IB Extended Essay

Mathematics

ON THE COUNTER-INTUITIVE BEHAVIOURS OF INSCRIBED SPHERES AND CUBES IN HIGHER DIMENSIONS

Research Question:

Why does the radius of an IK-Sphere tend to infinity as dimensionality increases and does it truly intersect with its bounding box in $n \geq 4$ dimensions?

Word Count: 3838

November 2021

To those who wonder about the extra-ordinary.

This essay was prepared using the LaTeX typesetting package.

Abstract

What would appear to be a sphere inside of a cube in our two or three dimensions completely inverts to a cube inside of an incomparably large sphere in higher dimensions. In this essay, we examine the curious nature of an Inscribed Kissing (IK) Sphere to understand why these counter-intuitive phenomena occur in higher dimensions and add rigour and reason to Barry Cipra's IK-Sphere problem from the first volume of *What's Happening in the Mathematical Sciences*. Throughout the essay, two methods of reasoning – an algebraic versus a geometric approach – will be used with varying degrees of certainty versus intuition gained in each. Finally, we answer the research question: **Why does the radius of an IK-Sphere tend to infinity as dimensionality increases and does it truly intersect with its bounding box in** $n \geq 4$ **dimensions?**

Contents

List of Figures[1]

[1]All illustrations and graphs in this essay were made by the author

Nomenclature

Linear Algebra

(x, y, z) Point

$\langle x, y, z \rangle$ Vector (Rectangular Notation)

Number sets

$\mathbb{N}$ Natural numbers

$\mathbb{R}$ Real numbers

Other symbols

$:$ Such That

$\blacksquare$ End of Proof

$\forall$ For All (e.g. $\forall A, B$ occurs)

$\in$ In (e.g. $n \in \mathbb{N}$)

$\square$ End of Subproof

$\mathbb{R}^n$ n-Dimensional Euclidean Space

Chapter 1

Introduction

1.1 A Deficit of Higher Dimensional Intuition

Our intuition was formed in three dimensions and is often misleading in higher dimensions. Consequently, we resort to unsound dimensional analogy where our observations from two or three dimensions are assumed to generalise to higher dimensions. In Volume 1 of the yearly release of *What's Happening in the Mathematical Sciences*, two such false higher dimensions conjectures proposed were presented in the chapter *Disproving the Obvious in Higher Dimensions* [4]. As an example for the reader to explore their own high-dimensional intuition, the end of the chapter featured a box titled *Here's Looking at Euclid*. Within the box, without much elaboration, was the IK-Sphere Problem.

1.2 The IK-Sphere Problem

Suppose a square with a side length of 2 units in two-dimensional space has unit circles (circles with a radius of 1 unit) on each of its vertices as in Figure 1.1[1].

[1] All illustrations and graphs in this essay were made by the author

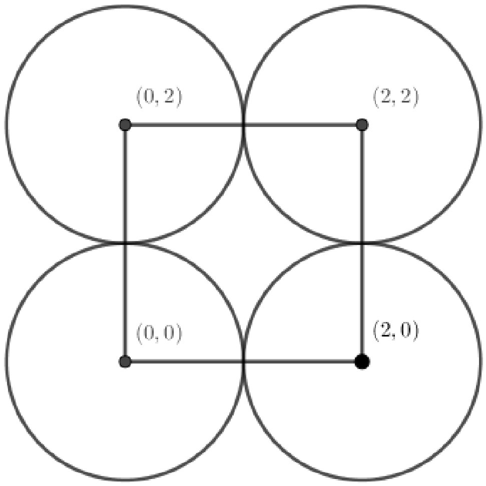


Figure 1.1: Setup for the 2D IK-Sphere problem

Now, draw a circle at the centre of the square that 'kisses' the unit circles (see Figure 1.2). This inner circle will be named an IK-Sphere (Inscribed Kissing Sphere) for the purposes of this essay.

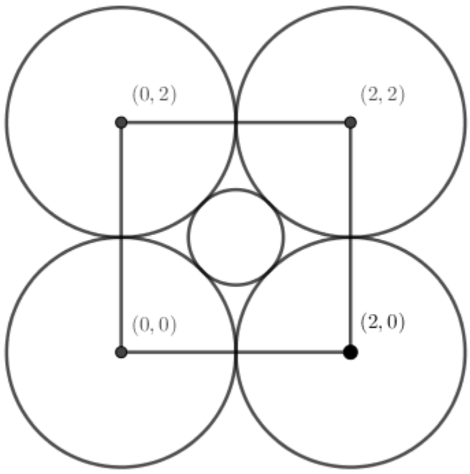


Figure 1.2: The 2D IK-Sphere.

The definition of a sphere and cube can be generalised across all dimensions so that we may extend the IK-Sphere into a higher dimensional space. We will name these the n-sphere and n-cube respectively.

Definition 1.2.1 (n-Sphere)**.** Created by all points equidistant (specified by a radius) from a common centre point, an n-sphere is a generalised 3D sphere (3-sphere) to n-dimensional space, For example, a circle is a 2-sphere, and a line is a 1-sphere.

Definition 1.2.2 (n-Cube)**.** An n-cube is created by extending an $(n-1)$-cube in a direction perpendicular to itself (see Figure 1.3) where all edges must be of the same length. An n-cube generalises the cube (3-cube) to n dimensions.

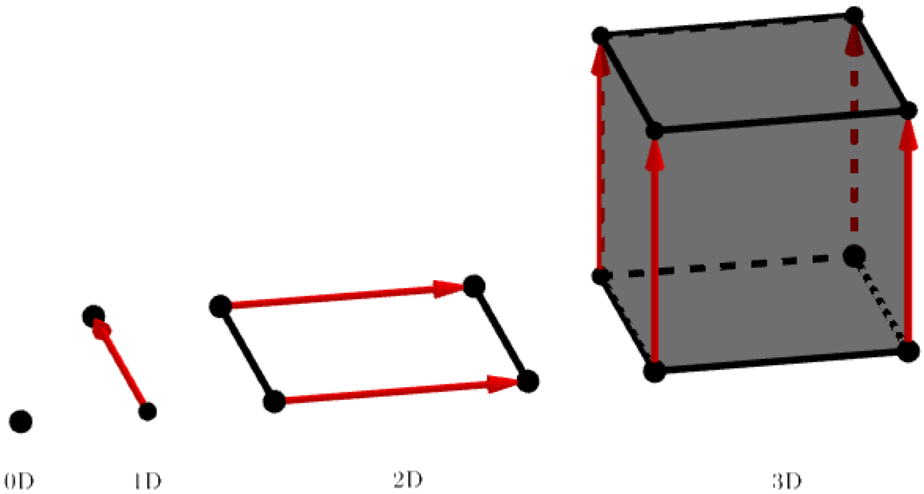


Figure 1.3: Extending an $(n-1)$-cube in a perpendicular direction to create an n-cube

Abiding by these definitions, we repeat the process to create an IK-Sphere in three dimensions (see Figure 1.4),

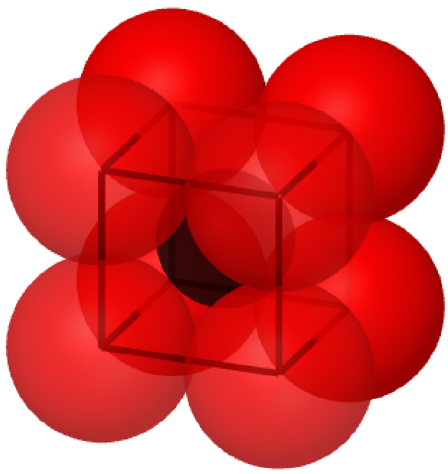


Figure 1.4: 3D IK-Sphere.

and find that the radius of the IK-Sphere increases (see Figure 1.5).

