# Exercises

CAL1 2025
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Please note that the exercise compendium may be subject to change during the semester.
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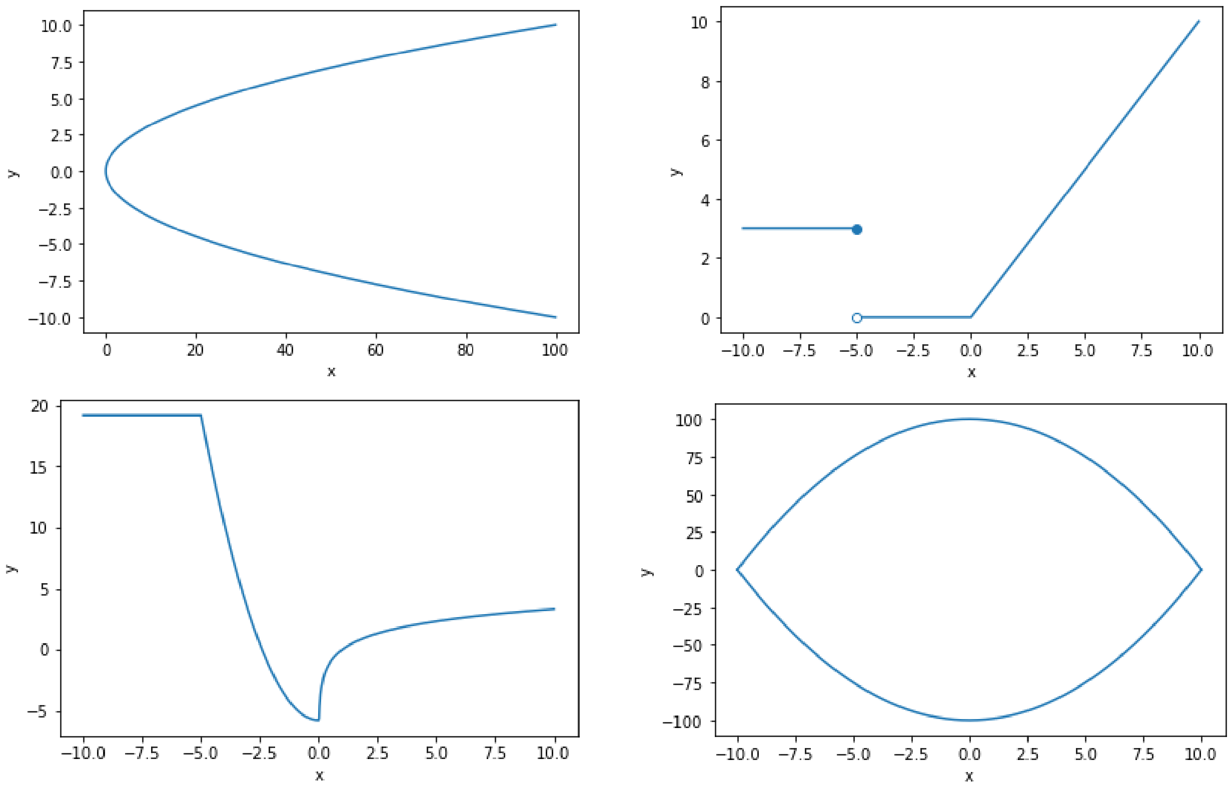
# A. Functions, limits and continuity

## A1. To function or not to function

Which of the graphs below are graphs of functions of $x$, and which are not? Make sure to give an argument for your answers.




## A2. Graphs of continuous functions

For the graphs in A1 that *are* graphs of functions, are the functions continuous? If not, what type of discontinuities are present?

## A3. Potential energy

According to Hooke's law, the potential energy $U$ possessed by a stretched spring is proportional to the square of its displacement, $x$. If $U = 1{,}045$ joules when $x = 10$ cm, what is $U$ when $x = 15$ cm?

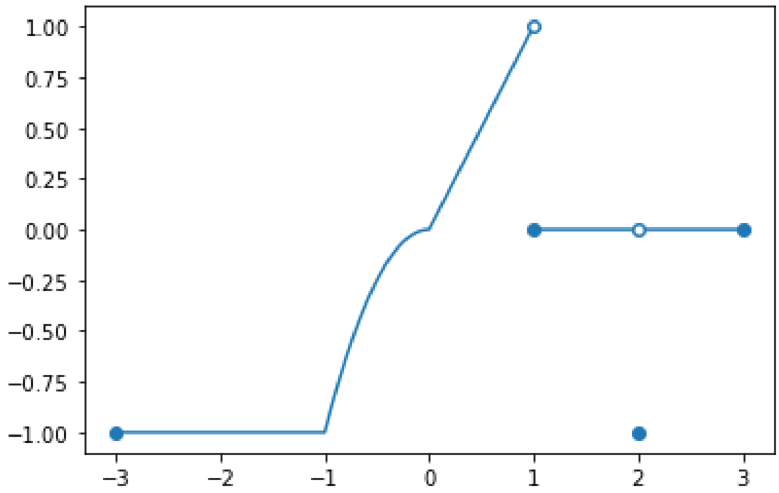
## A4. Composite functions

Complete the following table:

| $f(x)$ | $g(x)$ | $(f \circ g)(x)$ |
|---|---|---|
| $x^2$ | $\frac{1}{x}$ | |
| | $x^3$ | $\frac{1}{x^3 - 1}$ |
| $\ln(x)$ | | $2\ln(x)$ |
| | $\sin^2(x)$ | $\cos^2(x)$ |

## A5. True or false?

Which of the following statements about the function $f(x)$ graphed below are true, and which are false?




i. $\lim_{x \to -1} f(x)$ does not exist.
ii. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$.
iii. $f(x)$ is continuous at every point in the open interval $(1,2)$.
iv. $f(x)$ is continuous at every point in the open interval $(1,3)$.
v. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$.
vi. $\lim_{x \to 2} f(x) = 0$.
vii. $\lim_{x \to 1} f(x) = 0$.
viii. $f(x)$ is discontinuous at $x = 1$, and the discontinuity is removable.
ix. $f(x)$ is discontinuous at $x = 2$, and the discontinuity is removable.

