Preface

At the root of measure theory is a rigorous treatment of how to extend the notions of length (on the real line), area (on the Euclidean plane), and volume (in three-space) to a larger collection of sets than those seen in, say, courses on Calculus. We have seen in a first course in analysis that there exist sets with some exceptionally strange properties (the Cantor set being a classic example). We would like to be able to discuss these types of sets in the context of measuring their sizes (be it a length, area, volume, etc). While the historical impetus was to solve this problem for sets in $\mathbb{R}^n$, one finds that it is not hard to abstract the theory to more general sets. Accordingly we take up this abstract study. This is not just for the sake of abstraction. In fact one can find applications for abstract measure theory in, for example, probability theory (where the measure of a set corresponds to the likelihood of an element in that set being chosen under some random process) and physics (where one might want the measure of a set to correspond to the mass of some object occupying that space).

Last update: 2025 September 25

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Chapter 1

Preliminaries

§ 1.1. Set Theory

Definition 1.1.1. A *set* is a collection of different things; the things are called *elements* of the set.

Zermelo–Fraenkel set theory introduces eight different axioms which mathematics builds upon. Below is an introduction to the ones which are relevant for the material covered in this course.

Axiom 1 (Axiom of Extension)**.** *If the sets* X *and* Y *have the same members, then they are the same set. In other words:*

$$(\forall X)(\forall Y)\Big((\forall z)\big(z \in X \Leftrightarrow z \in Y\big) \implies X = Y\Big)$$

Axiom 2 (Axiom of Existence)**.** *There is a set such that no element is a member of it. In other words:*

$$(\exists A) : (\forall x)(x \notin A)$$

Note that the Axiom of Extension immediately proves that the set which contains no elements is unique; two sets are equal if they have the same members, hence two sets which contain no elements are equal.

Definition 1.1.2. The set which contains no elements is called the *empty set*, and is denoted by $\emptyset$.

Axiom 3 (Axiom of Pairing)**.** *Let* A *and* B *be sets. There exists the set* $\mathcal{C} := \{A, B\}$*. In other words:*

$$(\forall A)(\forall B)(\exists \mathcal{C}) : (A \in \mathcal{C} \wedge B \in \mathcal{C}).$$

Axiom 4 (Axiom of Union)**.** *Let* $\mathcal{C}$ *be a set. Then there exists a set* U *consisting of the elements* x *that are contained in at least one element of the set* $\mathcal{C}$*, that is:*

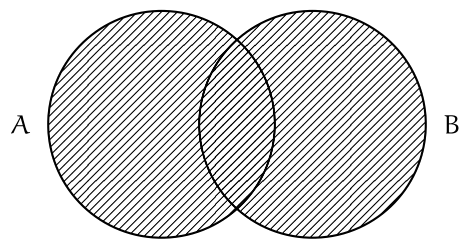
$$(\forall \mathcal{C})(\exists U) : (\forall x)(x \in U \Leftrightarrow (\exists C)(u \in C \wedge C \in \mathcal{C}))$$

Definition 1.1.3. Let $\mathcal{C}$ be a set, and let U be the set consisting of the elements x that are contained in at least one element of $\mathcal{C}$.

(1) The set U is called the *union of the set* $\mathcal{C}$ and is denoted:

$$U := \bigcup \mathcal{C}.$$

(2) If the set $\mathcal{C}$ consists of only two sets A and B, then we write $U := A \cup B$.



$A \cup B$

Axiom 5 (Axiom Schema of Specification). *Let A be a set, and let φ be a predicate containing the free variable x. There exists a subset B of the set A whose members are precisely the elements x of the set A such that the sentence $\varphi(x)$ is true. In other words:*

$$(\forall A)(\forall \varphi)(\exists B) : (B = \{x \in A \mid \varphi(x)\})$$

Proposition 1.1.1. *Let $\mathcal{C}$ be a nonempty set. There exists a set D consisting of the elements x that are contained in all elements of the set $\mathcal{C}$.*

