Journal of Statistical
Mechanics: Theory and
Experiment

PAPER

Exact solutions to macroscopic fluctuation theory through classical integrable systems

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

PAPER: STATPHYS 28

Exact solutions to macroscopic fluctuation theory through classical integrable systems

Kirone Mallick[1], **Hiroki Moriya**[2] **and Tomohiro Sasamoto**[3,*]

[1] Institut de Physique Théorique, CEA, CNRS, Université Paris–Saclay, F–91191 Gif-sur-Yvette cedex, France
[2] Laboratoire de Physique Théorique de la Matière Condensée, Sorbonne Université, 4 Place Jussieu, 75005 Paris, France
[3] Department of Physics, Tokyo Institute of Technology, Ookayama 2-12-1, Meguro-ku, Tokyo 152-8551, Japan
E-mail: sasamoto@phys.titech.ac.jp, kirone.mallick@ipht.fr and hiroki.moriya@sorbonne-universite.fr

Received 5 March 2024
Accepted for publication 7 May 2024
Published 3 July 2024

Online at stacks.iop.org/JSTAT/2024/074001
https://doi.org/10.1088/1742-5468/ad485e

Abstract. We give a short overview of recent developments in exact solutions for macroscopic fluctuation theory by using connections to classical integrable systems. A calculation of the cumulant generating function for a tagged particle is also given, agreeing with a previous result obtained from a microscopic analysis.

Keywords: current fluctuations, exclusion processes, large deviations in non-equilibrium systems, stochastic particle dynamics

Contents

[*] Author to whom any correspondence should be addressed.

1. Introduction

Fluctuations of non-equilibrium many-body systems have been a central subject in statistical physics for a long time. Whereas equilibrium systems are described by the Gibbs measure allowing us to clarify many intriguing and universal thermodynamic properties, no standard prescription for the stationary measure of non-equilibrium processes is available. Indeed, the study of stationary and time-dependent fluctuations out of equilibrium remains a very challenging problem.

There are several types of fluctuations on different time scales. This may be exemplified by a simple random walker on a one dimensional lattice, which starts at the origin and hops symmetrically to the left or to the right neighboring sites with probability 1/2. At time step n, the average and variance of the random walker are given by 0 and n

respectively. The latter implies that the typical fluctuation of the position of the walker is of order $O(n^{1/2})$. Then, according to the central limit theorem, this fluctuation on the scale $O(n^{1/2})$ is given by the Gaussian distribution.

We can also study the probability that the position of the random walker is on a scale of $O(n)$. Since typical fluctuations are of order $O(n^{1/2})$, such a probability should vanish when n tends to infinity. Nevertheless, one can investigate how this probability decays to 0 when n is large; it takes the form, $\mathbb{P}[X_n = ny] \simeq \mathrm{e}^{-\Phi(y)n}$ where X_n is the position of the walker at time n. This type of fluctuation is called a large deviation and the function $\Phi(y)$ is known as the rate function we wish to determine, in general. For the random walker the rate function is simply given by the binary entropy.

Large deviation theory is a huge subfield of probability [Var84, DZ09, Tou09] and its close connection to equilibrium statistical mechanics is known [Ell06]. More recently, large deviation theory has become one of the essential tools to study fluctuations of non-equilibrium systems. For instance, fluctuation theorems, which describe a symmetry property of generic non-equilibrium systems can be formulated in the language of large deviation [GC95, LS99].

Studying large deviation properties of a many particles system is a difficult problem. Around 2000, Bertini, De Sole, Gabrielli, Jona–Lasinio and Landim developed a framework, called *the macroscopic fluctuation theory* (MFT) [BDSG+02], to analyze large deviations. Their original scheme was formulated for diffusive systems and has been utilized to understand various properties of such processes [BSG+15]. It should be remarked that large deviations for certain stochastic models had been considered earlier: for example, the large deviation principle (LDP) for the symmetric simple exclusion process (SEP) was established in [KOV89]. Besides, in recent years, various extensions have been proposed, such as the ballistic MFT [DPSY23] and a quantum version of MFT [Ber21].

The MFT is based on a variational principle and the associated Euler–Lagrange equations, called hereafter *the MFT equations*, play a key role. These equations govern the large deviation properties of the system but they are often difficult to solve exactly because, in most cases, they are coupled nonlinear partial differential equations (PDEs). Until recently, solutions to the MFT equations had been available only for some stationary cases and for the case of independent particles.

In the last few years, there has been a significant progress for studying time dependent large deviation for interacting systems. This has been made possible thanks to the discovery of subtle connections between the MFT equations and some classical integrable systems. The purpose of this article is to give a brief account of these recent developments, preceded by some explanations about the basics of MFT. Our discussion of classical integrable systems will be mostly based on [MMS22]. We shall also extend our previous work by calculating, within the MFT framework, the large deviation of a tagged particle in the symmetric SEP; this purely macroscopic derivation will be shown to be identical to the microscopic formula that was obtained earlier by Bethe Ansatz [IMS21].

The rest of the paper is organized as follows. In section 2 we introduce a few models such as the SEP, reflective Brownian motions (RBM) and the Kipnis–Marchioro–Presutti (KMP) model. In section 3, we review the basics of MFT. In section 3.5, we

https://doi.org/10.1088/1742-5468/ad485e

discuss results for RBM. In section 4, we describe a few cases where the MFT equations are directly related to classical integrable systems, namely the KMP model and the Kardar–Parisi–Zhang (KPZ) equation with weak noise. However, the SEP requires a special discussion because the corresponding MFT equations are not manifestly integrable. In section 5, we present a general mapping from the MFT equations to the integrable AKNS system, which works for all interacting particles models with constant diffusivity and a quadratic mobility, including SEP ; basic ideas of the inverse scattering method (ISM) are then explained. Some details of the calculations of the cumulant generating function for a tagged particle are given in section 6. The article ends with concluding remarks in section 7.

2. Models and quantities

2.1. SEP

In the symmetric SEP, particles on a one-dimensional lattice attempt to hop from their current position to a neighboring site with probability $\mathrm{d}t$ during an infinitesimal short duration $\mathrm{d}t$. Because of hard-core volume exclusion, if the target site is already occupied, the transition is forbidden (see figure 1). A configuration of particles of SEP at time t is specified by the binary variables $\eta_i(t)$ for all $i \in \mathbb{Z}$, where $\eta_i(t) = 1$ if the site i is occupied and $\eta_i(t) = 0$ otherwise. The SEP has become one of the most standard models in non-equilibrium statistical physics and has been studied extensively [Lig73, Spo91]. The unique translation invariant stationary measure of SEP is given by the Bernoulli measure in which all sites are independent and each site is occupied by a particle with probability ρ where $0 < \rho < 1$. This parameter ρ corresponds to the density of particles in SEP, the only conserved quantity in the model. The hydrodynamic behavior of the density profile is given by the diffusion equation [KL99], as follows from the fact that the average density $\langle \eta_i(t) \rangle$ at site i satisfies the lattice diffusion equation, which can seen to be a consequence of the $su(2)$ symmetry of SEP [GKRV09]. The LDP for the density profile trajectory has been established in [KOV89].

