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From Lieb Liniger to Gross Pitaevskii : Rapidities distribution and phonons relaxation

PARITE / presentation du labo

Le laboratoire Charles Fabry (LCF) est une unité mixte de recherche entre l'Institude d'Optique Graduate School (IOGS) et le CNRS et en partenariat avec l'Université ParisSud. La recherche au sein du laboratoire couvre un grand nombre de domaines ayant comme point commum l'optique.

Le laboratoire regroupe 170 personnes dont 22 chercheurs CNRS, 25 enseignantschercheurs de l'IOGS et de Paris-Sud et 39 personnels de soutien à la recherche. L'autre moitié étant des doctorants et des postdocs. La recherche y est organisée en huits groupes selon le domaine : "Biophotonique", "Lasers", "Matériaux non linéaires et applications", "Nanophotonique et Electromagnétisme", "Optique atomique", "Optique quantique", "Optique XUV et Surfaces Optiques" et "Systèmes d'imagerie et physique des images". Le groupe de recherche "Optique atomique" porte sur l'étude des atomes froids et regroupe 5 expériences : "Optique atomique quantique", "Gaz sur réseaux", "Transport Quantique", "Gaz quantiques désordonnés et à intéraction variable", "Puce atomique", et une équipe de théoriciens.

J'ai effectué mon stage au sein de l'équipe Puce atomique composée de deux membres : Isabelle Bouchoule, chargée de recherche CNRS, et Léa Dubois, étudiante en première année de thèse. L'éxpérience de puce atomique consiste à étudier des gaz d'atomes froids confinés à une dimension.

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1 Introduction

The Lieb Liniger model is a quantum model describing 1D bosons interacting through a contact potential. This model is integrable and his classical equivalent, called Gross Pitaevskii or Non linear Schrodinger equation is also integrable, but regarding the classical meaning. Integrability imposes strong restrictions on the dynamics of the model:

After a unitary time evolution. One expects that a non integrable system will locally relax to a certain steady-state which can be characterised in terms of statistical physics. The corresponding ensemble depends though on the nature of the system. It will locally equilibrate to a Gibbs Ensemble,whose temperature is determined by the energy injected at t = 0, which is conserved during the time evolution : $\hat{\rho}_{GE} = \frac{1}{Z} e^{-\beta(\hat{H}-\mu\hat{N})}$

On the contrary, an integrable model will not relax to a Gibbs ensemble, but to a Generalized Gibbs ensemble. It means that we need an infinite set of conserved quantities $\hat{Q}_j$ to entirely describe the system : $\hat{\rho}_{GGE} = \frac{1}{Z} e^{-\sum_j \beta_j \hat{Q}_j}$ Where the infinitely many generalised temperatures β_j are in principle fixed matching the expectation values of the charges $\hat{Q}_j$ in terms of their value on the initial state. In this case,the presence of infinitely-many conserved charges will deeply constrain the dynamics of these models and many properties of the initial states are not washed out by the time-evolution,but remain encoded in the statistical nature of late-time steady state.

However, it' very important to precise that the existence of an infinite conserved charges is not a sufficient condition to be quantum integrable: indeed, any quantum model, including chaotic models, admits an infinite set of conserved charges, which are the eigenstates projectors. Another important point here, is that the infinite conserved charges could be described or generated with the distribution of rapidities $\rho(\lambda)$, that we will explain later.
But on the other hand, this condition remains sufficient to be classical integrable. And in classical integrable models, the conserved charges are the trace of a particular matrix, constructed to match the so called "zero curvature condition".

The first part of this internship was to exploit the classical integrability of Gross-Pitaevskii in order to find a link between the classical conserved charges and $\rho(\lambda)$, and then to check our results numerically.

The second part of this internship was about the study of phonons relaxation. Phonons correspond to collective excitations of the system, which include oscillation of the density. The relaxation of phonons has been studied in the recent article [ ] Citer article, in the quantum model of Lieb-Liniger. Because phonons are not exact eigenstates of the Hamiltonian(they are only at low energy), the

gas relaxes to a stationary state at very long times. But due to integrability, the system will not thermalize : it means this stationary state won't be a thermal state. They have characterized the stationary state of the gas after relaxation and computed its phonon population distribution. The goal of this second part is to test those results but in the classical limit.

2 Quantum integrability

An exact definition of quantum integrability does not make consensus. For example, one definition of integrable systems is to say that level spacing distribution follows a poissonian statistics, while for non-integrable systems, random matrix theory predicts the so-called Wigner-Dyson distribution. But the most common definition and accepted one is to say that a quantum model is integrable if it is solvable by using Bethe ansatz. Indeed, it means that we can, using the Yang-Baxter relation, get a so-called "transfer matrix" which can be used, to construct an infinite set of conserved quantities.

We will not consider and explain the mathematical details of the Bethe ansatz. The idea here is to directly apply it to the quantum integrable Lieb Liniger model, and to get a basic understanding of what it is and to introduce a key element for generating conserved quantities : the distribution of rapidities.

2.1 Lieb Liniger model

The Lieb Liniger model describes 1D bosons interacting through contact potential. We work here under the definition $\hbar = 1$ and $m = \frac{1}{2}$.

In second quantization, the hamiltonian reads :

$$\hat{H} = \int dx \left[\partial_x \psi^\dagger \partial_x \psi + g\, \psi^\dagger \psi^\dagger \psi \psi\right] \tag{1}$$

where the complex Bose field $\psi(t, x)$ satisfies the canonical commutation relations

$$[\psi(x), \psi^\dagger(y)] = \delta(x - y) \qquad [\psi(x), \psi(x')] = 0 \ .$$

And in first quantization, it reads :

$$H_{LL} = -\sum_j \frac{\partial^2}{\partial {x_j}^2} + g \sum_{i<j} \delta(x_i - x_j) \tag{2}$$

We focus here on the two body problem, since it allows to understand the important concepts and quantities of the Bethe ansatz. Then, we will generalize to the N body problem.

Thus, Schrodinger equation reads :

$$(-\frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} + g\delta(x_2 - x_1))\Psi = E\Psi \tag{3}$$

Let us consider the wave function in the spatial sector R : $x_1 \leq x_2$.

The wave function for $x_2 > x_1$ is then determined by Bose symmetry $\Psi(x_1, x_2) = \Psi(x_2, x_1)$. The interactions are repulsive for g > 0, and attractive for g < 0.
If $x_1 \neq x_2$, then $\delta(x_2 - x_1) = 0$ and the particles do not interact. If $x_1 = x_2$, the following boundary condition on R have to be fulfilled:

$$(-\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + g)\Psi_{(x_1=x_2)} = 0 \tag{4}$$

The Bethe ansatz consists in finding the wave function Ψ in a certain form :

$$\Psi(x_1, x_2) = a_1 e^{i(\lambda_1 x_1 + \lambda_2 x_2)} + a_2 e^{i(\lambda_2 x_1 + \lambda_1 x_2)} \tag{5}$$

λ_1 and λ_2 are called rapidities. We will show that the important quantities such as energy,eigenvalues, can be caracterized by the rapidities.

This ansatz is motivated by the fact that the solution should be a superposition of plane waves for $x_2 \neq x_1$, where the system is non-interacting, and that scattering of two identical particles in one dimension is very restrictive: momentum and energy conservation dictate that particles can either exchange momenta, or just pass through each other.

