Math 374 - Spring 2025 - Final Exam

- **Each question is worth 10 points.**
- **You may use a calculator and Mathematica, but only in the ways we've used these in class. In particular, don't use commands that directly solve problems with no process or steps, such as the Solve or Eigenvalue commands.**
- **You MUST show all work to receive credit.**
- **This is due Saturday, May 10, at noon in my box or hand. Do NOT email.**

1. Write **T** for TRUE, and **F** for FALSE for each statement below:

   (a) The eigenvalues of a tri-diagonal Toeplitz matrix are always real numbers.

   (b) The quadratic form of Newton's method for determining roots of functions can occasionally achieve cubic convergence.

   (c) The number $\frac{1}{7}$ can be expressed exactly as a 64 bit double IEEE 754 number.

   (d) The number $\frac{1001001}{50331648}$ can be expressed exactly as a 64 bit double IEEE 754 number.

   (e) If $T(4n) = 3T(n)$, then $T(n) = O(3^{\log_4(n)})$.

   (f) The IEEE 754 defined 64 bit LONG INT data type is useful for computing the first 10 perfect numbers.

   (g) A probability matrix with appropriate elements can have an eigenvalue equal to π.

   (h) Newton's method cannot be used to determine a root of $f(x) = x^4 + 2x^2 + 1$.

   (i) A regular matrix must have at least one positive eigenvector.

   (j) The Predator-Prey model can be expanded to 8 species, some of which prey on species which themselves prey on others.

   (k) In the basic SIR model of disease transmission, if $I = 0$ at some point t, then I can be greater than 0 at some future time.

2. In class, we saw the Jacobi and Gauss Seidel iterative methods for solving a linear system of the form $A\vec{x} = \vec{b}$. These methods partitioned the matrix A as: $A = L + D + U$, where D is a diagonal matrix, U is the part of A above the diagonal, and L is the part of A below the diagonal. The Jacobi method was derived as:

$$(L + D + U)\vec{x} = \vec{b} \quad \Rightarrow \quad D\vec{x} = \vec{b} - (U + L)\vec{x} \quad \Rightarrow \quad \mathbf{\tilde{x}_{n+1} = D^{-1}\tilde{b} - D^{-1}(U + L)\tilde{x}_n}.$$

The Gauss-Seidel method was derived as:

$$(L+D+U)\vec{x} = \vec{b} \quad \Rightarrow \quad (D+L)\vec{x} = \vec{b} - U\vec{x} \quad \Rightarrow \quad \mathbf{\tilde{x}_{n+1} = (D+L)^{-1}\tilde{b} - (D+L)^{-1}U\tilde{x}_n}.$$

(a) Prove, using your knowledge of Linear Algebra, that if A has a 0 on the diagonal, neither method above will produce a solution without somehow rearranging the rows of A and $\vec{b}$.

(b) Compose a NEW iterative method similar to the Jacobi or Gauss-Seidel method ($\vec{x}_{n+1} = ...\vec{x}_n$), which CAN find a solution of $A\vec{x} = \vec{b}$ without rearranging rows.

(c) Use your new method to determine a solution to the linear system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Continue to iterate until your result shows four decimal digit stability. How many iterations were required?

3. Use the Euler method to determine a numerical solution to the initial value problem: $\dfrac{dy}{dx} = -9y + \cos(e^x), \;\; y(0) = 1.2$ over the interval $0 \leq x \leq 5$. Choose one (1) value of Δx for which the method is UNstable, and one (1) value of Δx for which the method is STABLE. Print your solution curves for each case.

4. Single but Exciting Questions!

(a) Let $T(4n) = 3T(4n - 4)$. Determine $T(n)$ explicitly.

(b) Express the number -219.1963 as a 64 bit double precision number in IEEE 754 format.

(c) List the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 3 & 8 & 0 \\ 2 & 3 & 8 \\ 0 & 2 & 3 \end{bmatrix}$ using only an appropriate formula, and without computing the characteristic polynomial or solving any linear systems.

5. Consider the two equations $x^3 + y^2 = 2$ and $2x^2 + \sin(y) = 3$.

   (a) Research the Mathematica command "ContourPlot". Then plot both of these equations in the same plot on the interval $-3 \leq x, y \leq 3$. Turn in your plot.

   (b) If your graph is correct, you should see four points of intersection between the two equations. Using your graph to "suggest" good initial guesses, use an iterative method to determine the four points of intersection $(x_1, y_1), ...(x_4, y_4)$, stable to 4 decimal places.