Preface

At the root of measure theory is a rigorous treatment of how to extend the notions of length (on the real line), area (on the Euclidean plane), and volume (in three-space) to a larger collection of sets than those seen in, say, courses on Calculus. We have seen in a first course in analysis that there exist sets with some exceptionally strange properties (the Cantor set being a classic example). We would like to be able to discuss these types of sets in the context of measuring their sizes (be it a length, area, volume, etc). While the historical impetus was to solve this problem for sets in $\mathbb{R}^n$, one finds that it is not hard to abstract the theory to more general sets. Accordingly we take up this abstract study. This is not just for the sake of abstraction. In fact one can find applications for abstract measure theory in, for example, probability theory (where the measure of a set corresponds to the likelihood of an element in that set being chosen under some random process) and physics (where one might want the measure of a set to correspond to the mass of some object occupying that space).

Last update: 2025 September 27

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Chapter 1

Preliminaries

§ 1.1. Set Theory

Definition 1.1.1. A *set* is a collection of different things; the things are called *elements* of the set.

Zermelo–Fraenkel set theory introduces eight different axioms which mathematics builds upon. Below is an introduction to the ones which are relevant for the material covered in this course.

Axiom 1 (Axiom of Extension). *If the sets X and Y have the same members, then they are the same set. In other words:*

$$(\forall X)(\forall Y)\Big((\forall z)\big(z \in X \Leftrightarrow z \in Y\big) \implies X = Y\Big)$$

Axiom 2 (Axiom of Existence). *There is a set such that no element is a member of it. In other words:*

$$(\exists A) : (\forall x)(x \notin A)$$

Note that the Axiom of Extension immediately proves that the set which contains no elements is unique; two sets are equal if they have the same members, hence two sets which contain no elements are equal.

Definition 1.1.2. The set which contains no elements is called the *empty set*, and is denoted by $\emptyset$.

Axiom 3 (Axiom of Pairing). *Let A and B be sets. There exists the set $\mathcal{C} := \{A, B\}$. In other words:*

$$(\forall A)(\forall B)(\exists \mathcal{C}) : (A \in \mathcal{C} \wedge B \in \mathcal{C}).$$

Axiom 4 (Axiom of Union). *Let $\mathcal{C}$ be a set. Then there exists a set U consisting of the elements x that are contained in at least one element of the set $\mathcal{C}$, that is:*

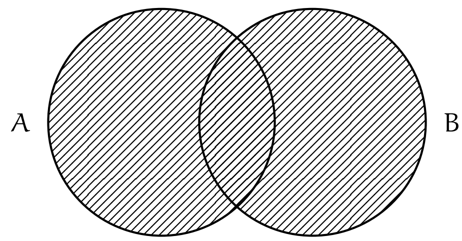
$$(\forall \mathcal{C})(\exists U) : (\forall x)(x \in U \Leftrightarrow (\exists C)(u \in C \wedge C \in \mathcal{C}))$$

Definition 1.1.3. Let $\mathcal{C}$ be a set, and let U be the set consisting of the elements x that are contained in at least one element of $\mathcal{C}$.

(1) The set U is called the *union of the set* $\mathcal{C}$ and is denoted:

$$U := \bigcup \mathcal{C}.$$

(2) If the set $\mathcal{C}$ consists of only two sets A and B, then we write $U := A \cup B$.



$A \cup B$

Axiom 5 (Axiom Schema of Specification)**.** *Let* A *be a set, and let* φ *be a predicate containing the free variable* x*. There exists a subset* B *of the set* A *whose members are precisely the elements* x *of the set* A *such that the sentence* $\varphi(x)$ *is true. In other words:*

$$(\forall A)(\forall \varphi)(\exists B) : (B = \{x \in A \mid \varphi(x)\})$$

Proposition 1.1.1. *Let* $\mathcal{C}$ *be a nonempty set. There exists a set* D *consisting of the elements* x *that are contained in all elements of the set* $\mathcal{C}$.

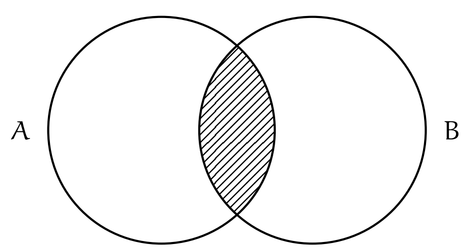
Proof. Let $E \in \mathcal{C}$. By the Axiom Schema of Specification we have $D := \{c \in E \mid (\forall X)(X \in \mathcal{C} \Rightarrow c \in X)\}$. □

Definition 1.1.4. Let $\mathcal{C}$ be a nonempty set, and let D be the set consisting of the elements x that are contained in all elements of the set $\mathcal{C}$.

(1) The set D is called the *intersection of the set* $\mathcal{C}$ and is denoted:

$$D := \bigcap \mathcal{C}.$$

(2) If the set $\mathcal{C}$ contains only two sets A and B, then we write $D := A \cap B$.