Inserting this ansatz in the previous boundary condition, we obtain :

$$\frac{a_1}{a_2} = \frac{i(\lambda_2 - \lambda_1) + g}{i(\lambda_2 - \lambda_1) - g} \tag{6}$$

Since $|\frac{a_1}{a_2}| = 1$, we can write :

$$\frac{a_1}{a_2} = e^{i\theta(\lambda_1 - \lambda_2)} \tag{7}$$

where

$$\theta(\lambda_1 - \lambda_2) = 2\arctan(\frac{\lambda_1 - \lambda_2}{g}) \tag{8}$$

Thus,the relation between the amplitudes a_1 and a_2 is a pure phase factor, where θ is the phase shift due to the scattering, considering the interactions. Even though the λ are quasi-momenta, they determine the total momentum and energy of the system :

$$P\Psi = (-i\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})\Psi = (\lambda_1 + \lambda_2)\Psi \tag{9}$$

$$H\Psi = (\lambda_1^2 + \lambda_2^2)\Psi \tag{10}$$

Until now the $\lambda_{1,2}$ are free parameters of our ansatz, and generally they are not the single-particle momenta as in the non-interacting case. To fix them, we need to apply periodic boundary conditions :

$$\Psi(0, x_2) = \Psi(x_2, L) \tag{11}$$

and to ensure continuity :

$$\frac{\partial\Psi(x_1,x_2)}{\partial x_1}\rfloor_{x_1=0} = \frac{\partial\Psi(x_2,x_1)}{\partial x_1}\rfloor_{x_1=L} \tag{12}$$

It reads that the λ have to satisfy the so called "Bethe equations" :

$$e^{i\lambda_1 L} = \frac{\lambda_1-\lambda_2+ig}{\lambda_1-\lambda_2-ig} \tag{13a}$$

$$e^{i\lambda_2 L} = \frac{\lambda_2-\lambda_1+ig}{\lambda_2-\lambda_1-ig} \tag{13b}$$

Another way to write these equations is to use the following identities :

$$ln(re^{i\theta}) = ln(r)+i\theta+i2n\pi \quad \arctan(z) = \frac{i}{2}(ln(1-iz)-ln(1+iz)) \tag{14}$$

where n in an integer

Then Bethe equations become :

$$\lambda_1 = \frac{2\pi}{L}m_1 - \frac{2}{L}\arctan(\frac{\lambda_1-\lambda_2}{g}) \tag{15a}$$

$$\lambda_2 = \frac{2\pi}{L}m_2 - \frac{2}{L}\arctan(\frac{\lambda_2-\lambda_1}{g}) \tag{15b}$$

The wavefunction is antisymmetric in the rapidities, which means that they cannot coincide : $\lambda_1 \neq \lambda_2$, and therefore the m_j are 'fermionic-like' quantum numbers. We often call them "Bethe Fermions".

Now, we can generalize these results to N particles : the general expression for the wavefunction on the spatial sector R ($x_1 \leq x_2 ..... \leq x_n$) reads :

$$\Psi(x_1,x_2,..,x_N) = A\sum_\sigma (-1)^{\lfloor\sigma\rfloor}\prod_{k<j}(\lambda_{\sigma(j)}-\lambda_{\sigma(k)}-ic)e^{i\sum_j \lambda_{\sigma(j)}} \tag{16}$$

where $\sum_\sigma$ extends over all N! permutations, A is a normalization factor and $(1)^{[\sigma]}$ is the sign of the permutation.

Or we can also write it :

$$|\{\lambda_j\}_{j=1}^N\rangle = \int d^N x\, \chi_N(x_1,...,x_N)\Psi^+(x_1)...\Psi^+(x_N)|0\rangle \tag{17}$$

where $\chi_N(x_1,\ldots,x_N) = \sum_P A(P)\prod_{j=1}^N e^{i\lambda_{P_j}x_j}$

Applying periodic boundary conditions leads to the Bethe equations, a set of N equations for N particles :

$$e^{i\lambda_j L} = \prod_{k\neq j}\frac{\lambda_j-\lambda_k+ig}{\lambda_j-\lambda_k-ig} \tag{18}$$

And the other way to write it :

$$\lambda_j = \frac{2\pi}{L} m_j - \frac{2}{L} \sum_k \arctan(\frac{\lambda_j - \lambda_k}{g}) \tag{19}$$

where $m_j = j - \frac{N+1}{2}$ are called the Fermionic quantum numbers. They are integers (half-integers) in the case of N odd (even).

Concerning the energy and the number of particles $\hat{N} = \int dx \Psi^+ \Psi$ one gets:

$$\hat{H}|\{\lambda_j\}_{j=1}^N\rangle = \left(\sum_{j=1}^N E(\lambda_j)\right)|\{\lambda_j\}_{j=1}^N\rangle \qquad , \qquad \hat{N}|\{\lambda_j\}_{j=1}^N\rangle = N|\{\lambda_j\}_{j=1}^N\rangle \tag{20}$$

where $E(\lambda) = \lambda^2$ is the single-particle energy.

2.2 Rapidities distribution $\rho(\lambda)$

In the repulsive regime of the quantum Lieb Liniger model, the rapidities are real and, rather than exactly tracking them, it's a clever idea to introduce a coarse-grain counting functions $\rho(\lambda)$, called the root density or the rapidities distribution.

Given a physical state, $L\rho(\lambda)d\lambda$ counts the number of rapidities laying around λ in a small window of length $d\lambda$. In a quantum integrable model, the rapidities distribution fully encodes the extensive expectation value of the conserved charges.

For what concerns the energy and number of particles, one has :

$$L^{-1} \langle\{\lambda_j\}_{j=1}^N|\hat{H}|\{\lambda_j\}_{j=1}^N\rangle = \int d\lambda\, E(\lambda)\rho(\lambda) + \mathcal{O}(L^{-1}) \tag{21a}$$

$$L^{-1} \langle\{\lambda_j\}_{j=1}^N|\hat{N}|\{\lambda_j\}_{j=1}^N\rangle = \int d\lambda\, \rho(\lambda) + \mathcal{O}(L^{-1})\,, \tag{21b}$$

There is another definition of the rapidities distribution, which is more convenient for our further purpose : λ correspond to the momentums in the very diluted, where the gas has expanded and relaxed in a box of very long length L. $\rho(\lambda)$ correspond to the asymptotic momentum distribution, after expansion. Asymptotic means here that we consider the limit of infite time expansion. voir si toujours ok dans boite ou pas : cf article Isabelle

This definition give a physical meaning to the rapidities and has a classical meaning whereas the definition with Bethe Ansatz has no sense in a classical model.

3 Integrability in classical systems

3.1 Gross-Pitaevskii as the semiclassical limit of Lieb-Liniger

Let us now explore the connection between the Lieb-Liniger model and its classical counterpart,the Non Linear Schrodinger equation. The aim here is do it properly, by using the formalism of path integral.

$\phi(t,x)$ is the classical complex field which obeys to Gross Pitaevskii equation. It's important to understand how to write the differents quantum quantities of the Lieb Liniger model in the classical one.

To do so,we have to take the semiclassical limit which can be seen as a high temperature limit. In order to see this, we define a rescaled quantum Hamiltonian via the Lieb Liniger Hamiltonian, and we reintroduce $\hbar$ and m.

$$\hat{H}' = \int dx \, \left(\partial_x \psi^\dagger \partial_x \psi + 2mg\hbar^2 \, \psi^\dagger \psi^\dagger \psi \psi \right) \tag{22}$$

Comparing with the first Hamiltonian , we have $\hat{H} = \frac{\hbar^2}{2m}\hat{H}'$, hence the equality of the density matrices $e^{-\beta_q \hat{H}} = e^{-\beta'_q \hat{H}'}$ with $\beta'_q = \frac{\hbar^2}{2m}\beta_q$.

Thus the limit $\hbar \to 0$ can be interpreted as a high temperature or small coupling limit for the rescaled Hamiltonian: we will pursue this second interpretation along this section. But before pursuing this interpretation, we need to briefly introduce the concept of euclidean time coordinate in statistical field theory : citer LIVRE

We recall that the evolution operator is $U(t) = e^{\frac{t}{i\hbar}H}$. Let us consider the case where t is a purely imaginary : $t = -i\tau$, where $\tau > 0$ is real and called euclidean time, and we get $U(-i\tau) = e^{-\frac{\tau}{\hbar}H}$. The transformation $(t \to i\tau)$ is called Wick rotation. One can remark that the evolution operator has the same form as the density matrix at thermoequilibrium at temperature T : $\rho_\beta = \frac{1}{Z(\beta)}U(-i\tau)$ where $\beta = \frac{1}{k_b T} = \frac{\tau}{\hbar} = i\frac{t}{\hbar}$
Thus, we have the analogy between an imaginary time and a finite temperature. When considering the formalism of path integral (see CITER REF),i.e writing the evolution operator as a path integral, the partition function could be written as the euclidean path integral on periodic trajectories (in imaginary time) of period τ : $Z(\beta) = Tr(e^{-\beta H}) = Tr(U(\tau))$

Now to apply the semiclassical limit, we will consider the quantum partition function with the rescaled Hamiltonian with an inverse temperature β'_q and a chemical potential μ. To reach the semiclassical limit of this expression, we supplement the real space with Euclidean time τ seen above.

