Relaxation of phonons in 1D Bose gases
or
From collective Bosons to indiviudal Bosons going through Fermions

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Outline

Hamiltonian of the Luttinger Liquid

- 1D gas
- Low energy
- Long wavelength

$$H_{LL} = \frac{v_s}{2} \int \left(\frac{\pi}{K} (\delta n)^2 + \frac{K}{\pi} (\partial x \theta)^2 \right) \mathrm{d}x.$$

$[\delta n(x), \theta(y)] = i\delta(x-y)$.
Luttinger parameter : $K = \pi \rho_0 / v_s$

Phonons : creation operator a_k^+

$$H_{LL} = \sum_{k \in (2\pi/L)Z, k \neq 0} v_s |k| a_k^+ a_k + \frac{P^2}{2mN}$$

$$a_k = \frac{\mathrm{sgn}(k)}{2} \int dx \left(-\frac{i\sqrt{2\pi}}{\sqrt{KL|k|}} \delta n(x) e^{-ikx} + \sqrt{\frac{2K|k|}{\pi L}} \theta(x) e^{-ikx} \right)$$

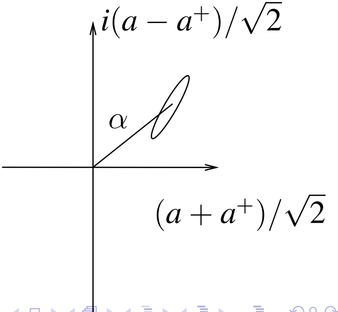
Production of out-of-equilibrium states

- **Parametric excitation** : $g = g_0 + \delta g \cos(\omega_{\rm mod} t) \Rightarrow$ 2-mode squeezing (Jaskula et al., Phys. Rev. Lett. **109**, 220401 (2013))
- **Linear drive** : $V(z,t) = V_0 \cos(k_0 x - \omega_0 t)$ (short time Bragg pulse)
- **Sudden modification of** g : $\Rightarrow$ squeezed phononic modes (Schemmer et al., Phys. Rev. A. **109**, 043604 (2018))
- **Sudden modification of** $V(z)$: $\Rightarrow$ Displaced phonons. (Federica Cataldini et al., arxiv :2111.13647 (2021))

Situation considered here : gaussian states

$$\hat{\rho} = \prod_k \hat{\rho}_k, \quad \hat{\rho}_k = D_k S_k e^{-\lambda_k a_k^+ a_k} S_k^{-1} D_k^{-1}$$

$$D_k = e^{\alpha_k a_k - \alpha_k^* a_k^+}, S_k = e^{\mu_k a_k^2 - \mu_k^* a_k^{+2}}$$



Outline

Chiral bosonic fields

$$\begin{cases} \varphi_R(x) = \sqrt{\frac{2\pi}{L}} \sum_{k>0} \frac{1}{\sqrt{k}} (e^{ikx} a_k - e^{-ikx} a_k^+) \\ \varphi_L(x) = \sqrt{\frac{2\pi}{L}} \sum_{k<0} \frac{1}{\sqrt{|k|}} (e^{ikx} a_k - e^{-ikx} a_k^+) \end{cases}$$

Heisenberg picture : $\varphi_R(x,t) = \varphi_R(x - ct)$,$\varphi_L(x,t) = \varphi_R(x + ct)$

$$\begin{cases} \partial_x \varphi_R(x) = \frac{2\pi}{\sqrt{K}} (\frac{1}{2} \delta n(x) + \frac{K}{2\pi} \partial_x \theta) \\ \partial_x \varphi_L(x) = \frac{2\pi}{\sqrt{K}} (\frac{1}{2} \delta n(x) - \frac{K}{2\pi} \partial_x \theta) \end{cases}$$

2 undependent fields : Right-movers and Left-movers

$$H_{LL} = H_R + H_L$$

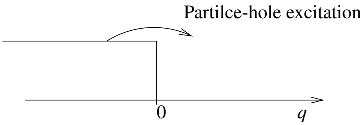
$$H_R = \int dx : (\partial_x \varphi_R)^2 := \sum_{k>0} c k a_k^+ a_k$$

The Rozhkov Fermions

- Consider right-movers only

The Rozhkov Fermions :

- $\psi_R^+(x) = \sqrt{\frac{2\pi}{L}} \sum_q e^{iqx} c_{Rq}^+$
- Linear dispersion : $H_R = \sum_q cq c_{Rq}^+ c_{Rq}$
- Ground state : semi-infinite Fermi sea



Bosonisation

$$H_R = \sum_{k>0} ck a_k^+ a_k = \sum_q cq c_{Rq}^+ c_{Rq}$$

Bosonisation rules

$$a_k^+ = \frac{i}{\sqrt{kL/(2\pi)}} \sum_q c_R^+(q+k) c_R(q)$$

$$\psi_R(x) =: e^{i\varphi_R(x)} :$$

Absence of relaxation within Luttinger Liquid

Luttinger Liquid : integrable system.
Relaxation ? Some other integrable systems do relax, towards a GGE.