$A \cap B$

Axiom 6. *Let* X *be a set. Then the set of all subsets of* X *exists.*

Definition 1.1.5. Let X be a set. Then the set of all subsets of the set X is called the *power set of the set* X and is denoted $\mathcal{P}(X)$.

Axiom 7 (Axiom of Infinity)**.** *There is a set that contains all natural numbers.*

Definition 1.1.6. An *index set* is a nonempty set whose elements label the elements of another set. We denote the collection of objects labeled by elements of I as $\{A_\alpha\}_{\alpha \in I}$.

With this definition and the Axiom of Infinity, we can extend the concept of unions and intersections to families of sets of arbitrary size.***

Definition 1.1.7. Let I be an index set, let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of sets, and write $\mathcal{X} := \{X_\alpha \mid \alpha \in I\}$. Define:

$$\bigcup_{\alpha \in I} X_i := \bigcup \mathcal{X},$$
$$\bigcap_{\alpha \in I} X_i := \bigcap \mathcal{X}.$$

Definition 1.1.8. *** Let $\{X_n\}_{n \in \mathbb{N}}$ be a family of sets indexed by the natural numbers.

(1) The *limit superior* of $\{X_n\}_{n \in \mathbb{N}}$ is:

$$\begin{aligned}\limsup X_n &:= \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} X_n \right) \\ &= \{x \mid x \in E_n \text{ for infinitely many } n\}.\end{aligned}$$

(2) The *limit inferior* of $\{X_n\}_{n \in \mathbb{N}}$ is:

$$\begin{aligned}\limsup X_n &:= \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} X_n \right) \\ &\{x \mid x \notin E_n \text{ for only finitely many } n\}\end{aligned}$$

Definition 1.1.9. Let A and B be two sets. Define the *difference of* A *and* B by $A \setminus B := \{x \in A \mid x \notin B\}$.

Note that the difference of two sets exists by the Axiom Schema of Specification.

Definition 1.1.10. Let a and b be two objects. The *ordered pair* (a, b) is defined by $(a, b) := \{\{a\}, \{a, b\}\}$.

Proposition 1.1.2. *Let* A *and* B *be sets. There exists a set* E *containing all ordered pairs* (a, b)*, where* $a \in A$ *and* $b \in B$.

Proof. Let (a, b) be any ordered pair such that $a \in A$ and $b \in B$. By definition $(a, b) = \{\{a\}, \{a, b\}\}$. Now $\{a\} \subset A$, so $\{a\} \subset A \cup B$, hence $\{a\} \in \mathcal{P}(A \cup B)$. Similarly, $\{a, b\} \subset A \cup B$, so $\{a, b\} \in \mathcal{P}(A \cup B)$. It follows that $\{\{a\}, \{a, b\}\} \subset \mathcal{P}(A \cup B)$, hence $\{\{a\}, \{a, b\}\} \in$

$\mathcal{P}(\mathcal{P}(A \cup B))$. We know all these sets are known to exist by the Axiom of Union and Axiom of Power Set. Then by the Axiom Schema of Specification, there exists a set:

$$E := \{t \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid (\exists a)(\exists b)(a \in A \wedge b \in B \wedge t = (a, b))\}.$$

Moreover, this set is unique by the Axiom of Extension, and is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. □

Definition 1.1.11. Let A and B be two sets, and let E be the set containing all ordered pairs (a, b) where $a \in A$ and $b \in B$. The set E is called the *Cartesian product of* A *and* B, and is denoted $E := A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Definition 1.1.12. Let X and Y be sets. A *relation* from X to Y is a subset R of $X \times Y$. We write xRy to mean $(x, y) \in R$.

(1) An *equivalence relation* on X is a relation which is reflexive, transitive, and symmetric. We write $x \sim y$ to mean xRy.

(2) A *function* (or *map*) from X to Y, denoted $f : X \to Y$, is a relation from X to Y such that, for every $x \in X$, there exists a unique $y \in Y$ such that xRy. We write $f(x) = y$ to mean xRy.

(3) A *partial order* on X is a relation which is reflexive, transitive, and antisymmetric. We write $x \leqslant y$ to mean xRy, and say that X is an *ordered set*.

Proposition 1.1.3. **** Let* X *be a set. The following are equivalent:*

(1) S *is a partition of* X*;*

(2) There exists an equivalence relation $\sim$ *on* X *such that the equivalence classes of* $\sim$ *are exactly the elements of* S.

Proof. □

Example 1.1.1. An equivalent relation on X partitions X into equivalence classes.

Example 1.1.2. Recall that, if $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *covariant functor* $T : \mathcal{C} \to \mathcal{D}$ is a map such that:

(i) If $A \in \text{obj}(\mathcal{C})$, then $T(A) \in \text{obj}(\mathcal{D})$.

(ii) If $f \in \text{Hom}_{\mathcal{C}}(A, A')$, then $T(f) \in \text{Hom}_{\mathcal{D}}(T(A), T(A'))$.

