Lax Pairs and AKNS Pairs for Integrable PDEs

III. Inverse Scattering Transform

Certain NPDEs are classified as integrable in the sense that their corresponding IVPs can be solved with the help of an IST. The idea behind the IST method is as follows: Each integrable NPDE is associated with a LODE (or a system of LODEs) containing a parameter λ (usually known as the spectral parameter), and the solution $u(x,t)$ to the NPDE appears as a coefficient (usually known as the potential) in the corresponding LODE. In the NPDE the quantities x and t appear as independent variables (usually known as the spatial and temporal coordinates, respectively), and in the LODE x is an independent variable and λ and t appear as parameters. It is usually the case that $u(x,t)$ vanishes at each fixed t as x becomes infinite so that a scattering scenario can be created for the related LODE, in which the potential $u(x,t)$ can uniquely be associated with some scattering data $S(\lambda,t)$. The problem of determining $S(\lambda,t)$ for all λ values from $u(x,t)$ given for all x values is known as the direct scattering problem for the LODE. On the other hand, the problem of determining $u(x,t)$ from $S(\lambda,t)$ is known as the inverse scattering problem for that LODE.

The IST method for an integrable NPDE can be explained with the help of the diagram

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{direct scattering for LODE at } t=0"] \arrow[d, "\text{solution to NPDE}"'] & S(\lambda,0) \arrow[d, "\text{time evolution of scattering data}"] \\
u(x,t) & S(\lambda,t) \arrow[l, "\text{inverse scattering for LODE at time } t"]
\end{tikzcd}$$

In order to solve the IVP for the NPDE, i.e. in order to determine $u(x,t)$ from $u(x,0)$, one needs to perform the following three steps:

(i) Solve the corresponding direct scattering problem for the associated LODE at $t=0$, i.e. determine the initial scattering data $S(\lambda,0)$ from the initial potential $u(x,0)$.

(ii) Time evolve the scattering data from its initial value $S(\lambda,0)$ to its value $S(\lambda,t)$ at time t. Such an evolution is usually a simple one and is particular to each integrable NPDE.

(iii) Solve the corresponding inverse scattering problem for the associated LODE at fixed t, i.e. determine the potential $u(x,t)$ from the scattering data $S(\lambda,t)$.

It is amazing that the resulting $u(x,t)$ satisfies the integrable NPDE and that the limiting value of $u(x,t)$ as $t \to 0$ agrees with the initial profile $u(x,0)$.

IV. The Lax Method

In 1968 Peter Lax introduced [15] a method yielding an integrable NPDE corresponding to a given LODE. The basic idea behind the Lax method is the following. Given a linear differential operator $\mathcal{L}$ appearing in the spectral problem $\mathcal{L}\psi = \lambda\psi$, find an operator $\mathcal{A}$ (the operators $\mathcal{A}$ and $\mathcal{L}$ are said to form a Lax pair) such that:

(i) The spectral parameter λ does not change in time, i.e. $\lambda_t = 0$.

(ii) The quantity $\psi_t - \mathcal{A}\psi$ remains a solution to the same linear problem $\mathcal{L}\psi = \lambda\psi$.

(iii) The quantity $\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}$ is a multiplication operator, i.e. it is not a differential operator.

From condition (ii) we get

$$\mathcal{L}\left(\psi_t - \mathcal{A}\psi\right) = \lambda\left(\psi_t - \mathcal{A}\psi\right), \tag{4.1}$$

and with the help of $\mathcal{L}\psi = \lambda\psi$ and $\lambda_t = 0$, from (4.1) we obtain

$$\mathcal{L}\psi_t - \mathcal{L}\mathcal{A}\psi = \lambda\psi_t - \mathcal{A}\left(\lambda\psi\right) = \partial_t\left(\lambda\psi\right) - \mathcal{A}\mathcal{L}\psi = \partial_t\left(\mathcal{L}\psi\right) - \mathcal{A}\mathcal{L}\psi = \mathcal{L}_t\psi + \mathcal{L}\psi_t - \mathcal{A}\mathcal{L}\psi, \tag{4.2}$$

where ∂_t denotes the partial differential operator with respect to t. After canceling the term $\mathcal{L}\psi_t$ on the left and right hand sides of (4.2), we get

$$\left(\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}\right)\psi = 0,$$

which, because of (iii), yields

$$\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L} = 0. \tag{4.3}$$

Note that (4.3) is an evolution equation containing a first-order time derivative, and it is the desired integrable NPDE. The equation (4.3) is often called a compatibility condition.