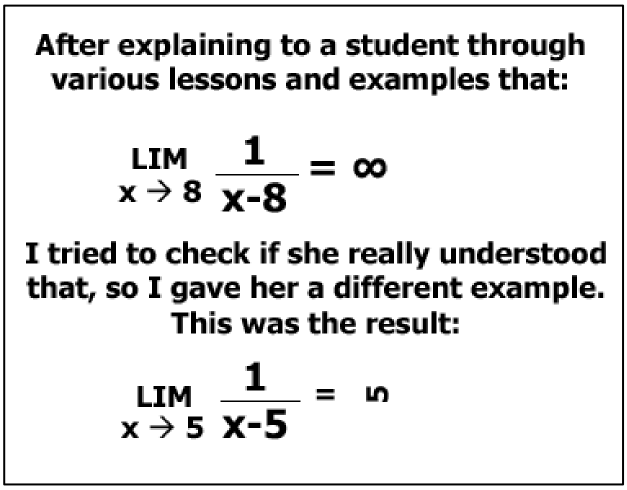
## A6. Continuous functions

Sketch the following functions and comment on whether there are discontinuities present.

i. $f(x) = x^2$
ii. $f(x) = x^{-1}$
iii. $f(x) = e^{-x}$
iv. $f(x) = \tan(x)$

## A7. An old math joke

An old math joke[1] goes like this:




Obviously, the second equation is wrong. But so is the first.

i. What error does the teacher make?
ii. Does over-analyzing a math joke take the fun out of it?

## A8. A mysterious function

Professor Mathson, cryptic as she is, tells you that the function $f(x)$ has the property:

$$\lim_{h\to 0} f(c+h) = f(c)$$

at every point $c$ in its domain. Explain, in no more than five words, what professor Mathson has really told you about the function $f(x)$.

## A9. The sandwich theorem

Use the sandwich theorem to show that

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

As a starting point, consider the fact that $-1 \leq \sin x \leq 1$ for all $x$ (*i.e.*, $\sin\ x$ never exceeds 1 and never subceeds -1).

[1] Unscrupulously stolen from joi.ito.com/weblog/2004/04/01/math-joke.html.

# B. Derivatives

## B1. Calculating derivatives

Calculate the following derivatives (using pen and paper):

| | | | | | |
|---|---|---|---|---|---|
| i. | $\frac{d}{dx}(x^2 - 4x^3)$ | ii. | $\frac{d}{dx}\sin(x^2)$ | iii. | $\frac{d}{dx}(\ln(x)\, e^x)$ |
| iv. | $\frac{d}{dx}\pi$ | v. | $\frac{d^2}{dx^2}x^{3/2}$ | vi. | $\frac{d}{dx}\left(1 + \frac{1}{x}\right)^3$ |
| vii. | $\frac{d}{dx}\sin(\cos(x))$ | viii. | $\frac{d^2}{dx^2}\sin(x)$ | ix. | $\frac{d}{dy}\left(y - \frac{1}{y}\right)$ |

## B2. The limit definition of a derivative

Find the derivative of $f(x) = 3x^2 + 7x - 1$ using the limit definition of a derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

and confirm that it matches what you expect.

## B3. The derivative of $x^x$

For some functions, like $f(x) = x^x$, it is not straight-forward to find the derivative using standard rules. In such cases, we usually have to have a really good idea! Recall that for $z > 0$, $z = e^{\ln z}$, and as such, we can rewrite the function $f(x) = e^{\ln f(x)}$. With this really good idea, calculate the derivative of $x^x$ (for $x > 0$).

## B4. A mysterious derivative

Professor Mathson (not her again!) tells you that the function $f$ has the property:

$$f = \frac{d^4 f}{dx^4}$$

She then asks you to suggest no less than *three* candidates for what the function $f$ might be.

## B5. L'Hôpital's rule

Where applicable, use l'Hôpital's rule to evaluate the following limits. Otherwise, find a different way of evaluating the limit:

| | | | | | |
|---|---|---|---|---|---|
| i. | $\lim_{x \to 0} \frac{\tan x}{x}$ | ii. | $\lim_{x \to -\infty} xe^x$ | iii. | $\lim_{x \to \infty} xe^x$ |

## B6. Big-Oh

In the study of the run time of algorithms, it is a well-known fact that, as $n$ approaches infinity, $O(\log n)$ algorithms are faster than $O(n^k)$ algorithms for any $k > 0$. Use l'Hôpital's rule to show this. For simplicity, use $\ln n$ instead of $\log n$.