(iii) If $f \in \text{Hom}_{\mathcal{C}}(A, A')$ and $g \in \text{Hom}_{\mathcal{C}}(A', A'')$, then $T(f) \in \text{Hom}_{\mathcal{D}}(T(A), T(A'))$ and $T(g) \in \text{Hom}_{\mathcal{D}}(T(A'), T(A''))$. In particular:

$$T(gf) = T(g)T(f).$$

(iv) For every $A \in \text{obj}(\mathcal{C})$, $T(\text{id}_A) = \text{id}_{T(A)}$.

A *contravariant functor* $T : \mathcal{C} \to \mathcal{D}$ is similarly defined, but (ii) and (iii) are instead defined as:

(ii) If $f \in \text{Hom}_{\mathcal{C}}(A, A')$, then $T(f) \in \text{Hom}_{\mathcal{D}}(T(A'), T(A))$.

(iii) If $f \in \mathrm{Hom}_{\mathcal{C}}(A, A')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(A', A'')$, then $T(f) \in \mathrm{Hom}_{\mathcal{D}}(T(A'), T(A))$ and $T(g) \in \mathrm{Hom}_{\mathcal{D}}(T(A''), T(A'))$. In particular:

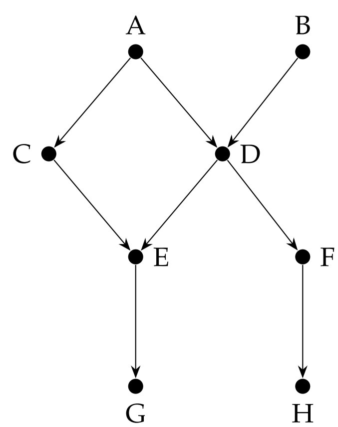
$$T(gf) = T(f)T(g).$$

Given $f : X \to Y$, the power set gives rise to two functors, the *contravariant power set functor* $\mathfrak{Set}^{\mathrm{op}} \to \mathfrak{Set}$ and the *covariant power set functor* $\mathfrak{Set} \to \mathfrak{Set}$. The first sends $f : X \to Y$ to the *preimage* function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$, whereas the second sends f to the *image* function $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$. The preimage is more well-behaved, commuting with boolean operations:

$$f^{-1}(E^c) = \left(f^{-1}(E)\right)^c,$$
$$f^{-1}\left(\bigcup_{\alpha \in I} E_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(E_\alpha),$$
$$f^{-1}\left(\bigcap_{\alpha \in I} E_\alpha\right) = \bigcap_{\alpha \in I} f^{-1}(E_\alpha).$$

This is not necessarily the case for the image function.

Example 1.1.3. ***(I don't like the prose here) Consider the directed acyclic graph:



Let $G = \{A, B, C, D, E, F, G, H\}$ be the set of vertices of our graph. We can define a partial ordering on G as follows: for any $x, y \in G$, we have $x \leqslant y$ if and only if there exists a directed path from x to y. However, note that not every element of G can be compared.

Definition 1.1.13. An ordering on a set X is said to be *linear* (or *total*) if for every $x, y \in X$, $x \leqslant y$ or $y \leqslant x$.

Definition 1.1.14. Let X be an ordered set. Let $A \subset X$.

(1) A is called *bounded above* if there exists an element $u \in X$ with $a \leqslant u$ for all $a \in A$. Such a u is called an *upper bound* for A. The set of upper bounds of A is denoted $\mathcal{U}_A = \{u \in X \mid u \text{ is an upper bound of } A\}$.

(2) A is called *bounded below* if there exists an element $v \in X$ with $v \leqslant a$ for all $a \in A$. Such a v is called a *lower bound* for A. The set of lower bounds of A is defined as $\mathcal{L}_A = \{v \in X \mid v \text{ is a lower bound of } A\}$.

(3) If A admits an upper bound u with $u \in A$, then u is called *the greatest element of* A.

(4) If A admits a lower bound v with $v \in A$, then v is called *the least element of* A.

(5) If l is the least element of $\mathcal{U}_A$, we write $l = \sup(A)$ and call it the *supremum of* A.

(6) If g is the greatest element of $\mathcal{L}_A$, we write $g = \inf(A)$ and call it the *infimum of* A.

(7) A *maximal element of* A is an element $m \in A$ such that if $a \geqslant m$, then $a = m$ (not necessarily unique).

(8) A *minimal element of* A is an element $n \in A$ such that if $a \leqslant n$, then $a = n$ (not necessarily unique).

Definition 1.1.15. If a linear order on a set X satisfies the property that every nonempty subset has a minimal element, then we say it is a *well-ordering*.

——————//——————

***include stuff HERE

Definition 1.1.16.

do general cartesian product
question is it nonempty?
axiom of choice
answer: no, axiom of choice says this choice function exists.
'shoes vs. socks' formulation of the axiom

——————//——————

§ 1.2. The Extended Real Numbers***

Definition 1.2.1. The *extended real number line* is the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, \infty\} = [-\infty, \infty]$.

The extended real number line has many unique properties.

- For all $x \in \mathbb{R}$, we have $-\infty < x < \infty$. This allows us to extend the ordering of $\mathbb{R}$ to $\overline{\mathbb{R}}$.
- Recall that since $\mathbb{R}$ is complete, every *bounded subset* admits a supremum. However, for the extended real number line, *every subset* of $\overline{\mathbb{R}}$ admits a supremum.

