One-dimensional Stark operators in the half-line

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Abstract

We obtain asymptotic formulas for the spectral data of perturbed Stark operators associated with the differential expression

$$-\frac{d^2}{dx^2} + x + q(x), \quad x \in [0, \infty), \quad q \in L^1(0, \infty),$$

and having either Dirichlet or Neumann boundary condition at the origin.

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1 Introduction and statement of results

Self-adjoint operators of the form

$$-\frac{d^2}{dx^2} + f(x) + q(x), \quad x \in (0, \infty),$$

with domain in $L^2(\mathbb{R}_+)$, occur naturally in the context of quantum-mechanical operators with spherical symmetry; here q plays the role of a small perturbation of f in some suitable sense. The spectral analysis of this kind of operators have attracted considerable attention for various choices of the dominant term f, usually in connection with well-known special functions. Most remarkable among them are the investigations concerning perturbed Bessel operators [1,3,4,10,16-19] (corresponding to $f(x) = l(l+1)x^{-2}$, $l \ge -1/2$), and perturbed harmonic oscillator in the half-line [7,8] (in this case $f(x) = x^2$); the latter is closely related to the spectral analysis of perturbed harmonic oscillator in the whole real line [5,6].

In this paper we consider self-adjoint operators associated with a differential expression of the form

$$\tau = -\frac{d^2}{dx^2} + x + q(x), \quad x \in [0, \infty),$$

acting in the space $L^2(\mathbb{R}_+)$, where q is a real-valued function that lies in $L^1(\mathbb{R}_+)$. Self-adjoint operators are defined by adjoining to τ a standard boundary condition (see Section 2) at the left endpoint. For the sake of brevity, we only consider Dirichlet ($\varphi(0) = 0$) and Neumann

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 $(\varphi'(0) = 0)$ boundary conditions; let H^D and H^N denote the corresponding self-adjoint operators.

Clearly the unperturbed case $q \equiv 0$ can be solved explicitly. For in this case a square-integrable solution to the associated eigenvalue problem is given by the Airy function of the first kind Ai(z) so

$$\sigma(H_0^D) = \{-a_k\}_{k \in \mathbb{N}} \text{ and } \sigma(H_0^N) = \{-a_k'\}_{k \in \mathbb{N}},$$

where a_k and a_k' denote the zeros of $\mathrm{Ai}(z)$ and its derivative $\mathrm{Ai}'(z)$, respectively. The corresponding set of norming constants $\{\nu_{0,k}^D\}_{k\in\mathbb{N}}$ and $\{\nu_{0,k}^N\}_{k\in\mathbb{N}}$ are then given by

$$\frac{1}{\nu_{0,k}^D} = \frac{\|\operatorname{Ai}(\cdot + a_k)\|^2}{\left(\operatorname{Ai}'(a_k)\right)^2} = 1 \quad \text{and} \quad \frac{1}{\nu_{0,k}^N} = \frac{\|\operatorname{Ai}(\cdot + a_k')\|^2}{\left(\operatorname{Ai}(a_k')\right)^2} = -a_k'.$$

The related results for arbitrary q are stated in Theorems 3.5, 3.6, 3.8 and 3.9. They can be summarized as follows:

Theorem. Suppose $q \in L^1(\mathbb{R}_+)$. Then the eigenvalues and norming constants of H^D , the operator associated with τ and boundary condition $\varphi(0) = 0$, satisfy

$$\lambda_k^D = \left(\frac{3}{2}\pi(k - \frac{1}{4})\right)^{2/3} \left(1 + O(k^{-1})\right) \quad and \quad \frac{1}{\nu_k^D} = 1 + o(1)$$

as $k \to \infty$. Similarly, the eigenvalues and norming constants of H^N corresponding to the boundary condition $\varphi'(0) = 0$ satisfy

$$\lambda_k^N = \left(\frac{3}{2}\pi(k - \frac{3}{4})\right)^{2/3} \left(1 + O(k^{-1})\right) \quad and \quad \frac{1}{\nu_k^N} = \left(\frac{3}{2}\pi(k - \frac{3}{4})\right)^{2/3} \left(1 + o(1)\right)$$

as $k \to \infty$.

The direct spectral problem for the one-dimensional Stark operator in the semi-axis, with Dirichlet boundary condition, has also been treated recently in [22], where the authors use transformation operator methods and their results are valid under the more restrictive assumption $q \in C^{(1)}[0,\infty) \cap L^1(\mathbb{R}_+, x^4dx)$, q(x) = o(x) as $x \to \infty$. The corresponding inverse spectral problem is discussed in [20].

Finally, it is worth mentioning that one-dimensional Stark operators have been studied mostly when defined on the whole real line, see for instance [2,11–13,21,25]. As it is well-known, Stark operators on the real line are characterized by the presence of resonances; see [14,15] for some recent developments on this subject.

2 Preliminaries

In what follows, we consider the differential expression

$$\tau = -\frac{d^2}{dx^2} + x + q(x), \quad x \in [0, \infty),$$

where $q \in L^1(\mathbb{R}_+)$ and it is real-valued.

