Step profile evolution

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I. NOTATIONS AND THE FULL ANSWER

I will use a bit different notations than in the original Zakharov Manakov paper [Zh. Eksp. Teor. Fiz. 71, 203-215]. Namely, from I will align with Its paper [Soviet Math. Dokl. Vol. 24 (1981) N3], there the defocusing NLS have the following form

$$i\partial_t\psi + \partial_{xx}\psi - 8|\psi|^2\psi = 0 \tag{1}$$

The initial profile is

$$\psi(x,0) = h\theta(\ell - |x|). \tag{2}$$

The leading asymptotic reads for $t \to \infty$ for the fixed ratio $x/(4t) = -\lambda_0$

$$\psi(x,t) = \frac{\sqrt{2\pi}e^{\pi i\nu_0}}{\Gamma(-\nu_0)}\frac{a(\lambda_0)}{b(\lambda_0)}e^{-\Phi(\lambda_0)}\frac{e^{ix^2/(4t)}}{(8it)^{\nu_0+1/2}} \tag{3}$$

Here

$$\nu_0 = \nu(\lambda_0), \qquad \nu(\lambda) = \frac{1}{\pi i}\ln\frac{1}{|a(\lambda)|}, \qquad \Phi(\lambda) = \int_{-\infty}^{\lambda}\ln|\zeta-\lambda|\partial_\zeta\nu(\zeta)d\zeta \tag{4}$$

Now let me compute $a(\lambda)$ and $b(\lambda)$ for the initial profile (2). The auxiliary linear problem reads

$$\frac{dF}{dx} = \begin{pmatrix} -i\lambda & 2i\psi(x) \\ -2i\psi^*(x) & i\lambda \end{pmatrix} F \tag{5}$$

The Jost solution F_- is defined by its behaviour for $x \to -\infty$

$$F_-(x) \sim e^{-i\lambda x\sigma_z}, \qquad x \to -\infty \tag{6}$$

reads

$$F_-(x) = \begin{cases} e^{-i\lambda x\sigma_z}, & x < -\ell \\ e^{-iMx}e^{-iM\ell}e^{i\lambda\ell\sigma_z}, & |x| < \ell \\ e^{-i\lambda x\sigma_z}e^{i\lambda\ell\sigma_z}e^{-2iM\ell}e^{i\lambda\ell\sigma_z}, & x > \ell \end{cases} \tag{7}$$

where

$$M = \begin{pmatrix} \lambda & -2h \\ 2h & -\lambda \end{pmatrix}. \tag{8}$$

Similar expressions can be obtained for the Jost solution F_+ normalized by its behaviour at $x \to \infty$

$$F_+(x) \sim e^{-i\lambda x\sigma_z}, \qquad x \to +\infty. \tag{9}$$

F_- and F_+ are related by transfer matrix

$$F_- = F_+T(\lambda) \qquad \Rightarrow \qquad T(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ \bar{b}(\lambda) & \bar{a}(\lambda) \end{pmatrix} = e^{i\lambda\ell\sigma_z}e^{-2iM\ell}e^{i\lambda\ell\sigma_z} \tag{10}$$

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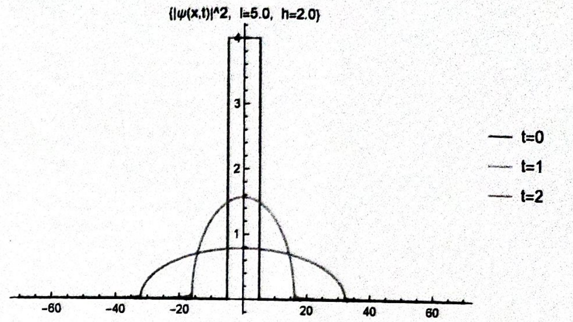

FIG. 1: Typical asymptotic behaviour. The blue rectangle is the initial profile

This gives

$$a(\lambda) = e^{2i\lambda\ell}\left(\cos\left(2\ell\sqrt{\lambda^2-4h^2}\right) - \frac{i\lambda\sin\left(2\ell\sqrt{\lambda^2-4h^2}\right)}{\sqrt{\lambda^2-4h^2}}\right), \qquad b(\lambda) = \frac{2ih\sin\left(2\ell\sqrt{\lambda^2-4h^2}\right)}{\sqrt{\lambda^2-4h^2}}. \tag{11}$$

Therefore

$$\nu(\lambda) = \frac{1}{2\pi i}\ln\frac{\lambda^2-4h^2}{\lambda^2-4h^2\cos(2\ell\sqrt{\lambda^2-4h^2})} \tag{12}$$

We see that $\nu(\lambda)$ is purely imaginary for all real λ, therefore

$$|\psi(x,t)|^2 = \frac{1}{8t}\frac{2\pi e^{\pi i\nu_0}}{\Gamma(\nu_0)\Gamma(-\nu_0)}\left|\frac{a(\lambda_0)}{b(\lambda_0)}\right|^2 \tag{13}$$

Taking into account that

$$\frac{2\pi e^{\pi i\nu_0}}{\Gamma(\nu_0)\Gamma(-\nu_0)} = i\nu_0(e^{2\pi i\nu_0}-1) = i\nu_0(1/|a(\lambda_0)|^2-1) = -i\nu_0\frac{|b(\lambda_0)|^2}{|a(\lambda_0)|^2} \tag{14}$$

we obtain

$$|\psi(x,t)|^2 = \frac{1}{16\pi t}\ln\frac{x^2-(8ht)^2\cos(\frac{\ell}{t}\sqrt{x^2-(8ht)^2})}{x^2-(8ht)^2} \tag{15}$$

We plot exemplary behaviour in Fig. (I)