Assuming periodic boundary conditions in the real space, the quantum problem is mapped into a classical partition function on a torus $(x,\tau) \in [0,L] \times [0,\beta'_{\rm q}]$.(see figure below)

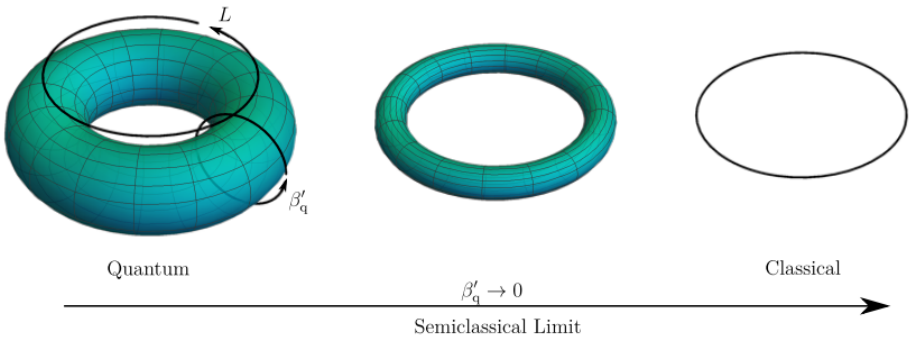

Figure 1: Graphical representation of the semiclassical limit of the thermal partition function. Within the path integral formalism, the partition function of the quantum Lieb-Liniger model is represented on a torus $(x,\tau) \in [0,L] \times [0,\beta'_{\rm q}]$.Increasing the temperature, i.e. $\beta'_{\rm q} \to 0$, the transverse dimension of the torus shrinks until the torus collapses on a one dimensional ring, namely the domain of the path integral describing the classical partition function of the Non-Linear Schrodinger equation.

The quantum partition function can then be written as

$$\mathcal{Z}_{\rm q} = \int \mathcal{D}\psi \exp\left[-\int_0^L dx \int_0^{\beta'_{\rm q}} d\tau \left\{ \frac{1}{2}(\psi^\dagger \partial_\tau \psi - \psi \partial_\tau \psi^\dagger) + \partial_x \psi^\dagger \partial_x \psi \right.\right. \tag{23}$$
$$\left.\left. - \mu \psi^\dagger \psi + 2m\hbar^2 g \psi^\dagger \psi^\dagger \psi \psi \right\}\right],$$

where $\psi(\tau,x)$ now denotes a classical field whose domain is the torus. On the other hand, we write the classical partition function as a path integral on a ring $x \in [0,L]$

$$\mathcal{Z} = \int \mathcal{D}\phi \exp\left[-\beta \int_0^L dx \, \left\{ \partial_x \phi^\dagger \partial_x \phi - \mu \phi^\dagger \phi + \phi^\dagger \phi^\dagger \phi \phi \right\} \right] \tag{24}$$

Now, in order to map the quantum partition function into a classical one, we have to take the limit $\hbar \to 0$, which could be seen as "shrinking" the imaginary time dimension $\beta'_{\rm q} \to 0$. To do so, let us consider the Fourier components of the field along the Euclidean time direction

$$\psi(\tau,x) = \sum_{n \in Z} \frac{e^{i2\pi n\tau/\beta'_q}}{\sqrt{2m\hbar^2 g}} \psi_n(x) \tag{25}$$

Above, for later use, we have suitably normalised the modes. We can see that it is dominated by the zero-frequency mode and therefore it is described by the classical NLS. Indeed, plugging this mode decomposition into the quantum action, we get

$$\mathcal{Z}_{\rm q} = \int \mathcal{D}\psi \exp\left[-\frac{\beta'_{\rm q}}{2m\hbar^2 g}\int_0^L dx \sum_n \left\{ (i2\pi/\beta'_{\rm q}) n\psi_n^\dagger\psi_n + \partial_x\psi_n^\dagger\partial_x\psi_n - \mu\psi_n^\dagger\psi_n \right\} + \sum_{n_1+n_2=n_3+n_4} \psi_{n_1}^\dagger\psi_{n_2}^\dagger\psi_{n_3}\psi_{n_4} \right] \tag{26}$$

and when $\beta'_{\rm q} \to 0$, this path integral is dominated by the zero-frequency mode since the $(i2\pi/\beta'_{\rm q})n\psi_n^\dagger\psi_n$ term tends to pin $\psi_{n\neq 0} \to 0$, while it vanishes for ψ_0 that is then free to fluctuate.

Hence, discarding the non-vanishing Matsubara frequencies we can approximate the partition function as

$$\mathcal{Z}_{\rm q} \simeq \int \mathcal{D}\psi_0 \exp\left[-\frac{\beta'_{\rm q}}{2m\hbar^2 g}\int dx \left\{ \partial_x\psi_0^\dagger\partial_x\psi_0 - \mu\psi_0^\dagger\psi_0 + \psi_0^\dagger\psi_0^\dagger\psi_0\psi_0 \right\} \right] \tag{27}$$

which is nothing else than the classical partition function provided the replacement $\phi(x) \to \psi_0(x)\,, \quad \beta \to \frac{\beta'_{\rm q}}{2m\hbar^2 g} = \frac{\beta_{\rm q}}{(2m)^2 g}$ Notice that the classical inverse temperature β is kept constant in the $\hbar \to 0$ limit, as it should be.
$\mathcal{Z}_{\rm q}$ can be useful to easily obtain some quantities using thermodynamic identities and it allows us to understand how the classical limit can be reached..

Another understanding of the classical limit : While the above derivation is quite formal, a more simple understanding of the classical regime applies when the system can be discribed by undependent modes. This is the case in the limit of ideal Bose gas, where modes are simply single particule states of well defined momentum. It is also the case deeply into the quasicondensate regime (citer reference), were the modes are the collective Bogoliubov modes. Then the classical limit of the quantum field theory corresponds to the case where the occupation of each mode is much larger than one.

3.2 The classical integrability of the Gross Pitaevskii: the inverse scattering method

In this section, we will exploit the classical integrability of Gross pitaevkii to extract the conserved quantities. To do so, we will apply the so called "inverse scattering method".

3.2.1 Zero curvature condition

The remarkable observation which led to the development of the inverse scattering transform is that the Gross Pitaevskii equation is equivalent to a com-

patibility condition for a linear scattering problem depending parametrically on a spectral parameter λ.

In the following, we will construct two matrices U_λ and V_λ,that depend on $\phi(x,t)$. And we will show that Gross Pitaevskii equation is in fact equivalent to a compatibility equation.

