$\left\{\begin{array}{l} 39 \longrightarrow 45 \quad \text{timestep utilisé.} \\ 38 \quad \text{non utilisé} \end{array}\right.$

⓪

I.1 The Inverse Scattering Transform

The Lax Pairs

$L\varphi = \lambda\varphi$

$\varphi_t = A\varphi$

$\lambda_t = 0 \to L_t + [L, A]$.

$$u_0 \longrightarrow R(u_0) = S(\lambda, 0)$$

Cauchy problem — time evol

$$u(t) \xleftarrow{\ I\ } S(\lambda, t)$$

Remark: $-v_{xx} + uv = \lambda v \qquad m = -\frac{v_x}{v} \qquad m_x = -\frac{(v_{xx}v - v_x^2)}{v^2}$

$$m_{xx} = -\frac{(v_{xxx}v + v_x^2 - 2v_{xx}v_x)v^2 - (v_{xx}v - v_x^2)2v_x v}{v^4}$$

$$= -\frac{v_{xxx}v^3 - v_x^2v^2 - 2v_{xx}v_xv^2 + 2v_x^3v}{v^4}$$

flemme — Stationary Schrödinger equa

$$-\varphi_{xx} + u\varphi = \lambda\varphi \longrightarrow \begin{pmatrix}\varphi \\ \varphi_x\end{pmatrix}_x = \begin{pmatrix}0 & 1 \\ -\lambda+u & 0\end{pmatrix}\begin{pmatrix}\varphi \\ \varphi_x\end{pmatrix}$$

ou $\left\{\begin{array}{l}\psi_x = -ik\psi + u\varphi \\ \varphi_x = \psi + ik\varphi\end{array}\right. \longrightarrow \varphi_{xx} = \psi_x + ik\varphi_x = -\cancel{ik\psi} + u\varphi + \cancel{ik\psi}$

$$= \underbrace{-k^2}_{\lambda}\varphi$$

$$\begin{pmatrix}\psi \\ \varphi\end{pmatrix}_x = \begin{pmatrix}-ik & u \\ 1 & ik\end{pmatrix}\begin{pmatrix}\psi \\ \varphi\end{pmatrix}$$

NLS $\quad iq_t + q_{xx} \mp 2|q|^2q = 0$

linear spectral problème

$$\mathcal{L}v = -v_{xx} + uv = \lambda v$$

$\left\{\begin{array}{l}\psi_x = \cancel{ik\,q + q^2} \\ q_x = \cancel{q^2} + ikq\end{array}\right.$

$\left\{\begin{array}{l}q_{xx} = 2r(-ikr + q^2) \\ \quad + ik(r + ikq)\end{array}\right.$

$= 2rq^2 - ik^2q$

$\left\{\begin{array}{l}q_x = \cancel{-ikq + qr} \\ r_x = \cancel{rq} + ikr\end{array}\right.$

$$q_{xx} = \cancel{(-ik)(-ikq + qr)} + q(rq + ikr) = -k^2q - ikqr + iq + q^2r$$

$$\begin{pmatrix} r \\ q \end{pmatrix}_x = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}\begin{pmatrix} r \\ q \end{pmatrix} \qquad \begin{pmatrix} r \\ q \end{pmatrix}_{xx} = \begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix}\begin{pmatrix} r \\ q \end{pmatrix} + \begin{pmatrix} r & ik \end{pmatrix}\begin{pmatrix} q \end{pmatrix}$$

$$\begin{pmatrix} r \\ q \end{pmatrix}_{xx} = \left(\begin{pmatrix} 0 & r^2 + ikq \\ -ikr + q^2 & 0 \end{pmatrix} + \underbrace{\begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}^2}_{\begin{pmatrix} -k^2 + rq & 0 \\ & -k^2 + rq \end{pmatrix}} \right)\begin{pmatrix} r \\ q \end{pmatrix}$$

detuning DNID

pompe detunne -20

$$\begin{pmatrix} r \\ q \end{pmatrix}_{xx} = \left\{ \begin{pmatrix} -k^2 + rq & r^2 + ikq \\ -ikr + q^2 & -k^2 + rq \end{pmatrix} \right\}\begin{pmatrix} r \\ q \end{pmatrix} = \begin{cases} r_{xx} = 2r^2 q + ikq^2 - k^2 r \\ q_{xx} = 2rq^2 - ikr^2 - k^2 q \end{cases}$$

$$\left(v^i_{,x} = A^i_j v^i \qquad v^i_{xx} = A^i_{j,x} v^i + A^i_j v^i_x \right.$$

$$\left. A^{ii} = \text{constante} \quad (A^{ii})^2 = -d \quad A^{ij} = q, r \right)$$

$$r_{xx} = -ikr_x + 2qq_x = -ik(-ikr + q^2) + 2q(r^2 + ikq)$$

$$= -k^2 r + ikq^2 + 2qr^2$$

$$q_{xx} = -k^2 q - ikr^2 + 2rq^2 = q(-ik + r) \qquad = (-ikq + qr)(-ik + r) + q(rq + i$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_x = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}\begin{pmatrix} q \\ r \end{pmatrix} \Rightarrow \begin{cases} q_x = -ikq + qr \\ r_x = rq + ikr \\ \quad = r(q + ik) \end{cases} \Rightarrow \begin{cases} q_{xx} = \cancel{\ldots} q + qr) \\ r_{xx} = (rq + ikr)(q + ik) + r(-ikq + q) \end{cases}$$

$$\Rightarrow \begin{cases} q_{xx} = q(-ik + r)(-ik + r) + qr(q + ik) = q(r^2 - 2ikqr - k^2 q + qrq + ikqr \\ r_{xx} = r(q + ik)^2 \qquad + qr(qr - ik) \end{cases}$$

...ttering for LODE at $t=0$ $\longrightarrow S(\lambda,0)$

...ution to NPDE

$$\begin{array}{ccc} & & S(\lambda,0) \\ & & \Big\downarrow \text{ time evolution of scattering data} \\ u(x,t) & \xleftarrow{\text{inverse scattering for LODE at time } t} & S(\lambda,t) \end{array}$$

Inverse Scattering Transform

(i) Solve the corresponding direct scattering problem for the associated LODE at $t=0$ ie determine the initial scattering data $S(\lambda,0)$ from the initial $u(x,0)$

(ii) Time evolution the scattering data from its initial value $S(\lambda,0)$ to its value $S(\lambda,t)$ at time t. Such an evolution is usually a simple one and is particular to each integrable NPDE.

(iii) Solve the corresponding inverse scattering problem for the associated LODE at fixed t ie determine the potential $u(x,t)$ from the scattering data $S(\lambda,t)$

The Lax Method Spectral problem $\mathcal{L}\Psi=\lambda\Psi \to \{\mathcal{L},A\}$
$\Psi_t = A\Psi$

(i) $\lambda_t = 0$ ie The spectral parameter λ does not change in time

(ii) The quantity $\Psi_t - A\Psi$ remain a solution to the same lineare $\mathcal{L}\Psi=\lambda\Psi$

(iii) The quantity $\mathcal{L}_t + \mathcal{L}A - A\mathcal{L}$ is a multiplication operator ie is not a differential opperator

(ii) $\mathcal{L}(\Psi_t - A\Psi) = \lambda(\Psi_t - A\Psi)$ (4,1)

$$\mathcal{L}\Psi = \lambda\Psi \oplus \lambda_t = 0 \oplus (4.1) \Longrightarrow \mathcal{L}\Psi_t - \mathcal{L}A\Psi = \lambda\Psi_t - A(\lambda\Psi) = \partial_t(\mathcal{L}\Psi) - A\mathcal{L}\Psi = \mathcal{L}_t\Psi + \mathcal{L}\Psi_t - A\mathcal{L}\Psi \quad (4.2)$$

where ∂_t denotes the partial differential opperator with respect to t

$$\Rightarrow -\mathcal{L}A\Psi = \mathcal{L}_t\Psi - A\mathcal{L}\Psi \Rightarrow \mathcal{L}_t + \underbrace{\mathcal{L}A - A\mathcal{L}}_{[\mathcal{L},A]} = 0 \quad (4,3)$$

an evolution equation containing a first order time derivative

- The integrable NPDE known as the defocusing/focusing NLS

$$iu_t + u_{xx} \mp 2|u|^2 u = 0$$

→ is associated with the system of first-order LODEs known as the Zakharov-Shabat system

