OPTIMIZATION AND NEWTON'S METHOD

CALI 2025

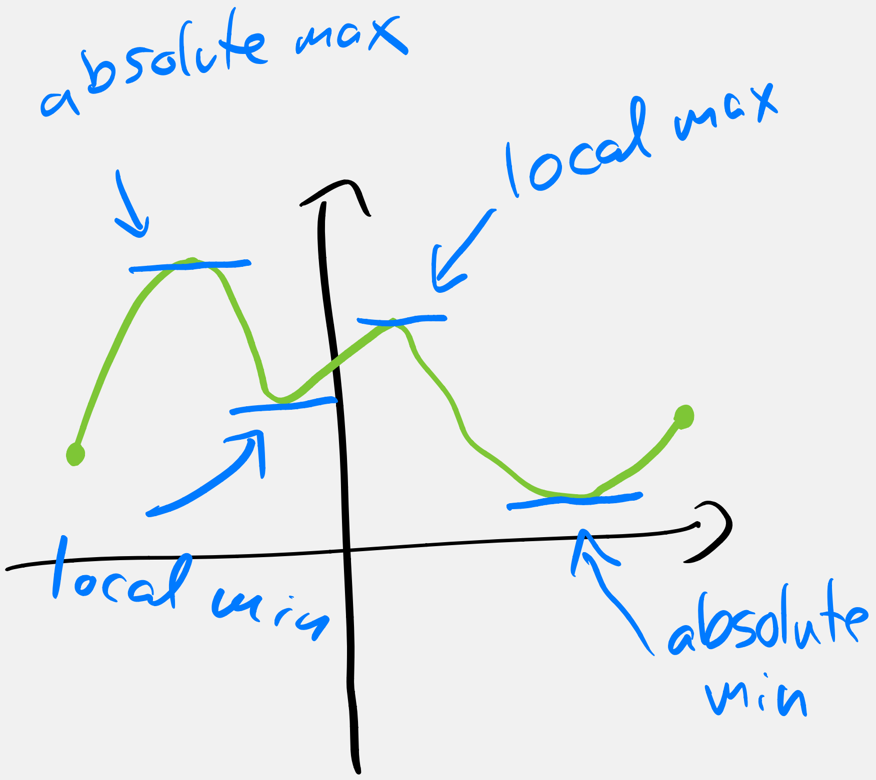
OPTIMIZATION AND NEWTON'S METHOD

- Optimization
  - Extreme values
  - Optimization problems
- Newton's method
  - … for finding roots
  - … for optimization

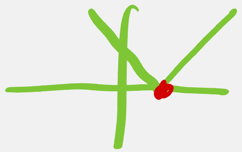
EXTREME VALUES

maxima & minima

$$\boxed{\frac{df}{dx} = 0}$$

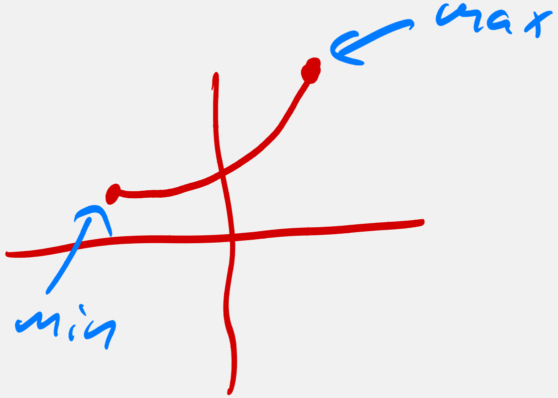


OTHER CANDIDATES FOR EXTREME VALUES



- points in the domain where $f'(x)$ is undefined

- end points of the domain



CRITICAL POINTS

- points where $f'(x)$ is zero or not defined

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OPTIMIZATION PROBLEMS

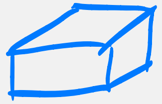
sustainability

- minimize packing material
- maximize profit
- maximize accuracy of ML model

OPTIMIZATION PROBLEMS

1. Read problem
2. Draw a picture
3. Introduce variables
4. Write an expression for the quantity to be optimized
5. Test critical points and end points