Having outlined the Lax method, let us now list the Lax pairs $(\mathcal{L}, \mathcal{A})$ corresponding to some known integrable NPDEs and their associated LODEs.

1. The integrable NPDE known as the Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{4.4}$$

is associated with the LODE known as the 1-D Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + u(x,t)\,\psi = \lambda\psi, \tag{4.5}$$

and the corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = -\partial_x^2 + u, \qquad \mathcal{A} = -4\partial_x^3 + 6u\partial_x + 3u_x. \tag{4.6}$$

2. The integrable NPDE known as the focusing nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2u = 0, \tag{4.7}$$

is associated with the system of first-order LODEs known as the Zakharov-Shabat system

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta, \\[2ex] \dfrac{d\eta}{dx} = i\lambda\eta - u(x,t)^*\,\xi, \end{cases} \tag{4.8}$$

where the asterisk denotes complex cojugation. The corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \begin{bmatrix} i\partial_x & -iu \\ -iu^* & -i\partial_x \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} 2i\partial_x^2 + i|u|^2 & -2iu\partial_x - iu_x \\ -2iu^*\partial_x - iu_x^* & -2i\partial_x^2 - i|u|^2 \end{bmatrix}. \tag{4.9}$$

3. The integrable NPDE known as the defocusing NLS equation

$$iu_t + u_{xx} - 2|u|^2u = 0, \tag{4.10}$$

is associated with the first-order system of LODEs

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta, \\[2mm] \dfrac{d\eta}{dx} = i\lambda\eta + u(x,t)^*\,\xi, \end{cases} \tag{4.11}$$

and the corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \begin{bmatrix} i\partial_x & -iu \\ iu^* & -i\partial_x \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} 2i\partial_x^2 - i|u|^2 & -2iu\partial_x - iu_x \\ 2iu^*\partial_x + iu_x^* & -2i\partial_x^2 + i|u|^2 \end{bmatrix}. \tag{4.12}$$

4. The integrable system of NPDEs

$$\begin{cases} iu_t + u_{xx} - 2u^2v = 0, \\ iv_t - v_{xx} + 2uv^2 = 0, \end{cases} \tag{4.13}$$

is associated with the first-order system of LODEs known as the AKNS system

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta, \\[2mm] \dfrac{d\eta}{dx} = i\lambda\eta + v(x,t)\,\xi, \end{cases} \tag{4.14}$$

and the corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \begin{bmatrix} i\partial_x & -iu \\ iv & -i\partial_x \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} 2i\partial_x^2 - iuv & -2iu\partial_x - iu_x \\ 2iv\partial_x + iv_x & -2i\partial_x^2 + iuv \end{bmatrix}. \tag{4.15}$$

Note that the case $v = -u^*$ in (4.13) yields the focusing NLS equation (4.7) and the case $v = u^*$ yields the defocusing NLS equation (4.10).

5. The integrable NPDE known as the focusing modified Korteweg-de Vries (mKdV) equation

$$u_t + 6u^2u_x + u_{xxx} = 0, \tag{4.16}$$

is associated with the first-order linear system given by

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta, \\[2mm] \dfrac{d\eta}{dx} = i\lambda\eta - u(x,t)\,\xi. \end{cases} \tag{4.17}$$

and the corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \begin{bmatrix} i\partial_x & -iu \\ -iu & -i\partial_x \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} -4\partial_x^3 - 6u^2\partial_x - 6uu_x & 6u_x\partial_x + 3u_{xx} \\ -6u_x\partial_x - 3u_{xx} & -4\partial_x^3 - 6u^2\partial_x - 6uu_x \end{bmatrix}. \tag{4.18}$$