In this article, we shall focus on the following quantities: the integrated current Q_t at the bond (0,1) for time $[0,t]$, which is defined as the total number of particles which have hopped from site 0 to site 1 minus the number of particles which have jumped from site 1 to site 0 during time interval $[0,t]$; we shall also consider a related quantity, which is the tagged particle position: we assume that at initial time there is a particle at the origin, we put a tag on it and we denote its position at time t by X_t. The LDP for these quantities was established in 2013 in [SS13]. Besides, for each site x, we consider the 'height' function $N(x,t)$, defined as

$$N(x,t) = Q_t + \begin{cases} -\sum_{y=1}^{x} \eta_y(t), & (x > 0), \\ 0, & (x = 0), \\ +\sum_{y=x+1}^{0} \eta_y(t), & (x < 0). \end{cases} \tag{2.1}$$

Thanks to the simple identity $\mathbb{P}[X_t > x] = \mathbb{P}[N(x,t) > 0]$, see the discussions in [IMS21], the study of X_t is equivalent to that of $N(x,t)$ (for all x).

J. Stat. Mech. (2024) 074001

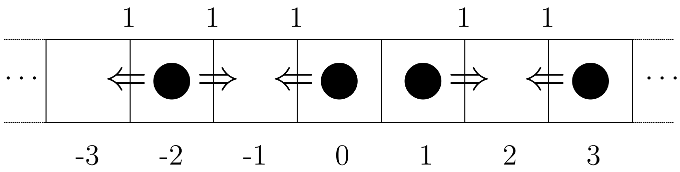

Figure 1. SEP.

For SEP, the rate function for the integrated current and the tagged particle position were calculated exactly by solving the model using Bethe Ansatz, combinatorial calculations and asymptotic analysis [DG09b, IMS21]. One of the aims of the present work is to extract these results from MFT, at the macroscopic level.

2.2. Reflective BM

One may consider a Brownian version of SEP, in which many Brownian particles reflect against each other symmetrically when they collide. This model, which can be obtained as a low density limit of SEP, will be called the RBM. The uniform stationary measure is given by the Poisson point process in which distances of neighboring two particles are independent Poisson distributions and the hydrodynamics is still governed by the diffusion equation. The integrated current Q_t, the position of the tagged particle X_t and the height function $N(x,t)$ can be defined and studied in this model. Thanks to the symmetric nature of reflections, trajectories of the reflecting Brownian particles are statistically the same as those of independent Brownian particles. Accordingly, fluctuations of certain quantities in the RBM, such as the current, can be related to those for independent Brownian particles. Although the RBM model may look elementary, the calculations of large deviations have some non-trivial and generic aspects that give very useful insights for studying interacting models [DG09a, KMS14, SD15].

2.3. KMP and related models

The KMP model [KMP82] is also defined on a one-dimensional lattice, but now the state at each site is a non-negative real variable E_i, which represents a quantity of energy (or mass) present at the site i. The stochastic time evolution rule is the following: two neighboring sites $(i, i+1)$ are randomly selected and the sum $E_i + E_{i+1}$ of the energies is uniformly redistributed between i and $i+1$. The uniform stationary measure is given by independent exponential distribution $\mathrm{e}^{-\beta E_i}$, which conforms to the usual canonical distribution. The hydrodynamics is again given by the diffusion equation [KMP82], as follows from the fact that the average density $\langle E_i(t)\rangle$ obeys the lattice diffusion equation (this can also be understood as a consequence of stochastic duality due to the existence of $su(1,1)$ symmetry for the model [GKRV09]). In the last decade or so various models which have the same $su(1,1)$ symmetry and related to the KMP model have been introduced and studied [FGK19]. For the KMP and related models, one can study the integrated current Q_t and the height function $N(x,t)$.

SEP and KMP are cousin models: both have a single conserved quantity, the hydrodynamics is the diffusion equation and some fluctuation properties can be studied in a similar manner. But there are also some important differences. One technical point is that for SEP the large deviation principle has been established a long time ago whereas it has not yet been proved for KMP, although some exact but non-rigorous results about large deviations have been derived in [BGL05]. Another difference is that for SEP many exact solutions using methods of quantum integrable systems such as Bethe Ansatz are available. In contrast, the KMP model has not been 'exactly solved' at the microscopic level. Nevertheless, some exact results for KMP, such as the hydrodynamics and stationary two point function, have been found by using the stochastic duality, that follows from the $su(1,1)$ symmetry (but this does not imply integrability), see for instance [CGRS16].

2.4. Typical fluctuations

In the Introduction, we recalled that the typical fluctuations of a random walker are of order $O(t^{1/2})$ with a Gaussian distribution. We shall now investigate the typical fluctuations of the current and of the tagged particle position for the interacting models discussed above.

For SEP, the typical fluctuation of a tagged particle was studied a long time ago in [Arr83]. For the case of uniform density ρ, the average and the variance are given by

$$\langle X_t \rangle = 0, \tag{2.2}$$

$$\langle X_T^2 \rangle \simeq \frac{2(1-\rho)}{\rho}\sqrt{\frac{T}{\pi}}. \tag{2.3}$$

Here, and in the following, T means a final time which is considered to be large. The behavior of the variance indicates that the typical fluctuation of the tagged particle position of SEP is on the scale of $O(T^{1/4})$, which is much smaller than the case of a single random walker. It is further known that the distribution of the tagged particle position on this scale is Gaussian [DMF02]. The results (2.2) and (2.3) about typical fluctuations for the tagged particle position can be related to those of the integrated current as

$$\langle Q_t \rangle = 0, \tag{2.4}$$

$$\langle Q_T^2 \rangle \simeq 2\rho(1-\rho)\sqrt{\frac{T}{\pi}}. \tag{2.5}$$

Typical fluctuations for other models can also be studied. For the few models in the previous subsections, the average and the variance for the stationary measure with uniform density ρ (particle density for SEP and RBM, or energy density for KMP) are summarized as [KM12]

$$\langle Q_t \rangle = 0, \tag{2.6}$$

$$\langle Q_T^2 \rangle \simeq \sigma(\rho)\sqrt{\frac{T}{\pi}}. \tag{2.7}$$

J. Stat. Mech. (2024) 074001

Here $\sigma(\rho)$ is called the mobility and is given by

$$\sigma(\rho) = \begin{cases} 2\rho(1-\rho), & \text{SEP}, \\ 2\rho, & \text{RBM}, \\ \rho^2, & \text{KMP}. \end{cases} \tag{2.8}$$

As explained in the next section, the MFT may be formulated for systems with given diffusivity $D(\rho)$ and mobility $\sigma(\rho)$. The diffusivity for all the three models above is a constant, equal to 1. The mobility which is quadratic in ρ is often easier to handle. This case, including all the above three models, may be treated in a unified way by introducing general quadratic $\sigma(\rho) = 2A\rho(B-\rho)$ [DG09a].

