# One-dimensional Bose gases: cold atoms as quantum simulators

Isabelle Bouchoule

CAPS Shool on ultracold atoms
Barcelona, November 2022

# 1D gazes in cold atoms experiments

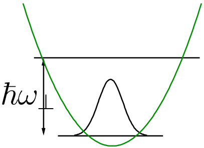
- Phycics very different from that in 2D and 3D
- Model system of quantum many-body systems : exact solutions, numerical techniques
- Occur in “real” life : carbone nano-tubes, nano-wires, elongated Josephson junctions, edge states ...

Cold atoms added value

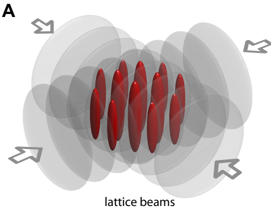
- Well controled systems
- Highly controlable
- Many different probes
- Dilute and ultra-cold systems : simple modelisation of the interactions

# Reaching 1D regime in cold atoms experiments

Cooling transverse degrees of freedom
Typical energy per atom $E/N \ll \omega_\perp$

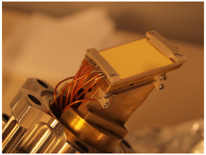



Tubes of a 2D optical lattice




From Haller et al (2009)

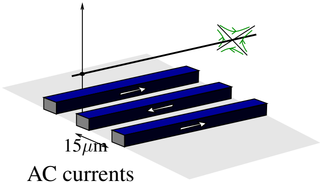
A single 1D gas on a chip



From Palaiseau

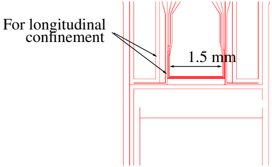
# Experimental setup : trapping atoms with an atom-chip

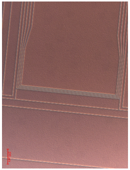
- Magnetic confinement of $^{87}$Rb atoms : $V = \mu_B|\mathbf{B}|$
- Cu micro-wires deposited on a substrate




AC currents
$\omega_\perp = 5 - 8$kHz

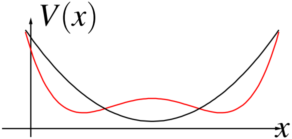
Chip design (wire edges shown)

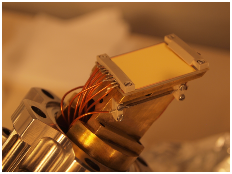


- Longitudinal confinement :

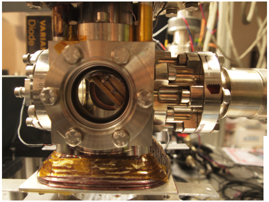
2 pairs of wires $\rightarrow$ control of terms $x, x^2, x^3, x^4$

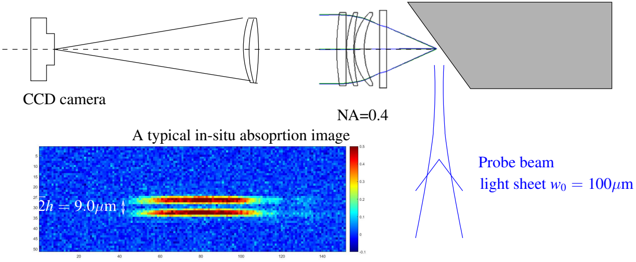


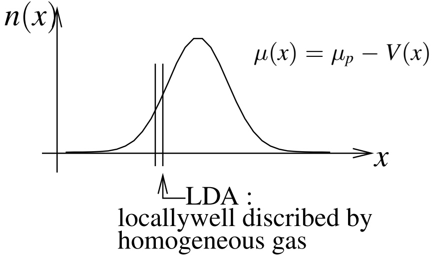
Fabrication : LPN, CNRS, S. Bouchoule

# Experimental setup : realizing and imaging 1D gases



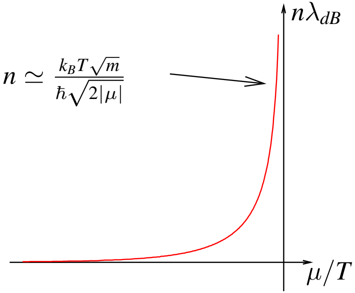



- $\mu/k_B \simeq 50 - 100$ nK
- $T \simeq$300 nK
- $\mu, T < \hbar\omega_\perp \Rightarrow$ 1D effective gas

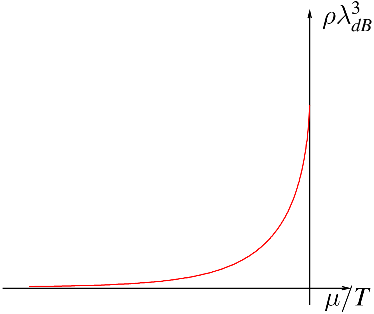


# Absence of Bose Einstein condensation in 1D

Gaz 1D




Gaz 3D




No BEC in an ideal 1D gas

Physics governed by interactions

Important role of fluctuations

# Outline

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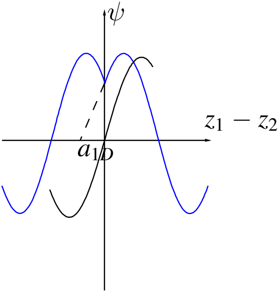
# Contact interactions in 1D

2 particules of mass $m$. Interaction : $V(z_1 - z_2) = g\delta(z_1 - z_2)$

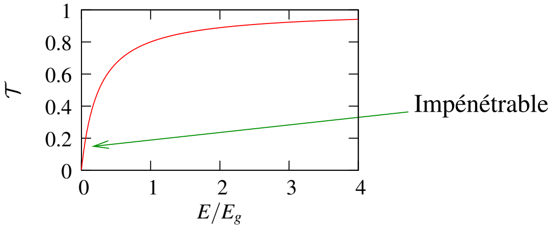



Boundary condition :
$\partial_z\psi(0^+) - \partial_z\psi(0^-) = \frac{mg}{\hbar^2}\psi(0)$

Symetric case : $\partial_z\psi(0) = -\psi(0)/a_{1D}$,
$a_{1D} = -2\hbar^2/(mg)$.
$E_g = mg^2/\hbar^2$

**Distinguishable particles : Transmission probability**

$\psi_k(z) = e^{ikz} + fe^{ik|z|}$,
energy : $E = \hbar^2k^2/m$
$\mathcal{T} = |1+f|^2$
$\mathcal{T} = \frac{\hbar^2k^2/m}{\hbar^2k^2/m + E_g/4}$



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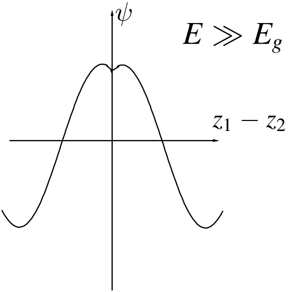


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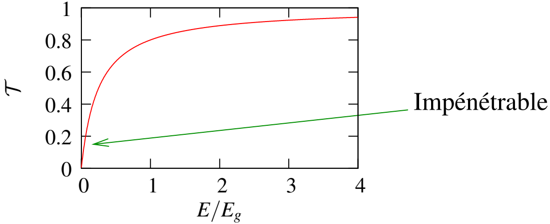

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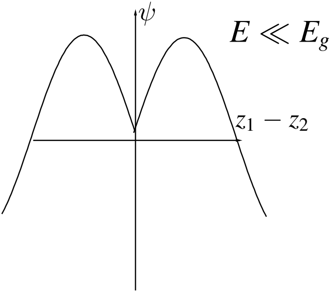


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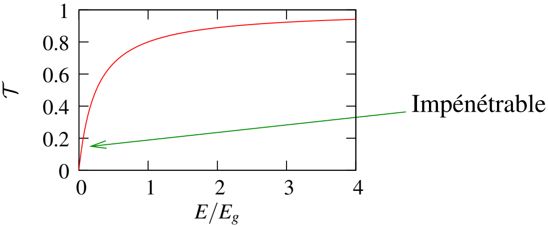
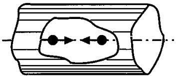


# From 3D to 1D

Transverse confinement : $\omega_\perp$, $a_\perp = \sqrt{2\hbar/(m\omega_\perp)}$

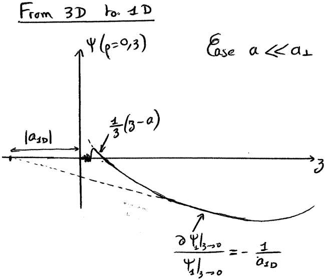
Energy $< 2\hbar\omega_\perp$

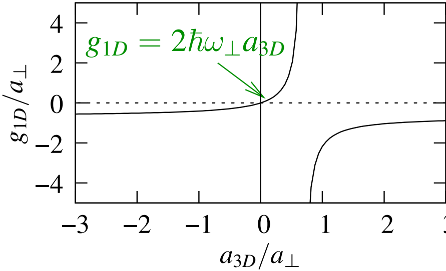
- $r \gg l_\perp$ : $\psi \simeq \phi_0(x,y)\cos(k(|z| - a_{1D})$
- $r \ll l_\perp$ : 3D-divergence $\psi(r) \underset{r\to 0}{\propto} (a_{3D}/r - 1)$



$$\boxed{a_{1D} = -\frac{a_\perp^2}{2a_{3D}}\left(1 - Ca_{3D}/a_\perp\right),}$$

$C = 1.46..$




Solved by M.Olshanii :

Phys, Rev. Lett., **81**, 938 (1998)

$$g_{1D} = -\frac{2\hbar^2}{ma_{1D}}$$

# Experimental observation of the CIR

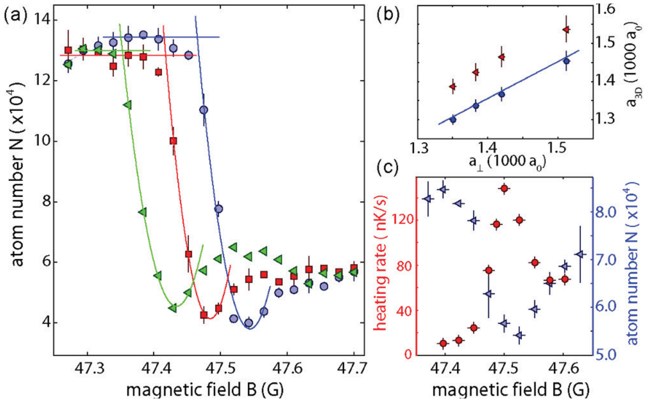


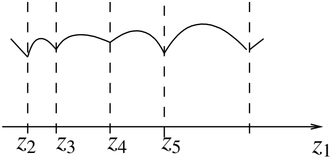

FIGURE – *CIR observed via atomic losses, taken from E. Haller et al. Phys. Rev. Lett., 104 :153203, (2010)*

# Outline

# ground state of N particules : hand-waving approach

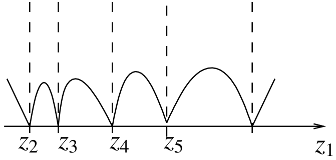
N identical bosons of mass $m$, density $n$.