6. The integrable NPDE known as the defocusing modified Korteweg-de Vries (mKdV) equation

$$u_t - 6u^2u_x + u_{xxx} = 0, \tag{4.19}$$

is associated with the first-order linear system given by

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta, \\[2ex] \dfrac{d\eta}{dx} = i\lambda\eta + u(x,t)\,\xi. \end{cases} \tag{4.20}$$

The corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \begin{bmatrix} i\partial_x & -iu \\ iu & -i\partial_x \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} -4\partial_x^3 + 6u^2\partial_x + 6uu_x & 6u_x\partial_x + 3u_{xx} \\ 6u_x\partial_x + 3u_{xx} & -4\partial_x^3 + 6u^2\partial_x + 6uu_x \end{bmatrix}. \tag{4.21}$$

7. The integrable NPDE known as the Dym equation

$$u_t = u^3 u_{xxx}, \tag{4.22}$$

is associated with the LODE

$$\frac{d^2\psi}{dx^2} = \frac{\lambda}{u(x,t)^2}\,\psi, \tag{4.23}$$

and the corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = u^2\partial_x^2, \qquad \mathcal{A} = 4u^3\partial_x^3 + 6u^2u_x\partial_x^2. \tag{4.24}$$

8. The integrable system of NPDEs

$$\begin{cases} u_t - 6uu_x + u_{xxx} + \dfrac{3}{2}\,vv_{xxx} + 3v_xv_{xx} - 6uvv_x - \dfrac{3}{2}\,u_xv^2 = 0, \\[2ex] v_t + v_{xxx} - 6uv_x - 6u_xv - \dfrac{15}{2}\,v^2v_x = 0, \end{cases} \tag{4.25}$$

is associated with the Jaulent equation

$$-\frac{d^2\psi}{dx^2} + u(x,t)\,\psi + k\,v(x,t)\,\psi = k^2\psi. \tag{4.26}$$

Writing the above LODE in the form $\mathcal{L}\phi = k\phi$ with

$$\begin{bmatrix} 0 & 1 \\ -\partial_x^2 + u(x,t) & v(x,t) \end{bmatrix} \begin{bmatrix} \psi \\ k\psi \end{bmatrix} = k \begin{bmatrix} \psi \\ k\psi \end{bmatrix}, \qquad \phi := \begin{bmatrix} \psi \\ k\psi \end{bmatrix}, \tag{4.27}$$

the corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \begin{bmatrix} 0 & 1 \\ -\partial_x^2 + u & v \end{bmatrix}, \tag{4.28}$$

$$\mathcal{A} = \begin{bmatrix} -4\partial_x^3 + \left(6u + \dfrac{3}{2}\,v^2\right)\partial_x + \left(3u_x - \dfrac{3}{2}\,vv_x\right) & 6v\partial_x + 3v_x \\[2ex] -6v\partial_x^3 - 3v_x\partial_x^2 + 6uv\partial_x + (6u_xv + 3uv_x) & -4\partial_x^3 + \left(6u + \dfrac{15}{2}\,v^2\right)\partial_x + \left(3u_x + \dfrac{15}{2}\,vv_x\right) \end{bmatrix}. \tag{4.29}$$

9. The integrable NPDE known as the sine-Gordon equation

$$u_{xt} = \sin u, \tag{4.30}$$

is associated with the linear system

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi - \dfrac{1}{2}\, u_x(x,t)\,\eta, \\ \dfrac{d\eta}{dx} = i\lambda\eta + \dfrac{1}{2}\, u_x(x,t)\,\xi. \end{cases} \tag{4.31}$$

The corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \begin{bmatrix} i\partial_x & \dfrac{iu_x}{2} \\ \dfrac{iu_x}{2} & -i\partial_x \end{bmatrix}, \tag{4.32}$$

$$\mathcal{A} = \frac{1}{8}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) dy \left\{ \cos\left(\frac{u(x,t)+u(y,t)}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\left(\frac{u(x,t)+u(y,t)}{2}\right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}. \tag{4.33}$$

