Abstract

This is an outline of many results related to the Darboux integral. Included are basic definitions, the integrability of continuous and monotone functions, the fundamental theorem of calculus, and limits of integrable functions. The culmination of this paper is proving Arzela's Theorem, which is a special case of the Bounded Convergence Theorem for the Lebesgue integral.

§ Basic Definitions and Results

Definition 1. Let $[a, b] \subseteq \mathbf{R}$.

(1) A *partition* P of the interval $[a, b]$ is a finite set of points $\{x_0, ..., x_n\} \subseteq [a, b]$ such that:

$$a = x_0 < x_1 < ..., x_{n-1} < x_n = b.$$

The set of partitions of an interval $[a, b]$ is denoted $B_{[a,b]}$.

(2) A refinement of a partition P is a partition Q such that $P \subset Q$. In this case, we say that Q is *finer* than P. Given two partitions $P_1, P_2 \subseteq [a, b]$, we call the union $P_1 \cup P_2$ their *common refinement*.

(3) The *change in* x_i is denoted $\Delta x_i := x_i - x_{i-1}$.

(4) The *mesh* of a partition P is denoted $\|P\| := \max_{i=1}^{n} \Delta x_i$.

Definition 2. Let $f : [a, b] \to \mathbf{R}$ be a bounded function and $P = \{x_0, ..., x_n\}$ a partition of $[a, b]$. We define:

(1) $M_j(f) = \sup_{x\in[x_{j-1},x_j]} f(x)$; and

(2) $m_j(f) = \inf_{x\in[x_{j-1},x_j]} f(x)$.

We call the numbers:

$$U(P, f) = \sum_{j=1}^{n} M_j(f)\Delta x_j, \qquad L(P, f) = \sum_{j=1}^{n} m_j(f)\Delta x_j,$$

the *upper Darboux sum* and, respectively, the *lower Darboux sum* of f over the partition P.

Proposition 1. *The sets* $\{U(P, f) \mid P \in B_{[a,b]}\}$ *and* $\{L(P, f) \mid P \in B_{[a,b]}\}$ *are bounded subsets of* $\mathbf{R}$.

Proof. Since f is bounded, we can find $m, M \in \mathbf{R}$ such that $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$. Then for any partition P we have $m \leqslant m_j(f) \leqslant M_j(f) \leqslant M$ for all $j = 1, 2, ..., |P| - 1$. Multiplying Δx_j throughout gives:

$$m\Delta x_j \leqslant m_j(f)\Delta x_j \leqslant m_j(f)\Delta x_j \leqslant M\Delta x_j$$

Summing from $j = 1$ to $|P| - 1$ gives:

$$m(b - a) \leqslant L(P, f) \leqslant U(P, f) \leqslant M(b - a).$$

Since P was arbitrary, the above sets are bounded. □

Proposition 2. *Let* $P, Q \in B_{[a,b]}$*. If* $P \subset Q$*, then* $L(P,f) \leqslant L(Q,f)$ *and* $U(Q,f) \leqslant U(P,f)$*.*

Proof. Since Q is obtained from P by adding finitely many points, by induction, we only need to prove the case when Q is obtained from P by adding one extra point. Let:

$$\begin{aligned} P &= \{x_0, x_1, ..., x_{k-1}, x_k, ..., x_n\}, \\ Q &= \{x_0, x_1, ..., x_{k-1}, y, x_k, ..., x_n\}, \end{aligned}$$

where $x_{k-1} < y < x_k$. Observe that:

$$\begin{aligned} L(P,f) &= \sum_{j=1}^{n} m_j(f)\Delta x_j \\ &\leqslant \sum_{j=1}^{k-1} m_j(f)\Delta x_j + \inf_{x\in[x_{k-1},y]} f(x)(y - x_{k-1}) \\ &\qquad + \inf_{x\in[y,x_k]} f(x)(x_k - y) + \sum_{j=k+1}^{n} m_j(f)\Delta x_j \\ &= L(Q,f). \end{aligned}$$

