WHAT IS ... THE INVERSE-SCATTERING TRANSFORM?

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Historical Background

- 1967: Gardner, Greene, Kruskal, and Miura discover a strange way to solve the initial-value problem for the Korteweg-de Vries (KdV) equation: $u_t + uu_x + u_{xxx} = 0$, $u(x,0) = u_0(x)$, $x \in \mathbb{R}$. Pretend that for each t, the function $V(x) := -u(x,t)$ is taken as a potential energy coefficient in the linear (time-independent) Schrödinger equation: $-6\phi''(x) + V(x)\phi(x) = E\phi(x)$ where E is the energy (eigenvalue). By direct calculations, GGKM showed that various quantities associated with this linear problem are either indpendent of t (L^2 energy eigenvalues E) or evolve in an elementary fashion (the so-called reflection coefficient) as u evolves according to KdV. Therefore, if the appropriate data can be calculated from given initial conditions and if the inverse problem of finding the potential V given the data can be solved, one has an algorithm for solving the KdV initial-value problem.
- 1968: Lax finds an operator-theoretic way to generalize the method of GGKM, introducing *Lax pairs.*
- 1971: Zakharov and Shabat apply Lax's method to solve the cubic nonlinear Schrödinger equation.
- 1974: Ablowitz, Kaup, Newell, and Segur write an influential paper emphasizing an analogy between the solution methods of GGKM, Lax, and ZS, and the solution of linear PDE by Fourier transform. They introduce the term "inverse-scattering transform".

The inverse-scattering transform is different in the details for different equations. Hence there is no such thing as "the" inverse-scattering transform. We will explore the easiest case today.

The inverse-scattering transform for the defocusing nonlinear Schrödinger equation

Lax pair and zero-curvature representation. The defocusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi = 0 \tag{1}$$

for a complex-valued field $\psi(x,t)$ can be viewed as a compatibility condition for the simultaneous linear equations of a Lax pair:

$$\frac{\partial \mathbf{w}}{\partial x} = \mathbf{U}\mathbf{w}, \quad \mathbf{U} = \mathbf{U}(x,t,\lambda) := \begin{bmatrix} -i\lambda & \psi \\ \psi^* & i\lambda \end{bmatrix} \tag{2}$$

and

$$\frac{\partial \mathbf{w}}{\partial t} = \mathbf{V}\mathbf{w}, \quad \mathbf{V} = \mathbf{V}(x,t,\lambda) := \begin{bmatrix} -i\lambda^2 - i\frac{1}{2}|\psi|^2 & \lambda\psi + i\frac{1}{2}\psi_x \\ \lambda\psi^* - i\frac{1}{2}\psi_x^* & i\lambda^2 + i\frac{1}{2}|\psi|^2 \end{bmatrix}. \tag{3}$$

In other words, the defocusing NLS equation is equivalent to the *zero-curvature condition*

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U}, \mathbf{V}] = \mathbf{0}$$

(the left-hand side is independent of λ and vanishes when the defocusing NLS equation holds for ψ).

We want to solve general initial-value problems, say of the form in which we seek $\psi(x,t)$ for $x \in \mathbb{R}$ and $t \geq 0$, such that $\psi(x,t)$ is smooth and decays to zero for large $|x|$ and satisfies a specified initial condition

$$\psi(x,0) = \psi_0(x)\,,$$

for some given function $\psi_0(x)$.

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Linearized theory. Consider first the initial-value problem for the linear Schrödinger equation:

$$i\psi_t + \frac{1}{2}\psi_{xx} = 0\,, \qquad \psi(x,0) = \psi_0(x)\,.$$

Taking the (direct) Fourier transform via

$$\hat{\psi}(k,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x,t)e^{-ikx}\,dx,$$

the initial value problem goes over into

$$i\hat{\psi}_t - \frac{k^2}{2}\hat{\psi} = 0, \quad \hat{\psi}(k,0) = \hat{\psi}_0(k).$$

This ordinary differential equation is solved in closed form as follows:

$$\hat{\psi}(k,t) = e^{-ik^2t/2}\hat{\psi}(k,0) = e^{-ik^2t/2}\hat{\psi}_0(k).$$

Thus, the time evolution of the Fourier transform is given by multiplication by a simple and explicit oscillatory factor. To complete the solution of the initial-value problem we have to invert the Fourier transform by evaluating another integral:

$$\psi(x,t) = \int_{-\infty}^{\infty} \hat{\psi}(k,t)e^{ikx}\,dk.$$

Even though the formulae look simple one should keep in mind that only rarely can the direct and inverse transform integrals be evaluated in closed form.

Nonetheless we have an algorithmic way to solve general initial-value problems for the linear Schrödinger equation:

(1) Map the given initial condition $\psi_0(x)$ to the direct transform thereof: $\psi_0 \mapsto \hat{\psi}_0$.
(2) Evolve the transform explicitly in time: $\hat{\psi}_0(k) \mapsto \hat{\psi}(k,t) := \hat{\psi}(k)e^{-ik^2t/2}$.
(3) $\hat{\psi}(k,t)$ is the direct transform of $\psi(x,t)$. Find $\psi(x,t)$ at a later time t by applying the inverse transform to $\hat{\psi}(k,t)$.

For the nonlinear problem we will have instead the inverse-scattering transform; it has the same kinds of benefits and shortcomings as the Fourier transform for linear problems:

- It gives an algorithmic way to solve initial-value problems with arbitrary given initial data.
- Just as the Fourier integrals can rarely be computed in closed form, the individual steps in the solution algorithm can rarely be carried out in closed form.

If the steps of the algorithm cannot be carried out in closed form, then what is the use? Well, in the case of Fourier transforms, we can analyze the solutions of initial-value problems with great precision in asymptotic regimes like long time limits due to the existence of analytical tools like the method of steepest descents and the method of stationary phase for asymptotic expansions of integrals. It turns out that similar methods exist at the integrable nonlinear level, so similar questions can be addressed as in the linear case.

The direct transform. Suppose that t is fixed and $\psi(x) = \psi(x,t)$ is a rapidly decreasing function of $x \in \mathbb{R}$. We will impose specific conditions on $\psi(x)$ as we go. We are interested in the properties of solutions $\mathbf{w} = \mathbf{w}(x;\lambda)$ of the equation (2) when λ is assumed to be a real number. Since $\psi(x)$ decays rapidly for large $|x|$, we have the following asymptotic behavior for the coefficient matrix:

$$\mathbf{U}(x,t,\lambda) = -i\lambda\sigma_3 + o(1), \quad |x| \to \infty, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This in turn suggests that when $|x|$ is large, we could approximate solutions $\mathbf{w}(x;\lambda)$ by the solutions $\mathbf{w}_0(x;\lambda)$ of the $\psi \equiv 0$ system:

$$\mathbf{w}_0(x;\lambda) = c_1 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix},$$

where c_1 and c_2 are arbitrary constants.

Jost solutions. We can isolate particular solutions of our linear system by insisting on certain values of the constants c_1 and c_2 in the limit $x \to -\infty$ or $x \to +\infty$. For example, we can look for a solution $\mathbf{w} = \mathbf{j}_-^{(1)}(x;\lambda)$ that satisfies a boundary condition of the form

$$\mathbf{j}_-^{(1)}(x;\lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1)\,, \qquad \text{as } x \to -\infty$$

which corresponds to insisting that $\mathbf{j}_-^{(1)}(x;\lambda)$ is asymptotically of the form $\mathbf{w}_0(x;\lambda)$ with $c_1 = 1$ and $c_2 = 0$ as $x \to -\infty$. Similarly, we can look for a solution $\mathbf{w} = \mathbf{j}_-^{(2)}(x;\lambda)$ that satisfies the boundary condition

$$\mathbf{j}_-^{(2)}(x;\lambda) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1)\,, \qquad \text{as } x \to -\infty$$

so we're taking $c_1 = 0$ and $c_2 = 1$ as $x \to -\infty$. These two solutions are called *Jost solutions* normalized as $x \to -\infty$. These two Jost solutions are linearly independent. Forming a 2×2 solution matrix from the column vectors:

$$\mathbf{J}_-(x;\lambda) := [\mathbf{j}_-^{(1)}(x;\lambda), \mathbf{j}_-^{(2)}(x;\lambda)]$$

the Wronskian of the Jost solutions is just the determinant of $\mathbf{J}_-$. Since the trace of the coefficient matrix $\mathbf{U}$ is zero, Abel's Theorem says that Wronskians are independent of x. Therefore we may compute the Wronskian of the Jost solutions for any x we like, and a good choice is the limit $x \to -\infty$ since this is where we have information about the columns of $\mathbf{J}_-(x;\lambda)$. Therefore,

$$\det(\mathbf{J}_-(x;\lambda)) = \lim_{x\to-\infty} \det(\mathbf{J}_-(x;\lambda)) = \lim_{x\to-\infty} \det\left(e^{-i\lambda x\sigma_3} + o(1)\right) = 1\,.$$