These two matrices read (we omit the explicit space-time dependence to simplify the notation) :

$$U_\lambda = U_0 + \lambda U_1 \tag{28a}$$
$$V_\lambda = V_0 + \lambda V_1 + \lambda V_2 \tag{28b}$$

where

$$U_0 = \sqrt{g}\begin{pmatrix} 0 & \phi^* \\ \phi & 0 \end{pmatrix} = \sqrt{g}(\phi^*\sigma_+ + \phi\sigma_-) \tag{29a}$$
$$U_1 = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2i}\sigma_3 \tag{29b}$$
$$V_0 = i\sqrt{g}\begin{pmatrix} \sqrt{g}|\phi|^2 & -\frac{\partial\phi^*}{\partial x} \\ \frac{\partial\phi}{\partial x} & -\sqrt{g}|\phi|^2 \end{pmatrix} = ig|\phi|^2\sigma_3 - i\sqrt{g}(\frac{\partial\phi^*}{\partial x}\sigma_+ - \frac{\partial\phi}{\partial x}\sigma_-) \tag{29c}$$
$$V_1 = -U_0 \tag{29d}$$
$$V_2 = -U_1 \tag{29e}$$

Here $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices, which obey the following commutations rules :

$$[\sigma_+, \sigma_3] = 2\sigma_+ \quad [\sigma_-, \sigma_3] = -2\sigma_- \quad [\sigma_+, \sigma_-] = \sigma_3 \quad [\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l$$

Now, let us calculate $\partial_t U_\lambda - \partial_x V_\lambda + [U_\lambda, V_\lambda]$:

The first term reads :

$$\partial_t U_\lambda = \sqrt{g}(\partial_t\phi^*\sigma_+ + \partial_t\phi\sigma_-) \tag{30}$$

Then, the second term reads :

$$\partial_x V_\lambda = ig(\phi^*\frac{\partial\phi}{\partial x} + \phi\frac{\partial\phi^*}{\partial x})\sigma_3 - i\sqrt{g}(\frac{\partial^2\phi^*}{\partial x^2}\sigma_+ - \frac{\partial^2\phi}{\partial x^2}\sigma_-) - \lambda\sqrt{g}(\frac{\partial\phi^*}{\partial x}\sigma_+ + \frac{\partial\phi}{\partial x}\sigma_-) \tag{31}$$

And the commutator reads :

$$\begin{aligned}[U_\lambda, V_\lambda] = ig(\phi^*\frac{\partial\phi}{\partial x} + \phi\frac{\partial\phi^*}{\partial x})\sigma_3 + 2ig\sqrt{g}\phi^*|\phi|^2\sigma_+ \\ -2ig\sqrt{g}\phi|\phi|^2\sigma_- + \lambda\sqrt{g}\frac{\partial\phi^*}{\partial x}\sigma_+ - \lambda\sqrt{g}\frac{\partial\phi}{\partial x}\sigma_-\end{aligned} \tag{32}$$

Thus,

$$\partial_t U_\lambda - \partial_x V_\lambda + [U_\lambda, V_\lambda] = \begin{pmatrix} 0 & c.c \\ -i\sqrt{g}\frac{\partial^2\phi}{\partial x^2} + \lambda\sqrt{g}\frac{\partial\phi}{\partial x} + 2ig\sqrt{g}\phi|\phi|^2 - \sqrt{g}\lambda\frac{\partial\phi}{\partial x} + \sqrt{g}\frac{\partial\phi}{\partial t} & 0 \end{pmatrix} \quad (33)$$

Thus, we find that the necessary and sufficient condition to have :

$$\partial_t U_\lambda - \partial_x V_\lambda + [U_\lambda, V_\lambda] = 0 \quad (34)$$

is that $\phi(x,t)$ obeys Gross-Pitaevskii equation. The above equation is called the zero curvature condition and has important consequences, and a geometrical interpretation, that we'll see further.

3.2.2 Conserved quantities

Here, we will show that this classical integrable model admits an infinity of conserved charges. This is the direct consequence of the zero curvature condition. We'll prove that the conserved charges are generated by the trace of a matrix T_λ,which is the propagator related to U_λ. That is to say we'll prove $\frac{dTr((T_\lambda)}{dt} = 0$, under some conditions.
To do so, we first derive an important consequence of the zero curvature condition, and then we'll apply it to our propagators.

Now, let's consider the differential form $A = A_\mu dx^\mu = V_\lambda dt + U_\lambda dx$ and a closed path Γ in space-time. In light of the compatibility equation, the differential form A is a closed form and by applying Stokes theorem, it gives :

$$\mathcal{P}\exp\left(\oint_{\mathcal{C}} A_\mu dx^\mu\right) = \mathcal{P}\exp\left(\int_{\mathcal{C}} DA\right) = \mathcal{I} \quad (35)$$

where $\mathcal{P}$ is the path ordering operator.

We propose to show the above relation : Let us consider a small rectangular formed by the 4 points $(x,t),(x+dx,t),(x+dx,t+dt),(x,t+dt)$. We want to compute $I = P\exp\int_{\mathcal{C}} DA$
The idea is to develop this integral at the second order on dx and dt.

To do so, let us introduce B(s), which is defined on $\mathcal{C}$ and which verify $B(0) = \mathcal{I}$ and $\frac{dB}{ds} = DAB$
Here s is the curviligne index, which goes from 0 to 1, and $DA = Vdt + Udx$
I is nothing but B(1). By developping B(1) at the second order on U and V, we get

$$B(1) = \mathcal{I} + \int_0^1 DA + \int_0^1 dA(s)\int_0^s DA(s') \quad (36)$$

And now we develop also the second term and the third term at the second order :

$$\int_0^1 DA = (\partial_x V_\lambda - \partial_t U_\lambda)dxdt \quad (37)$$

$$\int_0^1 dA(s) \int_0^s DA(s') = [V_\lambda, U_\lambda] dxdt \tag{38}$$

So it leads to

$$I = \mathcal{I} - (\partial_t U_\lambda - \partial_x V_\lambda + [U_\lambda, V_\lambda]) dxdt \tag{39}$$

And U_λ and V_λ satisfy the zero curvature condition, so we conclude that :

$$I = P \exp \int_{\mathcal{C}} DA = \mathcal{I} \tag{40}$$

Now, let's apply this property. To do so, we consider the closed path defined by the points (t_1, x_1), (t_1, x_2), (t_2, x_2), (t_2, x_1) (see below).

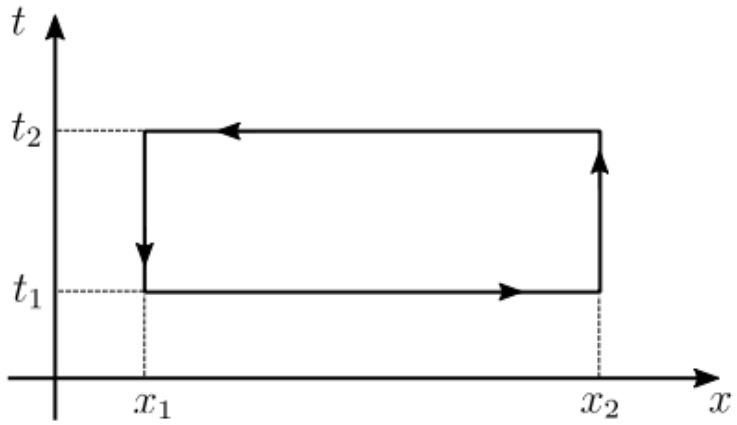

Figure 2: Closed path relative to the definition of the monodromy matrices

By definition, the propagator in space T_λ and the propagator in time S_λ read :

$$T_\lambda(t; x, y) = \mathcal{P} \exp \left(\int_x^y U_\lambda(t, x') dx' \right), \quad S_\lambda(t_1, t_2; x) = \mathcal{P} \exp \left(\int_{t_1}^{t_2} V_\lambda(t', x) dt' \right) . \tag{41}$$

We obtain by applying (35) to the path shown in figure 2 (see below), with the propagators above :

$$S_\lambda(t_2, t_1; x) T_\lambda(t_2; y, x) S_\lambda(t_1, t_2; y) T_\lambda(t_1; x, y) = I \tag{42}$$

Using $S_\lambda^{-1}(t_1, t_2; x) = S_\lambda(t_2, t_1; x)$ and $T_\lambda^{-1}(t; x, y) = T_\lambda(t; y, x)$, the above equation leads to :

$$T_\lambda(t_1; x, y;) \; = \; S_\lambda^{-1}(t_1, t_2; y) \, T_\lambda(t_2; x, y) \, S_\lambda(t_1, t_2; x) \tag{43}$$

And by taking the trace and permuting the first term, we get :

$$Tr(T_\lambda(t_1;x,y)) = Tr(T_\lambda(t_2;x,y)S_\lambda(t_1,t_2;x)S_\lambda^{-1}(t_1,t_2;y)) \tag{44}$$

As announced in the beginning of the section, under some conditions, we want to prove that the quantity $\tau_\lambda = Tr(T_{\lambda)}$ is time independent. To do so, we need two points in space-time such that $S_\lambda(t_1,t_2;x)S_\lambda^{-1}(t_1,t_2;y) = I$, or equivalently such that $V_\lambda(t,x) = V_\lambda(t,y)$. To ensure that, we choose periodic boundary conditions, i.e. $\phi(t,x+L) = \phi(t,x)$.