3. Macroscopic fluctuation theory

3.1. Coarse-grained description by Langevin equation

We now describe a coarse-grained description of a microscopic model, with definite values of D and σ, such as SEP, RBM and KMP. Since we are considering systems with only one conserved quantity ρ, we can write a continuity equation $\partial_t\rho + \partial_x j = 0$. If the current j obeys the Fourier's law or Fick's law, we have $j = -D\partial_x\rho$ with a constant D and the continuity equation becomes the diffusion equation. This corresponds to the hydrodynamic description of the models. To take into account the effects of fluctuation, we would add a noise to the current so that we put $j = -D\partial_x\rho + C\xi(x,t)$ where $\xi(x,t)$ is a Gaussian white noise with $\langle\xi(x,t)\rangle = 0, \langle\xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t')$. The strength of the noise C should be taken to be consistent with the size of the fluctuation of the stationary measure and is determined to be $C = \sqrt{\sigma(\rho)}$ [Spo91]. We may also consider the case where the diffusion constant D depends on the density ρ.

To summarize, we may suppose that a coarse grained-description with fluctuations of a microscopic model is given by the Langevin equation,

$$\partial_t\rho = \partial_x\left[D(\rho)\partial_x\rho + \sqrt{\sigma(\rho)}\xi(x,t)\right]. \tag{3.1}$$

Note that the height function at position X and at time T is given by

$$N(X,T) = \int_X^\infty \mathrm{d}x\rho(x,T) - \int_0^\infty \mathrm{d}x\rho(x,0). \tag{3.2}$$

3.2. MFT action

Starting from the Langevin equation (3.1) and using a standard way to represent noise in terms of a functional integral [MSR73], the height generating function $\langle \mathrm{e}^{\lambda N(X,T)}\rangle$ can be written as

$$\langle \mathrm{e}^{\lambda N(X,T)}\rangle = \int \mathcal{D}[\rho,H]\,\mathrm{e}^{S[\rho,H]}, \tag{3.3}$$

$$\text{with } S[\rho,H] = \lambda N(X,T) - \mathcal{F}_0[\rho(x,0)] - \int_0^T \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}x\,(H\partial_t\rho + \mathcal{H}), \tag{3.4}$$

J. Stat. Mech. (2024) 074001

where the function S is often called the MFT action. Here $N(X,T)$ is given in terms of $\rho(x,t)$ using (3.2) and the Hamiltonian $\mathcal{H}$ has the following expression

$$\mathcal{H}[\rho,H] = D(\rho)(\partial_x\rho)(\partial_x H) - \frac{1}{2}\sigma(\rho)(\partial_x H)^2 . \tag{3.5}$$

Finally, the free energy of the initial density profile, chosen as a local equilibrium state with density $\bar{\rho}(x)$, is given by

$$\mathcal{F}_0[\rho(x,0)] = \int_{-\infty}^{\infty} \mathrm{d}x \int_{\bar{\rho}(x)}^{\rho(x,0)} \mathrm{d}r \frac{2D(r)(\rho(x,0)-r)}{\sigma(r)}. \tag{3.6}$$

The derivation of these formulas is not very difficult. To obtain (3.3) one starts from the identity

$$\langle \mathrm{e}^{\lambda N(X,T)} \rangle = \left\langle \int \mathcal{D}[\rho]\, \mathrm{e}^{\lambda N(X,T)} \delta\left(\partial_t \rho - \partial_x \left[D(\rho)\partial_x\rho + \sqrt{\sigma(\rho)}\xi(x,t)\right]\right)\right\rangle. \tag{3.7}$$

Here δ is the Dirac delta function and the above expression simply means that we are summing over all possible density profile trajectories with the condition that the Langevin equation (3.1) should be satisfied. Next we write the delta function as an integral and then, recalling that the noise ξ is Gaussian with $\langle\xi(x,t)\rangle = 0, \langle\xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t')$, we can take the average with respect to the noise, and obtain (3.3). The term (3.6) comes from the initial condition, which is taken to be the local stationary measure with density $\bar{\rho}(x)$. The rate function for the density $\bar{\rho}$ is written in the form $f(\rho) - f(\bar{\rho}) - (\rho-\bar{\rho})f'(\bar{\rho})$ where f is the free energy density for the system with density ρ; then using the fluctuation-dissipation theorem $f''(\rho) = 2D(\rho)/\sigma(\rho)$, one finds (3.6) [Der07].

3.3. MFT equations

In principle the functional integral (3.3) contains the whole information of the system described by the Langevin equation (3.1). Large T behavior is dominated by the maximum of the action S and is expected to be the same for the original microscopic models.

Euler–Lagrange equations for the above MFT action are

$$\partial_t\rho = \partial_x\left[D(\rho)\partial_x\rho - \sigma(\rho)\partial_x H\right], \tag{3.8}$$

$$\partial_t H = -D(\rho)\partial_x^2 H - \sigma'(\rho)(\partial_x H)^2/2, \tag{3.9}$$

accompanied by the conditions at the initial and the final times:

$$H(x,T) = \lambda\theta(x-X), \tag{3.10}$$

$$H(x,0) = \lambda\theta(x) + f'(\rho(x,0)) - f'(\bar{\rho}(x)). \tag{3.11}$$

Here $f'(\rho) = \log\frac{\rho}{1-\rho}$ for SEP, $f'(\rho) = \log\rho$ for RBM and $f'(\rho) = -1/\rho$ for KMP model.

These coupled nonlinear partial PDEs are in general difficult to solve. Most of the works until recently have been restricted to studies for stationary situations and for non-interacting models like RBM. Besides, perturbative expansions have also been performed [KM12].

J. Stat. Mech. (2024) 074001

3.4. Stationary case

The first successes of the MFT were about the stationary case for systems with open boundaries. For SEP with open boundaries, the exact stationary measure can be constructed by the matrix product method [DEHP93]. Based on this representation the large deviation of the density profile was calculated in [DLS02]. At the same time, the same large deviation was calculated at the macroscopic level based on the MFT formulation [BDSG+03] (see also [TKL08]). These results turned out to be the same: the microscopic and the macroscopic results matched and this was an important touchstone in the development of the MFT framework. For other related works, see for instance [BD04, DDR04].

