FUNCTIONS, LIMITS AND CONTINUITY

CALI 2025

FUNCTIONS, LIMITS AND CONTINUITY

- Functions
- Limits
- Continuity

WHAT IS A FUNCTION?

$\underbrace{\text{Something}}$ depends on $\underbrace{\text{something else}}$

$\underbrace{\text{my height}}_{y}$ depends on $\underbrace{\text{my age}}_{x}$

y is a function x

$$y = f(x)$$

WHAT IS A FUNCTION?

Table

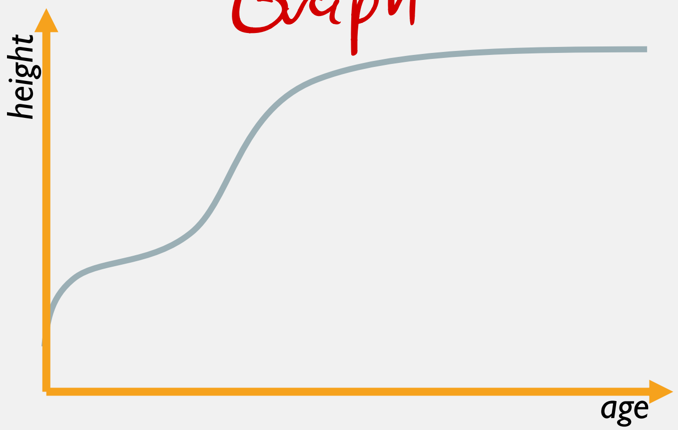
age / years	height / cm
0	51
5	102
10	112
15	159
20	182
25	182

height = f(age)

The height increases with age,
beginning at 51 cm and ending at 182 cm.
Two growth spurts appear: between the ages of 0 and 5 and again between 13 and 18.
After the age of 20, the height increases no more.