Zakharov-Shabat spectral

$$\underbrace{\begin{pmatrix} \xi \\ \eta \end{pmatrix}_x}_{\partial_x(\,)} = \begin{pmatrix} -i\lambda & u(x,t) \\ \underbrace{\pm u^*(x,t)}_{r} & i\lambda \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\Rightarrow \mathcal{L}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} i\partial_x & -iu \\ \pm u^* & -i\partial_x \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \times(-i)$$

$$\begin{cases} \partial_x \xi = -i\lambda\xi + u\eta \\ \partial_x\eta = i\lambda\eta \pm u^*\xi \end{cases} \Rightarrow \begin{matrix} i\lambda\xi = -\partial_x\xi + u\eta \\ -i\lambda\eta = -\partial_x\eta \pm u^*\xi \end{matrix} \Rightarrow \begin{matrix} \lambda\xi = i\partial_x\xi - iu\eta \\ \lambda\eta \overset{\times i}{=} -i\partial_x\eta \pm u\xi \end{matrix}$$

$$\mathcal{L} = \begin{pmatrix} i\partial_x & -iu \\ \pm u^* & -i\partial_x \end{pmatrix} \qquad A = \begin{bmatrix} 2i\partial_x^2 \mp i|u|^2 & -2iu\partial_x - iu_x \\ \pm 2iu^*\partial_x \pm iu_x^* & -2i\partial_x^2 \pm i|u|^2 \end{bmatrix}$$

defocussing / focussing

Rappel pour stationary Schrödinger equation $-\varphi_{xx} + u\varphi = \lambda\varphi$

$$\lambda = k^2 \qquad \begin{matrix} ik\psi = -\partial_x\psi + u\varphi \\ -ik\varphi = -\partial_x\varphi + \psi \\ ik\varphi = \partial_x\varphi - \psi \end{matrix} \Rightarrow ik\begin{pmatrix} \psi \\ -\varphi \end{pmatrix} = \begin{pmatrix} -\partial_x & u \\ 1 & -\partial_x \end{pmatrix}\begin{pmatrix} \psi \\ \varphi \end{pmatrix}$$

$$ik\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} -\partial_x & u \\ -1 & \partial_x \end{pmatrix}\begin{pmatrix} \psi \\ \varphi \end{pmatrix}$$

$$-k^2\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} \partial_x^2 - u & 0 \\ 0 & \partial_x^2 - u \end{pmatrix}\begin{pmatrix} \psi \\ \varphi \end{pmatrix} \Rightarrow \mathcal{L}\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \lambda\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = (\partial_x^2 + u)\begin{pmatrix} \psi \\ \varphi \end{pmatrix}$$

... spectrum $\mathbb{R}$ and associated eigenfunctions: Jost functions

$$L\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = \begin{pmatrix} i\partial_x & -iu \\ \pm u^* & -i\partial_x\end{pmatrix}\begin{pmatrix}\psi\\ \varphi\end{pmatrix} \xrightarrow[x\to\pm\infty]{u\to 0}$$

$$\varphi(x,\lambda) \sim \begin{pmatrix}1\\0\end{pmatrix} e^{+i\lambda x} \qquad \bar{\Phi}(x,\lambda) \sim \begin{pmatrix}0\\1\end{pmatrix} e^{-i\lambda x} \quad x\to-\infty$$

$$\psi(x,\lambda) \sim \begin{pmatrix}0\\1\end{pmatrix} e^{-i\lambda x} \qquad \bar{\psi}(x,\lambda) \sim \begin{pmatrix}1\\0\end{pmatrix} e^{i\lambda x} \quad x\to+\infty$$

Function with constant boundary conditions:

$$M(x,\lambda) = e^{i\lambda x}\Phi(x,\lambda)$$
$$N(x,\lambda) = e^{\mp i\lambda x}\Phi(x,\lambda)$$

Property: $\partial_x M(x,\lambda) = \left(-i\lambda\Phi(x,\lambda) + \Phi_x(x,\lambda)\right)e^{-i\lambda x}$

$$= \left(-i\lambda\psi(x,\lambda) + \left(i\lambda\varphi(x,\lambda)\right) \mp u^*\psi(x,\lambda)\right)e^{-i\lambda x}$$

$$= \mp u^*\psi(x,\lambda)\,e^{-\lambda x}$$

afterward

Classical field theory $H = \int dx\left[\frac{1}{2}|\partial\psi_x|^2 + \frac{k}{2}|\psi|^4\right]$

Non LS Classical: $i\psi_t = -\frac{1}{2}\psi_{xx} + k|\psi|^2\psi$ $\quad \{\psi(x), \psi(y)\} = \{\psi^*(x), \psi^*(y)\} = 0$

$\{\psi^*(x), \psi(y)\} = i\delta(x-y)$

Quantum mechanics

$$H = \int dx\left[\frac{1}{2}\psi^+_x\psi_x + \frac{k}{2}\psi^+\psi^+\psi\psi\right]$$

Solving the equation

$$\phi_x = J\phi\Lambda + U\phi$$
$$\phi_t = 2J\phi\Lambda^2 + 2U\phi\Lambda + (JU^2 - JU_x)\phi$$

$$\Lambda = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},\quad J = i\sigma = \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad U = i\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$$

→ Zakharov - Shabat system $\phi_{xt} = \phi_{tx} \Rightarrow U_t = -JU_{xx} + 2JU^2U \Leftrightarrow \begin{cases} iq_t = q_{xx} + 2qrq \\ -ir_t = -r_{xx} - 2qrr\end{cases}$

By setting $q = r^*$ or $q = -r^*$ the NLS attractive/repulsive interaction.

following Darboux transformation $\phi \to \phi[1] = \phi\Lambda - \sigma\phi$ $\qquad \varphi_x = J\varphi\Omega + U\varphi$

$U \to U[1] = U + [J,\sigma]$ $\qquad \varphi_t = 2J\varphi\Omega^2 + 2U\varphi\Omega + (JU^2 - JU_x)\varphi$

$\sigma = \varphi\Omega\varphi^{-1}$

which leaves the system invariant

Si $M(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} 0 & \mu(x') \\ r(x')\, e^{2i\lambda(x-x')} & 0 \end{pmatrix} M(x',\lambda')\, dx'$

$$\partial_x M(x,\lambda) = \begin{pmatrix} 0 & \mu(x) \\ r(x) & 0 \end{pmatrix} M(x,\lambda)$$

$$\Updownarrow$$

$$\left(-i\lambda \phi(x,\lambda) + \phi_x(x,\lambda)\right) e^{-i\lambda x} = \left(\cancel{-i\lambda\phi} + \cancel{i\lambda\phi} + r\psi\right) e^{-i\lambda x} = r$$

$$\partial_x \left(e^{-i\lambda x} \begin{pmatrix} \phi \\ \psi \end{pmatrix}(x,\lambda)\right) = e^{-i\lambda x}\left(-i\lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \begin{pmatrix} \phi \\ \psi \end{pmatrix}_x (x,\lambda)\right)$$

$$= e^{-i\lambda x} \underbrace{\left\{ \begin{pmatrix} -i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} + \begin{pmatrix} -i\lambda & \mu \\ r & i\lambda \end{pmatrix} \right\}}_{\begin{pmatrix} -2\lambda i & \mu \\ r & 0 \end{pmatrix}} \begin{pmatrix} \phi \\ \phi \end{pmatrix}(x,\lambda)$$

$$= \begin{pmatrix} -2\lambda i & \mu \\ r & 0 \end{pmatrix} e^{-i\lambda x} \begin{pmatrix} \phi \\ \psi \end{pmatrix}(x,\lambda)$$

$$M(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} -2\lambda i & \mu(x') \\ r(x') & 0 \end{pmatrix} M(x',\lambda)\, dx'$$