Furthermore, we have:

$$\begin{aligned} U(Q,f) = \sum_{j=1}^{k-1} M_j(f)\Delta x_j &+ \sup_{x\in[x_{k-1},y]} f(x)(y - x_{k-1}) \\ &+ \sup_{x\in[y,x_k]} f(x)(x_k - y) + \sum_{j=k+1}^{n} m_j(f)\Delta x_j \end{aligned}$$

Since $[x_{k-1}, y] \subseteq [x_{k-1}, x_k]$ and $[y, x_k] \subseteq [x_{k-1}, x_k]$, clearly $\sup_{x\in[x_{k-1},y]} f(x) \leqslant M_k(f)$ and $\sup_{x\in[y,x_k]} f(x) \leqslant M_k(f)$. This gives:

$$\begin{aligned} \sup_{x\in[x_{k-1},y]} f(x)(y - x_{k-1}) + \sup_{x\in[y,x_k]} f(x)(x_k - y) &\leqslant M_k(y - x_{k-1}) + M_k(x_k - y) \\ &= M_k(x_k - x_{k-1}) \end{aligned}$$

Thus $U(Q,f) \leqslant U(P,f)$. □

Definition 3. Let $f : [a,b] \to \mathbf{R}$ be a bounded function.

(1) The *upper Darboux integral* of f over $[a,b]$ is:

$$\overline{\int_a^b} f\,dx := \inf_{P\in B_{[a,b]}} U(P,f).$$

(2) The *lower Darboux integral* of f over $[a,b]$ is:

$$\underline{\int_a^b} f\,dx := \sup_{P\in B_{[a,b]}} L(P,f).$$

(3) We say that f is *Darboux integrable* on $[a,b]$ provided that:

$$\overline{\int_a^b} f dx = \underline{\int_a^b} f dx.$$

In this case, the common value of the upper and lower Darboux integrals is called the *Darboux integral* of f over $[a,b]$ and is denoted:

$$\int_a^b f dx.$$

(4) The set of Darboux integrable functions on $[a,b]$ is denoted $\mathcal{R}[a,b]$

Example 1. Let $f : [0,1] \to \mathbf{R}$ be defined by:

$$f(x) = \begin{cases} 0, & x \in [0,1] \setminus \mathbf{Q} \\ 1, & x \notin [0,1] \setminus \mathbf{Q}. \end{cases}$$

Note that:

$$\begin{aligned} \underline{\int_0^1} f(x)dx &= \sup_{P \in B_{[0,1]}} L(P,f) \\ &= 1, \end{aligned}$$

whilst:

$$\begin{aligned} \overline{\int_0^1} f(x)dx &= \inf_{P \in B_{[0,1]}} U(P,f) \\ &= 0. \end{aligned}$$

Since $\underline{\int_0^1} f(x)dx \neq \overline{\int_0^1} f(x)dx$, f is not Darboux integrable.

Theorem 3. *A bounded function* $f : [a,b] \to \mathbf{R}$ *is Darboux integrable if and only if for all* $\epsilon > 0$*, there exists a partition* $P \in B_{[a,b]}$ *such that* $U(P,f) - L(P,f) < \epsilon$.

Proof. ($\Rightarrow$) Let $\epsilon > 0$. Since f is Darboux integrable, we have:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \inf_{P \in B_{[a,b]}} U(P,f).$$

By the "infimum property", there exists $P_1 \in B_{[a,b]}$ such that:

$$\int_a^b f(x)dx \leqslant U(P_1,f) < \int_a^b f(x)dx + \frac{\epsilon}{2}.$$

Again, since f is Darboux integrable we have:

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \sup_{P \in B_{[a,b]}} L(P,f).$$

By the "supremum property", there exists $P_2 \in B_{[a,b]}$ such that:

$$\underline{\int_a^b} f(x)dx - \frac{\epsilon}{2} < L(P_1, f) \leqslant \underline{\int_a^b} f(x)dx.$$

Let P be the common refinement of P_1 and P_2. By Proposition 2, we have $L(P_2, f) \leqslant L(P, f)$ and $U(P, f) \leqslant U(P_1, f)$. With this we obtain:

$$\int_a^b f(x)dx - \frac{\epsilon}{2} < L(P, f) \leqslant \int_a^b f(x)dx \leqslant U(P, f) < \int_a^b f(x)dx + \frac{\epsilon}{2}.$$

Rearranging terms gives the desired result of $U(P, f) - L(P, f) < \epsilon$.

