Hydrodynamic Diffusion in Integrable Systems

Jacopo De Nardis,[1] Denis Bernard,[2] and Benjamin Doyon[3]

[1]*Département de Physique, Ecole Normale Supérieure, PSL Research University, CNRS, 24 rue Lhomond, 75005 Paris, France*

[2]*Laboratoire de Physique Théorique de l'Ecole Normale Supérieure de Paris, CNRS, ENS, PSL University & Sorbonne Université, 75005 Paris, France*

[3]*Department of Mathematics, King's College London, Strand WC2R 2LS London, United Kingdom*

(Received 30 July 2018; revised manuscript received 21 September 2018; published 17 October 2018)

We show that hydrodynamic diffusion is generically present in many-body, one-dimensional interacting quantum and classical integrable models. We extend the recently developed generalized hydrodynamic (GHD) to include terms of Navier-Stokes type, which leads to positive entropy production and diffusive relaxation mechanisms. These terms provide the subleading diffusive corrections to Euler-scale GHD for the large-scale nonequilibrium dynamics of integrable systems, and arise due to two-body scatterings among quasiparticles. We give exact expressions for the diffusion coefficients. Our results apply to a large class of integrable models, including quantum and classical, Galilean and relativistic field theories, chains, and gases in one dimension, such as the Lieb-Liniger model describing cold atom gases and the Heisenberg quantum spin chain. We provide numerical evaluations in the Heisenberg *XXZ* spin chain, both for the spin diffusion constant, and for the diffusive effects during the melting of a small domain wall of spins, finding excellent agreement with time-dependent density matrix renormalization group numerical simulations.

DOI: 10.1103/PhysRevLett.121.160603

Introduction.—The study of quantum systems far from equilibrium has received a large amount of attention in recent years. In the context of cold atom gases, unprecedented experimental control means that it is now possible to observe the behavior of many-body quantum systems in fully inhomogeneous and isolated setups [1–6]. The foremost theory for describing inhomogeneous many-body systems is hydrodynamics. Despite its long history, it has now seen a resurgence of interest, as it emerges in new contexts and finds new applications especially in the field of strongly correlated systems [7–12]. Of particular interest in the present Letter is the recent development of the hydrodynamic theory appropriate to integrable systems, dubbed generalized hydrodynamics (GHD) [13–18]. This theory applies to a large family of models, including the paradigmatic Heisenberg quantum chain for one-dimensional magnetism (see Refs. [15,17,19,20]), the Bose gas with delta-function interaction [the Lieb-Liniger (LL) model] [21], which describes gases of ultracold atoms confined to one-dimensional traps (see Refs. [22,23]), as well as classical integrable gases [24–34]. GHD has given rise to a raft of new exact, sometimes unexpected, results in the past few years: it provides exact descriptions of steady states fully out of equilibrium [13,15], it efficiently describes the famous quantum Newton cradle setup [5,35] where lack of thermalization is explicitly observed [22,36], it characterizes the transport of quantum entropy through a chain [37,38] and it gives exact expressions for Drude weights [17,39–41] characterizing ballistic transport in integrable models [42,43].

The dynamical observables in hydrodynamic theories are the conserved densities afforded by the microscopic dynamics. Let $Q_i = \int_{\mathbb{R}} dx \mathfrak{q}_i(x)$ be conserved charges, which commute with the Hamiltonian and with each other, and $\mathfrak{q}_i(x,t)$ and $\mathfrak{j}_i(x,t)$ their charge and current density operators,

$$\partial_t \mathfrak{q}_i(x,t) + \partial_x \mathfrak{j}_i(x,t) = 0. \tag{1}$$

Hydrodynamics assumes weak time and space modulations of expectation values of charge and current densities, and is formulated as a theory for the space-time evolution of their averages over the system's density matrix,

$$\partial_t \langle \mathfrak{q}_i(x,t) \rangle + \partial_x \langle \mathfrak{j}_i(x,t) \rangle = 0. \tag{2}$$