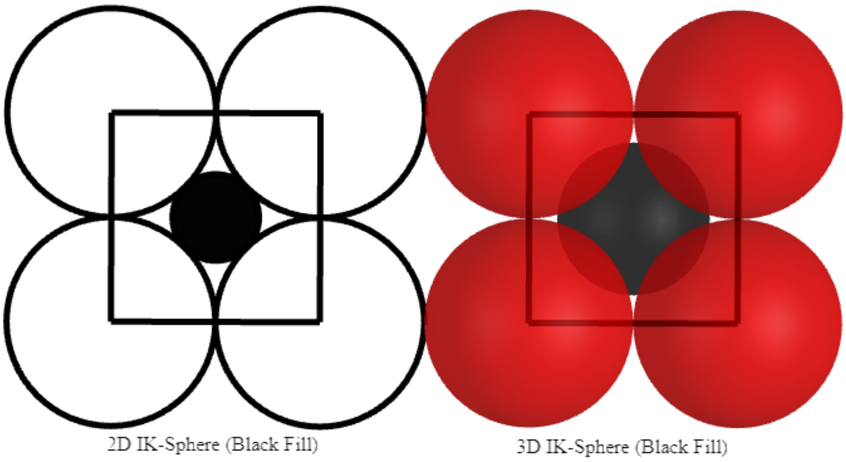


Figure 1.5: Comparing radius of 2D IK-Sphere to 3D IK-Sphere.

Without a rigorous explanation as to why, the *Here's Looking at Euclid* box states that as the number of dimensions increase, the radius of the IK-Sphere increases without bound – contrary to one's assumption that the radius will approach touching the bounding cube but never exceed it. Thus, this prompted the research question (RQ) to understand and verify the conclusions proposed about IK-Spheres:

RQ: Why does the radius of an IK-Sphere tend to infinity as dimensionality increases and does it truly intersect with its bounding box in $n \geq 4$ dimensions?

Chapter 2

Body: Algebraic Reasoning

> *Why does the radius of an IK-Sphere tend to infinity as dimensionality increases and does it truly intersect with its bounding box in $n \geq 4$ dimensions?*
>
> Research Question

Algebraic reasoning aims to abstract the IK-Sphere problem and formally generalise it to the unfamiliar fourth and above dimensions to answer the research question. Firstly, in section 2.1, we find an expression for the radius r_n of an IK-Sphere that depends on dimension n so that the limit as $n \to \infty$ may be taken and the resulting behaviour of r_n can be determined. Secondly, in section 2.2, the point(s) of intersection between the IK-Sphere and its bounding n-cube can be solved to verify whether the IK-Sphere truly expands outside of its bounding n-cube. In verifying whether an intersection exists not, one may gain confidence in the unusual behaviour of higher dimensional IK-Spheres, for it appears counter-intuitive that IK-Spheres would eventually expand far enough that they fully encapsulate their 'bounding' boxes. It may be proposed that there is a higher dimensional phenomena causing intersections to function differently, and as a result, the IK-Sphere indeed does remain contained within its bounding box. Thus, we must verify the higher dimensional intersection as will be shown in section 2.2.

2.1 The Limit of the IK-Sphere Radius as Dimension Tends to Infinity

The radius r of the IK-Sphere can be solved for algebraically and generalised by extending the Pythagorean Theorem.

Lemma 2.1.1 (Extended Pythagorean Theorem for n-cube)**.** *For an n-cube in n-dimensions, with a side length of a, the length of the diagonal c of the n-cube is given by*

$$c^2 = \sum_{i}^{n} a_i^2 \tag{2.1.1}$$

Proof. Given the original two-dimensional Pythagorean Theorem is proven, we may use mathematical induction to prove the lemma.

Base case ($n = 2$)

$$c^2 = a_1^2 + a_2^2$$

which is analogous to $c^2 = a^2 + b^2$. So, the lemma holds for $n = 3$.

Inductive hypothesis: Suppose the theorem holds for all values of n up to some k, $k \geq 3$.

$$c_k^2 = \sum_{i}^{k} a_i^2$$

Inductive step: Let $n = k + 1$.

$$c_{k+1}^2 = \sum_i^{k+1} a_i^2$$

By the regular Pythagorean Theorem,

$$c^2 = a^2 + b^2,$$

the k-dimensional length c_k and a new line of length a_{i+1} perpendicular to it create the $k + 1$ dimensional cube (see three-dimensional analogy in Figure 2.1).

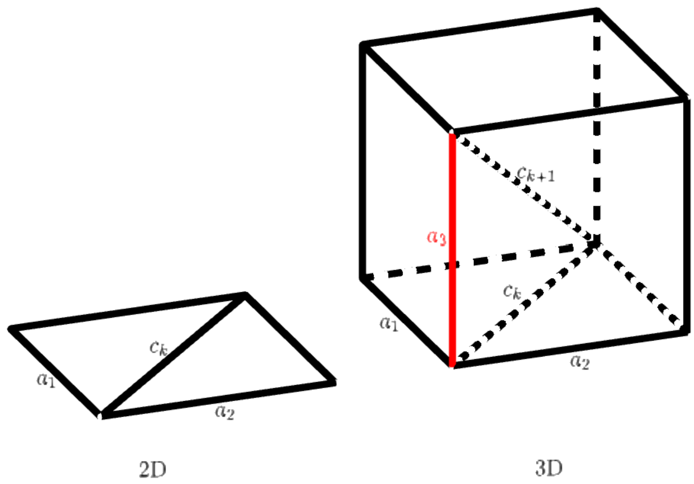


Figure 2.1: Two to three-dimensional depiction of inductive step.

Thus, the left hand side can be rewritten as

$$\begin{aligned} c_{k+1}^2 &= c_k^2 + a_{k+1}^2 \\ &= \sum_i^{k} a_i^2 + a_{k+1}^2 \\ &= \sum_i^{k+1} a_i^2 \end{aligned}$$

which is also our right side. So, the theorem holds for $n = k + 1$. By the principle of mathematical induction, the theorem holds for all $n \geq 3$, $n \in \mathbb{N}$. ■

Let us use a diagram for a 2D IK-Sphere to generalise the IK-Sphere radius r_n to n-dimensions. Using the Extended Pythagorean Theorem for an n-cube we can determine the length of a diagonal (see line $\overline{\mathrm{AB}}$ in Figure 2.2) of a quadrant of our 2-cube that includes a unit n-sphere radius and the radius of the IK-Sphere we wish to obtain.

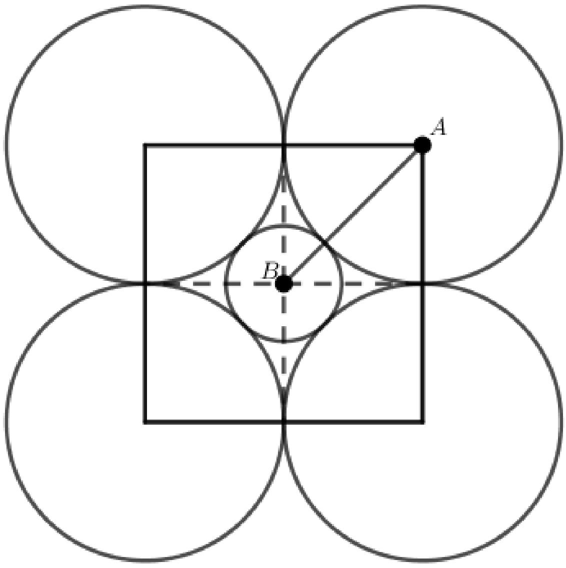


Figure 2.2: 2D IK-Sphere radius determined using Extended Pythagorean Theorem.

