Preface

These are my notes for Math 550 at Cal Poly.

Contents

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Chapter 1

Preliminaries

§ 1.1. Set Theory

Definition 1.1.1. A *set* (typically denoted with capital letters) is a collection of different things; the things are called *elements* (typically denoted with lowercase letters) of the set.

Zermelo–Fraenkel set theory introduces eight different axioms which mathematics builds upon. Below is an introduction to the ones which are relevant for the material covered in this course.

Axiom 1 (Axiom of Extension)**.** *If the sets* X *and* Y *have the same members, then they are the same set. In other words:*

$$(\forall X)(\forall Y)\Big((\forall z)\big(z \in X \Leftrightarrow z \in Y\big) \implies X = Y\Big)$$

Axiom 2 (Axiom of Existence)**.** *There is a set such that no element is a member of it. In other words:*

$$(\exists A) : (\forall x)(x \notin A)$$

Note that the Axiom of Extension immediately proves that the set which contains no elements is unique; two sets are equal if they have the same members, hence two sets which contain no elements are equal.

Definition 1.1.2. The set which contains no elements is called the *empty set*, and is denoted by $\emptyset$.

Axiom 3 (Axiom of Pairing)**.** *Let* A *and* B *be sets. There exists the set* $\mathcal{C} := \{A, B\}$*. In other words:*

$$(\forall A)(\forall B)(\exists \mathcal{C}) : (A \in \mathcal{C} \land B \in \mathcal{C}).$$

Axiom 4 (Axiom of Union)**.** *Let* $\mathcal{C}$ *be a set. Then there exists a set* U *consisting of the elements* x *that are contained in at least one element of the set* $\mathcal{C}$*, that is:*

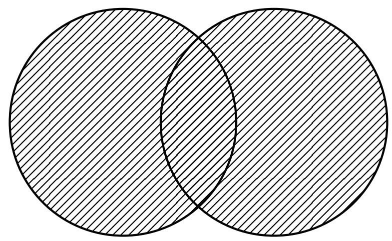
$$(\forall \mathcal{C})(\exists U) : (\forall x)(x \in U \Leftrightarrow (\exists C)(u \in C \land C \in \mathcal{C}))$$

Definition 1.1.3. Let $\mathcal{C}$ be a set, and let U be the set consisting of the elements x that are contained in at least one element of $\mathcal{C}$.

(1) The set U is called the *union of the set* $\mathcal{C}$ and is denoted:

$$U := \bigcup \mathcal{C}.$$

(2) If the set $\mathcal{C}$ consists of only two sets A and B, then we write $U := A \cup B$.

A  B

$A \cup B$

Axiom 5 (Axiom Schema of Specification)**.** *Let A be a set, and let φ be a predicate containing the free variable x. There exists a subset B of the set A whose members are precisely the elements x of the set A such that the sentence $\varphi(x)$ is true. In other words:*

$$(\forall A)(\forall \varphi)(\exists B) : (B = \{x \in A \mid \varphi(x)\})$$

Proposition 1.1.1. *Let $\mathcal{C}$ be a nonempty set. There exists a set D consisting of the elements x that are contained in all elements of the set $\mathcal{C}$.*

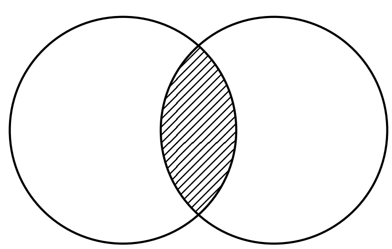
Proof. Let $E \in \mathcal{C}$. By the Axiom Schema of Specification we have $D := \{c \in E \mid (\forall X)(X \in \mathcal{C} \Rightarrow c \in X)\}$. □

Definition 1.1.4. Let $\mathcal{C}$ be a nonempty set, and let D be the set consisting of the elements x that are contained in all elements of the set $\mathcal{C}$.

(1) The set D is called the *intersection of the set* $\mathcal{C}$ and is denoted:

$$D := \bigcap \mathcal{C}.$$

(2) If the set $\mathcal{C}$ contains only two sets A and B, then we write $D := A \cap B$.

A  B

$A \cap B$

Axiom 6. *Let X be a set. Then the set of all subsets of X exists.*

Definition 1.1.5. Let X be a set. Then the set of all subsets of the set X is called the *power set of the set* X and is denoted $\mathcal{P}(X)$.

Axiom 7 (Axiom of Infinity)**.** *There is a set that contains all natural numbers.*

Definition 1.1.6. An *index set* is a nonempty set whose elements label the elements of another set. We denote the collection of objects labeled by elements of I as $\{A_\alpha\}_{\alpha \in I}$.