10. The integrable NPDE known as the sinh-Gordon equation

$$u_{xt} = \sinh u, \tag{4.34}$$

is associated with the linear system

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi - \dfrac{i}{2}\, u_x(x,t)\,\eta, \\ \dfrac{d\eta}{dx} = i\lambda\eta + \dfrac{i}{2}\, u_x(x,t)\,\xi. \end{cases} \tag{4.35}$$

The corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \begin{bmatrix} i\partial_x & -\dfrac{u_x}{2} \\ -\dfrac{u_x}{2} & -i\partial_x \end{bmatrix}, \tag{4.36}$$

$$\mathcal{A} = \frac{1}{8}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) dy \left\{ \cosh\left(\frac{u(x,t)+u(y,t)}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i\,\sinh\left(\frac{u(x,t)+u(y,t)}{2}\right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}. \tag{4.37}$$

11. The integrable NPDE in two-spatial dimensions known as the Kadomtsev-Petviashvili (KP) equation

$$(u_t - 6uu_x + u_{xxx})_x + 3\epsilon^2 u_{yy} = 0, \tag{4.38}$$

where $\epsilon = i$ for the KP I equation and $\epsilon = 1$ for the KP II equation, is related to the linear PDE

$$\psi_{xx} + \epsilon\psi_y + (\lambda - u)\psi = 0. \tag{4.39}$$

The corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = -\partial_x^2 - \epsilon\partial_y + u, \qquad \mathcal{A} = -4\partial_x^3 + 6u\partial_x + 3u_x - \frac{3\epsilon}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty} ds\right) u_y(s,y,t). \tag{4.40}$$

12. The coupled system of integrable NPDEs

$$\begin{cases} u_t = 10u_{xxx} + 6uu_x - 24v_x, \\ v_t = 3u_{xxxxx} + 3uu_{xxx} + 3u_x u_{xx} - 6uv_x - 8v_{xxx}, \end{cases} \tag{4.41}$$

contains two potentials $u(x,t)$ and $v(x,t)$ and is associated with the LODE

$$\frac{d^4\psi}{dx^4} + u(x,t)\,\frac{d^2\psi}{dx^2} + u_x(x,t)\,\frac{d\psi}{dx} + v(x,t)\,\psi = \lambda\psi. \tag{4.42}$$

The corresponding Lax pair $(\mathcal{L}, \mathcal{A})$ is given by

$$\mathcal{L} = \partial_x^4 + \partial_x u \partial_x + v, \qquad \mathcal{A} = -8\partial_x^3 - 6u\partial_x - 3u_x. \tag{4.43}$$

12. The coupled system of integrable NPDEs

$$\begin{cases} u_t + u_{xxx} + u_{yyy} = 3u_x v_{xx} + 3uv_{xxx} + 3u_y v_{yy} + 3uv_{yyy}, \\ v_{xy} = 0, \end{cases}, \tag{4.44}$$

where $u(x,y,t)$ and $v(x,y,t)$ are the two potentials is known as the Nizhnik-Veselov-Novikov system. It is related to the pair of LPDEs given by

$$\begin{cases} \psi_{xy} = u\psi, \\ \psi_t + \psi_{xxx} + \psi_{yyy} = 3v_{xx}\psi_x + 3v_{yy}\psi_y. \end{cases}, \tag{4.45}$$

V. The AKNS Method

In 1973 Ablowitz, Kaup, Newell, and Segur introduced [2,3] another method to determine an integrable NPDE corresponding to a LODE. This method is now known as the AKNS method, and the basic idea behind it is the following. Given a linear operator $\mathcal{X}$ associated with the first-order system $\theta_x = \mathcal{X}\theta$, we are interested in finding an operator $\mathcal{T}$ (the operators $\mathcal{T}$ and $\mathcal{X}$ are said to form an AKNS pair) such that:

(i) The spectral parameter λ does not change in time, i.e. $\lambda_t = 0$.