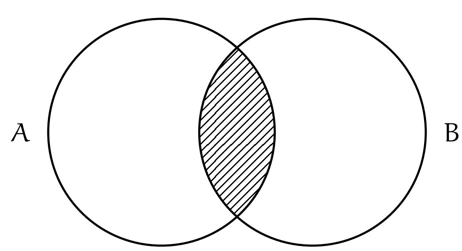
Proof. Let $E \in \mathcal{C}$. By the Axiom Schema of Specification we have $D := \{c \in E \mid (\forall X)(X \in \mathcal{C} \Rightarrow c \in X)\}$. $\square$

Definition 1.1.4. Let $\mathcal{C}$ be a nonempty set, and let D be the set consisting of the elements x that are contained in all elements of the set $\mathcal{C}$.

(1) The set D is called the *intersection of the set* $\mathcal{C}$ and is denoted:

$$D := \bigcap \mathcal{C}.$$

(2) If the set $\mathcal{C}$ contains only two sets A and B, then we write $D := A \cap B$.



$A \cap B$

Axiom 6. *Let X be a set. Then the set of all subsets of X exists.*

Definition 1.1.5. Let X be a set. Then the set of all subsets of the set X is called the *power set of the set* X and is denoted $\mathcal{P}(X)$.

Axiom 7 (Axiom of Infinity)**.** *There is a set that contains all natural numbers.*

Definition 1.1.6. An *index set* is a nonempty set whose elements label the elements of another set. We denote the collection of objects labeled by elements of I as $\{A_\alpha\}_{\alpha \in I}$.

With this definition and the Axiom of Infinity, we can extend the concept of unions and intersections to families of sets of arbitrary size.***

Definition 1.1.7. Let I be an index set, let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of sets, and write $\mathcal{X} := \{X_\alpha \mid \alpha \in I\}$. Define:

$$\bigcup_{\alpha \in I} X_i := \bigcup \mathcal{X},$$
$$\bigcap_{\alpha \in I} X_i := \bigcap \mathcal{X}.$$

Definition 1.1.8. *** Let $\{X_n\}_{n \in \mathbb{N}}$ be a family of sets indexed by the natural numbers.

(1) The *limit superior* of $\{X_n\}_{n \in \mathbb{N}}$ is:

$$\begin{aligned} \limsup X_n &:= \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} X_n \right) \\ &= \{x \mid x \in E_n \text{ for infinitely many } n\}. \end{aligned}$$

(2) The *limit inferior* of $\{X_n\}_{n \in \mathbb{N}}$ is:

$$\begin{aligned} \limsup X_n &:= \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} X_n \right) \\ &\{x \mid x \notin E_n \text{ for only finitely many } n\} \end{aligned}$$

Definition 1.1.9. Let A and B be two sets. Define the *difference of* A *and* B by $A \setminus B := \{x \in A \mid x \notin B\}$.

Note that the difference of two sets exists by the Axiom Schema of Specification.

Definition 1.1.10. Let a and b be two objects. The *ordered pair* (a, b) is defined by $(a, b) := \{\{a\}, \{a, b\}\}$.

Proposition 1.1.2. *Let* A *and* B *be sets. There exists a set* E *containing all ordered pairs* (a, b)*, where* $a \in A$ *and* $b \in B$*.*

Proof. Let (a, b) be any ordered pair such that $a \in A$ and $b \in B$. By definition $(a, b) = \{\{a\}, \{a, b\}\}$. Now $\{a\} \subset A$, so $\{a\} \subset A \cup B$, hence $\{a\} \in \mathcal{P}(A \cup B)$. Similarly, $\{a, b\} \subset A \cup B$, so $\{a, b\} \in \mathcal{P}(A \cup B)$. It follows that $\{\{a\}, \{a, b\}\} \subset \mathcal{P}(A \cup B)$, hence $\{\{a\}, \{a, b\}\} \in$

$\mathcal{P}(\mathcal{P}(A \cup B))$. We know all these sets are known to exist by the Axiom of Union and Axiom of Power Set. Then by the Axiom Schema of Specification, there exists a set:

$$E := \{t \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid (\exists a)(\exists b)(a \in A \wedge b \in B \wedge t = (a, b))\}.$$

Moreover, this set is unique by the Axiom of Extension, and is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. □

Definition 1.1.11. Let A and B be two sets, and let E be the set containing all ordered pairs (a, b) where $a \in A$ and $b \in B$. The set E is called the *Cartesian product of* A *and* B, and is denoted $E := A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Definition 1.1.12. Let X and Y be sets. A *relation* from X to Y is a subset R of $X \times Y$. We write xRy to mean $(x, y) \in R$.

