Math 550

Homework 1

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Exercise 1. Show that $\mathcal{B}_{\mathbf{R}}$ is generated by each of the following:

a. the open intervals: $\mathcal{E}_1 = \{(a,b) \mid a < b\}$,

b. the closed intervals: $\mathcal{E}_2 = \{[a,b] \mid a < b\}$,

c. the half-open intervals: $\mathcal{E}_3 = \{(a,b] \mid a < b\}$ or $\mathcal{E}_4 = \{[a,b) \mid a < b\}$,

d. the open rays: $\mathcal{E}_5 = \{(a,\infty) \mid a \in \mathbf{R}\}$ or $\mathcal{E}_6 = \{(-\infty,a) \mid a \in \mathbf{R}\}$,

e. the closed rays: $\mathcal{E}_7 = \{(a,\infty) \mid a \in \mathbf{R}\}$ or $\mathcal{E}_8 = \{(-\infty,a) \mid a \in \mathbf{R}\}$.

Proof. □

Exercise 2. Let $\mathcal{M}$ be an infinite σ-algebra.

a. $\mathcal{M}$ contains an infinite sequence of disjoint sets.

b. $\operatorname{card}(\mathcal{M}) \geqslant \mathfrak{c}$.

Exercise 3. An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions.

Proof. If $\mathcal{A}$ is a σ-algebra, then $\mathcal{A}$ is closed under countable unions, hence it is closed under countable *increasing* unions. Conversely, suppose $\mathcal{A}$ is closed under countable increasing unions. Let $\{E_n\}_{n=1}^{\infty}$ be a family of sets in $\mathcal{A}$. Define:

$$\begin{aligned} F_1 &= E_1, \\ F_2 &= E_1 \cup E_2, \\ &\vdots \\ F_n &= E_1 \cup E_2 \cup \ldots \cup E_n. \end{aligned}$$

Then clearly $F_1 \subset F_2 \subset F_3 \ldots$ Moreover, it is clear that $\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} F_n$. Conversely, let $x \in \bigcup_{n=1}^{\infty} F_n$. Then $x \in F_i$ for some $i \in \mathbf{N}$. By construction, $x \in E_1 \cup \ldots \cup E_i$, hence $x \in E_j$ for some $j < i$. Whence $x \in \bigcup_{n=1}^{\infty} E_n$, giving the inclusion $\bigcup_{n=1}^{\infty} F_n \subset \bigcup_{n=1}^{\infty} E_n$. Together, we have $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n \in \mathcal{A}$. Since $\{E_n\}_{n=1}^{\infty}$ was an arbitrary family of sets in $\mathcal{A}$, we can conclude $\mathcal{A}$ is a σ-algebra. □