Hence, we can subtract the unit circle radius to obtain the radius of the two-dimensional IK-Sphere:

$$r_2 = \sqrt{1^2 + 1^2} - 1.$$

Similarly, for a three-dimensional IK-Sphere, the length of the diagonal is the square root of the sum of three sides of an octant ($\frac{1}{8}$ of a cube) of the n-cube at a vertex (see Figure 2.3). From there, the process is the same and we subtract the unit sphere. Thus, we have an inscribed sphere radius r_3 of:

$$r_3 = \sqrt{1^2 + 1^2 + 1^2} - 1.$$

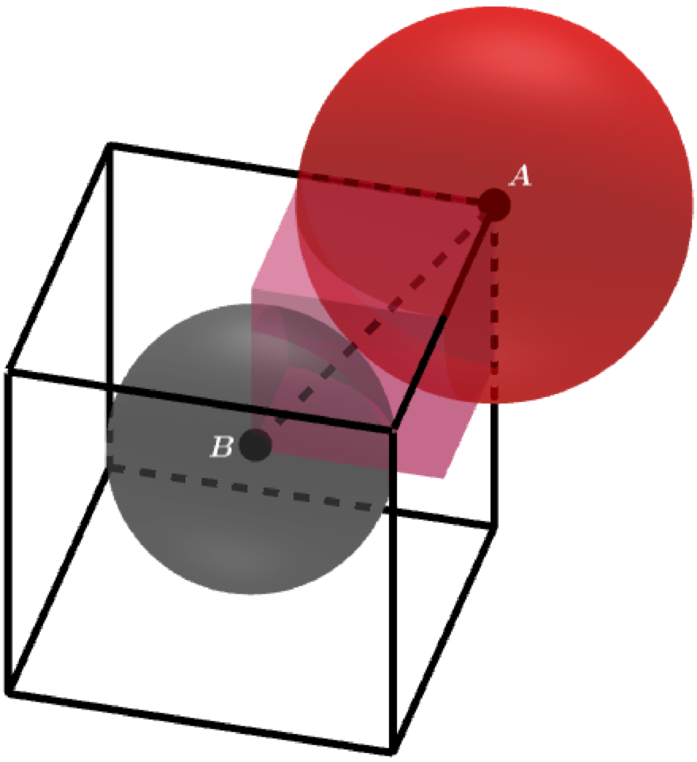


Figure 2.3: 3D IK-Sphere radius determined using Extended Pythagorean Theorem.

Corollary 2.1.1.1 (To the Extended Pythagorean Theorem for n-Cubes)**.** *The diagonal c of a unit n-cube is given by*

$$c = \sqrt{n} \tag{2.1.2}$$

Proof. Substituting $a = 1$ into the extended Pythagorean Theorem (see Equation 2.1.1), we have

$$\begin{aligned} c^2 &= \sum_{i}^{n} 1_i^2 \\ &= n \\ \therefore c &= \sqrt{n}. \end{aligned}$$

■

If the diagonal of a unit n-cube is given by $\sqrt{n}$ (see Equation 2.1.2), the IK-Sphere radius can be generalised across all n-dimensions by subtracting the unit sphere radius as follows:

$$r_n = \sqrt{n} - 1 \tag{2.1.3}$$

We may now proceed to the final step of proving that the radius of an IK-Sphere increases without bound.

Theorem 2.1.2. *As the dimension n tends to infinity, the radius r_n of the inscribed sphere also tends to infinity.*

Proof. Take the limit as n approaches infinity of the expression for r_n in terms of n (See Equation 2.1.3) and apply the Algebraic Limit Difference Theorem [2].

Algebraic Limit Difference Theorem: $\lim_{a\to b} x - y = \lim_{a\to b} x - \lim_{a\to b} y$. For example,

$$\lim_{x\to\infty} x - 7 = \lim_{x\to\infty} x - \lim_{x\to\infty} 7.$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty} \sqrt{n} - 1 &= \lim_{n\to\infty} \sqrt{n} - \lim_{n\to\infty} \sqrt{1} \\ &= \infty - 1 \\ &= \infty.\end{aligned}$$

$\therefore$ the radius of an IK-Sphere diverges to infinity as dimension n increases.

$\blacksquare$

Graphically (see Figure 2.4), we may observe the same result with less rigour. The domain has been restricted to $n \geq 2$ as $n = 2$ was the base case for the Extended Pythagorean Theorem for an n-Cube (see Lemma 2.1.1).

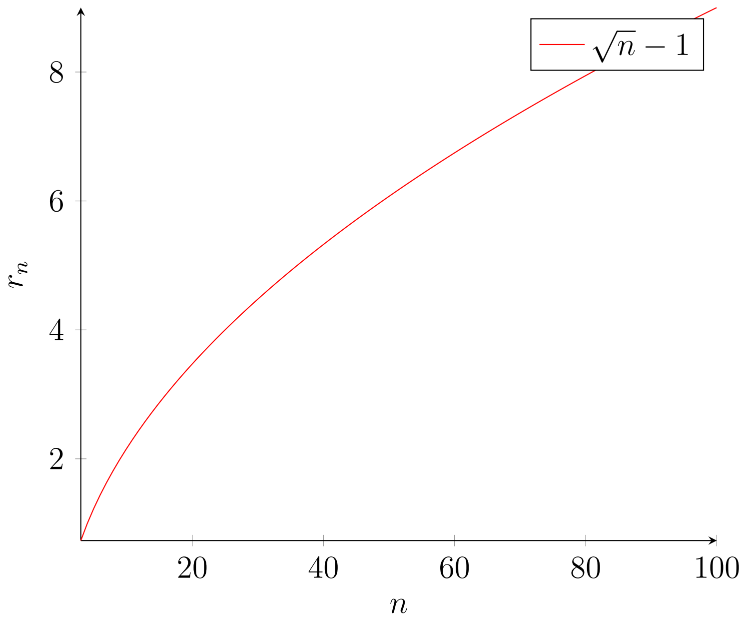


Figure 2.4: Radius increases without bound as dimensionality increases.

When $n < 1$ we obtain negative values for r_n which can accordingly be considered an undefined region. Investigation into fractional dimensions (such as those observed in fractals) could reveal a reason, however, that is currently beyond the scope of this essay. As the radius of an IK-Sphere has been proven algebraically, we will now proceed to verify whether or not the IK-Sphere intersects with its bounding box.

2.2 Verification of the IK-Sphere Intersection in Four Dimensions

The radius of an IK-Sphere has now been proven to increase without bound. As such, one may infer that the IK-Sphere bursts outside of its bounding box as dimensionality increases, implying that there exist intersection points between the IK-Sphere and its 'bounding' n-cube at higher dimensions. In fact, upon inspection, one may observe that at $n = 4$ dimensions,

$$r_4 = \sqrt{4} - 1 = 1. \tag{2.2.1}$$

The IK-Sphere just touches the sides of its bounding 4-cube with side length 2 units. Here, intuition takes hold once again and one may assume there exists a solvable intersection point between the IK-Sphere and its bounding 4-cube at the center of a single face of the 4-cube. However, another possibility arises in which higher dimensional intersections may operate differently and, as we originally assumed, the IK-Sphere may still exist within its bounding n-cube. We can verify the existence, or lack thereof, of an intersection point to decide which of these possibilities is true in higher dimensions.

To solve for the intersection point, we would need two equations in 4 coordinate variables (x, y, z, w): one for the 4-cube and another for the 4-sphere. From the definition of an n-sphere (see Definition 1.2.1) the following equation can be constructed:

Lemma 2.2.1. *In four-dimensional space with coordinates (x, y, z, w), a unit 4-sphere centered at the origin can be defined as*

$$x^2 + y^2 + z^2 + w^2 = 1. \tag{2.2.2}$$

where (x, y, z, w) is a point on the 4-sphere.