# C. Optimization and Newton's method

## C1. Extreme values

For each of the following functions, identify the position of all extreme values, if there are any. State whether they are minima or maxima, and whether they are local (relative) or global (absolute). Remember to check endpoints if the function is defined on a closed interval.

| | | | | | |
|---|---|---|---|---|---|
| i. | $-x^2 + 3x$ | ii. | $x^2 e^{-x}$ | iii. | $3x - 5$ |
| iv. | $\ln(x) - \sqrt{x}$ | v. | $\sqrt{x}$ | vi. | $\sin(x)$ |

## C2. The hydrogen atom

The hydrogen atom consists of a proton and an electron. The *radial distribution function* $P(r)$ tells us the probability that the distance between the proton and the electron is $r$. Given that

$$P(r) = \frac{4}{a_0^3} r^2 e^{-\frac{2r}{a_0}}$$

find the most probable distance between the proton and the electron (*i.e.*, the distance $r$ where the probability is largest). In the equation above, $a_0 = 52.9$ pm is a constant known as the Bohr radius.

## C3. Crossing a lake[2]

A woman on the shore of a circular lake with radius 3 km wants to arrive at the point diametrically opposite her in the shortest possible time. She can walk at a speed of 6 km/h and row a boat at 3 km/h. How should she proceed? In other words, should she (1) just walk around the lake, (2) just row across it or (3) row at an angle and end up some distance away from her destination and walk the rest of the way?

In solving this exercise, follow the five-step procedure presented in the lecture.

## C4. A game of chance

Professor Mathson (oh, come on!) offers you a game of chance. She rolls a six-sided dice as many times as you like. Every time the dice lands on 1, 2, 3, 4 or 5, she adds 100 Danish crowns to a pile on the table. You can stop after any number of dice rolls, and take the money on the table. Should the dice land on a 6, however, she takes the money and leaves (without even saying goodbye!). How many times do you want Professor Mathson to roll the dice? [HINT]

## C5. Estimating π

Use Newton's method to estimate the value of $\pi$. Take $\pi$ to be the solution to the equation $\sin(x) = 0$, and let your initial guess be $x_0 = 3$. Continue until you're satisfied.

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[2] This exercise is heavily inspired by exercise 4.6.32 in *Calculus: Concepts and Contexts* by James Stewart. Brooks/Cole. 3rd edition, 2006.

## C6. Newton’s method for optimization

Use Newton’s method for optimization to identify critical points in any one of the exercises C1-C4, you choose which one. Compare with your analytical solution.

# D. Integrals

## D1. Indefinite integrals

Calculate the following indefinite integrals. Check your answers using differentiation.

| | | | | | |
|---|---|---|---|---|---|
| i. | $\int (x+1)dx$ | ii. | $\int \sin\left(\frac{x}{4}\right)dx$ | iii. | $\int \sqrt{x}dx$ |
| iv. | $\int (x^2+2x)dx$ | v. | $\int e^{2x}dx$ | vi. | $\int (x^5-4x^2)dx$ |

## D2. Riemann sums

Use the definition of a Riemann sum,

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\cdot\frac{b-a}{n}$$

to estimate the area under the curve $f(x) = x^2$ on the interval [0,2]. Start with $n = 1$ and increase $n$ until you just can't stand it anymore. Then calculate the area under the curve as a definite integral instead.

## D3. Stirling's approximation

Stirling's approximation is an approximation used to estimate the value of $n!$ for large values of $n$:

$$\ln(n!) \approx n\ln(n) - n$$

In this exercise, we will justify Stirling's approximation.

i. Use logarithmic rules to write $\ln(n!)$ as the sum of $n$ terms.
ii. If any of the terms happen to be zero, get rid of them.
iii. Draw a graph of the function $f(x) = \ln x$.
iv. Consider the part of the graph between $x = 1$ and $x = n$. Write an integral to calculate the area below the graph in this interval.
v. Argue that this area is approximately equal to the sum in part i (think about Riemann sums).
vi. Evaluate the integral and use the result to justify the equation above.

This exercise shows that, although we usually use sums to estimate integrals, we can also use integrals to estimate sums.

## D4. Integration by substitution

Use integration by substitution to calculate the integrals:

| | | | |
|---|---|---|---|
| i. | $\int_0^{\pi} (\cos x)^2 \sin x \; dx$ | ii. | $\int_0^{2} xe^{x^2}\; dx$ |

## D5. Integration by parts

Use integration by parts to calculate the integrals:

i. $\displaystyle\int_{-\pi/2}^{\pi/2} x^2 \sin x \; dx$  ii. $\displaystyle\int_{-1}^{1} (3x-7)e^{x}\, dx$

## D6. The integral of ln(x)/x

Professor Mathson wants you to calculate the indefinite integral of the function $\ln(x)\,/x$. Demanding as she is, she is not satisfied that you do it in just one way, and wants you to do it in two different ways: Integration by substitution and integration by parts. Show, using both methods, that

$$\int \frac{\ln x}{x}\, dx = \frac{1}{2}(\ln x)^2 + C$$

[HINT]

## D7. Making a list, checking it twice

List at least three examples of integrals you would evaluate using integration by substitution and at least three examples of integrals you would evaluate using integration by parts. Don't just copy integrals from the slides or from exercises D4-6. That would be too boring.