The averages $\langle \mathfrak{q}_i(x,t) \rangle$ are seen as independent local-state variables, and the currents $\langle \mathfrak{j}_i(x,t) \rangle$ as functions of these state variables and their space derivatives,

$$\langle \mathfrak{j}_i(x,t) \rangle = \mathcal{F}_i(x,t) + \sum_j \mathcal{F}_{ij}(x,t) \partial_x \langle \mathfrak{q}_j(x,t) \rangle + \cdots. \tag{3}$$

The quantities $\mathcal{F}_i(x,t), \mathcal{F}_{ij}(x,t), \ldots$ are functions of all averages $\langle \mathfrak{q}_k(x,t) \rangle$ at the point x, t, whence their space-time dependence, and encode the model properties. The terms $\mathcal{F}_i(x,t)$ represent the *Euler scale* hydrodynamic theory [24]. The assumption is that all local fluid cells are, to leading approximation, at thermal equilibrium. This means that, locally, the reduced density matrix is stationary and it is then given by a (generalized) Gibbs ensemble (GGE), proportional to (see the reviews [44–46]) the following:

$$e^{-\sum_i \beta_i(x,t) Q_i}. \tag{4}$$

The Lagrange parameters $\beta_i(x,t)$ are fixed by the average values of the densities, $\langle \mathfrak{q}_i(x,t)\rangle$, and in turn determine the currents $\langle \mathfrak{j}_i(x,t)\rangle$: these are the equations of state. Euler hydrodynamics is time reversible and describes ballistic transport. It applies when variations in space and time occur at very large scales only. However, Euler hydrodynamics often develop instabilities such as large gradients and fails to describe the loss of large-scale structures over time [24,47], see [19,48,49] for recent observations in integrable systems.

Beyond the Euler scale, the terms involving spatial derivatives cannot be determined by the homogeneous, stationary thermodynamics of the gas. They are referred to as *constitutive relations* of the hydrodynamic theory, and must be fixed in an alternative fashion. Often, one takes into account various symmetries available and fixes them in a phenomenological fashion. Of particular importance are the terms of Navier-Stokes type, $\sum_j \mathcal{F}_{ij}(x,t)\partial_x\langle \mathfrak{q}_j(x,t)\rangle$: they give rise to diffusive effects, the irreversible processes by which large-scale structures are passed to mesoscopic-scale fluid cells and increase their entropy.

Euler hydrodynamics is very relevant to integrable models, as the infinite number of conserved quantities guarantees a large amount of ballistic transport. In integrable models, the homogeneous steady states [Eq. (4)] involve infinitely many Lagrange parameters. The equations of states are known exactly [13,15] by using the methods of the thermodynamic Bethe ansatz (TBA) [50–52] and its refinements [53–55]. GHD, as it is currently developed, is then a Euler hydrodynamics based on these equations of state. Its power lies in part in the fact that the underlying TBA description is extremely universal, taking as input only few properties that are readily available for almost all integrable models. It is however a crucial question to understand the diffusive, Navier-Stokes corrections to Euler hydrodynamics (other types of corrections due to the lattice in free fermionic theories where studied in Ref. [56]). In certain cases, spin and charge transport in quantum chains has been observed to show diffusive and other non-Eulerian behaviors [33,49,57–64]. Diffusion also occurs in gases of hard rods [24–27] due to fluctuating accumulations of displacements, whence similar effects might be expected in soliton gases [28–34]. Moreover it has been argued that what distinguishes interacting integrable models from free models should be diffusion [65]. Is there then diffusion in integrable models more generally? If so, how universal is it, what form does it take? How does it modify the ballistic transport?

In this Letter, we show that *there is generically diffusive transport in interacting integrable models* by providing an exact expression for the diffusion matrix. We use this result to write an *exact and universal expression* for the Navier-Stokes term to the Euler-GHD hydrodynamic theory. We derive these expressions in the Lieb-Liniger (LL) model by performing a microscopic calculation of the Kubo formula [Eq. (5)]. Our results agree with the known diffusion matrix in the hard rod gases [24,26,27], and we provide numerical checks for its validity in the anisotropic Heisenberg *XXZ* spin chain. Therefore we conjecture that our results are universal for every integrable models. We show that diffusive terms are responsible for positive entropy production, and we evaluate the diffusive, large-time corrections to the nonequilibrium currents in the partitioning protocol [60,66–82].