  Example 1.2.1.

  (i) In $\overline{\mathbb{R}}$, we have $\sup \mathbb{R} = \infty$.
  (ii) $\inf \mathbb{Q} \cap (-\infty, 0) = -\infty$.

(iii) $\inf \emptyset = \infty$. This follows from basic intuition: the infimum of a set is the greatest lower bound, so the infimum of the empty set is an element which is a lower bound for nothing. Hence it must be ∞.

- In $\mathbb{R}$, any sequence which is monotone and bounded converges. However, in $\overline{\mathbb{R}}$, *any* monotone sequence converges to a value in $\overline{\mathbb{R}}$.

- For any sequence $(x_n)_n$, we define:

$$\limsup x_n := \inf_{k \geqslant 1} \left(\sup_{n \geqslant k} x_n \right),$$
$$\liminf x_n := \sup_{k \geqslant 1} \left(\inf_{n \geqslant k} x_n \right).$$

  In $\mathbb{R}$, it could be the case that each is properly divergent, however in $\overline{\mathbb{R}}$ the limit superior and limit inferior *always* exist.

- We can define arithmetic with $-\infty$ and ∞. For $x \in \overline{\mathbb{R}}$:

$$\begin{aligned} x + \infty &= \infty, \\ x - \infty &= -\infty, \\ \infty + \infty &= \infty, \\ -\infty - \infty &= -\infty. \end{aligned}$$

  We don't give any meaning to $\infty - \infty$. If $x > 0$, we define:

$$\begin{aligned} x \cdot \infty &= \infty, \\ x \cdot (-\infty) &= -\infty. \end{aligned}$$

  If $x < 0$, we define:

$$\begin{aligned} x \cdot \infty &= -\infty, \\ x \cdot (-\infty) &= \infty. \end{aligned}$$

  Unless stated otherwise, we define $0 \cdot \pm\infty = 0$.

- Let X be any set and let $f : X \longrightarrow [0, \infty]$. We'd like to give some meaning to $\sum_{x \in X} f(x)$. Regardless of the cardinality of X, it is a theorem that

This whole section needs to be redone.

Proposition 1.2.1. ****(could be cleaned up) Every open set in $\mathbb{R}$ is at most a countable union of disjoint open intervals.*

Proof. Let $U \subset \mathbb{R}$ be open. Define a binary relation $\sim$ on U as follows: $x \sim y$ if and only if there exists an open interval I such that $\{x, y\} \subset I \subset U$. One can check that this is indeed an equivalence relation, hence U is partitioned by the equivalence classes of $\sim$.

We are now going to show that each equivalence class is an open interval. Let E be an equivalence class and let $I = (\inf E, \sup E)$. Clearly $E \subset I$. Let $x \in I$. Then $\inf E < x$ and

$x < \sup E$. We can find $u, v \in E$ such that $\inf E < u < x < v < \sup E$. Since $u, v \in E$, we can find an interval J such that $\{u, v\} \subset J \subset U$. But this means $x \in J$, hence $x \sim u$ and $x \sim v$. Thus $x \in E$, giving $I \subset E$ meaning that E is an open interval.

It remains to show that there are at most a countable number of equivalence classes. Note that each equivalence class contains a rational number, and no two classes can contain the same rational number (otherwise the equivalence classes wouldn't be disjoint). By the Axiom of Choice, we can find a choice function $f : U/{\sim} \longrightarrow \mathbb{Q}$ which is injective. Therefor there are at most countably many open intervals. □

§ 1.3. Metric Spaces

Definition 1.3.1. A *distance function* (or *metric*) on a nonempty set X is a map:

$$\rho : X \times X \longrightarrow [0, \infty)$$

satisfying for all $x, y, z \in X$:

(1) $\rho(x, x) = 0$;

(2) $\rho(x, y) = \rho(y, x)$;

(3) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

The pair (X, ρ) is called a *metric space*.

Distance functions let us talk about sets which centered at a point.

Definition 1.3.2. Let X be a metric space, $x \in X$ and $r > 0$.

(1) The *open ball of radius* r *centered at* x is the set $B(r, x) := \{y \in X \mid \rho(x, y) < r\}$.

(2) The *closed ball of radius* r *centered* x is the set $V(r, x) := \{y \in X \mid \rho(x, y) \leqslant r\}$.

(3) The *sphere of radius* r *centered at* x is the set $S(r, x) := \{y \in X \mid \rho(x, y) = r\}$.

Such distance functions induce a topology on our metric space X.

Definition 1.3.3. Let X be a metric space.

(1) We say $U \subset X$ is *open* if, for all $x \in U$, there exists $r > 0$ such that $B(r, x) \subset U$.

(2) We say $V \subset X$ is *closed* if V^c is open.

Given a set, we characterize its inside, the smallest closed set containing it, as well as its boundary.

Definition 1.3.4. Let X be a metric space and $E \subset X$.