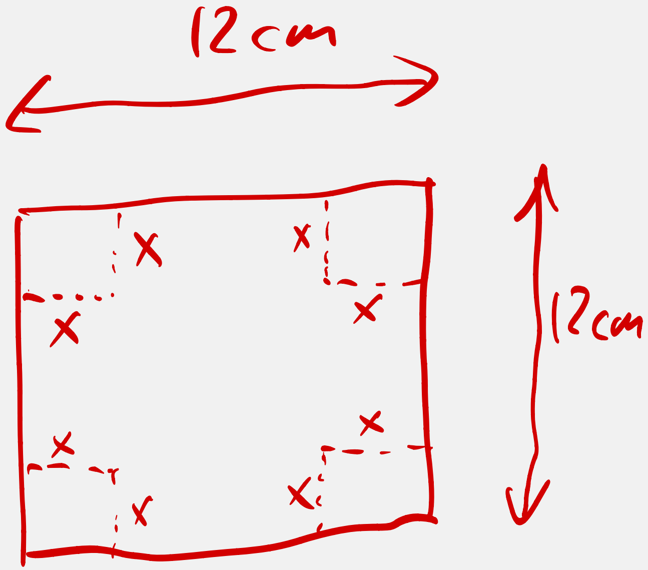
EXAMPLE 1: AN OPEN-TOP BOX



An open-top box is to be made by cutting small congruent squares from the corners of a 12-cm-by-12-cm sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

