence. Observe that:

$$\begin{aligned} |x_3 - x_2| &\leq \rho |x_2 - x_1| \\ |x_4 - x_3| &\leq \rho |x_3 - x_2| \leq \rho^2 |x_2 - x_1| \\ &\vdots \end{aligned}$$

Inductively, $|x_{n+1} - x_n| \leq \rho^{n-1} |x_2 - x_1|$. For $m > n$:

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \dots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n| \\ &\leq \rho^{m-2} |x_2 - x_1| + \rho^{m-3} |x_2 - x_1| + \dots + \rho^{n-1} |x_2 - x_1| \\ &= \rho^{n-1} |x_2 - x_1| (1 + \rho + \rho^2 + \dots + \rho^{m-n-1}) \\ &= \rho^{n-1} |x_2 - x_1| \frac{1 - \rho^{m-n}}{1 - \rho} \\ &\leq \frac{\rho^{n-1}}{1 - \rho} |x_2 - x_1| \\ &= \frac{\rho^n}{\rho(1 - \rho)} |x_2 - x_1|. \end{aligned}$$

Given $\epsilon > 0$, find N so large that $\frac{\rho^n}{\rho(1-\rho)} |x_2 - x_1| < \epsilon$. When $m, n \geq N$, then $|x_m - x_n| \leq \frac{\rho^n}{\rho(1-\rho)} |x_2 - x_1| < \epsilon$. \square

Examples

Sequences of Functions

Definitions

- (1) A function space is a set of functions between two fixed sets, denoted $\mathcal{F}(\Omega, X) = \{f \mid f : \Omega \rightarrow X\}$.
- (2) A sequence $(f_n)_n$ in $\mathcal{F}(\Omega, \mathbf{R})$ converges pointwise to $f \in \mathcal{F}(\Omega, \mathbf{R})$ if $(\forall x \in \Omega)((f_n(x))_n \rightarrow f(x))$.
In particular:

$$(\forall x \in \Omega)(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geq N \implies d(f_n(x), f(x)) < \epsilon).$$

- (3) A sequence $(f_n)_n$ in $\mathcal{F}(\Omega, \mathbf{R})$ converges uniformly to $f \in \mathcal{F}(\Omega, \mathbf{R})$ if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(\forall x \in \Omega)(n \geq N \implies d(f_n(x), f(x)) < \epsilon).$$

Equivalently (and sometimes preferably):

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geq N \implies \sup_{x \in \Omega} |f_n(x) - f(x)| < \epsilon).$$

Theorems/Propositions/Lemmas

- (1) Let $(f_n)_n \in \mathcal{F}(\Omega, \mathbf{R})^{\mathbf{N}}$.
- (i) If $(f_n)_n \rightarrow f$ uniformly, then $(f_n)_n \rightarrow f$ pointwise.
- (ii) If $(f_n)_n \rightarrow f$ and $(f_n)_n \rightarrow f'$ pointwise, then $f = f'$.

Proof. (i) Let $x \in \Omega$ be given. We have that $|f_n(x) - f(x)| \leq d_u(f_n, f)$. Since $(f_n)_n$ converges uniformly, $(d_u(f_n, f))_n \rightarrow 0$. Hence by "Lemma", $(f_n(x))_n \rightarrow f(x)$; i.e., $(f_n)_n$ converges pointwise.

(ii) Let $\epsilon > 0$ be given. Since $(f_n)_n \rightarrow f$, for all $x \in \Omega$ there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $|f_n(x) - f(x)| \leq \frac{\epsilon}{2}$. Since $(g_n)_n \rightarrow f$, for all $x \in \Omega$ there exists $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $|g_n(x) - g(x)| \leq \frac{\epsilon}{2}$. For $n \geq \max\{N_1, N_2\}$:

$$\begin{aligned} |f(x) - g(x)| &= |f(x) - f_n(x) + f_n(x) - g(x)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - g(x)| \\ &< \epsilon. \end{aligned}$$

This holds for all $\epsilon > 0$, hence $|f(x) - g(x)| = 0$; i.e., $f(x) = g(x)$. Since f and g are equal on every x , $f = g$. \square

- (2) $(f_n)_n \not\rightarrow f$ uniformly if and only if:

$$(\exists \epsilon > 0)(\exists (f_{n_k})_k)(\exists (x_k)_k) \ni |f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0.$$

Proof. (\Rightarrow) By definition, $(f_n)_n \not\rightarrow f$ uniformly if:

$$(\exists \epsilon_0 > 0)(\forall k \in \mathbf{N}) \ni (\exists n \in \mathbf{N})(\exists x \in \Omega)(n \geq N \wedge |f_n(x) - f(x)| \geq \epsilon_0).$$

