Bulchandani's construction of stationary states

J. Dubail

November 17, 2023

1 GHD-stationary solution in trapping potential

The GHD equation in a trap is

$$\partial_t \nu + v^{\rm eff} \partial_x \nu - (\partial_x V) \partial_\theta \nu = 0, \tag{1}$$

where $\nu(x, \theta, t)$ is the occupation ratio. Here the goal is to find the stationary solutions of that equation, that is

$$\nu(x, \theta) \qquad \text{s.t.} \qquad v^{\rm eff} \partial_x \nu - (\partial_x V) \partial_\theta \nu = 0. \tag{2}$$

I learned how to do that from Vir Bulchandani's talk in Marseille, see his paper 'Modified Enskog equation for hard rods', arXiv:2309.15846. He wrote the solution for the hard rod gas but the idea is completely general.

The general solution is parametrized by functions $s : [0,1] \to \mathbb{R}$. Let us chose a function s, and let us assume that we find the solution of the following self-consistent equation for $\nu(\theta)$ at every point x,

$$\frac{\theta^2}{2} - \mu + V(x) = s'(\theta) + \int \frac{d\lambda}{2\pi} \varphi(\theta - \lambda) \left[s(\nu(\lambda)) - \nu(\lambda) s'(\nu(\lambda)) \right]. \tag{3}$$

Notice that this is the Yang-Yang equation in the particular case $s(\nu) = -\nu \log \nu - (1-\nu)\log(1-\nu)$; the point here is that any other choice of function $s(\nu)$ works.

The claim is that the occupation ratio $\nu(x, \theta)$ constructed from Eq. (3) is stationary, namely it satisfies (2).

2 Why this works

The key point is that, actually, the Yang-Yang entropy plays absolutely no role in Euler-scale GHD. The Yang-Yang entropy can be replaced by any other entropy functional of the form

$$S[\rho] = \int s(\nu(\theta))\rho_s(\theta)d\theta, \tag{4}$$

which is also conserved by the GHD equations, as pointed out in SciPost Phys. 6, 070 (2019). That is, any microscopic model where the number of microstates (in a box of size L) corresponding to a continuous rapidity distribution $\rho(\theta)$ is

$$\#\text{microstates} \sim e^{LS[\rho]} \tag{5}$$

leads to the same GHD equations at the Euler scale. So we can write the thermodynamic Bethe Ansatz for an arbitrary choice of entropy functional $S[\rho]$, namely one can replace discrete sums over microstates —weighted by some generalized Boltzmann weight e^{-F}— by the functional integral

$$\sum_{\text{microstates}} e^{-F} \longrightarrow \int \mathcal{D}\rho \; e^{LS[\rho]} e^{-L\int f(\theta)\rho(\theta)d\theta}, \tag{6}$$

and then the saddle-point gives the TBA equation associated to s,

$$f(\theta) = s'(\nu(\theta)) + \int \frac{d\lambda}{2\pi}\varphi(\theta-\lambda)\left[s(\nu(\lambda)) - \nu(\lambda)s'(\nu(\lambda))\right]. \tag{7}$$

This self-consistent equation needs to be solved iteratively, just like the usual Yang-Yang equation. In this way the rapidity distribution $\rho(\theta)$, or the occupation function $\nu(\theta)$, is parameterized by the driving function $f(\theta)$.

Then it is easy to see that the GHD equation (1), written in terms of the (position-dependent) driving function, is:

$$(\partial_t f)^{\rm dr} + v^{\rm eff}(\partial_x f)^{\rm dr} - (\partial_x V)(\partial_\theta f)^{\rm dr} = 0. \tag{8}$$

For the particular choice $f(x,\theta) = \frac{\theta^2}{2} - \mu + V(x)$, one has $v^{\rm eff}(\partial_x f)^{\rm dr} - (\partial_x V)(\partial_\theta f)^{\rm dr} = v^{\rm eff}(\partial_x V)1^{\rm dr} - (\partial_x V)(\mathrm{id})^{\rm dr} = 0$.