…ering data : The reflection coefficient : $\phi = (\phi^{(1)}, \phi^{(2)})^t$ $\bar\phi$ linearly …

$$W(\phi,\bar\phi) = \begin{vmatrix} \phi^{(1)} & \bar\phi^{(1)} \\ \phi^{(2)} & \bar\phi^{(2)} \end{vmatrix} = \phi^{(1)}\bar\phi^{(2)} - \phi^{(2)}\bar\phi^{(1)} \qquad \partial_x W(\phi,\bar\phi) = \begin{vmatrix} \phi_x^{(1)} & \bar\phi^{(1)} \\ \phi_x^{(2)} & \bar\phi^{(2)} \end{vmatrix} + \begin{vmatrix} \phi^{(1)} & \bar\phi_x^{(1)} \\ \phi^{(2)} & \bar\phi_x^{(2)} \end{vmatrix}$$

$$W_x(\phi,\bar\phi) = W(\phi_x,\bar\phi) + W(\phi,\bar\phi_x) = 0$$

$$W(\phi_x,\bar\phi) = W(\bar\phi_x,\phi) = -W(\phi,\bar\phi_x)$$

$W(\phi,\bar\phi)$ independent de x $\qquad W(\phi,\bar\phi) = \lim_{x\to+\infty} W(\phi,\bar\phi) = \begin{vmatrix} e^{idx} & 0 \\ 0 & e^{-idx} \end{vmatrix} = 1$

$$\phi \sim \binom{1}{0} e^{+idx} \qquad \bar\phi \underset{x\to-\infty}{\sim} \binom{0}{1} e^{-idx}$$

De meme

$$W(\psi,\bar\psi) = \lim_{x\to+\infty} W(\psi,\bar\psi) = \begin{vmatrix} 0 & e^{idx} \\ e^{-idx} & 0 \end{vmatrix} = -1$$

$$\psi \underset{x\to+\infty}{\sim} \binom{0}{1} e^{-idx} \qquad \bar\psi \underset{x\to+\infty}{\sim} \binom{1}{0} e^{idx}$$

donc $W(\psi,\bar\psi) = -W(\phi,\bar\phi)$

$$\begin{vmatrix} \phi^{(1)} & \bar\phi^{(1)} \\ \phi^{(2)} & \bar\phi^{(2)} \end{vmatrix} = \begin{vmatrix} \bar\psi^{(1)} & \psi^{(1)} \\ \bar\psi^{(2)} & \psi^{(2)} \end{vmatrix} \iff \phi^{(1)}\bar\phi^{(2)} - \phi^{(2)}\bar\phi^{(1)} = \bar\psi^{(1)}\psi^{(2)} - \bar\psi^{(2)}\psi^{(1)}$$

ψ et $\bar\psi$ lineaire indep

$$\phi = b(d)\psi + a(d)\bar\psi$$
$$\bar\phi = \bar a(d)\psi + \bar b(d)\bar\psi(x,d)$$

$\cancel{\phi \underset{x\to-\infty}{\sim} \binom{1}{0} e^{+idx}}$

$$(b\psi^{(1)} + a\bar\psi^{(1)})(\bar a\psi^{(2)} + \bar b\bar\psi^{(2)})$$

$$\begin{vmatrix} b\psi^{(1)} + a\bar\psi^{(1)} & \bar a\psi^{(1)} + \bar b\bar\psi^{(1)} \\ b\psi^{(2)} + a\bar\psi^{(2)} & \bar a\psi^{(2)} + \bar b\bar\psi^{(2)} \end{vmatrix} = \left[b\bar a\cancel{\psi^{(1)}\psi^{(2)}} + |b|^2\psi^{(1)}\bar\psi^{(2)} + |a|^2\bar\psi^{(1)}\psi^{(2)} + \right] a\bar b\cancel{\bar\psi^{(1)}\bar\psi^{(2)}}$$

$$- \left[b\bar a\cancel{\psi^{(2)}\psi^{(1)}} + |b|^2\psi^{(2)}\bar\psi^{(1)} + |a|^2\bar\psi^{(2)}\psi^{(1)} + a\bar b\cancel{\bar\psi^{(2)}\bar\psi^{(1)}} \right]$$

$$\underset{+\infty}{\to} \begin{vmatrix} b\,0 + ae^{idx} & \bar b e^{+idx} \\ be^{-idx} & \bar a e^{-idx} \end{vmatrix} \quad \text{tout commute}$$

$\hookrightarrow a^2 - b^2 = 1$

$$= |b|^2\left(\psi^{(1)}\bar\psi^{(2)} + \bar\psi^{(1)}\psi^{(2)} - \psi^{(2)}\bar\psi^{(1)} - \bar\psi^{(2)}\psi^{(1)}\right)$$
$$\left(\bar\psi^{(1)}\psi^{(2)} - \bar\psi^{(2)}\psi^{(1)}\right)$$

aucun calcul en $\pm\infty$ $\qquad = -W(\psi,\bar\psi)$

$$W(\phi,\psi) = \cancel{b\psi + a} \; W(b\psi + a\bar{\psi}, \psi) = W(a\psi, \ldots$$

$$C_1 = C_1 - b\psi$$

$$= a\, W(\psi, \bar{\psi}) = a$$

$$W(\underbrace{e^{-idx}\phi}_{M}, \underbrace{e^{+idx}\psi}_{N}) = W(\phi,\psi)$$

$\underline{a(k) = W(\phi,\psi) = W(M,N)}$

$$W(\bar{M},\bar{N}) = W\left(e^{idx}\bar{\phi}, e^{-idx}\bar{\psi}\right) = W(\bar{\phi},\bar{\psi}) = W(\bar{a}\psi + \bar{b}\bar{\psi}, \bar{\psi})$$

$$= \bar{a}\, \underbrace{W(\psi,\bar{\psi})}_{-1} = -\bar{a}$$

$\underline{\bar{a}(k) = -W(\bar{\phi},\bar{\psi}) = -W(\bar{M},\bar{N})}$

$$W(\phi,\bar{\psi}) = W(b\psi + a\bar{\psi}, \bar{\psi}) = bW(\psi,\bar{\psi}) = -b$$

$\underline{b(k) = -W(\phi,\bar{\psi})}$

$$W(\bar{\phi},\psi) = W(\bar{a}\psi + \bar{b}\bar{\psi}, \psi) = \bar{b}\, W(\bar{\psi},\psi) = -b$$

$\underline{\bar{b}(k) = W(\bar{\phi},\psi)}$

$\int_a f(x,t)\,dt$ $\quad \frac{d}{dt}$ lin lazory

$= F(x,x) - F(x,a)$

$$\frac{d}{dx}\int_a^x f(x,t)\,dt = \frac{d}{dx}\Big(F(x,x) - F(x,a)\Big) =$$

$$= f(x,x) - \cancel{f(x,a)} + \int_a^x \partial_x f(x,t)\,dx$$

Leibniz

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = v'(x) f(x,v(x)) - u'(x) f(x,u(x)) + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} f(x,t)$$

$$\frac{d}{dx}\int_{-\infty}^{x}\begin{pmatrix} 0 & u(x') \\ r(x')\,e^{2id(x-x')} & 0\end{pmatrix}\Pi(x',d)\,dx' = \begin{pmatrix} 0 & u(x) \\ r(x) & 0\end{pmatrix}\Pi(x,d) + \int_{-\infty}^{x}\begin{pmatrix} 0 & u(x') \\ 2id\,r(x')\,e^{2id(x-x')} & 0\end{pmatrix}\Pi(x',d')\,dx'$$