1. Read the problem

2. Draw a picture



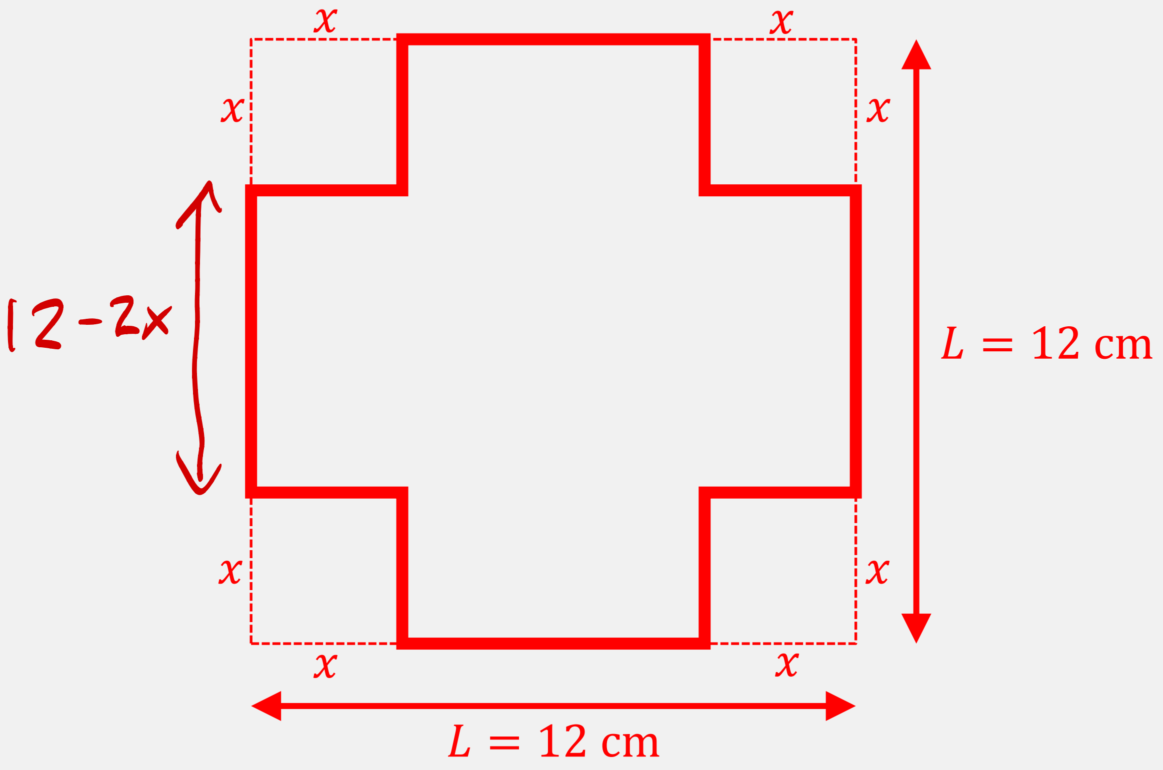
3. Introduce variables x, V

EXAMPLE 1: AN OPEN-TOP BOX

4. Write an equation

$V(x) = (L - 2x)^2 \cdot x$

$x \in [0; 6]$



EXAMPLE 1: AN OPEN-TOP BOX

5. Test the critical points and the endpoints

$$V = (L - 2x)^2 x = L^2 x - 4Lx^2 + 4x^3$$

$$\frac{dV}{dx} = \overbrace{L^2}^{c} \overbrace{- 8L}^{b} x + \overbrace{12}^{a} x^2 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8L \pm \sqrt{(-8L)^2 - 4 \cdot 12 \cdot L^2}}{2 \cdot 12} = \frac{8L \pm \sqrt{64L^2 - 48L^2}}{24}$$

$$= \frac{8L \pm \sqrt{16L^2}}{24} = \frac{8L \pm 4L}{24} = \begin{cases} \frac{L}{6} = 2\,\text{cm} \\ \frac{L}{2} = 6\,\text{cm} \end{cases}$$

end point

EXAMPLE 1: AN OPEN-TOP BOX

5. Test the critical points and the endpoints (cont.)

$$V = (L - 2x)^2 x$$

$V(2) = (12-4)^2 \cdot 2 = 64 \cdot 2 = 128 \text{ cm}^3$

max

choose 2cm

$V(0) = (12-0)^2 \cdot 0 = 0$

$V(6) = (12-12)^2 \cdot 6 = 0$

EXAMPLE II: A ONE-LITER CAN

You have been asked to design a one-liter can shaped like a right circular cylinder. What dimensions will use the least material?

1. Read the problem

2. Draw a picture

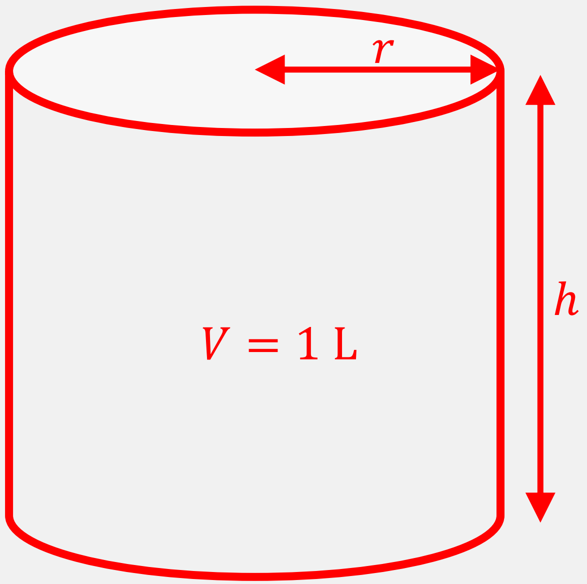
3. Introduce variables r, h, A

EXAMPLE II: A ONE-LITER CAN

4. Write an equation

$r \in (0; \infty)$

$$A = 2\pi r^2 + 2rh \cdot \pi$$



$$V = h\pi r^2 \iff h = \frac{V}{\pi r^2}$$

$$A(r) = 2\pi r^2 + 2\cancel{\pi} r \cdot \frac{V}{\cancel{\pi} r^{\cancel{2}}}$$

$$= 2\pi r^2 + \frac{2V}{r}$$

EXAMPLE II: A ONE-LITER CAN

5. Test the critical points and the endpoints

$$A = 2\pi r^2 + \frac{2V}{r}$$

$$h = \frac{V}{\pi r^2} \approx 10.8\text{ cm}$$

$$\frac{dA}{dr} = 4\pi r - \frac{2V}{r^2} = 0$$

$$4\pi r = \frac{2V}{r^2}$$

$$r^3 = \frac{V}{2\pi}$$

$$r = \sqrt[3]{\frac{V}{2\pi}} = \sqrt[3]{\frac{1}{2\pi}} \approx 5{,}4\text{ cm}$$