Description

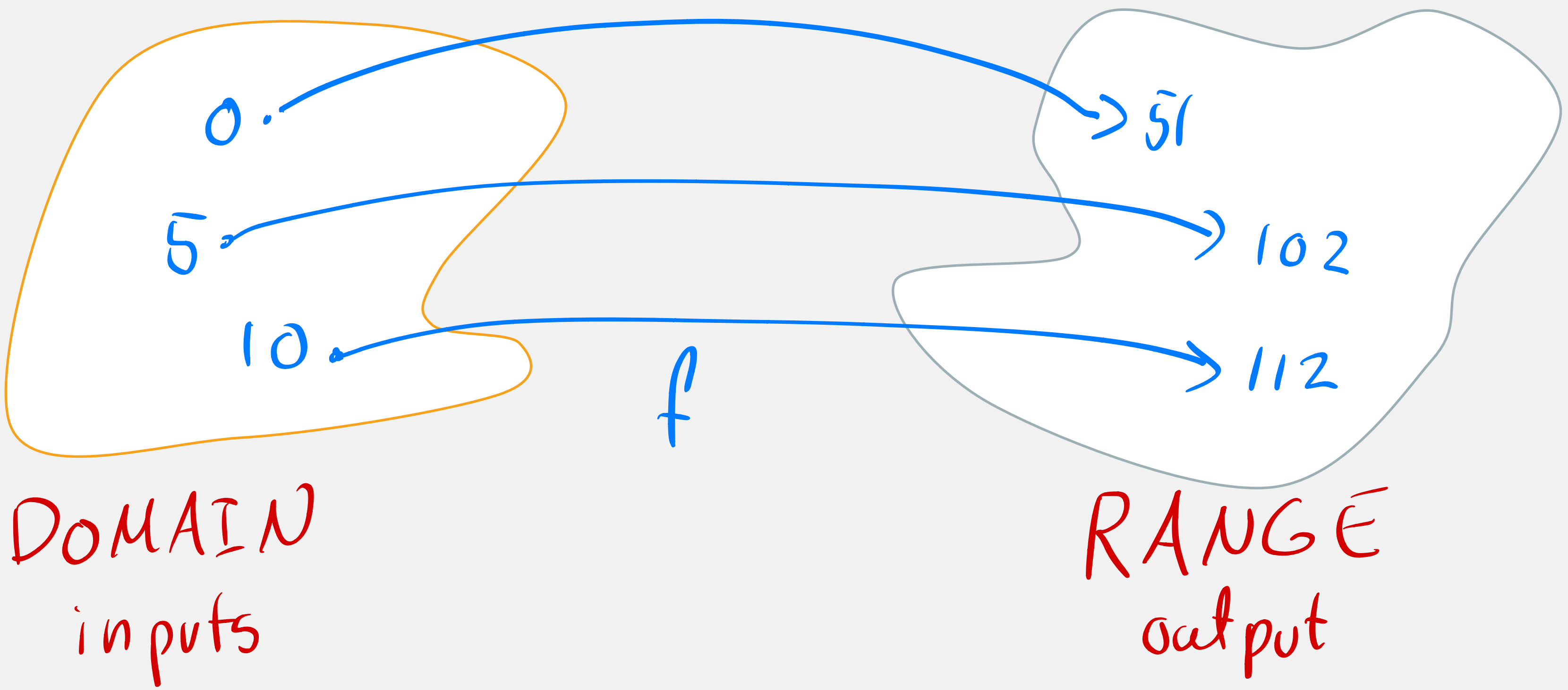
Graph



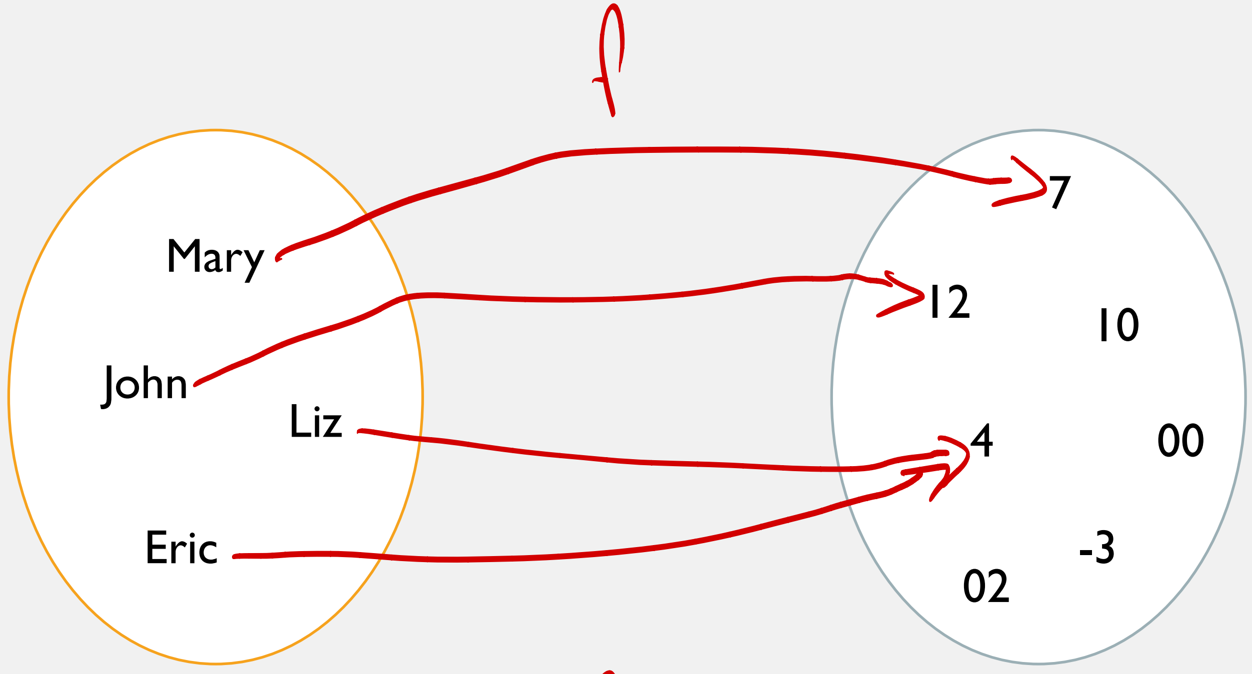
Formula

$$f(x) = \frac{a_1}{1 + \exp(b_1 x + c_1)} + \frac{a_2}{1 + \exp(b_2 x + c_2)}$$

DOMAIN AND RANGE