By standard theory (see e.g. [26, Ch. 6]), τ is in the limit-circle case at 0 and in the limit-point case at ∞ . Hence (the closure of) the minimal operator H' defined by τ is symmetric and has deficiency indices (1, 1). Also, there exists a solution $\psi(z, x)$ to the eigenvalue equation $\tau \varphi = z \varphi$, real entire as a function of $z \in \mathbb{C}$ for every $x \in [0, \infty)$, such that $\psi(z, \cdot) \in L^2(\mathbb{R}_+)$ for

every $z \in \mathbb{C}$. This function is unique up to multiplication by a zero-free, real entire function of the spectral parameter z.

The self-adjoint extensions H^{β} ($0 \le \beta < \pi$) of H' are determined by imposing the usual boundary condition at x = 0. Namely,

$$\mathcal{D}(H^{\beta}) = \left\{ \begin{aligned} \varphi \in L^{2}(\mathbb{R}_{+}) : \varphi, \varphi' \in AC_{loc}([0, \infty)), & \tau \varphi \in L^{2}(\mathbb{R}_{+}), \\ \cos(\beta)\varphi(0) - \sin(\beta)\varphi'(0) = 0 \end{aligned} \right\}, \quad H^{\beta}\varphi = \tau \varphi.$$

Since $x + q(x) \to \infty$ as $x \to \infty$, it follows that $\sigma(H^{\beta})$ has only eigenvalues of multiplicity one, possibly with a finite number of them being negative. Moreover,

$$\sigma(H^{\beta}) = \{ \lambda \in \mathbb{R} : \cos(\beta)\psi(\lambda, 0) - \sin(\beta)\psi'(\lambda, 0) = 0 \} \quad (0 \le \beta < \pi).$$

We henceforth suppose $\sigma(H^{\beta})$ is arranged as an increasing sequence, viz., $\sigma(H^{\beta}) = \{\lambda_k^{\beta}\}_{k \in \mathbb{N}}$ with $\lambda_k^{\beta} < \lambda_{k+1}^{\beta}$.

In what follows we use the notation $'=\partial_x$ and $=\partial_z$. Along with the spectrum $\{\lambda_k^\beta\}_{k\in\mathbb{N}}$ one has the corresponding set of norming constants $\{\nu_k^\beta\}_{k\in\mathbb{N}}$. In terms of $\psi(z,x)$, the norming constants for Dirichlet $(\beta=0)$ and Neumann $(\beta=\pi/2)$ boundary conditions are given by the formulas

$$\frac{1}{\nu_k^D} = \frac{\left\| \psi(\lambda_k^D, \cdot) \right\|^2}{\left[\psi'(\lambda_k^D, 0) \right]^2} = -\frac{\dot{\psi}(\lambda_k^D, 0)}{\psi'(\lambda_k^D, 0)} \quad \text{and} \quad \frac{1}{\nu_k^N} = \frac{\left\| \psi(\lambda_k^N, \cdot) \right\|^2}{\left[\psi(\lambda_k^N, 0) \right]^2} = \frac{\dot{\psi}'(\lambda_k^N, 0)}{\psi(\lambda_k^N, 0)},$$

respectively. The second part of these equations follows from the identity $W'(\eta, \dot{\eta}) = -\eta^2$, which is valid for any solution to $\tau \eta = z \eta$. We recall that the spectral data $(\{\mu_k^{\beta}\}_{k \in \mathbb{N}}, \{\nu_k^{\beta}\}_{k \in \mathbb{N}})$ are the poles and residues of the Weyl function associated with H^{β} , and they determine the potential q by virtue of the Borg–Marchenko uniqueness theorem [9].

As mentioned in the Introduction, the unperturbed case q=0 can be treated explicitly. A solution to the equation $-\varphi'' + (x-z)\varphi$, belonging to $L^2(\mathbb{R}_+)$, is

$$\psi_0(z,x) = \sqrt{\pi} \operatorname{Ai}(x-z),$$

where the factor $\sqrt{\pi}$ is included for convenience. It follows that

$$\sigma(H_0^D) = \{-a_k\}_{k \in \mathbb{N}} \quad \text{and} \quad \sigma(H_0^N) = \{-a_k'\}_{k \in \mathbb{N}}$$

respectively, where the zeros of Ai(z) and Ai'(z) obey the asymptotic formulas

$$-a_k = \left(\frac{3}{2}\pi(k - \frac{1}{4})\right)^{2/3} \left(1 + O(k^{-2})\right) \tag{1}$$

and

$$-a'_k = \left(\frac{3}{2}\pi(k - \frac{3}{4})\right)^{2/3} \left(1 + O(k^{-2})\right)$$

as $k \to \infty$ (see [23, §9.9(iv)]).

Lemma 2.1. There exists a constant $C_0 > 0$ such that

$$|\operatorname{Ai}(z)| \le C_0 \frac{g_A(z)}{1 + |z|^{1/4}}$$
 and $|\operatorname{Ai}'(z)| \le C_0 (1 + |z|^{1/4}) g_A(z)$,

for all $z \in \mathbb{C}$, where $g_A(z) = \exp(-\frac{2}{3}\operatorname{Re} z^{3/2})$.