Using this $\epsilon_0 > 0$:

$$\begin{aligned} k = 1 &\implies (\exists x_1 \in \Omega)(\exists n_1 \in \mathbf{N})(n_1 \geq 1 \wedge |f_{n_1}(x_1) - f(x_1)| \geq \epsilon_0) \\ k = n_1 + 1 &\implies (\exists x_2 \in \Omega)(\exists n_2 \in \mathbf{N})(n_2 \geq n_1 \wedge |f_{n_2}(x_2) - f(x_2)| \geq \epsilon_0) \\ &\vdots \end{aligned}$$

Inductively, we obtain sequences $(x_k)_k$ and $(f_{n_k})_k$ with $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0$. (\Leftarrow). \square

- (3) Let $(f_n)_n$ and $(g_n)_n$ be sequences in $\ell_\infty(\Omega)$ with $(f_n)_n \rightarrow f$ and $(g_n)_n \rightarrow g$ uniformly in Ω . Prove that $(f_n g_n)_n \rightarrow fg$ uniformly on Ω .

Proof. Let:

$$\begin{aligned} |f_n(x)| &\leq U \\ |g(x)| &\leq P. \end{aligned}$$

Given $\epsilon > 0$:

$$\begin{aligned} (\exists N_1 \in \mathbf{N}) \ni (\forall x \in \Omega)(\forall n \in \mathbf{N}) \left(n \geq N_1 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2P} \right), \\ (\exists N_2 \in \mathbf{N}) \ni (\forall x \in \Omega)(\forall n \in \mathbf{N}) \left(n \geq N_2 \implies |g_n(x) - g(x)| < \frac{\epsilon}{2U} \right). \end{aligned}$$

Then for all $n \geq \max\{N_1, N_2\}$ and $x \in \Omega$:

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\ &\leq |f_n(x)(g_n(x) - g(x))| + |g(x)(f_n(x) - f(x))| \\ &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ &< U \frac{\epsilon}{2U} + P \frac{\epsilon}{2P} \\ &= \epsilon. \end{aligned} \quad \square$$

Examples

- (1) Let $(f_n)_n \in \mathcal{F}([0, 1], \mathbf{R})^{\mathbf{N}}$ be defined by $f_n(x) = x^n$. Determine if f converges pointwise.

Solution. Given $x \in [0, 1)$, note that $(f_n(x))_n = (x^n)_n \rightarrow 0$ (geometric). Given $x = 1$, note that $(f_n(x))_n = (1^n)_n \rightarrow 1$. Define $f : [0, 1] \rightarrow \mathbf{R}$ by $f = \mathbf{1}_{\{1\}}$. Then $(f_n)_n \rightarrow f$.

- (2) Let $(f_n)_n \in \mathcal{F}(\mathbf{R}, \mathbf{R})^{\mathbf{N}}$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}$. Determine if $(f_n)_n$ converges pointwise.

Solution. If $x = 0$, then $(f_n(0))_n \rightarrow (0)_n \rightarrow 0$. If $x \neq 0$, note that:

$$\begin{aligned} |f_n(x)| &= \left| \frac{nx}{1+n^2x^2} \right| \\ &\leq \frac{n|x|}{n^2|x^2|} \\ &= \frac{1}{n|x|}. \end{aligned}$$

Since $\left(\frac{1}{n|x|}\right) \rightarrow 0$, $(f_n(x))_n \rightarrow 0$. Hence $(f_n)_n \rightarrow 0$.

- (3) Let $(h_n)_n \in \mathcal{F}([0, \infty), \mathbf{R})^{\mathbf{N}}$ be defined by $h_n(x) = x^{\frac{1}{n}}$. Determine if $(h_n)_n$ converges pointwise.

Solution. If $x > 0$, $(h_n(x)) = (x^{\frac{1}{n}})_n \rightarrow 1$. If $x = 0$, $(h_n(0))_n \rightarrow (0^{\frac{1}{n}})_n \rightarrow 0$. Define $h : [0, \infty) \rightarrow \mathbf{R}$ by $h = \mathbf{1}_{(0, \infty)}$. Then $(h_n)_n \rightarrow h$ pointwise.

- (4) Let $(h_n)_n \in \mathcal{F}([0, \infty), \mathbf{R})^{\mathbf{N}}$ be defined by $h_n(x) = e^{-nx}$. We have that $(h_n)_n \rightarrow \mathbf{1}_{\{0\}}$ pointwise. Does it converge uniformly?

Solution.

Series

Definitions

- (1) Let $(x_k)_k$ be a sequence of real numbers.
 - (i) The sequence of partial sums $(s_n)_n$ is $s_n := \sum_{k=1}^n x_k$.
 - (ii) If $(s_n)_n \rightarrow s$ in \mathbf{R} , we say the infinite series $\sum_{k=1}^{\infty} x_k$ converges and we write $\sum_{k=1}^{\infty} x_k = s$ or $\sum_{k=1}^{\infty} x_k < \infty$.
 - (iii) If $(s_n)_n$ diverges we say that the infinite series $\sum_{k=1}^{\infty} x_k$ diverges. If $(s_n)_n$ properly diverges to $\pm\infty$, we may write $\sum_{k=1}^{\infty} x_k = \pm\infty$.
- (2) A series $\sum x_k$ converges absolutely if $\sum |x_k| < \infty$.
- (3) An alternating series is an infinite series of the form $\sum_k (-1)^k b_k$, $b_k \geq 0$.