Diffusion in hydrodynamics.—We first recall how diffusion is accounted for in linear fluctuating hydrodynamics [24,83–86]. The general setting applies equally to conventional and generalized hydrodynamics. Its main objects are linear response and correlation functions of local charges and currents in homogeneous steady state. The static covariance matrix C of conserved charges is $C_{ij} = \int_{\mathbb{R}} dx \langle \mathfrak{q}_i(x,0)\mathfrak{q}_j(0,0)\rangle^{\mathrm{c}}$, where $\langle \mathfrak{q}_i(x,0)\mathfrak{q}_j(0,0)\rangle^{\mathrm{c}} = \langle \mathfrak{q}_i(x,0)\mathfrak{q}_j(0,0)\rangle - \langle \mathfrak{q}_i\rangle\langle \mathfrak{q}_j\rangle$. An important quantity in nonequilibrium physics, part of Euler hydrodynamics, is the Drude weight, or Drude matrix $D_{ij} = \lim_{t\to\infty}\int_{\mathbb{R}} dx \langle \mathfrak{j}_i(x,t)\mathfrak{j}_j(0,0)\rangle^{\mathrm{c}}$. If nonzero, it indicates the presence of ballistic transport. Diffusive spreading is not precluded by the presence of ballistic transport (nonzero Drude weight), and provides subleading corrections [65,87–89]. The effects of diffusion are encoded within the diffusion matrix $\mathfrak{D}_{ij}$, simply related to the Navier-Stokes terms in Eq. (3) by $\mathfrak{D}_{ij} = -2\mathcal{F}_{ij}$. By employing linear response theory, it can be shown [24,65,90] that the diffusion matrix, evaluated in a generic homogeneous steady state, can be related to the integrated current-current correlator in this state (Kubo formula),

$$(\mathfrak{D}C)_{ij} = \int_{\mathbb{R}} dt \left(\int_{\mathbb{R}} dx \langle \mathfrak{j}_i(x,t)\mathfrak{j}_j(0,0)\rangle^{\mathrm{c}} - D_{ij} \right). \tag{5}$$

Linear response theory also gives dynamical correlation functions in homogeneous steady state as $S_{ij}(k,t) = \int dx e^{-\mathrm{i}kx}\langle \mathfrak{q}_i(x,t)\mathfrak{q}_j(0,0)\rangle^{\mathrm{c}} = \exp[-\mathrm{i}kAt - \frac{1}{2}k^2\mathfrak{D}|t|]C$, valid at large t, small k, where the ballistic propagation matrix is $A_{ij} = \partial\mathcal{F}_i/\partial\langle \mathfrak{q}\rangle_j$. The term inside the exponent proportional to kt represents ballistic transport, and that involving $k^2|t|$ the diffusive broadening of order $\sqrt{t}$ around the ballistic path.

Exact diffusion in GHD.—In integrable models, stable quasiparticles exist whose scattering is elastic and factorized. As a consequence, a stationary homogeneous thermodynamic state (a GGE) in the thermodynamic limit can be fully characterized by a density of quasiparticles (microcanonical GGE [91]): the spectral density $\rho_{\mathrm{p}}(\theta)$, with spectral parameter θ encoding both momentum and the quasiparticle type. Each quasiparticle carries a quantity $h_i(\theta)$ of the charge Q_i,

$$\langle \mathfrak{q}_i \rangle = \int d\theta h_i(\theta)\rho_{\mathrm{p}}(\theta) \tag{6}$$