(Note: this calculation uses the fact that $\lambda \in \mathbb{R}$. Otherwise either $e^{i\lambda x}$ or $e^{-i\lambda x}$ is exponentially growing as $x \to -\infty$ depending on whether λ is in the upper or lower half-plane, and we cannot conclude that the product the growing exponential with the unspecified rate of decay $o(1)$, as will arise in computing the determinant, decays to zero.) Thus, for all $\lambda \in \mathbb{R}$ for which the Jost solutions exist they are linearly independent of each other, and the matrix $\mathbf{J}_-(x;\lambda)$ is therefore a fundamental solution matrix for the system. We can express the normalization condition for the Jost solutions in terms of $\mathbf{J}_-(x;\lambda)$ by the equation

$$\lim_{x\to-\infty} \mathbf{J}_-(x;\lambda) e^{i\lambda x\sigma_3} = \mathbb{I}\,, \qquad \lambda \in \mathbb{R}\,.$$

In a similar way, we can also construct Jost solutions that are normalized in the limit $x \to +\infty$, where again the coefficient $\psi(x)$ decays to zero and the coefficient matrix $\mathbf{U}$ can be approximated by $-i\lambda\sigma_3$. Thus, we may seek a solution $\mathbf{w} = \mathbf{j}_+^{(1)}(x;\lambda)$ that satisfies the boundary condition

$$\mathbf{j}_+^{(1)}(x;\lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1)\,, \qquad \text{as } x \to +\infty$$

and a solution $\mathbf{w} = \mathbf{j}_+^{(2)}(x;\lambda)$ that satisfies the boundary condition

$$\mathbf{j}_+^{(2)}(x;\lambda) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1)\,, \qquad \text{as } x \to +\infty$$

and we will find that for real λ the Wronskian determinant of the solution matrix

$$\mathbf{J}_+(x;\lambda) := [\mathbf{j}_+^{(1)}(x;\lambda), \mathbf{j}_+^{(2)}(x;\lambda)]$$

is exactly equal to 1, so $\mathbf{J}_+(x;\lambda)$ is also a fundamental solution matrix for our system of equations. Its normalization condition can be written in the form

$$\lim_{x\to+\infty} \mathbf{J}_+(x;\lambda) e^{i\lambda x\sigma_3} = \mathbb{I}\,.$$

Scattering matrix. The Jost solution matrices $\mathbf{J}_\pm(x;\lambda)$ are necessarily both nonsingular matrices satisfying the differential equation

$$\frac{\partial \mathbf{J}_\pm}{\partial x} = \mathbf{U}\mathbf{J}_\pm\,.$$

This is a 2×2 linear system of equations, and so can only have two linearly independent column vector solutions. We have, however, discussed four solutions: the two columns of $\mathbf{J}_-(x;\lambda)$ and the two columns of $\mathbf{J}_+(x;\lambda)$. These cannot all four be independent of each other. But, the columns of $\mathbf{J}_+(x;\lambda)$ form a basis of the space of solutions, as do those of $\mathbf{J}_-(x;\lambda)$. Therefore, we may write any column of $\mathbf{J}_+(x;\lambda)$ as a linear combination of the columns of $\mathbf{J}_-(x;\lambda)$. The constants involved in the linear combinations may depend on $\lambda\in\mathbb{R}$. We may write the linear combinations compactly in the form

$$\mathbf{J}_+(x;\lambda) = \mathbf{J}_-(x;\lambda)\mathbf{S}(\lambda), \tag{4}$$

for some 2×2 matrix $\mathbf{S}(\lambda)$. Indeed, the first column of this formula simply says that

$$\mathbf{j}_+^{(1)}(x;\lambda) = S_{11}(\lambda)\mathbf{j}_-^{(1)}(x;\lambda) + S_{21}(\lambda)\mathbf{j}_-^{(2)}(x;\lambda)\,,$$

and the second column says

$$\mathbf{j}_+^{(2)}(x;\lambda) = S_{12}(\lambda)\mathbf{j}_-^{(1)}(x;\lambda) + S_{22}(\lambda)\mathbf{j}_-^{(2)}(x;\lambda)\,,$$

so the matrix $\mathbf{S}(\lambda)$ contains the four constants involved in the linear combinations. The matrix $\mathbf{S}(\lambda)$ is called the *scattering matrix* associated with the coefficient $\psi(x)$ in the matrix $\mathbf{U}$.

Let us now draw a premature analogy with Fourier transform theory for linear initial-value problems. The first step of the solution algorithm in the linear case is the association of a function $\hat{\psi}_0(k)$ with the given initial condition $\psi_0(x)$, via the Fourier transform. Here we see another way, via the Lax pair for defocusing NLS, to associate some functions of an auxiliary variable λ with a given initial condition $\psi_0(x)$: make $\psi_0(x)$ the known function in the coefficient matrix $\mathbf{U}$ and solve the linear ODE system subject to the appropriate boundary conditions to find the Jost matrices $\mathbf{J}_\pm(x;\lambda)$. Then the matrix $\mathbf{S}(\lambda) := \mathbf{J}_-(x;\lambda)^{-1}\mathbf{J}_+(x;\lambda)$ is guaranteed to be indepdendent of x and to contain four functions of λ that can be viewed as "transforms" of $\psi_0(x)$.

Elementary properties of the scattering matrix. First consider taking determinants in the defining relation (4) and using the fact that the Jost matrices have determinant 1. Thus,

$$\det(\mathbf{S}(\lambda)) = 1\,,\qquad \lambda\in\mathbb{R}\,.$$

Next, note that the coefficient matrix $\mathbf{U}(x,t,\lambda)$ of the linear system (2) has the following symmetry

$$\mathbf{U}(x,t,\lambda)^* = \sigma_1\mathbf{U}(x,t,\lambda)\sigma_1,\quad \lambda\in\mathbb{R},\quad \sigma_1 := \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

It follows that if $\mathbf{w}(x;\lambda)$ is a solution of (2) for $\lambda\in\mathbb{R}$, then so also is $\sigma_1\mathbf{w}(x;\lambda)^*$. Considering the asymptotic behavior of the Jost solutions, this means that

$$\mathbf{j}_\pm^{(2)}(x;\lambda) = \sigma_1\mathbf{j}_\pm^{(1)}(x;\lambda)^*\,,\qquad \lambda\in\mathbb{R}\,.$$

From this it follows that the Jost matrices $\mathbf{J}_\pm(x;\lambda)$ satisfy

$$\mathbf{J}_\pm(x;\lambda)^* = \sigma_1\mathbf{J}_\pm(x;\lambda)\sigma_1\,.$$

Taking conjugates in the relation $\mathbf{S}(\lambda) = \mathbf{J}_-(x;\lambda)^{-1}\mathbf{J}_+(x;\lambda)$ we therefore get

$$\mathbf{S}(\lambda)^* = \sigma_1\mathbf{S}(\lambda)\sigma_1\,.$$

This latter relation implies that $\mathbf{S}(\lambda)$ can be written in the form

$$\mathbf{S}(\lambda) = \begin{bmatrix} a(\lambda)^* & -b(\lambda)^*\\ -b(\lambda) & a(\lambda)\end{bmatrix},\qquad \lambda\in\mathbb{R}$$

for some complex-valued functions $a(\lambda)$ and $b(\lambda)$. The condition that $\det(\mathbf{S}(\lambda)) = 1$ then implies that

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1\quad\text{or, equivalently}\quad |R(\lambda)|^2 + |T(\lambda)|^2 = 1$$

where the quantity $T(\lambda) := 1/a(\lambda)$ is sometimes called the *transmission coefficient* and the ratio $R(\lambda) := b(\lambda)/a(\lambda)$ is sometimes called the *reflection coefficient.* Indeed, from

$$\mathbf{J}_-(x;\lambda) = \mathbf{J}_+(x;\lambda)\mathbf{S}(\lambda)^{-1} = \mathbf{J}_+(x;\lambda)\begin{bmatrix} a(\lambda) & b(\lambda)^* \\ b(\lambda) & a(\lambda)^* \end{bmatrix},$$

we have in particular that

$$\mathbf{j}_-^{(1)}(x;\lambda) = a(\lambda)\mathbf{j}_+^{(1)}(x;\lambda) + b(\lambda)\mathbf{j}_+^{(2)}(x;\lambda)\,,$$

which means, upon dividing through by $a(\lambda)$, that there is for each $\lambda \in \mathbb{R}$ a solution $\mathbf{w}(x;\lambda) = a(\lambda)^{-1}\mathbf{j}_-^{(1)}(x;\lambda)$ satisfying

$$\mathbf{w}(x;\lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + R(\lambda)\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \to +\infty$$

and also

$$\mathbf{w}(x;\lambda) = T(\lambda)\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \to -\infty.$$

Therefore, interpreting $e^{i\lambda x}$ ($e^{-i\lambda x}$) as a "wave" propagating to the right (left), we see that a unit wave traveling to the left incident on the disturbance modeled by the potential $\psi(x)$ causes a reflected wave of complex amplitude $R(\lambda)$ and admits a transmitted wave of complex amplitude $T(\lambda)$. In the inverse-scattering transform theory, the reflection coefficient $R(\lambda)$ is a nonlinear analogue of the Fourier transform of $\psi(x)$.