# E. Integration techniques

## E1. Even and odd functions

State whether the following functions are even, odd or neither:

| | | | | | |
|---|---|---|---|---|---|
| i. | $x^2 \exp(-x^2)$ | ii. | $x^2 + x^3$ | iii. | $\frac{d}{dx}\lvert x\rvert$ |
| iv. | $\sin(x)\cos(x)$ | v. | $\sin(\cos(x))$ | vi. | $\sin(x^3)$ |
| vii. | $e^x$ | viii. | $\sin^5(x)$ | ix. | $1 + x^2$ |

## E2. The area of a circle

A circle of radius $r$ centered in at the origin is defined by the equation:

$$x^2 + y^2 = r^2$$

In this exercise, we will show that the area of a circle is $\pi r^2$.

i. The equation above cannot be expressed as $y = f(x) =$ some function of $x$ whose graph is a circle. Why not?

ii. Rewrite the equation to get a function $y = f(x)$ for the part of the circle in the first quadrant (*i.e.*, both $x$ and $y$ take only positive values). Hopefully, you reach the following expression:

$$f(x) = \sqrt{r^2 - x^2} \quad \text{with} \quad x > 0$$

iii. Write an integral that will allow you to calculate the area of this quarter circle. Think about your integration limits.

iv. Rewrite the integral using the appropriate trigonometric substitution. Draw the corresponding triangle and find expressions for $x$ and $dx$ in terms of $\theta$ and $d\theta$. Make sure you change both your integrand, $dx$ and your integration limits. You can find inspiration in the PowerPoint slides.

v. Reduce the expression. Hopefully, it reaches the form:

$$\int_0^{\frac{\pi}{2}} r^2 \cos^2\theta \, d\theta$$

vi. Use the following trigonometric identity to linearize the expression:

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

vii. Evaluate the integral and multiply the result by 4 to get the area of the entire circle. With any luck, you should get $\pi r^2$.

## E3. Trigonometric substitution

Use the method of trigonometric substitution to show that

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

You can either find inspiration in the PowerPoint slides or challenge yourself by making the substitution $x = \tan\ u$, find the relationship between $dx$ and $du$, substitute and consider what $1 + \tan^2 u$ is equal to. For an extra challenge, don't look up the derivative of the tangent function when you need it, but derive it using the quotient rule.

## E4. Estimating π (again)

In exercise C5, we used Newton's method to get an estimate of $\pi$. Now, let's use numerical integration for the very same task. For this purpose, we take advantage of the fact that

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$

and thus

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = 1$$

This means that $\tan^{-1}(1) = \frac{\pi}{4}$ or $\pi = 4\tan^{-1}(1)$. Given that $\tan^{-1}(0) = 0$, and based on your work in exercise E3, we can define $\pi$ as:

$$\pi = 4\tan^{-1}(1) = 4(\tan^{-1}(1) - \tan^{-1}(0)) = \int_0^1 \frac{4}{1+x^2}\,dx$$

Use Simpson's rule to estimate $\pi$. Choose $n$ as you wish, as long as it is an even number. How large is the relative error? Did you do better than in exercise C5?

## E5. Integral tables

Evaluate the following integrals using an integral table. All integrals can be solved using the table in [T], but you are welcome to use another resource. Always remember to check whether you're integrating an odd function across a symmetric interval before you begin.

| | | | | | |
|---|---|---|---|---|---|
| i. | $\int_0^{\pi} \sin(3x)\cos(3x)\,dx$ | ii. | $\int_0^{\frac{\pi}{4}} \tan(x)\,dx$ | iii. | $\int_1^5 xe^{3x}\,dx$ |
| iv. | $\int_{-1}^{1} e^{-x^2}\left(\frac{d}{dx}e^{-x^2}\right)dx$ | v. | $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1+\cos(2x)}$ | vi. | $\int_{-\pi}^{\pi} x^2 \sin 7x\,dx$ |

Note: When using integral tables, you may come across the secant (sec), cosecant (csc) and cotangent (cot) functions. These functions complement the standard trigonometric functions as $\sec(x) = 1/\cos(x)$, $\csc(x) = 1/\sin(x)$ and $\cot(x) = 1/\tan(x)$.

# F. Improper integrals

## F1. Improper integrals

Evaluate the following improper integrals, all of which converge:

| | | | | | |
|---|---|---|---|---|---|
| i. | $\displaystyle\int_1^\infty \frac{3}{x^2}\,dx$ | ii. | $\displaystyle\int_{-2}^2 \frac{1}{\sqrt{\lvert x\rvert}}\,dx$ | iii. | $\displaystyle\int_{-\infty}^0 e^z\,dz$ |
| iv. | $\int_0^1 \ln x\ \,dx$ [HINT] | v. | $\displaystyle\int_1^\infty \frac{dx}{x^\pi}$ | vi. | $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ |

## F2. The direct comparison test

Use the direct comparison test to determine whether the following improper integrals converge or diverge:

| | | | | | |
|---|---|---|---|---|---|
| i. | $\displaystyle\int_1^\infty \frac{e^x}{x}\,dx$ | ii. | $\displaystyle\int_\pi^\infty \frac{\cos^2(x)}{x^2}\,dx$ | iii. | $\displaystyle\int_1^\infty \frac{e^{-x}}{x}\,dx$ |

## F3. Approximating a tricky integral

In this exercise, we will attempt to estimate the value of the improper integral

$$\int_0^\infty e^{-x^2}dx$$

We know for a fact that the integrand has no elementary antiderivative, and as such, calculating the value of the integral is no easy task. Using methods that are beyond the scope of this course, it can be shown that this particular definite integral equals $\sqrt{\pi}/2$. We will try to see if we can estimate this number.

The idea is the following: We split the integral in two parts:

$$\int_0^\infty e^{-x^2}dx = \int_0^a e^{-x^2}dx + \int_a^\infty e^{-x^2}dx$$

We then evaluate the first term using Simpson's rule and eliminate the second term using the direct comparison test. If $a$ is too small, the second term will be large and the result inaccurate; if $a$ is too large, we will need a lot of terms in the Simpson calculation to get something accurate. Thus, we need a good compromise value of $a$.

i. Argue that $\int_a^\infty e^{-x^2}dx \leq \int_a^\infty e^{-ax}dx$ for any positive value of $a$.
ii. Evaluate the integral $\int_a^\infty e^{-ax}dx$ and decide on a value of $a$ such that you feel confident that $\int_a^\infty e^{-x^2}dx$ is small enough to be negligible.
iii. Use Simpson's rule to evaluate the first term. Play around with the number of intervals, $n$, until you feel confident that the estimate is reasonably accurate.

iv. With your chosen values of $a$ and $n$, estimate $\int_0^\infty e^{-x^2}\, dx$.

v. Multiply your result by 2, and then square it. How close do you get to $\pi$?