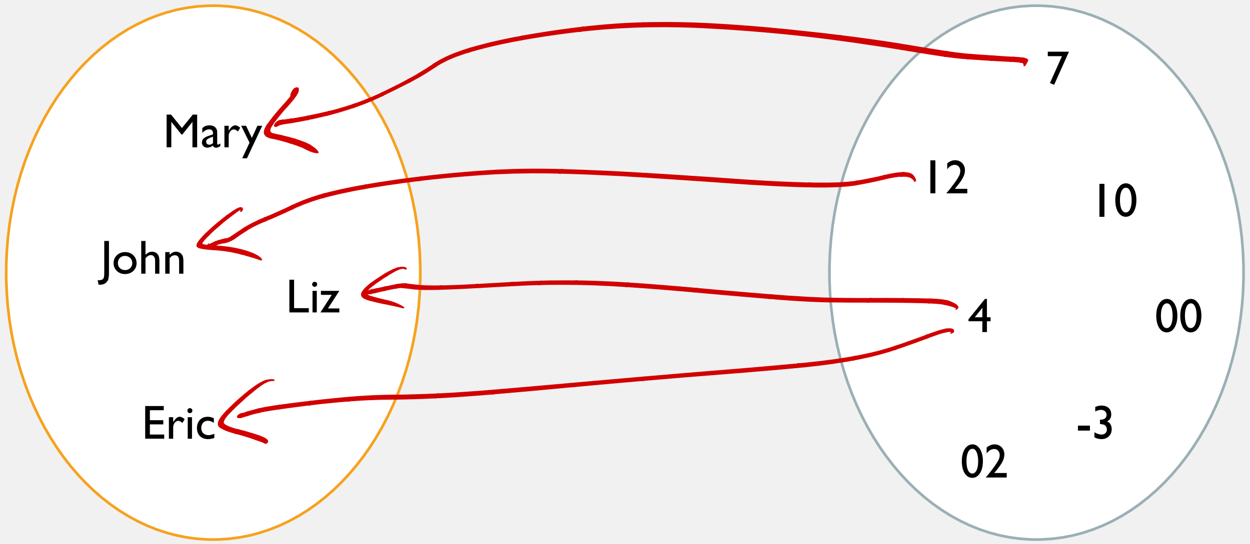
THE VERTICAL LINE TEST



Grades is a function of students

THE VERTICAL LINE TEST

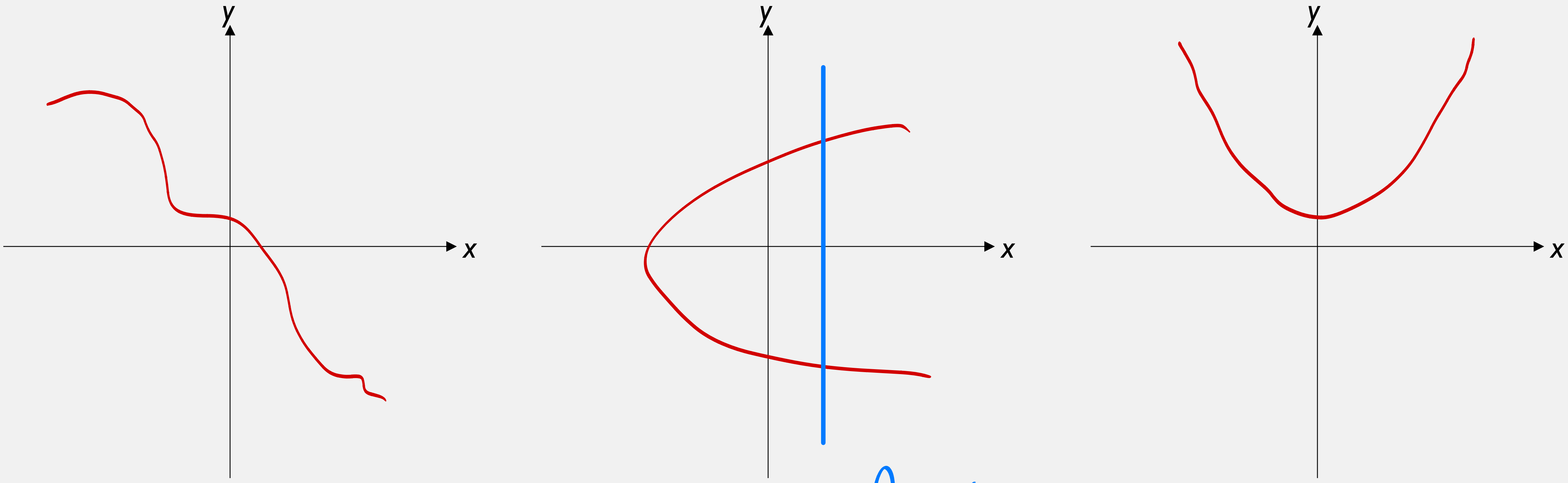
A function *f* can only have one value for each *x* in its domain