Definition 1 (Direct transform)**.** *For a suitable function $\psi : \mathbb{R} \to \mathbb{C}$ decaying as $x \to \pm\infty$, the direct transform for the defocusing NLS equation is the mapping $\psi \mapsto R$ associating to ψ its reflection coefficient $R = R(\lambda)$, $\lambda \in \mathbb{R}$.*

Example: barrier initial data. As a simple example, suppose that $\psi(x)$ is the piecewise-constant function

$$\psi(x) = \begin{cases} 0\,, & |x| > L\,, \\ B\,, & |x| \le L\,. \end{cases}$$

Thus, ψ represents a barrier of width $2L$ and complex "height" $B \in \mathbb{R}$. Then, for $|x| > L$ we have the simple diagonal system:

$$\frac{\partial \mathbf{w}}{\partial x} = -i\lambda\sigma_3\mathbf{w}\,, \qquad |x| > L\,,$$

and for $|x| \le L$ we have a non-diagonal but still constant-coefficient system:

$$\frac{\partial \mathbf{w}}{\partial x} = \mathbf{U}_B(\lambda)\mathbf{w}, \quad \mathbf{U}_B(\lambda) := \begin{bmatrix} -i\lambda & B \\ B^* & i\lambda \end{bmatrix}.$$

We may construct the Jost solutions piecewise, and join them at $x = \pm L$ by continuity.

The general solution for $|x| > L$ is as we found earlier, $\mathbf{w} = \mathbf{w}_0(x;\lambda)$. Thus, the Jost matrix $\mathbf{J}_-(x;\lambda)$ satisfies exactly

$$\mathbf{J}_-(x;\lambda) = e^{-i\lambda x\sigma_3}\,, \qquad x < -L\,.$$

Similarly, the Jost matrix $\mathbf{J}_+(x;\lambda)$ satisfies exactly

$$\mathbf{J}_+(x;\lambda) = e^{-i\lambda x\sigma_3}\,, \qquad x > L\,.$$

The general solution of the differential equation for $|x| < L$ can be written in terms of the matrix exponential $\exp(x\mathbf{U}_B(\lambda))$. Thus, for some constant matrix $\mathbf{C} = \mathbf{C}(\lambda)$, we may write the Jost matrix $\mathbf{J}_-(x;\lambda)$ for $|x| < L$ in the form

$$\mathbf{J}_-(x;\lambda) = e^{x\mathbf{U}_B(\lambda)}\mathbf{C}(\lambda), \quad |x| < L,$$

and we may find $\mathbf{C}(\lambda)$ by demanding continuity of $\mathbf{J}_-(x;\lambda)$ at $x = -L$:

$$\lim_{x\uparrow -L}\mathbf{J}_-(x;\lambda) = e^{i\lambda L\sigma_3} \quad \text{and} \quad \lim_{x\downarrow -L}\mathbf{J}_-(x;\lambda) = e^{-L\mathbf{U}_B(\lambda)}\mathbf{C}(\lambda)$$

so matching gives

$$\mathbf{C}(\lambda) = e^{L\mathbf{U}_B(\lambda)}e^{i\lambda L\sigma_3}.$$

Now, $\mathbf{J}_-(x;\lambda)$ and $\mathbf{J}_+(x;\lambda)$ are both known at $x = L$, so the scattering matrix may be calculated explicitly:

$$\mathbf{S}(\lambda) = \mathbf{J}_-(L;\lambda)^{-1}\mathbf{J}_+(L;\lambda) = \mathbf{C}(\lambda)^{-1}e^{-L\mathbf{U}_B(\lambda)} \cdot e^{-i\lambda L\sigma_3} = e^{-i\lambda L\sigma_3}e^{-2L\mathbf{U}_B(\lambda)}e^{-i\lambda L\sigma_3}.$$

An interesting calculation is to consider the expansion of the scattering matrix elements in the small-amplitude limit, $B \to 0$. Expanding the matrix exponential in Taylor series gives:

$$\begin{aligned}
\mathbf{S}(\lambda) &= e^{-i\lambda L\sigma_3}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}\left(2i\lambda L\sigma_3 - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0\end{bmatrix}\right)^n\right\}e^{-i\lambda L\sigma_3} \\
&= e^{-i\lambda L\sigma_3}\left\{e^{2i\lambda L\sigma_3} - \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=1}^{n}(2i\lambda L\sigma_3)^{k-1}\begin{bmatrix} 0 & 2LB \\ 2LB^* & 0\end{bmatrix}(2i\lambda L\sigma_3)^{n-k}\right\}e^{-i\lambda L\sigma_3} + O(B^2) \\
&= e^{-i\lambda L\sigma_3}\left\{e^{2i\lambda L\sigma_3} - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0\end{bmatrix}\sum_{n=1}^{\infty}\frac{1}{n!}(2i\lambda L\sigma_3)^{n-1}\sum_{k=1}^{n}(-1)^{k-1}\right\}e^{-i\lambda L\sigma_3} + O(B^2) \\
&= \mathbb{I} - \begin{bmatrix} 0 & 2LB \\ 2LB^* & 0\end{bmatrix}\sum_{m=0}^{\infty}\frac{(-1)^m(2\lambda L)^{2m}}{(2m+1)!} + O(B^2) \\
&= \mathbb{I} - \begin{bmatrix} 0 & B\sin(2\lambda L)/\lambda \\ B^*\sin(2\lambda L)/\lambda & 0\end{bmatrix} + O(B^2).
\end{aligned}$$

Therefore, in the limit $B \to 0$, the conjugate reflection coefficient takes the form

$$R(\lambda)^* = \frac{b(\lambda)^*}{a(\lambda)^*} = -\frac{S_{12}(\lambda)}{S_{11}(\lambda)} = \frac{B}{\lambda}\sin(2\lambda L) + O(B^2).$$

Noting the Fourier transform of the "barrier" initial condition $\psi(x)$:

$$\hat{\psi}(k) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(x)e^{-ikx}\,dx = \frac{B}{2\pi}\int_{-L}^{L}e^{-ikx}\,dx = \frac{B}{\pi k}\sin(kL)\,,$$

it is easy to see that in fact

$$R(\lambda)^* = 2\pi\hat{\psi}(2\lambda) + O(B^2), \quad B \to 0,$$

so just as in the small-amplitude limit the defocusing NLS equation formally goes over into the linear Schrödinger equation, it makes sense that the calculation of the reflection coefficient is essentially a nonlinear analogue of the Fourier transform.

Time evolution of the reflection coefficient. If the potential $\psi(x)$ is replaced by a time-dependent potential $\psi(x,t)$ that satisfies the defocusing NLS equation (1), then, as $\psi(x,t)$ evolves in time (within a suitable function space), the reflection coefficient $R(\lambda)$ becomes time dependent as well: $R(\lambda) = R(\lambda;t)$. The analogy with Fourier transforms is further strengthened at this point because it turns out that $R(\lambda;t)$ evolves in time in the same elementary way as does the Fourier transform under the linear Schrödinger equation.

To see this, we consider how the Jost matrices vary in time with $\psi(x,t)$; we now write them as $\mathbf{J}_\pm(x,t;\lambda)$.

Lemma 1. *Let $\psi(x,t)$ be a solution of the defocusing NLS equation* (1) *that decays rapidly to zero as $|x| \to \infty$, and let $\mathbf{J}_\pm(x,t;\lambda)$ be the corresponding Jost matrices defined for each t. Then, the matrices $\mathbf{W}_\pm(x,t;\lambda) := \mathbf{J}_\pm(x,t;\lambda)e^{-i\lambda^2 t\sigma_3}$ are simultaneous fundamental matrix solutions of the compatible linear problems* (2) *and* (3) *of the Lax pair.*

Proof. Obviously we have $\det(\mathbf{W}_\pm(x,t;\lambda)) = 1$ as a consequence of $\det(\mathbf{J}_\pm(x,t;\lambda)) = 1$. Since $e^{-i\lambda^2 t\sigma_3}$ is independent of x, it is equally obvious that $\mathbf{W}_\pm(x,t;\lambda)$ satisfy (2) as this is true by definition of $\mathbf{J}_\pm(x,t;\lambda)$. Since the zero-curvature (compatibility) condition holds, there exists a simultaneous fundamental solution matrix of (2) and (3) for every $\lambda \in \mathbb{R}$. Every such matrix must have the form $\mathbf{J}_+(x,t;\lambda)\mathbf{C}_+(t;\lambda)$ or $\mathbf{J}_-(x,t;\lambda)\mathbf{C}_-(t;\lambda)$ because the Jost matrices already are fundmental solution matrices of (2). Substituting into (3) one sees that the matrices $\mathbf{C}_\pm(t;\lambda)$ must satisfy the differential equations

$$\frac{\partial \mathbf{C}_\pm}{\partial t}(t;\lambda) = \mathbf{J}_\pm(x,t;\lambda)^{-1}\mathbf{V}(x,t,\lambda)\mathbf{J}_\pm(x,t;\lambda)\mathbf{C}_\pm(t;\lambda) - \mathbf{J}_\pm(x,t,\lambda)^{-1}\frac{\partial \mathbf{J}_\pm}{\partial t}(x,t;\lambda)\mathbf{C}_\pm(t;\lambda).$$

