Thermalization of a Trapped One-Dimensional Bose Gas via Diffusion

Alvise Bastianello,[1] Andrea De Luca,[2] Benjamin Doyon,[3] and Jacopo De Nardis[4]

[1]*Institute for Theoretical Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands*
[2]*Laboratoire de Physique Théorique et Modélisation (CNRS UMR 8089), Université de Cergy-Pontoise,*
F-95302 Cergy-Pontoise, France
[3]*Department of Mathematics, King's College London, Strand WC2R 2LS, London, United Kingdom*
[4]*Department of Physics and Astronomy, University of Ghent, Krijgslaan 281, 9000 Gent, Belgium*

For a decade the fate of a one-dimensional gas of interacting bosons in an external trapping potential remained mysterious. We here show that whenever the underlying integrability of the gas is broken by the presence of the external potential, the inevitable diffusive rearrangements between the quasiparticles, quantified by the diffusion constants of the gas, eventually lead the system to thermalize at late times. We show that the full thermalizing dynamics can be described by the generalized hydrodynamics with diffusion and force terms, and we compare these predictions to numerical simulations. Finally, we provide an explanation for the slow thermalization rates observed in numerical and experimental settings: the hydrodynamics of integrable models is characterized by a continuity of modes, which can have arbitrarily small diffusion coefficients. As a consequence, the approach to thermalization can display prethermal plateau and relaxation dynamics with long polynomial finite-time corrections.

In the past two decades, low-temperature gases of bosonic or fermionic atoms emerged as the best experimental platform from which to study many-body phenomena and probe their nonequilibrium dynamics [1–3]. A great deal of attention has been dedicated to the problem of thermalization in isolated systems, such as a single gas of bosonic atoms interacting via contact repulsion. The experiment in 2006 dubbed the "quantum Newton cradle" [4] represented a turning point. The apparent lack of thermalization at late times for a nonequilibrium one-dimensional cold atomic gas in an external trap started numerous research paths that focused on the role of the underlying integrability in the isolated dynamics. It was then later clarified that integrable models do not thermalize to standard Gibbs ensembles but to generalized Gibbs ensembles (GGE) [1,5,6] where entropy is maximized given the constraints of all local and quasilocal conserved quantities. As the integrability is weakly broken by the external trap, one therefore expects to cross over from a fast relaxation to a generalized Gibbs state to a slow prethermal decay to the final thermal ensemble. In the past years, the study of prethermalization dynamics has received many contributions, and, for homogeneous systems, a quite comprehensive understanding has now been achieved [7–17]. However, a full description of the (pre) thermalization mechanism in inhomogeneous systems is still lacking. While a recent experimental work showed that multiple one-dimensional gases coupled to each other or in which transitions to higher dimension are allowed do indeed thermalize at late times [18,19], the fate of a single isolated one-dimensional gas remains to be understood.

In this Letter, we show that, in the presence of an external force and diffusive spreading, a single gas described by an integrable Hamiltonian and an external trapping potential does thermalize. It does so due to the interplay of the external force that breaks integrability and the diffusion spreading that redistributes the quasiparticle momenta and increases the local thermodynamic entropy. Previous works have studied the hydrodynamics of an integrable gas in the presence of the external force but without accounting for diffusive terms [20–24]. Here, we show that diffusive corrections play an essential role in the late-time dynamics.

Lieb-Liniger gas in an external potential.—Our model of interest is the Lieb-Liniger (LL) model [25] for a gas of bosons interacting with contact repulsion. The Hamiltonian is given by

$$\hat{H}_{LL} = \int dx \left[\frac{\hbar^2 \partial_x \psi^\dagger(x) \partial_x \psi(x)}{2m} + c\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) \right],$$

with corresponding bosonic operators $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$. In the following, we set the mass of the bosons such that $\hbar^2/2m = 1$. The model is known to describe state-of-the-art cold atom experiments [4,26–33] and also to be integrable [25]. As a consequence, all physical initial states [34–39] eventually relax to a GGE density matrix $\hat{\varrho} \sim e^{-\sum_j \hat{Q}_j \beta_j}$, where the set of Lagrange multipliers β_j is fixed by the initial condition.