(1) The *interior of* E is the set:

$$E^o := \bigcup \{U \subset E \mid \ U \text{ is open}\}.$$

(2) The *closure of* E is the set:

$$\overline{E} := \bigcap \{C \supset E \mid \ C \text{ is closed}\}.$$

(3) The *boundary of* E is the set:

$$\partial E = \overline{E} \setminus E^{o}.$$

Proposition 1.3.1. *A countable subset of* $\mathbb{R}$ *has an empty interior.*

Proof. Let A be a subset of $\mathbb{R}$ which has a nonempty interior. Then A contains a non-trivial interval. Hence $\text{card}(A) \geqslant \mathfrak{c} > \aleph_0$. Thus A is uncountable. □

Example 1.3.1. Let $E = \mathbb{Q} \cap (0, 1)$. We have $E^{o} = \emptyset$ and $\overline{E} = [0, 1]$.

Definition 1.3.5. Let X be a metric space and $E \subset X$.

(1) E is *dense in* X if $\overline{E} = X$.

(2) E is *nowhere dense* if $(\overline{E})^{o} = \emptyset$.

Intuitively, a set is nowhere dense if we can't fit an open ball inside it.

Definition 1.3.6. Let (X, d) be a metric space.

(1) The collection of open sets of X is denoted τ_X.

(2) A *base* for τ_X is a family of open subsets $\mathcal{B} \subseteq \tau_X$ such that:

$$(\forall U \in \tau_X)(\forall x \in U)(\exists B \in \mathcal{B}) : x \in B \subseteq U.$$

Equivalently, for all $U \in \tau_X$, we can write $U = \bigcup_{i \in I} B_i$, where $\{B_i\}_{i \in I} \subseteq \tau_X$.

(3) X is *second countable* if it has a countable base.

Definition 1.3.7. Let (X, ρ) be a metric space. X is *separable* if there exists a countable dense subset.

Proposition 1.3.2. *** *Let* (X, ρ) *be a metric space.* X *is second countable if and only if* X *is separable.*

Proof. □

With a metric we can talk about the distance from a point to a set, as well as a way to define if a set is bounded.

Definition 1.3.8. Let (X, ρ) be a metric space, $x \in X$, and $E \subset X$.

(1) The distance from x to E is $\rho(x, E) := \inf_{y \in E} \rho(x, y)$.

(2) The *diameter* of E is $\text{diam}(E) = \sup_{x,y \in E} \rho(x, y)$.

(3) A subset of a metric space is *bounded* if its diameter is finite.

With a notion of distance, we can talk about sequence convergence.

Definition 1.3.9. Let (X, ρ) be a metric space and $(x_n)_n$ a sequence in X. We say $(x_n)_n$ *converges to a point* x_0 if $\lim_{n \to \infty} \rho(x_n, x) = 0$.

Proposition 1.3.3. *Let* (X, ρ) *be a metric space,* $E \subset X$, *and* $x \in X$. *The following are equivalent:*

(1) $x \in \overline{E}$;

(2) For every $r > 0$, $B(r, x) \cap E \neq \emptyset$;

(3) There is a sequence $(x_n)_n$ *in* E *which converges to* x

We can even talk about continuity of functions between metric spaces.

Definition 1.3.10. Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces and $f : X_1 \to X_2$.

(1) We say f is *continuous* if:

$$(\forall c \in X)(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x \in X_1)(\rho_1(x, y) < \delta \implies \rho_2(f(x), f(y)) < \epsilon).$$

(2) We say f is *uniformly continuous* if:

$$(\forall \epsilon > 0)(\exists \delta > 0) : (\forall x, y \in X_1)(\rho_1(x, y) < \delta \implies \rho_2(f(x), f(y)) < \epsilon).$$

(3) We say f is *Lipschitz* if:

$$(\exists C \geqslant 0) : (\forall x, y \in X)(\rho_2(f(x), f(y)) \leqslant C \cdot \rho_1(x, y)).$$

Note that δ may depend on our point c in the case that f is continuous. A function is uniformly continuous if δ does not depend on any two points in its domain.

Proposition 1.3.4. *Let* (X_1, ρ_1) *and* (X_2, ρ_2) *be metric spaces.* f *is continuous if and only if the preimage of every open set in* X_2 *is open in* X_1.

Similarly to how every Cauchy sequence in $\mathbb{R}$ converges, we can talk about metric spaces which guarantee that same property.

Definition 1.3.11.

(1) Let $(x_n)_n$ be a sequence in a metric space (X, ρ). We say $(x_n)_n$ is ρ-*Cauchy* if:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N}) : (\forall n, m \in \mathbb{N})(n > m \geqslant N \implies \rho(x_n, x_m) < \epsilon).$$

(2) A metric space X is called *complete* if every Cauchy sequence converges to a point in X.

We can generalize the notion of a set being closed and bounded in $\mathbb{R}^n$.

Proposition 1.3.5. *Let* X *be a metric space. The following are equivalent:*

(1) Every open cover of X *admits a finite subcover;*

(2) Every sequence in X *admits a convergent subsequence;*

(3) X *is complete and totally bounded.*

Definition 1.3.12. We say a metric space X is *compact* if it satisfies any of the properties in Proposition 1.3.5.