(the integral being implicitly accompanied by a sum over quasiparticle types in case there are many of them). For instance, in the repulsive LL model, there is a single type of quasiparticle and we choose θ to be the velocity, $h_0(\theta) = 1$ for the actual particle density, $h_1(\theta) = p(\theta) = m\theta$ for the momentum and $h_2(\theta) = E(\theta) = m\theta^2/2$ for the energy. The interaction is fully encoded within the two-body scattering kernel $T(\theta,\alpha) = (2\pi \mathrm{i})^{-1} d\log S(\theta,\alpha)/d\theta$ [92], where $S(\theta,\alpha)$ is the two-body scattering matrix between particles of spectral parameters θ and α. In the LL model, we have $T(\theta,\alpha) = c/(\pi[(\alpha-\theta)+c^2])$ where c is the coupling constant between the bosonic particles. The TBA gives exact quasiparticle spectral densities in thermal states and GGEs [51–53,93], and, in GHD, $\rho_{\mathrm{p}}(\theta;x,t)$ describes the local state [Eq. (4)], via Eq. (6).

At the Euler scale, the GHD equation is [13,15] this:

$$\partial_t \rho_{\mathrm{p}}(\theta;x,t) + \partial_x(v^{\mathrm{eff}}(\theta;x,t)\rho_{\mathrm{p}}(\theta;x,t)) = 0. \tag{7}$$

The *effective velocity* $v^{\mathrm{eff}}(\theta;x,t)$ represents the velocity of the quasiparticle of spectral parameter θ, which is the group velocity [94] renormalized by the interactions with the other particles inside the local state at x, t. It solves the linear integral equation $p'(\theta)v^{\mathrm{eff}}(\theta) = E'(\theta) + (2\pi)\int d\alpha \rho_{\mathrm{p}}(\alpha)T(\theta,\alpha)[v^{\mathrm{eff}}(\alpha) - v^{\mathrm{eff}}(\theta)]$. The root of Eq. (7) is the expression $\mathcal{F}_i = \int d\theta v^{\mathrm{eff}}(\theta)\rho_{\mathrm{p}}(\theta)h_i(\theta)$ for the Euler term of the constitutive relation, Eq. (3). In fact, at the Euler scale, every local average $\langle \mathcal{O}(x,t)\rangle$ is evaluated within the GGE described by $\rho_{\mathrm{p}}(x,t;\theta)$, and indeed, in a GGE, the currents take the simple form $\langle \mathfrak{j}_i \rangle = \int d\theta h_i(\theta)v^{\mathrm{eff}}(\theta)\rho_{\mathrm{p}}(\theta)$. Beyond the Euler scale, Eq. (6) stays true by definition but averages of other local observables have corrections that depend on the first derivative of the state variable, and in particular average currents get modified by the diffusion matrix, via Eq. (3).

The static covariance C, the Drude weight D and the ballistic propagation matrix A were evaluated exactly in Ref. [40] in the full generality of Euler GHD. Three ingredients are involved: (i) the *occupation function* $n(\theta) = \rho_{\mathrm{p}}(\theta)/\rho_{\mathrm{s}}(\theta)$, where $\rho_{\mathrm{s}}(\theta) = p'(\theta)/(2\pi) + \int d\alpha T(\theta,\alpha)\rho_{\mathrm{p}}(\alpha)$ is the state density; (ii) the *dressing* $h^{\mathrm{dr}}(\theta)$ of scalar spectral functions $h(\theta)$, a state-dependent modification of $h(\theta)$ expressed as the solution to the linear integral equation $h^{\mathrm{dr}}(\theta) = h(\theta) + \int d\alpha T(\alpha,\theta)n(\alpha)h^{\mathrm{dr}}(\alpha)$; and (iii) the *statistical factor* $f(\theta)$, equal to $1-n(\theta)$ if the quasiparticle has fermionic statistics (such as in the LL model), $1+n(\theta)$ for bosonic statistics, 1 for classical statistics, and $n(\theta)$ for radiative modes (such as in classical field theory) [95].