($\Leftarrow$) Let $\epsilon > 0$. By our hypothesis, we can find a partition P such that $U(P, f) - L(P, f) < \epsilon$. Since $\overline{\int_a^b} f(x)dx \leqslant U(P, f)$ and $L(P, f) \leqslant \underline{\int_a^b} f(x)dx$, we have:

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leqslant U(P, f) - L(P, f) < \epsilon.$$

Thus $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$; i.e., f is Darboux integrable. □

Proposition 4. *The set $\mathcal{R}[a, b]$ is an $\mathbf{R}$-vector space.*

Proof. Instead of verifying all the vector space axioms for $\mathcal{R}[a, b]$, we will use the fact that $Bd([a, b])$ is an $\mathbf{R}$-vector space and show that $\mathcal{R}[a, b]$ is a subspace. Note that $\mathcal{R}[a, b] \neq \emptyset$, as it is intuitively obvious that the zero function is integrable. Let $f, g \in \mathcal{R}[a, b]$ and $c \in \mathbf{R}$. We proceed by cases on c.

Suppose $c > 0$. By our definitions of $M_j(f)$ and $m_j(f)$, we have:

$$\begin{aligned} M_j(f + cg) &\leqslant M_j(f) + cM_j(g), \\ m_j(f + cg) &\geqslant m_j(f) + cm_j(g). \end{aligned}$$

So given a partition Q, we have:

$$\begin{aligned} U(Q, f + cg) &= \sum_{j=1}^{|Q|-1} M_j(f + cg)\Delta x_j \\ &\leqslant \sum_{j=1}^{|Q|-1} M_j(f)\Delta x_j + c \sum_{j=1}^{|Q|-1} M_j(g)\Delta x_j \\ &= U(Q, f) + cU(Q, g). \end{aligned}$$

It follows similarly that $L(Q, f + cg) \geqslant L(Q, f) + cL(Q, g)$. Since $f \in \mathcal{R}[a, b]$, there exists $P_1 \in B_{[a,b]}$ such that $U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$. Since $g \in \mathcal{R}[a, b]$, there exists $P_2 \in B_{[a,b]}$ such that $U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2c}$. Let P be the common refinement of P_1 and P_2. We have:

$$\begin{aligned} U(P, f + cg) - L(P, f + cg) &\leqslant U(P, f) + cU(P, g) - L(P, f) - cL(P, g) \\ &= \big(U(P, f) - L(P, f)\big) + c\big(U(P, g) - L(P, g)\big) \\ &\leqslant \big(U(P_1, f) - L(P_1, f)\big) + \big(U(P_2, g) - L(P_2, g)\big) \\ &< \frac{\epsilon}{2} + c \cdot \frac{\epsilon}{2c} \\ &= \epsilon. \end{aligned}$$

Thus $f + cg \in \mathcal{R}[a, b]$ when $c > 0$. For $c = 0$, the claim is trivial. For $c < 0$, our definitions of $M_j(f)$ and $m_j(f)$ instead give:

$$\begin{aligned} M_j(f + cg) &\leqslant M_j(f) + cm_j(g) \\ m_j(f + cg) &\geqslant m_j(f) + cM_j(g). \end{aligned}$$

Using a similar refinement argument as our $c > 0$ case, we lead to the conclusion that $f + cg \in \mathcal{R}[a, b]$ for $c < 0$. Since $f + cg \in \mathcal{R}[a, b]$ for all $c \in \mathbf{R}$, we have that $\mathcal{R}[a, b]$ is a subspace of $\mathrm{Bd}\big([a, b]\big)$. □

Corollary 5. **** Let* $f, g \in \mathcal{R}[a, b]$ *and* $c \in \mathbf{R}$. *We have:*