And moreover, we also need ϕ to vanish at ∞ : so we impose open boundary conditions, i.e. $\phi(t,0) = \phi(t,L) = 0$.

Finally, we can conclude that :

$$Tr(T_\lambda(t_1;x,y)) = Tr(T_\lambda(t_2;x,y)) \quad \Rightarrow \frac{dTr((T_\lambda)}{dt} = 0 \tag{45}$$

Thus, we finally found that the infinite set of conserved quantities are $\tau_\lambda = Tr(T_\lambda)$.

Remark : one sees that $T_\lambda(t;x,y)$ and $V_\lambda(t,x)$ verify this equation :

$$\dot{T_\lambda} = [V, T_\lambda]\,. \tag{46}$$

They form what we call a Lax pair. In fact, there exist another more general formalism that exploit integrability through this Lax Pair. Knowing the Lax Pair, it is possible to construct the conserved quantities through it. Having a Lax pair formulation of integrability is very convenient, but inspiration is needed to find it and it is not at all unique. For more details about it, the reader could look at [ ]reference a citer

4 Classical rapidities distribution

The aim of this part is to establish a link between the classical infinite conserved quantities τ_λ obtained through the inverse scattering process and the rapidities distribution $\rho(\lambda)$, whose definition has been given in section 2.2

4.1 Unveiling the link between $\rho(\lambda)$ and the classical conserved quantities τ_λ

4.1.1 Under Markov approximation

$\rho(\lambda)$ has been defined as the momentum distribution for a gaz which has relaxed in a large box of length L, in the very diluted regime. We will see that fulfilling this condition implies that the typical length governing the evolution of T_λ with x is much larger than the typical correlation length of ϕ. This will allow us to estimate the matrix $T_\lambda(L)$ using a markovian approximation. Then, we will be able to unveil the link with the classical conserved charges τ_λ.

As seen before, T_λ obeys:

$$\partial_x T_\lambda(t,x) = U_\lambda(t,x)T_\lambda(t,x) \tag{47}$$

Let us go into an interaction picture, introducing $\tilde{T_\lambda} = T_\lambda^0 T_\lambda$ where T_λ^0 is obtained with (5) when $\phi = 0$:

$T_\lambda^0 = T_\lambda(\phi = 0) = \begin{pmatrix} e^{-i\frac{\lambda x}{2}} & 0 \\ 0 & e^{i\frac{\lambda x}{2}} \end{pmatrix}$

Then (52) become $\partial_x \tilde{T}_\lambda(t,x) = U_\lambda(\tilde{t},x)\tilde{T}_\lambda(t,x)$ with

$$\tilde{U}_\lambda = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}e^{-i\lambda x}\phi \\ \frac{1}{\sqrt{2}}e^{i\lambda x}\phi^* & 0 \end{pmatrix} \tag{48}$$

Now, we drop the tilde to simplify the notations. Then, we go back to (56) and we integrate it between 0 and dx :

$$T_\lambda(t,dx) = T_\lambda(t,0) + \int_0^{dx} dx' U_\lambda(t,x')T_\lambda(t,x') \tag{49}$$

And then by replacing $T_\lambda(t,x')$ in the integral above, we get :

$$T_\lambda(t,dx) = T_\lambda(t,0) + \int_0^{dx} dx' U_\lambda(t,x')T_\lambda(t,0) + \int_0^{dx} dx' U_\lambda(t,x') \int_0^{x'} dx'' U_\lambda(t,x'')T_\lambda(t,x'') \tag{50}$$

This equation can be greatly simplified by the following estimation of length scales: the matrix U given in (43) evolves in x with a typical correlation length l_c, which is the correlation length of ϕ and which is of the order of the De Broglie wavelength corresponding to the typical energy per atoms.
On the other hand, the amplitude of ϕ is very small since we consider that the gas has relaxed into a very large box. Thus the elements of U_λ are very small, which means that the matrix $T_\lambda(x)$ evolves very slowly in x : the typical evolution length of $T_\lambda(x)$, L_c, is very large. If the size L of the box in which we let the gas relaxed is large enough, the two lengths will obey the markovian approximation : $L_c \gg l_c$

This approximation enables us to consider a step dx which fulfill $dx \ll L_c$ and $dx \gg l_c$. The first condition permits to replace $T_\lambda(x'')$ by $T_\lambda(0)$ in the right hand side of Eq.(45), while the second permits to extend the integral over x" to ∞, such that eq.(45) becomes :

$$T_\lambda(t, dx) = [I_d + \int_0^{dx} dx' U_\lambda(x') + \int_0^{dx} dx' U_\lambda(x') \int_0^{\infty} dx'' U_\lambda(x'')] T_\lambda(t, 0) \quad (51)$$

And in order, to have a more condensed formula, we write with the matrix coefficients :

$$T_\lambda(t, dx) = [I_d + \begin{pmatrix} a_\lambda(dx) & b^*_\lambda(dx) \\ b_\lambda(dx) & a^*_\lambda(dx) \end{pmatrix}] T_\lambda(t, x = 0) \quad (52)$$

Now, let's specify these matrix elements using (60) :

$$b_\lambda(dx) = \frac{1}{\sqrt{2}} \int_0^{dx} dx' \ \phi^*(x') e^{i\lambda x'} = \frac{1}{\sqrt{2}} \sum_k \phi_k \int_0^{dx} dx' \ e^{ix'(\lambda - k)} \quad (53)$$

This term contains a sum of random complex number ϕ_k, such that for N sufficiently large, the real parts and the imaginary parts will annihilate. Thus, this coefficient $b_\lambda(x)$ is not relevant ,and we will forget it .

We focus now on $a^*_\lambda(dx)$:

$$\begin{aligned} a^*_\lambda(dx) &= \frac{1}{2} \int_0^{dx} dx' \int_0^{\infty} dx'' \ \phi(x') \phi^*(x'') e^{i\lambda(x'' - x')} \\ &= \frac{1}{2} \int_0^{dx} dx' \int_0^{\infty} d\epsilon \ \phi(x') \phi^*(x' - \epsilon) e^{-i\lambda\epsilon} \\ &= \frac{1}{2} \sum_{k,k'} \phi_k \phi^*_{k'} \int_0^{dx} dx' e^{ix'(k - k')} \int_0^{\infty} d\epsilon \ e^{-i\epsilon(\lambda - k)} \\ &= \frac{dx}{2} \sum_k |\phi_k|^2 \int_0^{\infty} d\epsilon \ e^{-i\epsilon(\lambda - k)} \end{aligned} \quad (54)$$

Now, let's explicit the above integral :

$$\begin{aligned}\int_0^\infty d\epsilon \;\; e^{-i(\lambda-k)\epsilon} &= \lim_{\eta\to 0^+}\int_0^\infty d\epsilon \;\; \mathrm{e}^{-i(\lambda-k)\epsilon-\eta\epsilon} \\ &= \lim_{\eta\to 0^+}\frac{1}{\eta+i(\lambda-k)} \\ &= \lim_{\eta\to 0^+}\frac{\eta-i(\lambda-k)}{\eta^2+(\lambda-k)^2} \\ &= \lim_{\eta\to 0^+}\frac{\eta}{\eta^2+(\lambda-k)^2}-\lim_{\eta\to 0^+}\frac{i(\lambda-k)}{\eta^2+(\lambda-k)^2} \\ &= \pi\delta(\lambda-k)-i\mathrm{V.P}\frac{1}{(\lambda-k)}\end{aligned}$$

Thus

$$a_\lambda^*(dx)=\frac{dx}{2}\sum_k|\phi_k|^2[\pi\delta(\lambda-k)-iV.P(\frac{1}{\lambda-k})] \tag{55}$$