EXAMPLE II: A ONE-LITER CAN

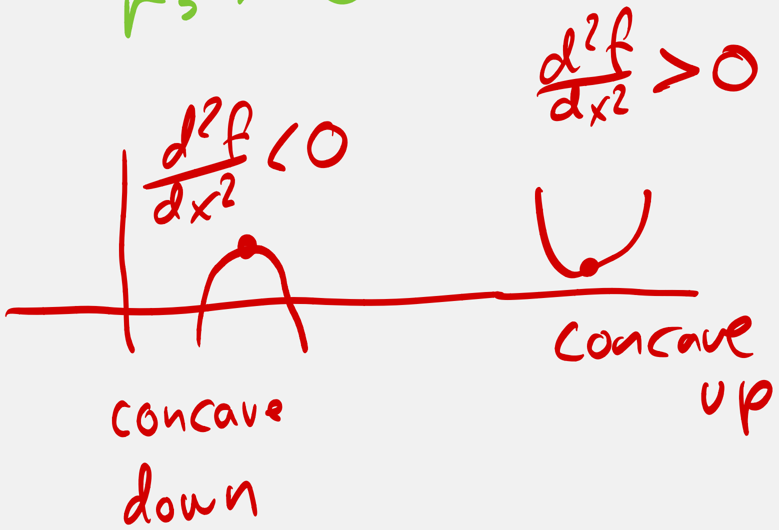
5. Test the critical points and the endpoints (cont.)

Is $r = \sqrt[3]{\frac{V}{2\pi}}$ a minimum? Yes

since $r, V > 0$

$$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(4\pi r - \frac{2V}{r^2}\right) = 4\pi + \frac{4V}{r^3} > 0$$

"2nd derivative test"



FIRST AND SECOND DERIVATIVES

	$\frac{df}{dx}$	$\frac{d^2f}{dx^2}$
Minimum	0	> 0
Maximum	0	< 0

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SOLVING EQUATIONS

$$3x - 6 = 0$$
$$3x = 6$$
$$x = 2$$

done!

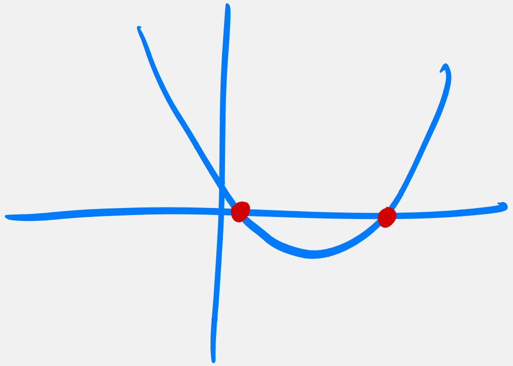
$$x^3 + 2x^2 - 3x + 7 = 0$$
$$\cos x = 3x$$
$$xe^x = 1$$

many equations don't have analytical solutions!