- Weak correlations




Energy : $E/N \simeq E_{\text{int}} \simeq gn$

- Strong correlations




Energy : $E/N \simeq E_{\text{kin}} \simeq \hbar^2 n^2/m$

Highly correlated ( hard-core) : $\hbar^2 n^2/m \ll gn \Rightarrow \boxed{\gamma = mg/(n\hbar) \gg 1}$

3D : opposit case

Highly correlated regime :

- Excited states : $\psi(z_i = z_j) \simeq 0$ as long as $(E/N) \ll E_g$.
- Fermionization : bijection with a fermionic gas (Peculiar to 1D )
  $\psi_{\text{B}} = \prod_{j<i} sgn(x_j - x_i)\psi_{\text{F}}(z_1, ..., z_N)$
- Fermionization also applicable with an external potential

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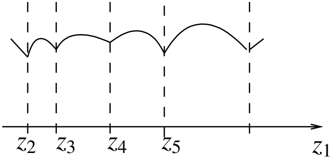


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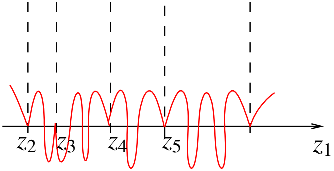


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# N particules : Bethe Ansatz

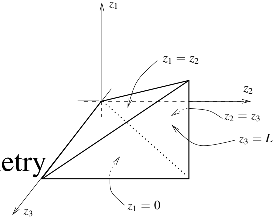
Lieb-Liniger hamiltonian (1963) : $\boxed{H = \sum_i -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z_i^2} + g\sum_{i<j}\delta(z_i - z_j)}$

Sub-space $z_1 < z_2 < ... < z_N$.

Plane wave $e^{ik_1 z_1} e^{ik_2 z_2} ... e^{ik_N z_N}$

Bethe-Ansatz : $\psi = \sum_\sigma a(\sigma) e^{ik_{\sigma(1)} z_1} e^{ik_{\sigma(2)} z_2} ... e^{ik_{\sigma(N)} z_N}$

- $\sigma$ et $\nu$ : exchange of rapidities between $i$ and $i+1$
  Boundary condition at $z_{i+1} = z_i$ and bosonic symmetry
  $\Rightarrow a(\sigma) = -a(\nu) e^{2i\,\mathrm{Atan}(\hbar^2(k_{\sigma(i)} - k_{\sigma(i+1)})/mg)}$.




Same factor for different transposition combinaisons : Yang-Baxter

$$\boxed{a(\sigma) \propto \prod_{i<j}\left(1 + \frac{i\,gm/\hbar^2}{k_{\sigma(j)} - k_{\sigma(i)}}\right) \propto (-1)^{|\sigma|}\prod_{i<j}(k_{\sigma(j)} - k_{\sigma(i)} + i\,gm)}$$

$$\psi(\{z_i\}) = \sum_\sigma \prod_{i<j}\left(1 + \frac{i\,gm\,\mathrm{sgn}(z_i - z_j)}{\hbar^2(k_{\sigma(j)} - k_{\sigma(i)})}\right) e^{i\sum k_{\sigma(i)} z_i}$$

Total momentum : $\sum_i k_i$, Energy : $E = \sum_i k_i^2/2$

# Rapidities : asymptotic momenta after expansion

- Hand-waving argument

$$\psi(x_1,\dots,x_N) \propto \underbrace{e^{i(k_1x_1+k_2x_2+\dots)}}_{\text{Incoming wave}} + \sum_{\sigma\neq \mathrm{Id}} e^{i\Phi(\sigma)} \underbrace{e^{i(k_{\sigma(1)}x_1+k_{\sigma(2)}x_2+\dots)}}_{\text{out--going wave}}$$

- Wave-packet point of view (Campbelle et al. 2015)

$$|\psi_0\rangle = \int_{\theta_1>\theta_2>\dots>\theta_N} d\theta_1 d\theta_2 \dots d\theta_N \, \langle\{\theta_a\}|\psi_0\rangle \, |\{\theta_a\}\rangle$$

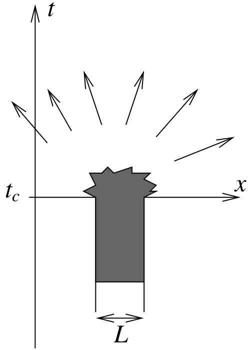
Wave function :

$$\psi(t;x_1,\dots,x_N) \propto \int d\theta_1\dots d\theta_N \, \langle\{\theta_a\}|\psi_t\rangle \, e^{i\sum_a x_a\theta_a} e^{-it\sum\theta_a^2/2}$$

**Large time : stationnary phase approximation**

$$P_{\exp}(x_1,\dots,x_N) = \frac{1}{t_{\exp}^N} \left|\langle\{x_1/t_{\exp},\dots,x_N/t_{\exp}\}|\psi_0\rangle\right|^2$$

# Asymptotic momenta as defining rapidities




**Rapidities**

Expansion to infinity

Asymptotic momenta = **rapidities**

$\Pi(k)dk =$# rapidities in $[k, k + dk]$

Integrability of Lieb-Liniger

$\rightarrow \Pi(k)$ conserved (independent on $t_c$)

**Defining the rapidities as the asymptotic momenta**

- Gives an experimental protocol to measure them
- Implemented experimentally (Wilson et al. (2020))

# experimental measure

"Observation of dynamical fermionization" : Wilson et al., Science 27 Mar 2020 : Vol. 367, Issue 6485, pp. 1461-1464

## Experimental parameters : 1D gases in Tonks regime

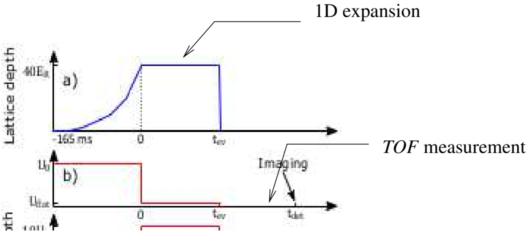
Rubidium 87 BEC in a crossed dipole trap
Atoms loaded into blue detuned 2D-optical lattice (depth 40 $E_R$)
$\Rightarrow$ undependant 1D tubes

- $N_{\text{at/tube}} = 0 - 26$
- $\gamma = mg/(\hbar^2 m) = 4.2 - \infty, \bar{\gamma} = 8.5$
- Longitudinal trapping : controled by the crossed dipole beams. Initial value : $\omega_z/(2\pi) = 18$ Hz

## Numerics

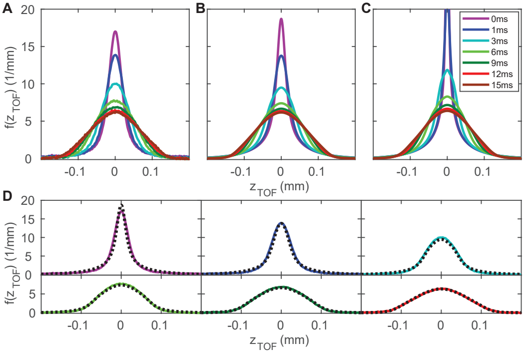
Assume hard-core Bosons. Discretises space.
Use Jordan-Wigner : $a_{i,\text{Bosons}} = (-1)^{N_{[0,i]}} c_{i,\text{Fermions}}$

# Experimental sequence




- **1D expansion during $t_{\mathrm{ev}}$ :**
  $\Rightarrow$ momentum distribution converges for long enough $t_{\mathrm{ev}}$ to rapidity distribution
- **Measure of momentum distribution using time-of-flight :**
  - Switch off of transverse confinement $\rightarrow$ cut-off of interactions
  - Balistic flight : at large time $n(x) \propto \bar{n}(v = x/t_{\mathrm{TOF}})$ .
    $t_{\mathrm{TOF}} = t_{\mathrm{det}} - t_{\mathrm{ev}} = 70\mathrm{ms} - \mathrm{t}_{\mathrm{ev}}$

# Results




**A** : Experimental results **B** : Full numerical simulation

**C** : Numerical calculation of momentum distribution evolution during $t_{\mathrm{ev}}$ (shown is $\bar{n}(v = z_{\mathrm{TOF}}/t_{\mathrm{det}})$)

Descripancy **C** and **A** : initial size ($t_{\mathrm{det}}$ not long enough), finite resolution (4.8 $\mu$m)

**D** : Comparison **A** and **B**

# Outline

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# Bethe Ansatz with periodic boundary condition

## Periodic boundary conditions

$$e^{ik_aL}\prod_{b\neq a} e^{i\phi(k_a-k_b)} = (-1)^{N-1}, \qquad a = 1,\dots,N$$

Bethe equations :

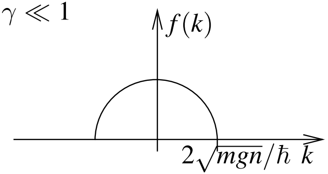
$$k_a + \frac{1}{L}\sum_{b\neq a} 2\arctan\left(\frac{k_a-k_b}{c}\right) = p_a, \text{ where } \begin{cases} p_a \in \frac{2\pi}{L}\mathbb{Z} & N \text{odd} \\ p_a \in \frac{2\pi}{L}(\mathbb{Z}+\frac{1}{2}) & N \text{even.} \end{cases}$$

$p_a$ : Bethe-Fermions

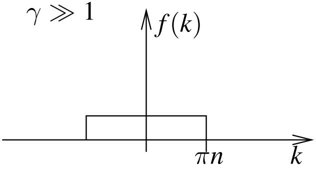
Ground state : $p_a = (-N/2, -N/2+1, \dots, N/2)2\pi/L$

Rapidities distribution : $f(k)$

$\gamma \ll 1$




$\gamma \gg 1$

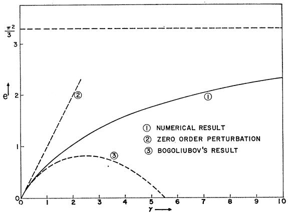



$E/N = g\rho/2$

$E/N = \rho^2\frac{\pi^2}{3}$

# Ground state properties

- Ground state energy



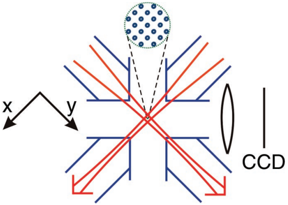

$$\gamma \ll 1 \quad , E \simeq Ngn/2$$
$$\gamma \gg 1 \quad , E \simeq E \simeq N\frac{\pi^2}{3}n^2$$