## F4. The limit comparison test

Use the limit comparison test to determine whether the following improper integrals converge or not.

i. $\displaystyle\int_5^\infty \frac{\sqrt{x} + \tan^{-1} x}{3x^2 - x - 1}\, dx$ ii. $\displaystyle\int_e^\infty \frac{2x^x - e^x}{x^{x-1} + e^{x+1}}\, dx$ iii. $\int_1^\infty \sin\left(\frac{1}{x^2}\right)\, dx$ [HINT]

# G. Infinite series

## G1. Geometric series

State whether the following geometric series converge or diverge by identifying $a$ and $r$, and calculate the sum if they converge:

i. $\sum_{n=1}^{\infty} \pi^{n-1}$ ii. $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{n-1}$ iii. $\sum_{n=1}^{\infty} \frac{1}{3^{n-2}}$

## G2. Estimating an infinite series

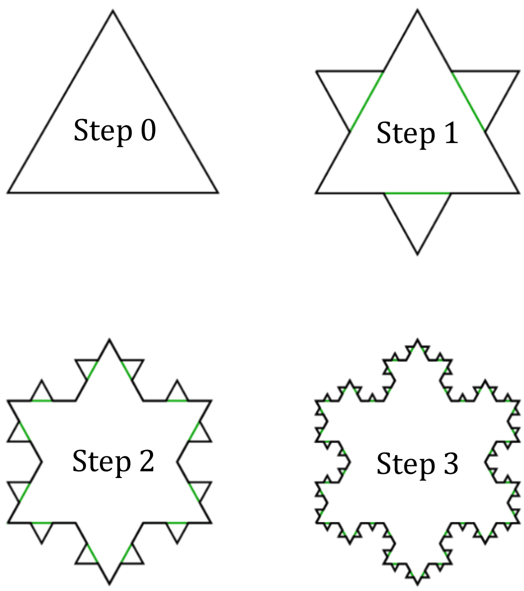
Consider the infinite series

$$\sum_{n=1}^{\infty} e^{-n}$$

Estimate the sum by calculating the $n$th partial sum $s_n$ (choose your own $n$) and estimate the remainder $R_n$. Use this to estimate the value of the series. Compare with the exact value, equal to $1/(e-1)$.

## G3. The snowflake curve[3]

To construct the *snowflake curve*, start with an equilateral triangle with sides of length 1. Then divide each side into three equal parts, construct an equilateral triangle on the middle part, and the delete the middle part. Then repeat for each side in the resulting polygon, and repeat indefinitely – see the figure.[4]




[3] This exercise is heavily inspired by exercise 3 in the *Focus on problem solving* section at the end of chapter 8 of *Calculus: Concepts and Contexts* by James Stewart. Brooks/Cole. 3rd edition, 2006.

[4] Figure taken from https://en.wikipedia.org/wiki/Koch_snowflake (yes, I just cited Wikipedia)

First, we show that the total length of the polygon diverges. Let $N_n$, $L_n$ and $P_n$ be the number of sides, the length of a side and the total length of the polygon at step $n$.

i. Write an equation for $N_n$ in terms of $n$ (consider the fact that the number of sides increases by a factor of 4 every time $n$ increases by one). [HINT]
ii. Write an equation for $L_n$ in terms of $n$ (consider the fact that the length of a side decreases by a factor of 3 every time $n$ increases by one).
iii. Then write an equation for $P_n = N_n L_n$ and show that the sequence $P_n \to \infty$.

Now we show that the total area of the polygon converges. Let $T_n$ and $a_n$ be the number of new triangles and the area of a new triangle at step $n$.

iv. How many new triangles are added in step $n$ (*i.e.*, what is $T_n$?)? [HINT]
v. Write an equation for $a_n$ in terms of $n$ and $a_0$ (consider the fact that the area of a triangle decreases by a factor of 9 every time $n$ increases by one).
vi. Now write the total area as a series of the form:

$$A_n = a_0 + \sum_{k=1}^{n} T_k a_k$$

vii. Show that the series converges, and find the area of the infinite snowflake in terms of $a_0$.

So, in conclusion, the snowflake curve is infinitely long, but encloses a finite area. This is actually a real problem with many shapes – in particular, when trying to measure the length of the coastline of an island or a country. Similar to the snowflake curve, as precision is increased, the area tends to a finite value, but the coastline may tend to infinity! This is known as the *coastline paradox*.

# H. Taylor and Maclaurin series

## H1. Estimating $e$

Use the Maclaurin expansion of $e^x$ to estimate the value of $e$. Decide for yourself how many terms you include. How close do you get?

## H2. Taylor polynomials

For the following functions, find the Taylor polynomials of order 2 generated by $f$ at $a$. Use plotting software to confirm that your polynomials are good descriptions around $a$.

i. $f(x) = \ln x\,,\ \ a = 1$  ii. $f(x) = \sqrt{x},\ \ a = 3$  iii. $f(x) = 1/x,\ \ a = 2$

## H3. The Maclaurin series of the sinc function

As we know, the function $\sin(x)/x$ is defined at every point except $x = 0$, where it is discontinuous.

i. What type of discontinuity is present at $x = 0$?
ii. What is $\lim\limits_{x\to 0} \sin(x)/x$?