Since $\mathbf{C}_\pm(t;\lambda)$ are matrices indpendent of x, we can choose any value of x in this equation (they are all equivalent by compatibility). Letting $x \to \pm\infty$ and taking into account the t-independent boundary conditions

$$\lim_{x\to\pm\infty}\mathbf{J}_\pm(x,t;\lambda)e^{i\lambda x\sigma_3} = \mathbb{I},$$

as well as the fact that (due to decay of ψ and ψ_x as $|x| \to \infty$)

$$\lim_{x\to\pm\infty} \mathbf{V}(x,t,\lambda) = -i\lambda^2\sigma_3,$$

we see that in fact

$$\frac{\partial \mathbf{C}_\pm}{\partial t}(t;\lambda) = -i\lambda^2\sigma_3\mathbf{C}_\pm(t;\lambda)$$

a particular solution of which is obviously $\mathbf{C}_\pm(t;\lambda) = e^{-i\lambda^2 t\sigma_3}$. □

Since $\mathbf{W}_\pm(x,t;\lambda) = \mathbf{J}_\pm(x,t;\lambda)e^{-i\lambda^2 t\sigma_3}$ solve (3), by substitution it follows that the Jost matrices themselves satisfy

$$\frac{\partial \mathbf{J}_\pm}{\partial t} = i\lambda^2\mathbf{J}_\pm\sigma_3 + \mathbf{V}\mathbf{J}_\pm\,.$$

This turns out to be enough information to obtain a differential equation in t for the scattering matrix $\mathbf{S}(\lambda;t)$. Solving (4) for $\mathbf{S}(\lambda) = \mathbf{S}(\lambda;t)$ and differentiating with respect to t gives

$$\begin{aligned}\frac{\partial \mathbf{S}}{\partial t} = \frac{\partial}{\partial t}(\mathbf{J}_-^{-1}\mathbf{J}_+) &= \mathbf{J}_-^{-1}\frac{\partial \mathbf{J}_+}{\partial t} - \mathbf{J}_-^{-1}\frac{\partial \mathbf{J}_-}{\partial t}\mathbf{J}_-^{-1}\mathbf{J}_+ \\ &= \mathbf{J}_-^{-1}\left(i\lambda^2\mathbf{J}_+\sigma_3 + \mathbf{V}\mathbf{J}_+\right) - \mathbf{J}_-^{-1}\left(i\lambda^2\mathbf{J}_-\sigma_3 + \mathbf{V}\mathbf{J}_-\right)\mathbf{J}_-^{-1}\mathbf{J}_+ \\ &= i\lambda^2\mathbf{S}\sigma_3 - i\lambda^2\sigma_3\mathbf{S} \\ &= i\lambda^2[\mathbf{S},\sigma_3],\end{aligned}$$

or, in terms of the functions $a = a(\lambda;t)$ and $b = b(\lambda;t)$,

$$\begin{bmatrix} a_t^* & -b_t^* \\ -b_t & a_t \end{bmatrix} = \begin{bmatrix} 0 & 2i\lambda^2 b^* \\ -2i\lambda^2 b & 0 \end{bmatrix} \quad\Leftrightarrow\quad a(\lambda;t) = a(\lambda;0), \quad b(\lambda;t) = b(\lambda;0)e^{2i\lambda^2 t}.$$

In particular, it follows from the definition $R(\lambda) := b(\lambda)/a(\lambda)$ that the reflection coefficient $R(\lambda;t)$ evolves explicitly in time.

Theorem 1 (Time evolution of the reflection coefficient)**.** *Suppose that $\psi = \psi(x,t)$ is a smooth solution of* (1) *with suitable decay as $x \to \pm\infty$. Then the corresponding reflection coefficient $R(\lambda;t)$ satisfies*

$$R(\lambda;t) = R(\lambda;0)e^{2i\lambda^2 t}\,.$$

Inverse transform.

Integral equations for Jost solutions. While we've constructed the Jost solutions explicitly in a simple example, we have not yet understood how they can be found more generally. The Jost solutions are supposed to be particular vector solutions, for $\lambda \in \mathbb{R}$, of the linear differential equation (2) that also satisfy certain boundary conditions as $x \to \pm\infty$. How can we specify such solutions precisely? The central idea is the same as that which arises in the proof of existence of unique solutions for initial-value problems for differential equations: replace the differential equations by integral equations that build in the required auxiliary conditions.

Let us describe how to find $\mathbf{j}_-^{(1)}(x;\lambda)$ in some detail. Recall that the relevant boundary condition is in this case that

$$\mathbf{j}_-^{(1)}(x;\lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1)\,, \quad \text{as } x \to -\infty$$

for each real λ. The key idea is to recall the Fundamental Theorem of Calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

and use this to integrate both sides of the differential equation taking into account the boundary condition at $x = -\infty$. The problem is that $\mathbf{j}_-^{(1)}(x;\lambda)$ has no limit as $x \to -\infty$, so the boundary terms will not be well-defined. However, if we write the components of $\mathbf{j}_-^{(1)}(x;\lambda)$ exactly in the form

$$\mathbf{j}_-^{(1)}(x;\lambda) = \begin{bmatrix} e^{-i\lambda x}u(x;\lambda) \\ e^{i\lambda x}v(x;\lambda) \end{bmatrix}$$

then from the differential equation (2) satisfied by $\mathbf{j}_-^{(1)}(x;\lambda)$ we deduce the differential equations

$$\frac{\partial u}{\partial x}(x;\lambda)=e^{2i\lambda x}\psi(x)v(x;\lambda)\,,\qquad \frac{\partial v}{\partial x}(x;\lambda)=e^{-2i\lambda x}\psi(x)^*u(x;\lambda)\,,$$

and now we have

$$\lim_{x\to-\infty}u(x;\lambda)=1\,,\qquad \lim_{x\to-\infty}v(x;\lambda)=0\,,$$

for all real λ. Now we are in a position to apply the Fundamental Theorem of Calculus. Integrating from $-\infty$ to x we have

$$u(x;\lambda)=1+\int_{-\infty}^x e^{2i\lambda y}\psi(y)v(y;\lambda)\,dy\,,\qquad v(x;\lambda)=\int_{-\infty}^x e^{-2i\lambda y}\psi(y)^*u(y;\lambda)\,dy\,.$$

Let's substitute the second equation into the first:

$$u(x;\lambda)=1+\int_{-\infty}^x e^{2i\lambda y}\psi(y)\int_{-\infty}^y e^{-2i\lambda z}\psi(z)^*u(z;\lambda)\,dz\,dy$$

This gives us a closed equation for $u(x;\lambda)$ that incorporates the boundary condition at $x=-\infty$. To analyze this equation, it is useful to exchange the order of integration using the formula

$$\int_{-\infty}^x\int_{-\infty}^y f(y,z)\,dz\,dy=\int_{-\infty}^x\int_z^x f(y,z)\,dy\,dz\,,$$

which gives

$$u(x;\lambda)=1+\int_{-\infty}^x K(x,z;\lambda)u(z;\lambda)\,dz$$

where the kernel is

$$K(x,z;\lambda):=\psi(z)^*\int_z^x e^{2i\lambda(y-z)}\psi(y)\,dy,\quad x>z.$$

One way to think about solving this integral equation for u is to use iteration. That is, we think of the integral equation as a mapping turning a function $u_n(x;\lambda)$ into another function $u_{n+1}(x;\lambda)$:

$$u_{n+1}(x;\lambda):=1+\int_{-\infty}^x K(x,z;\lambda)u_n(z;\lambda)\,dz\,.$$

A reasonable way to start the iteration, in view of the boundary condition at $x=-\infty$, is to choose $u_0(x;\lambda)\equiv 1$. Then, it is easy to see that

$$u_1(x;\lambda)=1+\int_{-\infty}^x K(x,z_1;\lambda)\,dz_1\,,$$

$$u_2(x;\lambda)=1+\int_{-\infty}^x K(x,z_1;\lambda)\,dz_1+\int_{-\infty}^x K(x,z_2;\lambda)\int_{-\infty}^{z_2}K(z_2,z_1;\lambda)\,dz_1\,dz_2\,,$$

and so on. Based on these calculations, we may make an inductive hypothesis that

$$u_n(x;\lambda)=\sum_{k=0}^n I_k(x;\lambda)\,,$$

where I_k denotes the k-fold integral

$$I_k(x;\lambda):=\int_{-\infty}^x K(x,z_k;\lambda)\int_{-\infty}^{z_k}K(z_k,z_{k-1})\cdots\int_{-\infty}^{z_2}K(z_2,z_1;\lambda)\,dz_1\,\cdots\,dz_k\,.$$

By definition $I_0=1$. This certainly gives the correct iterates for $n=1$ and $n=2$. It is a direct matter to check that the recurrence step is consistent with this formula for $u_n(x;\lambda)$, which completes an inductive proof that $u_n(x;\lambda)$ is indeed given by the claimed formula.