- Zero-distance correlation function $g^{(2)}(0) = \langle \psi^+ \psi^+ \psi \psi \rangle / n^2$
  Helman-Feynman theorem : $g^{(2)}(0) = \frac{2}{Ln^2}\frac{dE}{dg}$

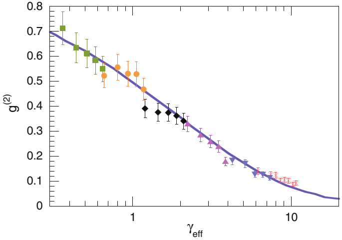
$$g^{(2)}(0) \underset{\gamma \to \infty}{\to} 0$$

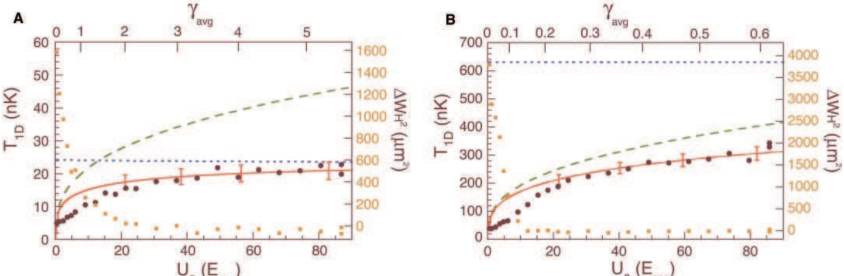
# Bethe-Ansatz : résultats expérimentaux

Kinoshita et al., Science **305** , 1125 (2004), Kinoshita et al., Phys. Rev. Lett. **95**, 190406 (2005)




Photoassociation measurement



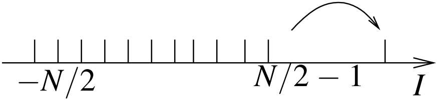
# Ansatz de Bethe : excited states

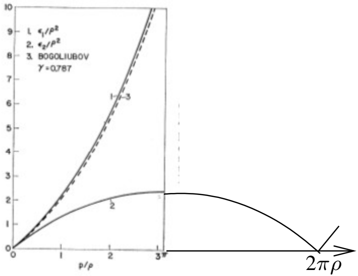
Excitations : modification of the distribution $\{p_a\}$

2 branches of elementary excitations :

Type I :

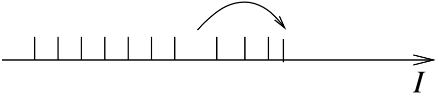
$\{p_a\}\frac{L}{2\pi} =$
$-N/2, -N/2+1, ..., N/2-1, N/2+p$




Type II :

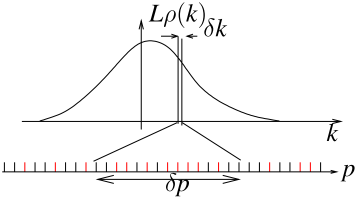
$\{p_a\}\frac{L}{2\pi} = -N/2, ..., j, j+2, N/2+1$



# Outline

# Rapidity distribution and Thermodynamic Bethe-Ansatz

Very large system :
$N$ large, many rapidities in each small interval $k, k+dk$




$$\boxed{\rho(k) = \frac{1}{L}\frac{\#\text{ rapidities in }[k,k+\delta k]}{\delta k}}$$

Density of available rapidities : $\rho_s(k) = \frac{1}{2\pi}\frac{\delta p}{\delta k}$

$$\rho_s(k) = \frac{1}{2\pi}\left(1 + g\int dk' \frac{\rho(k')}{g^2+(k-k')^2}\right)$$

**Occupation factor :** $\nu(k) = \rho(k)/\rho_s(k)$

**Generalized Eigenstate Thermalization Hypothesis**

All micro-states with the same coarse-grained rapidity distribution identical with respect to local observables

Local observable : 0

$$\boxed{\langle\{k_i\}|O|\{k_i\}\rangle = \langle O\rangle_{[\rho]}}$$

# Relaxation in the Lieb-Liniger model

Out-of-equilibrium state : $$|\psi\rangle = \sum_{\{k_i\}} c_{\{k_i\}} e^{i\sum_i k_i^2/2t} |\{k_i\}\rangle$$

$c_{\{k_i\}}$ peaked on $\{k_i\}$ whose distribution is close to $\rho(k)$.

Local observable $O$ : $$\langle O\rangle \underset{t\to\infty}{\to} \sum_{\{k_i\}} |c_{\{k_i\}}|^2 \langle O\rangle_{\{k_i\}} = \langle O\rangle_\infty$$

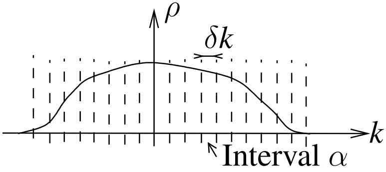
$$\text{GETH} \Rightarrow \boxed{\langle O\rangle_\infty = \langle O\rangle_{[\rho]}}$$

**System after relaxation described only by $\rho(k)$**

Any diagonal ensemble peaked around $\rho(k)$ can serve to describe local properties of the system

- A single Bethe-Ansatz state
- A GGE in terms of the Bethe-Fermions $\{p_i\}$
- A GGE in terms of rapidities

# GGE in terms of rapidities




## Quasi-locallity

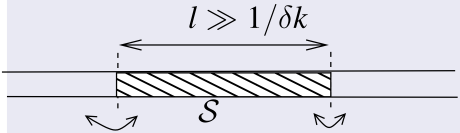
$Q_\alpha = \#$ rapidities in $[k_\alpha, k_\alpha + \delta k]$

$Q_\alpha = \int dx\, q_\alpha(x)$

$\uparrow$

extends over $\simeq 1/\delta k$

## Small sub-system




$Q_\alpha = Q_\alpha^{(\mathcal{S})} + Q_\alpha^{(\mathrm{res})}$

For each $\alpha$ :

rest of system = reservoir of rapidities

GGE for $\mathcal{S}$ : $p_{\{k_i\}} \propto e^{-\sum_\alpha f_\alpha \langle Q_\alpha^{(\mathcal{S})}\rangle_{\{k_i\}}}$, $\{f_\alpha\}$ : Lagrange multiplyiers

## GGE : generalized Lagrange multiplyier function

$p_{\{k_i\}} \propto e^{-\sum_i f(k_i)}$ Can be sampled by Monte-Carlo

# Link between the $f$-function and $\rho(k)$

General thermodynamic relation : $$f_\alpha = \frac{\partial S}{\partial Q_\alpha}, \qquad f(k) = \frac{\delta}{\delta \rho} S$$

Entropy $S = \log(\#$ Bethe-Ansatz states of rapidity distribution $\rho(k))$
Computed by Yang and Yang :

$$S[\rho] := \int_{-\infty}^{\infty} (\rho_s \log \rho_s - \rho \log \rho - (\rho_s - \rho) \log(\rho_s - \rho))\, dk.$$

$$f(\theta) = \log\left(\frac{\rho_s(\theta)}{\rho(\theta)} - 1\right) - \int \frac{d\theta'}{2\pi} \Delta(\theta - \theta') \log\left(1 - \frac{\rho(\theta')}{\rho_s(\theta')}\right)$$

$$\Delta(\theta - \theta') = \frac{2mg/\hbar}{(mg/\hbar)^2 + (\theta - \theta')^2}$$

# Outline

# Relaxation in a chaotic system : thermal state

## Chaotic system : Eigenstate Thermalization Hypothesis

Local observable $O$

For an eigenstate $|i\rangle$ : $\langle i|O|i\rangle = \mathcal{O}(E_i, N_i)$

All eigenstates with same $E$, same $N$ (within small width), identical

## Relaxation towards a thermal state

After relaxation, local properties of the system defined by $E, N$.

System well described by any peaked diagonal ensemble.

Gibbs ensemble : $p_i \propto e^{-(E_i/T-(\mu/T)N_i)}$

Breaking of integrability in 1D experiments :

- Transverse degree of freedom $\Rightarrow$ 3-body effective term
- Effect of the longitudinal potential
- Coupling between several 1D gases

Effect of preparation scheme

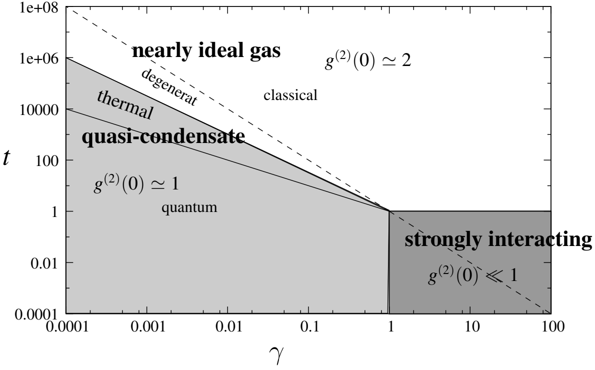
# Lieb-Liniger at thermal equilibrium

Gibbs ensemble : $p_{\{k_i\}} \propto e^{-(E_{\{k_i\}}/T-(\mu/T)N_{\{k_i\}})}$

$E_{\{k_i\}} = \sum_i k_i^2/2, \quad N_{\{k_i\}} = \sum_i 1 \quad \Rightarrow \quad f(k) = k^2/(2T) - (\mu/T)$

$f(k) \quad \Rightarrow$ thermal rapidity distribution $\rho(k)$ (Yang and Yang, 1969)

$\Rightarrow$ Equation of state $n(T,\mu)$, and all thermodynamic quantities $g^{(2)}(0)$ computed via Hellman-Feynman




$F = E - TS$

$\left.\frac{dF}{dg}\right)_T = \langle \frac{dH}{dg} \rangle$

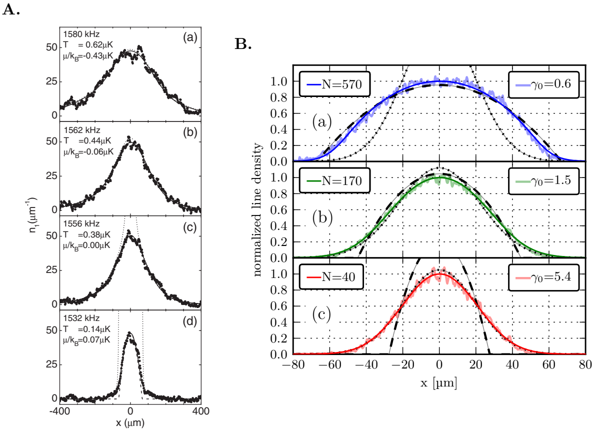
$$\boxed{\begin{aligned} \gamma &= \frac{mg}{\hbar^2 n} \\ t &= \frac{k_B T}{mg^2/\hbar^2} \end{aligned}}$$