The main result of this Letter is a general, exact expression for the diffusion matrix in integrable models entering the Kubo formula [Eq. (5)] on a generic stationary state. We have derived it in the LL model based on natural conjectures on the matrix elements (form-factors) of conserved densities [96]. We find the diffusion matrix

$$(\mathfrak{D}C)_{ij} = \int \frac{d\alpha}{2}\rho_{\mathrm{p}}(\theta)f(\theta)\rho_{\mathrm{p}}(\alpha)f(\alpha)|v^{\mathrm{eff}}(\theta) - v^{\mathrm{eff}}(\alpha)| \times [\mathfrak{A}_i(\theta,\alpha) - \mathfrak{A}_i(\alpha,\theta)][\mathfrak{A}_j(\theta,\alpha) - \mathfrak{A}_j(\alpha,\theta)] \tag{8}$$

where $\mathfrak{A}_i(\theta,\alpha) = h_i^{\mathrm{dr}}(\theta)T^{\mathrm{dr}}(\theta,\alpha)(\rho_{\mathrm{s}}(\theta))^{-1}$. The dressed differential scattering phase $T^{\mathrm{dr}}(\theta,\alpha)$ is the dressing of $T(\theta,\alpha)$ as a vector field in its first argument θ, solving $T^{\mathrm{dr}}(\theta,\alpha) = T(\theta,\alpha) + \int d\omega T(\theta,\omega)n(\omega)T^{\mathrm{dr}}(\omega,\alpha)$. Expression (8) depends on the state via ρ_{p}, ρ_{s}, the effective velocity and the dressing operation. Writing matrices in their dual integral-kernel form, $\mathfrak{D}_{ij} = (h_i, \mathfrak{D}h_j) = \int \mathrm{d}\theta \mathrm{d}\alpha h_i(\theta)\mathfrak{D}(\theta,\alpha)h_j(\alpha)$, and using matrix kernel multiplication $(AB)(\alpha,\theta) = \int d\gamma A(\alpha,\gamma)B(\gamma,\theta)$, it is possible to extract an expression for the diffusion kernel $\mathfrak{D} = (1-nT)^{-1}\rho_{\mathrm{s}}\tilde{\mathfrak{D}}\rho_{\mathrm{s}}^{-1}(1-nT)$ where n, ρ_{s}, and 1 are seen as diagonal integral kernels, with $\rho_{\mathrm{s}}(\theta)^2\tilde{\mathfrak{D}}(\theta,\alpha) = [w(\theta)\delta(\theta-\alpha) - W(\theta,\alpha)]$, where $W(\theta,\alpha) = \rho_{\mathrm{p}}(\theta)f(\theta)[T^{\mathrm{dr}}(\theta,\alpha)]^2|v^{\mathrm{eff}}(\theta) - v^{\mathrm{eff}}(\alpha)|$ and $w(\theta) = \int d\alpha W(\alpha,\theta)$ (with the parametrization choice such that T is symmetric). We can now give a space and time dependence to the local GGE state $\rho_{\mathrm{p,s}}(\theta) \to \rho_{\mathrm{p,s}}(\theta;x,t)$ and the hydrodynamic Eq. (7) are then modified by a Navier-Stokes term:

$$\boxed{\partial_t \rho_{\mathrm{p}} + \partial_x(v^{\mathrm{eff}}\rho_{\mathrm{p}}) = \frac{1}{2}\partial_x(\mathfrak{D}\partial_x\rho_{\mathrm{p}})} \tag{9}$$

where $(\mathfrak{D}\partial_x\rho_{\mathrm{p}})(\theta;x,t) = \int d\alpha \mathfrak{D}(\theta,\alpha;x,t)\partial_x\rho_{\mathrm{p}}(\alpha;x,t)$. Although derived in the LL model, the expression [Eq. (8)] is expressed in complete generality, as a function of the differential scattering kernel $T(\theta,\alpha)$ and the statistical factor $f(\theta)$. We conjecture that it applies to any integrable model, including Galilean and relativistic quantum and classical field theory, quantum chains, and classical gases [24,27,97,98]. For a gas of hard rods, where $T(\theta,\alpha)$ is a constant and $f(\theta) = 1$, the diffusion operator is known exactly [24,26,27], and we have verified that Eq. (8) reproduces it correctly (see [99]). In free (fermionic or bosonic) models, the differential scattering phase is zero, and thus no diffusion occurs. This confirms the proposition made in [65], and applies for instance to the infinite-coupling (Tonks-Girardeau) limit of the LL model. Expanding at large coupling c [100], the leading term of the diffusion matrix is in $1/c^2$, and we observe that it exactly agrees with the diffusion matrix for the hard rod gas. Often there is a choice of spectral parameter such that $T(\theta,\alpha) = T(\alpha,\theta)$, for instance when α is the velocity (rapidity), in most Galilean (relativistic) models. Symmetry of T implies symmetry of T^{dr}, simplifying Eq. (8). As a consequence, one has $\int d\theta p'(\theta)\mathfrak{D}(\theta,\alpha) = 0$. This sum rule also follows from Galilean (relativistic) invariance, as then the current of mass (energy) is the momentum density. It is the defining property of a "Markov operator" with respect to the measure $dp(\theta)$, extending the observation made for the hard rods [24,26,27,32].

