Fluctuations of the rapidity distribution

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April 10, 2025

In the coarse-grained Lieb-Liniger model, expectation values w.r.t an ensemble are given by

$$\langle \hat{A} \rangle_w \,=\, \frac{\int \mathcal{D}\rho \, e^{L(S[\rho]-W[\rho])} \, A[\rho]}{\int \mathcal{D}\rho \, e^{L(S[\rho]-W[\rho])}} \tag{1}$$

where $A[\rho]$ is the expectation value of $\hat{A}$ in an eigenstate with rapidity distribution ρ, $\mathcal{D}\rho$ denotes the functional integral over all possible rapidity distributions, L is the length of the system, $S[\rho]$ is the Yang-Yang entropy, which is of the form

$$S[\rho] \,=\, \int s(\nu(\theta))\rho_{\rm s}(\theta) d\theta \,, \tag{2}$$

and the weight functional that defines the ensemble is

$$W[\rho] \,=\, \int w(\theta)\rho(\theta) d\theta \,. \tag{3}$$

Since the system size L is assumed to be large, the expectation value (5) can be evaluated by saddle point, leading to the Yang-Yang equation,

$$w(\theta) \,=\, \frac{\delta S[\rho]}{\delta \rho(\theta)}. \tag{4}$$

This fixes all one-point functions. If we are interested in two-point functions, $\langle \hat{A}\hat{B} \rangle^{\rm conn.} = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$, then

$$\langle \hat{A}\hat{B} \rangle^{\rm conn.}_w \,=\, \int d\theta \int d\theta' \, \frac{\delta A}{\delta \rho(\theta)} \frac{\delta B}{\delta \rho(\theta')} \langle \delta\rho(\theta)\delta\rho(\theta') \rangle \tag{5}$$

where

$$\begin{aligned} \langle \delta\rho(\theta)\delta\rho(\theta') \rangle \;&=\; \frac{\int \mathcal{D}\delta\rho \, e^{\frac{L}{2}\int d\lambda \int d\lambda' \frac{\delta^2 S_{\rm YY}}{\delta\rho(\lambda)\delta\rho(\lambda')} \delta\rho(\lambda)\delta\rho(\lambda')} \delta\rho(\theta)\delta\rho(\theta')}{\int \mathcal{D}\delta\rho \, e^{\frac{L}{2}\int d\lambda \int d\lambda' \frac{\delta^2 S_{\rm YY}}{\delta\rho(\lambda)\delta\rho(\lambda')} \delta\rho(\lambda)\delta\rho(\lambda')}} \\ &=\; \frac{1}{L} \left[-\frac{\delta^2 S_{\rm YY}}{\delta\rho\delta\rho} \right]^{-1} (\theta, \theta'). \end{aligned} \tag{6}$$

After a calculation (see below), I arrive at the following formula,

$$\langle \delta\rho(\theta)\delta\rho(\theta')\rangle = \frac{1}{L}\int \frac{[\delta(\theta-\lambda)+\nu(\theta)C(\theta,\lambda)]\,[\delta(\theta'-\lambda)+\nu(\theta')C(\theta',\lambda)]}{-s''(\nu(\lambda))}\rho_{\rm s}(\lambda)d\lambda\,, \tag{7}$$

with

$$C(\theta,\lambda) = [\frac{1}{2\pi}\Delta(\lambda-.)]^{\rm dr}(\theta). \tag{8}$$

This result is compatible with formula (1.1) in arXiv:1705.08141 by Herbert and Benjamin, who say that it is 'an immediate consequence of the (generalized) thermodynamic Bethe ansatz formalism'. To check the equivalence with my formula, one specifies $s(\nu) = -\nu\log\nu - (1-\nu)\log(1-\nu)$ so that $\frac{1}{-s''(\nu)} = \nu(1-\nu)$, and then one uses the fact that

$$h^{\rm dr}(\lambda) = \int d\theta\, h(\theta)\,[\delta(\theta-\lambda)+\nu(\theta)C(\theta,\lambda)]\,.$$

Derivation. To first order we have

$$\begin{aligned}\delta(S-W) &= \int\big[s'(\nu(\theta))(\delta\rho(\theta)-\nu(\theta)\delta\rho_{\rm s}(\theta)) + s(\nu(\theta))\delta\rho_{\rm s}(\theta) - w(\theta)\delta\rho(\theta)\ \big]\,d\theta \\ &= \int\big[(s'(\nu)-w)\delta\rho + (s(\nu)-\nu s'(\nu)\delta\rho_{\rm s}\ \big]\,d\theta\end{aligned}$$

and, remarkably, to second order

$$\delta(S-W) = \int s''(\nu)\,(\delta\nu)^2\,\rho_{\rm s}(\theta)d\theta\,.$$

It is a miracle that the result is diagonal in $\delta\nu(\theta)$. There must be a deeper meaning to this, but for now I don't know what it is. Since $\delta^2 S$ is diagonal, it can be inverted straighforwardly:

$$\begin{aligned}\langle\delta\nu(\theta)\delta\nu(\theta')\rangle &= \frac{1}{L}\left[-\frac{\delta^2(S-W)}{\delta\nu\delta\nu}\right]^{-1}(\theta,\theta') \\ &= \frac{1}{L}\,\frac{1}{-s''(\nu(\theta))\,\rho_{\rm s}(\theta)}\delta(\theta-\theta') \end{aligned} \tag{9}$$

where the last δ is the Dirac delta function, not a differential. We also have

$$\delta\rho(\theta) = \int [\delta(\theta-\lambda)+\nu(\theta)C(\theta,\lambda)]\,\rho_{\rm s}(\lambda)\delta\nu(\lambda)d\lambda \tag{10}$$

where $C(\theta,\lambda)$ is defined in (8). This follows as usual from the definition of the dressing. Putting everything together, one gets formula (7).