The iterates $u_n(x;\lambda)$ form partial sums of an infinite series. The question at hand is the convergence of this infinite series. Now, since $\lambda \in \mathbb{R}$,

$$\begin{aligned}
|K(x,z;\lambda)| &= |\psi(z)| \left| \int_z^x e^{2i\lambda(y-z)} \psi(y)\, dy \right| \\
&\leq |\psi(z)| \int_z^x |e^{2i\lambda(y-z)}| |\psi(y)|\, dy \\
&= |\psi(z)| \int_z^x |\psi(y)|\, dy \\
&\leq |\psi(z)| \cdot \|\psi\|_1 \,,
\end{aligned}$$

where $\|\psi\|_1$ denotes the L^1-norm:

$$\|\psi\|_1 := \int_{-\infty}^{\infty} |\psi(y)|\, dy \,.$$

For this upper bound on K to be useful, we have to introduce the first technical assumption we will make on $\psi(x)$: that it is an absolutely integrable function on $(-\infty, \infty)$, *i.e.* $\psi \in L^1(\mathbb{R})$. This is a restriction on the rate of decay we will require on $\psi(x)$ as $x \to \pm\infty$. This estimate gives us a corresponding estimate for the terms $I_k(x;\lambda)$: by the triangle inequality,

$$\begin{aligned}
|I_k(x;\lambda)| &\leq \int_{-\infty}^x |K(x,z_k;\lambda)| \int_{-\infty}^{z_k} |K(z_k, z_{k-1};\lambda)| \cdots \int_{-\infty}^{z_2} |K(z_2, z_1;\lambda)|\, dz_1 \, \cdots \, dz_k \\
&\leq \|\psi\|_1^k \int_{-\infty}^x |\psi(z_k)| \int_{-\infty}^{z_k} |\psi(z_{k-1})| \cdots \int_{-\infty}^{z_2} |\psi(z_1)|\, dz_1 \, \cdots \, dz_k \,.
\end{aligned}$$

Now, the last line can be simplified. We claim that

$$\int_{-\infty}^x |\psi(z_k)| \int_{-\infty}^{z_k} |\psi(z_{k-1})| \cdots \int_{-\infty}^{z_2} |\psi(z_1)|\, dz_1 \, \cdots \, dz_k = \int_0^m \int_0^{m_k} \cdots \int_0^{m_2} dm_1 \, \cdots \, dm_k \,,$$

where

$$m := \int_{-\infty}^x |\psi(y)|\, dy \,.$$

This can be verified easily by making the change of variables

$$m_j := \int_{-\infty}^{z_j} |\psi(y)|\, dy \,, \qquad j = 1, \ldots, k$$

in each of the integrals on the right-hand side. Furthermore, a direct calculation shows that

$$\int_0^m \int_0^{m_k} \cdots \int_0^{m_2} dm_1 \, \cdots \, dm_k = \frac{m^k}{k!} \,.$$

Therefore, we have the estimate

$$|I_k(x;\lambda)| \leq \frac{1}{k!} \left(\|\psi\|_1 \int_{-\infty}^x |\psi(y)|\, dy \right)^k \,.$$

It follows that the infinite series

$$u(x;\lambda) = \sum_{k=0}^{\infty} I_k(x;\lambda)$$

converges absolutely by comparison with an exponential series; indeed we have

$$|u(x;\lambda)| \leq \sum_{k=0}^{\infty} |I_k(x;\lambda)| \leq \exp\left(\|\psi\|_1 \int_{-\infty}^x |\psi(y)|\, dy \right) \,.$$

Note that by replacing x by $+\infty$ in the estimates for $|I_k(x;\lambda)|$ we can even show that the convergence is uniform for all $x \in \mathbb{R}$, and by doing so in the above upper bound for $|u(x;\lambda)|$ we see that the function $u(x;\lambda)$ is uniformly bounded as a function of $x \in \mathbb{R}$:

$$\|u\|_\infty := \sup_{x \in \mathbb{R}} |u(x;\lambda)| \leq \exp\left(\|\psi\|_1^2 \right)$$

for each $\lambda \in \mathbb{R}$.

Now we have a uniformly bounded function $u(x;\lambda)$ satisfying the integral equation

$$u(x;\lambda) = 1 + \int_{-\infty}^{x} K(x,z;\lambda)u(z;\lambda)\,dz$$

for each $\lambda \in \mathbb{R}$ and each $x \in \mathbb{R}$. From the Lebesgue Dominated Convergence Theorem, it follows that $u(x;\lambda)$ has well-defined limiting values as $x \to \pm\infty$:

$$u(-\infty;\lambda) = 1\,, \qquad u(+\infty;\lambda) = 1 + \int_{-\infty}^{\infty} K(+\infty,z;\lambda)u(z;\lambda)\,dz\,.$$

Furthermore, since $e^{-2i\lambda x}\psi(x)^* u(x;\lambda)$ is then an absolutely integrable function, the function $v(x;\lambda)$ defined as an antiderivative thereof:

$$v(x;\lambda) = \int_{-\infty}^{x} e^{-2i\lambda y}\psi(y)^* u(y;\lambda)\,dy$$

is an absolutely continuous and uniformly bounded function that also has well-defined limiting values as $x \to \pm\infty$:

$$v(-\infty;\lambda) = 0\,, \qquad v(+\infty;\lambda) = \int_{-\infty}^{\infty} e^{-2i\lambda y}\psi(y)^* u(y;\lambda)\,dy\,.$$

Putting back the exponential factors:

$$\mathbf{j}_-^{(1)}(x;\lambda) = \begin{bmatrix} e^{-i\lambda x}u(x;\lambda) \\ e^{i\lambda x}v(x;\lambda) \end{bmatrix}$$

we see that we have constructed a vector solution $\mathbf{w} = \mathbf{j}_-^{(1)}(x;\lambda)$ of the differential equation (2) satisfying

$$\mathbf{j}_-^{(1)}(x;\lambda) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \to -\infty,$$

and

$$\mathbf{j}_-^{(1)}(x;\lambda) = a(\lambda)\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + b(\lambda)\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \to +\infty,$$

where

$$a(\lambda) = 1 + \int_{-\infty}^{\infty} K(+\infty,z;\lambda)u(z;\lambda)\,dz\,, \qquad b(\lambda) = \int_{-\infty}^{\infty} e^{-2i\lambda y}\psi(y)^* u(y;\lambda)\,dy\,.$$

Analytic properties of Jost solutions. So far, we have been considering $\lambda \in \mathbb{R}$. But when we go back and look at what we have done, we can see that some of what we have done works for more general complex values of λ as well. Suppose that $\lambda = \alpha + i\beta$. Then our convergence analysis for the infinite series representing $u(x;\lambda)$ begins as before with an estimate of the kernel K:

$$\begin{aligned} |K(x,z;\lambda)| &= |\psi(z)| \left| \int_z^x e^{2i\lambda(y-z)}\psi(y)\,dy \right| \\ &\le |\psi(z)| \int_z^x \left| e^{2i\lambda(y-z)} \right| |\psi(y)|\,dy \\ &= |\psi(z)| \int_z^x e^{-2\beta(y-z)}|\psi(y)|\,dy. \end{aligned}$$

Now, in the integrand $y \ge z$, so if $\beta \ge 0$, we will have again the estimate

$$|K(x,z;\lambda)| \le |\psi(z)| \cdot \|\psi\|_1\,.$$

Therefore, the existence of $u(x;\lambda)$ as a uniformly bounded function of x satisfying

$$\|u\|_\infty \le \exp\left(\|\psi\|_1^2\right)$$

is guaranteed as long as $\Im(\lambda) \ge 0$ by the same arguments as before. Moreover, the formulae for the limiting values of u as $x \to \pm\infty$ remain valid. The formula for $v(x;\lambda)$ in terms of $u(x;\lambda)$ also remains valid for $\Im(\lambda) \ge 0$ since the exponential factor in the integrand is decaying in the direction of $x \to -\infty$. The limiting statement $v(-\infty;\lambda)=0$ therefore also remains valid for $\Im(\lambda)\geq 0$. In fact, we even get an exponential rate of decay for v:

$$\begin{aligned}|v(x;\lambda)| &\leq \int_{-\infty}^{x} e^{2\beta y}|\psi(y)||u(y;\lambda)|\,dy\\ &\leq e^{2\beta x}\int_{-\infty}^{x}|\psi(y)||u(y;\lambda)|\,dy\\ &\leq e^{2\beta x}\|\psi u\|_1\\ &\leq e^{2\beta x}\|u\|_\infty\|\psi\|_1\\ &\leq e^{2\beta x}\|\psi\|_1\exp\left(\|\psi\|_1^2\right)\,.\end{aligned}$$

This is $O(e^{2\beta x})$ as $x\to-\infty$. However, if $\Im(\lambda)>0$ we can no longer deduce a limiting value for $v(x;\lambda)$ in the limit $x\to+\infty$. (Why not?)