Smooth cross-overs

Kheruntsyan et al. (2003)

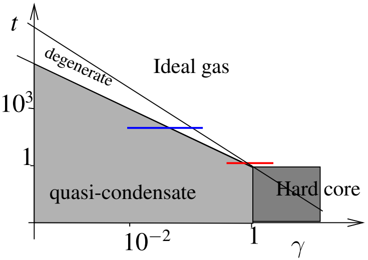
# Yang-Yang density profiles : experimental tests

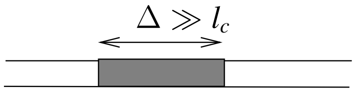
Presence of a smooth longitudinal confinement : local equilibrium
Local density approximation : $\mu(z) = \mu_0 - V(z)$




**A.** From van Amerongen et al. (2008), **B** from Vogler et al. (2013)

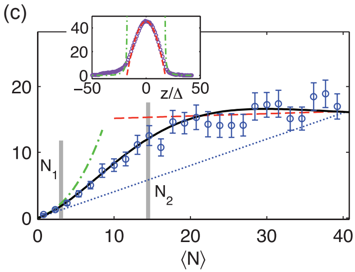
# Derivative of the equation of state : density fluctuations

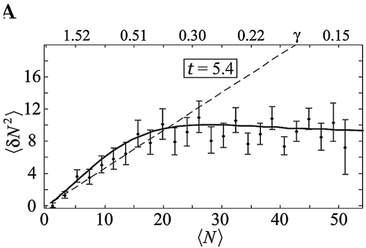



Gas in the pixel : described by GE

$$\langle \delta N^K \rangle = (k_B T)^{K-1} \Delta \left( \frac{\partial^{K-1} n}{\partial^{K-1} \mu} \right)_T$$

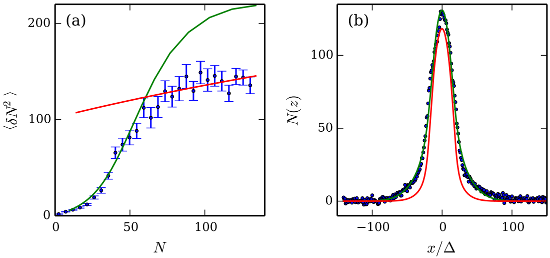



Armijo et al. 2010, Jacqmin et al. 2011

# Evidence for non-thermal long-lived states

Analysis of density profile : $\Rightarrow T_{\text{profile}} \simeq 140$ nK
Analysis of fluctuations in the center : $\Rightarrow T_{\text{fluctus}} \simeq 60$ nK




**Interpretation :** low-energy, long wave-length collective excitations (phonons) at lower energy than particles at higher energy

## Plausible explanation

Effect of losses

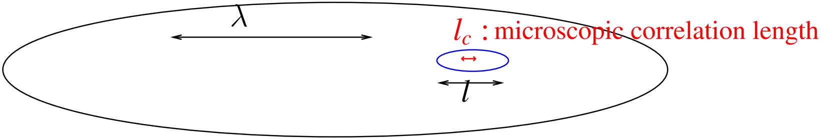
Johnson et al. (2017)

# Outline

# Outline

# Hydrodynamic approach : scale separation

Dynamics at long wavelength $\lambda$ and at large temporal scale $t$




Scales separations : $\begin{cases} \lambda \gg l_c \\ t \gg t_{\text{relax}} \end{cases}$ $\Rightarrow$ "elementary" cells of size $l$ with $\lambda \gg l \gg l_c$

## Fluid cells

Each cell : described by a homogeneous and relaxed gas

$\Rightarrow$ Described by $\{q_i\}$
Global conserved charge : $Q_i = \int d^d r \, q_i(r)$

# Generic system : Conventional Hydrodynamics (CHD)

**Properties of "elementary" cells (system obeying ETH)**

Entirely determined by :

- $n$ : atomic density
- $nmu$ : momentum per unit volume ($mu$ : momentum/atom)
- $nmu^2/2 + ne$ : energy per unit volume ($e$ : internal energy/atom )

Local values of global conserved quantities

Time evolution : $\partial_t q + \partial_x j_q = 0$, with $j_q = j_q^{(0)}(n, u, e) + \cancel{A\partial_x n} + \ldots$

$$\left\{ \begin{array}{rcl} \partial_t n + \partial_x(un) & = & 0 \\ \partial_t(mnu) + \partial_x(u(mnu) + P) & = & -n\partial_x V \\ \partial_t(n\frac{mu^2}{2} + ne) + \partial_x(u(n\frac{mu^2}{2} + ne + nV) + uP) & = & 0. \end{array} \right.$$

Pressure : $P = P(n, e)$ equation of state

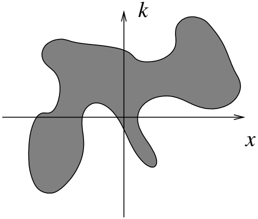
System described by : $n(x), u(x), e(x)$

3rd equation equivalent to isentropic deformation

# GHD for the Lieb-Liniger model

At a given $x$, state described by the function $k \to \rho(k,x)$

Evolution of $\rho(k,x)$ :




**Generalized Hydrodynamics Equations :**

$$\partial_t \rho + \partial_x (v_{\mathrm{eff}} \rho) = (\partial_x V/\hbar)\, \partial_k \rho$$

$$v_{\mathrm{eff}}(k) = \hbar k/m - \int dk'\, \rho(k') \frac{2mg/\hbar^2}{(mg/\hbar^2)^2 + (k-k')^2} \left(v_{\mathrm{eff}}(k) - v_{\mathrm{eff}}(k')\right)$$

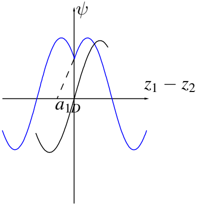
Real space density distribution $n(x) = \int dk \rho(k,x)$

O. A. Castro-Alvaredo, B. Doyon et al. , Phys. Rev. X **6** , 041065 (2016)

B. Bertini, et al., Phys. Rev. Lett. **117** , 207201 (2016)

# Physical interpretation of $v_{\text{eff}}$ : a naive picture

- **From two-body ...**

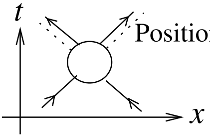



stationary state :

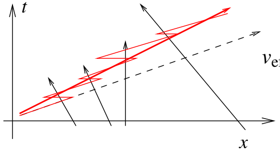
$$\psi_k(z) = \cos(k|z| - \phi_k), \ \tan(\phi_k) = \frac{mg}{2k}$$

$$\Delta_k = -\partial_k \phi_k = -2(mg/\hbar^2)/((mg/\hbar^2)^2 + 4k^2)$$

Wave packet evolution




- **...to many body**



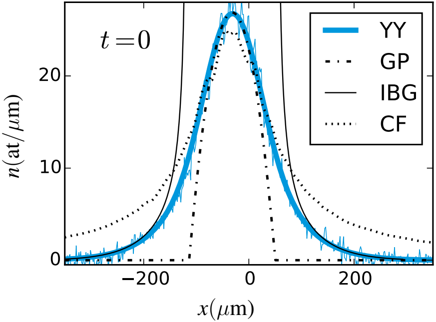

$$v_{\text{eff}}(k) = \hbar k/m - \int \underbrace{dk' \, \rho(k') \left(v_{\text{eff}}(k) - v_{\text{eff}}(k')\right)}_{\text{Collisions per unit time}} \underbrace{\frac{2mg/\hbar^2}{(mg/\hbar^2)^2 + (k-k')^2}}_{\text{position shift}}$$

# Outline

# Test of GHD in the atom-chip setup

Schemmer et al. (2019)

Initial state : equilibrium state in a harmonic potential




GP : $\mu = gn$

IBG : ideal Bose gas

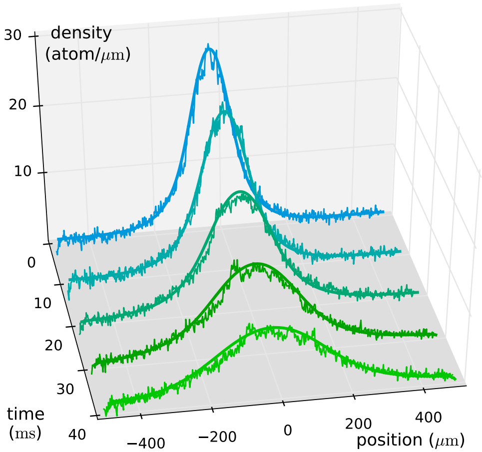
CF : Classical field

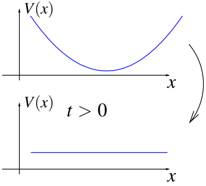
Profile well described by Yang-Yang thermodynamics :

$n(x) = n_{\mathrm{YY}}(T, \mu(x)), \; \mu(x) = \mu_0 - \frac{1}{2}m\omega^2 x^2$

Cloud explores sevral regimes : no simple theory applies

# Expansion in the 1D guide

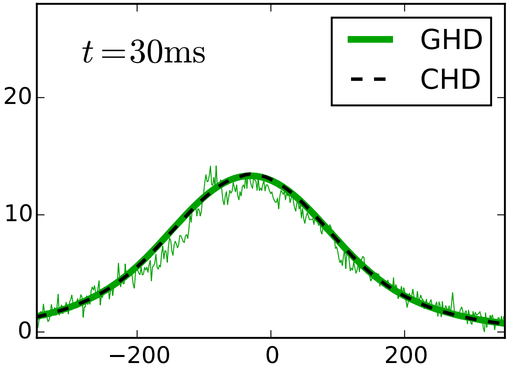



excellent agreement with GHD

# Expansion from a harmonic potential : validity of CHD

GHD is also in very good agreement with CHD




CHD calculations : made using Yang-Yang equation of state

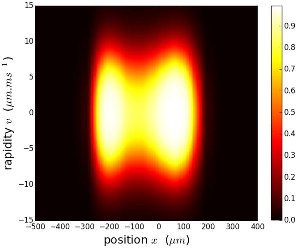
Miracle of harmonic potential

# Finding a situation where GHD and CHD differ

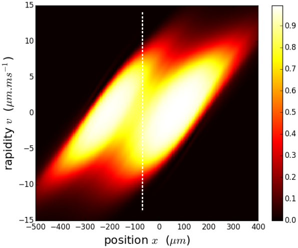
**Classical Hydrodynamics :**
gas described locally by a Gibbs Ensemble
**Rapidity distribution** for a Gibbs Ensemble : **single peaked**

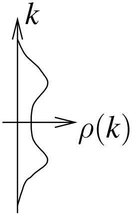
$\Rightarrow$ Finding situation where local distribution of rapidities become double-peaked
Occurs during the expansion from a doubly-peaked cloud.