Proof. This is a simple proof using vectors. Consider the vector $\vec{\mathbf{r}}$ originating from the center of the 4-sphere. For this vector to be on a unit 4-sphere, the magnitude of $\vec{\mathbf{r}}$ must equal 1. In notation, this is

$$\|\vec{\mathbf{r}}\| = 1.$$

Expanding $\|\vec{\mathbf{r}}\|$ then squaring both sides, we can obtain the required result:

$$\sqrt{x^2 + y^2 + z^2 + w^2} = 1$$
$$\therefore x^2 + y^2 + z^2 + w^2 = 1$$

$\blacksquare$

However, an explicit equation for the entire 4-cube is not readily available. As such, we will use a single face of the 4-cube, or rather, a plane in 4-dimensions with restrictions $0 \leq x, y, z, w \leq 2$. Let this plane be denoted by Π.

Definition 2.2.1 (n-Plane)**.** An $(n-1)$-dimensional space extending to infinity in all $n-1$ dimensions embedded in n-dimensional space. All vectors $\vec{\mathbf{v}}$ contained within a plane will satisfy the condition $\vec{\mathbf{v}} \cdot \vec{\mathbf{n}} = 0$ for a normal vector $\vec{\mathbf{n}}$ to the plane. Examples include: a line in $\mathbb{R}^2$, a plane in $\mathbb{R}^3$, and allowing x, y and z to vary through $\mathbb{R}^4$ while keeping w fixed.

Lemma 2.2.2. *A 4-plane can be represented by the equation*

$$ax + by + cz + dw = k \tag{2.2.3}$$

where a, b, c, d, k are constants and (x, y, z, w) is a point on Π.

Proof. Let the vector $\vec{\mathbf{r}}$ denote a position vector $\langle x, y, z, w\rangle$ from the origin to any point on Π; $\vec{\mathbf{a}}$ be a position vector to a fixed point $\langle x_0, y_0, z_0, w_0\rangle$ contained within Π; $\vec{\mathbf{n}}$ be a normal direction vector $\langle a, b, c, d\rangle$ to Π; and $\vec{\mathbf{v}}$ be the direction vector $\langle x-x_0, y-y_0, z-z_0, w-w_0\rangle$ contained within Π. See Figure 2.5 for a diagram of these variables in $\mathbb{R}^3$.

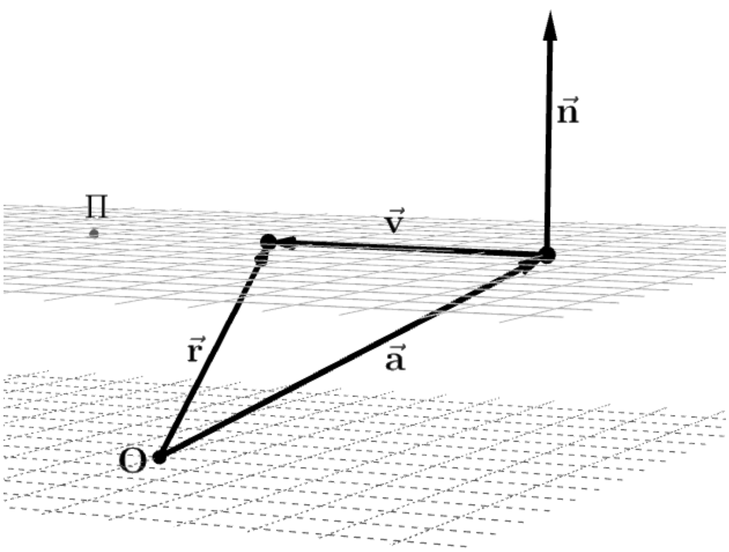


Figure 2.5: A plane Π containing a point and vector, and a normal vector to Π.

According to the definition of an n-plane (see Definition 2.2.1), the vector $\vec{\mathbf{v}}$ is in Π if and only if

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{n}} = 0.$$

Expanding $\vec{\mathbf{v}}$ and $\vec{\mathbf{n}}$ we have

$$\langle x - x_0, y - y_0, z - z_0, w - w_0 \rangle \cdot \langle a, b, c, d \rangle = 0.$$

Then, by calculating the dot product and rearranging, we can obtain the form $ax + by + cz + dw = k$:

$$\begin{aligned} a(x - x_0) + b(y - y_0) + c(z - z_0) + d(w - w_0) &= 0 \\ ax + by + cz + dw - ax_0 - by_0 - cz_0 - dw_0 &= 0 \\ ax + by + cz + dw &= ax_0 + by_0 + cz_0 + dw_0 \end{aligned}$$

Letting $ax_0 + by_0 + cz_0 + dw_0 = k$ gives

$$ax + by + cz + dw = k$$

as required. ■

Corollary 2.2.2.1 (to the equation of an n-plane)**.** *The equation of a plane can also be written as*

$$\vec{\mathbf{r}} \cdot \vec{\mathbf{n}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{n}}.$$

where the vector $\vec{\mathbf{r}}$ denotes the position vector $\langle x, y, z, w \rangle$ to any point on Π; $\vec{\mathbf{a}}$ is the position vector to a fixed point $\langle x_0, y_0, z_0, w_0 \rangle$ within Π; and $\vec{\mathbf{n}}$ is the normal direction vector $\langle a, b, c, d \rangle$ to Π.

Proof. Upon inspection of the form

$$ax + by + cz + dw = ax_0 + by_0 + cz_0 + dw_0$$

before simplifying to $ax + by + cz + dw = k$ as done in the proof of Lemma 2.2.2, it may be noticed that

$$\begin{aligned} ax + by + cz + dw &= \langle x, y, z, w \rangle \cdot \langle a, b, c, d \rangle \\ &= \vec{\mathbf{r}} \cdot \vec{\mathbf{n}} \end{aligned}$$

and

$$\begin{aligned} ax_0 + by_0 + cz_0 + dw_0 &= \langle x_0, y_0, z_0, w_0 \rangle \cdot \langle a, b, c, d \rangle \\ &= \vec{\mathbf{a}} \cdot \vec{\mathbf{n}} \end{aligned}$$

Thus, having

$$ax + by + cz + dw = ax_0 + by_0 + cz_0 + dw_0,$$

implies the vector notation form of the same equation, namely

$$\begin{aligned} ax + by + cz + dw &= ax_0 + by_0 + cz_0 + dw_0 \\ \implies \vec{\mathbf{r}} \cdot \vec{\mathbf{n}} &= \vec{\mathbf{a}} \cdot \vec{\mathbf{n}}. \end{aligned}$$

■

We now have two equations: a 4-plane (see Equation 2.2.3) and a unit 4-sphere (see Equation 2.2.2):

4-plane:

$$ax + by + cz + dw = k;$$

unit 4-sphere:

$$x^2 + y^2 + z^2 + w^2 = 1.$$

However, before solving for an intersection in four variables, we need to find the constants a, b, c, d, and k. From the definition of the n-cube (see Definition 1.2.2), we can deduce that a face of the 4-cube will have either x, y, z or w constant. Let us take w constant, and choose the face which intersects with $(0, 0, 0, 0)$, giving $w = 0$. Thus, a point on Π will be of the form $(x, y, z, 0)$ where $0 \leq x, y, z \leq 2$ so as to restrict it to be a face of the bounding 4-cube. We can now simplify the equation of Π as long as we maintain that x, y, and z satisfy $0 \leq x, y, z \leq 2$.