(1) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$*;*

(2) $\int_a^b \big(f(x) + g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

Proof. (1) For $c > 0$ we have:

$$\begin{aligned} \int_a^b cf(x)dx &= \overline{\int_a^b} cf(x)dx \\ &= \inf_{P\in B_{[a,b]}} U(P, cf) \\ &= c \inf_{P\in B_{[a,b]}} U(P, f) \\ &= c\overline{\int_a^b} f(x)dx \\ &= c\int_a^b f(x)dx. \end{aligned}$$

For $c < 0$ we have:

$$\begin{aligned}
\int_a^b cf(x)dx &= \overline{\int_a^b} cf(x)dx \\
&= \inf_{P \in B_{[a,b]}} U(P, cf) \\
&= \inf_{P \in B_{[a,b]}} cL(P, f) \\
&= c \sup_{P \in B_{[a,b]}} L(P, f) \\
&= c \underline{\int_a^b} f(x)dx \\
&= c \int_a^b f(x)dx.
\end{aligned}$$

□

Corollary 6. *Let* $f, g \in \mathcal{R}[a, b]$ *and suppose* $f(x) \leqslant g(x)$ *for all* $x \in [a, b]$. *Then* $\int_a^b f(x)dx \leqslant \int_a^b g(x)dx$.

Proof. Since g and f are integrable, by Proposition 4, the difference $g - f$ is also integrable. Moreover, since Darboux sums are greater than or equal to zero, we have $\int_a^b g(x)-f(x)dx \geqslant 0$. Again, by Proposition 4, $\int_a^b g(x) - f(x)dx = \int_a^b g(x)dx - \int_a^b f(x)dx \geqslant 0$, which establishes the claim. □

Corollary 7. *If* $f \in \mathcal{R}[a, b]$, *then there exists* $M \geqslant 0$ *such that* $\int_a^b f(x)dx \leqslant M(b - a)$.

Proof. Note that $\mathcal{R}[a, b] \subseteq \mathrm{Bd}([a, b])$ is a subspace, hence f is bounded. From this, there exists $M \geqslant 0$ such that $\sup_{x \in [a,b]} f(x) \leqslant M$. By Corollary 6, we have $\int_a^b f(x)dx \leqslant \int_a^b Mdx = M(b - a)$. □

§ Conditions for Integrability

Theorem 8. *Let* $f : [a, b] \to \mathbf{R}$ *be continuous. Then* $f \in \mathcal{R}[a, b]$.

Proof. Let $\epsilon > 0$. Since f is continuous on $[a, b]$, it is uniformly continuous. Find $\delta > 0$ such that, for all $x, y \in [a, b]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. Let $P = \{x_0, ..., x_n\}$ be a partition of $[a, b]$ such that $\|P\| < \delta$ —that is, for each $1 \leqslant i \leqslant n$ we have $x_i - x_{i-1} < \delta$. Now for each $1 \leqslant i \leqslant n$, f is continuous on the closed intervals $[x_{i-1}, x_i]$. By the Extreme Value Theorem, there exists $x_{M_i}, x_{m_i} \in [x_{i-1}, x_i]$ such that $\sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_{M_i})$ and

$\inf_{x\in[x_{i-1},x_i]} f(x) = f(x_{m_i})$ for each $1 \leqslant i \leqslant n$. With this, observe that:

$$\begin{aligned}
U(P,f) - L(P,f) &= \sum_{j=1}^{n} \sup_{x\in[x_{j-1},x_j]} f(x)\Delta x_j - \sum_{j=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)\Delta x_j \\
&= \sum_{j=1}^{n} f(x_{M_j})\Delta x_j - \sum_{j=1}^{n} f(x_{m_j})\Delta x_j \\
&= \sum_{j=1}^{n} \big(f(x_{M_j}) - f(x_{m_j})\big)\Delta x_j.
\end{aligned}$$

Since $x_{M_j}, x_{m_j} \in [x_{j-1}, x_j]$, and since $\|P\| < \delta$, we have that $|x_{M_j} - x_{m_j}| < \delta$ for each $1 \leqslant j \leqslant n$. The uniform continuity of f finally gives:

$$\begin{aligned}
U(P,f) - L(P,f) &= \sum_{j=1}^{n} \big(f(x_{M_j}) - f(x_{m_j})\big)\Delta x_j \\
&< \sum_{j=1}^{n} \frac{\epsilon}{b-a}\Delta x_j \\
&= \epsilon.
\end{aligned}$$

Thus $f \in \mathcal{R}[a,b]$. □

Theorem 9. *Let* $f : [a,b] \to \mathbf{R}$ *be monotone. Then* $f \in \mathcal{R}[a,b]$.