Proof. Define $\zeta = \frac{2}{3}z^{3/2}$ with branch cut along \mathbb{R}_- . According to [23, §9.7(ii)], the function $\operatorname{Ai}(z)$ satisfies the asymptotic expansions

$$\operatorname{Ai}(z) = \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}} \left[1 + O(\zeta^{-1}) \right], \quad |\operatorname{arg}(z)| \le \pi - \delta,$$
 (2)

and

$$\operatorname{Ai}(-z) = \frac{1}{\sqrt{\pi}z^{1/4}} \left[\sin\left(\zeta + \frac{\pi}{4}\right) + O\left(\zeta^{-1}e^{|\operatorname{Im}\zeta|}\right) \right], \quad |\operatorname{arg}(z)| \le \frac{2\pi}{3} - \delta, \tag{3}$$

as $|z| \to \infty$. These expansions are uniform for any given small $\delta > 0$ and $|z| \ge 1$. In what follows we set $\delta = \pi/3$. Since Ai(z) is an entire function, it follows that there exists $C_0 > 0$ such that

$$|\operatorname{Ai}(z)| \le \frac{C_0}{1 + |z|^{1/4}} \times \begin{cases} \exp(-\frac{2}{3}\operatorname{Re} z^{3/2}), & \arg(z) \in [-\frac{2\pi}{3}, \frac{2\pi}{3}], \\ \exp(\frac{2}{3}|\operatorname{Im}(-z)^{3/2}|), & \arg(z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi]. \end{cases}$$

Thus, the bound on Ai(z) follows after noticing that $|\operatorname{Im}(-z)^{3/2}| = |\operatorname{Re} z^{3/2}|$ and $|\operatorname{Re} z^{3/2}| = -\operatorname{Re} z^{3/2}$ if $\operatorname{arg}(z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi]$. The bound on Ai'(z) follows an analogous argument so the details are omitted.

Lemma 2.1 clearly implies

$$|\psi_0(z,x)| \le C_0 \frac{e^{-\frac{2}{3}\operatorname{Re}(x-z)^{3/2}}}{1+|x-z|^{1/4}} \quad \text{and} \quad |\psi_0'(z,x)| \le C_0 (1+|x-z|^{1/4}) e^{-\frac{2}{3}\operatorname{Re}(x-z)^{3/2}}$$
 (4)

with $(x, z) \in \mathbb{R}_+ \times \mathbb{C}$. Later we will make use of a linearly independent solution to $-\varphi'' + (x - z)\varphi$. An obvious choice is given by the Airy function of the second kind

$$\theta_0(z,x) = \sqrt{\pi} \operatorname{Bi}(x-z).$$

However, it will be more convenient to use an independent solution of the form

$$\theta_{\pm}(z,x) = \theta_0(z,x) \mp i\psi_0(z,x) = 2\sqrt{\pi}e^{\mp i\pi/6} \operatorname{Ai}((x-z)e^{\mp i2\pi/3})$$

(in the context of the present work any of these two functions is equally good). According to [23, §9.2(iv)], one has $W(\psi_0(z), \theta_{\pm}(z)) \equiv 1$. Moreover, since $\text{Re}(ze^{\pm i2\pi/3})^{3/2} = -\text{Re}\,z^{3/2}$, we have the bounds

$$|\theta_{\pm}(z,x)| \le 2C_0 \frac{e^{\frac{2}{3}\operatorname{Re}(x-z)^{3/2}}}{1+|x-z|^{1/4}} \quad \text{and} \quad |\theta'_{\pm}(z,x)| \le 2C_0 (1+|x-z|^{1/4}) e^{\frac{2}{3}\operatorname{Re}(x-z)^{3/2}}.$$
 (5)

Lemma 2.2. The map $x \mapsto g_A(x-z)$, $x \in \mathbb{R}_+$, is decreasing whenever $z \in \mathbb{C} \setminus \mathbb{R}$. If $z \in \mathbb{R}$, then $g_A(x-z)$ is constant (equal to 1) for $x \in [0,z]$ and decreasing for $x \in (z,\infty)$.

Proof. Suppose $z \in \mathbb{C}_-$. A simple computation shows that, given $x \in \mathbb{R}$, there exists a unique $\gamma \in (0, \pi)$ such that

$$x - z = \frac{|\operatorname{Im} z|}{\sin \gamma} e^{i\gamma}.$$

Then,

$$\operatorname{Re}(x-z)^{3/2} = |\operatorname{Im} z|^{3/2} \frac{\cos \frac{3}{2} \gamma}{(\sin \gamma)^{3/2}}.$$

The right hand side of the last equation is decreasing as a function of γ . But the map $x \mapsto \gamma$ is also decreasing so the map $x \mapsto \text{Re}(x-z)^{3/2}$ is increasing. This in turn implies the assertion. Clearly, a similar reasoning works if $z \in \mathbb{C}_+$. The statement is obvious for $z \in \mathbb{R}$.