(ii) The quantity $\theta_t - \mathcal{T}\theta$ is also a solution to $\theta_x = \mathcal{X}\theta$, i.e. we have $(\theta_t - \mathcal{T}\theta)_x = \mathcal{X}(\theta_t - \mathcal{T}\theta)$.

(iii) The quantity $\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X}$ is a (matrix) multiplication operator, i.e. it is not a differential operator.

Having outlined the AKNS method, let us now list the AKNS pairs $(\mathcal{X}, \mathcal{T})$ corresponding to some known integrable NPDEs and their associated linear ODEs.

1. For the KdV equation (4.4) and the associated 1-D Schrödinger equation written as the first-order system $\theta_x = \mathcal{X}\theta$, as

$$\begin{bmatrix} \psi_x \\ \psi \end{bmatrix}_x = \begin{bmatrix} 0 & u(x,t) - \lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \qquad \theta := \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \tag{5.1}$$

we have the corresponding AKNS pair $(\mathcal{X},\mathcal{T})$ given by

$$\mathcal{X}=\begin{bmatrix}0 & u-\lambda\\ 1 & 0\end{bmatrix},\qquad \mathcal{T}=\begin{bmatrix}u_x & -4\lambda^2+2\lambda u+2u^2-u_{xx}\\ 4\lambda+2u & -u_x\end{bmatrix}. \tag{5.2}$$

2. The Zakharov-Shabat system (4.8) and the focusing NLS equation (4.7) correspond to the AKNS pair $(\mathcal{X},\mathcal{T})$ given by

$$\mathcal{X}=\begin{bmatrix}-i\lambda & u\\ -u^* & i\lambda\end{bmatrix},\qquad \mathcal{T}=\begin{bmatrix}-2i\lambda^2+i|u|^2 & 2\lambda u+iu_x\\ -2\lambda u^*+iu_x^* & 2i\lambda^2-i|u|^2\end{bmatrix}. \tag{5.3}$$

3. The defocusing NLS equation (4.10) and the associated linear system (4.11) correspond to the AKNS pair $(\mathcal{X},\mathcal{T})$ given by

$$\mathcal{X}=\begin{bmatrix}-i\lambda & u\\ u^* & i\lambda\end{bmatrix},\qquad \mathcal{T}=\begin{bmatrix}-2i\lambda^2-i|u|^2 & 2\lambda u+iu_x\\ 2\lambda u^*-iu_x^* & 2i\lambda^2+i|u|^2\end{bmatrix}. \tag{5.4}$$

4. The focusing mKdV equation (4.16) and the associated linear system (4.17) correspond to the AKNS pair $(\mathcal{X},\mathcal{T})$ given by

$$\mathcal{X}=\begin{bmatrix}-i\lambda & u\\ -u & i\lambda\end{bmatrix},\qquad \mathcal{T}=\begin{bmatrix}-4i\lambda^3+2i\lambda u^2 & 4\lambda^2u+2i\lambda u_x-u_{xx}-2u^3\\ -4\lambda^2u+2i\lambda u_x+u_{xx}+2u^3 & 4i\lambda^3-2i\lambda u^2\end{bmatrix}. \tag{5.5}$$

5. The sine-Gordon equation (4.30) and the associated linear system (4.31) correspond to the AKNS pair $(\mathcal{X},\mathcal{T})$ given by

$$\mathcal{X}=\begin{bmatrix}-i\lambda & -\dfrac{1}{2}\,u_x\\ \dfrac{1}{2}\,u_x & i\lambda\end{bmatrix},\qquad \mathcal{T}=\frac{i}{4\lambda}\begin{bmatrix}\cos u & \sin u\\ \sin u & -\cos u\end{bmatrix}. \tag{5.6}$$

6. The sinh-Gordon equation (4.34) and the associated linear system (4.35) correspond to the AKNS pair $(\mathcal{X},\mathcal{T})$ given by

$$\mathcal{X}=\begin{bmatrix}-i\lambda & -\dfrac{i}{2}\,u_x\\ \dfrac{i}{2}\,u_x & i\lambda\end{bmatrix},\qquad \mathcal{T}=\frac{i}{4\lambda}\begin{bmatrix}\cosh u & i\sinh u\\ i\sinh u & -\cosh u\end{bmatrix}. \tag{5.7}$$