Student is not a function of grade

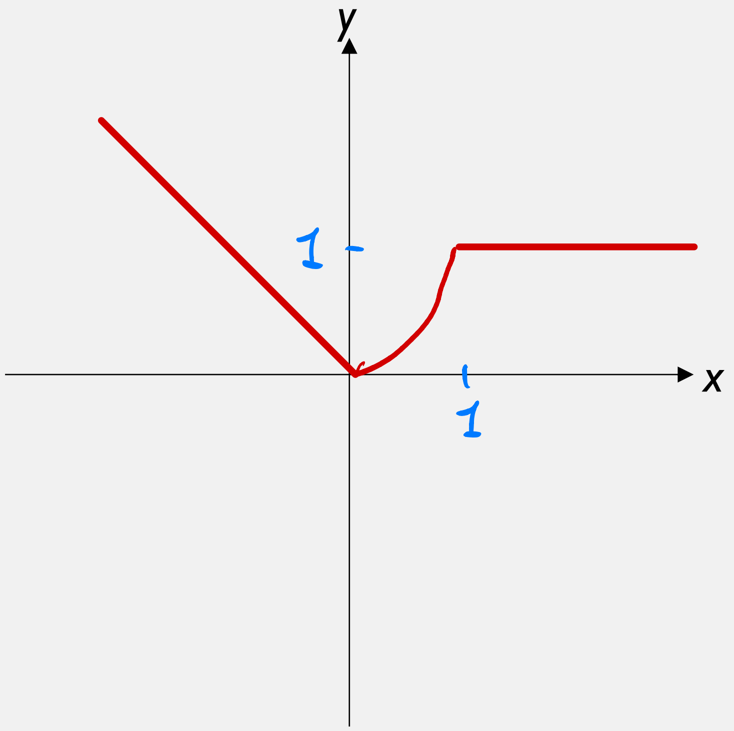
THE VERTICAL LINE TEST

No vertical line can intersect the graph of a function more than once



NOT a function

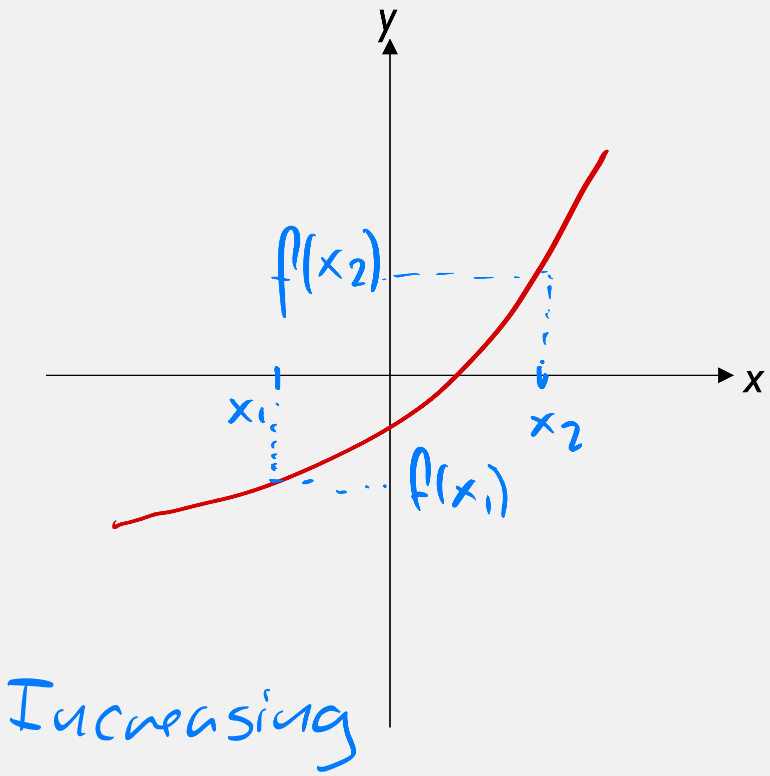
PIECEWISE-DEFINED FUNCTIONS



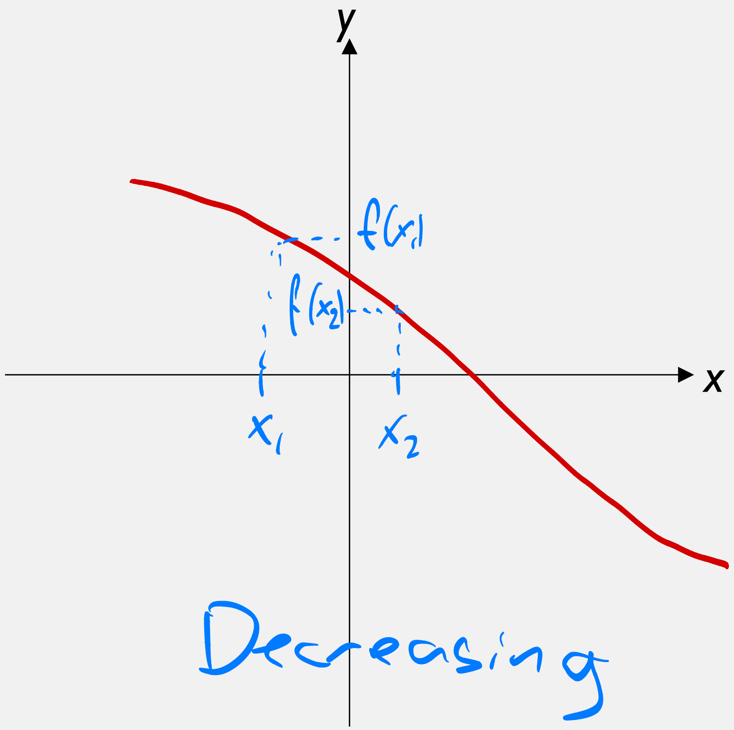
$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

```
if x<0:
    return -x
else if x≤1:
    return x^2
else: return 1
```