3 Main results

3.1 Adding a perturbation

We look for a solution to the eigenvalue equation $\tau \varphi = z \varphi$, with $q \in L^1(\mathbb{R}_+)$, that is real entire with respect to the spectral parameter $z \in \mathbb{C}$ and lies in $L^2(\mathbb{R}_+)$. To this end we introduce the auxiliary function

$$\omega(z) = \int_0^\infty \frac{|q(x)|}{\sqrt{1 + |x - z|}} dx.$$

Clearly, $\omega(z)$ is well defined for all $z \in \mathbb{C}$. Moreover, $\omega(z)$ is well defined under the weaker assumption $q \in L^1(\mathbb{R}_+, (1+x)^{-1/2}dx)$. However, our hypothesis on q give us control on the decay of $\omega(z)$ as it is shown next.

Lemma 3.1. Assume $q \in L^1(\mathbb{R}_+)$. Then $\omega(z) \to 0$ as $z \to \infty$.

Proof. Given $\varepsilon > 0$, choose $x_* > 0$ and $\mu_* > x_*$ such that

$$\int_{x_*}^{\infty} |q(x)| \, dx < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{\sqrt{\mu_* - x_*}} < \frac{\varepsilon}{2 \|q\|_1}.$$

Suppose $|\operatorname{Im}(z)| > \mu_*$. Then $|x - z| > \mu_*$ for any x > 0. Hence,

$$\frac{1}{\sqrt{1+|x-z|}} \le \frac{1}{\sqrt{\mu_*}} < \frac{\epsilon}{2 \|q\|_1}$$

for all $x \in \mathbb{R}_*$, which in turn implies $\omega(q, z) < \epsilon$. A similar reasoning applies when $|\operatorname{Im}(z)| \le \mu_*$ and $\operatorname{Re}(z) < -\mu_*$. Finally, suppose that $|\operatorname{Im}(z)| \le \mu_*$ and $\operatorname{Re}(z) > \mu_*$. Since $\omega(q, z) \le \omega(q, \operatorname{Re}(z))$, it suffices to consider $z = \mu \in \mathbb{R}$ with $\mu > \mu_*$. Then,

$$\omega(q,\mu) < \frac{1}{\sqrt{1+|x_*-\mu_*|}} \int_0^{x_*} |q(x)| \, dx + \int_{x_*}^{\infty} |q(x)| \, dx < \varepsilon.$$

Thus, we have shown that $\omega(q,z) < \varepsilon$ whenever $|\text{Re}(z)| + |\text{Im}(z)| > \mu_*$.

In what follows C denotes a generic positive constant.

Proposition 3.2. Suppose $q \in L^1(\mathbb{R}_+, (1+x)^{-1/2}dx)$. Then, the eigenvalue equation $\tau \varphi = z\varphi$ admits a solution $\psi(z, x)$, real entire with respect to z, such that:

(i) $\psi(z,x)$ solves the Volterra integral equation

$$\psi(z,x) = \psi_0(z,x) - \int_x^\infty J_0(z,x,y)q(y)\psi(z,y)dy, \tag{6}$$

where

$$J_0(z, x, y) = \psi_0(z, y)\theta_0(z, x) - \psi_0(z, x)\theta_0(z, y),$$

and satisfies the estimates

$$|\psi(z,x)| \le Ce^{C\omega(z)} \frac{g_A(x-z)}{1+|x-z|^{1/4}} \quad and$$

$$|\psi(z,x) - \psi_0(z,x)| \le C\omega(z)e^{C\omega(z)} \frac{g_A(x-z)}{1+|x-z|^{1/4}}.$$
(7)

(ii) Moreover, $\psi'(z,x)$ obeys the equation

$$\psi'(z,x) = \psi'_0(z,x) - \int_x^\infty \partial_x J_0(z,x,y) q(y) \psi(z,y) dy$$

and satisfies the estimates

$$|\psi'(z,x) - \psi_0'(z,x)| \le C\omega(z)e^{C\omega(z)}(1 + |x-z|^{1/4})g_A(x-z).$$
(8)

Proof. For $n \in \mathbb{N}$ define

$$\psi_n(z,x) = -\int_x^\infty J_0(z,x,y)q(y)\psi_{n-1}(z,y)dy.$$

Then,

$$|\psi_n(z,x)| \le \int_x^\infty |J_0(z,x,y)| |q(y)| |\psi_{n-1}(z,y)| dy.$$
 (9)

Next, we note that

$$J_0(z, x, y) = \pm i \left[\psi_0(z, x) \theta_+(z, y) - \psi_0(z, y) \theta_+(z, x) \right]$$