The large scale dynamics of a one dimensional bose gas can be accurately described via the so-called generalized hydrodynamics introduced in the past years [40,41], and the resulting predictions have even received an experimental confirmation [32]. In contrast to standard hydrodynamics, generalized hydrodynamics is built on the infinite number of local and quasilocal conserved quantities $\hat{Q}_j = \int dx \hat{q}_j(x)$ characterizing integrable systems. Equivalently, each stationary state of the system within fluid cells of the hydrodynamics can be described by a set of stable quasiparticles with density ρ_θ per unit rapidity θ and per unit length. Within a scattering theory description of the state, a quasiparticle at rapidity θ is identified with an asymptotic particle of velocity θ (see, e.g., [42]). The quantity ρ_θ is in a one-to-one correspondence with a set of expectation values of conserved quantities: $\int d\theta \rho_\theta h_{j,\theta} = \langle \hat{q}_j \rangle$, with $h_{j,\theta}$ the single-particle eigenvalues of the quasiparticle with rapidity θ.

Because of the interactions leading to scattering phases, the effective physical energy and momentum of quasiparticles depend nontrivially on the local thermodynamic state. Given the derivative of the scattering phase between two quasiparticles with different rapidities, $T_{\theta,\alpha}$, any single-particle function $h_{j,\theta}$ is dressed via the linear transformation $h_j^{\mathrm{dr}} = (\mathbf{1} - Tn)^{-1} \cdot h$, with $n_\theta = 2\pi\rho_\theta/(k'_\theta)$ the fermionic filling function and $(\mathbf{1} - Tn)_{\theta,\alpha} = \delta_{\theta,\alpha} - T_{\theta,\alpha}n_\alpha$ the dressing matrix (here and in the following we denote the derivative with respect to rapidities as $\partial_\theta h_\theta = h'_\theta$). The effective quasiparticle momentum k_θ and energy ε_θ are obtained by integrating the dressed derivatives: $k' = (\mathbf{1} - Tn)^{-1} \cdot (k^{\mathrm{bare}})'$ (and similarly for the energy). For the LL model, $T_{\theta,\alpha} = 2c/[(\theta - \alpha)^2 + c^2]$, and $(k_\theta^{\mathrm{bare}})' = 1$, $(\varepsilon_\theta^{\mathrm{bare}})' = 2\theta$.

Without external forces, the large scale evolution of any local density of conserved quantities can be written in terms of the evolution of the local density of quasiparticles $\rho_\theta(x, t)$. This involves a convective (Euler scale) and a diffusive part [43–47], and it reads $\partial_t \rho_\theta = -\partial_x(v_\theta^{\mathrm{eff}}\rho_\theta) + \partial_x(\mathfrak{D}_\theta^\alpha \partial_x \rho_\alpha)$, where the repeated rapidity index α is summed over (summation over repeated indices is assumed in what follows). The effective velocity $v_\theta^{\mathrm{eff}} = \varepsilon'_\theta/k'_\theta$ gives the convective motion of densities [48,49], while the nondiagonal diffusion kernel, which vanishes in noninteracting systems [50], describes both a diffusive spreading of the quasiparticle motion and a redistribution of quasiparticle velocities (see below). Both quantities are a nonlinear functional of the densities $\rho_\theta(x, t)$ fixed by the equation of state and by the explicit form of the diffusion kernel.

The effect of an external trap is described by the Hamiltonian

$$\hat{H}_{\mathrm{trap}} = \hat{H}_{\mathrm{LL}} + \hat{V} \tag{1}$$

with the potential $\hat{V} = \int dx V(x)\hat{q}_0(x)$ and the particle density $\hat{q}_0(x) = \psi^\dagger(x)\psi(x)$. The contribution of the

trapping potential to the dynamics of the quasiparticle densities can be treated perturbatively in the degree of smoothness of the external potential $V(x)$. It was first shown in Ref. [20] that, at the Euler scale, this amounts to adding the simple force term $[\partial_x V(x)]\partial_\theta\rho_\theta$. We here claim that the addition of this single term also gives the correct hydrodynamic equation up to diffusive scales:

$$\partial_t\rho_\theta = -\partial_x(v_\theta^{\mathrm{eff}}\rho_\theta) + \partial_x(\mathfrak{D}_\theta^\alpha\partial_x\rho_\alpha) + (\partial_x V(x))\partial_\theta\rho_\theta \tag{2}$$

That is, this equation holds up to order $O(L^{-3})$ corrections, with L the minimal length of spatial inhomogeneity of the gas.