(1) An *equivalence relation* on X is a relation which is reflexive, transitive, and symmetric. We write $x \sim y$ to mean xRy.

(2) A *function* (or *map*) from X to Y, denoted $f : X \to Y$, is a relation from X to Y such that, for every $x \in X$, there exists a unique $y \in Y$ such that xRy. We write $f(x) = y$ to mean xRy.

(3) A *partial order* on X is a relation which is reflexive, transitive, and antisymmetric. We write $x \leqslant y$ to mean xRy, and say that X is an *ordered set*.

Proposition 1.1.3. **** Let X be a set. The following are equivalent:*

(1) S is a partition of X;

(2) There exists an equivalence relation $\sim$ on X such that the equivalence classes of $\sim$ are exactly the elements of S.

Proof. □

Example 1.1.1. An equivalent relation on X partitions X into equivalence classes.

Example 1.1.2. Recall that, if $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *covariant functor* $T : \mathcal{C} \to \mathcal{D}$ is a map such that:

(i) If $A \in \text{obj}(\mathcal{C})$, then $T(A) \in \text{obj}(\mathcal{D})$.

(ii) If $f \in \text{Hom}_{\mathcal{C}}(A, A')$, then $T(f) \in \text{Hom}_{\mathcal{D}}(T(A), T(A'))$.

(iii) If $f \in \text{Hom}_{\mathcal{C}}(A, A')$ and $g \in \text{Hom}_{\mathcal{C}}(A', A'')$, then $T(f) \in \text{Hom}_{\mathcal{D}}(T(A), T(A'))$ and $T(g) \in \text{Hom}_{\mathcal{D}}(T(A'), T(A''))$. In particular:

$$T(gf) = T(g)T(f).$$

(iv) For every $A \in \text{obj}(\mathcal{C})$, $T(\text{id}_A) = \text{id}_{T(A)}$.

A *contravariant functor* $T : \mathcal{C} \to \mathcal{D}$ is similarly defined, but (ii) and (iii) are instead defined as:

(ii) If $f \in \text{Hom}_{\mathcal{C}}(A, A')$, then $T(f) \in \text{Hom}_{\mathcal{D}}(T(A'), T(A))$.

(iii) If $f \in \mathrm{Hom}_{\mathcal{C}}(A, A')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(A', A'')$, then $T(f) \in \mathrm{Hom}_{\mathcal{D}}(T(A'), T(A))$ and $T(g) \in \mathrm{Hom}_{\mathcal{D}}(T(A''), T(A'))$. In particular:

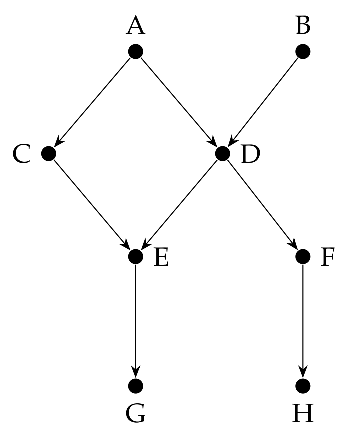
$$T(gf) = T(f)T(g).$$

Given $f : X \to Y$, the power set gives rise to two functors, the *contravariant power set functor* $\mathfrak{Set}^{\mathrm{op}} \to \mathfrak{Set}$ and the *covariant power set functor* $\mathfrak{Set} \to \mathfrak{Set}$. The first sends $f : X \to Y$ to the *preimage* function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$, whereas the second sends f to the *image* function $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$. The preimage is more well-behaved, commuting with boolean operations:

$$\begin{aligned} f^{-1}\left(E^c\right) &= \left(f^{-1}(E)\right)^c, \\ f^{-1}\left(\bigcup_{\alpha \in I} E_\alpha\right) &= \bigcup_{\alpha \in I} f^{-1}(E_\alpha), \\ f^{-1}\left(\bigcap_{\alpha \in I} E_\alpha\right) &= \bigcap_{\alpha \in I} f^{-1}(E_\alpha). \end{aligned}$$

This is not necessarily the case for the image function.