(the choice of sign is irrelevant). Then, recalling (4) and (5), (9) yields

$$|\psi_n(z,x)| \le 2C_0^2 \frac{g_A(x-z)}{1+|x-z|^{1/4}} \int_x^\infty \frac{|q(y)|}{1+|y-z|^{1/4}} g_\#(y-z) |\psi_{n-1}(z,y)| \, dy + 2C_0^2 \frac{g_\#(x-z)}{1+|x-z|^{1/4}} \int_x^\infty \frac{|q(y)|}{1+|y-z|^{1/4}} g_A(y-z) |\psi_{n-1}(z,y)| \, dy.$$

where $g_{\#}(z) := 1/g_A(z)$. We claim that every $\psi_n(z,x)$ is real entire with respect to the spectral parameter and satisfies the estimate

$$|\psi_n(z,x)| \le \frac{4^n}{n!} C_0^{2n+1} \omega^n(z) \frac{g_A(x-z)}{1+|x-z|^{1/4}}.$$
 (10)

From this it will follow that

$$\psi(z,x) = \sum_{n=0}^{\infty} \psi_n(z,x)$$

converges uniformly on compact subsets of \mathbb{C} to the solution with the desired properties. First, consider n = 1. Then, we have

$$|\psi_1(z,x)| \le 2C_0^3 \frac{g_A(x-z)}{1+|x-z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1+|y-z|^{1/4})^2} g_\#(y-z) g_A(y-z) dy + 2C_0^3 \frac{g_\#(x-z)}{1+|x-z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1+|y-z|^{1/4})^2} (g_A(y-z))^2 dy.$$

Clearly, $(g_A g_\#)(x-z) \equiv 1$. Also, Lemma 2.2 implies $g_A(y-z) \leq g_A(x-z)$ for all $y \in [x, \infty)$. Hence,

$$\begin{aligned} |\psi_1(z,x)| &\leq 2C_0^3 \frac{g_A(x-z)}{1+|x-z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1+|y-z|^{1/4})^2} dy \\ &\qquad \qquad + 2C_0^3 \frac{(g_\# g_A g_A)(x-z)}{1+|x-z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1+|y-z|^{1/4})^2} dy, \end{aligned}$$

that is.

$$|\psi_1(z,x)| \le 4C_0^3 \frac{g_A(x-z)}{1+|x-z|^{1/4}} \int_x^\infty \frac{|q(y)|}{(1+|y-z|^{1/4})^2} dy.$$

For arbitrary $n \in \mathbb{N}$ we use the identity

$$\int_{x}^{\infty} \int_{y_{1}}^{\infty} \cdots \int_{y_{n-1}}^{\infty} \prod_{l=1}^{n} h(y_{l}) dy_{1} \cdots dy_{n} = \frac{1}{n!} \left[\int_{x}^{\infty} h(y) dy \right]^{n}$$

$$\tag{11}$$

to obtain

$$|\psi_n(z,x)| \le \frac{4^n}{n!} C_0^{2n+1} \frac{g_A(x-z)}{1+|x-z|^{1/4}} \left[\int_x^\infty \frac{|q(y)|}{(1+|y-z|^{1/4})^2} dy \right]^n \tag{12}$$

which in turn implies (10). Then (i) follows after a suitable choice for the constant C.

The proof of (ii) is omitted since it arises from an analogous reasoning.

Clearly, (7) implies that $\psi(z,x)$ so constructed belongs to the domain of the maximal operator H.

The asymptotic analysis of the norming constants depends also on the following estimates.

Proposition 3.3. Suppose $q \in L^1(\mathbb{R}_+)$. Then, $\psi(z,x)$ satisfies

$$\left|\dot{\psi}(z,x) - \dot{\psi}_0(z,x)\right| \le Ce^{C\|q\|} \left(\left(1 + |x-z|^{1/4}\right)\omega(z) + \frac{\|q\|}{1 + |x-z|^{1/4}} \right) g_A(x-z). \tag{13}$$

Also,

$$\left|\dot{\psi}'(z,x) - \dot{\psi}'_0(z,x)\right| \le Ce^{C\|q\|} \left(\left(1 + |x-z|^{1/4}\right) \|q\|^2 + \frac{|x-z|}{1 + |x-z|^{1/4}} \omega(z) \right) g_A(x-z). \tag{14}$$

Proof. From (6) we see that $\dot{\psi}(z,x)$ is a solution to the integral equation

$$\dot{\psi}(z,x) = \dot{\psi}_0(z,x) - \int_x^\infty \partial_z J_0(z,x,y) q(y) \psi(z,y) dy - \int_x^\infty J_0(z,x,y) q(y) \dot{\psi}(z,y) dy.$$

Let $\{\eta_k(z,x)\}_{k\in\mathbb{N}}$, be solutions to the recursive equation

$$\eta_k(z,x) = -\int_x^\infty \partial_z J_0(z,x,y) q(y) \psi_{k-1}(z,y) dy - \int_x^\infty J_0(z,x,y) q(y) \eta_{k-1}(z,y) dy,$$

where $\{\psi_k(z,x)\}_{k\in\mathbb{N}}$ are defined in the proof of Proposition 3.2 and $\eta_0(z,x) := \dot{\psi}_0(z,x)$. Using induction one can show that

$$|\eta_k(z,x)| \le \frac{4^k C_0^{2k+1}}{k!} \left((1+|x-z|^{1/4}) \left(\int_x^\infty \frac{|q(y)|}{(1+|y-z|^{1/4})^2} dy \right)^k + \frac{2k}{1+|x-z|^{1/4}} \left(\int_x^\infty |q(y)| dy \right)^k \right) g_A(x-z),$$

hence

$$|\eta_k(z,x)| \le \frac{4^k C_0^{2k+1}}{k!} \left((1+|x-z|^{1/4})\omega(z)^k + \frac{2k}{1+|x-z|^{1/4}} \|q\|^k \right) g_A(x-z).$$