Finally, (64) become :

$$T_\lambda(t,dx)=[I_d+\begin{pmatrix}a_\lambda(dx) & 0\\ 0 & a_\lambda^*(dx)\end{pmatrix}]T_\lambda(t,0) \tag{56}$$

$$T_\lambda(t,dx)-T_\lambda(t,0)=\begin{pmatrix}a_\lambda(dx) & 0\\ 0 & a_\lambda^*(dx)\end{pmatrix}T_\lambda(t,0) \tag{57}$$

And thanks to the Markov approximation, we can translate the last equation. We replace dx by $x+dx$ and 0 by x:

$$T_\lambda(t,x+dx)-T_\lambda(t,x)=\begin{pmatrix}a_\lambda(x) & 0\\ 0 & a_\lambda^*(x)\end{pmatrix}T_\lambda(t,x) \tag{58}$$

$$\partial_x T_\lambda(t,x)\sim\begin{pmatrix}a_\lambda(x) & 0\\ 0 & a_\lambda^*(x)\end{pmatrix}T_\lambda(t,x) \tag{59}$$

Then, by integrating the last equation, we get :

$$T_{\lambda[1,1]}(t,L)\sim e^{\frac{L}{2}\sum_k|\phi_k|^2\pi\delta(\lambda-k)}e^{-i\frac{L}{2}\sum_k|\phi_k|^2V.P(\frac{1}{\lambda-k})} \tag{60}$$

Now, since $\sigma_1 U_\lambda\sigma_1=U_\lambda^*$, the diagonal entries of T_λ are complex conjugate : $Tr(T_\lambda)=2Re(T_{\lambda[1,1]})$

Thus, (55) is equivalent to :

$$Tr(T_\lambda)=e^{\frac{L}{2}\sum_k|\phi_k|^2\pi\delta(\lambda-k)}\cos(\frac{L}{2}\Omega(\lambda)) \tag{61}$$

where $\Omega(\lambda)=\sum_k|\phi_k|^2V.P(\frac{1}{\lambda-k})$

Now, can we establish a link with $\rho(\lambda)$?

As we can see, $T_{\lambda[1,1]}$ contains an oscillating term and a term with the momentum distribution ϕ_k^2.

By definition, $L\rho(\lambda)d\lambda$ counts the number of rapidities laying around λ in a small window of length $d\lambda$. And we already know (see for more details []citer la bonne ref) that rapidities correspond to the momentums of atoms, in the dituted regime, where the gas has expanded and relaxed in box of very long length L.

It means that we can deduce

$$\sum_k |\phi_k|^2 \delta(\lambda - k) \sim \rho(\lambda) \tag{62}$$

Ans as showed in the previous section, the infinite conserved quantities are $\tau_\lambda = Tr(T_\lambda)$.

Thus, we can conclude that

$$\rho(\lambda) = \lim_{L\to\infty} \frac{2\ln|[T_\lambda(0,L)]_{1,1}|}{\pi L} \quad . \tag{63}$$

Or

$$[T_\lambda(0,L)]_{1,1} = e^{\frac{L}{2}\pi\rho(\lambda)} e^{i\frac{L}{2}\Omega(\lambda)} \tag{64}$$

Equations (58) and (59) are our main results. They establish the link between the classical conserved quantities and $\rho(\lambda)$

4.1.2 Momentum distribution in the very diluted regime

In order to test further our result numerically, we need to compute the rapidities distribution using the definition given at the end of 2.1 .

As said, we consider the following situation : a gas of bosons 1D which has expanded and relaxed, at thermoequilibrium , strongly diluted, with neglected interactions : so that we neglect the interaction term in Gross Pitaevskii equation.

Thus, we get the Hamiltonian of a free field :

$$\begin{aligned} H &= \int_0^L dx[\phi^*(x)\partial_x^2\phi(x) - \mu|\phi|^2] \\ H &= \sum_{k,k'} \phi_k^*\phi_{k'}(k'^2 - \mu)\int_0^L dx e^{ix(k'-k)} \\ H &= L\sum_k |\phi_k|^2(k^2 - \mu) \end{aligned} \tag{65}$$

And since $\mu < 0$, let's take $\mu' = -\mu > 0$ and drop the ':

$$H = \sum_k h(k) = \sum_k L|\phi_k|^2(k^2 + \mu) \tag{66}$$

And ϕ_k are random numbers, which are distributed according a gaussian, centered on 0 and of width $\sqrt{\frac{2}{k^2-\mu}}$:

$$p[\phi_k] = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi \;\; e^{\left[|\phi_k|^2(k^2+\mu)\right]} \tag{67}$$

On the other hand, we have $\rho(\lambda) \sim \sum_k |\phi_k|^2 \delta(\lambda - k) \sim \langle |\phi_\lambda|^2 \rangle$

To continue, we need to explicit $< |\phi_\lambda|^2 >$. Using the fact that H is quadratic, to apply the equipartition theorem:

$$\langle |\phi_k|^2 \rangle = \frac{k_b T}{2L(k^2 + \mu)} \tag{68}$$

Therefore, according our definition, we get

$$\rho(\lambda) \sim \frac{k_b T}{2L(\lambda^2 + \mu)} \tag{69}$$

4.1.3 Recovering the semi-circle distribution in the ground state

Let us use our formula to compute $\rho(\lambda)$ when considering a homogeneous field : ϕ is a constant and the matrix U_λ is also constant

$$\partial_x T_\lambda(t, x) = U_\lambda T_\lambda(t, x) \tag{70}$$

Then, we obtain the following sytem for the matrix elements T_{11} and T_{21} :

$$\dot{T_{11}}(x) = \phi e^{-i\lambda x} T_{21}(x) \tag{71a}$$
$$\dot{T_{21}}(x) = \phi^* e^{i\lambda x} T_{11}(x) \tag{71b}$$

And by taking the derivative of the first equation and inserting the second one, we finally get :

$$\ddot{T_{11}} - i\lambda \dot{T_{11}} - |\phi|^2 T_{11} = 0 \tag{72}$$

It leads to

$$T_{11}(L) = e^{\frac{iL\lambda}{2}} e^{\frac{L}{2}\sqrt{4|\phi|^2 - \lambda^2}} \tag{73}$$

(the term $e^{\frac{-iL\lambda}{2}} e^{\frac{-L}{2}\sqrt{4|\phi|^2-\lambda^2}}$ vanishes when considering $L \to \infty$)

And finally, we extract $\rho(\lambda)$:

$$\rho(\lambda) = \lim_{L\to\infty} \frac{2 \ln |[T_\lambda(0, L)]_{1,1}|}{\pi L} = \frac{1}{\pi}\sqrt{4|\phi|^2 - \lambda^2} \tag{74}$$

Thus we recover the predicted semi-circle distribution by Lieb in his article citer [] when considering the ground state of the Lieb-liniger model.

4.2 Markov approximation

Let's explicit further the Markov approximation we used in the previous section. First one needs to find the correlation length l_c of $\phi(x,t)$. To do so, we recall that we consider a perfect gas of bosons at thermodynamic equilibrium at temperature T with no interactions.

We treat it as a classical field : it means the modes are not quantized and there are occupied by a high number of bosons. The mean occupation of a mode k is

$$\langle n(k)\rangle = \langle \phi_k^2\rangle = \frac{k_b T}{2L(k^2+\mu)} \tag{75}$$

The quantity of interest is the correlation function $\langle \phi(x)\phi^*(x-\epsilon)\rangle$.
By applying Residu theorem, we get :

$$\begin{aligned} \langle \phi(x)\phi^*(x-\epsilon)\rangle &= \frac{k_b T}{4\pi}\int_{-\infty}^{+\infty} dk\ \frac{1}{(k^2+\mu)} e^{-ik\epsilon} \\ \langle \phi(x)\phi^*(x-\epsilon)\rangle &= \frac{k_b T}{4}\frac{e^{\epsilon\sqrt{\mu}}}{\sqrt{\mu}} \end{aligned} \tag{76}$$

Thus $l_c = \frac{1}{\sqrt{\mu}}$.