A set may can have more than one metric defined on it.

Definition 1.3.13. Let X be a metric space. We say ρ_1 and ρ_2 are *equivalent metrics on* X if $\mathrm{id} : (X, \rho_1) \to (X, \rho_2)$ and $\mathrm{id}^{-1} : (X, \rho_2) \to (X, \rho_1)$ are Lipschitz.

Chapter 2

Measures

Ideally in $\mathbb{R}$, we would like to a have a function μ that assigns each $E \subset \mathbb{R}$ to a number $\mu(E) \in [0, \infty]$. Such a function μ should possess the "essence" of length:

1. The unit cube should have a measure equal to one —that is, $\mu([0, 1)) = 1$;
2. If E and F are congruent subsets of $\mathbb{R}$ (that is, F is a translation, rotation, and/or reflection of E), then $\mu(E) = \mu(F)$;
3. If $E_1, E_2, ...$ are pairwise disjoint, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$.

Unfortunately, a function possessing all of these traits does not exist.

Theorem 2.0.1. *A function* $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ *satisfying (1), (2), and (3) from above does not exist.*

Proof. Suppose such a function μ does exist. By (1) we have $\mu([0, 1)) = 1$. Define a relation on $[0, 1)$ as follows: $x \sim y$ if and only if $x - y \in \mathbb{Q}$. This is an equivalence relation. Using the Axiom of Choice, let N be a set which contains exactly one element from each equivalence class. Now for each $q \in \mathbb{Q} \cap [0, 1)$, define:

$$N_q = \{x + q \mid N \cap [0, 1 - q)\} \cup \{x + (q - 1) \mid x \in N \cap [1 - q, 1)\}.$$

That is, the points in N_q are obtained by "rotating" the points in N by a rational number q. Clearly $N_q \subset [0, 1)$, hence $\bigcup_{q \in \mathbb{Q} \cap [0,1)} N_q \subset [0, 1)$. We have the following three properties:

(a) For any $q \in \mathbb{Q}$, $\mu(N) = \mu(N_q)$

(b) If $r, s \in \mathbb{Q}$, $r \neq s$, then $N_r \cap N_s = \emptyset$.

(c) For every $y \in [0, 1)$, there exists a $q \in \mathbb{Q}$ such that $y \in N_q$.

Property (a) follows from the fact that N and N_r are congruent, hence:

$$\begin{aligned}
\mu(N) &= \mu\big(N \cap [0, 1 - q)\big) + \mu\big(N \cap [1 - q, 1)\big) \\
&= \mu\big(\{x + q \mid x \in N \cap [0, 1 - q)\}\big) + \mu\big(\{x + q - 1 \mid x \in N \cap [1 - q, 1)\}\big) \quad \text{"Rotation"} \\
&= \mu(N_r).
\end{aligned}$$

Property (b) follows by contradiction —if $x \in N_r \cap N_s$, then $x - r$ and $x - s$ would be distinct elements of N. However, $(x - r) - (x - s) = s - r \in \mathbb{Q}$, so $x - r$ and $x - s$ are in the same equivalence class. But this contradicts the way which we've constructed N. For property (c), given $y \in [0, 1)$, let $x \in N$ satisfy $x \sim y$. Since x and y are in the same equivalence class, we can find $q \in \mathbb{Q}$ such that:

$$y = \begin{cases} x + q, & x < y \\ x - (q - 1) & x > y \end{cases}.$$

Hence $y \in N_q$, giving $[0,1) \subset N_q \subset \bigcup_{q \in \mathbb{Q} \cap [0,1)} N_q$. We've obtained the equality:

$$\bigcup_{q \in \mathbb{Q} \cap [0,1)} N_q = [0,1),$$

and by property (a) this becomes:

$$\bigsqcup_{q \in \mathbb{Q} \cap [0,1)} N_q = [0,1).$$

Finally, with property (a), we have:

$$\begin{aligned} 1 &= \mu([0,1)) \\ &= \mu\left(\bigsqcup_{q \in \mathbb{Q} \cap [0,1)} N_q \right) \\ &= \sum_{q \in \mathbb{Q} \cap [0,1)} \mu(N_q) \\ &= \sum_{n=1}^{\infty} \mu(N). \end{aligned}$$

This is impossible, hence there does not exist a μ satisfying (1), (2), and (3). □

One might guess that this inconsistency is a result of assuming countable additivity as opposed to finite additivity. This is not the case, and in dimensions $n \geqslant 3$ assuming a weak form of μ leads to the following result:

Theorem 2.0.2 (Banach-Tarski)**.** *Let* U *and* V *be arbitrary bounded open sets in* $\mathbb{R}^n$ *for* $n \geqslant 3$. *There exist* $k \in \mathbb{N}$ *and subsets* $E_1, ..., E_k$, $F_1, ..., F_k$ *of* $\mathbb{R}^n$ *such that:*

- *the* E_j*'s are disjoint and their union is* U*;*
- *the* F_j*'s are disjoint and their union is* V*;*
- E_j *is congruent to* F_j *for all* $1 \leqslant j \leqslant k$.