Mais j'ai $\partial_x\Big(\underbrace{e^{-idx}\begin{pmatrix}\phi\\ \psi\end{pmatrix}(x,d)}_{\Pi(x,d)}\Big) = \begin{pmatrix} -2di & u(x) \\ r(x) & 0\end{pmatrix}\Pi(x,d)$

$$\neq \partial_x\int_{-\infty}^{x}\begin{pmatrix} 0 & u(x') \\ r(x')\,e^{2id(x-x')} & 0\end{pmatrix}\Pi(x',d')\,dx'$$

$$= \begin{pmatrix} 0 & u(x) \\ r(x) & 0\end{pmatrix}\Pi(x,d)$$

Sauf si $\begin{pmatrix} -2di & 0 \\ 0 & 0\end{pmatrix}\Pi(x,d) + \int_{-\infty}^{x}\begin{pmatrix} 0 & u(x') \\ 2id\,r(x')\,e^{2id(x-x')} & 0\end{pmatrix}\Pi(x',d')\,dx'$

... and zero-curvature representation $\quad i\psi_t + \frac{1}{2}\psi_{xx} \mp |\psi|^2\psi = 0 \quad$ (NLS)

defocusing (upper sign), focusing (lower sign)

$$\partial_x v = U v \qquad U = U(x,t,\lambda) \doteq \begin{bmatrix} -i\lambda & u(x,t) \\ r(x,t) & i\lambda \end{bmatrix}$$

defocusing

$$\mathcal{L} v = \lambda v \qquad \mathcal{L} = \mathcal{L}(x,t,\lambda) \doteq \begin{bmatrix} i\partial_x & -iu(x,t) \\ \pm u^*(x,t) & -i\partial_x \end{bmatrix} \qquad u = \psi \quad r = \pm\psi$$

~~$v_t = T v \qquad T = T(x,t,\lambda) \doteq \Big[-i\lambda^2 \mp i\frac{1}{2}|u|^2 \; - 2iu\partial_x +$~~

$$v_{,t} = V v \qquad V = V(x,t,\lambda) = \begin{bmatrix} -i\lambda^2 \mp i\frac{1}{2}|u|^2 & \lambda u + i\frac{1}{2}u_x \\ \pm \lambda u^* \pm i\frac{1}{2}u^*_x & i\lambda^2 \pm i\frac{1}{2}|u|^2 \end{bmatrix}$$

$$v_{,t} = A v \qquad A = A(x,t,\lambda) = \begin{bmatrix} i\partial_x^2 \mp \frac{i|u|^2}{2} & -iu\partial_x - i\frac{1}{2}u_x \\ \pm iu^*\partial_x \pm \frac{1}{2}iu^*_x & -i\partial_x^2 \pm \frac{1}{2}i|u|^2 \end{bmatrix}$$

$$\mathcal{L}_{,t} + [\mathcal{L}, A] = 0 \quad \text{et} \quad U_{,t} - V_{,x} + [U, V] = 0$$

<u>Linearized theory</u> $\quad i\psi_{,t} + \frac{1}{2}\psi_{,x,x} = 0 \quad \psi(x,0) = \psi_0(x)$

$$\hat\psi(k,t) \doteq \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x,t)\, e^{-ikx}\, dx \qquad i\hat\psi_t - \frac{k^2}{2}\hat\psi = 0 \qquad \hat\psi(k,0) = \hat\psi_0(k)$$

$$\hat\psi(k,t) = e^{-ik^2t/2}\hat\psi(k,0) = e^{-ik^2t/2}\hat\psi_0(k) \qquad \psi(x,t) = \int_{-\infty}^{\infty} \hat\psi(k,t)\, e^{ikx}\, dk$$

<u>The direct transform</u> $\quad U(x,t,\lambda) \underset{|x|\to\infty}{=} -i\lambda\sigma_3 + o(1) \qquad \sigma_3 \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$w = w(x;\lambda) \qquad w_{,x} = U w \underset{|x|\to\infty}{\Longrightarrow} w_{0,x} = -i\lambda\sigma_3 w_0 \Longrightarrow w_0(x;\lambda) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\lambda x} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\lambda x}$$

Jost solutions. $f_-^{(1)}(x;d) \underset{x\to-\infty}{=} \begin{bmatrix}1\\0\end{bmatrix} e^{-idx} + o(1)$ $\quad f_-^{(2)} = \overline{f_-^{(1)}}$ "

$$f_-^{(2)}(x;d) \underset{x\to-\infty}{=} \begin{bmatrix}0\\1\end{bmatrix} e^{idx} + o(1)$$

$\underline{J}(x;d) = [f_-^{(1)}(x,d), f_-^{(1)}(x;d)]$ $\quad W(\underline{J}(x;d)) = W(f_-^{(1)}, f_-^{(2)})(x,d)$

$$\underline{\partial_x W(f_-^{(1)}, f_-^{(2)})} = W(\partial_x f_-^{(1)}, f_-^{(2)}) + W(f_-^{(1)}, \partial_x f_-^{(2)})$$

$$W(f_-^{(1)}, \partial_x f_-^{(2)}) = W(\overline{f_-^{(1)}}, \partial_x \overline{f_-^{(1)}}) = W(f_-^{(1)}, \partial_x f_-^{(1)})$$

<u>Abels's Formula</u>: y_1 et y_2 solutions of $y'' + p(x)y' + q(x)y = 0$ $\quad W(y_1, y_2)(x) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2'\end{pmatrix}$

$$W_{,x}(y_1, y_2)(x) = \underbrace{p(x)}_{-tr(P(x))} W(y_1, y_2)(x) \qquad W(x) = W(x_0) \exp\int_{x_0}^{x} p(t)\,dt$$

$$y'' + p(x)y' + q(x)y = 0 \Rightarrow \begin{pmatrix} y \\ y_{,x}\end{pmatrix}_{,x} = \underbrace{\begin{pmatrix} 0 & 1 \\ -q(x) & -p(x)\end{pmatrix}}_{P(x)} \begin{pmatrix} y \\ y_{,x}\end{pmatrix}$$

$\rightarrow W(f_-^{(1)}, f_-^{(2)})$ $\quad \cancel{W_{,x}(f_{-,x}^{(1)}, f_-^{(1)}}$ $\quad W_{,x}(f_-^{(1)}, f_-^{(2)}) = W(f_{-,x}^{(1)}, f_-^{(2)}) + W(f_-^{(1)}, f_{-,x}^{(2)})$

$$W_{,x}(f_-^{(1)}, f_-^{(2)}) = W(U f_-^{(1)}, f_-^{(2)}) + W(f_-^{(1)}, U f_-^{(2)})$$

- $\partial_x w = Uw \quad (1)$

<u>Wronskian</u>: $w_1, \dots, w_n$ solutions of (1) $\quad W(\{w_n\})$ $\quad W_{,x}(\{w\}) = \sum W(\{w_i : \partial w_i, w\})$

$$W_{,x}(\{w\}) = \sum_{i=1}^{n} W(w_1, \dots, \partial_x w_i, \dots w_n) = \sum_{i=1}^{n} W(w_1, \dots, Aw_i, \dots, w_n)$$

$(A) = \varepsilon^{i_1 \dots i_n} A^1_{i_1} \dots A^n_{i_n} =$

$$\det_B(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{j=1}^{n} X_{\sigma(j),j}$$

$$\det_B(x_1, \dots, \underset{i}{a+db}, \dots, x_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{\substack{j=1 \\ j \neq i}}^{n} X_{\sigma(j),j} \times \underbrace{X_{\sigma(i),i}}_{a_{\sigma(i)} + d b_{\sigma(i)}}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{\substack{j=1 \\ j \neq i}} X_{\sigma(j),j} \times a_{\sigma(j)} + d \dots$$

$$= \det_B(x_1, \dots, a, \dots, x_n) + d \det_B(x_1, \dots, b, \dots, x_n)$$

$$\det_B(x_1, \dots, Ax_i, \dots, x_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{\substack{j=1 \\ j \neq i}} X_{\sigma(j),j} \times \Big(\sum_k A_{\sigma(i)k} x_k\Big)$$