It follows that

$$\dot{\psi}(z,x) = \sum_{k=0}^{\infty} \eta_k(z,x)$$

(the convergence being uniform on compact subsets of \mathbb{C}) which in turn implies the assertion. The proof of the second inequality follows from an analogous reasoning.

3.2 Dirichlet boundary condition

Define the contours

$$\mathcal{E}^m := \left\{ z \in \mathbb{C} : |\zeta| = \left(m + \frac{1}{4}\right)\pi \right\}, \quad \mathcal{E}_k := \left\{ z \in \mathbb{C} : \left| \zeta - \left(k - \frac{1}{4}\right)\pi \right| = \frac{\pi}{2} \right\}, \quad m, k \in \mathbb{N}.$$

In view of (1), every \mathcal{E}_k encloses one and only one zero of $\operatorname{Ai}(-\lambda)$, at least for k sufficiently large.

Lemma 3.4. There exists $m_0, k_0 \in \mathbb{N}$ such that, for every $m \geq m_0$ and $k \geq k_0$, the following statement holds true:

$$\frac{g_A(-z)}{1+|z|^{1/4}} < 8\sqrt{\pi} |\text{Ai}(-z)|,$$
 (15)

whenever $z \in \mathcal{E}^m$ or $z \in \mathcal{E}_k$.

Proof. Let us begin by recalling (2) and (3) in more precise terms:

$$Ai(z) = \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}} \left[1 + W_1(z) \right], \quad |\arg(z)| \le \frac{2\pi}{3}, \quad |z| \ge 1, \tag{16}$$

$$Ai(-z) = \frac{1}{\sqrt{\pi}z^{1/4}} \left[\sin\left(\zeta + \frac{\pi}{4}\right) + W_2(z) \right], \quad |\arg(z)| \le \frac{\pi}{3}, \quad |z| \ge 1, \tag{17}$$

where the functions $W_1(z)$ and $W_2(z)$ satisfy

$$\left| \frac{W_1(z)}{\zeta^{-1}} \right| \le D_1, \quad |\arg(z)| \le \frac{2\pi}{3}, \quad |z| \ge 1,$$

$$\left| \frac{W_2(z)}{\zeta^{-1}e^{|\operatorname{Im}\zeta|}} \right| \le D_2, \quad |\arg(z)| \le \frac{\pi}{3}, \quad |z| \ge 1.$$
(18)

There exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $z \in \mathcal{E}_k$ implies $\operatorname{Re} z \geq 1$ and $\operatorname{arg}(z) \in (-\frac{\pi}{3}, \frac{\pi}{3})$ so $\operatorname{arg}(-z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi]$. Since in this case $|\operatorname{Im} z^{3/2}| = -\operatorname{Re}(-z)^{3/2}$, one has

$$\frac{g_A(-z)}{1+|z|^{1/4}} = \frac{e^{|\operatorname{Im}\zeta|}}{1+|z|^{1/4}} \le \frac{e^{|\operatorname{Im}(\zeta+\frac{\pi}{4})|}}{|z|^{1/4}}$$

for all $z \in \mathcal{E}_k$ and $k \ge k_0$. By a well-known result (see [24, Ch. 2, Lemma 1]),

$$|w - n\pi| \ge \frac{\pi}{4} \implies e^{|\operatorname{Im} w|} < 4 |\sin w|$$

for all integer n. Hence,

$$\frac{g_A(-z)}{1+|z|^{1/4}} < 4 \frac{\left|\sin(\zeta + \frac{\pi}{4})\right|}{|z|^{1/4}} \tag{19}$$

for all $z \in \mathcal{E}_k$ and $k \ge k_0$. On the other hand, since $\left|\sin(\zeta + \frac{\pi}{4})\right| \ge d > 0$ for all $z \in \mathcal{E}_k$ and $k \ge k_0$, (17) implies

$$|\operatorname{Ai}(-z)| \ge \frac{\left|\sin(\zeta + \frac{\pi}{4})\right|}{\sqrt{\pi} |z|^{1/4}} \left|1 - \frac{|W_2(z)|}{\left|\sin(\zeta + \frac{\pi}{4})\right|}\right|.$$

However,

$$\frac{|W_2(z)|}{|\sin(\zeta + \frac{\pi}{4})|} \le \frac{e^{|\operatorname{Im} \zeta|}}{|\zeta|} \frac{D_2}{d},$$

and note that $|\operatorname{Im} \zeta| \leq \pi/2$ if $z \in \mathcal{E}_k$. Thus, by increasing k_0 if necessary, we have

$$|\text{Ai}(-z)| \ge \frac{\left|\sin(\zeta + \frac{\pi}{4})\right|}{2\sqrt{\pi}|z|^{1/4}},$$

for all $z \in \mathcal{E}_k$ with $k \geq k_0$.