7. The AKNS system (4.14) and the associated system of integrable NPDEs correspond to the AKNS pair $(\mathcal{X},\mathcal{T})$ given by

$$\mathcal{X}=\begin{bmatrix}-i\lambda & u\\ v & i\lambda\end{bmatrix},\qquad \mathcal{T}=\begin{bmatrix}-2i\lambda^2-iuv & 2\lambda u+iu_x\\ 2\lambda v-iv_x & 2i\lambda^2+iuv\end{bmatrix}. \tag{5.8}$$

8. The Jaulent equation (4.26) can be written as the first-order system $\theta_x=\mathcal{X}\theta$, which is given by

$$\begin{bmatrix}\psi_x\\ \psi\end{bmatrix}_x=\begin{bmatrix}0 & u(x,t)+k\,v(x,t)-k^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_x\\ \psi\end{bmatrix},\qquad \theta:=\begin{bmatrix}\psi_x\\ \psi\end{bmatrix}, \tag{5.9}$$

For (5.9) and the associated integrable NPDEs (4.25) we have the corresponding AKNS pair $(\mathcal{X},\mathcal{T})$ is given by

$$\mathcal{X}=\begin{bmatrix}0 & u+kv-k^2\\ 1 & 0\end{bmatrix},\qquad \mathcal{T}=\begin{bmatrix}v_xk+\left(u_x+\frac{3}{2}\,vv_x\right) & \mathcal{T}_{12}\\ 4k^2+2vk+\left(2u+\frac{3}{2}\,v^2\right) & -v_xk-\left(u_x+\frac{3}{2}\,vv_x\right)\end{bmatrix}, \tag{5.10}$$

where we have defined

$$\mathcal{T}_{12}:=-4k^4+2vk^3+\left(2u+\frac{1}{2}\,v^2\right)k^2+\left(v_{xx}+\frac{3}{2}\,v^3\right)k+\left(u_{xx}+2u^2+\frac{3}{2}\,uv^2-\frac{3}{2}\,v_x^2-\frac{3}{2}\,vv_{xx}\right). \tag{5.11}$$

9. For the Dym equation (4.22) and the associated first-order system $\theta_x=\mathcal{X}\theta$, which is equivalent to (4.23) and given by

$$\begin{bmatrix}\psi_x\\ \psi\end{bmatrix}_x=\begin{bmatrix}0 & \dfrac{\lambda}{u(x,t)^2}\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_x\\ \psi\end{bmatrix},\qquad \theta:=\begin{bmatrix}\psi_x\\ \psi\end{bmatrix}, \tag{5.12}$$

we have the corresponding AKNS pair $(\mathcal{X},\mathcal{T})$ given by

$$\mathcal{X}=\begin{bmatrix}0 & \dfrac{\lambda}{u(x,t)^2}\\ 1 & 0\end{bmatrix},\qquad \mathcal{T}=\begin{bmatrix}2\lambda u_x & \dfrac{4\lambda^2}{u}-2\lambda u_{xx}\\ 4\lambda u & -2\lambda u_x\end{bmatrix}. \tag{5.13}$$

10. The 2-component Camassa-Holm (CH) equation

$$\begin{cases}\rho\rho_t+(u\rho)_x=0,\\ m_t+2u_xm+um_x+\sigma\rho\rho_x=0,\end{cases} \tag{5.14}$$

where $\rho(x,t)$ and $u(x,t)$ are the two potentials, $m:=u-u_{xx}$ and $\sigma_1=0,1$ and $\sigma=\pm1$, is associated with the LODE given by

$$\frac{d^2\psi}{dx^2}=\left(\frac{\sigma_1}{4}+\lambda\,m(x,t)-\sigma\lambda^2\rho(x,t)^2\right)\psi. \tag{5.15}$$