3.5. Reflective Brownian motion

The large deviation for time dependent case is more challenging. For RBM, however, even the time dependent large deviations are accessible in a rather simple manner [KMS15]. Let us put the values of D and σ for RBM, i.e. $D = 1, \sigma(\rho) = 2\rho$ in (3.8) and (3.9). The MFT equations are still coupled non-linear equations, but applying the canonical Cole–Hopf transformation,

$$Q = \rho \mathrm{e}^{-H}, \quad P = \mathrm{e}^{H}, \tag{3.12}$$

the equations are decoupled into the diffusion and anti-diffusion equations,

$$\partial_t Q = \partial_{xx} Q, \; \partial_t P = -\partial_{xx} P. \tag{3.13}$$

These linear equations are readily solved by Fourier transformations, leading to explicit formulas for large deviation properties of the RBM [KMS15].

4. MFT equations and classical integrable systems

4.1. KMP

For the KMP model, the mobility is $\sigma(\rho) = \rho^2$ and the MFT equations read

$$\partial_t \rho = \partial_x^2 \rho - \partial_x \left(\rho^2 \partial_x H\right), \tag{4.1}$$

$$\partial_t H = -\partial_x^2 H - \rho \left(\partial_x H\right)^2. \tag{4.2}$$

Equations for $\rho, \partial_x H$ are nothing but the derivative nonlinear Schrödinger equation [KN78], which is a classical integrable system [AS81]. Exploiting this fact, the MFT equations for the KMP model, with the initial condition $\rho(x) = \delta(x)$, was solved in [BSM22b]. This may be considered as the first exact solution to the MFT equations in the time dependent regime for an interacting model. But, because the KMP model has not been exactly solved at the microscopic level, the validity of the obtained result can not be checked easily.

4.2. Weak noise KPZ

In this article, we mainly focus on the MFT for diffusive systems. But similar equations also appear in related systems, for example when one considers the large deviation properties of the KPZ equation in the small noise limit. It had been noticed in [JKM16] that the optimal path equations for the weak noise KPZ are in fact integrable and a full solution for this case, using the ISM, was obtained in [KLD21].

5. Solution to the MFT equation for SEP

5.1. Mapping to a classical integrable system

For the case of SEP, the MFT equation itself is not a classical integrable system. In [MMS22], we found a transformation of the MFT equations for SEP to a classical integrable system. The mapping reads

$$u(x,t) = \frac{1}{\sigma'(\rho)}\frac{\partial}{\partial x}\sigma(\rho)\exp\left[-\int_{-\infty}^{x}\mathrm{d}y\sigma'(\rho)\partial_y H/2\right], \tag{5.1}$$

$$v(x,t) = -\frac{2}{\sigma'(\rho)}\frac{\partial}{\partial x}\exp\left[\int_{-\infty}^{x}\mathrm{d}y\sigma'(\rho)\partial_y H/2\right]. \tag{5.2}$$

In fact the transformation works for any quadratic σ, and the MFT equations become

$$\partial_t u(x,t) = \partial_{xx}u(x,t) - 2u(x,t)^2 v(x,t), \tag{5.3}$$

$$\partial_t v(x,t) = -\partial_{xx}v(x,t) + 2u(x,t)v(x,t)^2. \tag{5.4}$$

They are still coupled non-linear PDEs but now they are exactly in the form of a well-known classical integrable system, known as the AKNS (Ablowitz–Kaup–Newell–Segur) system [AS81]. This opens up the possibility to study MFT equations of models with quadratic σ using the standard machinery of classical integrable systems. Note that in the low density limit, in which σ becomes simply 2ρ, the y integrals in the exponent of the transformation can be performed and they become equivalent to the canonical Cole–Hopf transformation.

The above coupled PDEs should be solved under certain initial and final time conditions. In this article, we focus on the two sided stationary initial condition where $\bar{\rho}(x) = \rho_-\theta(-x) + \rho_+\theta(x)$. After the transformation, the conditions (3.10) and (3.11) for the MFT equations become

$$u(x,0) = g_u\delta(x),\ v(x,T) = g_v\delta(x-X) \tag{5.5}$$

with some coefficients g_u and g_v. The function v becomes a δ function at final time is related to the fact that we are interested in the integrated current. On the other hand, the condition that u becomes proportional to a δ function at $t=0$ is nontrivial and is very important in order to solve the problem exactly. However, the amplitudes g_u, g_v can not be determined from the transformation alone and must be fixed later.

It had been known that the MFT equations for SEP may be mapped to the Landau–Lifshitz equations (LLE) for classical spin chains [TKL08], which are classically integrable. However, the initial and final time conditions for the LLE do not seem easy to handle.

The transformations (5.1) and (5.2) were discovered while trying to find a mapping from MFT equations of SEP to a tractable classical integrable system ; they may also be obtained by combining some known mappings between soliton equations [KLD22].

5.2. Inverse scattering applied to MFT

Once the problem is mapped to a classical integrable system, one can try to use all available theoretical tools to derive exact solutions. Here, the ISM [AS81] turns out to be particularly useful. The basic strategy of the ISM is to reformulate the non-linear PDE as two auxiliary linear problems, one in space and the other in time. The first linear equation is regarded as a scattering problem, for which one can determine the scattering amplitudes. The second linear equation ensures that the time evolution of these scattering amplitudes is simple and fixes their values at any arbitrary time. Finally, inverse scattering allows us to reconstruct the original variables at the final time.

However, in our problem, there is an important difference from the usual setup of the ISM. Normally, given the initial condition, we try to determine the variables at an arbitrary time. But, here, in the MFT case, we do not have full information about the initial conditions. Rather, we have one condition at the initial time and another at the final time. But, still, we can apply the basic strategy of ISM to our problem. We may first solve the scattering problems and determine the scattering amplitudes both at initial and final times in terms of unknown functions. Since they have to be related by a simple time evolution, the unknown functions should satisfy some consistency conditions. In our case, these conditions can be written as a scalar Riemann-Hilbert problem, which may be solved rather easily, leading to explicit expressions of the unknown functions, which, in turn, yield the large deviation properties we are looking for.

In the next section, we apply these ideas to the calculation of the cumulant generating function of the height $N(X,t)$ for an arbitrary value of the position X. This is an extension of [MMS22], in which only the $X=0$ case was studied.