Initial cloud



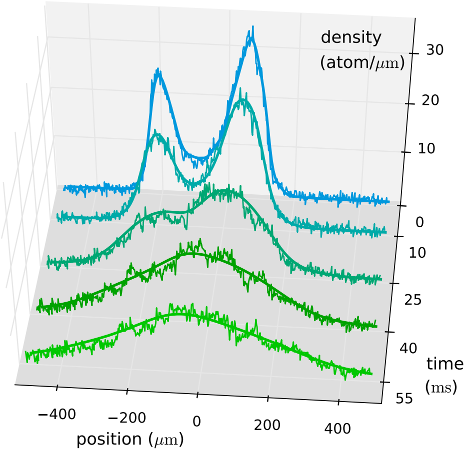

After some expansion

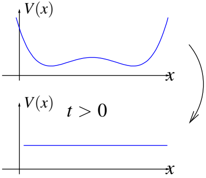


# Expansion from a double-well

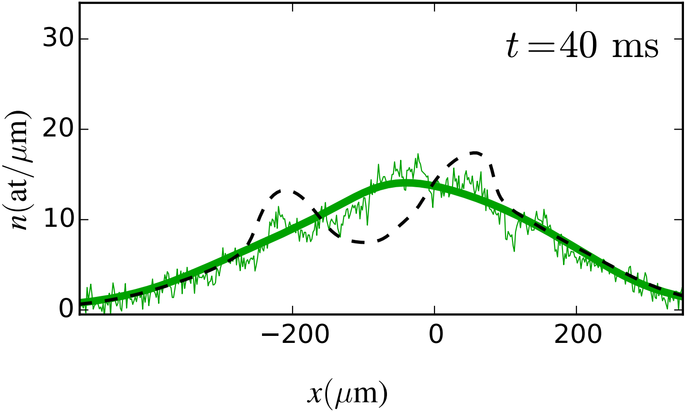
Initial cloud : equilibrium state in a double-well potential




Very good agreement with GHD

# Expansion from a double-well : Failure of CHD

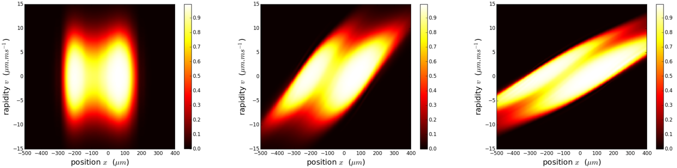



CHD : very different from GHD.

CHD predicts a choc at time $t \gtrsim 40$ ms

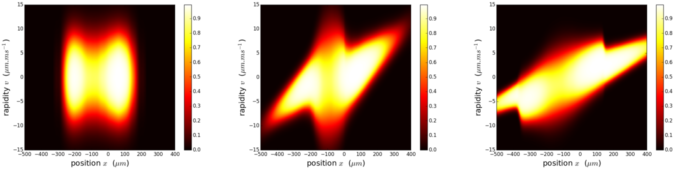
# GHD vs CHD : evolution of rapidities distributions

Generalized HydroDynamics (GHD) :




$t = 0$ $t = 25$ ms $t = 55$ ms

Conventional hydrodynamics (CHD) :



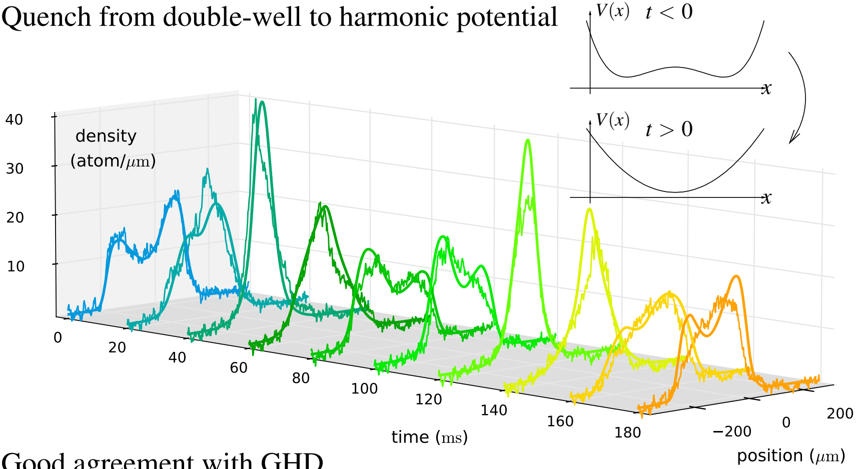

$t = 0$ $t = 25$ ms $t = 55$ ms

Plotted : occupation factor $\nu(v, x)$, $v = \hbar k/m$

# Newton's Craddle like experiment

'A quantum newton's cradle', Kinoshita *et al.* (2006)

Quench from double-well to harmonic potential



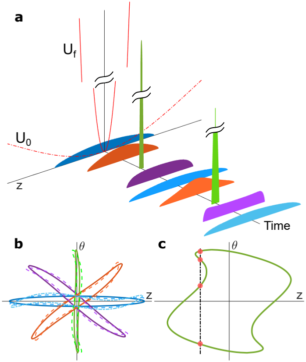

Good agreement with GHD

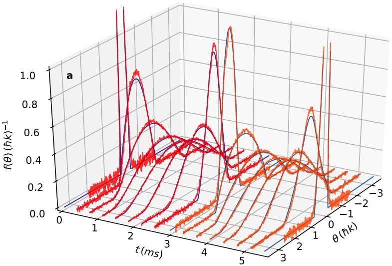
CHD : predicts a choc at $t \simeq 30$ms

# Another test of GHD : strong quench of confining potential

Malvania, N., Zhang, Y., Le, Y., Dubail, J., Rigol, M. and Weiss, D. S. , Science **373**, 1129 (2021).

- 2D optical lattice $\Rightarrow$ Collection of 1D gases
- Strong interaction regime ($\gamma \simeq 1$), $T \simeq 0$
- Small atom number per tube : $\simeq 20$
- Measured quantity : rapidity distribution



# Conserved quantities by GHD

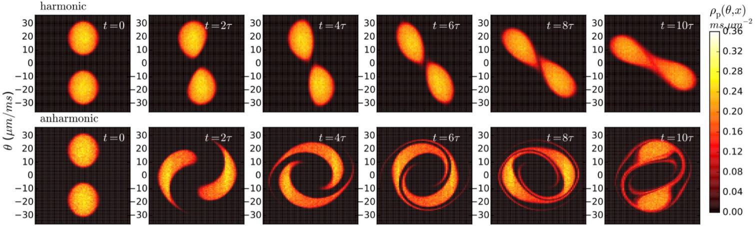
## Another form of GHD equation : conservation of $\nu$ along a trajectory

$$\partial_t \nu + v_{\text{eff}} \partial_x \nu - (\partial_x V/\hbar)\, \partial_k \nu = 0$$

## Quantities conserved by GHD

$$S[f] = \int dxdk \rho(x,k,t) f(\nu(x,k,t))$$

In particular, total entropy conserved $\boxed{\Rightarrow \text{Prevent relaxation}}$




Calculations from Caux et al. (2019)

# Outline

# Beyond Euler scale : diffusion terms

de Nardis et al (2018))

Higher order in $1/l$ in the evaluation of the flux of rapidities :

$$j_{\rho(k)}(x) = \underset{\substack{\uparrow \\ v^{\text{eff}}\rho}}{F[\rho]} + A[\rho]\partial_x\rho(k) + \dots$$

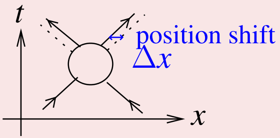
$$\partial_t\rho + \partial_x\left(v^{\text{eff}}\rho\right) - (\partial_x V)\partial_k\rho = \frac{1}{2}\partial_x\left(\mathfrak{D}\partial_x\rho\right)$$

In absence of $V(z)$ : $\boxed{\Rightarrow \text{relaxation to a GGE of Lieb-Liniger}}$

## Thermalization in presence of $V(z)$

Diffusion + $V(z)$ $\Rightarrow$ breakdown of integrability

Hand-waving argument (Mazets (2010))




Before collision : $\pm v$

After collision : $E_{\text{pot}} \simeq \partial_x^2 V(\Delta x)^2$

$v'^2 = v^2 - E_{\text{pot}}$

# Slow modification of the coupling constant

$g$ varies slowly in space and time (Bastianello et al. (2019).)
Rapidities not conserved. But the Bethe-Fermions momenta $\{p_a\}$ are.

$\{p_a\}$, $p_a \in \frac{2\pi}{L}\mathbb{Z}$ label the eigenstates

- $t = 0$ : relaxed state $\Rightarrow |\psi\rangle = |\{p_a^0\}\rangle$

$g$ varies $\Rightarrow$ correlations to other BA states :

$$\langle\{p_a'\}|\psi\rangle(dt) = \frac{dg}{dt}\left(\frac{d}{dg}\langle\{p_a'\}|\right)\{p_a^0\}\rangle dt$$

But $\langle\{p_a\}|\psi\rangle = 1 + o(dt)$
**Relaxation $\Rightarrow$ coherences killed** $\boxed{\Rightarrow |\psi\rangle(dt) = |\{p_a\}\rangle}$

## Generalized GHD equation

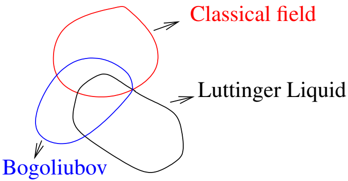
$$\partial_t\rho + \partial_x\left(v^{\text{eff}}\rho\right) - (\partial_x V)\partial_k\rho + \partial_k\left(\frac{(\partial_t c)F^{\text{dr}} + (\partial_x c)\Lambda^{\text{dr}}}{1^{\text{dr}}}\rho\right) = 0$$

# Outline

# Different approximate descriptions

Need for simpler approximate descriptions :

- obtain analytical results for correlation functions
- obtain results for short time dynamics
- describe phenomena like sound waves that are meaningfull at short time
- ...




Classical field : ignores "particles". weak interacting regime, energy not too low.