We can write the above LODE as a first-order system as $\theta_x=\mathcal{X}\theta$ with

$$\begin{bmatrix}\psi_x\\ \psi\end{bmatrix}_x=\begin{bmatrix}0 & \dfrac{\sigma_1}{4}+\lambda\,m(x,t)-\sigma\lambda^2\rho(x,t)^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_x\\ \psi\end{bmatrix},\qquad \theta:=\begin{bmatrix}\psi_x\\ \psi\end{bmatrix}. \tag{5.16}$$

The AKNS pair $(\mathcal{X},\mathcal{T})$ corresponding to (5.14) and (5.16) is given by

$$\mathcal{X}=\begin{bmatrix}0 & \dfrac{\sigma_1}{4}+\lambda\,m(x,t)-\sigma\lambda^2\rho^2\\ 1 & 0\end{bmatrix}, \tag{5.17}$$

$$\mathcal{T}=\begin{bmatrix}-\dfrac{1}{2}\,u_x & \sigma u\rho^2\lambda^2-\left(\dfrac{1}{2}\sigma\rho^2+um\right)\lambda+\dfrac{1}{4}\left(2u_{xx}+2m-\sigma_1u\right)+\dfrac{\sigma_1}{2\lambda}\\ \dfrac{1}{2\lambda}-u & \dfrac{1}{2}\,u_x\end{bmatrix}. \tag{5.18}$$

11. The Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{5.19}$$

where κ is a constant, is known to be integrable if $[m(x,t)+\kappa] > 0$ with $m := u - u_{xx}$. It is associated with the LODE

$$\frac{d^2\psi}{dx^2} = \left(\frac{1}{4} - \frac{m(x,t)+\kappa}{2\lambda}\right)\psi. \tag{5.20}$$

We can write (5.20) as a first-order system of LODEs as $\theta_x = \mathcal{X}\theta$ with

$$\begin{bmatrix} \psi_x \\ \psi \end{bmatrix}_x = \begin{bmatrix} 0 & \dfrac{1}{4} - \dfrac{m(x,t)+\kappa}{2\lambda} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \qquad \theta := \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \tag{5.21}$$

the corresponding AKNS pair $(\mathcal{X}, \mathcal{T})$ is given by

$$\mathcal{X} = \begin{bmatrix} 0 & \dfrac{1}{4} - \dfrac{m+\kappa}{2\lambda} \\ 1 & 0 \end{bmatrix}, \qquad \mathcal{T} = \begin{bmatrix} -\dfrac{1}{2}\,u_x & \dfrac{(m+\kappa)u}{2\lambda} + 4(2u_{xx} - u + 2m + 2\kappa) - \dfrac{\lambda}{4} \\ -u - \lambda & \dfrac{1}{2}\,u_x \end{bmatrix}. \tag{5.22}$$

12. The integrable NPDE known as the Degasperis-Procesi equation

$$m_t + um_x + 3u_x m = 0, \qquad m := u - u_{xx}, \tag{5.23}$$

is associated with the LODE

$$\frac{d^3\psi}{dx^3} - \frac{d\psi}{dx} = \lambda\, m(x,t)\,\psi. \tag{5.24}$$

We can write (5.24) as a first-order system of LODEs as $\theta_x = \mathcal{X}\theta$ with

$$\begin{bmatrix} \psi_{xx} \\ \psi_x \\ \psi \end{bmatrix}_x = \begin{bmatrix} 0 & 1 & \lambda m(x,t) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{xx} \\ \psi_x \\ \psi \end{bmatrix}, \qquad \theta := \begin{bmatrix} \psi_{xx} \\ \psi_x \\ \psi \end{bmatrix}. \tag{5.25}$$

The AKNS pair $(\mathcal{X}, \mathcal{T})$ corresponding to (5.23) and (5.25) is given by

$$\mathcal{X} = \begin{bmatrix} 0 & 1 & \lambda m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{xx} \\ \psi_x \\ \psi \end{bmatrix}, \qquad \mathcal{T} = \begin{bmatrix} \dfrac{1}{\lambda} - u_x & 0 & u_x - \lambda um \\ -u & \dfrac{1}{\lambda} & u \\ \dfrac{1}{\lambda} & -u & u_x \end{bmatrix}. \tag{5.26}$$