Proof. Without loss of generality, suppose f is non-constant and non-decreasing on $[a,b]$. Let $\epsilon > 0$ and let $P = \{x_0, ..., x_n\}$ be a partition of $[a,b]$ with $\|P\| < \frac{\epsilon}{f(b)-f(a)}$. Since f is non-decreasing, note that for all $1 \leqslant j \leqslant n$ we have:

$$\begin{aligned}
M_j(f) &= \sup_{x\in[x_{j-1},x_j]} f(x) = f(x_j), \\
m_j(f) &= \inf_{x\in[x_{j-1},x_j]} f(x) = f(x_{j-1}).
\end{aligned}$$

This gives:

$$\begin{aligned}
U(P,f) - L(P,f) &= \sum_{j=1}^{n} M_j(f)\Delta x_j - \sum_{j=1}^{n} m_j(f)\Delta x_j \\
&= \sum_{j=1}^{n} f(x_j)\Delta x_j - \sum_{j=1}^{n} f(x_{j-1})\Delta x_j \\
&= \sum_{j=1}^{n} \big(f(x_j) - f(x_{j-1})\big)\Delta x_j.
\end{aligned}$$

Since $\|P\| < \frac{\epsilon}{f(b)-f(a)}$, this means $\Delta x_j < \frac{\epsilon}{f(b)-f(a)}$ for each $1 \leqslant j \leqslant n$. Finally:

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{j=1}^{n} \big(f(x_j) - f(x_{j-1})\big)\Delta x_j \\ &< \frac{\epsilon}{f(b)-f(a)} \sum_{j=1}^{n} \big(f(x_j) - f(x_{j-1})\big) \\ &= \epsilon. \end{aligned}$$

Thus $f \in \mathcal{R}[a,b]$. □

Lemma 1. *Let* $[c,d] \subseteq [a,b] \subset \mathbf{R}$*. If* $f \in \mathcal{R}[a,b]$*, then* $f \in \mathcal{R}[c,d]$*.*

Proof. Let $\epsilon > 0$. Find $P \in B_{[a,b]}$ such that $U(P,f) - L(P,f) < \epsilon$. Let $P' = P \cup \{c,d\}$ and $P_1 = P' \cap [c,d]$. Note that P' is a refinement of P, giving the inequality:

$$U(P',f) - L(P',f) \leqslant U(P,f) - L(P,f)$$

Moreover, since P_1 is a partition of $[c,d] \subseteq [a,b]$, it must be the case that:

$$U(P_1,f) - L(P_1,f) \leqslant U(P',f) - L(P',f).$$

Together:

$$\begin{aligned} U(P_1,f) - L(P_1,f) &\leqslant U(P',f) - L(P',f) \\ &\leqslant U(P,f) - L(P,f) \\ &< \epsilon. \end{aligned}$$

Thus $f \in \mathcal{R}[c,d]$. □

Theorem 10. *Let* $f : [a,b] \to \mathbf{R}$ *and let* $c \in (a,b)$*. The following are equivalent:*

(1) $f \in \mathcal{R}[a,b]$*;*

(2) $f \in \mathcal{R}[a,c]$ *and* $f \in \mathcal{R}[c,b]$*.*

Moreover, we have in this case that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$*.*

Proof. ($\Rightarrow$) If $f \in \mathcal{R}[a,b]$, apply Lemma 1 to $[a,c]$ and $[c,b]$. Then $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$.