With this definition and the Axiom of Infinity, we can extend the concept of unions and intersections to families of sets of arbitrary size.

Definition 1.1.7. Let I be an index set, let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of sets, and write $\mathcal{X} := \{X_\alpha \mid \alpha \in I\}$. Define:

$$\bigcup_{\alpha \in I} X_i := \bigcup \mathcal{X},$$
$$\bigcap_{\alpha \in I} X_i := \bigcap \mathcal{X}.$$

Definition 1.1.8. Let $\{X_n\}_{n \in \mathbb{N}}$ be a family of sets indexed by the natural numbers.

(1) The *limit superior* of $\{X_n\}_{n \in \mathbb{N}}$ is:

$$\begin{aligned}\limsup X_n &:= \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} X_n \right) \\ &= \{x \mid x \in E_n \text{ for infinitely many } n\}.\end{aligned}$$

(2) The *limit inferior* of $\{X_n\}_{n \in \mathbb{N}}$ is:

$$\begin{aligned}\limsup X_n &:= \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} X_n \right) \\ &\{x \mid x \notin E_n \text{ for only finitely many } n\}\end{aligned}$$

Definition 1.1.9. Let A and B be two sets. Define the *difference of* A *and* B by $A \setminus B := \{x \in A \mid x \notin B\}$.

Note that the difference of two sets exists by the Axiom Schema of Specification.

Definition 1.1.10. Let a and b be two objects. The *ordered pair* (a, b) is defined by $(a, b) := \{\{a\}, \{a, b\}\}$.

Proposition 1.1.2. *Let* A *and* B *be sets. There exists a set* E *containing all ordered pairs* (a, b)*, where* $a \in A$ *and* $b \in B$*.*

Proof. Let (a, b) be any ordered pair such that $a \in A$ and $b \in B$. By definition $(a, b) = \{\{a\}, \{a, b\}\}$. Now $\{a\} \subseteq A$, so $\{a\} \subseteq A \cup B$, hence $\{a\} \in \mathcal{P}(A \cup B)$. Similarly, $\{a, b\} \subseteq A \cup B$, so $\{a, b\} \in \mathcal{P}(A \cup B)$. It follows that $\{\{a\}, \{a, b\}\} \subseteq \mathcal{P}(A \cup B)$, hence $\{\{a\}, \{a, b\}\} \in$

$\mathcal{P}(\mathcal{P}(A \cup B))$. We know all these sets are known to exist by the Axiom of Union and Axiom of Power Set. Then by the Axiom Schema of Specification, there exists a set:

$$E := \{t \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid (\exists a)(\exists b)(a \in A \wedge b \in B \wedge t = (a, b))\}.$$

Moreover, this set is unique by the Axiom of Extension, and is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. □

Definition 1.1.11. Let A and B be two sets, and let E be the set containing all ordered pairs (a, b) where $a \in A$ and $b \in B$. The set E is called the *Cartesian product of* A *and* B, and is denoted $E := A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Definition 1.1.12. Let X and Y be sets. A *relation* from X to Y is a subset R of $X \times Y$. We write xRy to mean $(x, y) \in R$.

(1) An *equivalence relation* on X is a relation which is reflexive, transitive, and symmetric. We write $x \sim y$ to mean xRy.

(2) A *function* (or *map*) from X to Y, denoted $f : X \to Y$, is a relation from X to Y such that, for every $x \in X$, there exists a unique $y \in Y$ such that xRy. We write $f(x) = y$ to mean xRy.

(3) A *partial order* on X is a relation which is reflexive, transitive, and antisymmetric. We write $x \leqslant y$ to mean xRy, and say that X is an *ordered set*.

Example 1.1.1. An equivalent relation on X partitions X into equivalence classes.

Example 1.1.2. Recall that, if $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *covariant functor* $T : \mathcal{C} \to \mathcal{D}$ is a map such that:

(i) If $A \in \operatorname{obj}(\mathcal{C})$, then $T(A) \in \operatorname{obj}(\mathcal{D})$.

(ii) If $f \in \operatorname{Hom}_{\mathcal{C}}(A, A')$, then $T(f) \in \operatorname{Hom}_{\mathcal{D}}(T(A), T(A'))$.

(iii) If $f \in \operatorname{Hom}_{\mathcal{C}}(A, A')$ and $g \in \operatorname{Hom}_{\mathcal{C}}(A', A'')$, then $T(f) \in \operatorname{Hom}_{\mathcal{D}}(T(A), T(A'))$ and $T(g) \in \operatorname{Hom}_{\mathcal{D}}(T(A'), T(A''))$. In particular:

$$T(gf) = T(g)T(f).$$

(iv) For every $A \in \operatorname{obj}(\mathcal{C})$, $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$.