The Jost solution $\mathbf{j}_-^{(1)}(x;\lambda)$ therefore exists for all λ in the upper half-plane as well as for real λ. From the conditions that $u(-\infty;\lambda)=1$ and $v(x;\lambda)=O(e^{2\beta x})$ that hold true for $\Im(\lambda)>0$, we see that

$$\mathbf{j}_-^{(1)}(x;\lambda)=O(e^{\beta x})\,,\quad x\to-\infty,$$

and therefore represents a solution that, for $\Im(\lambda)>0$, decays exponentially to zero as $x\to-\infty$.

Now another important feature of $\mathbf{j}_-^{(1)}(x;\lambda)$ comes from thinking of the dependence of $u(x;\lambda)$ on λ with $\Im(\lambda)>0$ for fixed $x\in\mathbb{R}$. The kernel $K(x,z;\lambda)$ is an entire analytic function of λ for each fixed x and z. Furthermore, if $\Im(\lambda)>0$, then the terms $I_k(x;\lambda)$ are analytic functions of λ for each $x\in\mathbb{R}$. Now since the infinite series representing $u(x;\lambda)$ converges uniformly with respect to λ in the upper half-plane, and since the partial sums are analytic functions of λ there, it follows that in fact $u(x;\lambda)$ is itself an analytic function of λ in the upper half-plane, for each $x\in\mathbb{R}$. From here it is easy to see that both components of the Jost solution vector $\mathbf{j}_-^{(1)}(x;\lambda)$ are analytic functions of λ in the upper half-plane for each fixed $x\in\mathbb{R}$.

How does $\mathbf{j}_-^{(1)}(x;\lambda)$ behave as $\lambda\to\infty$ with $\Im(\lambda)>0$ (holding $x\in\mathbb{R}$ fixed)? We can start with the uniform upper bound for $u(x;\lambda)$:

$$\sup_{x\in\mathbb{R},\Im(\lambda)\geq 0}|u(x;\lambda)|\leq\exp\left(\|\psi\|_1^2\right)\,.$$

Then, from the integral formula relating $v(x;\lambda)$ to $u(x;\lambda)$ we have

$$e^{2i\lambda x}v(x;\lambda)=\int_{-\infty}^{x}e^{2i\lambda(x-y)}\psi(y)^*u(y;\lambda)\,dy\,,$$

so

$$|e^{2i\lambda x}v(x;\lambda)|\leq\exp\left(\|\psi\|_1^2\right)\int_{-\infty}^{x}e^{2\beta(y-x)}|\psi(y)|\,dy\,.$$

The integrand tends to zero as $\beta\to+\infty$ for almost every y, and since $\beta\geq 0$ guarantees that

$$e^{2\beta(y-x)}|\psi(y)|\chi_{(-\infty,x)}(y)\leq|\psi(y)|$$

($\chi_S(y)$ denotes the characteristic function of a set $S\subset\mathbb{R}$, equal to 1 for $y\in S$ and equal to zero otherwise) with the upper bound being independent of β and absolutely integrable, the Lebesgue Dominated Convergence Theorem says that

$$\lim_{\Im(\lambda)\to+\infty}e^{2i\lambda x}v(x;\lambda)=0$$

for each $x\in\mathbb{R}$. Note also that if we make the further assumption that $\psi\in L^\infty(\mathbb{R})$, that is, $\psi(x)$ is a uniformly bounded function, then we can also see that

$$|e^{2i\lambda x}v(x;\lambda)|\leq\exp\left(\|\psi\|_1^2\right)\int_{-\infty}^{x}e^{2\beta(y-x)}|\psi(y)|\,dy\leq\|\psi\|_\infty\exp\left(\|\psi\|_1^2\right)\int_{-\infty}^{x}e^{2\beta(y-x)}\,dy=\frac{1}{2\beta}\|\psi\|_\infty\exp\left(\|\psi\|_1^2\right)\,,$$

so we get a *rate* of decay of $O(1/\beta)$.

Next, consider the difference $u(x;\lambda)-1$ as given in terms of $v(x;\lambda)$:

$$u(x;\lambda)-1=\int_{-\infty}^{x}e^{2i\lambda y}\psi(y)v(y;\lambda)\,dy\,.$$

By our above result, the integrand tends to zero for almost every y in the limit $\Im(\lambda) \to +\infty$. Moreover, we have seen that whenever $\Im(\lambda) \geq 0$,

$$|e^{2i\lambda x}v(x;\lambda)| \leq \|\psi\|_1 \exp\left(\|\psi\|_1^2\right) ,$$

so that with $\psi \in L^1(\mathbb{R})$ the Lebesgue Dominated Convergence Theorem again applies, and shows that

$$\lim_{\Im(\lambda)\to+\infty} u(x;\lambda) = 1$$

for all $x \in \mathbb{R}$.

Therefore, it follows under the hypothesis that $\psi \in L^1(\mathbb{R})$ that

$$\lim_{\Im(\lambda)\to+\infty} e^{i\lambda x}\mathbf{j}_-^{(1)}(x;\lambda) = \begin{bmatrix}1\\0\end{bmatrix}$$

for each $x \in \mathbb{R}$. If $\psi \in L^\infty(\mathbb{R})$ as well, then the rate of decay in the second row is $O(1/\beta)$.

Other Jost solutions. Analyticity properties of the scattering matrix elements. Parallel analysis applies to the other Jost solutions: $\mathbf{j}_-^{(2)}(x;\lambda)$, $\mathbf{j}_+^{(1)}(x;\lambda)$, and $\mathbf{j}_+^{(2)}(x;\lambda)$. Namely, under the same hypotheses on ψ as above:

- $\mathbf{j}_-^{(2)}(x;\lambda)$ is continuous for $\Im(\lambda) \leq 0$ and analytic for $\Im(\lambda) < 0$, and satisfies
$$\mathbf{j}_-^{(2)}(x;\lambda) = O(e^{-\Im(\lambda)x}), \qquad \text{as } x \to -\infty$$
for $\Im(\lambda) < 0$ (exponential decay for negative x) and
$$\lim_{\Im(\lambda)\to-\infty} e^{-i\lambda x}\mathbf{j}_-^{(2)}(x;\lambda) = \begin{bmatrix}0\\1\end{bmatrix}$$
for all $x \in \mathbb{R}$.
- $\mathbf{j}_+^{(1)}(x;\lambda)$ is continuous for $\Im(\lambda) \leq 0$ and analytic for $\Im(\lambda) < 0$, and satisfies
$$\mathbf{j}_+^{(1)}(x;\lambda) = O(e^{\Im(\lambda)x}), \qquad \text{as } x \to +\infty$$
for $\Im(\lambda) < 0$ (exponential decay for positive x) and
$$\lim_{\Im(\lambda)\to-\infty} e^{i\lambda x}\mathbf{j}_+^{(1)}(x;\lambda) = \begin{bmatrix}1\\0\end{bmatrix}$$
for all $x \in \mathbb{R}$.
- $\mathbf{j}_+^{(2)}(x;\lambda)$ is continuous for $\Im(\lambda) \geq 0$ and analytic for $\Im(\lambda) > 0$, and satisfies
$$\mathbf{j}_+^{(2)}(x;\lambda) = O(e^{-\Im(\lambda)x}), \qquad \text{as } x \to +\infty$$
for $\Im(\lambda) > 0$ (exponential decay for positive x) and
$$\lim_{\Im(\lambda)\to+\infty} e^{-i\lambda x}\mathbf{j}_+^{(2)}(x;\lambda) = \begin{bmatrix}0\\1\end{bmatrix}$$
for all $x \in \mathbb{R}$.

Now consider the first column of the scattering relation $\mathbf{J}_-(x;\lambda) = \mathbf{J}_+(x;\lambda)\mathbf{S}(\lambda)^{-1}$:

$$\mathbf{j}_-^{(1)}(x;\lambda) = a(\lambda)\mathbf{j}_+^{(1)}(x;\lambda) + b(\lambda)\mathbf{j}_+^{(2)}(x;\lambda).$$

Let's make both sides of this equation into matrices by adjoining as a second column to both sides the same vector, $\mathbf{j}_+^{(2)}(x;\lambda)$:

$$\left[\mathbf{j}_-^{(1)}(x;\lambda), \mathbf{j}_+^{(2)}(x;\lambda)\right] = \left[a(\lambda)\mathbf{j}_+^{(1)}(x;\lambda) + b(\lambda)\mathbf{j}_+^{(2)}(x;\lambda), \mathbf{j}_+^{(2)}(x;\lambda)\right].$$