Now let's find the typical length evolution of T_λ. We propose here two ways to do it :

We recall that $\hbar = m = 1$ Let's derive it from T_λ: (Note that we reintroduce g in our expression)

$$\begin{aligned} T_{\lambda_{(1,1)}}(L) &\sim e^{L\pi\rho(\lambda)} \sim e^{L/L_c} \\ with\ L_c \sim \frac{1}{\rho(\lambda)} &\sim \frac{1}{g\langle|\phi_k|^2\rangle} \sim \frac{(k^2+\mu)}{gT} \sim \frac{\mu}{gT} \end{aligned} \tag{77}$$

We can also write it with the mean density n_0,by linking μ to n_0 :

$$\begin{aligned} \langle n_0\rangle = \frac{\langle N\rangle}{L} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\ n(k) \\ \langle n_0\rangle = \frac{k_b T}{4L\pi\mu}\int_{-\infty}^{+\infty} \sqrt{\mu}dx \frac{1}{(x^2+1)} \\ \langle n_0\rangle &= \frac{k_b T}{4L\sqrt{\mu}} \end{aligned} \tag{78}$$

Thus the markov approximation become : $gT \ll \mu^{\frac{3}{2}}$ or $\sqrt{g\langle n_0\rangle}\langle n_0\rangle \ll T$

4.3 Numerical simulation

To compute numerically the evolution of the matrix T_λ,we can simply discretizing the path-ordered integral in (41) and expressing it in terms of a matrix product :

$$T_\lambda(t_0; x, y) \overset{\Delta x \to 0}{=} W_{\Delta x}(t, x)\, W_{\Delta x}(t, x + \Delta x)\, \ldots W_{\Delta x}(t, y - \Delta x) \tag{79}$$

where $W_{\Delta x}(t, x) \equiv e^{\Delta x U_\lambda}$

Then, we can extract $\rho(\lambda)$ with

$$\rho(\lambda) = \lim_{L \to \infty} \frac{2 \log |[T_\lambda(0, L)]_{1,1}|}{\pi L} \ . \tag{80}$$

We consider here a system on a finite volume of size L and then take the limit $L \to \infty$ at finite energy density.The system is initially prepared in a thermal state, which is , for a free field,easily generated in the Fourier space, where the modes are independently gaussianly distributed as seen before.

Note that the root density is self-averaging and therefore for sufficiently large L, a single configuration should be sufficient. However, the numerical cost for a large L is huge. So to avoid that, we've decided to average Eq. (80) over many field configurations so that finite size fluctuations should be suppressed.

Thus, the numerical protocol consists in three steps:

- Sample the initial free field configuration from a thermal ensemble.
- Let each field configuration evolve
- Repeat the above steps for several initial field configurations and average over them.

The goal here is to check our formula for $\rho(\lambda)$ is correct through the Markov approximation w .

Here, a plot that compare our numerical simulation of $\rho(\lambda)$ expected vs $\rho(\lambda)$ obtained :

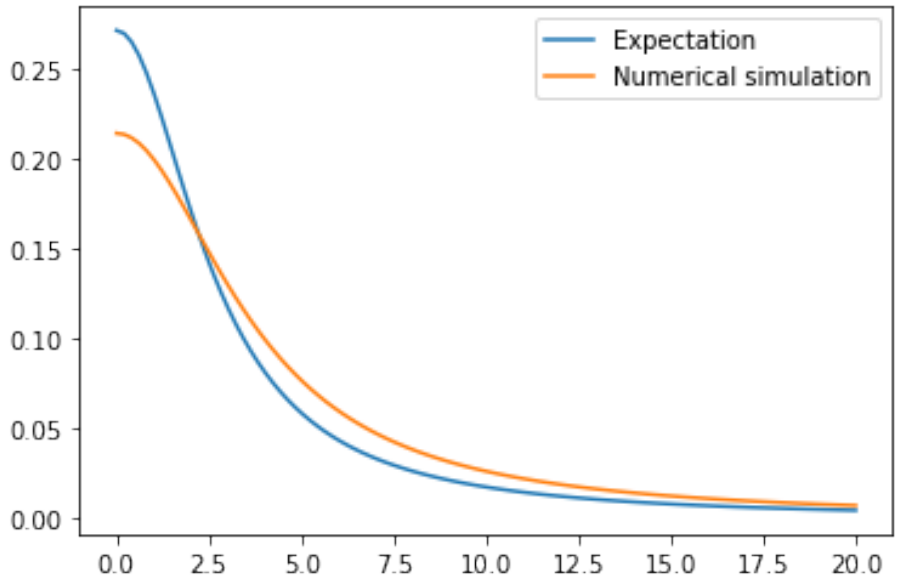

Figure 3 : Comparison between expected value of $\rho(\lambda)$ and the numerical simulated one for L=10

We can clearly see that the numerical $\rho(\lambda)$ obtained has a lorentzian shape, as expected. However it doesn't fit exactly with the expected analytical value we've obtained.

This difference is due to the finite size of the system. To get a better fit, we have to increase the length L of the system but it would increase consequently the numerical cost. So instead,we've just increase the number of initial configurations of the field and then average. And the more we average, better the fit is.

5 Relaxation of phonons

As we will see, in the limit of small k or low energy, interactions change the spectrum of excitations : it will not be possible anymore to limit the excitation to a single particle; the excitations of the system are collective, including all particles and including oscillation of the bosons density. Those collectives excitations are called phonons.

The relaxation of phonons has been studied in the recent article [ ] Citer article. They have investigated a Lieb-Liniger gas initially prepared in an out-

of-equilibrium state that is an excited coherent state for a single phonon mode.

Because phonons are not exact eigenstates of the Hamiltonian(they are only at low energy), the gas relaxes to a stationary state at very long times. But due to integrability, the system will not thermalize : it means this stationary state won't be a thermal state. They have characterized the stationary state of the gas after relaxation and computed its phonon population distribution. The idea here is to test those results in the classical limit, where exact numerical results can be obtained.

5.1 Quasi-condensate

A 1D Bose-gas can never be in a strictly condensed state as demonstrated by the Mermin-Wagner theorem, long-wavelength phase fluctuations prevent long range order to appear in the system. However, when interactions start to play a role, they can suppress density fluctuations while the phase fluctuations are large [citer article]: the one particle correlation function $g_1(x-x') = \langle \phi^\dagger(x)\phi(x')\rangle$ goes to 0 as $|x-x'|$ goes to ∞ (we consider here translation invariance).

This state is called a quasi-condensate and can be achieved at low enough temperature. Such quasi-condensate can then be described by writing the field ϕ in terms of a density ρ and phase θ fields : $\phi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$

The Madelung transformation $(\phi, \phi^\dagger) \to (\rho, \theta)$ is a canonical transformation at the classical level :

$$\{\rho(x), \theta(x')\} = i\delta(x-x') \tag{81}$$

Inserting this in the gross pitaevskii equation and separating the real and imaginary part, we get the so called hydrodynamical equations

$$\partial_t \rho = -\Delta(\theta\rho) \tag{82a}$$

$$\partial_t \theta = -\frac{1}{2}(\nabla\theta)^2 + \frac{1}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - g\rho - \mu \tag{82b}$$

The term $\frac{1}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ is the so-called quantum pressure. If it is neglected, we obtain Euler equation $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{\partial P}{\partial x}$, by introducing the local fluid velocity $v = \nabla\theta$ and the pressure $P = \frac{g\rho^2}{2}$.