Lemma 2.2.3. *The equation for a plane tangent to the face of an n-cube with coordinates of the form $(x, y, z, 0)$ can be simplified from the form $ax + by + cz + dw = k$ to*

$$w = 0. \tag{2.2.4}$$

Proof. Notice that the normal vector must be perpendicular to the xyz-plane. Thus, we can deduce that $\vec{\mathbf{n}} = \langle 0, 0, 0, 1\rangle$ because it is the unit direction vector for the w axis. Thus, substituting in the normal vector, we have

$$(0)x + (0)y + (0)z + (1)w = k.$$

However, k is still unknown. To get rid of k, a slight trick may be employed. We know from the equation of a plane $\vec{\mathbf{r}} \cdot \vec{\mathbf{n}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{n}}$ (see Corollary 2.2.2.1) that $k = \vec{\mathbf{a}} \cdot \vec{\mathbf{n}}$ where $\vec{\mathbf{n}}$ is the normal vector $\langle a, b, c, d\rangle$ to Π and $\vec{\mathbf{a}}$ is a position vector to a fixed point on Π.

We wish for our plane to intersect any point satisfying $0 \leq x, y, z \leq 2$ and $(x, y, z, 0)$; therefore, we let $\vec{\mathbf{a}} = \langle 1, 1, 1, 0 \rangle$, giving

$$\begin{aligned}(0)x + (0)y + (0)z + (1)w &= \vec{\mathbf{a}} \cdot \vec{\mathbf{n}} \\ (0)x + (0)y + (0)z + (1)w &= \langle 1, 1, 1, 0 \rangle \cdot \langle 0, 0, 0, 1 \rangle \\ \implies w &= 0.\end{aligned}$$

Therefore, the equation of Π simplifies to

$$w = 0.$$

■

We then determine the specific equation of the IK-sphere centered at $(1, 1, 1, 1)$ in the middle of the 4-cube. When the vector $\vec{\mathbf{r}}$ from the center of the IK-sphere has a magnitude $\|\vec{\mathbf{r}}\| = 0$, its components $\langle x, y, z, w \rangle = \langle 1, 1, 1, 1 \rangle$ because they must be the center coordinates of the 4-sphere because a vector with magnitude of zero is a point. Thus, we obtain the equation for our IK-sphere centered at $(1, 1, 1, 1)$. Recall that $r_4 = 1$ (see Equation 2.2.1), giving

$$(x-1)^2 + (y-1)^2 + (z-1)^2 + (w-1)^2 = 1. \tag{2.2.5}$$

The coordinate(s) of intersection between the plane and IK-sphere where $0 \leq x, y, z \leq 2$ can now be solved for. Substituting $w = 0$ from the equation of the plane Π (see Equation 2.2.4) into the equation of the IK-sphere (see Equation 2.2.5) and simplifying,

$$\begin{aligned}(x-1)^2 + (y-1)^2 + (z-1)^2 + 1 &= 1 \\ \implies (x-1)^2 + (y-1)^2 + (z-1)^2 &= 0,\end{aligned}$$

we can solve for x, y, and z because for any arbitrary value α,

$$\forall \alpha \in \mathbb{R},\ (\alpha - 1)^2 \geq 0.$$

And since $(x-1)^2 + (y-1)^2 + (z-1)^2 = 0$, the only choice is that

$$\begin{cases} (x-1)^2 = 0 \\ (y-1)^2 = 0 \\ (z-1)^2 = 0 \end{cases}$$

$$\begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$$

Thus, the intersection between the IK-sphere and its bounding 4-cube's $w = 0$ face occurs at the only one solution, $(1, 1, 1, 0)$, which is directly in the center of the face as expected.

In response to the research question, it has been verified that an IK-Sphere does share a coordinate with its bounding n-cube at $n = 4$ dimensions. Additionally, we provided algebraic reasoning for the radius of an IK-Sphere tending to infinity by proving that the limit of the expression for the radius of an IK-Sphere in terms of n approaches infinity as $n \to \infty$. However, a more visual high dimensional intuition can be provided through geometric reasoning as detailed in the following chapter.

Chapter 3

Body: Geometric Reasoning

Why does the radius of an IK-Sphere tend to infinity as dimensionality increases and does it truly intersect with its bounding box in $n \geq 4$ dimensions?

Research Question

Geometric reasoning can aid in developing a better visual intuition for higher dimensional phenomena. This chapter aims to reason why the IK-Sphere radius diverges in terms of space consumption (volume). We analyse the volumes of the IK-Sphere, the unit n-spheres surrounding the IK-Sphere, and the bounding n-cube with a side length of 2 units on which the unit n-spheres are situated, and determine the effects each has on the other.

Definition 3.0.1 (Volume)**.** Here, volume has been generalised across n dimensions as the number of unit n-cubes of space a shape occupies. For example, 'length' is volume in one-dimensional space, 'area' is volume in two-dimensional space, as is 'volume' in three-dimensional space.

3.1 The Volume of the n-Sphere

The volume of an n-sphere is given as [8]:

$$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} r^n \tag{3.1.1}$$

As the derivation requires spherical coordinates and multi-variable calculus, it will be avoided. In place of a derivation, the following are examples of its use in deriving the well known two and three-dimensional volumes:

1. Two Dimensional Case: Let $n = 2$,

$$\begin{aligned} V_2(r) &= \frac{\pi^{\frac{2}{2}}}{\left(\frac{2}{2}\right)!} r^2 \\ &= \pi r^2. \end{aligned}$$

2. Three Dimensional Case: Let $n = 3$,

$$V_3(r) = \frac{\pi^{\frac{3}{2}}}{\left(\frac{3}{2}\right)!} r^3$$

We may use the gamma function (extended factorial function for all real values excluding integer negative values [3]) where $\Gamma(x+1) = x!$ as documented by [5] to calculate $\left(\frac{3}{2}\right)!$.

$$\begin{aligned} \left(\frac{3}{2}\right)! &= \Gamma\left(\frac{3}{2} + 1\right) \\ &= \frac{3}{4}\sqrt{\pi} \\ \therefore V_3(r) &= \frac{\sqrt{\pi}^3}{\frac{3}{4}\sqrt{\pi}} r^3 \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

3.2 Implications of the Higher Dimensional Unit Sphere on the IK-Sphere

First, we must determine how the space consumption of the unit n-sphere changes as dimensionality n increases. As $n \to \infty$, a unit n-sphere's volume will tend to zero [1], resulting in the volume (and hence, the radius) of the IK-Sphere, which expands to fill the space between the surrounding unit n-spheres, eventually expanding to infinity.

Lemma 3.2.1. *As dimensionality increases a unit sphere's volume converges to zero.*

Proof. First, notice that r_n will always equal 1 because it is a unit sphere and that $r^n = 1^n$ will always equals 1 for all $n \in \mathbb{N}$. Thus, we can simplify the expression (Equation 3.1.1) on the right hand side to

$$V_n = \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}.$$

Next, the limit of the right hand side can be taken as n approaches infinity:

$$\lim_{n\to\infty} \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}.$$

To evaluate this limit we can use the squeeze theorem whereby if the factors of an expression tend towards a limit, then that expression itself must also tend to the same limit. Letting $m = \frac{n}{2}$ allows for a simpler expansion of the expression, namely

$$\begin{aligned}\frac{\pi^m}{m!} &= \left(\frac{\pi}{m} \times \frac{\pi}{m-1} \times ... \times \frac{\pi}{1}\right)\\ &= \left(\frac{\pi}{1} \times ... \times \frac{\pi}{m-1} \times \frac{\pi}{m}\right).\end{aligned}$$