Theorems/Propositions/Lemmas

- (1) Let $(x_k)_k$ be a sequence and let $k_0 \in \mathbf{N}$. Then $\sum_{k=1}^{\infty} x_k$ converges if and only if $\sum_{k>k_0}^{\infty} x_k$ converges. In the case of convergence, $\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{k_0} x_k + \sum_{k>k_0}^{\infty} x_k$.

Proof. (\Rightarrow) Suppose $\sum_{k=1}^{\infty} x_k = s$. Then $\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{k_0} x_k + \sum_{k=k_0+1}^{\infty} x_k = s$. Rearranging gives $\sum_{k=k_0+1}^{\infty} x_k = s - \sum_{k=1}^{k_0} x_k$. Since $\sum_{k=1}^{k_0} x_k < \infty$, it must be that $\sum_{k=k_0+1}^{\infty} x_k < \infty$. (\Leftarrow) Now suppose $\sum_{k=k_0+1}^{\infty} x_k = s$. Since $\sum_{k=1}^{k_0} x_k < \infty$, we have that $\sum_{k=1}^{\infty} x_k = s + \sum_{k=1}^{k_0} x_k$; i.e., the infinite series is convergent. \square

- (2) (Divergence Test) If $\sum_{k=1}^{\infty} x_k$ converges then $(x_k)_k \rightarrow 0$.

Proof. Suppose $\sum_{k=0}^{\infty} x_k = s$. Then $(s_n)_n \rightarrow s$. We have $x_n = s_n - s_{n-1}$. Taking the limit on both sides gives $(x_n)_n \rightarrow 0$. \square

- (3) Let $(x_k)_k$ be a sequence. The following are equivalent:

- (i) $\sum_{k=1}^{\infty} x_k$ converges.
- (ii) $(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\exists m, n \in \mathbf{N})(m > n \geq N \Rightarrow |\sum_{k=n+1}^m x_k| < \epsilon)$.
- (iii) $(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni |\sum_{k>N} x_k| < \epsilon$.
- (iv) $(\sum_{k>n} x_k)_n \rightarrow 0$.

Proof. (1) \iff (2). Let $s_n = \sum_{k=1}^n x_k$. Note that $s_m - s_n = \sum_{k=n+1}^m x_k$. So $\sum_{k=1}^{\infty} x_k$ converges if and only if $(s_n)_n$ converges if and only if $(s_n)_n$ is Cauchy. (3) \iff (4) This follows from definitions. (1) \implies (3) Suppose $(s_n)_n \rightarrow s$. Then:

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N}) \ni (\forall n \in \mathbf{N})(n \geq N \implies |s_n - s| < \epsilon).$$

But $s = s_n + \sum_{k>N} x_k$. So $|s - s_n| < \epsilon$ is equivalent to $|\sum_{k>N} x_k| < \epsilon$. (3) \implies (1) Since $|\sum_{k>N} x_k| < \epsilon$, it converges. This is a tail, hence $\sum_{k=1}^{\infty} x_k$ converges. \square

(4) Let $s_n = \sum_{k=1}^{\infty} x_k$ with $x_k \geq 0$ for all k . Then $\sum_{k=1}^{\infty} x_k$ converges if and only if $(s_n)_n$ is bounded.

Proof. (\implies) If $\sum_{k=1}^{\infty} x_k$ converges then $(s_n)_n$ converges, hence $(s_n)_n$ is bounded. (\impliedby) If $(s_n)_n$ is bounded and increasing, then by MCT $(s_n)_n$ converges, hence $\sum_{k=1}^{\infty} x_k$ converges. \square

(5) (Comparison Test) Let $(x_k)_k$ and $(y_k)_k$ be sequences with $0 \leq x_k \leq y_k$.

- (i) If $\sum_{k=1}^{\infty} y_k < \infty$, then $\sum_{k=1}^{\infty} x_k < \infty$ with $\sum_{k=1}^{\infty} x_k \leq \sum_{k=1}^{\infty} y_k$.
- (ii) If $\sum_{k=1}^{\infty} x_k = \infty$, then $\sum_{k=1}^{\infty} y_k = \infty$.

Proof. <https://www.math.uci.edu/~ndonalds/math2b/notes/11-4.pdf> \square

(6) * (Limit Comparison) Let $(x_k)_k$ and $(y_k)_k$ be sequences of positive terms.

- (i) If $\sum y_k < \infty$ and $\limsup \frac{x_k}{y_k} < \infty$, then $\sum x_k < \infty$.
- (ii) If $\sum y_k = \infty$ and $\liminf \frac{x_k}{y_k} > 0$, then $\sum x_k = \infty$.

Proof. \square