6. Cumulant generating function for general *X*

6.1. Linear problems

By following the standard procedures of ISM [AS81], we first consider the auxiliary linear problems,

$$\partial_x \Psi(x,t;k) = U(x,t;k)\,\Psi(x,t;k), \tag{6.1}$$

$$\partial_t \Psi(x,t;k) = V(x,t;k)\,\Psi(x,t;k), \tag{6.2}$$

J. Stat. Mech. (2024) 074001

where

$$U(x,t;k) = -ik\sigma_3 + v\sigma_+ + u\sigma_-, \tag{6.3}$$
$$V(x,t;k) = 2k^2\sigma_3 + 2ik[v\sigma_+ + u\sigma_-] + [uv\sigma_3 - \partial_x v\sigma_+ + \partial_x u\sigma_-], \tag{6.4}$$

with $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right), \sigma_+ = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right), \sigma_- = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$. The zero-curvature condition, $U_t - V_x + [U,V] = 0$, which is the compatibility condition of the above two linear problems, gives the AKNS equation (5.4). This means that one may solve the original AKNS equation (5.4) by studying the two linear problems (6.1) and (6.2).

Because $U(x,t)$ is vanishing as $x \to \pm\infty$, equation (6.1) may be thought of as a scattering problem. We may consider two solutions with the boundary conditions,

$$\phi(x;k) \sim \begin{pmatrix}1\\0\end{pmatrix} \mathrm{e}^{-\mathrm{i}kx}, \quad \bar{\phi}(x;k) \sim \begin{pmatrix}0\\-1\end{pmatrix} \mathrm{e}^{\mathrm{i}kx}, \quad \text{as} \quad x \to -\infty, \tag{6.5}$$

$$\psi(x;k) \sim \begin{pmatrix}0\\1\end{pmatrix} \mathrm{e}^{\mathrm{i}kx}, \quad \bar{\psi}(x;k) \sim \begin{pmatrix}1\\0\end{pmatrix} \mathrm{e}^{-\mathrm{i}kx}, \quad \text{as} \quad x \to +\infty, \tag{6.6}$$

and define the scattering amplitudes $a(k), \bar{a}(k), b(k), \bar{b}(k)$ as coefficients of linear relations between them,

$$\phi(x;k) = a(k)\bar{\psi}(x;k) + b(k)\psi(x;k), \tag{6.7}$$
$$\bar{\phi}(x;k) = \bar{b}(k)\bar{\psi}(x;k) - \bar{a}(k)\psi(x;k). \tag{6.8}$$

They satisfy the relation

$$a(k)\bar{a}(k) + b(k)\bar{b}(k) = 1. \tag{6.9}$$

6.2. Scattering amplitudes in terms of unknown functions

In our case, because the initial condition for u is a δ function, one can solve the scattering problem easily, see [MMS22], as

$$a(k,0) = 1 + g_u\hat{v}_+(k), \quad b(k,0) = g_u, \tag{6.10}$$
$$\bar{a}(k,0) = 1 + g_u\hat{v}_-(k), \quad \bar{b}(k,0) = -[\hat{v}(k) + g_u\hat{v}_+(k)\hat{v}_-(k)], \tag{6.11}$$

in terms of the unknown functions,

$$\hat{v}_\pm(k) = \int_{\mathbb{R}_\pm} v(x,0)\,\mathrm{e}^{2\mathrm{i}kx}\mathrm{d}x \tag{6.12}$$

and $\hat{v}(k) := \hat{v}_+(k) + \hat{v}_-(k)$, which are the half and full Fourier transforms of v respectively.

In the same way, because the final condition for v is a δ function, one can solve the scattering problem easily as

$$a(k,T) = 1 + g_v\hat{u}_+(k), \quad b(k,T) = [\hat{u}(k) + g_v\hat{u}_+(k)\hat{u}_-(k)]\,\mathrm{e}^{-2\mathrm{i}kX}, \tag{6.13}$$
$$\bar{a}(k,T) = 1 + g_v\hat{u}_-(k), \quad \bar{b}(k,T) = -g_v\mathrm{e}^{2\mathrm{i}kX}, \tag{6.14}$$

in terms of the unknown functions,

$$\hat{u}_{\pm}(k) := \int_{\mathbb{R}_{\mp}} u(x+X,T)\,\mathrm{e}^{-2\mathrm{i}kx}\mathrm{d}x \tag{6.15}$$

and $\hat{u}(k) := \hat{u}_+(k) + \hat{u}_-(k)$, which are the half and full Fourier transforms of u respectively. The functions $\hat{u}_{\pm}(k)$ are analytic in upper and lower complex half-plane.

In addition, thanks to $V(x,t;k) \sim 2k^2\sigma_3$ for $x \to \pm\infty$, the linear problem for the time evolution (6.2) is simplified and the time evolution of the scattering amplitudes become trivial. We have

$$a(k,t) = a(k,0), \quad b(k,t) = b(k,0)\,\mathrm{e}^{-4k^2t}, \tag{6.16}$$

$$\bar{a}(k,t) = \bar{a}(k,0), \quad \bar{b}(k,t) = \bar{b}(k,0)\,\mathrm{e}^{4k^2t}. \tag{6.17}$$

6.3. Determination of the product $g_u g_v$

Up to now, the weights of delta functions in the initial and final conditions (5.5) have been treated as general parameters. To express them in terms of original model parameters, let us consider the linear problem,

$$\partial_x\Omega(x,t) = U(x,t;0)\,\Omega(x,t). \tag{6.18}$$

Noticing that this is nothing but the $k=0$ case of (6.1), the scattering amplitudes for $k=0$ can be written in terms of Ω as

$$\lim_{\substack{x\to+\infty\\ y\to-\infty}} \Omega(x,t)\,\Omega^{-1}(y,t) = \begin{pmatrix} a(0,t) & -\bar{b}(0,t) \\ b(0,t) & \bar{a}(0,t) \end{pmatrix}. \tag{6.19}$$