$X_{k,i}$

$$(Ax)^i = A^i_j x^j = \sum_j A_{ij} x_j$$

(A

$$= \cancel{\sum_{\sigma \in S_n}} \sum_k \cancel{A}$$

$$= \varepsilon^{i_1 \dots i_n} X^1_{i_1} \dots A^i_j X^{j i_i} \dots X^n_{i_n}$$

$$= \sum_k x_k \underbrace{\sum_{\sigma \in S_n} \varepsilon(\sigma) \Big(\prod_{\substack{j=1 \\ j \neq i}} (X_{\sigma(j),j}) \times A_{\sigma(i),k}\Big)}$$

$$\sum_k x_k \quad \det(X_1, \dots, (A)_k, \dots x_n)$$

$$\sum_i \cancel{\mathrm{Tr}}\ \det(x_1 \dots \cancel{A}x_i, \dots x_n)$$

$=$

- Dans "Determinants - Systèmes linéaire de Denis Vekemans"

Propositie 2 l'ensemble des n-formes linéaire alternés de E un $\mathbb{K}$e espace vectore de dimention 1 $\mathbb{K}$ pas de "caracteristique 2" est de dimension 1

Algebre Gourdon page 145 $\quad f_u : E^n \to \mathbb{K} \quad (x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} \underbrace{f(x_1, \dots, x_{i-1}, \underbrace{u(x_i)}_{\mathcal{L}(E)}, x_{i+1}, \dots, x_n)}_{\mathcal{A}^n(E)}$

$f_u = \mathrm{trac}(u) f$

$\partial_x \omega = A\omega \qquad W_{\partial x}(\{\omega_i\}) = \det(\{\omega_i\})$

$$W_{,x}(\{\omega_i\}) = \sum_{i=1}^{n} W(\{\omega_1, \dots, A(\omega_i), \dots \omega_n\}$$

$$= \mathrm{trac}(a)\, W(\{\omega_i\})$$

or

On défine U comme avant $\mathrm{trace}\ U = 0$ donc $W_{,x}\{J_-(x,d)\} = 0$

donc $W(J_-(x,d)) = \lim_{x \to -\infty} W(J_-(x,d)) = \lim_{x \to -\infty} \det(e^{-idx\sigma_x} + o(1)) = 1$

Si $d \in \mathbb{R}$ sinon ⚠

$$\lim_{x \to -\infty} J_-(x;d) e^{idx\sigma_x} = 1 \quad d \in \mathbb{R}.$$

$+o(1)$; $j_+^{(1)}(x;d) = \begin{pmatrix}0\\1\end{pmatrix} e^{idx} + o(1)$ $\quad J_+(x;d) \doteq (j_+^{(1)}(x;d) \;\; j_+^{(2)}(x;d))$
$x \to +\infty$

$w(J_+(x;d)) = +1 \qquad \lim_{x\to+\infty} J_+(x,d) e^{idx\sigma_x} = \mathbb{1}$

Scattering matrix $\quad J_{\pm,x} = UJ_\pm \qquad j_-^{(1)}$ et $j_+^{(2)}$ linéairement indépendantes

$$J_+(x;d) = J_-(x;d)S(d)$$

↳ scattering matrix associated with the coefficient $\psi(x)$ in U

Then $S(d) \doteq J_-(x;d)^{-1} J_+(x;d) \qquad \det(S(d)) = 1 \quad d \in \mathbb{R}$

pour la défocusing

$$U(x,t,d)^* = \sigma_1 U(x,t,d)\sigma_1 \; , \; d \in \mathbb{R} \; , \; \sigma_1 \doteq \begin{bmatrix}0&1\\1&0\end{bmatrix} \qquad U(x,t,d) = \begin{pmatrix}-id & u(x,t)\\ r(x,t) & id\end{pmatrix}$$

$$= \sigma_1 \begin{pmatrix}u(x,t) & -id\\ id & r(x,t)\end{pmatrix} = \begin{pmatrix}id & r(x,t)\\ u(x,t) & -id\end{pmatrix}$$

$\cancel{\text{def}}$ pas $\dagger$

$$\begin{pmatrix}-id & u\\ u^* & id\end{pmatrix}^* = \begin{pmatrix}id & u^*\\ u & -id\end{pmatrix} = \sigma_1 \underbrace{\begin{pmatrix}-id & u\\ u^* & id\end{pmatrix}}_{U} \sigma_1$$

en focusing c'est faux

$$\sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \sigma_2 \begin{pmatrix}\cancel{\pm}id & u\\ \pm u^* & id\end{pmatrix} \sigma_2 = \sigma_2 \begin{pmatrix}iu & -d\\ -d & iu^*\end{pmatrix}$$

$$= \begin{pmatrix}id & \mp u^*\\ -u & -id\end{pmatrix}$$

Ça marche que pour la défocusing
⚠

donc que pour la défocusing $\quad U = \begin{pmatrix}-id & u\\ u^* & id\end{pmatrix} \qquad U^* = \sigma_1 U \sigma_1$

$$\Rightarrow j_\pm^{(1)}(x,d) = \sigma_1 j_\pm^{(1)}(x;d)^* \quad d \in \mathbb{R} \qquad (Uj_\pm)^* = \cancel{j^* U^*} = j_\pm^* \sigma_1 U \sigma_1$$

$\cancel{(\sigma_1^{-1} U \sigma_1 j_\pm =}$

asymptotic solution

$$J_\pm^*(x;d)^{\cancel{*}} = \sigma_1 J(x;d)\sigma_1$$

$S(d) = J_-(x;d)^* J_+(x;d)$ $\quad S^*(d) = J_+^{*-1}(x;d) J_+ \ldots$

$*$ -conjugu

$= \sigma_1 J_+(x;d) J_-^{-1}(x,d)\sigma_1$

~~commutation~~ ?

$$\Rightarrow S(d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad S^*(d) = \begin{pmatrix} \cancel{a^*} & \cancel{c^*} \\ \cancel{b^*} & \cancel{d^*} \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

$$\Rightarrow \sigma_1 S(d) \sigma_1 = \sigma_1 \begin{pmatrix} b & a \\ d & c \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

$$S(d) = \begin{pmatrix} a(d)^* & -b(d)^* \\ -b(d) & a(d) \end{pmatrix} \qquad d \in \mathbb{R}$$

$$|a(d)|^2 - |b(d)|^2 = 1 \quad " \quad |R(d)|^2 + |T(d)|^2 = 1$$

$R(d) = \dfrac{b(d)}{a(d)}$, $\quad T(d) = 1/a(d)$

$$\text{et } J_-(x;d) = J_+(x;d) S(d)^{-1} = J_+(x;d) \begin{bmatrix} a(d) & b(d)^* \\ b(d) & a(d) \end{bmatrix}$$

$$w(x;d) = a(d)^{-1} J_-^{(1)}(x;d) \qquad w(x;d) \underset{x\to+\infty}{=} \begin{bmatrix} e^{-idx} \\ 0 \end{bmatrix} + R(d) \begin{bmatrix} 0 \\ e^{idx} \end{bmatrix} + o(1)$$

$$\underset{x\to-\infty}{=} T(d) \begin{bmatrix} e^{-idx} \\ 0 \end{bmatrix} + o(1)$$

Definition : (Direct transformation)

$$\psi(x) = \begin{cases} 0 & |x|>L \\ B & |x|\leq L \end{cases}$$

$$w_{,x} = -id\sigma_x w \quad |x|>L \qquad w_{,x} = U_B(d)\, w \qquad U_B(d) = \begin{bmatrix} -id & B \\ B^* & id \end{bmatrix}$$

$$J_-(x;d) = e^{-idx\sigma_3}, \; x<-L \quad ; \quad J_+(x;d) = e^{-idx\sigma_x}, \; x>L$$