($\Leftarrow$) Let $\epsilon > 0$. Since $f \in \mathcal{R}[a,c]$, find a partition $P_1 \in B_{[a,c]}$ such that $U(P_1,f) - L(P_1,f) < \frac{\epsilon}{2}$. Since $f \in \mathcal{R}[c,b]$, find a partition $P_2 \in B_{[c,b]}$ such that $U(P_2,f) - L(P_2,f) < \frac{\epsilon}{2}$. Let $P := P_1 \cup P_2$. Then $P \in B_{[a,]}$, and in particular we have:

$$\begin{aligned} U(P,f) &= U(P_1,f) + U(P_2,f), \\ L(P,f) &= L(P_1,f) + L(P_2,f). \end{aligned}$$

From this, we can see:

$$\begin{aligned} U(P,f) - L(P,f) &= U(P_1,f) + U(P_2,f) - L(P_1,f) - L(P_2,f) \\ &= \big(U(P_1,f) - L(P_1,f)\big) + \big(U(P_2,f) - L(P_2,f)\big) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $f \in \mathcal{R}[a,b]$.

Let ϵ, P, P_1, and P_2 be given as above. From the inequalities:

$$L(P_1,f) \leqslant \int_a^c f(x)dx \leqslant U(P_1,f),$$
$$L(P_2,f) \leqslant \int_c^b f(x)dx \leqslant U(P_2,f),$$
$$L(P,f) \leqslant \int_a^b f(x)dx \leqslant U(P,f),$$

we obtain the following result:

$$\left|\int_a^b f(x)dx - \left(\int_a^c f(x)dx - \int_c^b f(x)dx\right)\right| < \epsilon.$$

Hence $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

□

Theorem 11. *Let* $f : [a,b] \to \mathbf{R}$ *be bounded. Suppose that for all* $c \in (a,b)$, *we have* $f \in \mathcal{R}[a,c]$. *Then* $f \in \mathcal{R}[a,b]$.

Proof. Since f is bounded, there exists $K \geqslant 0$ such that $\sup_{x\in[a,b]}|f(x)| \leqslant K$. Equivalently, for all $x \in [a,b]$, we have $-K < f(x) < K$. Hence $\sup_{x\in[a,b]} f(x) \leqslant K$ and $\inf_{x\in[a,b]} \geqslant -K$. Thus $\sup_{x\in[x_n,b]} f(x) - \inf_{x\in[x_n,b]} f(x) \leqslant K - (-K) = 2K$.

Let $\epsilon > 0$. Pick c sufficiently close to b so that $b-c < \frac{\epsilon}{4K}$. Since $f \in \mathcal{R}[a,c]$, find $P_1 \in B_{[a,c]}$ so that $U(P_1,f) - L(P_1,f) < \frac{\epsilon}{2}$. Suppose $P_1 = \{x_0, x_1, ..., x_n\}$, where (by convention) $x_0 = a$ and $x_n = c$. Define $P := \{x_0, x_1, ..., x_n, b\}$. Then P is a partition of $[a,b]$, and we can see:

$$\begin{aligned}\left(\sup_{x\in[x_n,b]} f(x) - \inf_{x\in[x_n,b]} f(x)\right)(b-c) &\leqslant 2K(b-c)\\ &< \frac{\epsilon}{2}.\end{aligned}$$

This finally gives:

$$\begin{aligned}U(P,f) - L(P,f) &= \left(U(P_1,f) + \sup_{x\in[x_n,b]} f(x)(b-c)\right) - \left(L(P_1,f) + \inf_{x\in[x_n,b]} f(x)(b-c)\right)\\ &= \Big(U(P_1,f) - L(P_1,f)\Big) + \left(\sup_{x\in[x_n,b]} f(x) - \inf_{x\in[x_n,b]} f(x)\right)(b-c)\\ &\leqslant \Big(U(P_1,f) - L(P_1,f)\Big) + 2K(b-c)\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\ &= \epsilon.\end{aligned}$$

Thus $f \in \mathcal{R}[a,b]$. □

Example 2. Let $f : [0,1] \to \mathbf{R}$ be defined by:

$$f(x) = \begin{cases} 0, & x = 0 \\ \sin\frac{1}{x}, & x \in (0,1]. \end{cases}$$

Since f is continuous on $[c,1]$ for all $c > 0$, Theorem 8 gives that $f \in \mathcal{R}[c,1]$. Then by Theorem 11, we have $f \in \mathcal{R}[0,1]$.