Entropy production.—A fundamental property of diffusion is that it causes an increase of the entropy of the local fluid cells. This is not in contradictions with unitary evolution of isolated systems: diffusion is indeed the passage of entropy from small-scale cells to large-scale structures. We therefore consider the total entropy of all fluid cells, $S = \int dxs(x)$. The entropy of integrable models takes different forms depending on the statistics of the quasiparticles but in general it is $s = \int d\theta\rho_{\mathrm{s}}g$, where g, seen as a function of n, satisfies $\partial^2 g/\partial n^2 = 1/(nf)$. We have shown (see [101]) that if the operator $\mathfrak{D}C$ is positive [which is the case, Eq. (8)], the total fluid-cell entropy production is positive, since the entropy density $s(x,t)$ satisfies the equation

$$\partial_t s + \partial_x j_s = (\sigma, \mathfrak{D}C\sigma)/2, \tag{10}$$

with the choice of $\sigma(x,t)$ such that $(1-Tn)^{-1}\sigma = \partial_x n/(nf)$ and with the explicit expression of the entropy current given in [101].

Sketch of proof.—The derivation of Eq. (8) in the LL model is based on the form-factors [102] expansion of the dynamical current-current correlation function in the Kubo formula (5) into intermediate states of particles $\{\theta_{\mathrm{p}}^a\}$ and holes $\{\theta_{\mathrm{h}}^a\}$ microscopic excitation above the GGE stationary state $|\rho_{\mathrm{p}}\rangle$:

$$\begin{aligned}\langle \mathfrak{j}_i(x,t)\mathfrak{j}_j(0,0)\rangle^{\mathrm{c}} &= \sum_{n=1}^{\infty}\frac{1}{(n!)^2}\prod_{a=1}^{n}\left(\int \bar{d}\theta_{\mathrm{p}}^a \bar{d}\theta_{\mathrm{h}}^a e^{i([k(\theta_{\mathrm{p}}^a)-k(\theta_{\mathrm{h}}^a)]x-[\varepsilon(\theta_{\mathrm{p}}^a)-\varepsilon(\theta_{\mathrm{h}}^a)]t)}\right)\\ &\quad\times\langle\rho_{\mathrm{p}}|\mathfrak{j}_i|\{\theta_{\mathrm{p}}^\bullet,\theta_{\mathrm{h}}^\bullet\}\rangle\langle\{\theta_{\mathrm{p}}^\bullet,\theta_{\mathrm{h}}^\bullet\}|\mathfrak{j}_j|\rho_{\mathrm{p}}\rangle\end{aligned} \tag{11}$$

where $k(\theta)$ and $\varepsilon(\theta)$ are appropriate momentum and energy function satisfying $\varepsilon'(\theta) = v^{\mathrm{eff}}(\theta)k'(\theta)$ and the integration measures $\bar{d}\theta_{\mathrm{p,h}}^a \equiv d\theta_{\mathrm{p,h}}^a\rho_{h,p}(\theta_{\mathrm{p,h}}^a)$. The continuity of Eq. (1) implies the structure