By setting $k=0$ in (6.10) and (6.14), we find $g_u = b(0,0)$ and $g_v = -\bar{b}(0,T)$. Besides, (6.18) admits an explicit representation in terms of (ρ,H) as

$$\Omega(x,t) = \begin{pmatrix} \mathrm{e}^{\int_{-\infty}^{x}\mathrm{d}y\ (1-\rho)\partial_y H} & \mathrm{e}^{\int_{-\infty}^{x}\mathrm{d}y\ \rho\partial_y H} \\ -(1-\rho)\,\mathrm{e}^{\int_{-\infty}^{x}\mathrm{d}y\ \rho\partial_y H} & \rho\mathrm{e}^{\int_{-\infty}^{x}\mathrm{d}y\ (1-\rho)\partial_y H} \end{pmatrix}, \tag{6.20}$$

as confirmed by direct checking. Taking into account the boundary conditions, $\rho \sim \rho_{\pm}$ and $H - \lambda/2 \sim \pm\lambda/2$ for $x \to \pm\infty$, and using (6.19), we obtain

$$a(0,t) = \left[1 + \left(\mathrm{e}^{\lambda} - 1\right)\rho_-\right]\mathrm{e}^{\Lambda/2-\lambda/2}, \quad b(0,t) = -\left(r_- - \mathrm{e}^{-\lambda}r_+\right)\mathrm{e}^{-\Lambda/2+\lambda/2}, \tag{6.21}$$

$$\bar{a}(0,t) = \left[1 + \left(\mathrm{e}^{-\lambda} - 1\right)\rho_+\right]\mathrm{e}^{-\Lambda/2+\lambda/2}, \quad \bar{b}(0,t) = \left(\mathrm{e}^{\lambda} - 1\right)\mathrm{e}^{\Lambda/2-\lambda/2}, \tag{6.22}$$

with $\Lambda = \int_{-\infty}^{\infty}\mathrm{d}x\ (1-2\rho)\partial_x H$ being a conserved quantity and $r_{\pm} = \rho_{\pm}(1-\rho_{\mp})$.

From this we conclude that the product $g_u g_v$ is given, in terms of the parameters of the model, by

$$g_u g_v = -b(0,t)\,\bar{b}(0,t) = \left(\mathrm{e}^{\lambda} - 1\right)\rho_-(1-\rho_+) + \left(\mathrm{e}^{-\lambda} - 1\right)\rho_+(1-\rho_-). \tag{6.23}$$

The RHS of (6.23) is an important parameter and is denoted by ω in the following.

6.4. Riemann-Hilbert problem

As mentioned in section 5.2, the solution $u(x,T)$ at final time can be determined by using the consistency condition (6.16) for the scattering amplitudes, given in (6.10) and (6.13). Writing down the relation (6.9) at time T by using $b(k,T) = b(k,0)\mathrm{e}^{-4k^2T} = g_u\mathrm{e}^{-4k^2T}$, we find

$$[1 + g_v\hat{u}_+(k)][1 + g_v\hat{u}_-(k)] = 1 + g_ug_v\mathrm{e}^{-4k^2T+2\mathrm{i}kX}. \tag{6.24}$$

This is a scalar Riemann–Hilbert problem, where an analytic function is factorized into a product of functions analytic in the upper and lower half planes, and can be easily solved as

$$1 + g_v\hat{u}_\pm(k) = \mathrm{e}^{Z_\pm(k)} \tag{6.25}$$

where the function $Z_\pm(k)$ is given by

$$Z_\pm(k) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(-\omega\mathrm{e}^{-4\left(k-\frac{iX}{4T}\right)^2T-X^2/4T}\right)^n}{n}\mathrm{erfc}\left[\mp i\sqrt{4nT}\left(k-\frac{iX}{4T}\right)\right]. \tag{6.26}$$

Note that a special value at $k=0$ is evaluated as

$$\frac{Z'_\pm(0)}{(\pm 2i)\sqrt{T}} = -\sum_{n=1}^{\infty}(-\omega)^n\left[\frac{\mathrm{e}^{-n\xi^2}}{\sqrt{n\pi}} \pm \xi\mathrm{erfc}\left(\mp\sqrt{n}\xi\right)\right] \tag{6.27}$$

where $\xi = X/\sqrt{4T}$.

The reduction of the problem to a scalar Riemann–Hilbert factorization is one of the main technical simplifications. A condition equivalent to (6.24) was already obtained in [GPR+22], by an approach that did not use inverse scattering.

6.5. Cumulant generating function

The cumulant generating function is related to the height function by [BSM22a]

$$\sqrt{T}\mu'(\lambda) = N(X,T), \tag{6.28}$$

which may be rewritten as

$$\sqrt{T}\mu'(\lambda) = \int_X^\infty \mathrm{d}x\,[\rho(x,T) - \rho_+] - \int_0^\infty \mathrm{d}x\,[\rho(x,0) - \rho_+] - \rho_+X. \tag{6.29}$$

In terms of the ISM variables, this is written as

$$\sqrt{T}\mu'(\lambda) = -\int_X^\infty \mathrm{d}x\int_x^\infty \mathrm{d}y\;\mathrm{e}^{\Lambda}u(y,T) - \rho_+(1-\rho_+)\int_0^\infty \mathrm{d}x\int_x^\infty \mathrm{d}y\;\mathrm{e}^{-\Lambda}v(y,0) - \rho_+X. \tag{6.30}$$

Rewriting in terms of the half Fourier transforms, we obtain

$$\begin{aligned}\sqrt{T}\mu'(\lambda) &= -\mathrm{e}^{\Lambda}\int_0^\infty \mathrm{d}x\ xu(x+X,T) - \rho_+(1-\rho_+)\mathrm{e}^{-\Lambda}\int_0^\infty \mathrm{d}x\ xv(x,0) - \rho_+ X \\ &= -\mathrm{e}^{\Lambda}\frac{\hat{u}_-'(0)}{(-2i)} - \rho_+(1-\rho_+)\mathrm{e}^{-\Lambda}\frac{\hat{v}_+'(0)}{(2i)} - \rho_+ X \\ &= -\mathrm{e}^{\Lambda}\frac{\bar{a}'(0)}{(-2i)g_v} - \rho_+(1-\rho_+)\mathrm{e}^{-\Lambda}\frac{a'(0)}{(2i)g_u} - \rho_+ X \\ &= -\mathrm{e}^{\Lambda}\frac{Z_-'(0)\bar{a}(0)}{(-2i)g_v} - \rho_+(1-\rho_+)\mathrm{e}^{-\Lambda}\frac{Z_+'(0)a(0)}{(2i)g_u} - \rho_+ X. \end{aligned} \tag{6.31}$$

In the third equality, we used the conserved quantities $a(k) = 1 + g_v\hat{u}_+(k) = 1 + g_u\hat{v}_+(k) = \mathrm{e}^{Z_+(k)}$ and $\bar{a}(k) = 1 + g_v\hat{u}_-(k) = 1 + g_u\hat{v}_-(k) = \mathrm{e}^{Z_-(k)}$. This shows that the cumulant generating function depends only on conserved quantities. Plugging the special values (6.27), we obtain

$$\begin{aligned}\mu'(\lambda) = &-g_u\mathrm{e}^{\Lambda}\bar{a}(0)\sum_{n=1}^{\infty}(-\omega)^{n-1}\left[\frac{\mathrm{e}^{-n\xi^2}}{\sqrt{n\pi}} + \xi\mathrm{erf}\left(\sqrt{n}\xi\right) - \xi\right] \\ &- \rho_+(1-\rho_+)g_v\mathrm{e}^{-\Lambda}a(0)\sum_{n=1}^{\infty}(-\omega)^{n-1}\left[\frac{\mathrm{e}^{-n\xi^2}}{\sqrt{n\pi}} + \xi\mathrm{erf}\left(\sqrt{n}\xi\right) + \xi\right] - 2\rho_+\xi. \end{aligned} \tag{6.32}$$