Example 1.1.3. ***(I don't like the prose here) Consider the directed acyclic graph:



Let $G = \{A, B, C, D, E, F, G, H\}$ be the set of vertices of our graph. We can define a partial ordering on G as follows: for any $x, y \in G$, we have $x \leqslant y$ if and only if there exists a directed path from x to y. However, note that not every element of G can be compared.

Definition 1.1.13. An ordering on a set X is said to be *linear* (or *total*) if for every $x, y \in X$, $x \leqslant y$ or $y \leqslant x$.

Definition 1.1.14. Let X be an ordered set. Let $A \subset X$.

(1) A is called *bounded above* if there exists an element $u \in X$ with $a \leqslant u$ for all $a \in A$. Such a u is called an *upper bound* for A. The set of upper bounds of A is denoted $\mathcal{U}_A = \{u \in X \mid u \text{ is an upper bound of } A\}$.

(2) A is called *bounded below* if there exists an element $v \in X$ with $v \leqslant a$ for all $a \in A$. Such a v is called a *lower bound* for A. The set of lower bounds of A is defined as $\mathcal{L}_A = \{v \in X \mid v \text{ is a lower bound of } A\}$.

(3) If A admits an upper bound u with $u \in A$, then u is called *the greatest element of* A.

(4) If A admits a lower bound v with $v \in A$, then v is called *the least element of* A.

(5) If l is the least element of $\mathcal{U}_A$, we write $l = \sup(A)$ and call it the *supremum of* A.

(6) If g is the greatest element of $\mathcal{L}_A$, we write $g = \inf(A)$ and call it the *infimum of* A.

(7) A *maximal element of* A is an element $m \in A$ such that if $a \geqslant m$, then $a = m$ (not necessarily unique).

(8) A *minimal element of* A is an element $n \in A$ such that if $a \leqslant n$, then $a = n$ (not necessarily unique).

Definition 1.1.15. If a linear order on a set X satisfies the property that every nonempty subset has a minimal element, then we say it is a *well-ordering*.

——————//——————

***include stuff HERE

Definition 1.1.16.

do general cartesian product
question is it nonempty?
axiom of choice
answer: no, axiom of choice says this choice function exists.
'shoes vs. socks' formulation of the axiom

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§ 1.2. The Extended Real Numbers***

Definition 1.2.1. The *extended real number line* is the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, \infty\} = [-\infty, \infty]$.

The extended real number line has many unique properties.

- For all $x \in \mathbb{R}$, we have $-\infty < x < \infty$. This allows us to extend the ordering of $\mathbb{R}$ to $\overline{\mathbb{R}}$.
- Recall that since $\mathbb{R}$ is complete, every *bounded subset* admits a supremum. However, for the extended real number line, *every subset* of $\overline{\mathbb{R}}$ admits a supremum.

  Example 1.2.1.

  (i) In $\overline{\mathbb{R}}$, we have $\sup \mathbb{R} = \infty$.
  (ii) $\inf \mathbb{Q} \cap (-\infty, 0) = -\infty$.

(iii) $\inf \emptyset = \infty$. This follows from basic intuition: the infimum of a set is the greatest lower bound, so the infimum of the empty set is an element which is a lower bound for nothing. Hence it must be ∞.

- In $\mathbb{R}$, any sequence which is monotone and bounded converges. However, in $\overline{\mathbb{R}}$, *any* monotone sequence converges to a value in $\overline{\mathbb{R}}$.
- For any sequence $(x_n)_n$, we define:

$$\limsup x_n := \inf_{k \geqslant 1} \left(\sup_{n \geqslant k} x_n \right),$$
$$\liminf x_n := \sup_{k \geqslant 1} \left(\inf_{n \geqslant k} x_n \right).$$

  In $\mathbb{R}$, it could be the case that each is properly divergent, however in $\overline{\mathbb{R}}$ the limit superior and limit inferior *always* exist.