INCREASING AND DECREASING FUNCTIONS



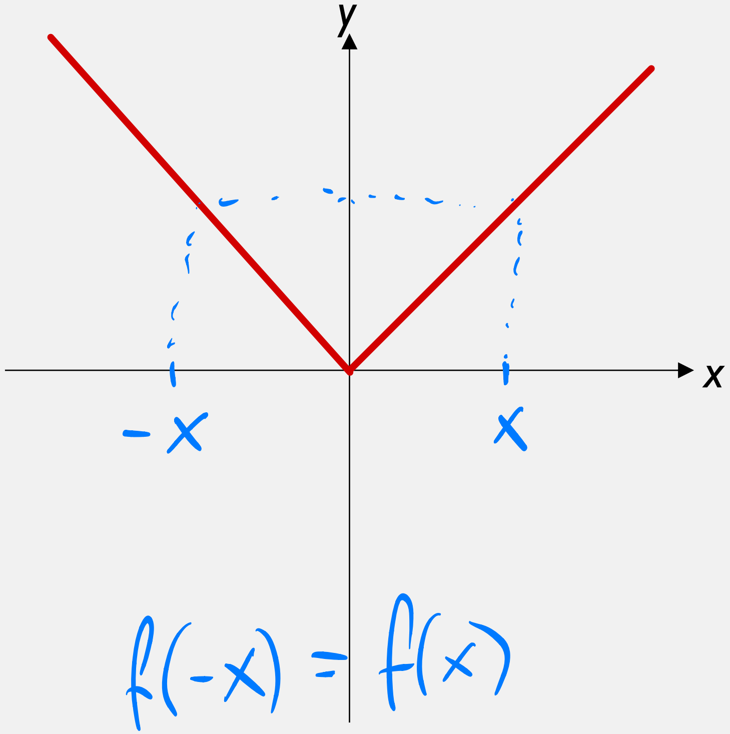
$f(x_1) < f(x_2)$ when $x_1 < x_2$



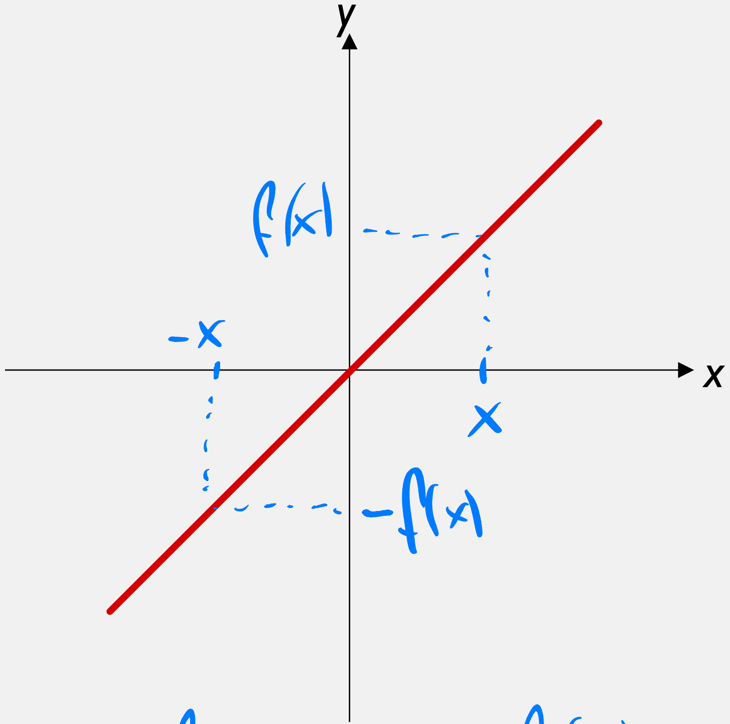
$f(x_1) > f(x_2)$ when $x_1 < x_2$

EVEN AND ODD FUNCTIONS

Even



$$f(-x) = f(x)$$

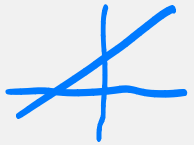


$$f(-x) = -f(x)$$

COMMON FUNCTIONS

- Linear

$f(x) = ax + b$



- Polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

- Trigonometric

$f(x) = \sin(x), \cos(x), \tan(x), \ldots$

- Exponential

$f(x) = e^x, 2^x, 10^x, \ldots$

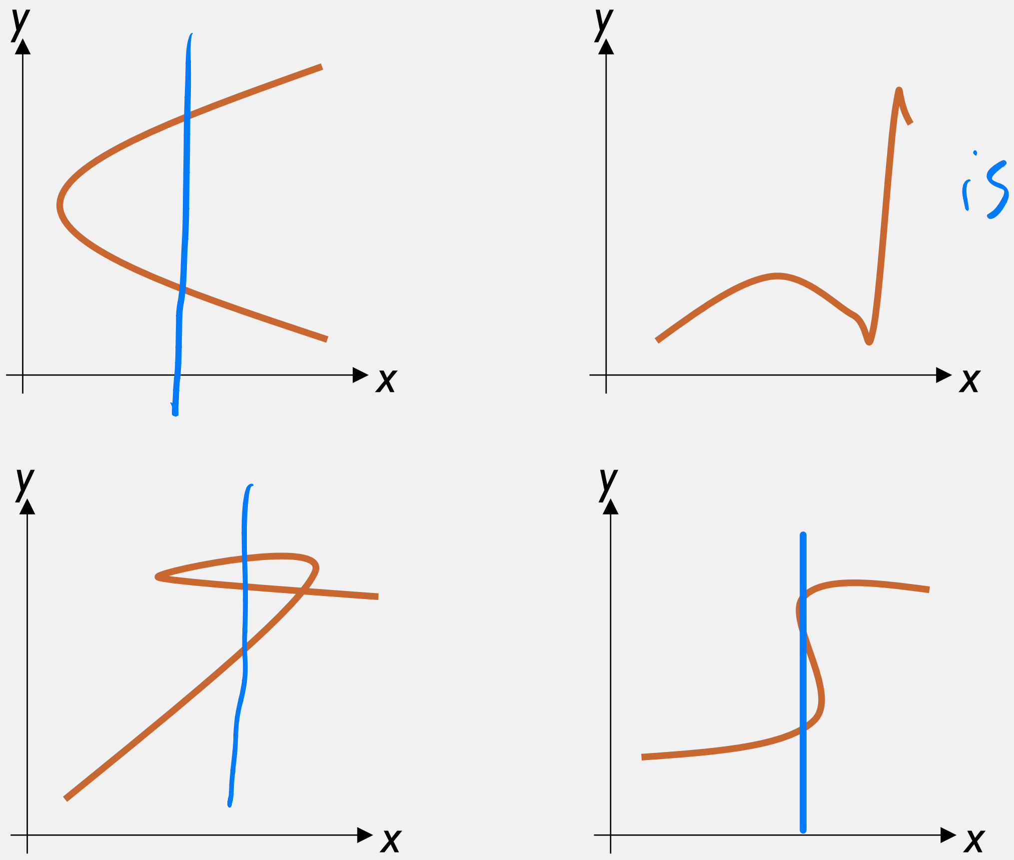
- Logarithmic

$f(x) = \ln(x), \log_2(x), \log_{10}(x), \ldots$

EXAMPLE FUNCTIONS

Function	Domain	Range	Notes
$f(x)=x^2$	$(-\infty;\infty)$	$[0;\infty)$	Even $(-3)^2=3^2$
$f(x)=\frac{1}{x}$	$(-\infty;0)\cup(0;\infty)$	$(-\infty;0)\cup(0;\infty)$	Odd
$f(x)=\sqrt{x}$	$[0;\infty)$	$[0;\infty)$	Increasing