- **In bosonic point of view**. Natural Generalized Gibbs ensemble

$$\hat{\rho} = \prod_k e^{\lambda_k a_k^+ a_k}$$

This is the GGE of T.Langen et al., Science **348**, 6231 (2015)

- **In Fermionic point of view**. Natural GGE :

$$\hat{\rho} = \prod_q (1 + e^{\lambda_q c_q^+ c_q})$$

No relaxation with Luttinger Liquid

No prefered point of view

Beyond Luttinger Liquid necessary to learn about relaxation

Non Linear Luttinger Liquid : Rozhkov fermions are beter quasiparticle

v depends on $n \Rightarrow$ beyond Luttinger Liquid

Non Linear Luttinger Liquid (from Dima Gangardt)

Fermions : quadratic term in relation dispersion : $\Rightarrow$ masse m^*

$$\frac{m}{m^*} = \frac{1}{2\sqrt{K}} \left(1 + \frac{n}{v} \frac{\partial v}{\partial n} \right)$$

Hamiltonian term :

$$H_3 = - \int \frac{1}{2m^*} \left(\psi_R^\dagger \nabla^2 \psi_R + \psi_R^\dagger \nabla^2 \psi_R \right) dx$$

Rozhkov fermions have a longer lifetime than phonons

$\Rightarrow$ Rozhkov Fermions are better quasi-particles

Outline

Lieb-Liniger model : Bethe-Ansatz fermions

Lieb-Liniger model : 1D Bosons with contact interactions

$$H_{\mathcal{L}\mathcal{L}} = \int dx \left(-\frac{\hbar^2}{2m} \psi^+(x) \partial_x^2 \psi(x) + \frac{g}{2} \psi^+ \psi^+ \psi \psi \right)$$

Eigenstates (Bethe-Ansatz form) : $|\{\theta_a\}\rangle = |\{p_a\}\rangle$

$$\theta_a + \frac{1}{L} \sum_{b \neq a} 2 \arctan\left(\frac{\theta_a - \theta_b}{mc} \right) = p_a, \text{where} \begin{cases} p_a \in \frac{\hbar 2\pi}{L} \mathbb{Z} & \text{for} \quad N \quad \text{odd} \\ p_a \in \frac{\hbar 2\pi}{L} (\mathbb{Z} + \frac{1}{2}) & \text{for} \quad N \end{cases}$$

$\{\theta_a\}$: rapidities (all different). $E = \sum \theta_a^2 / (2m)$
$\{p_a\}$: Bethe Fermions. Creation of a Bethe Ansatz Fermion : b_q^+

$P = \sum \theta_a = \sum p_a$

Rozhkov fermions are the Bethe-Ansatz fermions

- Hardcore Bosons case

Let us consider $\delta n_k = (1/\sqrt{L}) \int dx e^{ikx} \delta n(x)$

From Luttinger Liquid theory

$$\delta n_k = \frac{1}{\sqrt{L}} \left(\sum_q c^+_{Rq} c_{R(q+k)} + \sum_q c^+_{Lq} c_{L(q+k)} \right)$$

From BEthe Ansatz

$$\delta n_k = \frac{1}{\sqrt{L}} \sum_q b^+_q b_{q+k}$$

Close to ground state : right and left border of the Fermi sea contribute

- General case

Adiabatic increase of g until hardcore regime is reached
Distribution of Rozhkov fermions unchanged
Distribution of Bethe-Ansatz fermions unchanged

Outline

Relaxation of phonons

Relaxation within the Lieb-Liniger model

For local obsevrable :

$\hat{\rho} \underset{t\to\infty}{\rightarrow} \hat{\rho}_{GGE}$, where $\hat{\rho}_{GGE} = \sum_{\theta_a} |\{\theta_a\}\rangle\langle\{\theta_a\}| e^{\sum_a f(\theta_a)}$

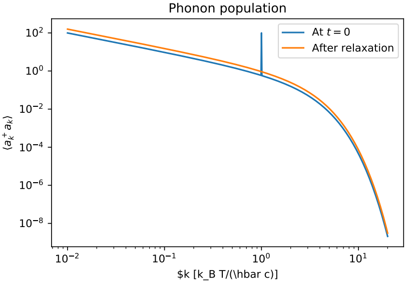
$\hat{\rho}_{GGE}$ very close to $\hat{\rho}_{GGE}^{(RF)}$, where $\hat{\rho}_{GGE}^{(RF)} = \sum_{p_a} |\{p_a\}\rangle\langle\{p_a\}| e^{\sum_a g(p_a)}$

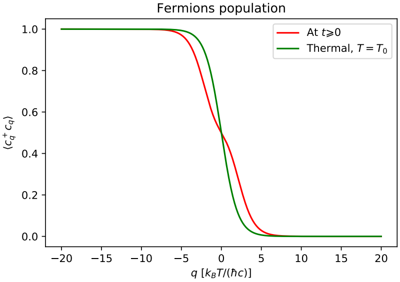
- **Initial out-of-equilibrium state** gaussian in phonons representation
- Wick theorem and bosonization rules $\Rightarrow$ **Rozhkov Fermions populations**
- $\Rightarrow$ $\hat{\rho}_{\mathbf{GGE}}^{(\mathbf{RF})}$ **known** : state after relaxation
- Compute properties of collective modes after relaxation using bosonization rules

Example of calculation

- Out-of-equilibrium initial state : phonon at k_0 displaced

$$\hat{\rho} = e^{\alpha_k a_{k_0} - \alpha_k^* a_{k_0}^+} \hat{\rho}_{T_0} e^{-\alpha_k a_{k_0} + \alpha_k^* a_{k_0}^+}$$





correlation between phonons

Correlation between phonons population

$$C_{q,q'} = \langle a_q^+ a_q a_{q'}^+ a_{q'} \rangle - \langle a_q^+ a_q \rangle \langle a_{q'}^+ a_{q'} \rangle$$

Bosonization rule + Wick theorem $\Rightarrow$

$$C_{q,q'} = \langle a_q^+ a_q \rangle^2 \delta_{q,q'} + \frac{1}{L} \mathcal{C}(q, q') \text{ where } \mathcal{C} = \sum_{i=1,6} C_i$$

Outline

Conclusion

The delicate points, not yet well established

- Equivalence between Rozhkov Fermions and Bethe-Ansatz Fermions
- Equivalence, up to order $1/L$ between $\hat{\rho}_{GGE}$ and $\hat{\rho}_{GGE}^{(RF)}$