From a scaling analysis, at this diffusive order possible corrections to Eq. (2) could come from three different sources: (i) the second derivative of the potential $\partial_x^2 V(x)$; (ii) mixed derivatives, $\partial_x V(x)\partial_x\rho_\alpha$; and (iii) a second power first derivative of the potential $[\partial_x V(x)]^2$. We now argue that none of these terms contributes. First, we recall that the definitions of densities of the charges $\hat{q}_j(x)$ are ambiguous by the addition of total derivative terms, and as shown in [51], we can make use of this freedom to enforce PT (parity and time) symmetry. With PT symmetry, we can argue that terms proportional to $\partial_x^2 V$ are excluded in Eq. (2). Specifically, consider the Heisenberg time-evolution equation for any conserved density $\hat{q}_j(x)$. In order to study their hydrodynamic evolution, we "Taylor expand" the potential around each point y, $V(x) = V(y) + (x - y)\partial_y V(y) + (x - y)^2\partial_y^2 V(y)/2 + \cdots$. We argue that the latter quadratic term does not contribute to Eq. (2), as $\int dx(x - y)^2 [\hat{q}_0(x), \hat{q}_j(y)]$ has a vanishing expectation on any stationary state by PT symmetry. Second, no mixed derivatives appear as the charge density $\hat{q}_0(x)$ is "ultralocal." That is, in terms of the field $\psi(x)$, $\hat{q}_0(x)$ does not contain any derivative, and, as a consequence, the number of particles does not generate any flow in space $[\int dx\hat{q}_0(x), \hat{q}_i(y)] = 0$. A change of flow due to a spatial variation of the local Hamiltonian would affect the diffusion kernel, this being the only source of the mixed derivatives. Because of ultralocality, no such change occurs. Third, there may be additional Fermi golden rule terms [13–15] leading to additional thermalization effects. These arise from the second-order terms in the perturbation for the evolution of the densities $\hat{q}_j(y)$. In our case, at given position y, the leading integrability-breaking perturbation is $\partial_y V(y)\hat{B}_y$, where $\hat{B}_y = \int dx(x - y)\hat{q}_0(x)$. This gives Fermi golden rule corrections to the evolution of $q_j(y)$ proportional to $(\partial_y V)^2 \int_{-t}^{t} ds[\hat{B}_y(s), [\hat{B}_y(t), \hat{q}_j(y, t)]]$ to be evaluated in a stationary, homogeneous state at position y. In the present case, we recognize $\hat{B}_y$ as a Galilean boost and therefore $d\hat{B}_y(s)/ds = \hat{P}$, the total conserved momentum. Thus, after integration by parts, we have, up to terms that vanish at large t, $\int_{-t}^{t} ds s\langle[\hat{P}, \cdots]\rangle = 0$, vanishing on any homogeneous local state. Any integrable model with these properties in a trapping potential is expected to obey Eq. (2), which is the focus of this Letter. For more details, see [52]. The general

case, along with a more rigorous demonstration, will be addressed in an upcoming publication.

Thermalization via diffusion.—Canonical thermalization to a Gibbs ensemble can only be reached at late times by breaking the conservation of all local conserved quantities, except for total energy and total particle number, and by ensuring that dynamics is irreversible. It can be shown that Gibbs ensembles are stationary solutions of the purely convective flow, the first and third terms on the right-hand side in Eq. (2) [52]. However, such an evolution is completely reversible [21,53] and does not generate entropy. In practice, the root density ρ_θ is roughened by the convective evolution and cannot relax. Relaxation may happen in a weak sense as local observables may become stationary [21,53] [see also Fig. 1(d)]. The determination of such a steady state is a nontrivial task. Physically, however, diffusion occurs, which modifies this picture. Diffusion, the second term in Eq. (2), increases entropy while preserving all conserved quantities. We will now show that the combined effect of diffusion together with the inhomogeneous potential $V(x)$ induces thermalization.