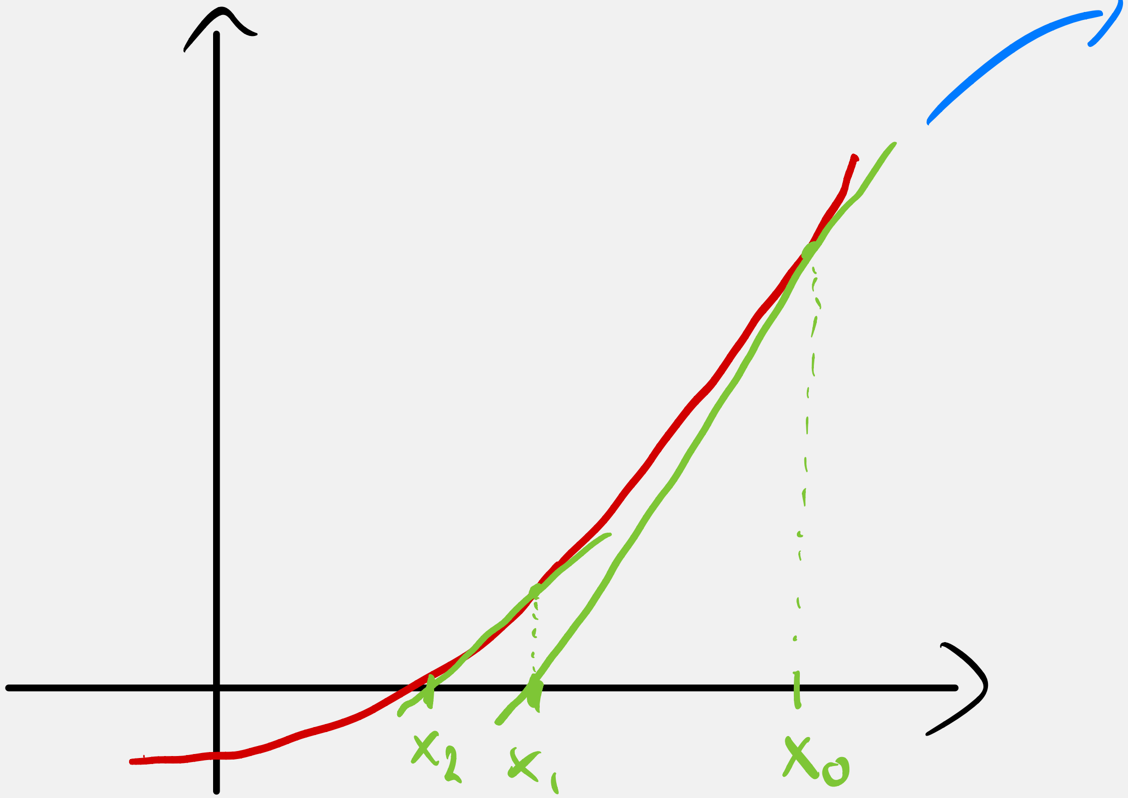
NEWTON'S METHOD
OR THE NEWTON-RAPHSON METHOD

Find approximate solutions to $\underline{f(x) = 0}$

roots



PROCEDURE FOR NEWTON'S METHOD



Equation for tangent

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$-f(x_0) = f'(x_0)(x_1 - x_0)$$

$$-\frac{f(x_0)}{f'(x_0)} = x_1 - x_0$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

PROCEDURE FOR NEWTON'S METHOD

General formula

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

ESTIMATING $\sqrt{2}$

$x^2 - 2 = 0$

$\sqrt{2}^2 - 2 = 2 - 2 = 0$

$\sqrt{2}$ is solution

is a root of

$f(x) = x^2 - 2$

$f'(x) = 2x$

$x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1	-1	2	$1 - \frac{-1}{2} = 1.5$
1	1.5	0.25	3	$1.5 - \frac{0.25}{3} = 1.4166\ldots$
2	$1.41\overline{6}$	$0.006\overline{94}$	$2.8\overline{3}$	$= 1.414216\ldots$
3	$1.414216\ldots$			

$\sqrt{2} = 1.41421\ldots$

SOLVING WEIRD EQUATIONS

$\cos x = 3x \Rightarrow \cos x - 3x = 0$

$f(x) = \cos x - 3x$

$f'(x) = -\sin x - 3$

Guess $x_0 = 1$

	A	B	C	D	E
1	n	xn	f(xn)	f'(xn)	xn+1
2	0	1	-2,4597	-3,84147	0,3597
3	1	0,3597	-0,14309	-3,35199	0,31701
4	2	0,31701	-0,00086	-3,31173	0,31675
5	3	0,31675	-3,2E-08	-3,31148	0,31675
6					

$x \approx 0.31675$

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NEWTON'S METHOD FOR OPTIMIZATION

Say we don't want to find roots, but critical points

$f'(x) = 0$

Then Newton's method can be modified:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$$