- We can define arithmetic with $-\infty$ and ∞. For $x \in \overline{\mathbb{R}}$:

$$\begin{aligned} x + \infty &= \infty, \\ x - \infty &= -\infty, \\ \infty + \infty &= \infty, \\ -\infty - \infty &= -\infty. \end{aligned}$$

  We don't give any meaning to $\infty - \infty$. If $x > 0$, we define:

$$\begin{aligned} x \cdot \infty &= \infty, \\ x \cdot (-\infty) &= -\infty. \end{aligned}$$

  If $x < 0$, we define:

$$\begin{aligned} x \cdot \infty &= -\infty, \\ x \cdot (-\infty) &= \infty. \end{aligned}$$

  Unless stated otherwise, we define $0 \cdot \pm\infty = 0$.

- Let X be any set and let $f : X \longrightarrow [0, \infty]$. We'd like to give some meaning to $\sum_{x \in X} f(x)$. Regardless of the cardinality of X, it is a theorem that

This whole section needs to be redone.

Proposition 1.2.1. ****(could be cleaned up) Every open set in $\mathbb{R}$ is at most a countable union of disjoint open intervals.*

Proof. Let $U \subset \mathbb{R}$ be open. Define a binary relation $\sim$ on U as follows: $x \sim y$ if and only if there exists an open interval I such that $\{x, y\} \subset I \subset U$. One can check that this is indeed an equivalence relation, hence U is partitioned by the equivalence classes of $\sim$.

We are now going to show that each equivalence class is an open interval. Let E be an equivalence class and let $I = (\inf E, \sup E)$. Clearly $E \subset I$. Let $x \in I$. Then $\inf E < x$ and

$x < \sup E$. We can find $u, v \in E$ such that $\inf E < u < x < v < \sup E$. Since $u, v \in E$, we can find an interval J such that $\{u, v\} \subset J \subset U$. But this means $x \in J$, hence $x \sim u$ and $x \sim v$. Thus $x \in E$, giving $I \subset E$ meaning that E is an open interval.

It remains to show that there are at most a countable number of equivalence classes. Note that each equivalence class contains a rational number, and no two classes can contain the same rational number (otherwise the equivalence classes wouldn't be disjoint). By the Axiom of Choice, we can find a choice function $f : U/\sim \longrightarrow \mathbb{Q}$ which is injective. Therefor there are at most countably many open intervals. □

§ 1.3. Metric Spaces

Definition 1.3.1. A *distance function* (or *metric*) on a nonempty set X is a map:

$$\rho : X \times X \longrightarrow [0, \infty)$$

satisfying for all $x, y, z \in X$:

(1) $\rho(x, x) = 0$;

(2) $\rho(x, y) = \rho(y, x)$;

(3) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

The pair (X, ρ) is called a *metric space*.

Distance functions let us talk about sets which centered at a point.

Definition 1.3.2. Let X be a metric space, $x \in X$ and $r > 0$.

(1) The *open ball of radius* r *centered at* x is the set $B(r, x) := \{y \in X \mid \rho(x, y) < r\}$.

(2) The *closed ball of radius* r *centered* x is the set $V(r, x) := \{y \in X \mid \rho(x, y) \leqslant r\}$.

(3) The *sphere of radius* r *centered at* x is the set $S(r, x) := \{y \in X \mid \rho(x, y) = r\}$.

Such distance functions induce a topology on our metric space X.

Definition 1.3.3. Let X be a metric space.

(1) We say $U \subset X$ is *open* if, for all $x \in U$, there exists $r > 0$ such that $B(r, x) \subset U$.

(2) We say $V \subset X$ is *closed* if V^c is open.

Given a set, we characterize its inside, the smallest closed set containing it, as well as its boundary.

Definition 1.3.4. Let X be a metric space and $E \subset X$.