Bogoliubov : collective modes, weak interactions, small perturbations

Luttinger Liquid : Long wave-length, low-enegy collective excitations

# Outline

# Classical field approach

Ignore the quantum nature of the field operator $\psi(r)$
$\Rightarrow \psi(r)$ complex field, $\{\psi(r), \psi^*(r')\} = i\delta(r - r')$

$$H = \int d^{(d)}(r) \frac{\hbar^2}{2m} \partial_r \psi^* \partial_r \psi + \frac{g}{2} \int d^{(d)} r |\psi(r)|^4 + \int d^{(d)} r V(r) |\psi|^2$$

Lagrangian

$$L = i \int d^{(d)}(r) \left( \psi^* \partial_t \psi - \partial_t \psi^* \, \psi \right) - H$$

Evolution equation : Gross-Pitaevski equation

$$i\partial_t \psi = -\frac{\hbar^2}{2m} \Delta \psi + V(r) \psi + g|\psi|^2 \psi$$

Classical field = non-linear field physics

# Classical field as truncated Wigner

## Wigner distribution

$$W(\{\psi_r,\psi_r^*\}) = \int \prod_r d^2\lambda_r \frac{e^{-(\lambda_r\psi_r^* + \lambda_r^*\psi_r)}}{\pi^2}\chi(\{\lambda_r,\lambda_r^*\})$$

$$\chi(\{\lambda_r,\lambda_r^*\}) = \mathrm{Tr}\left\{\rho \exp\left[\sum_r (\lambda_r\hat{\psi}_r^\dagger - \lambda_r^*\hat{\psi}_r)\right]\right\}$$

$$\langle\psi(x)+\psi^+(x)\rangle = \int d^2\psi_r W(\{\psi_r,\psi_r^*\})$$

Exact evolution : $\frac{\partial W}{\partial t} = \left.\frac{\partial W}{\partial t}\right|_{\mathrm{Kin}} + \left.\frac{\partial W}{\partial t}\right|_{\mathrm{Nonlin}}$

$$\left.\frac{\partial W}{\partial t}\right|_{\mathrm{Kin}} = \frac{i\hbar}{2m}\sum_r \left\{\frac{\partial}{\partial\psi_r}\partial_r^2\psi_r - \frac{\partial}{\partial\psi_r^*}\partial_r^2\psi_r^*\right\} W$$

$$\left.\frac{\partial W}{\partial t}\right|_{\mathrm{Nonlin}} = \frac{i\tilde{g}}{\hbar}\sum_r \left\{\frac{1}{4}\left(\frac{\partial^3}{\partial^2\psi_r\partial\psi_r^*}\psi_r - \frac{\partial^3}{\partial^2\psi_r^*\partial\psi_r}\psi_r^*\right) + \left(\frac{\partial}{\partial\psi_r}\psi_r - \frac{\partial}{\partial\psi_r^*}\psi_r^*\right)|\psi_r|^2\right\}$$

Ignore 3rd order derivatives ⇒ GPE

# Classical field at thermal equilibrium : UV catastrophy

High $k$ Fourier components of $\psi$ : interactions negligible

For $k > K$, $H \simeq \sum_k \frac{\hbar^2 k^2}{2m} |\psi_k|^2$

At thermal equilibrium $\Rightarrow$ $\boxed{\langle|\psi_k|^2\rangle = \frac{k_B T}{\hbar^2 k^2/(2m) - \mu} \simeq \frac{k_B T}{\hbar^2 k^2/(2m)}}$

- **In 3D :** Density diverges (black body catastrophy)
  $L^3 \int_K^{K_c} \frac{k^2}{2\pi^2} dk \langle|\psi_k|^2\rangle \underset{K_c\to\infty}{\to} \infty \Rightarrow$ Cutoff requiered
- **In 1D :** $L \int_K^{\infty} \frac{1}{2\pi} dk \langle|\psi_k|^2\rangle$ is finite $\Rightarrow$ No cutoff requiered (for $n$)

- Classical field plagued by over-estimation of the effect of high $k$ terms
- Curring : cut-off, improved methods
- Effect less important in 1D

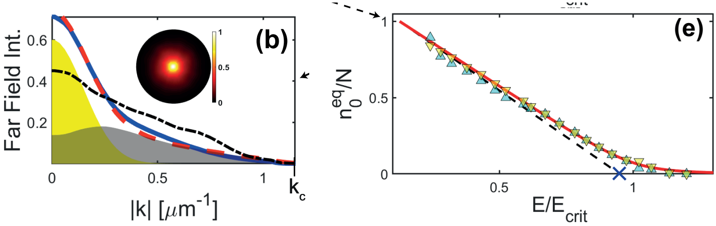
# Classical Bose-Einstein condensation in higher dimension

Thermal equilibrium at $T$ and $\mu$, harmonic potential :

$$\int_0^K d^{(d)}k d^{(d)}r \frac{k_B T}{\hbar^2 k^2/(2m) + V(r) - \mu} \underset{\mu \to 0}{\to} \text{ finite value for } d > 1$$

$\Rightarrow$ accumulation of energy in the lowest ($k = 0$) mode : BEC !

Recent result : K. Baudin et al. , Phys. Rev. Lett. 125, 244101 (2020)
Transverse field in a multimode optical fiber.

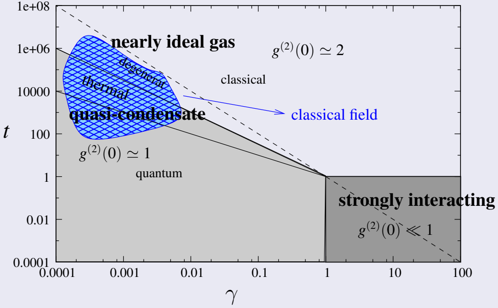


# Non linear classical field

- Account for Kostrelitz-Thouless transition in 2D
- Account for qBEC crossover in 1D

## Expected validity domain in 1D

No general answer : depends on the observable




Accounts for the crossover ideal Bose gas - quasi-condensate

# Thermal correlation functions : back to Shrödinger problem

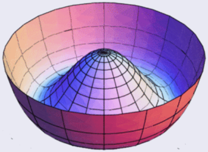
Homogeneous system with perdiodic boundary conditions :

$$H = \int_0^L dz \frac{\hbar^2}{2m} |\partial_z \psi|^2 + \frac{g}{2} \int dz |\psi(z)|^4 - \mu \int_0^L dz |\psi|^2$$

Example : calculation of $\langle \psi^*(z)\psi(0)\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\psi e^{-\beta H[\psi]} \psi^*(z)\psi(0)$

## Corresponding quantum Shrödinger problem

$H_q = \frac{p_x^2 + p_y^2}{2M} + V(x,y)$
$M = \hbar^3 \beta / m$
$V(x,y) = \hbar\beta \left( \frac{g}{2}(x^2+y^2)^2 - \mu(x^2+y^2) \right)$



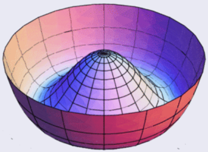
$$\langle \psi^*(z)\psi(0)\rangle = \frac{Tr\left(e^{-(L-z)H_q}(x-iy)e^{-zH_q}(x+iy)\right)}{Tr\left(e^{-LH_q}\right)} \simeq \langle (x-iy)e^{-zH_q}(x+iy)\rangle_0$$

- Equation of state
- density flutuations, 1-body correlation function, ...

# Poperties of the thermal classical field

## Equivalent Shrödinger problem

$$H_q = \frac{p_x^2+p_y^2}{2M} + V(x,y)$$
$$M = \hbar^3\beta/m$$
$$V(x,y) = \hbar\beta\left(\frac{g}{2}(x^2+y^2)^2 - \mu(x^2+y^2)\right)$$



| **Quasi-BEC** | **Ideal Bose gas** |
|---|---|
| $\mu \simeq gn$ | $\mu \simeq -\frac{T}{2\sqrt{\lvert mu\rvert}}$ |
| $g^{(1)}(z) \simeq e^{-\lvert z\rvert T/(2n)}$ | $g^{(1)}(z) \simeq e^{-\lvert z\rvert T/n}$ |
| $\langle\delta n^2\rangle \ll n^2$ | $\langle\delta n^2\rangle \simeq n^2$ |

$$\langle\psi^*(0)\psi(0) = \langle(x^2+y^2)\rangle_0 = \langle r^2\rangle_0$$
$$\langle\psi^*(z)\psi(0) = \langle re^{-i\theta}e^{-zH_q}re^{i\theta}\rangle_0$$

$$\longleftrightarrow$$

$$T_{\text{c.o.}} = \hbar n\sqrt{gn/m}$$
$$t \simeq \gamma^{-3/2}$$

# Thermal classical field : quantitative analysis

## Case where $V(z)$ is present

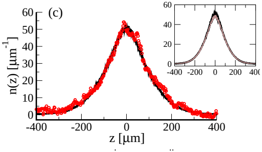
Method to sample the Gibbs ensemble : stochastic equation

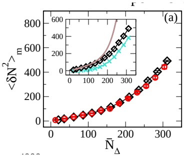
$$i\partial_t \psi = -i\Gamma \left( -\frac{1}{2}\partial_z^2 \psi + V(z)\psi(z) + g|\psi|^2\psi - \mu\psi \right) + \eta$$

$$\langle \eta^*(z', t')\eta(z, t) \rangle = 2k_B T \Gamma \delta(z - z')\delta(t - t')$$

Adding many effects

- Cut-off, populated transverse states
- Inflation of the transverse wave-function with interactions




Very good agreement with experiments
S. P. Cockburn et al. , PRA (2011)

# Classical field out-of-equilibrium dynamcis

$$i\partial_t\psi = -\frac{\hbar^2}{2m}\partial_z^2\psi + g|\psi|^2\psi + V\psi$$

**Madelung trannsformation** : $\psi = \sqrt{n}e^{i\theta}, \quad \partial_z(\theta) = (m/\hbar)v$

$$\begin{cases} \partial_t n + \partial_z(vn) = 0 \\ \partial_t v + v\partial_z v = -g\partial_z n + \underbrace{\frac{1}{2}\partial_z\left(\frac{\partial_z^2\sqrt{n}}{\sqrt{n}}\right)}_{\text{quantum pressure}} \end{cases}$$

- **Quantum pressure neglected : usual** $(T=0)$**hydrodynamics.** dynamics in harmonic traps for inverted parabola initial shape
- **In general.** Apparition of shock waves, Wave turbulence, route to relaxation, . . .