In order to proceed, we shall first write down the form of the diffusion matrix $\mathfrak{D}$. This is given by $\mathfrak{D} = R^{-1}\widetilde{\mathfrak{D}}R$, where $\widetilde{\mathfrak{D}}_{\theta,\alpha} = \delta_{\theta,\alpha}\int d\gamma (k'_\gamma/k'_\alpha)^2 W_{\gamma,\alpha} - W_{\theta,\alpha}$, with the matrix of dressings $R_{\theta,\alpha} = 2\pi(\delta_{\theta,\alpha} - n_\theta T_{\theta,\alpha})/k'_\theta$ with jump rates given by $W_{\theta,\alpha} = 1/2(2\pi T^{dr}_{\theta,\alpha}/k'_\theta)^2\rho_\theta(1-n_\theta)$ $|v^{eff}_\theta - v^{eff}_\alpha|$. The only conserved quantity T^{dr} is given in terms of the bare one by $T^{dr} = (\mathbf{1} - Tn)^{-1} \cdot T$. The kernel $\widetilde{\mathfrak{D}}_{\theta,\alpha}$ has the structure of a Markov matrix, and it describes the emergent stochastic process of random momentum exchanges between quasiparticles with jump rates given by $W_{\theta,\alpha}$ and where the only conserved quantity in the process is the total quasiparticle momentum, as $\int d\theta \mathfrak{D}^\alpha_\theta = 1^\theta \mathfrak{D}^\alpha_\theta = 0$, as it follows from $1^\theta[R^{-1}]_{\theta,\alpha} = (k'_\alpha)^2/(2\pi)$. The simple fact that the unit vector 1_θ is the *only* zero eigenvector of the diffusion operator $\mathfrak{D}$ ensures that the dynamics induced by Eq. (2) with $V'(x) \neq 0$ relaxes to a thermal ensemble. Namely, the large time limit of $\rho_\theta(x,t)$ in Eq. (2) is such that, for any local operator $\hat{\mathcal{O}}$ acting at position x, whose average is, by the hydrodynamic approximation, $\langle\hat{\mathcal{O}}(x,t)\rangle = \int d\theta \rho_\theta(x,t)h^{\mathcal{O}}_\theta$ for some functional $h^{\hat{\mathcal{O}}}_\theta$, we have $\lim_{t\to\infty}\langle\hat{\mathcal{O}}(x,t)\rangle = \text{Tr}[\hat{\mathcal{O}}\hat{\varrho}_{th}]$, with the thermal density matrix $\hat{\varrho}_{th} \propto \exp[-\beta(\hat{H}_{trap} - \mu\hat{Q}_0)]$. The final inverse temperature β and chemical potential μ are fixed by the initial total number of particles and energy.

To prove this fact, one first notices that the total density $N = \int dx \int d\theta \rho_\theta(x,t)$ and total energy $E = \int dx \int d\theta \rho_\theta(x,t)[\varepsilon^{bare}_\theta + V(x)]$ are preserved by the evolution, Eq. (2). The latter can be easily shown using that $v^{eff}_\theta = (k'_\theta)^{-1}[(\mathbf{1}-Tn)^{-1}(\varepsilon^{bare})']_\theta$. Then one proceeds to notice that the diffusive term in Eq. (2) produces an increase of the total thermodynamic entropy of the state
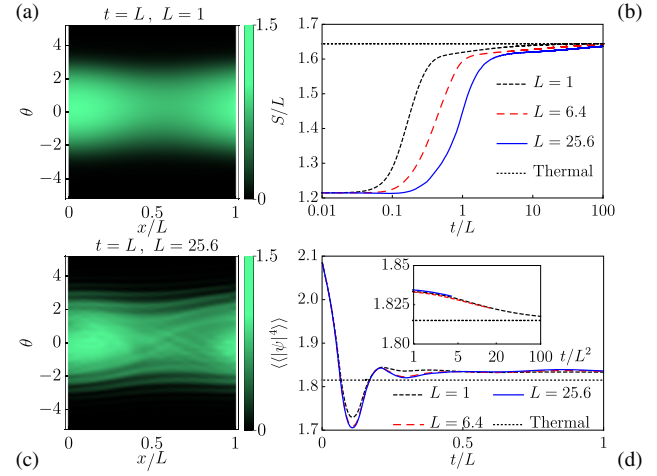