(1) The *interior of* E is the set:

$$E^o := \bigcup\{U \subset E \mid U \text{ is open}\}.$$

(2) The *closure of* E is the set:

$$\overline{E} := \bigcap\{C \supset E \mid C \text{ is closed}\}.$$

(3) The *boundary of* E is the set:

$$\partial E = \overline{E} \setminus E^{o}.$$

Proposition 1.3.1. *A countable subset of* $\mathbb{R}$ *has an empty interior.*

Proof. Let A be a subset of $\mathbb{R}$ which has a nonempty interior. Then A contains a non-trivial interval. Hence $\mathrm{card}(A) \geqslant \mathfrak{c} > \aleph_0$. Thus A is uncountable. $\square$

Example 1.3.1. Let $E = \mathbb{Q} \cap (0, 1)$. We have $E^{o} = \emptyset$ and $\overline{E} = [0, 1]$.

Definition 1.3.5. Let X be a metric space and $E \subset X$.

(1) E is *dense in* X if $\overline{E} = X$.

(2) E is *nowhere dense* if $(\overline{E})^{o} = \emptyset$.

Intuitively, a set is nowhere dense if we can't fit an open ball inside it.

Definition 1.3.6. Let (X, d) be a metric space.

(1) The collection of open sets of X is denoted τ_X.

(2) A *base* for τ_X is a family of open subsets $\mathcal{B} \subseteq \tau_X$ such that:

$$(\forall U \in \tau_X)(\forall x \in U)(\exists B \in \mathcal{B}) : x \in B \subseteq U.$$

Equivalently, for all $U \in \tau_X$, we can write $U = \bigcup_{i \in I} B_i$, where $\{B_i\}_{i \in I} \subseteq \tau_X$.

(3) X is *second countable* if it has a countable base.

Definition 1.3.7. Let (X, ρ) be a metric space. X is *separable* if there exists a countable dense subset.

Proposition 1.3.2. *** *Let* (X, ρ) *be a metric space.* X *is second countable if and only if* X *is separable.*

Proof. $\square$

With a metric we can talk about the distance from a point to a set, as well as a way to define if a set is bounded.

Definition 1.3.8. Let (X, ρ) be a metric space, $x \in X$, and $E \subset X$.

(1) The distance from x to E is $\rho(x, E) := \inf_{y \in E} \rho(x, y)$.

(2) The *diameter* of E is $\mathrm{diam}(E) = \sup_{x,y \in E} \rho(x, y)$.

(3) A subset of a metric space is *bounded* if its diameter is finite.

With a notion of distance, we can talk about sequence convergence.

Definition 1.3.9. Let (X, ρ) be a metric space and $(x_n)_n$ a sequence in X. We say $(x_n)_n$ *converges to a point* x_0 if $\lim_{n \to \infty} \rho(x_n, x) = 0$.

Proposition 1.3.3. *Let* (X, ρ) *be a metric space,* $E \subset X$*, and* $x \in X$*. The following are equivalent:*

(1) $x \in \overline{E}$;

(2) For every $r > 0$, $B(r, x) \cap E \neq \emptyset$;

(3) There is a sequence $(x_n)_n$ *in* E *which converges to* x

We can even talk about continuity of functions between metric spaces.

Definition 1.3.10. Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces and $f : X_1 \to X_2$.

(1) We say f is *continuous* if:

$$(\forall c \in X)(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X_1)(\rho_1(x, y) < \delta \implies \rho_2(f(x), f(y)) < \epsilon).$$

(2) We say f is *uniformly continuous* if:

$$(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x, y \in X_1)(\rho_1(x, y) < \delta \implies \rho_2(f(x), f(y)) < \epsilon).$$

Note that δ may depend on our point c in the case that f is continuous. A function is uniformly continuous if δ does not depend on any two points in its domain.

Proposition 1.3.4. *Let* (X_1, ρ_1) *and* (X_2, ρ_2) *be metric spaces.* f *is continuous if and only if the preimage of every open set in* X_2 *is open in* X_1.

Similarly to how every Cauchy sequence in $\mathbb{R}$ converges, we can talk about metric spaces which guarantee that same property.

Definition 1.3.11.