$$\langle\rho_{\mathrm{p}}|\mathfrak{j}_i|\{\theta_{\mathrm{p}}^\bullet,\theta_{\mathrm{h}}^\bullet\}\rangle = \left(\sum_a[\varepsilon(\theta_{\mathrm{p}}^a)-\varepsilon(\theta_{\mathrm{h}}^a)]\right)f_i(\{\theta_{\mathrm{p}}^\bullet,\theta_{\mathrm{h}}^\bullet\}) \tag{12}$$

for the form-factors on a nontrivial background (similar to finite-temperature form factors [103–105]). According to general principles, f_i has simple "kinematic poles" at $\theta_{\mathrm{p}}^a = \theta_{\mathrm{h}}^b$ [103,106–109]. The integral over x in Eq. (5) gives $\delta\left(\sum_a[k(\theta_{\mathrm{p}}^a)-k(\theta_{\mathrm{h}}^a)]\right)$. At $n=1$, this imposes equality of particle and hole momentum. Interpreting a hole excitation as an outgoing particle, this equality is interpreted as ballistic propagation. The exact value of the one particle-hole pair matrix element at equal particle and hole momenta is known [105,110], and gives the Drude weight [41] (see also [101]). At $n=2$, with two particles and two holes, the integral over time in Eq. (5) gives the extra factor $\delta\left(\sum_a[\varepsilon(\theta_{\mathrm{p}}^a)-\varepsilon(\theta_{\mathrm{h}}^a)]\right)$. The two delta functions are now simultaneous conservation of momentum and energy in a two-body scattering process. In $1+1$ dimension, this imposes equality of the sets of incoming and outgoing momenta. The combination of the "energy-conservation" factor $\sum_a[\varepsilon(\theta_{\mathrm{p}}^a)-\varepsilon(\theta_{\mathrm{h}}^a)]$ in Eq. (12) and equality of the sets of incoming (particle) and outgoing (hole) momenta imply that it is the kinematic poles of the functions f_i that provide a nonzero result at $n=2$. Evaluating this using known kinematic residues, we obtain Eq. (8). For $n>2$, the energy-conservation factor gives zero against $\delta\left(\sum_a[\varepsilon(\theta_{\mathrm{p}}^a)-\varepsilon(\theta_{\mathrm{h}}^a)]\right)$, as the set of incoming and outgoing momenta are no longer conserved with more than two particles. See Ref. [99] for full details of the computation.

Discussion.—The proof we sketched above has a natural interpretation: the contribution to diffusion comes from two-body scattering events between quasiparticle excitations above the local GGE state. Like the effective propagation velocity, the scattering amplitude is renormalized by the local fluid state; thus the diffusion matrix [Eq. (8)] involves the dressed scattering kernel $T^{\mathrm{dr}}(\theta,\alpha)$. Although the ballistic matrix A is diagonal in the quasiparticles momenta—quasiparticles propagate ballistically in a homogeneous GGE—the diffusion matrix $\mathfrak{D}$ is not diagonal, as to first order beyond the Euler scale elastic two-body scattering events are important, reminiscent of how two-body scattering terms in Boltzmann's equation lead to diffusion in standard hydrodynamics. Higher-body scattering processes are expected to take place at subdiffusive orders.

Numerical evaluations.—It is instructive to consider near-homogeneous and stationary situations, $\rho_{\mathrm{p}}(\theta;x,t) \sim \rho_{\mathrm{p}}^{\mathrm{sta}}(\theta)$ and $\partial_x\rho_{\mathrm{p}} \ll 1$. In terms of the occupation function $n(\theta;x,t)$ Eq. (9) becomes

$$\partial_t n + v^{\mathrm{eff}}\partial_x n = \frac{1}{2}\tilde{\mathfrak{D}}\partial_x^2 n + O[(\partial_x n)^2], \tag{13}$$

where the operator $\tilde{\mathfrak{D}}(\theta,\alpha)$, introduced above Eq. (9), is evaluated on the stationary state in the linear approximation. Equation (13) represents a diffusive spreading correcting the ballistic propagation. Let us consider the *partitioning protocol* for the construction of nonequilibrium steady states, where two semi-infinite baths, initially independently thermalized, are then connected and let to evolve unitarily, see Fig. 1. The solution at the Euler scale in integrable models is a continuum of contact singularities,