$f(x)=\sqrt{1-x^2}$	$[-1;1]$	$[0;1]$	Even
$f(x)=e^{-2x}$	$(-\infty;\infty)$	$(0;\infty)$	Decreasing

WHICH OF THE GRAPHS ARE GRAPHS OF FUNCTIONS?



is a function

COMPOSITE FUNCTIONS

$$(f \circ g)(x) = f(g(x))$$

$$f(x) = \sin(x)$$

$$g(x) = 1 - x^2$$

$$f(g(x)) = \sin(1 - x^2)$$

FUNCTIONS, LIMITS AND CONTINUITY

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$(a+b)^2 = a^2 + b^2 + 2ab$

$(a-b)^2 = a^2 + b^2 - 2ab$

$(a+b)(a-b) = a^2 - b^2$

LIMITS

$$f(x) = \frac{x^2 - 1}{x - 1}$$

How does $f(x)$ behave near $x = 1$?

Trick

$$\frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

if $x \neq 1$

$$= (x+1)$$

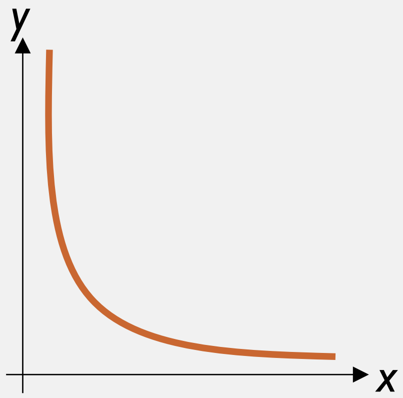
$$\lim_{x \to 1} (x+1) = 1+1 = 2$$

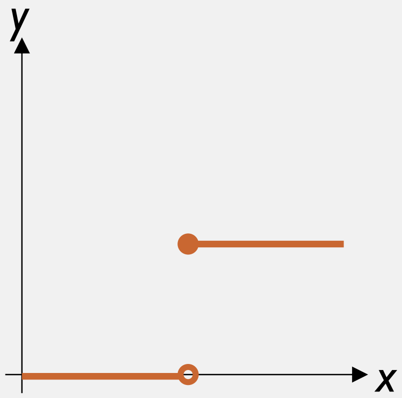
x	$f(x)$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

The limit of $f(x)$ when x approaches 1 is 2

WHAT IS THE LIMIT?