Given the answers in parts i and ii, we can easily define a continuous function, known as the sinc function, as follows:

$$\operatorname{sinc}(x) = \begin{cases} \sin(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

We wish to find the Maclaurin expansion of the sinc function.

iii. For $x \neq 0$, find the Maclaurin series of $\operatorname{sinc}(x)$ based on the Maclaurin expansion of $\sin(x)$. Write it using both sigma notation and by writing out the first few terms.
iv. Confirm that the Maclaurin series from iii. works (converges to the right value) at $x = 0$.

## H4. Solving equations

Find approximate solutions to the following equations using series expansions. Then use Newton's method to do the same. Compare the results.

i. $e^{-x} = x$  ii. $\cos(2x) = x$

## H5. The Maclaurin series of the arctangent function

Recall that the Maclaurin series of $1/(1+x)$ is:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

i. Use this to find a series expansion of $1/(1+x^2)$. Write the result using both sigma notation and by writing out the first few terms.

ii. Now recall that $1/(1+x^2)$ is the derivative of $\tan^{-1} x$ (exercise E3). Use this to show that the Maclaurin series of $\tan^{-1} x$ is: **[HINT]**

$$f(x) = \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

iii. With $f(x) = \tan^{-1} x$, argue that $f^{(2n)}(0) = 0$ if $n$ is an integer. (*i.e.*, all even-order derivatives are 0 at $x = 0$). **[HINT]**

iv. Show that the odd-order derivatives at $x = 0$ follow the equation:

$$f^{(2n+1)}(0) = (-1)^n \cdot (2n)!$$

v. What is $\frac{d^9}{dx^9} \tan^{-1}(x)$ at $x = 0$?

vi. The Maclaurin series of $\tan^{-1} x$ above converges for $|x| \leq 1$. In particular, we can use it to calculate $\tan^{-1} 1 = \pi/4$ (exercise E4). Based on this, write an infinite series that converges to $\pi$. This series is known as the Leibniz formula for $\pi$.

# I. Partial derivatives

## I1. Calculating partial derivatives

Calculate all first-order partial derivatives of the following functions:

i. $f(x,y,z) = xye^{z}$ ii. $f(x_1,x_2) = x_2\cos(x_1)$ iii. $f(x,y) = x^{y}$

iv. $f(x,y) = 5xe^{-y} + 3y$ v. $f(x_1,x_2,x_3,\dots,x_n) = \sum_{k=1}^{n} x_k^2$

## I2. Gradient vectors

For the functions in parts iii and iv of exercise I1, calculate the gradient vector at the point $(x,y) = (2,2)$.

## I3. Critical points

Find the critical points of the function $f(x,y) = x^3 - y^2 - 2xy + 6$. Are they saddle points, maxima or minima?

## I4. Gradient ascent

The gradient descent procedure is designed to find minima. How, asks Professor Mathson with a smirk, would one need to change the procedure to find maxima? And, she adds, with an even larger smirk, could the gradient descent procedure ever get stuck in a maximum?

## I5. Gradient descent

Use the gradient descent method with a learning rate of $\eta = 0.1$ to find the minimum of the function $f(x,y) = (x-2y)^2 + 2(x-3)y$. Start at $(x,y) = (1.1, 0.9)$ and stop when both partial derivatives are less than 0.05.

## I6. The importance of the learning rate

Consider the function $f(x,y) = x^2 + y^2 + x - y$. We want to find a minimum of this function using the gradient descent procedure

- starting at $(x_0,y_0) = (1,1)$
- with a convergence criterion that the magnitude of both partial derivatives must be less than 0.01.

How many steps it takes to converge depends on the learning rate. For this function, using learning rates between 0.1 and 1.2, we get convergence in the number of steps indicated below:

| Learning rate | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Steps to converge | 26 | 12 | 7 | 4 | 1 | 4 | 7 | 12 | 26 | Diverges | | |

i. Show that with a learning rate of 0.5, the procedure converges in a single step.
ii. Similarly, show that with a learning rate of 1.0, the procedure never converges.

# J. Multiple integrals

## J1. Evaluating multiple integrals

Evaluate the following integrals:

i. $$\int_0^2 \int_{-1}^1 (x - y)\, dydx$$

ii. $$\int_{-1}^2 \int_0^{\frac{\pi}{2}} y \sin x\, dxdy$$

iii. $$\int_1^4 \int_1^e \frac{\ln x}{xy}\, dxdy$$

iv. $$\int_0^\pi \int_0^{\sin x} y\, dydx$$

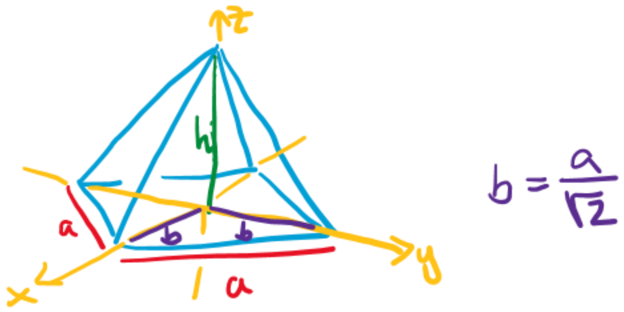
In part iv, sketch the region of integration.

## J2. The volume of a pyramid

In this exercise, we will derive the equation for the volume of a pyramid,

$$V = \frac{1}{3} a^2 h$$

where $a$ is the side length of the base and $h$ is the height, as indicated on the sketch below.