Theorem 12. *Let* $[a,b] \subset \mathbf{R}$ *and* $f \in \mathcal{R}[a,b]$. *Suppose* $\operatorname{im}(f) \subset [c,d]$ *and let* $g : [c,d] \to \mathbf{R}$ *be a continuous function. Then the composition* $g \circ f \in \mathcal{R}[a,b]$.

Proof. Let $\epsilon > 0$. Since g is continuous on $[c,d]$, there exists $K \geqslant 0$ such that $\sup_{x\in[c,d]} |g(x)| \leqslant K$. Similarly, the continuity of g on $[c,d]$ implies g is uniformly continuous. So there exists $\delta \in \left(0, \frac{\epsilon}{b-a+2K}\right)$ such that, for all $s,t \in [c,d]$, $|s-t| < \delta$ implies $|g(s)-g(t)| < \frac{\epsilon}{b-a+2K}$. Since $f \in \mathcal{R}[a,b]$, there exists a partition $P = \{x_0, x_1, ..., x_n\} \in B_{[a,b]}$ such that $U(P,f) - L(P,f) < \delta^2$. Write $\{1,2,...,n\} = A \cup B$, where $A := \{k \mid M_k - m_k < \delta\}$ and $B := \{k \mid M_k - m_k \geqslant \delta\}$. For $i \in A$, note that for every $x,y \in [x_{i-1}, x_i]$ we have:

$$\begin{aligned} |f(x) - f(y)| &\leqslant M_i(f) - m_i(f) \\ &< \delta. \end{aligned}$$

Since $f(x), f(y) \in \operatorname{im}(f) \subseteq [c,d]$, by the uniform continuity of g it follows that $|g(f(x)) - g(f(y))| < \frac{\epsilon}{b-a+2K}$. In particular:

$$\begin{aligned} \sum_{i\in A}(M_i(g\circ f) - m_i(g\circ f))\Delta x_i &= \sum_{i\in A}\left(\sup_{x,y\in[x_{i-1},x_i]} g(f(x)) - g(f(y))\right)\Delta x_i \\ &\leqslant \frac{\epsilon}{b-a+2K}\sum_{i\in A}\Delta x_i \end{aligned}$$

For $i \in B$, note that:

$$\begin{aligned} \delta\sum_{i\in B}\Delta x_i &\leqslant \sum_{i\in B}(M_i(f) - m_i(f))\Delta x_i \\ &\leqslant U(P,f) - L(P,f) \\ &< \delta^2. \end{aligned}$$

Simplifying gives $\sum_{i\in B}\Delta x_i < \delta < \frac{\epsilon}{b-a+2K}$. With all of this, and using the fact that $M_i(g\circ f) - m_i(g\circ f) \leqslant 2K$, we can see:

$$\begin{aligned} U(P, g\circ f) - L(P, g\circ f) &= \sum_{i\in A}(M_i(g\circ f) - m_i(g\circ f))\Delta x_i + \sum_{i\in B}(M_i(g\circ f) - m_i(g\circ f))\Delta x_i \\ &\leqslant \frac{\epsilon}{b-a+2K}\sum_{i\in A}\Delta x_i + 2K\sum_{i\in B}\Delta x_i \\ &\leqslant \frac{\epsilon}{b-a+2K}(b-a) + 2K\delta \\ &< \frac{\epsilon(b-a)}{b-a+2K} + \frac{2K\epsilon}{b-a+2K} \\ &= \epsilon. \end{aligned}$$

Thus $g \circ f \in \mathcal{R}[a,b]$. □

Corollary 13. *Let* $f, g \in \mathcal{R}[a, b]$. *Then* $f^2 \in \mathcal{R}[a, b]$, $|f| \in \mathcal{R}[a, b]$, $\min\{f, g\} \in \mathcal{R}[a, b]$, *and* $\max\{f, g\} \in \mathcal{R}[a, b]$.