Using (6.21) and (6.22), we find

$$-g_u\mathrm{e}^{\Lambda}\bar{a}(0) - \rho_+(1-\rho_+)g_v\mathrm{e}^{-\Lambda}a(0) = \omega'(\lambda) \tag{6.33}$$

and

$$\frac{g_u\mathrm{e}^{\Lambda}\bar{a}(0) - \rho_+(1-\rho_+)g_v\mathrm{e}^{-\Lambda}a(0)}{1+\omega} - 2\rho_+ = -\frac{\mathrm{e}^{\lambda}\rho_-}{1+(\mathrm{e}^{\lambda}-1)\rho_-} - \frac{\mathrm{e}^{-\lambda}\rho_+}{1+(\mathrm{e}^{-\lambda}-1)\rho_+}. \tag{6.34}$$

Therefore, integrating $\mu'(\lambda)$ w.r.t. λ, we conclude that the cumulant generating function is of the form:

$$\mu(\lambda) = -\sum_{n=1}^{\infty}\frac{(-\omega)^n}{n^{3/2}}\left[\frac{\mathrm{e}^{-n\xi^2}}{\sqrt{\pi}} + \sqrt{n}\xi\mathrm{erf}\left(\sqrt{n}\xi\right)\right] - \xi\log\frac{1+(\mathrm{e}^{\lambda}-1)\rho_-}{1+(\mathrm{e}^{-\lambda}-1)\rho_+}. \tag{6.35}$$

This agrees with the formula in [IMS21] which was found by directly studying the SEP microscopically[4].

In this paper we focused on the calculation of the cumulant generating function, we can also study optimal profiles of ρ and H at initial and final times as in [MMS22].

[4] Possible appearance of solitons for general x are not discussed in this paper because they are known not to change the final expression of the cumulant generating function for a few similar models [BSM22a].

J. Stat. Mech. (2024) 074001

7. Concluding remarks

In this article, after recalling some basic facts about MFT, we present a recent exact solution of the MFT equations, found by connecting them to a classical integrable system. We have also performed an original calculation of the cumulant generating function for the tagged particle position, which matches perfectly with the formula obtained by microscopic calculations in [IMS21].

We also emphasize the fact that the transformation of [MMS22] can be applied to MFT equations for all models with quadratic mobility $\sigma(\rho)$ which include not only SEP but also KMP and related models. For SEP, the large deviations of the same physical observables (current, height, tagged particle position) have been derived directly from microscopic models. The results agree with the predictions of MFT obtained by solving the MFT equations. On the other hand for the KMP model, an exact analysis of the microscopic model has not yet been performed. Thus, the solution of the MFT equation for the KMP model provides an indispensable tool to study large deviation of the model and might also give further motivation to study the microscopic model directly and exactly.

After the first few works on the subject, there have been several works in various directions [BSM22a, SSM23, SM23, HKD23, KD23, KLD23, Tsa, Bet23, Bet24]. It now seems obvious that this new approach to study large deviations of interacting models will provide a wider and deeper understanding of fluctuations of non-equilibrium systems.

Acknowledgments

The authors thank Aurélien Grabsch, Pierre Rizkallah, Pierre Illien and Olivier Bénichou for useful discussions on related problems. The work of K M has been supported by the Project RETENU ANR-20-CE40-0005-01 of the French National Research Agency (ANR). H M would like to acknowledge all the support of LPTMC and Sorbonne University. Part of this article was written during stays of T S at the Isaac Newton Institute of Mathematical Sciences and at Institut des Hautes Études Scientifiques. The work of T S has been supported by JSPS KAKENHI Grant Nos. JP21H04432 and JP22H01143.

References

[Arr83] Arratia R 1983 The motion of a tagged particle in the simple symmetric exclusion system on Z *Ann. Probab.* **11** 362–73

[AS81] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Society for Industrial and Applied Mathematics)

[BD04] Bodineau T and Derrida B 2004 Current fluctuations in nonequilibrium diffusive systems: an additivity principle *Phys. Rev. Lett.* **92** 180601

[BDSG+02] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C 2002 Macroscopic fluctuation theory for stationary non-equilibrium states *J. Stat. Phys.* **107** 635–75

[BDSG+03] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C 2003 Large deviations for the boundary driven symmetric simple exclusion process *Math. Phys. Anal. Geom.* **6** 231–67

[Ber21] Bernard D 2021 Can the macroscopic fluctuation theory be quantized? *J. Phys. A: Math. Theor.* **54** 433001

[Bet23] Bettelheim E 2023 Whitham approach to certain large fluctuation problems in statistical mechanics (arXiv:2307.09030)

[Bet24] Bettelheim E and Meerson B 2024 Complete integrability of the problem of full statistics of non-stationary mass transfer in the simple inclusion process (arXiv:2403.19536)

[BGL05] Bertini L, Gabrielli D and Lebowitz J L 2005 Large deviations for a stochastic model of heat flow *J. Stat. Phys.* **121** 843–85

[BSG+15] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C 2015 Macroscopic fluctuation theory *Rev. Mod. Phys.* **87** 593–636

[BSM22a] Bettelheim E, Smith N R and Meerson B 2022 Full statistics of nonstationary heat transfer in the Kipnis-Marchioro-Presutti model *J. Stat. Mech.* 093103

[BSM22b] Bettelheim E, Smith N R and Meerson B 2022 Inverse scattering method solves the problem of full statistics of nonstationary heat transfer in the Kipnis-Marchioro-Presutti model *Phys. Rev. Lett.* **128** 130602

[CGRS16] Carinci G, Giardina C, Redig F and Sasamoto T 2016 A generalized asymmetric exclusion process with $U_q(sl_2)$ stochastic duality *Probab. Theory Relat. Fields* **166** 887

[DDR04] Derrida B, Douçot B and Roche P-E 2004 Current fluctuations in the one-dimensional symmetric exclusion process with open boundaries *J. Stat. Phys.* **115** 717–48

[DEHP93] Derrida B, Evans M R, Hakim V and Pasquier V 1993 Exact solution of a 1D exclusion model using a matrix formulation *J. Phys. A: Math. Gen.* **26** 1493–517

[Der07] Derrida B 2007 Non-equilibrium steady states: fluctuations and large deviations of the density and of the current *J. Stat. Mech.* P 07023