FIG. 1. Left: (a,c) quasiparticle density $\rho_\theta(x,t)$ at $t = L$ for a Lieb-Liniger gas, Eq. (1), inside the trap $V(x) = 1/2(1 - \cos 2\pi x/L)$, in a circle with length $L = 1$ (a) and $L = 25.6$, (c), initialized at finite temperature equilibrium in the potential $V_0(x) = 2\sin(2\pi x/L)$ (see text for the values of the other parameters). Right: Log plot of the thermodynamic entropy density $\mathcal{S}(t)/L$ (b) converging to the thermal entropy for the given quench and of the integrated correlator, (d)$\langle\langle|\psi^4|\rangle\rangle = \int_0^L dx\langle(\psi^\dagger)^2(\psi)^2\rangle/L$. The inset is a log plot showing the approach to the thermal value at times of order L^2.

$S = \int dx \sum_j \beta_j(x)\langle\hat{q}_j(x)\rangle - f$, with f being the total free energy. The entropy growth on a generic GGE state was found in [42,51], and it can be written as

$$\frac{dS(t)}{dt} = \frac{1}{2}\int dx \sum_{j,k}[\partial_x\beta_j(x)]h^\theta_j(\mathfrak{D}C)_{\theta,\alpha}h^\alpha_k[\partial_x\beta_k(x)], \quad (3)$$

with the matrix of susceptibilities $[RCR^{-1}]_{\theta,\alpha} = \delta_{\theta,\alpha}\rho_\theta(1-n_\theta)$. As the only zero eigenvalue of the $\mathfrak{D}$ operator is the unit vector, the only states where entropy does not increase are the ones where all $\beta_{j>0}$ are constant in space, namely any GGE state with an arbitrary spatially modulated chemical potential $\sim \exp[-\sum^m_{j=1}\hat{Q}_j\beta_j - \int dx\tilde{\mu}(x)\hat{q}_0(x)]$, with all β_j constants in space. However, for $m > 1$ and generic $\tilde{\mu}(x)$, these states are not invariant under the convective part of Eq. (2) when $V(x) \neq 0$, and therefore they are not stationary. We conclude therefore that thermal states with $\tilde{\mu} = \mu - V(x)$ and $\beta_1 = \beta$ are the only stationary solutions of Eq. (2) (see [52] for more details).

Results in the Lieb-Liniger gas.—Equation (2) is a typical convection-diffusion equation. A common way to numerically solve these types of equations is the Crank-Nicolson method [54] [52]. The role of the diffusion term in Eq. (2) is to smooth the functions $\rho_\theta(x,t)$ in the x space and, primarily, in the θ space (see Fig. 1), a fact that was previously noticed in [55] (see also [21,22]).

We first study the following nonequilibrium settings: we consider periodic boundary conditions in $x \in [0, L]$. We prepare the gas in the thermal state with inverse temperature $\beta = c = 0.3$ and with chemical potential given
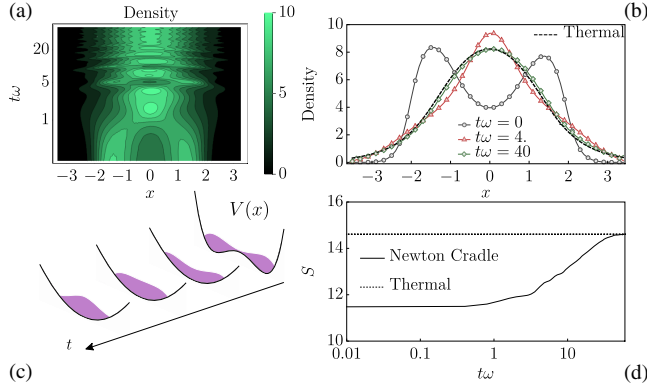
FIG. 2. Study of the dynamics predicted by Eq. (2) for a release of a cold atomic gas into a harmonic potential $V(x) = 2x^2$ from a double-well potential (see text for the values of all parameters). (a) Plot of the density profile $\langle \hat{q}_0(x,t) \rangle$ as a function of t and x and (b) at different times compared to its value in the thermal Gibbs ensemble. (d) Global thermodynamic entropy as a function of time reaching the thermal value at late times. (c) Cartoon picture of the experimental setting.