<u>Pour $|x|<L$</u> $\quad J_-(x;d) = e^{xU_B(d)}C(d)$, 4 $C(d)$? pour $x=-L$

$$\lim_{x\uparrow -L} J_-(x;d) = e^{idL\sigma_3} \text{ and } \lim_{x\downarrow -L} J_-(x;d) = e^{-LU_B(d)}C(d) \Rightarrow C(d) = e^{LU_B(d)} e^{idL\sigma_3}$$

$$S(d) = J_-(L;d)^{-1} J_+(L;d) = C(d)^{-1} e^{-LU_B(d)} e^{-idL\sigma_3} = e^{-idL\sigma_3} e^{-2LU_B(d)} e^{-idL\sigma_3}$$

$$S(d) \underset{B\to 0}{\Longrightarrow} e^{-idL\sigma_3} \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left(2idL\sigma_3 - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} \right)^n \right\} e^{-idL\sigma_3}$$

$$\sum_{j=1}^{n} \binom{n}{j} (2idL\sigma_3)^j \left(-\begin{pmatrix} 0 & 2LB \\ 2LB^* & 0 \end{pmatrix} \right)^{n-j}$$

$$(a+b)^n = \sum_k \binom{n}{k} a^k b^{n-k}$$

faux σ_3 et σ_2 ne commutent pas

$=$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$$

$$\cancel{\sum_{n=0}^{\infty} \frac{1}{n!} (2idL\sigma_3)^n (1 - (}$$

$$\sigma_j \sigma_k = \delta_{jk} I + i\epsilon_{jkl}\sigma_l \Longrightarrow \begin{cases} \sigma_3\sigma_1 = i\epsilon_{312}\sigma_2 \\ \sigma_3\sigma_2 = -i\,\sigma_1 \end{cases}$$

$$\left(2idL\sigma_3 - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} \right)^n = (2idL\sigma_3)^n + \sum_{k=1}^{n} (2idL\sigma_3)^{k-1} \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} (2idL\sigma_3)^{n-k} + O(B^2)$$

$$\underset{B\to 0}{=} e^{-idL\sigma_3} \left\{ e^{2idL\sigma_3} - \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} (2idL\sigma_3)^{k-1} \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} (2idL\sigma_3)^{n-k} \right\} e^{-idL\sigma_3} + O(B^2)$$

$$= e^{-idL\sigma_3} \left\{ e^{2idL\sigma_3} - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} \sum_{n=1}^{\infty} \frac{1}{n!} (2idL\sigma_3)^{n-1} \sum_{k=1}^{n} (-1)^{k-1} \right\} e^{-idL\sigma_3} + O(B^2)$$

$$= \mathbb{1} - e^{-idL\sigma_3} \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} \frac{e^{2idL\sigma_3}}{2idL}\sigma_3 \sum_{k=1} (-1)^{k-1} \; e^{2idL\sigma_3} / 2idL\sigma_3 + O(B)$$

$$\begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix} = a\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_1} + b\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_2}$$

$g = a + ib$

$$= (\quad)\sigma_3 + 2\begin{pmatrix} 0 & \overbrace{a-ib}^{y^*} \\ \underbrace{-a-ib}_{-y} & 0 \end{pmatrix}$$

$$\sigma_3^{k-1} = \begin{cases} 1 & k \in 2\mathbb{N}+1 \\ \sigma_3 & k \in 2\mathbb{N} \end{cases}$$

$$= (a\sigma_1 + b\sigma_2)\sigma_3 + 2i\begin{pmatrix} & ia-b \\ ia-ib & \end{pmatrix}$$

$\sigma_3\sigma_1 = \sigma_1\sigma_3 + 2i\sigma_2$

$$\sigma_3(a\sigma_1 + b\sigma_2) = a\sigma_1\sigma_3 + 2ai\sigma_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\sigma_3\sigma_2 = \sigma_2\sigma_3 - 2i\sigma_1$

σ

$$+ b\sigma_2\sigma_3 - 2bi\underbrace{\sigma_1}_{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}$$

$$\sum_{k=1}^{n} (2idL\sigma_3)^{k-1}\begin{pmatrix} 0 & B2L \\ B^*2L & 0 \end{pmatrix}(2idL\sigma_3)^{n-k}$$

$$= \sum_{m=0}^{n}\begin{pmatrix} 0 & 2LB \\ 2LB^* & 0 \end{pmatrix}(2idL\sigma_3)^{n-1} + \sum_{m=0}^{n}(2idL)^{2m-1}\underbrace{\sigma_3\begin{pmatrix} 0 & 2LB \\ 2LB^* & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 2LB \\ 2LB^* & 0 \end{pmatrix}\sigma_3 + 2\begin{pmatrix} 0 & 2LB \\ -2LB^* & 0 \end{pmatrix}}(2idL\sigma_3)^{n-(2m)}$$

$$+\begin{pmatrix} 0 & 2LB \\ 2LB^* & 0 \end{pmatrix}$$

$$a\sigma_3(a\sigma_1 + b\sigma_2) = (a\sigma_1 + b\sigma_2)\sigma_3 + \Big(2a\,i\underbrace{\sigma_2}_{\begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix}} + 2b\,i\underbrace{\sigma_1}_{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}\Big)$$

$$= (a\sigma_1 + b\sigma_2)\sigma_3 + 2\begin{pmatrix} 0 & a-ib \\ -a-ib & 0 \end{pmatrix}$$

$$\sigma_3\begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix}\sigma_3 + 2\begin{pmatrix} 0 & y^* \\ -y & 0 \end{pmatrix}$$

$$(\ldots\sigma_3)^{k-1}\begin{pmatrix}0 & 2LB\\ 2LB^* & 0\end{pmatrix}(2idL\sigma_3)^{n-k}$$

$$= \sum_{m=1}^{\lfloor n/2\rfloor}(2idL\sigma_3)^{2m-1}\begin{pmatrix}0 & 2LB\\ 2LB^* & 0\end{pmatrix}(2idL\sigma_3)^{n-2m} + \sum_{m=0}^{\lfloor n/2\rfloor}(2idL\sigma_3)^{2m+1-1}\begin{pmatrix}0 & 2LB\\ 2LB^* & 0\end{pmatrix}(2idL\sigma_3)^{n-(2m+1)}$$

$$= \sum_{m=1}^{\lfloor n/2\rfloor}(2idL)^{2m-1}\underbrace{\sigma_3\begin{pmatrix}0 & 2LB\\ 2LB^* & 0\end{pmatrix}}(2idL\sigma_3)^{n-2m} + \sum_{m=0}^{\lfloor m/2\rfloor}(2idL)^{2m}\begin{pmatrix}0 & 2LB\\ 2LB^* & 0\end{pmatrix}(2idL\sigma_3)^{n-(2m+1)}$$

$$\begin{pmatrix}0 & 2LB\\ +2LB^* & 0\end{pmatrix}\sigma_3 + 2\begin{pmatrix}0 & 2LB\\ -2LB^* & 0\end{pmatrix}$$

$$= \begin{pmatrix}0 & 2LB\\ 2LB^* & 0\end{pmatrix}\left\{\sum_{m=1}^{\lfloor n/2\rfloor}\underbrace{(2idL\sigma_3)^{2m-1}(2idL\sigma_3)^{n-2m}}_{(2idL\sigma_3)^{n-1}} + \sum_{m=0}^{\lfloor n/2\rfloor}\underbrace{(2idL\sigma_3)^{2m}(2idL\sigma_3)^{n-(2m+1)}}_{(2iL\sigma_3)^{n-1}}\right\}$$

$$+ \sum_{m=1}^{\lfloor n/2\rfloor}(2idL)^{n-1}\, 2\begin{pmatrix}0 & 2LB\\ -2LB^* & 0\end{pmatrix}\sigma_3^{n-2m}$$