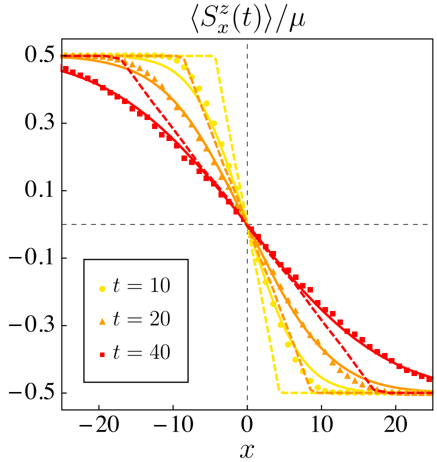

FIG. 1. Plot of the magnetization profile $\langle S^z_x(t)\rangle/\mu$ at different times with the evolution given by an *XXZ* chain with anisotropy $\Delta = \cos(\pi/7)$ and with initial conditions given by the (small) domain wall initial state with density matrix given by $\hat{\rho}_0 \propto \prod_{x<0} e^{\mu S^z_x} \otimes \prod_{x>0} e^{-\mu S^z_x}$ and $\mu = 0.05$, namely $\langle S^z_x(t)\rangle = \mathrm{Tr}[e^{iH_{\mathrm{XXZ}}t}\hat{\rho}_0 e^{-iH_{\mathrm{XXZ}}t} S^z_x]$. Continuous lines are the predictions of Eq. (13), dashed lines are the predictions of Euler GHD (7) and dots are tDMRG numerical simulation reported in Ref. [49].

one for each value of θ [13,111]. Such singularities are a feature of the Euler scale, and are smoothed out at shorter space-time scales by diffusive spreading effects, which, upon integration over θ, give rise to $1/\sqrt{t}$ corrections to local observables. We compared with exact time-dependent density matrix renormalization group (tDMRG) numerics in the *XXZ* Heisenberg chain $H_{XXZ} = \sum_x (S^+_x S^-_{x+1} + S^-_x S^+_{x+1})/2 + \Delta S^z_x S^z_{x+1}$ (with S^α_x, $\alpha = +, -, z$ the spin-1/2 operator at position x) in the gapless regime $|\Delta| < 1$. We find that corrections due to the diffusive term dramatically *improve* the Euler-scale predictions [15], see Fig. 1. We also report some additional numerical checks in Ref. [101].

Conclusion.—We derived large-scale hydrodynamics equations accounting for diffusive effects in integrable models. These equations complete the Euler-scale hydrodynamic approach introduced originally in Refs. [13,15] and allow us to access shorter time and length scales. We checked our results by reproducing analytically the known expression in classical hard rod gases and by numerical comparisons with tDMRG numerical data for an *XXZ* spin chain, finding excellent agreement. Extension to different models such as the gapped *XXZ* chain and the Fermi-Hubbard model [60], the dynamics of integrable spin chains with weak coupling to external environment [112–114], the effects of an external trapping potential on diffusive phenomena and thermalization [14,22,36] and super-diffusive transport in the presence of isotropic interactions [49,85,115] are under current investigation.

The authors kindly acknowledge Christoph Karrasch and Marko Ljubotina for providing tDMRG data and Enej Ilievski, Tomaz Prosen, Andrea De Luca, and Koenraad Schalm for stimulating discussions and valuable comments. J. D. N. acknowledges support from LabEx ENS-ICFP:ANR-10-LABX-0010/ANR-10-IDEX-0001-02 PSL*. D. B. acknowledges support from ANR Contract No. ANR-14-CE25-0003. B. D. thanks the École Normale Supérieure de Paris for an invited professorship, where a large part of this work was carried out, and he also thanks the Centre for Non-Equilibrium Science (CNES) and the Thomas Young Centre (TYC). All the authors acknowledge hospitality and funding from the Erwin Schrödinger Institute in Vienna (Austria) during the program "Quantum Paths" and the GGI in Florence (Italy) during the program "Entanglement in Quantum Systems."

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