Math 548

Portfolio Assignment 1

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Problem 1 (S24, P5)**.** Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be an orthogonal projection to the 2-dimensional plane P spanned by the vectors $\vec{v} = (2, 0, 1, 0)$ and $\vec{w} = (-1, 0, 2, 0)$.

(a) Find (with proof) all the eigenvalues and eigenvectors, along with their geometric and algebraic multiplicities.

(b) Find the matrix T with respect to the standard basis. Is this matrix diagonalizable? Why or why not?

Proof. (a) Since T is a projection onto P, note that for any vector $v \in \mathbf{R}^4$, we have that $T(v) \in P$. Since T acts as the identity map on any vector in P, it must be the case that $T^2(v) = T(v)$.

Let w be an eigenvector of T. Since $T^2(v) = T(v)$ for all $v \in \mathbf{R}^4$, we get:

$$\begin{aligned} T(w) &= \lambda w, \\ T(w) &= T^2(w) = \lambda^2 w. \end{aligned}$$

It must be the case that $\lambda^2 = \lambda$; i.e., $\lambda = 0$ or $\lambda = 1$. Note that $\lambda = 1$ corresponds to the subspace P, while $\lambda = 0$ corresponds to the subspace $P^\perp$. Since P is 2-dimensional, the geometric multiplicity of $\lambda = 1$ is 2. Moreover, because $\dim \mathbf{R}^4 = \dim P + \dim P^\perp$, we get that $\dim P^\perp = 2$, hence $\lambda = 0$ has a geometric multiplicity of 2.

Observe that:

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{cases} 2x_1 + x_3 = 0 \\ -x_1 + 2x_3 = 0 \end{cases}$$

$$\implies \begin{cases} 2x_1 + x_3 = 0 \\ -5x_1 = 0 \end{cases}$$

$$\implies \begin{cases} 2x_1 = -x_3 \\ x_1 = 0 \end{cases}$$

Hence $x_1 = x_3 = 0$. Since x_2 and x_4 are free variables, we have:

$$P^\perp = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since the eigenbasis:

$$\mathcal{B}_\lambda = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

has a dimension of 4, T is diagonalizable. Because T is diagonalizable, the algebraic multiplicities of its eigenvalues equals their geometric multiplicities.

(b) Let $\mathcal{B}$ denote the standard basis in $\mathbf{R}^4$ and define:

$$S = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since T is diagonalizable, this means:

$$\begin{aligned} [T]_{\mathcal{B}} &= S \cdot [T]_{\mathcal{B}_\lambda} \cdot S^{-1} \\ &= \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & 0 & \frac{1}{5} & 0 \\ -\frac{1}{5} & 0 & \frac{2}{5} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

□

Problem 2. Let L be the line in $\mathbf{R}^2$ defined by $y = -3x$, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that orthogonally projects onto L and then stretches along L by a factor of two.

(a) Find the eigenvalues and an eigenbasis $\mathcal{B}$ for T.

(b) Determine the matrix for T with respect to the basis $\mathcal{B}$.

(c) Determine the matrix for T with respect to the standard basis.

Proof. (a) Note that $\mathbf{R}^2 = L \oplus L^\perp$. Hence for any $v \in \mathbf{R}^2$, we can write $v = u + w$, where $u \in L$ and $w \in L^\perp$. This means:

$$\begin{aligned} T(v) &= T(u + w) \\ &= T(u) + T(w) \\ &= T(u) \\ &= 2u. \end{aligned}$$

Applying T to both sides yields:

$$\begin{aligned} T^2(v) &= 2T(u) \\ &= 2T(v). \end{aligned}$$

Whence $T^2 = 2T$. If x is an eigenvector of T, then $T(x) = \lambda x$ and $T(x) = \frac{1}{2}T^2(x) = \frac{1}{2}\lambda^2 x$. Thus $\lambda = \frac{1}{2}\lambda^2$, giving $\lambda = 2$ and $\lambda = 0$ as eigenvectors.

Note that $L = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$. Solving the equation $(1, -3) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (0, 0)$ gives $L^\perp = \operatorname{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$. This means the eigenbasis for T is $\mathcal{B} = \left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$.

(b) Since $\dim \mathcal{B} = \dim \mathbf{R}^2$, the matrix for T will be diagonal. In particular, the diagonal entries for $[T]_{\mathcal{B}}$ will be the eigenvalues of T; i.e., $[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

(c) Let $\mathcal{E}$ denote the standard basis in $\mathbf{R}^2$. Since T is diagonalizable, we have $[T]_{\mathcal{B}} = P^{-1} \cdot [T]_{\mathcal{E}} \cdot P$, where $P = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$. Solving for $[T]_{\mathcal{E}}$ yields:

$$\begin{aligned} [T]_{\mathcal{E}} &= P \cdot [T]_{\mathcal{B}} \cdot P^{-1} \\ &= \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{10} & -\frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{9}{5} \end{pmatrix}. \end{aligned}$$

□

Problem 3. Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

(a) Determine whether A is diagonalizable, and if so, give its diagonal form along with a diagonalizing matrix.

(b) Compute A^{42}. Remember to show all work.