We place the pyramid in a coordinate system as indicated with the orange arrows in the sketch above.

i. Consider the part of the coordinate system where $x \geq 0$ and $y \geq 0$. Argue that the side of the pyramid in the part of the coordinate system must be expressed by a function $z = f(x, y)$ that fulfills the following:
   - $f(0,0) = h$
   - $f(x, 0) = h - \frac{h}{b} x$
   - $f(0, y) = h - \frac{h}{b} y$

ii. Given that $f(x, y)$ represents a plane, it must have the form $f(x, y) = A + Bx + Cy$ (a linear function in both $x$ and $y$). Based on this, and on your arguments in i., write an expression for $f(x, y)$.

iii. Write a double integral that expresses the volume of the quarter-pyramid and multiply by 4 to get the volume of the entire pyramid. Evaluate the integral. [HINT]

## J3. Monte Carlo integration

Use Monte Carlo integration to evaluate

$$\int_0^2 \int_0^1 (4 - x - y)\, dydx$$

and compare with the exact value.

## J4. A double integral that equals a geometric series[5]

Consider the improper integral

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dxdy$$

i. Why is it improper?

ii. The integrand $1/(1 - xy)$ can be rewritten as an infinite (geometric) series – how so?

iii. Use the series expansion of $1/(1 - xy)$ to evaluate the integral and show that

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dxdy = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

iv. Estimate the left-hand side using Monte Carlo integration and estimate the right-hand side with the same method as in exercise G2. Compare with the exact value of $\pi^2/6$.

[5] This exercise is heavily inspired by exercise 5 in the *Focus on problem solving* section at the end of chapter 12 of *Calculus: Concepts and Contexts* by James Stewart. Brooks/Cole. 3rd edition, 2006.

# K. Differential equations

## K1. Solving differential equations

Solve the following differential equations by separation of variables:

i. $\frac{dz}{dt} = \sqrt{zt}$ ii. $\frac{dy}{dx} = y^2 \sin(x)$ iii. $(x^2 + 1)\frac{dy}{dx} = xy$

## K2. Administering a drug

Suppose a drug is administered intravenously at a constant rate of 5 mg per minute. The drug is quickly and uniformly distributed throughout the bloodstream. The rate at which the drug is eliminated from the body is proportional to the amount of drug present in the body at any time, with a constant of proportionality $k = 0.02 \text{ min}^{-1}$. Set up and solve a differential equation whose solution is the amount of the drug in the bloodstream as a function of time.

## K3. Linear differential equations

A first-order *linear* differential equation is a differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of $x$. It cannot be solved using separation of variables, so we must do it in a different way. In this exercise, we will show that the solution is

$$y(x) = \frac{1}{v(x)} \int v(x)\, Q(x) dx$$

where $v(x) = e^{\int P(x) dx}$.

i. Multiply the differential equation with $v(x)$.
ii. Take the derivative of $v(x) \cdot y$ to show that
$$\frac{d}{dx}(v(x) \cdot y) = v(x)\frac{dy}{dx} + v(x)P(x)y$$
iii. Use the result in ii to simplify your expression from i.
iv. You should now have an equation of the form $\frac{d}{dx}A = B$. Integrate both sides to get $A = \int B dx$.
v. Isolate $y$ to get the expression for the solution above.
vi. Solve the following differential equation:

$$\frac{dy}{dx} - \frac{3}{x}y = x^3, \qquad x > 0$$

## K4. Euler's method

Use Euler's method to solve the differential equation $y' = x + 1$ with $y(0) = 0$. Also solve the differential equation analytically and plot the results together.

# Hints for selected exercises

## C4.

From statistics, we know that the expected winnings after $n$ rolls is:

$$\begin{aligned}&\text{expected winnings}\\&\qquad= (\text{what you win after } n \text{ rolls if you don't land on a 6})\\&\qquad\times (\text{probability of not landing on a 6})^n\end{aligned}$$

Use that as the function to be maximized. [BACK]

## D6.

When doing integration by substitution, simply let $f(u) = u$. When doing integration by parts, you may stumble upon something that has the form $A = B - A$. If you want to know $A$, simply rearrange the equation. [BACK]

## F1iv.

You can use l'Hôpital's rule to get rid of the term $\lim_{a\to 0^+} a \ln a$. [BACK]

## IF4iii.

It is not obvious what the function $g(x)$ should be. However, we have established that

$$\lim_{z\to 0} \frac{\sin(z)}{z} = 1$$

Use the structure of the limit above to decide on a function $g(x)$. [BACK]

## G3i.

If $b_n = cb_{n-1}$, then $b_n = c^n b_0$. So, what is $N_n$ in terms of $N_0$ if $N_n = 4N_{n-1}$? [BACK]

## G3iv.

$T_n$ is equal to the number of sides in step $n - 1$. (Why?) [BACK]

## H5ii.

As you integrate, you will need to figure out what the integration constant $C$ should be. One way of doing this is by ensuring that the series expansion converges to the right value at $x = 0$ ($\tan^{-1} 0 = 0$). [BACK]

## H5iii.

Think of the parity of $f(x)$. Is it even or odd – or not at all? What does that mean for the value of $f(0)$? And for the derivatives? [BACK]

## J2iii.

Your integration area is a triangle defined by $y = 0$ and $y = b - x$. [BACK]