by $\mu - V_0(x)$, with $\mu = 2.5$ and $V_0 = 2\sin(2\pi x/L)$, such that $\langle \psi^\dagger \psi \rangle \sim O(1)$, namely the effective coupling $\gamma(x) = c/\langle \psi^\dagger \psi \rangle \sim O(10^{-1})$ for $x \in [0, L]$. At $t = 0^+$, we switch on the potential $V(x) = 1/2[1 - \cos(2\pi x/L)]$. In Eq. (2), one is free to rescale $x \to xL$ and $t \to tL$. This way the convective and the force term remain unaltered, but the diffusive part is rescaled by a factor $1/L$. In Fig. 1, we show the increase of the total entropy after the quench in the cases $L = 1$ and $L = 25.6$. For the larger value of L indeed we observe a prethermal plateau for the evolution of local observables (see Fig. 1). As discussed above, such a plateau corresponds to the stationary state attained by the purely convective evolution. The slow drift can be interpreted as the effect of the spatial imbalance of modes with small diffusion constants, which produces a diffusive small entropy increase in Eq. (3) and which is even more suppressed by larger values of L. Indeed, Fig. 1 clearly shows how short-intermediate timescales $t \sim L$ are dominated by the convective dynamics, while diffusive thermalizing dynamics takes place at scales $t \sim L^2$. Such a "prethermal plateau" was indeed also observed in numerical simulation of a trapped hard-rods gas in [22], and it is here therefore theoretically explained.

We then proceed to study a nonequilibrium setting that is more relevant for cold atoms experiments (see Fig. 2). We prepare a gas of bosons in a double-well potential as in Ref. [32]. We consider an initial gas with $c = 0.5$ at thermal equilibrium with temperature $T = 10$ and $\mu = 4$ inside the double-well trap $V_0(x) = x^4 - 4x^2 + x^3/10$. We then release the gas in the harmonic trap $V(x) = \omega x^2/2$ with $\omega = 4$. The initial density and temperature at the minima of the trap $V_0(x)$ is such that $\gamma \sim O(10^{-2})$ and $T/T_d \sim O(10^{-1})$, with quantum degeneracy temperature [56] $T_d = \langle \psi^\dagger \psi \rangle^2$ (in units $\hbar^2/2m = 1$ and Boltzmann's

constant $k_B = 1$), in order to be within typical experimental regimes.

By simulating the dynamics of the gas by Eq. (2), we find that diffusion eventually suppresses the oscillations of the gas inside the harmonic trap and global entropy reaches thermal equilibrium after around 50 oscillations inside the trap, much longer than the accessible experimental times in atom chips [32].

Finally, we stress that diffusive terms in Eq. (2) are roughly proportional to $\gamma^{-2} n_\theta(1 - n_\theta)$ and therefore thermalization times, which grow as the inverse of this, can become arbitrarily large as interaction γ and/or inverse temperature β are increased, as for example in the experimental setting in [4].