$$= \begin{pmatrix}0 & 2LB\\ 2LB^* & 0\end{pmatrix}(2iL\sigma_3)^{n-1}\left(\sum_{m=1}^{\lfloor n/2\rfloor}1 + \sum_{m=0}^{\lfloor n/2\rfloor}1\right) + \cancel{(2iL\sigma_3)}^{n-1}\Big($$

$$+\begin{pmatrix}0 & 2LB\\ -2LB^* & 0\end{pmatrix}(2iL\sigma_3)^{n-1}\sigma_3^{1-2m}\sum_{m=1}^{\lfloor n/2\rfloor}1$$

$$\sigma_3\begin{pmatrix}0 & y\\ y^* & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & y\\ y^* & 0\end{pmatrix} = \begin{pmatrix}0 & y\\ -y^* & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & y\\ y^* & 0\end{pmatrix}\sigma_3 = \begin{pmatrix}0 & y\\ y^* & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -y\\ y^* & 0\end{pmatrix}$$

$$\sigma_3\begin{pmatrix}0 & y\\ y^* & 0\end{pmatrix} = -\begin{pmatrix}0 & y\\ y^* & 0\end{pmatrix}\sigma_3$$

on sait que $\begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} \sigma_3 = -\sigma_3 \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}$

alors $\sum_{m=1}^{[n/2]} (2id L \sigma_3)^{2m-1} \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} (2iL\sigma_3)^{n-2m}$

$= \cancel{\sum_{m=1}^{[n/2]}} (-) \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} \sum_{m=1}^{[n/2]} (2iL\sigma_3)^{n-1}$

(cote) $\sum_{k=1}^{n} (-1)^{k-1} = (-1)^{-1} \underbrace{\sum_{k=1}^{n} (-1)^{k}}_{-1+1}$

$\sum_{m=0}^{[n/2]} (2idL\sigma_3)^{2m} \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} (2iL\sigma_3)^{n-(2m+1)} = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix} (2idL\sigma_3)^{2n-2} \sum_{m=0}^{[n/2]} (1)$

$\sum_{k=1}^{n} \cancel{(-1)^{k-1}}$

- On commence par $k-1$ paire $\to 1$

$2m-1$

$n \in 2\mathbb{N} \to 0$

$n \in 2\mathbb{N}+1 \to 1$ $\Big\} \Rightarrow \sum_{k=1}^{n} (-1)^{k-1}$

$(i)^{2m+1-1}$ $\quad \sum_{n=1} \frac{1}{n!} (2idL\sigma_3)^{n-1} \sum_{k=1}^{n} (-1)^{k-2}$

$= \underbrace{\sum_{m=0}^{\infty} (-1)^m \frac{(2dL\sigma)^{2m+1}}{(2m+1)!}}_{\sin(2dL)} / 2dL$

$e^{ia(\vec{n}\cdot\vec{\sigma})} = I\cos a + i(\vec{n}\cdot\vec{\sigma})\sin a$

$\sum_{n=0}^{+\infty} \frac{1}{n!} \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0 \end{bmatrix} (-idL\sigma_3)^n = \cancel{\sum_{m=0}} \cos(dL)[\quad] + i\sigma_3 \sin(dL)[\quad]$

$e^{-i\overset{a}{dL}\sigma_3} = \cos(dL) + i(\sigma_3)\sin(-dL)$

$$S(\lambda) = \mathbb{1} + \begin{bmatrix} 0 & B\sin(2\lambda L)/\lambda \\ B^*\sin(2\lambda L)/\lambda & 0 \end{bmatrix} + \mathcal{O}(B^2)$$

- $\psi(x,t)$ solution defocusing NLS et $\psi(x,t) \xrightarrow[|x|\to+\infty]{} 0$ $\quad J_\pm(x,t;\lambda)$ Jost matrix

$W_\pm(x,t;\lambda) \doteq J_\pm(x,t;\lambda)e^{-i\lambda^2 t\sigma_3}$

$$\det\left(W_\pm(x,t;\lambda)\right) = \det\left(J_\pm(x,t;\lambda)\right) = 1$$

(2) $w_{,x} = Uw$ $\qquad$ (3) $w_{,t} = Vw$ $\qquad$ $J_\pm$ solution of (2)

$J_\pm(x,t;\lambda)C_\pm(t;\lambda)$ or $-$ solution de 3 parce que $J_\pm$ forme in ch (2)

$$J_\pm C_{\pm,t} + J_{\pm,t}C_\pm = VJ_\pm C_\pm$$

$$C_{\pm,t} = J_\pm^{-1}VJ_\pm C_\pm - J_\pm^{-1}J_{\pm,t}C_\pm$$

$$\lim_{x\to\pm\infty} J_\pm(x,t;\lambda)e^{i\lambda x\sigma_3} = \mathbb{1} \quad \text{et} \quad \lim_{x\to\pm\infty} V = -i\lambda^2\sigma_3$$

$$C_{\pm,t} = e^{i\lambda x\sigma_3}\left(-i\lambda^2\sigma_3\right)e^{-i\lambda x\sigma_3} + 0$$

$$= -i\lambda^2\sigma_3 C_\pm(t;\lambda) \qquad C_\pm(t;\lambda) = e^{-i\lambda^2 t\sigma_3}$$

$\to$ $W_\pm(x,t;\lambda) = J_\pm(x,t;\lambda)e^{-i\lambda^2 t\sigma_3}$ $\to$ solu (3)

$$J_{\pm,t} = i\lambda^2 J_\pm\sigma_3 + VJ_\pm$$

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial t}\left(J_-^{-1}J_+\right) = J_-^{-1}\frac{\partial J_+}{\partial t} + \underbrace{\frac{\partial J_-^{-1}}{\partial t}}_{-J_-^{-1}\frac{\partial J_-}{\partial t}J_-^{-1}} J_+ = J_-^{-1}\frac{\partial J_+}{\partial t} - J_-^{-1}\frac{\partial J_-}{\partial t}J_-^{-1}J_+$$

$$= J_-^{-1}\left(i\lambda^2 J_+\sigma_3 + VJ_+\right) - J_-^{-1}\left(i\lambda^2 J_-\sigma_3 + VJ_-\right)J_-^{-1}J_+$$

$$= i\lambda^2 S\sigma_3 - i\lambda^2\sigma_3 S = i\lambda^2[S,\sigma_3]$$

$$\Longrightarrow R(\lambda,t) = R(\lambda,0)e^{2i\lambda^2 t}$$

$$f_-^{(1)} = \begin{bmatrix} e^{-i\lambda x} u(x;\lambda) \\ e^{i\lambda x} v(x;\lambda) \end{bmatrix} \qquad \partial_x f_-^{(1)} = \begin{bmatrix} (-i\lambda u(x;\lambda) + u_{,x}) e^{-i\lambda x} \\ (i\lambda v(x,\lambda) + v_{,x}) e^{i\lambda x} \end{bmatrix}$$

$$\parallel$$

$$\begin{pmatrix} -i\lambda (e^{-i\lambda x} u) + \psi e^{+i\lambda x} v \\ \psi^* e^{-i\lambda x} u + i\lambda e^{i\lambda x} v \end{pmatrix}$$

$$\Longrightarrow \frac{\partial u}{\partial x} = e^{2i\lambda x} \psi(x) v(x;\lambda) \qquad \frac{\partial v}{\partial x} = e^{-2i\lambda x} \psi(x)^* u(x;\lambda)$$

$$\lim_{x\to-\infty} u(x;\lambda) = 1 \qquad \lim_{x\to-\infty} v(x,\lambda) = 0$$

$$u(x;\lambda) = 1 + \int_{-\infty}^{x} e^{2i\lambda y} \psi(y) v(y;\lambda)\, dy \qquad v(x,\lambda) = \int_{-\infty}^{x} e^{-2i\lambda y} \psi(y)^* u(y,\lambda)\, dy$$