# Small perturbations : linearized approximation

q-BEC regime : $\delta\rho = (\rho - \rho_0) \ll \rho_0$, $\partial_x\theta \ll n_0$, $\{\delta n, \theta\} = \delta(z - z')$

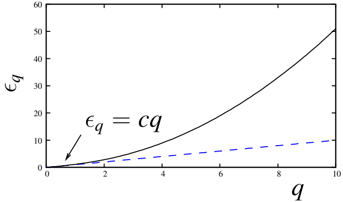
To first order in $\delta\rho$, $\partial_x\theta$ : $\begin{cases} \partial_t \delta n + n_0 \partial_z^2 \theta = 0 \\ \partial_t \theta = -g\partial_z \delta n + \frac{1}{4n}\partial_z^2 \delta n \end{cases}$

Hamiltonian $\boxed{H_B \simeq \int \left[ \frac{\hbar^2}{8mn_0}(\partial_x \delta n)^2 + \frac{g}{2}\delta n^2 + \frac{\hbar^2 n_0}{2m}(\partial_x\theta)^2 \right] dx.}$

$$\begin{cases} n_{q,c} = \sqrt{2/L}\int dz \delta n(z)\cos(qz) \\ \theta_{q,c} = \sqrt{2/L}\int dz \theta(z)\cos(qz) \end{cases}$$

$$H_q = A_q n_q^2 + B_q \theta_q^2$$

$$\epsilon_q = \sqrt{q^2/2(q^2/2 + 2gn_0)}$$




## Usual formulation : traveling waves

$b_q = u_q B_q - v_q B^*_{-q}$, $B_q = \int dz e^{iqz}/\sqrt{L}(\delta n(z)/(2\sqrt{n}) + i\sqrt{n}\theta)$

$$H_q = \sum_q \epsilon_q b_q^* b_q$$

# No long range order in 1D

Thermal equilibrium in the quasi-BEC regime

Suppose there is long-range order : $\Rightarrow \langle \theta^2 \rangle$ finite

$$\langle \theta^2(0) \rangle = \left\langle \left( \sum_q \sqrt{\frac{2}{L}} \theta_{c,q} \right)^2 \right\rangle$$

Thermal equilibirum : $\frac{\hbar^2 q^2 n_0}{2m} \langle \theta_{c,q}^2 \rangle = k_B T$

$$\Rightarrow \langle \theta^2(0) \rangle = \sum_q \frac{2}{L} \frac{2mk_B T}{\hbar^2 q^2} \simeq \frac{1}{\pi} \frac{2mk_B T}{\hbar^2} \int_0^\infty \frac{dq}{q^2} \to \infty$$

No long range order in 1D due to thermally excited long wave-length phonons

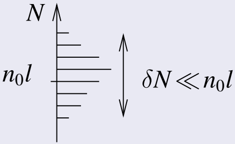
# Outline

# Quantum treatment : Bogoliubov approach

Keep trace of the discretness of the atoms : $[\psi(z), \psi(z')] = \delta(z - z')$

Quantify the previous collective modes

## Density-phase representation in quantum physics


assume $N : -\infty \to \infty$

$\Rightarrow \theta$ conjugate variable $[N, \theta] = i$

Mora and Castin (2003)

## Bogoliubov modes

$$H_{\mathrm{Bogo}} = \sum_q \epsilon_q b_q^+ b_q + \mathrm{cste}, \quad [\mathrm{b_q}, \mathrm{b_q^+}] = 1$$

$$\begin{pmatrix} b_q \\ b_{-q}^+ \end{pmatrix} = \begin{pmatrix} u_q & v_q \\ v_q & u_q \end{pmatrix} \begin{pmatrix} B_q \\ B_{-q}^+ \end{pmatrix}, \quad \begin{cases} u_q = \cosh(\theta_q/2) \\ v_q = \sinh(\theta_q/2) \end{cases}$$

$$\tanh \theta_q = \mu/(\mu + \frac{q^2}{2m}), \quad B_q = \frac{\delta n_q}{2\sqrt{n}} + i\sqrt{n}\theta_q$$

# Using Bogoliubov approach

- **At thermal equilibrium**
  - Equation of state. Quantum corrections
  - correlation functions. With quantum corrections
- **Ou-of-equilibrium dynamics**
  - Preparation of out-of-equilibrium states
  - Time evolution

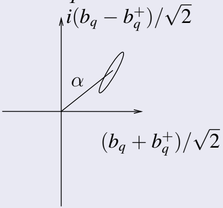
## Role of gaussian states

At thermal equilibrium, state gaussian (in terms of $b_q, b_q^+$)

Perturbation linear or quadratic in $\delta n$, $\theta$

$\Rightarrow$ state of the system stays gaussian

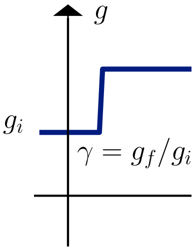
- Displacement
- Squeezing (2 modes squeezing)

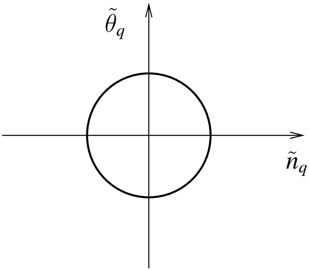


# Effect of an interaction quench

- **Mode of wave vector $q$ (stationnary mode)**

$$\begin{cases} \tilde{n}_q = n_q(A_q/(4B_q))^{1/4} \\ \tilde{\theta}_q = \theta_q(B_q/(4A_q))^{1/4} \end{cases} \Rightarrow \quad H_q = \hbar\omega_q\left(\frac{\tilde{n}_q^2}{2} + \frac{\tilde{\theta}_q^2}{2}\right) \qquad \begin{aligned} A_q &= g/2 \\ B_q &= \hbar^2 k^2 n_0/(2m) \end{aligned}$$




Squeezing and subsequent breathing of each mode

# Effect of an interaction quench

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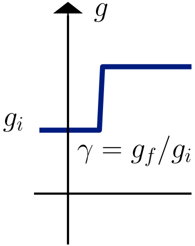


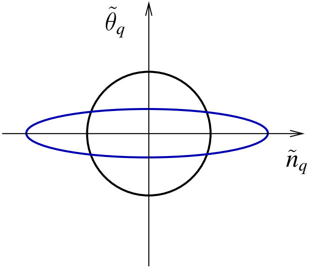


Squeezing and subsequent breathing of each mode

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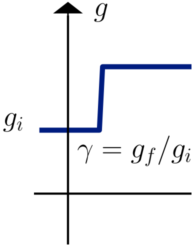
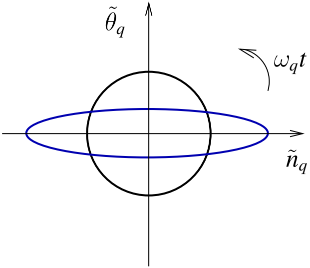
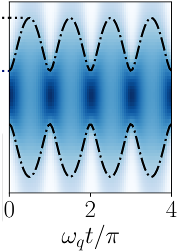



$$\langle\theta_q^2\rangle = \langle\theta_q^2\rangle_0\left(1 + (\gamma - 1)\sin^2(cq)\right)$$

Squeezing and subsequent breathing of each mode

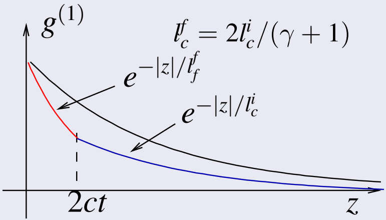
# Effect on the first order correlation function

$g^{(1)}(z) = \langle \psi_z^+ \psi_0 \rangle \simeq n_0 \langle e^{i(\theta(z)-\theta(0))} \rangle$ FT of momentum distribution

Gaussian distributions of $\theta \Rightarrow \boxed{g^{(1)}(z) = n_0 e^{-\langle (\theta(z)-\theta(0))^2 \rangle /2}}$

- $\theta(z) - \theta(0)$ get values $\gg 1$ : **All modes mixed**
- Thermal equilibrium :
  $\langle \theta_q^2 \rangle = m k_B T/(n_0 q^2)$ , $g^{(1)}(z) = n_0 e^{-|z|/l_c}$ , $l_c = \hbar^2 n_0/(2 m k_B T)$

## Effect of an interaction strength : light cone phenomena

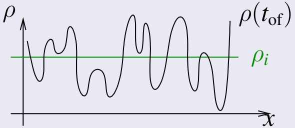



- Breathing of modes not visible
- Apparent thermalisation

for $t \gtrsim l_c/c$

# Density ripples : a filter for Bogoliubov modes

## Density ripples appearing after a free evolution time




Free evolution during $t_{\text{of}}$ after interactions turned off

Phase fluctuations $\Rightarrow$ density ripples

**Power spectrum :**

$$\langle|\rho(q)|^2\rangle \simeq n_0^2 \int dx e^{-iqx} e^{-\langle(\theta(0)-\theta(-\hbar q t_{\text{of}}/m)+\theta(x-\hbar q t_{\text{of}}/m)-\theta(x))^2\rangle/2}$$

## Low wavelength $q(\hbar t_{\text{of}})/m \ll l_c$

Exponential can be expanded

$$\langle|\rho(q)|^2\rangle = 4n_0^2\langle\theta_q^2\rangle \sin^2\left(\frac{\hbar q^2 t_{\text{of}}/2}{m}\right)$$

$$\langle|\rho(q)|^2\rangle \text{ resolves } \langle\theta_q^2\rangle$$

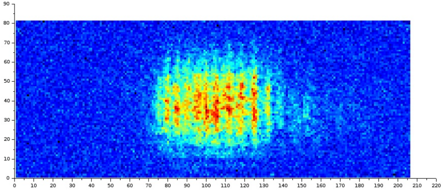
Experimental implementation : first in Vienna group, Manz et al. 2010

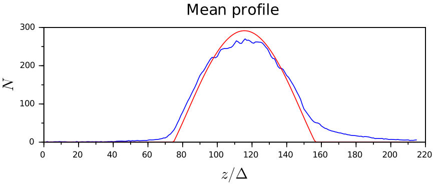
# density ripples measurement. Experiment at Palaiseau

- Trapping potential suddenly turned off

transverse expansion $\rightarrow$ instantaneous switching off of interactions

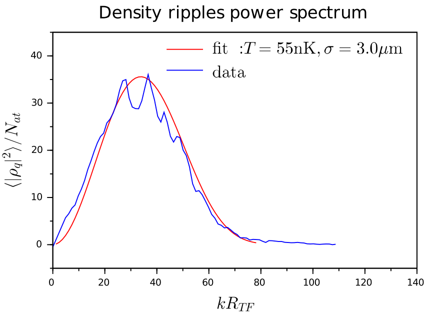
- 8 ms time of flight $\rightarrow$ apparition of density ripples