A *contravariant functor* $T : \mathcal{C} \to \mathcal{D}$ is similarly defined, but (ii) and (iii) are instead defined as:

(ii) If $f \in \operatorname{Hom}_{\mathcal{C}}(A, A')$, then $T(f) \in \operatorname{Hom}_{\mathcal{D}}(T(A'), T(A))$.

(iii) If $f \in \operatorname{Hom}_{\mathcal{C}}(A, A')$ and $g \in \operatorname{Hom}_{\mathcal{C}}(A', A'')$, then $T(f) \in \operatorname{Hom}_{\mathcal{D}}(T(A'), T(A))$ and $T(g) \in \operatorname{Hom}_{\mathcal{D}}(T(A''), T(A'))$. In particular:

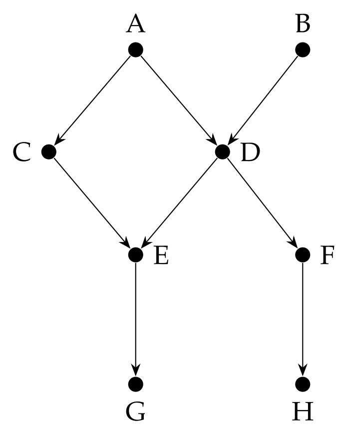
$$T(gf) = T(f)T(g).$$

Given $f : X \to Y$, the power set gives rise to two functors, the *contravariant power set functor* $\mathfrak{Set}^{\mathrm{op}} \to \mathfrak{Set}$ and the *covariant power set functor* $\mathfrak{Set} \to \mathfrak{Set}$. The first sends $f : X \to Y$ to the *preimage* function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$, whereas the second sends f to the *image* function $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$. The preimage is more well-behaved, commuting with boolean operations:

$$f^{-1}(E^c) = \left(f^{-1}(E)\right)^c,$$
$$f^{-1}\left(\bigcup_{\alpha\in I} E_\alpha\right) = \bigcup_{\alpha\in I} f^{-1}(E_\alpha),$$
$$f^{-1}\left(\bigcap_{\alpha\in I} E_\alpha\right) = \bigcap_{\alpha\in I} f^{-1}(E_\alpha).$$

This is not necessarily the case for the image function.

Example 1.1.3. Consider the directed acyclic graph:



Let $G = \{A, B, C, D, E, F, G, H\}$ be the set of vertices of our graph. We can define a partial ordering on G as follows: for any $x, y \in G$, we have $x \leqslant y$ if and only if there exists a directed path from x to y. However, note that not every element of G can be compared.

Definition 1.1.13. An ordering on a set X is said to be *linear* (or *total*) if for every $x, y \in X$, $x \leqslant y$ or $y \leqslant x$.

Definition 1.1.14. Let X be an ordered set. Let $A \subseteq X$.

(1) A is called *bounded above* if there exists an element $u \in X$ with $a \leqslant u$ for all $a \in A$. Such a u is called an *upper bound* for A. The set of upper bounds of A is denoted $\mathcal{U}_A = \{u \in X \mid u \text{ is an upper bound of } A\}$.

(2) A is called *bounded below* if there exists an element $v \in X$ with $v \leqslant a$ for all $a \in A$. Such a v is called a *lower bound* for A. The set of lower bounds of A is defined as $\mathcal{L}_A = \{v \in X \mid v \text{ is a lower bound of } A\}$.

(3) If A admits an upper bound u with $u \in A$, then u is called *the greatest element of* A.

(4) If A admits a lower bound v with $v \in A$, then v is called *the least element of* A.

(5) If l is the least element of $\mathcal{U}_A$, we write $l = \sup(A)$ and call it the *supremum of* A.

(6) If g is the greatest element of $\mathcal{L}_A$, we write $g = \inf(A)$ and call it the *infimum of* A.

(7) A *maximal element of* A is an element $m \in A$ such that if $a \geqslant m$, then $a = m$ (not necessarily unique).

(8) A *minimal element of* A is an element $n \in A$ such that if $a \leqslant n$, then $a = n$ (not necessarily unique).

Definition 1.1.15. If a linear order on a set X satisfies the property that every nonempty subset has a minimal element, then we say it is a *well-ordering*.

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include stuff HERE

Definition 1.1.16.

do general cartesian product
question is it nonempty?
axiom of choice
answer: no, axiom of choice says this choice function exists.
'shoes vs. socks' formulation of the axiom

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