$$= 1 + \int_{-\infty}^{x} e^{2i\lambda y} \psi(y) \int_{-\infty}^{x} e^{-2i\lambda z} \psi^*(z) u(z,\lambda)\, dz\, dy$$

boundary condition at $x = -\infty$ $\qquad \displaystyle\int_{-\infty}^{x} \int_{-\infty}^{y} f(y,z)\, dz\, dy = \int_{-\infty}^{x} \int_{z}^{x} f(y,z)\, dy\, dz$

$$u(x;\lambda) = 1 + \int_{-\infty}^{x} K(x,z;\lambda) u(z;\lambda)\, dz \qquad K(x,z;\lambda) \doteq \psi^*(z) \int_{z}^{x} e^{2i\lambda(y-z)} \psi(y)\, dy$$

kernel $\qquad x > z$

$$u_{n+1}(x;\lambda) \doteq 1 + \int_{-\infty}^{x} K(x,z;\lambda) u_n(z;\lambda)\, dz \qquad u_1(x;\lambda) \doteq 1 + \int_{-\infty}^{x} K(x;z_1;\lambda)\, dz_1$$

$$u_2(x;\lambda) = 1 + \int_{-\infty}^{x} K(x,z_1;\lambda)\, dz_1 + \int_{-\infty}^{x} K(x,z_2;\lambda) \int_{-\infty}^{z_2} K(z_2,z_1;\lambda)\, dz_1\, dz_1$$

$$u_n(x;\lambda) = \sum_{k=0}^{n} I_k(x;\lambda) \qquad I_k(x;\lambda) \doteq \int_{-\infty}^{x} K(x,z_k;\lambda) \int_{-\infty}^{z_k} K(z_k,z_{k-1};\lambda) \cdots \int_{-\infty}^{z_2} K(z_2,z_1;\lambda)\, dz_1\, dz_2 \cdots dz_k$$

$$I_0 = 1$$

$|K(x,z;\lambda)| \leq |\psi(z)| \cdot \|\psi\|_1 \qquad |I_k(x;\lambda)| \leq \|\psi\|_1^k \int_{-\infty}^{x} |\psi(z_k)| \int_{-\infty}^{z_k} |\psi(z_{k-1})| \, d \cdots \int_{-\infty}^{z_2} |\psi(z_1)| \, dz_1 \ldots dz_k$

$$\int_{-\infty}^{x} |\psi(z_k)| \int_{-\infty}^{z_k} |\psi(z_{k-1})| \cdots \int_{-\infty}^{z_2} |\psi(z_1)| \, dz_1 \ldots dz_k$$

$$= \int_0^m \int_0^{m_k} \cdots \int_0^{m_2} dm_1 \ldots dm_k = \frac{m^k}{k!}$$

$m_j = \int_{-\infty}^{z_j} |\psi(y)| \, dy \qquad j = 1, \ldots, k$

$m = \int_{-\infty}^{x} |\psi(y)| \, dy$

$dm_j = |\psi(z_j)| \, dz_j$

$$\Rightarrow |I_k(x;\lambda)| \leq \frac{1}{k!} \left(\|\psi\|_1 \int_{-\infty}^{x} |\psi(y)| \, dy \right)^k$$

$u = \sum_{k=0}^{\infty} I_k \qquad |u(x;\lambda)| \leq \sum_{k=0}^{\infty} |I_k(x;\lambda)| \leq \exp\left(\|\psi\|_1 \int_{-\infty}^{x} |\psi(y)| \, dy \right)$

$$\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x;\lambda)| \leq \exp\left(\|\psi\|_1^2 \right)$$

$u(x;\lambda) = 1 + \int_{-\infty}^{x} K(x,z;\lambda) \, u(z;\lambda) \, dz$

$u(-\infty;\lambda) = 1 \qquad u(+\infty,\lambda) = 1 + \int_{-\infty}^{+\infty} K(+\infty, z;\lambda) \, u(z;\lambda) \, dz$

$v(x;\lambda) = \int_{-\infty}^{x} e^{-2i\lambda y} \psi(y)^* u(y;\lambda) \, dy \qquad v(-\infty;\lambda) = 0 \qquad v(+\infty;\lambda) = \int_{-\infty}^{+\infty} e^{-2i\lambda y} \psi(y)^* u(y;\lambda) \, dy$

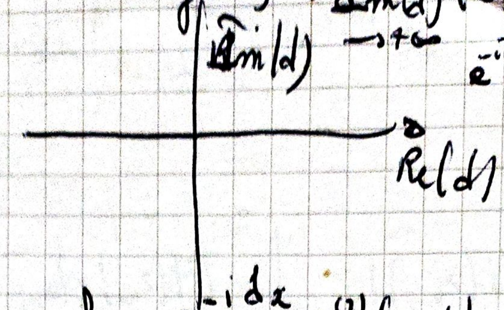
$f_-^{(1)}(x;\lambda) \underset{x \to -\infty}{=} \begin{pmatrix} e^{-i\lambda x} \\ 0 \end{pmatrix} + o(1) \qquad f_-^{(1)}(x;\lambda) = \begin{pmatrix} e^{-i\lambda x} & u(x;\lambda) \\ e^{i\lambda x} & v(x;\lambda) \end{pmatrix}$

$f_-^{(1)}(x,\lambda) \underset{x \to +\infty}{=} a(\lambda) \begin{pmatrix} e^{-i\lambda x} \\ 0 \end{pmatrix} + b(\lambda) \begin{pmatrix} 0 \\ e^{i\lambda x} \end{pmatrix} + o(1) \quad x$

- Ad folio
  M

$$e^{idx} f_-^{(1)}(x;d) \xrightarrow[\operatorname{Im}(d)\to+\infty]{} \begin{pmatrix}1\\0\end{pmatrix}$$

$$e^{-idx} f_{\oplus}^{(2)}(x;d) = \begin{pmatrix}0\\1\end{pmatrix}$$

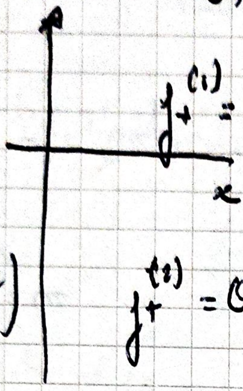


$$\lim_{\operatorname{Im}(d)\to+\infty} e^{idx} f_-^{(1)}(x;d) = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\lim_{\operatorname{Im}(d)\to+\infty} e^{-idx} f_-^{(2)}(x;d) = \begin{pmatrix}0\\1\end{pmatrix}$$

$$f_-^{(1)}(x;d) = \underset{x\to+\infty}{O}(e^{\beta x})$$

$$f_-^{(2)}(x,d) = \underset{x\to+\infty}{O}(e^{-\operatorname{Im}(d)x})$$



$$f_+^{(1)} = O(e^{\operatorname{Im}(d)x})$$

$$f_+^{(2)} = O(e^{-\operatorname{Im}(d)})$$

$$\lim_{\operatorname{Im}(d)\to-\infty} e^{idx} f_{\oplus}^{(1)}(x;d) = \begin{pmatrix}1\\0\end{pmatrix}$$

$$M(d,x) \doteq \begin{pmatrix} \dfrac{e^{idx}}{a(d)} f_-^{(1)}(x;d) \,;\, e^{-idx} f_{\oplus}^{(1)}(x;d) \\ e^{idx} f_{\oplus}^{(1)} \,;\, \dfrac{e^{-idx}}{a(d^*)} f_-^{(2)} \end{pmatrix}$$

$$[f_-^{(1)}; f_+^{(2)}] = [a(d) f_+^{(1)} + b f_+^{(1)}, f_+^{(2)}]$$

$$a = \det[f_-^{(1)}, f_+^{(2)}] = \xrightarrow[\operatorname{Im}(d)\to+\infty]{}$$

$$e^{-idx}\begin{pmatrix}1\\0\end{pmatrix} \xrightarrow[\operatorname{Im}(d)\to+\infty]{} \begin{pmatrix}0\\e^{+idx}\end{pmatrix}$$