We expand the density and phase field around their background values as

$$\rho(x,t) = \rho_0 + \delta\rho(x,t) \tag{83a}$$

$$\theta(x,t) = \theta_0 + \delta\theta(x,t) \tag{83b}$$

The density fluctuations are assumed small $|\frac{\delta\rho}{\rho_0}| \ll 1$ while only the gradient of the phase fluctuation is assumed small $|\frac{\partial_x \delta\theta}{\theta_0}| \ll 1$

It's true only if we verify $\frac{T}{n_0 g\sqrt{n_0}} \ll 1$

We deduce the stationnary and uniform solutions : $\rho_0 = \frac{\mu}{g}$ and $\theta_0 = 0$

Then, by keeping only the first order in $\delta\rho$ and $\partial_x\delta\theta$ in the hydrodynamical equations, we obtain linear (and therefore trivially solvable by taking fourier Transform) coupled equations :

$$\partial_t \delta\theta \simeq \frac{1}{2\rho_0}\Delta\delta\rho - g\delta\rho \tag{84a}$$

$$\partial_t \delta\rho \simeq -\rho_0 \Delta\delta\theta \tag{84b}$$

Now, we propose to establish a quantum equivalent of our transformation, in order to obtain useful results and then we will go back to our classical model. Indeed, these transformation give the idea of a very simple canonical but quantum transformation which, remarkably, maps our equations for a quasi-condensate into the equations for the Bogoliubov modes of a condensate: the field

$$\hat{B} = \frac{\hat{\delta\rho}}{2\sqrt{\rho_0}} + i\sqrt{\rho_0}\hat{\delta\theta} \tag{85}$$

has bosonic commutation relations :

$$[\hat{B}(x), \hat{B}^\dagger(x')] = \delta(x - x') \tag{86}$$

and it obeys the standard Bogoliubov equations

$$i\partial_t \begin{pmatrix} \hat{B} \\ \hat{B}^\dagger \end{pmatrix} = \mathcal{L}_{\mathrm{GP}} \begin{pmatrix} \hat{B} \\ \hat{B}^\dagger \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{2}\Delta - \mu + 2g\rho_0 & g\rho_0 \\ -g\rho_0 & -\left(-\frac{1}{2}\Delta - \mu + 2g\rho_0\right) \end{pmatrix} \begin{pmatrix} B \\ B^\dagger \end{pmatrix}. \tag{87}$$

This allow the decomposition in bogolyubov modes :

$$\hat{\delta\rho} = \sqrt{\frac{\rho_0}{L}} \sum_k f_k^- (\hat{a_k} e^{ikx} + \hat{a_k}^\dagger e^{-ikx}) \tag{88a}$$

$$\hat{\delta\theta} = \frac{1}{2i\sqrt{\rho L}} \sum_k f_k^+ (\hat{a_k} e^{ikx} - \hat{a_k}^\dagger e^{-ikx}) \tag{88b}$$

where we have introduced the functions $f_k^{+/-}$ such that :

$$f_k^+ = \left(\frac{E_k}{E_k + 2\mu}\right)^{\frac{1}{4}} \tag{89a}$$

$$f_k^- = \frac{1}{f k^+} \tag{89b}$$

where

$$E_k = \frac{k^2}{2}$$

With this procedure, the hamiltonian is diagonalized : $\hat{H} = \sum_k \omega_k \hat{a_k}^\dagger \hat{a_k}$. And the dispersion of the Bogoliubov quasi-particles is found to be

$$\omega_k = \sqrt{\frac{k^2}{2}(\frac{k^2}{2} + 2\mu)} \tag{90}$$

It is linear for $k \ll \frac{1}{\Gamma}$,on this regime excitations are phonons. And it is quadratic for $k \gg \frac{1}{\Gamma}$, here excitations are free particles. Note that we have introduced the healing length $\Gamma = \frac{1}{\sqrt{\mu}}$ which separates these two regimes.

Now, we can easily go back to our classical model : in fact if $k_B T \gg E_k$, the population of those modes is large so that a classical field approximation where the annihilation and creation operators in (105) can be replaced by the conjugate complex numbers a_k and a_k^*. Thus $\delta\rho$ and $\delta\theta$ are real functions : $\delta\rho_{-k} = \delta\rho_k^*$ and $\delta\theta_{-k} = \delta\theta_k^*$

We also have the inverse relation:

$$\delta\rho_k = \sqrt{\frac{\rho_0}{L}} f_k^- (a_k + a_{-k}^*) \tag{91a}$$

$$\delta\theta_k = -\frac{i}{2\sqrt{L\rho_0}} f_k^+ (a_k - a_{-k}^*) \tag{91b}$$

And

$$a_k = \sqrt{\frac{L}{2}} \ \ (2i\sqrt{\rho_0} f_k^+ \delta\theta_k + \frac{1}{\sqrt{\rho_0}} f_k^- \delta\rho_k) \tag{92}$$

The linearized hydrodynamical equations (101) could also be obtained with the hamiltonian :

$$H = \int_0^L dx \ \ [\frac{\rho_0}{2}(\nabla\delta\theta)^2 + \frac{1}{8\rho_0}(\nabla\delta\rho)^2 + \frac{g}{2}\delta\rho^2] \tag{93}$$

And his Fourier transform give the bogolyubov hamiltonian.

5.2 Gross pitaevskii simulation: splitting method

The idea here is to observe the relaxation of phonons, giving a certain initial state.

The initial state is a thermal field, easily generated in the fourier space,in which we add a phonon in a certain mode. To do so, we simply deform the

thermal field by $\phi(x, t = 0) \rightarrow \phi(x, t = 0)(1 + \epsilon \cos k_{ph} x)$

Then we let this field will evolving through Gross-Pitaevskii equation and we'll study how the energy in the modes evolve, after relaxation,that is to say after long time.

The exact updating with Gross-Pitaevskii is obtained through :

$\phi(x, t + dt) = U_{GP}(dt)\phi(x, t)$ And $U_{GP} = \mathcal{T} e^{\int_t^{t+dt} dt' H_{GP}(x,t')}$. Where $\mathcal{T}$ is a time ordering operator.

The splitting method consists in propagating the Gross-Pitaevskii equation in the following manner :

$$U_{GP}(dt) \simeq e^{-i\frac{dt}{2}V(t)} e^{-idtK} e^{-i\frac{dt}{2}V(t)} \tag{94}$$

where K is the kinetic operator $K = \frac{\partial^2}{\partial x^2}$ and $V(t)$ the non linear operator $:V(t) = g|\phi|^2 - \mu$. In the momentum space, the kinetic operator is simply : $K = \frac{-k^2}{2}$
So that, working in position or in momentum representation (we switch from one to the other by Fourier Transform), the three sub steps of the propagation reduce just to multiplications by scalars :

$$\phi(x, t + dt) = e^{-i\frac{dt}{2}V(t)} TF^{-1}[e^{-idtK} TF[e^{-i\frac{dt}{2}V(t)}\phi(x, t)]] \tag{95}$$

The method is very efficient because the error committed at each step is of third order in the step duration (we can see it using Backer-haussdorf formula).

6 Conclusion

In this work, we have seen that in quantum integrable models, an infinite set of conserved charges can be generated knowing the distribution of rapidities $\rho(\lambda)$. Then we focused on the classical counterpart : using the inverse scattering process, we showed that Gross Pitaevskii equation is equivalent to a compatibility condition for a linear scattering problem depending parametrically on a spectral parameter λ, called the zero curvature condition.

Fulfilling this condition, and under periodic and open boundaries conditions, we found that the infinite conserved charges are the trace of the matrix T_λ, which is the propagator related to U_λ.
Then, the core of our work was to establish a link between $\rho(\lambda)$ and those classical conserved charges. To do so, we used the definition of $\rho(\lambda)$ as being the distribution of momentums in the very diluted regime, where the gas has expanded and relaxed in a very large box of length L. Then we showed that fulfilling this condition implied that the typical length governing the evolution of T_λ with x is much larger than the typical correlation length of ϕ. This allowed us to estimate the matrix $T_\lambda(L)$ using a markovian approximation and to finally find the link between the classical conserved charges and $\rho(\lambda)$.

The second part of our work was to study the relaxation of phonons. In light of [ ] Citer article, the goal was to characterize the stationary state of the gas after relaxation and compute its phonon population distribution. However, at this time, I got some trouble with the numerical simulations, the results I get are not coherent with what we should obtain. I hope to find the reason, and then the goal is to compare with the quantum results. Another goal could be also to compute the distribution of rapidites after relaxation and to see if we get some remarkable behavior.

7 References

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