Now take determinants of both sides using linearity of the determinant in the first column to simplify the right-hand side:

$$\det\left[\mathbf{j}_-^{(1)}(x;\lambda), \mathbf{j}_+^{(2)}(x;\lambda)\right] = a(\lambda)\det\left[\mathbf{j}_+^{(1)}(x;\lambda), \mathbf{j}_+^{(2)}(x;\lambda)\right] + b(\lambda)\det\left[\mathbf{j}_+^{(2)}(x;\lambda), \mathbf{j}_+^{(2)}(x;\lambda)\right] = a(\lambda)$$

with the final result obtained by noting that the coefficient of $b(\lambda)$ is obviously the determinant of a singular matrix and that the coefficient of $a(\lambda)$ is simply $\det(\mathbf{J}_+(x;\lambda))$, which we previously evaluated by taking a limit as $x \to +\infty$. This gives a formula for the scattering coefficient $a(\lambda)$ in terms of the Jost solutions $\mathbf{j}_-^{(1)}(x;\lambda)$ and $\mathbf{j}_+^{(2)}(x;\lambda)$. Since both of these have analytic continuations into the upper half-plane, we learn that $a(\lambda)$ is an analytic function in the upper half-plane as well. Moreover, from the asymptotic behavior of the relevant Jost solutions for large λ, we have

$$\lim_{\Im(\lambda)\to+\infty} a(\lambda) = 1\,.$$

A formula for $b(\lambda)$ may also be obtained by taking determinants, this time after adjoining $\mathbf{j}_+^{(1)}(x;\lambda)$ as a first column to both sides of the scattering relation. This gives

$$b(\lambda) = \det\left[\mathbf{j}_+^{(1)}(x;\lambda), \mathbf{j}_-^{(1)}(x;\lambda)\right]\,.$$

Now as one of the relevant Jost functions is analytic in the upper half-plane while the other is analytic in the lower half-plane, we learn that, generally speaking, the scattering coefficient $b(\lambda)$ while defined for $\lambda \in \mathbb{R}$, generally does not have an analytic continuation into either of the half-planes.

We now have the following important result.

Lemma 2. *Suppose that $\psi \in L^1(\mathbb{R})$, and let $a(\lambda) = S_{22}(\lambda)$ be the corresponding element of the scattering matrix. Then $a(\lambda) \neq 0$ for $\Im\{\lambda\} \geq 0$.*

Proof. For real λ, from the identity $\mathbf{S}(\lambda)^* = \sigma_1\mathbf{S}(\lambda)\sigma_1$ and the fact that $\det(\mathbf{S}(\lambda)) = 1$ one easily obtains the inequality $|a(\lambda)|^2 \geq 1$.

Now suppose that $a(\lambda_0) = 0$ for some λ_0 with $\Im\{\lambda_0\} > 0$. From the Wronskian formula for $a(\lambda)$ it follows that the Jost solutions $\mathbf{j}_-^{(1)}(x;\lambda_0)$ and $\mathbf{j}_+^{(2)}(x;\lambda_0)$ are proportional to each other. But because $\Im(\lambda_0) > 0$, $\mathbf{j}_-^{(1)}(x;\lambda_0)$ is exponentially decaying as $x \to -\infty$ while $\mathbf{j}_+^{(2)}(x;\lambda_0)$ is exponentially decaying as $x \to +\infty$. This implies the existence of a nonzero solution $\mathbf{w}$ of (2) that decays exponentially as $|x| \to \infty$. Now the linear system (2) can be written in the form of an eigenvalue problem:

$$L\mathbf{w} = \lambda\mathbf{w}, \quad L := i\sigma_3\frac{d}{dx} + \begin{bmatrix} 0 & -i\psi(x) \\ i\psi(x)^* & 0 \end{bmatrix}$$

and a key point is that the matrix differential operator L is *self-adjoint* on the space of square-integrable vector functions $\mathbf{w}(x)$ equipped with the Euclidean inner product

$$\langle \mathbf{w}, \mathbf{v}\rangle := \int_{\mathbb{R}} \left[w_1(x)v_1(x)^* + w_2(x)v_2(x)^*\right]\,dx.$$

However, $\lambda = \lambda_0$ is a complex eigenvalue of L, which contradicts selfadjointness. □

Riemann-Hilbert problem. Let's recombine the columns of the Jost matrices in order to obtain matrices whose columns extend together into the same half-plane:

$$\mathbf{M}(\lambda;x) := \begin{cases} \left[\dfrac{e^{i\lambda x}}{a(\lambda)}\mathbf{j}_-^{(1)}(x;\lambda), e^{-i\lambda x}\mathbf{j}_+^{(2)}(x;\lambda)\right]\,, & \Im(\lambda) > 0\,, \\[2ex] \left[e^{i\lambda x}\mathbf{j}_+^{(1)}(x;\lambda), \dfrac{e^{-i\lambda x}}{a(\lambda^*)^*}\mathbf{j}_-^{(2)}(x;\lambda)\right]\,, & \Im(\lambda) < 0\,. \end{cases}$$

The notation we are using is meant to suggest that we are now changing our point of view and viewing λ as the basic (complex) variable, and x as the parameter. From the relations

$$\mathbf{j}_\pm^{(2)}(x;\lambda)^* = \sigma_1\mathbf{j}_\pm^{(1)}(x;\lambda)\,, \qquad \lambda \in \mathbb{R}\,,$$

we get by analytic continuation that

$$\mathbf{j}_-^{(2)}(x;\lambda^*)^* = \sigma_1\mathbf{j}_-^{(1)}(x;\lambda)\,, \qquad \Im(\lambda) \geq 0\,,$$

and

$$\mathbf{j}_+^{(2)}(x;\lambda^*)^* = \sigma_1\mathbf{j}_+^{(1)}(x;\lambda)\,, \qquad \Im(\lambda) \leq 0\,.$$

From these it follows easily that (by definition, really)

$$\mathbf{M}(\lambda;x) = \sigma_1\mathbf{M}(\lambda^*;x)^*\sigma_1\,.$$

From the definition it follows that $\det(\mathbf{M}(\lambda;x))=1$ at each point where $\mathbf{M}$ is defined.

Lemma 3 (Analyticity of $\mathbf{M}(\lambda;x)$)**.** *For every $x\in\mathbb{R}$, the elements of the matrix $\mathbf{M}(\lambda;x)$ are all analytic functions of λ for $\Im\{\lambda\}\neq 0$.*

Proof. This follows from the analyticity properties of the Jost solutions and of the function $a(\lambda)$, because $a(\lambda)\neq 0$ for $\Im\{\lambda\}>0$ according to Lemma 2. $\square$

Note that $\mathbf{M}(x;\lambda)$ has not actually been defined for $\lambda\in\mathbb{R}$. This is not an omission, but rather reflects the reality of the situation; the matrix $\mathbf{M}(\lambda;x)$ generally does not extend to the real λ-axis as an analytic (or even continuous) function. This is because there is a mismatch in the boundary values taken by $\mathbf{M}(\lambda;x)$ as λ approaches the real axis from the upper and lower half-planes. Let

$$\mathbf{M}_\pm(\lambda;x):=\lim_{\epsilon\downarrow 0}\mathbf{M}(\lambda\pm i\epsilon;x)\,,\qquad \lambda\in\mathbb{R}\,,$$

and let $\mathbf{m}_\pm^{(1)}(\lambda;x)$ and $\mathbf{m}_\pm^{(2)}(\lambda;x)$ denote the corresponding columns. (The subscript notation on $\mathbf{M}$ and its columns differs in meaning from that on $\mathbf{J}$ and its columns.) Thus,

$$\mathbf{m}_+^{(1)}(\lambda;x)=\frac{e^{i\lambda x}}{a(\lambda)}\mathbf{j}_-^{(1)}(x;\lambda)\,,\qquad \mathbf{m}_+^{(2)}(\lambda;x)=e^{-i\lambda x}\mathbf{j}_+^{(2)}(x;\lambda)\,,$$

and

$$\mathbf{m}_-^{(1)}(\lambda;x)=e^{i\lambda x}\mathbf{j}_+^{(1)}(x;\lambda)\,,\qquad \mathbf{m}_-^{(2)}(\lambda;x)=\frac{e^{-i\lambda x}}{a(\lambda)^*}\mathbf{j}_-^{(2)}(x;\lambda)\,,$$

all for $\lambda\in\mathbb{R}$. From the scattering relation

$$\mathbf{J}_-(x;\lambda)=\mathbf{J}_+(x;\lambda)\begin{bmatrix}a(\lambda) & b(\lambda)^*\\ b(\lambda) & a(\lambda)^*\end{bmatrix}\,,$$

it easily follows that

$$\mathbf{m}_+^{(1)}(\lambda;x)=\frac{e^{i\lambda x}}{a(\lambda)}\left(a(\lambda)\mathbf{j}_+^{(1)}(x;\lambda)+b(\lambda)\mathbf{j}_+^{(2)}(x;\lambda)\right)=\mathbf{m}_-^{(1)}(\lambda;x)+e^{2i\lambda x}\frac{b(\lambda)}{a(\lambda)}\mathbf{m}_+^{(2)}(\lambda;x)\,,$$

and

$$\mathbf{m}_-^{(2)}(\lambda;x)=\frac{e^{-i\lambda x}}{a(\lambda)^*}\left(-b(\lambda)^*\mathbf{j}_+^{(1)}(x;\lambda)+a(\lambda)^*\mathbf{j}_+^{(2)}(x;\lambda)\right)=e^{-2i\lambda x}\frac{b(\lambda)^*}{a(\lambda)^*}\mathbf{m}_-^{(1)}(\lambda;x)+\mathbf{m}_+^{(2)}(x;\lambda)\,.$$