This result means one can cut up a ball the size of a pea into a finite number of pieces, and rearrange them to form a ball the size of the earth! Ultimately, the problem comes from the fact we defined μ on *every possible* subset of $\mathbb{R}^n$. We can't hope to do this, since as we showed there exists bizarre sets which cannot be measured. Instead, we will define μ on a special class of subsets of $\mathbb{R}^n$.

§ 2.1. σ-Algebras

Definition 2.1.1. Let X be a set. An *algebra of sets* $\mathcal{A} \subset \mathcal{P}(X)$ is a collection of subsets of X which is:

(1) closed under finite union: if $E_1, E_2, ..., E_n \in \mathcal{A}$, then $\bigcup_{i=1}^n E_n \in \mathcal{A}$;

(2) closed under complement: if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

We used the ordered pair $(X, \mathcal{A})$...If $\mathcal{A}$ is closed under countable union, then we say $\mathcal{A}$ is a *σ-algebra*.

Proposition 2.1.1. *Let $\mathcal{A}$ be an algebra of sets.*

(1) $\mathcal{A}$ is closed under finite intersection.

(2) $\mathcal{A}$ is closed under set difference.

(3) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.

Proof. (1) Let $E_1, E_2, ..., E_n \in \mathcal{A}$. Then $\bigcap_{i=1}^n E_i = \left(\bigcup_{i=1}^n E_i^c\right)^c \in \mathcal{A}$. (2) If $E_1, E_2 \in \mathcal{A}$, then $E_1 \setminus E_2 = E_1 \cap E_2^c \in \mathcal{A}$. (3) For any $E \in \mathcal{A}$, we have $\emptyset = E \cap E^c \in \mathcal{A}$ and $X = E \cup E^c \in \mathcal{A}$. □

Proposition 2.1.2. *Let $\mathcal{A}$ be an algebra of sets. The following are equivalent:*

(1) $\mathcal{A}$ is a σ-algebra.

(2) $\mathcal{A}$ is closed under countable disjoint union.

Proof. If $\mathcal{A}$ is a σ-algebra, then it is closed under countable union, hence it must be closed under countable disjoint union. Conversely, suppose $\mathcal{A}$ is closed under countable disjoint union. Let $\{E_n\}_{n=1}^\infty$ be a family of sets in $\mathcal{A}$. Define:

$$\begin{aligned} F_1 &= E_1, \\ F_2 &= E_2 \setminus E_1 \\ F_3 &= E_3 \setminus (E_1 \cup E_2) \\ &\vdots \\ F_n &= E_n \setminus (E_1 \cup E_2 \cup ... \cup E_{n-1}). \end{aligned}$$

Let $i < j$ be natural numbers and consider $F_i \cap F_j$. If $x \in F_i$, then $x \in E_i$. Since $i < j$, we have $x \in E_1 \cup ... \cup E_{j-1}$, hence $x \notin F_j$. Thus $F_i \cap F_j = \emptyset$. Inductively, $\{F_n\}_{n=1}^\infty$ is a countable family of disjoint sets.

Clearly $\bigcup_{n=1}^\infty F_n \subset \bigcup_{n=1}^\infty E_n$. Conversely, let $x \in \bigcup_{n=1}^\infty E_n$. Find the smallest natural number j such that $x \in E_j$. Then $x \notin E_i$ for $i < j$. Hence $x \in E_j \setminus (E_1 \cup ... \cup E_{j-1}) = F_j \subset \bigcup_{n=1}^\infty F_n$.

Since $\mathcal{A}$ is closed under countable disjoint unions, we have $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n \in \mathcal{A}$. Since $\{E_n\}_{n=1}^\infty$ was an arbitrary family of sets, this means $\mathcal{A}$ is a σ-algebra. □

Proposition 2.1.3. *If $\{\mathcal{E}_i\}_{i \in I}$ is a family of σ-algebras, then $\bigcap_{i \in I} \mathcal{E}_i$ is a σ-algebra.*

Proof. Let $\{E_n\}_{n=1}^\infty$ be a family of sets contained in $\bigcap_{i \in I} \mathcal{E}_i$. Then each $\{E_n\}_{n=1}^\infty$ is contained in $\mathcal{E}_i$. Since $\mathcal{E}_i$ is a σ-algebra for every $i \in I$, we have $\bigcup_{n=1}^\infty E_n \in \mathcal{E}_i$ for every $i \in I$, which means $\bigcup_{n=1}^\infty E_n \in \bigcap_{i \in I} \mathcal{E}_i$.