Numerical simulations in the classical limit.— Continuous quantum models are notoriously hard to simulate with state-of-the-art tensor network techniques despite recent progress [57–59]; methods based on the Bethe ansatz reach a higher number of particles but are still limited [45]. We therefore benchmark the hydrodynamic equation, Eq. (2), with the classical field limit of the Lieb-Liniger gas, valid in the high mode occupation limit. Given the rescaling of the coupling $c = \hbar$, inverse temperature $\beta = \beta_{NLS}\hbar$, and the mass such that $\hbar^2/(2m) = 1$, in the limit $\hbar \to 0$ the quantum system is well described by the classical nonlinear Schrödinger model (NLS) [60] (see also [61–63]) initialized in an inhomogeneous thermal ensemble and evolving with deterministic equations of motion. Here, extensive numerical simulations can be carried out [64–69]. We numerically solve Eq. (2) for the Lieb-Liniger gas, and we extrapolated the behavior at $\hbar = 0$ to compare with numerical Monte Carlo simulations of the NLS equation [70] (see Fig. 3). We consider the same quench of Fig. 1
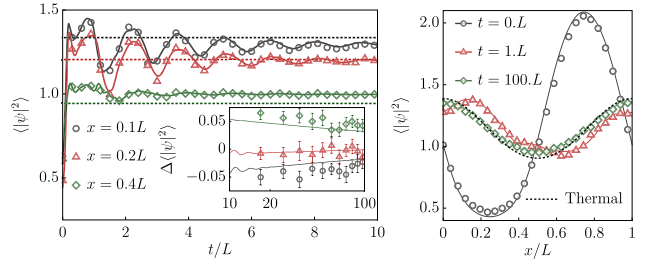

FIG. 3. Comparison between the predictions of the hydrodynamic Eq. (2) (continuous lines) and Monte Carlo numerical simulation of the NLS model (symbols, error bars are reported only in the enlarged inset since they are negligible on the plot scale) for the same quench as in Fig. 1 with $L = 25.6$. Dotted lines are the final values at thermal equilibrium. Left: evolution of the density profile $\langle |\psi|^2 \rangle = \langle (\psi^\dagger \psi)(x, t) \rangle$ at some given positions x as function of time. Inset shows the (diffusively) vanishing difference from the final thermal expectation value as function of time $\Delta \langle |\psi|^2 \rangle \equiv \langle (\psi^\dagger \psi)(x, t) \rangle - \langle (\psi^\dagger \psi)(x) \rangle_{GE}$. Right: density profile for all $x \in [0, L]$ at different times. The numerical simulations have been performed with the same method used in Ref. [70].

with $L = 25.6$ in order to suppress extra effects to the dynamics due to corrections in Eq. (2) with higher derivative terms. Due to the large value of L, the thermalization timescales are large (see Fig. 1). Nevertheless, we observe a small drift toward the thermal value for the profile of bosonic density, correctly reproduced by Eq. (2).

Conclusion.—In this Letter, we have shown that the hydrodynamic equation, Eq. (2), including diffusion and an external force, has Gibbs thermal states as the only stationary attractor states. This implies that interparticle diffusion is the leading mechanism for thermalization in the presence of integrability-breaking terms, and, as expected, no thermalization is present in the noninteracting limit of hard-core bosons [71–73] where there is no diffusion [43,44,50]. This is in agreement with previous works on perturbed classical integrable models [74,75] where it was noticed that, while chaotic behavior is achieved at short times in many-body systems close to integrability, thermalisation occurs after having diffusively spread within the classical phase space, and it manifests itself on much longer timescales (see also [76] for a previous study on the Bose gas and [77] in the Gross-Pitaevskii equation). Moreover, in integrable models, there exists an infinite number of modes, in particular the ones with large momentum, with the arbitrarily small diffusion constant $\mathfrak{D}_\theta^\alpha$. Therefore, local observables that couple to some density of quasiparticles thermalize with diffusive thermalization drifts whose effective diffusion constants are not bounded from below and can take quite small values, therefore leading to very large observable-dependent thermalization times. Our work gives a comprehensive explanation for the apparent lack of thermalization on finite timescales observed in experimental [4] and numerical [22] settings. It also provides a quantitative prediction for the timescales necessary to see thermalization up to a generic precision. Moreover, while we here focus on a purely one-dimensional gas, several extensions of Eq. (2) can be introduced to account for experimental effects such as losses [78], dephasing noise [70], and crossover to three-dimensional geometries [19,79]. Finally, it is an interesting question for the future to relate the growth of thermodynamic entropy of Eq. (3) to the well-known growth of entanglement entropy after quantum quenches [80–85].

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