The proof concerning \mathcal{E}^m is analogous: Suppose $m_0 = k_0$. Then, by the previous argument, (15) holds for $z \in \mathcal{E}^m$ within the sector $\arg(z) \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, for $m \geq m_0$. Within the sector $\arg(-z) \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$, we have $(\eta := \frac{2}{3}(-z)^{3/2})$

$$\frac{g_A(-z)}{1+|z|^{1/4}} \le \frac{e^{-\operatorname{Re}\eta}}{|z|^{1/4}}$$

and, due to (16),

$$|\operatorname{Ai}(-z)| \ge \frac{e^{-\operatorname{Re}\eta}}{2\sqrt{\pi}|z|^{1/4}} |1 - |W_1(-z)||.$$

Finally, using (18) —and increasing m_0 if required—, we have $1 - |W_1(-z)| \ge 1/4$ whenever $|z| \ge m_0$.

Theorem 3.5. Suppose $q \in L^1(\mathbb{R}_+)$. Then, the eigenvalues of H^D satisfy

$$\lambda_k^D = \left(\frac{3}{2}\pi(k - \frac{1}{4})\right)^{2/3} \left(1 + O(k^{-1})\right), \quad k \to \infty.$$

Proof. Abbreviate

$$\psi_0(z) := \psi_0(z,0), \quad \psi(z) := \psi(z,0), \quad \mu_k := -a_k.$$

Since $\sup_{z\in\mathbb{C}}\omega(z)<\infty$, Proposition 3.2 yields

$$|\psi(z) - \psi_0(z)| \le C\omega(z) \frac{g_A(-z)}{1 + |z|^{1/4}},$$

after redefining the constant C. Due to Lemma 3.1, there exists $k_1 \in \mathbb{N}$ such that $\omega(z) \leq (8C)^{-1}$ whenever $|z| \geq (\frac{3}{2}\pi(k_1 + \frac{1}{4}))^{2/3}$. Then, by Lemma 3.4, there exists $k_2 \geq k_1$ such that

$$|\psi(z) - \psi_0(z)| < |\psi_0(z)| \tag{20}$$

for all $z \in \mathcal{E}^{k_2}$; k_2 can be assumed large enough so \mathcal{E}^{k_2} encloses all the (finitely many) negative zeros of $\psi(z)$. Increase k_2 (if necessary) to ensure that (20) holds true for z on every contour \mathcal{E}_n whenever $n \geq n_2$. Then, in view of Rouché's theorem, we obtain

$$\left| \frac{2}{3} (\lambda_k^D)^{3/2} - \frac{2}{3} (-a_k)^{3/2} \right| \le \pi$$

for sufficiently large k, whence the asymptotics for the eigenvalues follows.

Theorem 3.6. Suppose $q \in L^1(\mathbb{R}_+)$. Then the Dirichlet norming constants ν_k^D satisfies

$$\frac{1}{\nu_k^D} = 1 + o(1)$$

as $k \to \infty$.

Proof. Abbreviate

$$\Delta_1(\lambda) := \frac{\psi'(\lambda, 0) - \psi'_0(\lambda, 0)}{\sqrt{\pi} \operatorname{Ai}'(-\lambda)}, \quad \Delta_2(\lambda) := \frac{\dot{\psi}(\lambda, 0) - \dot{\psi}_0(\lambda, 0)}{\sqrt{\pi} \operatorname{Ai}'(-\lambda)}.$$

It is straightforward to see that

$$-\frac{\dot{\psi}(\lambda_k^D, 0)}{\psi'(\lambda_k^D, 0)} = 1 - \frac{\Delta_1(\lambda_k^D)}{1 + \Delta_1(\lambda_k^D)} - \frac{\Delta_2(\lambda_k^D)}{1 + \Delta_1(\lambda_k^D)}$$

so it suffices to show that

$$\Delta_1(\lambda_k^D) \to 0$$
 and $\Delta_2(\lambda_k^D) \to 0$

as $k \to \infty$.

From Theorem 3.5 we obtain $\lambda_k^D = -a_k + O(k^{-1/3})$ thus

$$\sqrt{\pi} \operatorname{Ai}'(-\lambda_k^D) = (-1)^{k-1} \left(\frac{3}{2}\pi(k-\frac{1}{4})\right)^{1/6} (1+o(1)) = (-1)^{k-1} (\lambda_k^D)^{1/4} (1+o(1))$$

as $k \to \infty$. On the other hand, from (8), we have

$$\left| \psi'(\lambda_k^D, 0) - \psi'_0(\lambda_k^D, 0) \right| \le C\omega(\lambda_k^D) e^{C\omega(\lambda_k^D)} \left| \lambda_k^D \right|^{1/4}$$

hence the assertion on $\Delta_1(\lambda_k^D)$ holds true since $\omega(\lambda_k^D) \to 0$ as $k \to \infty$ due to Lemma 3.1. Finally, (13) implies the corresponding assertion on $\Delta_2(\lambda_k^D)$.