Similarly, let $F \in \bigcap_{i \in I} \mathcal{E}_i$. Then $F \in \mathcal{E}_i$ for every $i \in I$. Since $\mathcal{E}_i$ is a σ-algebra for every $i \in I$, we have $F^c \in \mathcal{E}_i$ for every $i \in I$, which means $F^c \in \bigcap_{i \in I} \mathcal{E}_i$. Thus $\bigcap_{i \in I} \mathcal{E}_i$ is a σ-algebra. $\square$

Definition 2.1.2. Let $\mathcal{E}$ be any nonzero collection of subsets of X. The *σ-algebra generated by* $\mathcal{E}$ is $\mathfrak{M}(\mathcal{E}) := \bigcap\{\mathcal{A} \mid \mathcal{E} \subset \mathcal{A}, \mathcal{A} \text{ is a σ-algebra}\}$.

For a metric space X, it'd be nice if we could measure all the open sets.

Definition 2.1.3. Let X be a metric space and τ_X the set containing all of its open sets. The *Borel σ-algebra on* X is the set $\mathcal{B}_X := \mathfrak{M}(\tau_X)$. Members of $\mathcal{B}_X$ are called *Borel sets*.

Exposition on appreciating whats inside the borel sets. define was F sigma and G delta sets are. talk about why they are in it and also combinations of them

Proposition 2.1.4. *$\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:*

(1) *the open intervals:* $\mathcal{E}_1 = \{(a, b) \mid a < b\}$,

(2) *the closed intervals:* $\mathcal{E}_2 = \{[a, b] \mid a < b\}$,

(3) *the half-open intervals:* $\mathcal{E}_3 = \{(a, b] \mid a < b\}$ *or* $\mathcal{E}_4 = \{[a, b) \mid a < b\}$,

(4) *the open rays:* $\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}$ *or* $\mathcal{E}_6 = \{(-\infty, a) \mid a \in \mathbb{R}\}$,

(5) *the closed rays:* $\mathcal{E}_7 = \{(a, \infty) \mid a \in \mathbb{R}\}$ *or* $\mathcal{E}_8 = \{(-\infty, a) \mid a \in \mathbb{R}\}$.

Proof. We are required to show for each $\mathcal{E}_i$ that $\mathfrak{M}(\mathcal{E}_i) = \mathcal{B}_{\mathbb{R}}$.

(1) Let $I \in \mathcal{E}_1$ be an open interval. Then $I \in \mathcal{B}_{\mathbb{R}}$ because I is an open subset of $\mathbb{R}$. From this we have $\mathcal{E}_1 \subset \mathcal{B}_{\mathbb{R}}$, but because $\mathcal{B}_{\mathbb{R}}$ is a σ-algebra containing $\mathcal{E}_1$, and $\mathfrak{M}(\mathcal{E}_1)$ is the smallest σ-algebra containing $\mathcal{E}_1$, we have $\mathfrak{M}(\mathcal{E}_1) \subset \mathcal{B}_{\mathbb{R}}$. Conversely, let $U \in \tau_{\mathbb{R}}$. Then U is at most the countable union of disjoint open intervals, hence $U \in \mathfrak{M}(\mathcal{E}_1)$. From this we have $\tau_{\mathbb{R}} \subset \mathfrak{M}(\mathcal{E}_1)$, but because $\mathfrak{M}(\mathcal{E}_1)$ is a σ-algebra containing $\tau_{\mathbb{R}}$, and $\mathfrak{M}(\tau_{\mathbb{R}})$ is the smallest σ-algebra containing $\tau_{\mathbb{R}}$, we have $\mathfrak{M}(\tau_{\mathbb{R}}) = \mathcal{B}_{\mathbb{R}} \subset \mathfrak{M}(\mathcal{E}_1)$.

(2) Let $I \in \mathcal{E}_1$ be an open interval. Then I^c is a closed interval, hence it is contained in $\mathfrak{M}(\mathcal{E}_2)$. Whence $\mathfrak{M}(\mathcal{E}_1) \subset \mathfrak{M}(\mathcal{E}_2)$. Conversely, let $C \in \mathcal{E}_2$ be a closed interval. Then C can be written as the countable intersection of open sets. This means $C \in \mathfrak{M}(\mathcal{E}_1)$. Whence $\mathfrak{M}(\mathcal{E}_2) \subset \mathfrak{M}(\mathcal{E}_1)$. Thus $\mathfrak{M}(\mathcal{E}_2) = \mathfrak{M}(\mathcal{E}_1) = \mathcal{B}_{\mathbb{R}}$. $\square$

Given two sets X_1 and X_2 with respective σ-algebras $\mathcal{M}_1$ and $\mathcal{M}_2$, we'd like to understand what a σ-algebra on $X_1 \times X_2$ looks like. For countably many pairs, we have the following intuitive definition.

Definition 2.1.4. Let $\{X_n\}_{n\in\mathbb{N}}$ be a family of sets. If $\mathcal{M}_n$ is a σ-algebra on X_n for each n, we define $\bigotimes_{n\in\mathbb{N}} \mathcal{M}_n$ as the σ-algebra generated by $\bigotimes_{n\in\mathbb{N}} \mathcal{M}_n := \{\prod_{n\in\mathbb{N}} X_n \mid X_n \in \mathcal{M}_n\}$.

Note that this definition assumes each X_i is indexed by a *countable* family of sets. For an uncountable family of sets $\{X_\alpha\}_{\alpha\in I}$