As can be seen with π as a constant on the numerator, the fractions are being multiplied by increasingly smaller fractions in the product $(... \times \frac{\pi}{m-2} \times \frac{\pi}{m-1} \times \frac{\pi}{m})$. As a result, as m

increases, this product will approach 0. More rigorously, by observing that

$$\begin{aligned}
\frac{\pi^m}{m!} &= \frac{\pi}{1} \times \frac{\pi}{2} \times \cdots \frac{\pi}{m-1} \times \frac{\pi}{m} \\
&\leq \frac{\pi}{1} \times \frac{\pi}{1} \times \cdots \frac{\pi}{1} \times \frac{\pi}{m} \\
&\to 0
\end{aligned}$$

approaches zero and also,

$$0 \leq \frac{\pi^m}{m!}$$

Thus, by the squeeze theorem,

$$\begin{aligned}
0 \leq & \frac{\pi^m}{m!} \leq \frac{\pi}{1} \times \frac{\pi}{1} \times \cdots \frac{\pi}{1} \times \frac{\pi}{m} \\
\therefore \lim_{m\to\infty} 0 \leq \lim_{m\to\infty} & \frac{\pi^m}{m!} \leq \lim_{m\to\infty} \frac{\pi}{1} \times \frac{\pi}{1} \times \cdots \frac{\pi}{1} \times \frac{\pi}{m} \\
0 \leq \lim_{m\to\infty} & \frac{\pi^m}{m!} \leq 0 \\
\therefore \lim_{m\to\infty} & \frac{\pi^m}{m!} = 0
\end{aligned}$$

Therefore, as $\frac{n}{2} \to \infty$ (because $m = \frac{n}{2}$), i.e. as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} = 0$$

as required. ■

Graphically, the limit of V_n as $n \to 0$ can be observed (see Figure 3.1). Note that the dotted lines indicate integer dimensions because fractional dimensions are undefined. Because the radius of the IK-Sphere expanding to 'kiss' the unit spheres centered on the vertices of the bounding n-cube, if the volume of the unit n-spheres converges to zero as dimensionality increases, that would explain why the radius reaches out to infinity to 'kiss' them – it never can 'kiss' them. Recall that the distance between the centre of

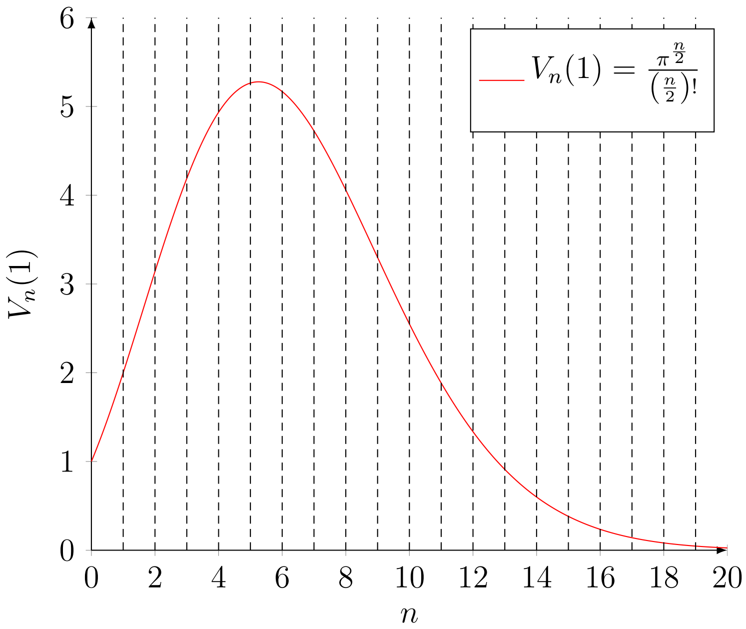


Figure 3.1: Volume of unit n-sphere tends to zero as dimension n decreases.

the IK-Sphere and any corner of the n-cube gets arbitrarily large according to $\sqrt{n}$ (see Equation 2.1.2). In this case, we would expect the volume of the IK-Sphere to diverge to infinity alongside its radius as it expands to touch the unit n-spheres surrounding it. However, it may also be noticed that until $n = 5$, the volume of the unit n-sphere increases. In contrast, this would imply that the volume of the IK-Sphere could decrease for $n \leq 5$. Nevertheless, the volume ultimately tends to infinity.

Theorem 3.2.2. *As the number of dimensions $n \to \infty$, the volume of an IK-Sphere will tend to infinity to match the decreasing volume of its surrounding unit n-spheres and increasing diameter of its bounding n-cube.*

Proof. We are required to evaluate the limit as r_n and n approach infinity of the IK-Sphere's volume. Expressed in limit form, this is

$$\lim_{r,n\to\infty} V_n(r) = \lim_{r,n\to\infty} \frac{\pi^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} r^n$$

In this case, r_n is dependent on n and can be written as $r_n = \sqrt{n} - 1$ (see Equation 2.1.3) as we derived in Chapter 1. Therefore, the limit of two variables can be converted to a one variable limit and so becomes easier to solve. Substituting $r_n = \sqrt{n} - 1$, the volume function can be rewritten as

$$V_n(r_n) = \lim_{n\to\infty} \frac{\pi^{\frac{n}{2}} \left(\sqrt{n} - 1\right)^n}{\left(\frac{n}{2}\right)!}.$$

Let $m = \frac{n}{2}$ and expand the right hand side expression (without the limit notation) as a product to get

$$\frac{\pi^m \left(\sqrt{2m} - 1\right)^{2m}}{m!} = \frac{\pi(\sqrt{2m} - 1)^2}{m} \times \frac{\pi(\sqrt{2m} - 1)^2}{m-1} \times \cdots \times \frac{\pi(\sqrt{2m} - 1)^2}{1}. \tag{3.2.1}$$

The last factor has a limit at infinity when $m \to \infty$ as follows:

1. Claim:

$$\lim_{m\to\infty} \pi \left(\sqrt{2m} - 1\right)^m = \infty.$$

Subproof. Notice that

$$\pi \left(\sqrt{2m} - 1\right)^m > \sqrt{2m} - 1$$

for all $m > 0, m \in \mathbb{N}$.

We can see that $\lim_{m\to\infty} \sqrt{2m} - 1 = \infty$, thus implying that no matter how large $\sqrt{2m} - 1$ is (i.e. $m \to \infty$), $\pi(\sqrt{2m} - 1)^m$ will always be larger. So, when $m \to \infty$, $\pi(\sqrt{2m} - 1)^m > \sqrt{2m} - 1 \implies \lim_{m\to\infty} \pi(\sqrt{2m} - 1)^m = \infty$. □

We must ensure that the other factors

$$\frac{\pi(\sqrt{2m} - 1)^2}{m} \times \frac{\pi(\sqrt{2m} - 1)^2}{m-1} \times \frac{\pi(\sqrt{2m} - 1)^2}{m-2} \times \cdots \tag{3.2.2}$$

do not equal zero, otherwise $0 \times \infty$ is undefined [6]. Thus, realise that the denominators

$m, m-1, m-2, \cdots$, all approaching infinity are equivalent to evaluating whether or not

$$\lim_{m\to\infty} \frac{\pi\left(\sqrt{2m}-1\right)^2}{m}$$

approaches zero. We may begin by expanding the power:

$$\lim_{m\to\infty} \frac{\pi\left(\sqrt{2m}-1\right)^2}{m} = \lim_{m\to\infty} \frac{\pi\left(2m-2\sqrt{2m}+1\right)^2}{m}.$$