(1) Let $(x_n)_n$ be a sequence in a metric space (X, ρ). We say $(x_n)_n$ is ρ-*Cauchy* if:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N}) : (\forall n, m \in \mathbb{N})(n > m \geqslant N \implies \rho(x_n, x_m) < \epsilon).$$

(2) A metric space X is called *complete* if every Cauchy sequence converges to a point in X.

We can generalize the notion of a set being closed and bounded in $\mathbb{R}^n$.

Proposition 1.3.5. *Let* X *be a metric space. The following are equivalent:*

(1) Every open cover of X *admits a finite subcover;*

(2) Every sequence in X *admits a convergent subsequence;*

(3) X *is complete and totally bounded.*

Definition 1.3.12. We say a metric space X is *compact* if it satisfies any of the properties in Proposition 1.3.5.

Chapter 2

Measures

Ideally to measure the size of a set, we'd like a function

$$\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$$

to have the following properties:

(1) The unit cube should have a measure equal to one —that is, $\mu\left(\{x \in \mathbb{R}^n \mid 0 \leqslant x_j < 1\}\right) = 1$;

(2) If E and F are congruent (that is, F is a translation, rotation, and/or reflection), then $\mu(E) = \mu(F)$;

(3) If $E_1, E_2, \dots$ are pairwise disjoint, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$.

This is impossible.

Theorem 2.0.1. *A function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ satisfying (1), (2), and (3) from above does not exist.*

Proof. By (1) we have $\mu([0, 1)) = 1$. Let $x, y \in [0, 1)$. Define a relation on $[0, 1)$ as follows: $x \sim y$ if and only if $x - y \in \mathbb{Q}$. This is an equivalence relation. Using the Axiom of Choice, let N be a set which contains exactly one element from each equivalence class. For $q \in \mathbb{Q} \cap [0, 1)$, define:

$$N_q = \{x + q \mid x \in N \cap [0, 1 - q)\} \cup \{x + q - 1 \mid x \in N \cap [1 - q, 1)\}$$

That is, N_q is a rotation of N by q units, in which the shifted part that sticks out past $[0, 1)$ is shifted back to zero. (include picture of circle here) With this, note that:

$$\begin{aligned} \mu(N) &= \mu\big(N \cap [0, 1 - q)\big) + \mu\big(N \cap [1 - q, 1)\big) \\ &= \mu\big(\{x + q \mid x \in N \cap [0, 1 - q)\}\big) + \mu\big(\{x + q - 1 \mid x \in N \cap [1 - q, 1)\}\big) \quad \text{Translation} \\ &= \mu(N_r). \end{aligned}$$

Now let $s, r \in \mathbb{Q} \cap [0, 1)$. Claim: if $r \neq s$, then $N_r \cap N_q = \emptyset$. Suppose not, that is, let $x \in N_r \cap N_s$. This means $x - r$ and $x - s$ would be distinct elements of N. However, notice that $(x - r) - (x - s) = s - r \in \mathbb{Q}$. Hence $x - r$ and $x - s$ are in the same equivalence class, which contradicts the way we've defined N. (We showed that any two rational rotations of N are disjoint)

Now let $y \in [0, 1)$. Let $x \in N$ satisfy $x \sim y$. Since x and y are contained in the same equivalence class, we can find $q \in \mathbb{Q}$ such that:

$$y = \begin{cases} x + q, & x < y \\ x - (q - 1) & x > y \end{cases}.$$

Then $y \in N_q$. (We've shown that if y is in the same equivalence class as a representative $x \in N$, then y is a rotation of N of some rational number). □

Relaxing (3) to be only finitely additive does not help either.

Theorem 2.0.2 (Banach-Tarski Paradox)**.** *Write this stuff out.*

This is a true statement...

We can't hope to measure every set of $\mathcal{P}(\mathbb{R}^n)$. Some sets are not measurable. We need to restrict μ to some subset of $\mathcal{P}(\mathbb{R}^n)$. We need to leave some sets out.

§ 2.1. σ-Algebras

Definition 2.1.1. Let X be a set. An *algebra* of sets is a subset $\mathcal{A} \subset \mathcal{P}(X)$ which is closed under finite unions and complements. If $\mathcal{A}$ is closed under countable unions, we say $\mathcal{A}$ is a σ-*algebra*.