Proof. Simply take $g(x) = x^2$ or $g(x) = |x|$, then applying Theorem 12 gives $|f|$ and f^2 as integrable. Moreover, since:

$$\begin{aligned} \max\{f, g\} &= \frac{1}{2}(f + g + |f - g|) \\ \min\{f, g\} &= \frac{1}{2}(f + g - |f - g|), \end{aligned}$$

we have $\min\{f, g\} \in \mathcal{R}[a, b]$ and $\max\{f, g\} \in \mathcal{R}[a, b]$. □

Corollary 14. *Let* $[a, b] \subset \mathbf{R}$. *The set* $\mathcal{R}[a, b]$ *is an* $\mathbf{R}$*-algebra.*

Proof. Let $f, g \in \mathcal{R}[a, b]$ and $c \in \mathbf{R}$. We showed in Proposition 4 that $\mathcal{R}[a, b]$ is an $\mathbf{R}$-vector space. We can write $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$, hence by Corollary 13 we have $fg \in \mathcal{R}[a, b]$. □

Corollary 15. *Let* $f \in \mathcal{R}[a, b]$. *Then* $\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx$.

Proof. Corollary 13 proved that $|f| \in \mathcal{R}[a, b]$, and we have for all $x \in [a, b]$ that $-|f(x)| \leqslant f(x) \leqslant |f(x)|$. Corollary 6 gives $-\int_a^b |f(x)|dx \leqslant \int_a^b f(x)dx \leqslant \int_a^b |f(x)|dx$, whence $\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx$. □

§ Integration and Differentiation

§ Limits of Darboux Integrable Functions

Example 3.

Proposition 16. *Let* $(f_n)_n$ *be a sequence in* $\mathcal{R}[a, b]^{\mathbf{N}}$ *which converges uniformly to* $f : [a, b] \to \mathbf{R}$. *Then* $f \in \mathcal{R}[a, b]$.

Proof. We must first show that f is bounded. Let $\epsilon = 1$. Since $(f_n)_n \to f$ uniformly, find N sufficiently large so that $\|f_N - f\| < 1$. Note that $f_N \in \mathcal{R}[a, b]$, hence it is bounded. This means there exists $M \geqslant 0$ such that $\|f_N\| \leqslant M$. Together, for all $x \in [a, b]$ we have:

$$\begin{aligned} |f(x)| &\leqslant |f_N(x)| - |f_N(x) - f(x)| \\ &\leqslant \|f_N\| - \|f_N - f\| \\ &< M - 1. \end{aligned}$$

Thus f is bounded.

Now let $\epsilon > 0$. Since $(f_n)_n \to f$ uniformly, find N large so that $\|f_N - f\| < \frac{\epsilon}{3(b-a)}$. Since

$f_N \in \mathcal{R}[a, b]$, there exists $P \in B_{[a,b]}$ such that $U(P, f_N) - L(P, f_N) < \frac{\epsilon}{3}$. We can see that:

$$\begin{aligned} U(P, f) - U(P, f_N) &= \sum_{j=1}^{|P|-1} M_j(f)\Delta x_j - \sum_{j=1}^{|P|-1} M_j(f_N)\Delta x_j \\ &= \sum_{j=1}^{|P|-1} \big(M_j(f) - M_j(f_N)\big)\Delta x_j \\ &\leqslant \sum_{j=1}^{|P|-1} M_j(f - f_N)\Delta x_j \\ &\leqslant \|f - f_N\| \sum_{j=1}^{|P|-1} \Delta x_j \\ &= \|f - f_N\|\,(b - a) \\ &< \frac{\epsilon}{3}. \end{aligned}$$

It follows similarly that $L(P, f_N) - L(P, f) \leqslant \|f - f_N\|\,(b - a)$. Together:

$$\begin{aligned} U(P, f) - L(P, f) &\leqslant |U(P, f) - L(P, f)| \\ &\leqslant |U(P, f) - U(P, f_N)| + |U(P, f_N) - L(P, f_N)| + |L(P, f_N) - L(P, f)| \\ &< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Thus $f \in \mathcal{R}[a, b]$. □

Lemma 2. *Let* $f : [a, b] \to \mathbf{R}$ *be bounded and positive-definite function. For each* $\epsilon > 0$*, there exists a continuous function* $g \in C([a, b])$ *satisfying* $0 \leqslant g \leqslant f$*. Moreover:*

$$\underline{\int_a^b} f(x)dx \leqslant \int_a^b g(x)dx + \epsilon.$$

Proof. □