Then using L'Hospital's Rule and calculating the derivatives of the numerator and denominator with respect to m, the limit can be evaluated to be 2π as follows:

$$\begin{aligned}
\lim_{m\to\infty} \frac{\pi\left(2m-2\sqrt{2m}+1\right)^2}{m} &= \lim_{m\to\infty} \frac{2\pi m - 2\pi\sqrt{2m}+\pi}{m} \\
&= \lim_{m\to\infty} \frac{\mathrm{d}}{\mathrm{d}m}\left[\frac{2\pi m - 2\pi\sqrt{2m}+\pi}{m}\right] \\
&= \lim_{m\to\infty} 2\pi \quad \underbrace{-2\pi\left(\frac{\sqrt{2}}{2}\cdot\frac{1}{\sqrt{m}}\right)}_{\to 0 \text{ because } \lim_{x\to\infty}\frac{1}{x}=0} \\
&= \lim_{m\to\infty} 2\pi \\
&= 2\pi.
\end{aligned}$$

Therefore, as long as $m > 0$ (otherwise division by zero and $0 \times \infty$ are undefined [6]) then $V_n(r_n) \to \infty$ due to the last multiplied term (see Equation 3.2.1) approaching infinity and none of the preceding terms approaching zero. Thus, the limit as $n \to \infty$ of $V_n(r_n)$ is ∞. ■

Upon examining the graph of the IK-Sphere Volume as n increases (see Figure 3.2), there is no decrease in volume for $n \leq 5$ as was expected due to the volume of the unit n-sphere increasing before $n = 5$ (see Figure 3.1). Hence, another factor is contributing to the constant increase in volume. It may be noticed, however, that the volume increase after $n = 5$ becomes much steeper. This suggests that the surrounding unit sphere volumes increasing before $n = 5$ are having some effect on the volume of the IK-Sphere. Consequently, we may now realise that the IK-Sphere volume also depends on the volume of its bounding n-cube on whose vertices the unit spheres are situated on.

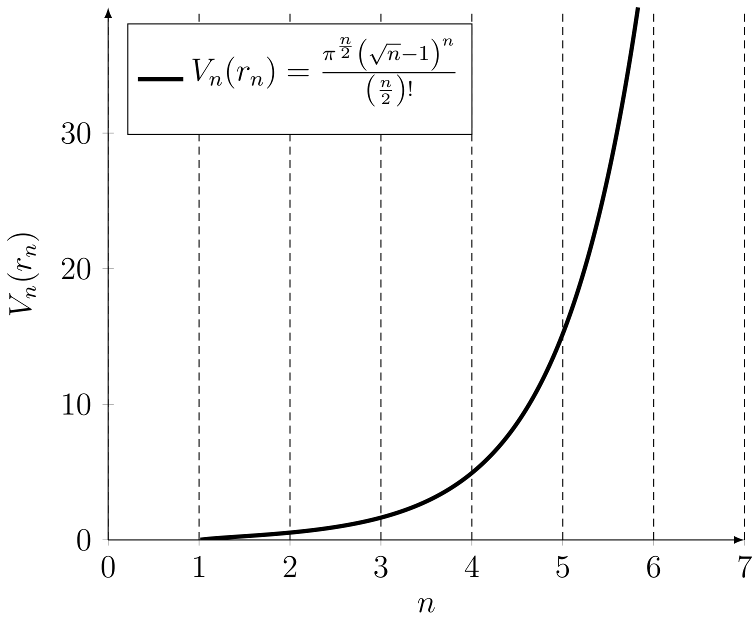


Figure 3.2: Volume of IK-sphere tends to infinity as dimension n decreases.

In order to come to a conclusion, we may graph the volumes of the IK-Sphere, unit n-sphere and bounding n-cube to visualise how the IK-Sphere expands to fill the space between its surrounding n-spheres (see Figure 3.3). The volume of the bounding n-cube with side length 2 units is given by $V_n = 2^n$ and again, the dotted lines indicate integer dimensions because fractional dimensions are undefined.

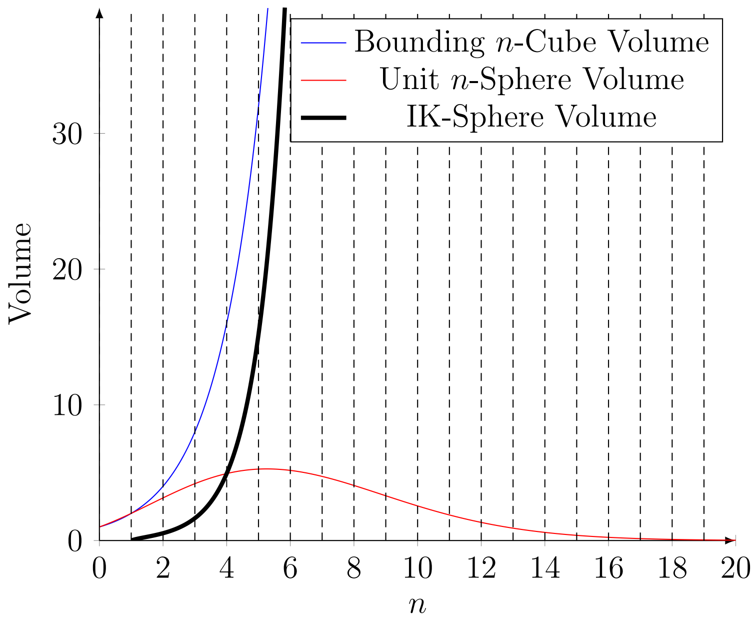


Figure 3.3: Volume of IK-sphere tends to infinity as dimension n decreases.

As can be seen, the increase in the unit n-sphere volume until $n = 5$ (red line) causes the IK-Sphere volume (bold-black line) to increase less rapidly. However, the exponential increase in the bounding n-cube volume (blue line) offsets the increase in unit n-sphere volume that would have caused the IK-Sphere volume to decrease for $n < 5$. As such, the IK-Sphere volume is seen to be always increasing while somewhat at a slower rate before $n = 5$. Nevertheless, after $n = 5$, the combination of the surrounding unit n-cube volumes tending to zero and the volume of the bounding n-cube on which they are situated tending to infinity causes the IK-Sphere volume to diverge to infinity. Thus, providing a geometric reason to answer why the radius of an IK-Sphere tends to infinity as dimensionality increases.

Chapter 4

Conclusion

4.1 Conclusion

In conclusion, to answer the research question:

RQ: Why does the radius of an IK-Sphere tend to infinity as dimensionality increases and does it truly intersect with its bounding box in $n \geq 4$ dimensions?

There were two parts to the question. Firstly, we answer why the radius of an IK-Sphere tends to infinity as dimensionality increases: The algebraic reason would be that when the limit of the expression for the radius r_n of an IK-Sphere in terms of dimension n was evaluated as n approached infinity, the radius tended to infinity as follows (see Theorem 2.1.2):

$$r_n = \sqrt{n} - 1$$
$$\lim_{n \to \infty} \sqrt{n} - 1 = \infty.$$

However, such abstraction was not very useful in understanding higher dimensions. As an aid to one's higher dimensional intuition, the geometric reason was that as the dimension increased, the space, determined by the volume of the unit n-spheres the IK-Sphere 'kisses', between unit spheres became infinitely large because the unit n-spheres' volumes approached zero (see Lemma 3.2.1) and the volume of the bounding n-cube on which those spheres are situated also tended to infinity. As a result, the radius of the IK-Sphere expanded to fill the space and its volume also tended to infinity (see Theorem 3.2.2) in an attempt to 'kiss' its bounding unit spheres.

The second part of the question was whether the IK-Sphere truly intersected with its 'bounding' box in $n > 4$ dimensions or if it was merely higher dimensional phenomena making it seem so. We found (see Chapter 2: Section 2.2) that the IK-Sphere in 4 dimensions intersected at the coordinate $(1, 1, 1, 0)$, directly in the center of the face of its bounding cube's $w = 0$ face, just as expected. Thus, we also conclude that there is truth to an IK-Sphere intersecting with its bounding n-cube as dimensionality increases past $n = 4$.