$$f(x) = \frac{1}{x}$$

$$g(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\lim_{x \to \infty} f(x) = 0$$

$\lim_{x \to 1} g(x)$ doesn't exist

THE SANDWICH THEOREM

Assume $g(x) \le f(x) \le h(x)$ in some interval containing x_0, except possibly at $x = x_0$

Assume $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L$

Then $\lim_{x \to x_0} f(x) = L$

THE SANDWICH THEOREM

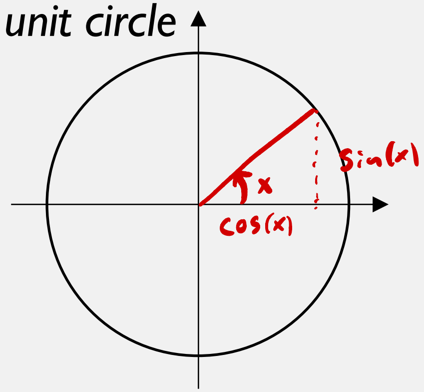
$$-|x| \le \sin(x) \le |x|$$

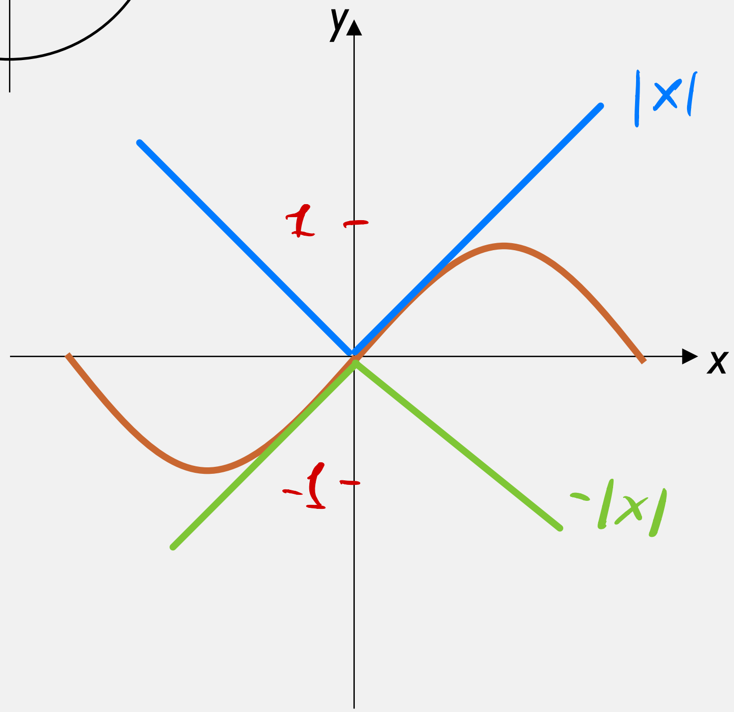
$$\lim_{x \to 0} |x| = 0$$

$$\lim_{x \to 0} -|x| = 0$$

$$\Downarrow$$

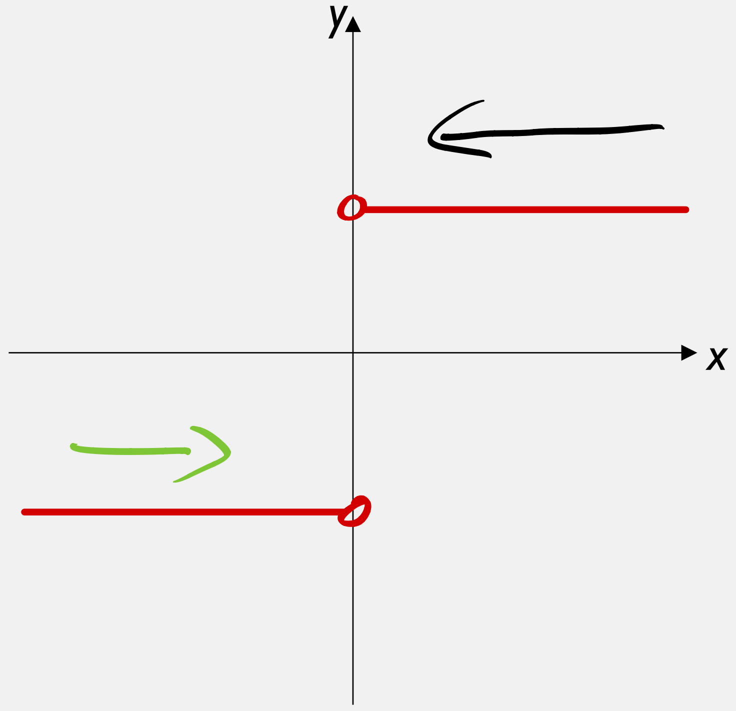
$$\lim_{x \to 0} \sin(x) = 0$$





ONE-SIDED LIMITS

$$f(x) = \frac{|x|}{x} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \\ \text{undefined} & x = 0 \end{cases}$$