3.3 Neumann boundary condition

The analysis of the asymptotic behavior of $\sigma(H^N)$ does not differ much from the Dirichlet case. We start by defining the contours

$$\mathcal{F}^m := \left\{ z \in \mathbb{C} : |\zeta| = \left(m - \frac{1}{4}\right)\pi \right\}, \quad \mathcal{F}_k := \left\{ z \in \mathbb{C} : \left| \zeta - \left(k + \frac{1}{4}\right)\pi \right| = \frac{\pi}{2} \right\}, \quad m, k \in \mathbb{N}.$$

As expected, \mathcal{F}_k encloses exactly one zero of $\mathrm{Ai}'(-\lambda)$ for sufficiently large values of k.

Lemma 3.7. There exists $m_0, k_0 \in \mathbb{N}$ such that, for every $m \geq m_0$ and $k \geq k_0$, the following statement holds true:

$$(1+|x-z|^{1/4})g_A(-z) < 16\sqrt{\pi} |\operatorname{Ai}'(-z)|,$$

whenever $z \in \mathcal{F}^m$ or $z \in \mathcal{F}_k$.

The proof of this assertion is nearly identical to the proof of Lemma 3.4, except that it relies on the identities

$$\operatorname{Ai}'(z) = -z^{1/4} \frac{e^{-\zeta}}{2\sqrt{\pi}} [1 + W_3(z)], \quad |\arg(z)| \le \frac{2\pi}{3}, \quad |z| \ge 1,$$

$$\operatorname{Ai}'(-z) = \frac{z^{1/4}}{\sqrt{\pi}} \left[\sin \left(\zeta - \frac{\pi}{4} \right) + W_4(z) \right], \quad |\arg(z)| \le \frac{\pi}{3}, \quad |z| \ge 1,$$

where the functions $W_3(z)$ and $W_4(z)$ satisfy

$$\left|\frac{W_3(z)}{\zeta^{-1}}\right| \le D_1, \quad |\arg(z)| \le \frac{2\pi}{3}, \quad |z| \ge 1,$$

$$\left| \frac{W_4(z)}{\zeta^{-1} e^{|\operatorname{Im} \zeta|}} \right| \le D_2, \quad |\operatorname{arg}(z)| \le \frac{\pi}{3}, \quad |z| \ge 1.$$

The details are therefore omitted.

Theorem 3.8. Suppose $q \in L^1(\mathbb{R}_+)$. Then, the eigenvalues of H^N satisfy

$$\lambda_k^N = \left(\frac{3}{2}\pi(k - \frac{3}{4})\right)^{2/3} \left(1 + O(k^{-1})\right), \quad k \to \infty.$$

Proof. Since it is similar to the proof of Theorem 3.5, we only hint at the main departure from it. Recalling that $\sup_{z\in\mathbb{C}}\omega(z)<\infty$, (8) implies

$$|\psi'(z) - \psi_0'(z)| \le C\omega(z)(1+|z|^{1/4})g_A(-z)$$

for certain positive constant C. Because of Lemma 3.7, there exists $k_1 \in \mathbb{N}$ such that

$$|\psi'(z) - \psi_0'(z)| < |\psi_0'(z)|$$

for all $z \in \mathcal{F}^{k_1}$ and $z \in \mathcal{F}_k$ for every $k > k_1$, hence $\left| (\lambda_k^N)^{3/2} - (-a_k')^{3/2} \right| < \pi$ for all k large enough.

Theorem 3.9. Suppose $q \in L^1(\mathbb{R}_+)$. Then the Neumann norming constants ν_k^N satisfies

$$\frac{1}{\nu_k^N} = \left(\frac{3}{2}\pi(k - \frac{3}{4})\right)^{2/3} (1 + o(1))$$

as $k \to \infty$.

Proof. The argument goes along the lines of the proof of Theorem 3.6. Define

$$\Delta_3(\lambda) := \frac{\dot{\psi}'(\lambda,0) - \dot{\psi}_0'(\lambda,0)}{\sqrt{\pi}\lambda\operatorname{Ai}(-\lambda)}, \quad \Delta_4(\lambda) := \frac{\psi(\lambda,0) - \psi_0(\lambda,0)}{\sqrt{\pi}\operatorname{Ai}(-\lambda)}.$$

Then

$$\frac{\dot{\psi}'(\lambda_k^N, 0)}{\psi(\lambda_k^N, 0)} = \lambda_k^N \frac{1 + \Delta_3(\lambda_k^N)}{1 + \Delta_4(\lambda_k^N)}$$

so we only need to prove that

$$\Delta_3(\lambda_k^D) \to 0$$
 and $\Delta_4(\lambda_k^D) \to 0$

as $k \to \infty$. But this follows from (7) and (14).

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