AN OPEN-TOP BOX

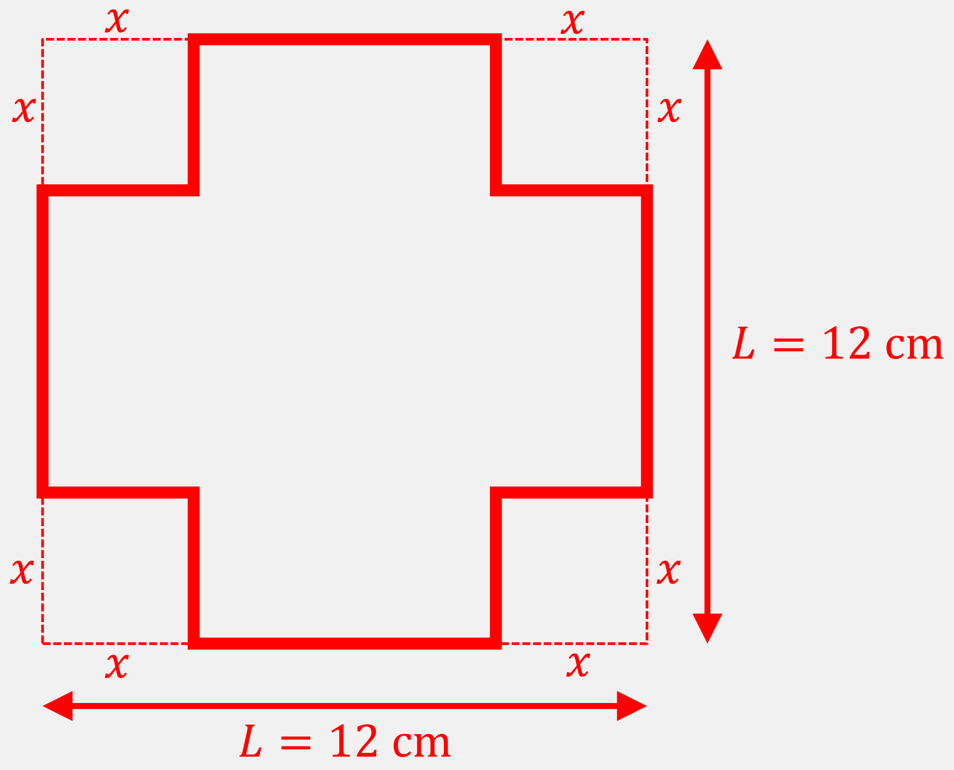
$$V = L^2x - 4Lx^2 + 4x^3$$

$$V'(x) = L^2 - 8Lx + 12x^2$$

$$= 144 - 96x + 12x^2$$

$$V''(x) = -8L + 24x$$

$$= -96 + 24x$$



AN OPEN-TOP BOX

$V'(x) = 144 - 96x + 12x^2$

$V''(x) = -96 + 24x$

```
def dv(x):
    return 144-96*x+12*x**2

def ddv(x):
    return -96+24*x

def next_step(x):
    return x-dv(x)/ddv(x)
```

```
x = 1
for i in range(10):
    x = next_step(x)
    print(x)
```

✓ 0.0s

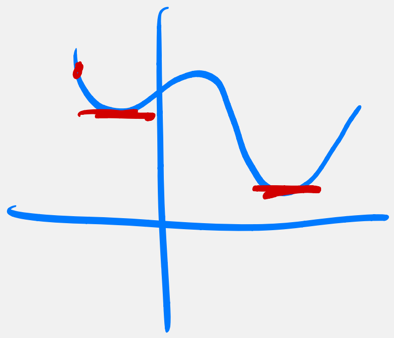
```
1.8333333333333335
1.9935897435897438
1.9999897599737855
1.9999999999737856
2.0
2.0
2.0
2.0
2.0
2.0
```

or

```
x = 1
while abs(dv(x)) > 1e-6:
    x = next_step(x)
    print(x)
```

✓ 0.0s

```
1.8333333333333335
1.9935897435897438
1.9999897599737855
1.9999999999737856
```



A NOTE OF CAUTION

- Newton's method
  - may diverge
  - may find a different solution than the one you want