Statistical analysis on $\simeq 50$ images

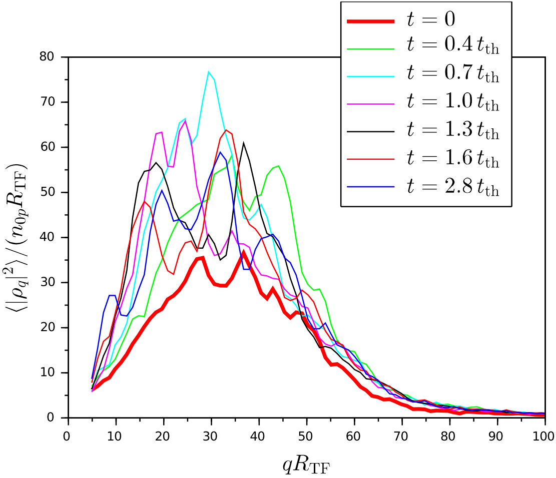
$\Rightarrow$ extract power spectrum $\langle|\rho_q|^2\rangle$




Imaging resolution $\sigma$ :
affected by transvserse size of cloud

# Density ripples power spectrum after the interaction quench

- At $t = 0$, quench of the interaction strength by a factor 3
- Measure of density ripples power spectrum after different evolution times $t$

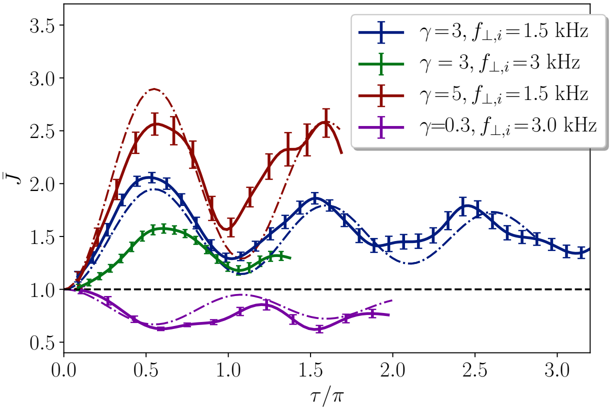



$t_{\rm th} = l_c^f/c$ : *thermalisation time* of the $g^{(1)}$ function

# Breathing of collective modes

Unveil breathing of each mode : $\tau = cqt$, $c$ evaluated at cloud center

$$J(q,\tau) = \langle|\rho_{t_f}(q)|^2\rangle(t)/\langle|\rho_{t_f}(q)|^2\rangle_i$$

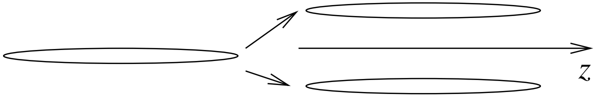



- theory curves : $\times$ 0.5
- Frequency in good agreement
- Damping : good agreement (pure dephasing effect)
- Amplitude : smaller than expected

# Measure of the evolution of $g^{(1)}$

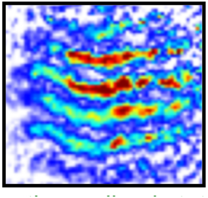
Experiment in Vienna (Schmiedmayer's group)

- **Splitting in 2 of a 1D gas**



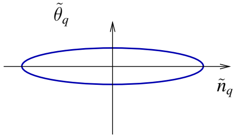

- **Measuring relative phase** $\theta(z) = \theta_1(z) - \theta_2(z)$

Small time of flight
$\Rightarrow$ interferences
$\Rightarrow \theta(z)$



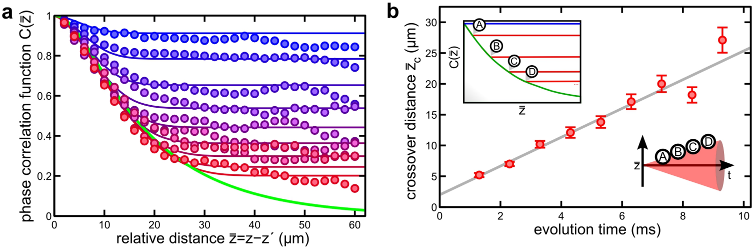
- **Initial state**

Fluctuations of $\delta n$ :
shot noise $\langle |\delta n_q|^2 \rangle = n_0$



# Measure of the evolution of $g^{(1)}$ : light-cone behavior

Experiment in Vienna (Schmiedmayer's group)




Ressemble to thermalization

Langen et al. (2013)

# Outline

# Phonons : description for any $\gamma$

Density : $\rho = \rho_0 + \delta\rho$, $\delta\rho \ll \rho_0$
Current density : $J = \rho_0 V$

Energy :
$H = \int dx \left( \frac{1}{2}\frac{\partial\mu}{\partial\rho}\delta\rho^2 + \frac{m}{2\rho_0}J^2 \right)$

Dynamics : hydrodynamics equations
Speed of sound : $v = \sqrt{\rho_0 \frac{\partial\mu}{\partial\rho}/m}$
Chiral current densities :
$$\begin{cases} J_R = (v\delta\rho + J)/2 \\ J_L = (v\delta\rho - J)/2 \end{cases}$$

## Quantization : Luttinger Liquid

$V = i\frac{\partial\theta}{\partial x}$, $\quad [\delta\rho(x), \theta(x')] = i\delta(x - x')$

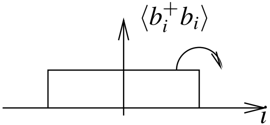
$$H = H_{LL} = \hbar \sum_{n\neq 0} v|k_n| a_n^+ a_n \; , k_n = (2\pi/L)n$$

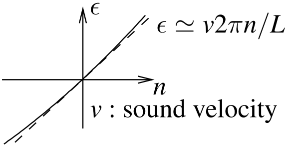
$J_R(x) = \frac{v\sqrt{K}}{L} \sum_{n>0} \sqrt{n}(e^{ik_n x} a_n + e^{-ik_n x} a_n^+)$ , $\quad K = \pi\rho_0/v$

# Low-energy eigenstates

$b_n^+$ : create a Fermion of momentum $2n\pi/L$

- Ground state : Fermi sea



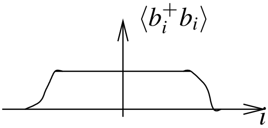

## Low energy excitations : particle-hole excitation

close to Fermi sea border : $c_{R,n}^+ = b_{N/2+n}^+$ , $c_{L,n}^+ = b_{-N/2-n}^+$ .

$c_{R,n}^+ c_{R,-m}|0\rangle$ : energy $E = \epsilon_n - \epsilon_m$

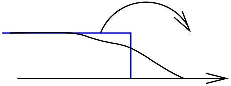
$$\epsilon_n = vn2\pi/L + \frac{1}{2m^*}\left(\frac{2\pi n}{L}\right)^2 \text{ with } m^* = m\left(1 + \frac{\rho_0}{v}\frac{\partial v}{\partial \rho}\right)$$

Several excitations :
deformed borders of the Fermi Sea



# Bosonization of the Bethe Fermions

Semi-infinite Fermi sea



## Collective Bosons

$$A^+_{R,n} = \frac{1}{\sqrt{n}} \left( \quad \sum_{l\in\mathbb{Z}+\frac{1}{2}} c^+_{R,n+l} c_{R,l} \quad \right), n > 0$$

$$[A_{R,n}, A^+_{R,n'}] = \delta_{n,n'}$$

## Inverse relation

Chiral field : $\varphi_{\mathrm{R}}(x) = -i \sum_{n>0} \frac{1}{\sqrt{n}} \left( e^{i2\pi nx/L} A_{\mathrm{R},n} - e^{-i2\pi nx/L} A^+_{\mathrm{R},n} \right)$

$$c^+_R(x) c_R(y) = \frac{1}{L} : e^{-i\varphi_R(x)} :: e^{i\varphi_R(y)} :$$

## Hamiltonian

Linear approximation : $H \simeq \sum_l v(2\pi l/L) c^+_{R,l} c_{R,l} - E_0$

In terms of collective bosons : $H = \sum_{n>0} v(2\pi n/L) A^+_{R,n} A_{R,n}$

# Collective bosons = Phonons

Long wave-length Fourier components of $\delta\rho$ and $J$ : $\delta\rho_n, J_n$

Kozlowski,J. Math. Phys. 52, 083302 (2011), De Nardis and Panfil, J. Stat. Mech. (2018)

Within the subspace of low-energy states (up to $\log L/L$ corrections) :

$$(\delta\rho)_n \simeq \left(\sqrt{K}\sum_{l\in\mathbb{Z}+\frac{1}{2}} c^{+}_{\mathrm{R},n+l}c_{\mathrm{R},l} + \sqrt{K}\sum_{l\in\mathbb{Z}+\frac{1}{2}} c^{+}_{\mathrm{L},-n+l}c_{\mathrm{L},l}\right)$$

$$J_n \simeq \left(v\sqrt{K}\sum_{l\in\mathbb{Z}+\frac{1}{2}} c^{+}_{\mathrm{R},n+l}c_{\mathrm{R},l} - v\sqrt{K}\sum_{l\in\mathbb{Z}+\frac{1}{2}} c^{+}_{\mathrm{L},-n+l}c_{\mathrm{L},l}\right)$$

**Chiral current density $J_R = (v\delta\rho + J)/2$**

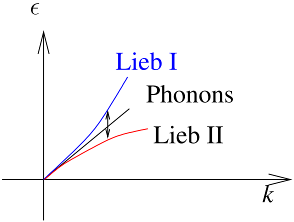
$$J_R(x) = \frac{v\sqrt{K}}{L}\sum_{n>0}\sqrt{n}\left(e^{i\frac{2\pi x}{L}n}A_{\alpha,n} + e^{-i\frac{2\pi x}{L}n}A^{+}_{\alpha,n}\right)$$

$\Rightarrow A^{+}_{R,n} = a^{+}_n$ creates a phonon

**Link between phonons and eigenstates of Bethe-Ansatz solution**

# Relaxation of phonons

Phonons : coherent superposition of many Bethe-Ansatz states




Superposition of many Bethe-Ansatz states of different energy

$\Rightarrow$ Relaxation

Bozonisation machinery

$\Rightarrow$ compute relaxed state for any initial phononic gaussian state

## Example : initial state translationnally invariant

$$\langle J_{\mathrm{R}}(x) J_{\mathrm{R}}(y)\rangle_0 = -\frac{\rho_0 v}{4\pi}\partial_x^2 g(x-y).$$

State after relaxation

$$\langle J_{\mathrm{R}}(x) J_{\mathrm{R}}(y)\rangle_\infty = \frac{\pi\rho_0 v}{L^2}\exp\left(2g(x-y)\right).$$