These can be written respectively as

$$\mathbf{M}_+(\lambda;x)\begin{bmatrix}1\\ -e^{2i\lambda x}\frac{b(\lambda)}{a(\lambda)}\end{bmatrix}=\mathbf{M}_-(\lambda;x)\begin{bmatrix}1\\0\end{bmatrix}\quad\text{and}\quad \mathbf{M}_+(\lambda;x)\begin{bmatrix}0\\1\end{bmatrix}=\mathbf{M}_-(\lambda;x)\begin{bmatrix}-e^{-2i\lambda x}\frac{b(\lambda)^*}{a(\lambda)^*}\\ 1\end{bmatrix}\,,$$

which can be combined into a single matrix equation as

$$\mathbf{M}_+(\lambda;x)\begin{bmatrix}1 & 0\\ -e^{2i\lambda x}\frac{b(\lambda)}{a(\lambda)} & 1\end{bmatrix}=\mathbf{M}_-(\lambda;x)\begin{bmatrix}1 & -e^{-2i\lambda x}\frac{b(\lambda)^*}{a(\lambda)^*}\\ 0 & 1\end{bmatrix}.$$

The latter is easily rewritten in terms of the reflection coefficient $R(\lambda)=b(\lambda)/a(\lambda)$ as follows:

Lemma 4 (Jump condition for $\mathbf{M}(\lambda;x)$)**.** *The boundary values $\mathbf{M}_\pm(\lambda;x)$ are related by the following jump condition:*

$$\mathbf{M}_+(\lambda;x)=\mathbf{M}_-(\lambda;x)\mathbf{D}(\lambda;x),\quad \lambda\in\mathbb{R},$$

where the so-called jump matrix is

$$\mathbf{D}(\lambda;x)=\begin{bmatrix}1-|R(\lambda)|^2 & -e^{-2i\lambda x}R(\lambda)^*\\ e^{2i\lambda x}R(\lambda) & 1\end{bmatrix}\,,\qquad \lambda\in\mathbb{R}.$$

Finally, we have the following property of $\mathbf{M}(\lambda;x)$.

Lemma 5 (Normalization of $\mathbf{M}(\lambda;x)$)**.** *For every $x \in \mathbb{R}$, the matrix $\mathbf{M}(\lambda;x)$ satisfies*

$$\lim_{|\Im(\lambda)|\to+\infty} \mathbf{M}(\lambda;x) = \mathbb{I}\,.$$

Proof. This follows from the definition of $\mathbf{M}(\lambda;x)$ with the use of the asymptotic conditions satisfied by the Jost solutions as $\Im(\lambda) \to \infty$ and the function $a(\lambda)$. ☐

In Lemma 3, Lemma 4, and Lemma 5, we have isolated three properties of the matrix $\mathbf{M}(\lambda;x)$ that we have deduced from scattering theory. The remarkable thing is that these properties are sufficient (in most cases) to determine the matrix $\mathbf{M}(\lambda;x)$ from the reflection coefficient $R(\lambda)$, and from $\mathbf{M}(\lambda;x)$ it is easy to extract the potential $\psi(x)$ whose reflection coefficient is $R(\lambda)$. This is the process of *inverse scattering.* Indeed, if the matrix $\mathbf{M}(\lambda;x)$ can be recovered for each x, then from the differential equation satisfied by $\mathbf{j}_+^{(2)}(x;\lambda)$ we can easily deduce a differential equation satisfied by $\mathbf{m}^{(2)}(\lambda;x)$ for $\Im(\lambda) > 0$:

$$\frac{\partial \mathbf{m}^{(2)}}{\partial x} = \begin{bmatrix} -2i\lambda & \psi(x) \\ \psi(x)^* & 0 \end{bmatrix} \mathbf{m}^{(2)}\,.$$

Recalling that $\mathbf{m}^{(2)}(\lambda;x) \to [0,1]^T$ as $\Im(\lambda) \to +\infty$, we see from the first row that

$$o(1) = -2i\lambda M_{12}(\lambda;x) + \psi(x)(1+o(1))$$

as $\Im(\lambda) \to +\infty$. Taking a limit gives us a formula for $\psi(x)$ in terms of $\mathbf{M}(\lambda;x)$:

$$\psi(x) = 2i \lim_{\Im(\lambda)\to+\infty} \lambda M_{12}(\lambda;x)\,.$$

The problem of finding the matrix $\mathbf{M}(\lambda;x)$ satisfying the three properties listed in Lemmas 3–5 given a suitable reflection coefficient $R(\lambda)$ is called a *Riemann-Hilbert problem.*

Summary: the inverse-scattering transform solution algorithm for the initial-value problem for the defocusing NLS equation.

Theorem 2. *Let $\psi_0(x)$ be a suitable complex-valued function of x decaying to zero as $x \to \pm\infty$. Let $R_0(\lambda)$ denote the corresponding reflection coefficient under the direct transform: $\psi_0 \mapsto R_0$. Then, the solution of the initial value problem for the defocusing NLS equation*

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi = 0, \quad \lim_{x\to\pm\infty} \psi(x,t) = 0, \quad \psi(x,0) = \psi_0(x)$$

is given by the formula

$$\psi(x,t) = 2i \lim_{\lambda\to\infty} \lambda M_{12}(\lambda;x,t)\,,$$

where $\mathbf{M}(\lambda;x,t)$ is the solution of the following Riemann-Hilbert problem: find a 2×2 matrix $\mathbf{M}(\lambda;x,t)$ with the following properties:

- **Analyticity:** *$\mathbf{M}(\lambda;x,t)$ is an analytic function of λ for $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*
- **Jump Condition:** *The matrix $\mathbf{M}(\lambda;x,t)$ takes continuous boundary values $\mathbf{M}_\pm(\lambda;x,t)$ on the real axis from $\mathbb{C}_\pm$, and they are related by the condition*

  $$\mathbf{M}_+(\lambda;x,t) = \mathbf{M}(\lambda;x,t)\mathbf{D}(\lambda;x,t), \quad \lambda \in \mathbb{R},$$

  where

  $$\mathbf{D}(\lambda;x,t) := \begin{bmatrix} 1-|R_0(\lambda)|^2 & -e^{-2i(\lambda x+\lambda^2 t)}R_0(\lambda)^* \\ e^{2i(\lambda x+\lambda^2 t)}R_0(\lambda) & 1 \end{bmatrix}.$$

- **Normalization:** *As $\lambda \to \infty$, $\mathbf{M}(\lambda;x,t) \to \mathbb{I}$.*

This problem has a unique solution for all $(x,t) \in \mathbb{R}^2$ if R_0 is a suitable function of $\lambda \in \mathbb{R}$.

Further Comments

Here are some final points:

- The basic method applies to solve many other nonlinear initial-value problems, for various equations (so-called *integrable systems* or *soliton equations*) that can be represented in the form of a compatibility condition between two or more linear equations involving a complex parameter λ. Some famous examples (in addition to NLS and KdV) include:
  - The sine-Gordon equation
  - The Toda lattice equations
  - The Kadomtsev-Petviashvili (KP) equation
  - The three-wave resonant interaction equations
  - The Benjamin-Ono equation
  - The Painlevé equations.
- In other problems eigenvalues can occur and these lead to poles of $\mathbf{M}$ that have to be built into the Riemann-Hilbert problem. The eigenvalues correspond to exact solutions called *solitons.* This can occur even for defocusing NLS under nonvanishing boundary conditions, when a gap appears in the (real, by selfadjointness) continuous spectrum that can contain discrete spectrum corresponding to "dark" solitons.
- The explicit dependence on the parameters (x, t) in the Riemann-Hilbert problem (via the oscillatory factors $e^{\pm 2i(\lambda x+\lambda^2 t)}$) actually allows the success of a family of asymptotic methods for Riemann-Hilbert problems that are natural generalizations of the famous methods for asymptotic analysis of Fourier integrals (stationary phase, steepest descent).
- In general, a matrix Riemann-Hilbert problem can be analyzed by using singular integral operators with Cauchy kernels to transform it into an equivalent system of singular integral equations. Although the kernels are singular, the integral operators are typically Fredholm of index zero. If the norms are sufficiently small, the singular integral equations can be solved (and hence so can the Riemann-Hilbert problem) by iteration and the Contraction Mapping Principle.
- A subject of current research is the generalization of these methods to two new areas:
  - Problems with more than one space dimension (like KP and Davey-Stewartson) can be solved by inverse-scattering transforms involving $\overline{\partial}$-problems in place of Riemann-Hilbert problems. Non-analyticity is smeared out over two-dimensional regions of the complex plane.
  - Problems on domains other than $\mathbb{R}^n$ with boundary conditions of various types (Dirichlet, Neumann, mixed) can be treated within a framework developed by Fokas and his collaborators. This technique reveals a key role played by linearizable Dirichlet-to-Neumann maps and shows that some boundary conditions are "integrable" while others are not.