# Answers to most of the exercises

**... with no guarantee of correctness whatsoever**

A1. Top: no, yes. Bottom: yes, no. A2. Top: -, jump discontinuity. Bottom: continuous, -. A3. 2351 joules. A4. $1/x^2$, $1/(x-1)$, $x^2$, $1-x$. A5. i. False. ii. False. iii. True. iv. False. v. True. vi. True. vii. False. viii. False. ix. True. A6. i. None. ii. At $x=0$. iii. None. iv. Many. A7. i. The limit is not defined. ii. Inconclusive. A8. The function is continuous. A9. -. B1. i. $2x-12x^2$. ii. $2x\cos(x^2)$. iii. $e^x(\ln x+x^{-1})$. iv. 0. v. $3/(4\sqrt{x})$. vi. $-3x^{-2}(1+x^{-1})^2$. vii. $-\cos(\cos x)\sin x$. viii. $-\sin x$. ix. $1+y^{-2}$. B2. $6x+7$. B3. $x^x(1+\ln x)$. B4. -. B5. i. 1. ii. 0. iii. ∞. B6. -. C1. i. $x=3/2$. ii. $x=0$ and $x=2$. iii. None. iv. $x=4$. v. $x=0$. vi. $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\ldots$. C2. $r=a_0=52.9$ pm. C3. Just walk. C4. 5 or 6. C5. 3.14159 (2 iterations). C6. -. D1. i. $x^2/2+x+C$. ii. $-4\cos(x/4)+C$. iii. $2x^{3/2}/3+C$. iv. $x^3/3+x^2+C$. v. $e^{2x}/2+C$. vi. $x^6/6-4x^3/3+C$. D2. $S_1=8, S_2=5, S_3\approx 4.15$, and so on, integral $=8/3\approx 2.67$. D3. -. D4. i. 2/3. ii. $(e^4-1)/2\approx 26.8$. D5. i. 0. ii. $13e^{-1}-7e\approx 14.2$. D6. -. D7. -. E1. i. Even. ii. Neither. iii. Odd. iv. Odd. v. Even. vi. Odd. vii. Neither. viii. Odd. ix. Even. E2. -. E3. -. E4. 3.14157 with $n=4$. E5. i. 0. ii. $\approx 0.35$. iii. $\approx 5{,}085{,}138$. iv. 0. v. 1. vi. 0. F1. i. 3. ii. $\approx 5.66$. iii. 1. iv. $-1$. v. $\approx 0.47$. vi. 0.5. F2. i. Diverges. ii. Converges. iii. Converges. F3. i. -. ii. 0.0004 for $a=3$. iii. -. iv. 0.8862 for $a=3$ and $n=8$. v. 3.14144 for $a=3$ and $n=8$. F4. i. Converges. ii. Diverges. G1. i. Diverges. ii. Converges to 5/3. iii. Converges to 4.5. G2. Using $n=5$, $S\approx 0.5827$. G3. i. $N_n=3\cdot 4^n$. ii. $L_n=1/3^n$. iii. $P_n=3\cdot(4/3)^n$. iv. $T_n=3\cdot 4^n/4$. v. $a_n=a_0/9^n$. vi. $8a_0/5$. H1. 2.71806 for $n=6$. H2. i. $(x-1)-(x-1)^2/2$. ii. $\sqrt{3}+(x-3)/(2\sqrt{3})-(x-3)^2/(8\cdot 3^{3/2})$. iii. $1/2-(x-2)/4+(x-2)^2/8$. H3. i. Removable. ii. 1. iii. $\sum(-1)^n x^{2n}/(2n+1)!$. iv. -. H4. i. Taylor to 2nd order: 0.5858, Newton: $x_3=0.5671$ with $x_0=1$. ii. Taylor to 2nd order: 0.5000, Newton: $x_3=0.5149$ with $x_0=1$. H5. i. $\sum(-1)^n x^{2n}$. ii. -. iii. -. iv. -. v. 40320. vi. $4(1-1/3+1/5-1/7+1/9-1/11+\cdots)$. I1. i. $f_x=ye^z, f_y=xe^z, f_z=xye^z$. ii. $f_{x_1}=-x_2\sin x_1, f_{x_2}=\cos x_1$. iii. $f_x=yx^{y-1}, f_y=x^y\ln x$. iv. $f_x=5e^{-y}, f_y=-5xe^{-y}+3$. $f_{x_k}=2x_k$. I2. $\nabla f\approx[4,2.77]$ and $\nabla f\approx[0.68,1.65]$. I3. Saddle point at (0,0), maximum at (-2/3, 2/3). I4. -. I5. After 4 iterations, $(x_4,y_4)=(1.03504{,}1.01056)$ with $\nabla f|_{(x_4,y_4)}=(0.04896{,}0.0144)$. I6. -. J1. i. 4. ii. 1.5. iii. ln(2). iv. π/4. J2. i. -. ii. $h-hx/b-hy/b$. iii. $4\int_0^b\int_0^{b-x}(h-hx/b-hy/b)\,dy\,dx$. J3. Depends on the random numbers. With 10 random points, I got 5.173, but it will be different every time. J4. i. -. ii. $\sum(xy)^k$. iii. -. iv. LHS: depends on the random numbers. With 1,000,000 random points, I got 1.6434, but it will be different every time. RHS: 1.6452 using $n=10$. Exact value ≈ 1.6449. K1. i. $z=(t^3+6Ct^{3/2}+9C^2)/9$. ii. $y=(\cos x+C)^{-1}$. iii. $y=C\sqrt{x^2+1}$. K2. $y(t)=250(1-e^{-0.02t})$. K3. vi. $v(x)=x^{-3}$ and $y(x)=x^4+Cx^3$. K4. Analytical solution is $y=x^2/2+x$. Euler, step size 0.1, after 10 steps, $y(1)=1.45$ (analytical value 1.5).