[DG09a] Derrida B and Gershenfeld A 2009 Current fluctuations in one dimensional diffusive systems with a step initial profile *J. Stat. Phys.* **137** 978–1000

[DG09b] Derrida B and Gershenfeld A 2009 Current fluctuations of the one dimensional symmetric simple exclusion process with step initial condition *J. Stat. Phys.* **136** 1–15

[DLS02] Derrida B, Lebowitz J L and Speer E R 2002 Exact free energy functional for a driven diffusive open stationary nonequilibrium system *Phys. Rev. Lett.* **89** 030601

[DMF02] De Masi A and Ferrari P A 2002 Flux fluctuations in the one dimensional nearest neighbors symmetric simple exclusion process *J. Stat. Phys.* **107** 677–83

[DPSY23] Doyon B, Perfetto G, Sasamoto T and Yoshimura T 2023 Ballistic macroscopic fluctuation theory *SciPost Phys.* **15** 136

[DZ09] Dembo A and Zeitouni O 2009 *Large Deviations Techniques and Applications* (*Stochastic Modelling and Applied Probability*) (Springer)

[Ell06] Ellis R S 2006 *Entropy, Large Deviations and Statistical Mechanics* (*Classics in Mathematics*) (Springer)

[FGK19] Frassek R, Giardinà C and Kurchan J 2019 Non-compact quantum spin chains as integrable stochastic particle processes *J. Stat. Phys.* **180** 135–71

[GC95] Gallavotti G and Cohen E G D 1995 Dynamical ensembles in nonequilibrium statistical mechanics *Phys. Rev. Lett.* **74** 2694

[GKRV09] Giardin C, Kurchan J, Redig F and Vafayi K 2009 Duality and hidden symmetries in interacting particle systems *J. Stat. Phys.* **135** 25–55

[GPR+22] Grabsch A, Poncet A, Rizkallah P, Illien P and Bénichou O 2022 Exact closure and solution for spatial correlations in single-file diffusion *Sci. Adv.* **8** eabm5043

[HKD23] Hartmann A K, Krajenbrink A and Le Doussal P 2023 Probing the large deviations for the Beta random walk in random medium (arXiv:2307.15041)

[IMS21] Imamura T, Mallick K and Sasamoto T 2021 Distribution of a tagged particle position in the one-dimensional symmetric simple exclusion process with two-sided Bernoulli initial condition *Commun. Math. Phys.* **384** 1409–44

[JKM16] Janas M, Kamenev A and Meerson B 2016 Dynamical phase transition in large-deviation statistics of the Kardar-Parisi-Zhang equation *Phys. Rev.* E **94** 032133

[KD23] Krajenbrink A and Le Doussal P 2023 The weak noise theory of the O'Connell-Yor polymer as an integrable discretisation of the nonlinear Schrödinger equation (arXiv:2307.01172)

[KL99] Kipnis C and Landim C 1999 *Scaling Limits of Interacting Particle Systems* (Springer)

[KLD21] Krajenbrink A and Le Doussal P 2021 Inverse scattering of the Zakharov-Shabat system solves the weak noise theory of the Kardar-Parisi-Zhang equation *Phys. Rev. Lett.* **127** 064101

[KLD22] Krajenbrink A and Le Doussal P 2022 Inverse scattering solution of the weak noise theory of the Kardar-Parisi-Zhang equation with flat and Brownian initial conditions *Phys. Rev.* E **105** 054142

[KLD23] Krajenbrink A and Doussal P 2023 Crossover from the macroscopic fluctuation theory to the Kardar-Parisi-Zhang equation controls the large deviations beyond Einstein's diffusion *Phys. Rev.* E **107** 014137

[KM12] Krapivsky P L and Meerson B 2012 Fluctuations of current in nonstationary diffusive lattice gases *Phys. Rev.* E **86** 031106

[KMP82] Kipnis C, Marchioro C and Presutti E 1982 Heat flow in an exactly solvable model *J. Stat. Phys.* **27** 65–74

[KMS14] Krapivsky P L, Mallick K and Sadhu T 2014 Large deviations in single file diffusion *Phys. Rev. Lett.* **113** 078101

[KMS15] Krapivsky P L, Mallick K and Sadhu T 2015 Tagged particle in single-file diffusion *J. Stat. Phys.* **160** 885–925

[KN78] Kaup D J and Newell A C 1977 An exact solution for a derivative nonlinear Schrödinger equation *J. Math. Phys.* **19** 798–801

[KOV89] Kipnis C, Olla S and Varadhan S R S 1989 Hydrodynamics and large deviation for simple exclusion process *Commun. Pure. Appl. Math.* **42** 115–37

[Lig73] Liggett T M 1973 An infinite particle system with zero range interactions *Ann. Prob.* **1** 240–53

[LS99] Lebowitz J L and Spohn H 1999 A Gallavotti-Cohen-Type symmetry in the large deviation functional for stochastic dynamics *J. Stat. Phys.* **95** 333–65

[MMS22] Mallick K, Moriya H and Sasamoto T 2022 Exact solution of the macroscopic fluctuation theory for the symmetric exclusion process *Phys. Rev. Lett.* **129** 040601

[MSR73] Martin P C, Siggia E D and Rose H A 1973 Statistical dynamics of classical systems *Phys. Rev.* A **8** 423–37

[SD15] Sadhu T and Derrida B 2015 Large deviation function of a tracer position in single file diffusion *J. Stat. Mech.* P 09008

[SM23] Smith N R and Meerson B 2023 Macroscopic fluctuation theory of local time in lattice gases (arXiv:2311.15286)

[Spo91] Spohn H 1991 *Large Scale Dynamics of Interacting Particles* (Springer)

[SS13] Varadhan S R S Sethuraman S 2013 Large deviations for the current and tagged particle in 1D symmetric simple exclusion *Ann. Probab.* **41** 1461–512

[SSM23] Schorlepp T, Sasorov P and Meerson B 2023 Short-time large deviations of the spatially averaged height of a Kardar-Parisi-Zhang interface on a ring *J. Stat. Mech.* 123202

[TKL08] Tailleur J, Kurchan J and Lecomte V 2008 Mapping out-of-equilibrium into equilibrium in one-dimensional transport models *J. Phys. A: Math. Theor.* **41** 505001

[Tou09] Touchette H 2009 The large deviation approach to statistical mechanics *Phys. Rep.* **478** 1–69

[Tsa] Tsai L-C 2022 Integrability in the weak noise theory (arXiv:2204.00614)

[Var84] Varadhan S R S 1984 *Large Deviations and Applications* (Society for Industrial and Applied Mathematics)