4.2 Implications and Further Investigation

The conclusion has implications relating to what is known as the curse of dimensionality for high dimensional data sets in machine learning [9]. We found that as we increase dimensions, that the amount of space between points increases drastically (see Figure 3.2 and 'Bounding n-Cube Volume' in Figure 3.3). As a result, when data has too many variables, the machine learning algorithm has trouble pinpointing attributes and patterns because the data is so vastly spread out. One may investigate possible dimensional elimination techniques to mitigate this issue. Additionally, the conclusion has implications for error correcting codes. For example, when a CD or DVD has a scratch on it, changing the encoded data, an error correcting code is used to fill in and 'repair' the missing data. If each coordinate in high dimensional space is a piece of data, the space is separated into as many spheres as possible and the data read from the disk is corrected to the coordinate of the center of the sphere it exists within. Thus, the closer together we can get the spheres, the more accurate data corrections we can achieve. However, we learned that the space between spheres can become increasingly larger as dimension increases which means that as dimensions increase, even more spheres can be packed around each other. Until now, this sphere packing problem has only been proven up to eight dimensions and in 24 dimensions [7]. With our newfound intuition of higher dimensions, could we prove the remaining dimensions? It is unlikely. However, such an investigation could reveal more phenomena to add to our bank of higher dimensional intuition.

Bibliography

[1] KB Athreya. "Unit ball in high dimensions". In: *Resonance* 13.4 (2008), pp. 334–342. DOI: https://doi.org/10.1007/s12045-008-0014-0. URL: https://link.springer.com/article/10.1007%2Fs12045-008-0014-0.

[2] Leonard Bakker. *Math 341 Lecture 8, 2.3: The Algebraic and Order Limit Theorems.* 2017. URL: https://math.byu.edu/~bakker/M341/Lectures/Lec08.pdf.

[3] James Bonnar. "The gamma function". In: Applied Research Press, 2014. ISBN: 1677804866.

[4] Barry Cipra. "What's Happening in the Mathematical Sciences". In: vol. 1. American Mathematical Society, 1993, p. 25. ISBN: 9780821889992.

[5] Detlef Gronau. "Why is the gamma function so as it is". In: *Teaching Mathematics and Computer Science* 1 (2003), pp. 43–53. URL: https://imsc.uni-graz.at/gronau/TMCS_1_2003.pdf.

[6] Saeed Al-Hajjar. "Indeterminate forms and their behaviours". In: *WSEAS Transactions on Mathematics* 7.11 (2008), pp. 661–662.

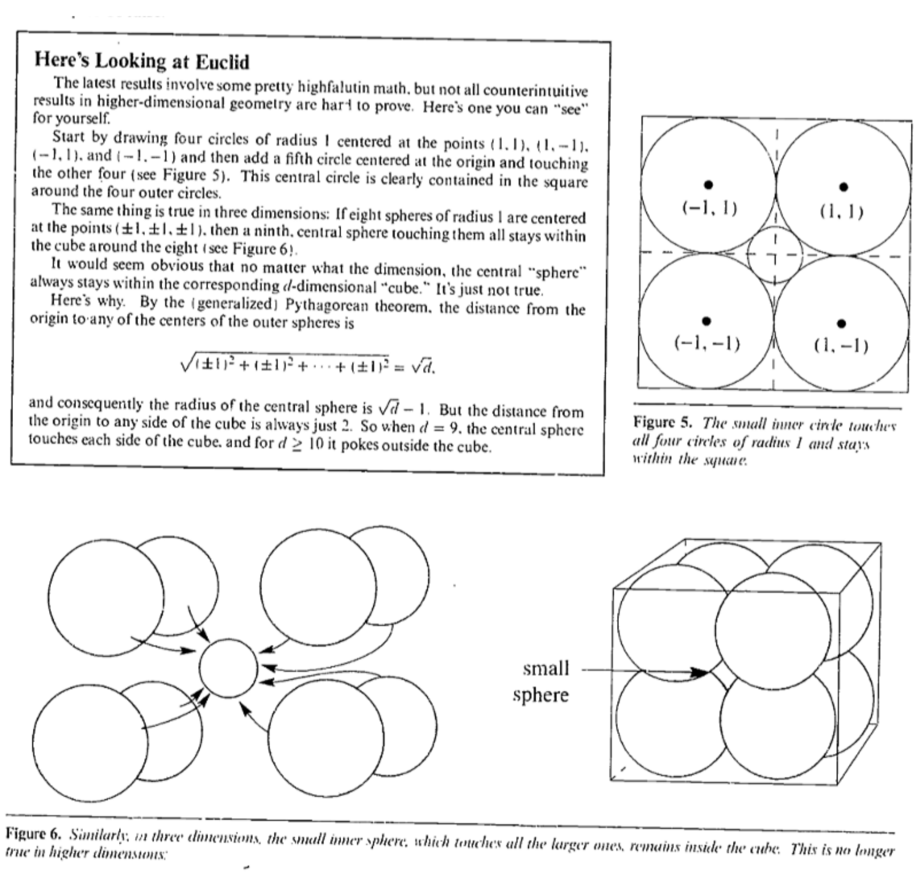
[7] Erica Klarreich. "Sphere packing solved in higher dimensions". In: *Quanta Magazine* (2016). URL: http://reprints.gravitywaves.com/UED/Klarreich-2016_Sphere%20Packing%20Solved%20in%20Higher%20Dimensions_Quanta%20Magazine.pdf.

[8] Alex Lopez-Ortiz. *Formula for the Surface Area of a sphere in Euclidean N-Space.* 1998. URL: https://cs.uwaterloo.ca/~alopez-o/math-faq/node75.html.

[9] Nir Raviv et al. "perm2vec: Graph permutation selection for decoding of error correction codes using self-attention". In: *arXiv preprint arXiv:2002.02315* (2020). DOI: 10.1109/JSAC.2020.3036951. URL: https://arxiv.org/abs/2002.02315.

Appendix A

Original IK-Spheres Clipping



Here's Looking at Euclid

The latest results involve some pretty highfalutin math, but not all counterintuitive results in higher-dimensional geometry are hard to prove. Here's one you can "see" for yourself.

Start by drawing four circles of radius 1 centered at the points $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$ and then add a fifth circle centered at the origin and touching the other four (see Figure 5). This central circle is clearly contained in the square around the four outer circles.

The same thing is true in three dimensions: If eight spheres of radius 1 are centered at the points $(\pm 1, \pm 1, \pm 1)$, then a ninth, central sphere touching them all stays within the cube around the eight (see Figure 6).

It would seem obvious that no matter what the dimension, the central "sphere" always stays within the corresponding d-dimensional "cube." It's just not true.

Here's why. By the (generalized) Pythagorean theorem, the distance from the origin to any of the centers of the outer spheres is

$$\sqrt{(\pm 1)^2 + (\pm 1)^2 + \cdots + (\pm 1)^2} = \sqrt{d},$$

and consequently the radius of the central sphere is $\sqrt{d} - 1$. But the distance from the origin to any side of the cube is always just 2. So when $d = 9$, the central sphere touches each side of the cube, and for $d \geq 10$ it pokes outside the cube.

Figure 5. *The small inner circle touches all four circles of radius 1 and stays within the square.*

Figure 6. *Similarly, in three dimensions, the small inner sphere, which touches all the larger ones, remains inside the cube. This is no longer true in higher dimensions.*

Figure A.1: Original IK-Spheres in *What's Happening in the Mathematical Sciences* (Attributed to Barry Cipra [4])