$\lim_{x \to 0^+} f(x) = 1$ right hand limit

$\lim_{x \to 0^-} f(x) = -1$ left hand limit

ONE-SIDED LIMITS

$$\boxed{\lim_{x \to x_0} f(x) = L} \iff \boxed{\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L}$$

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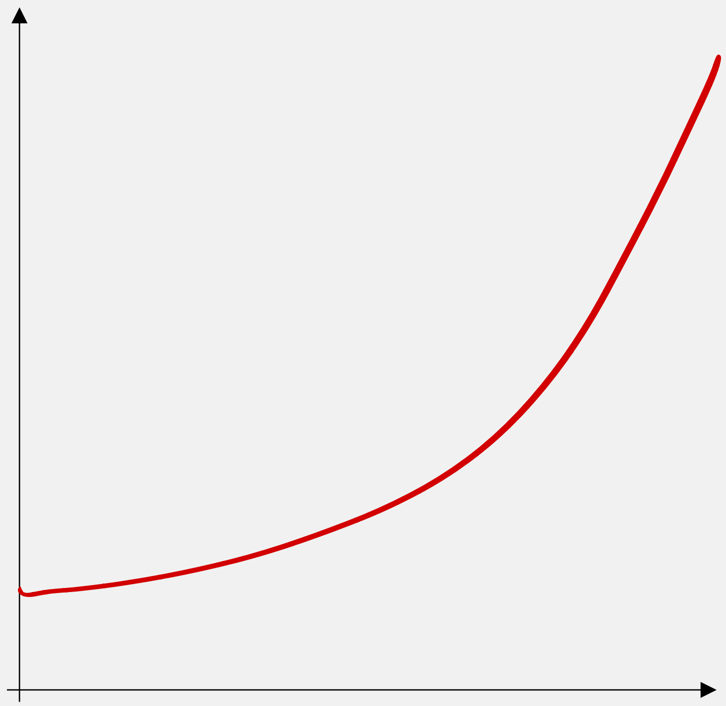
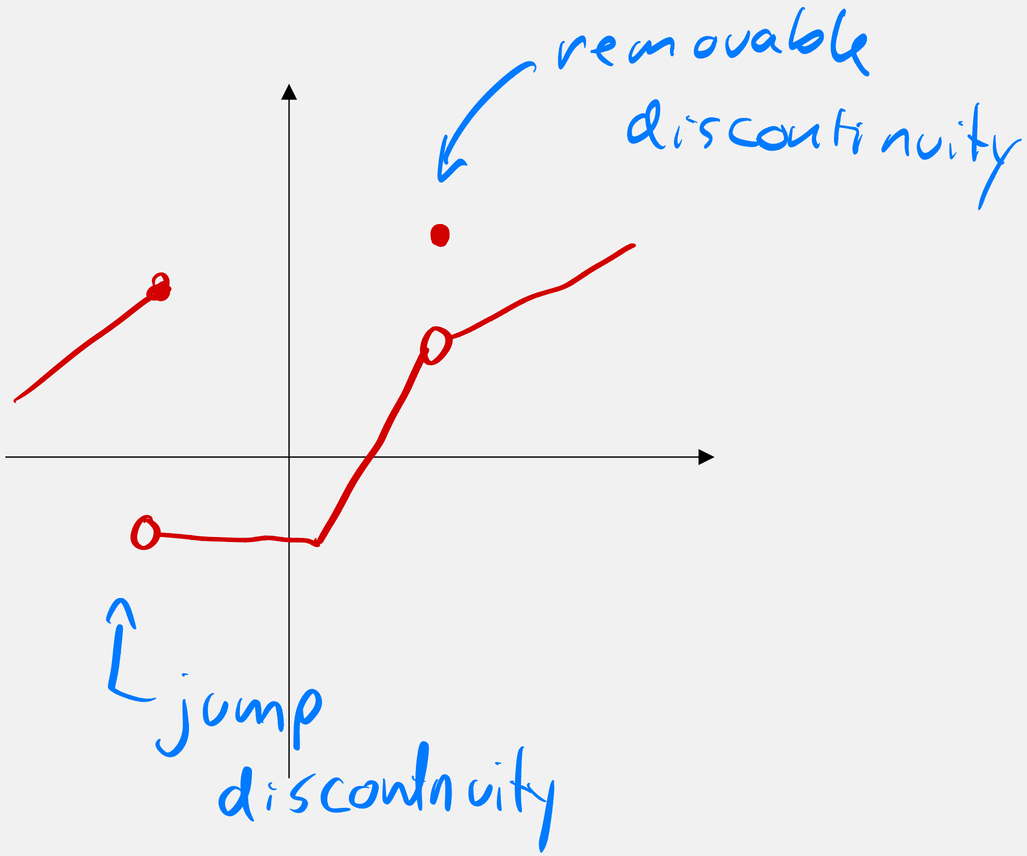
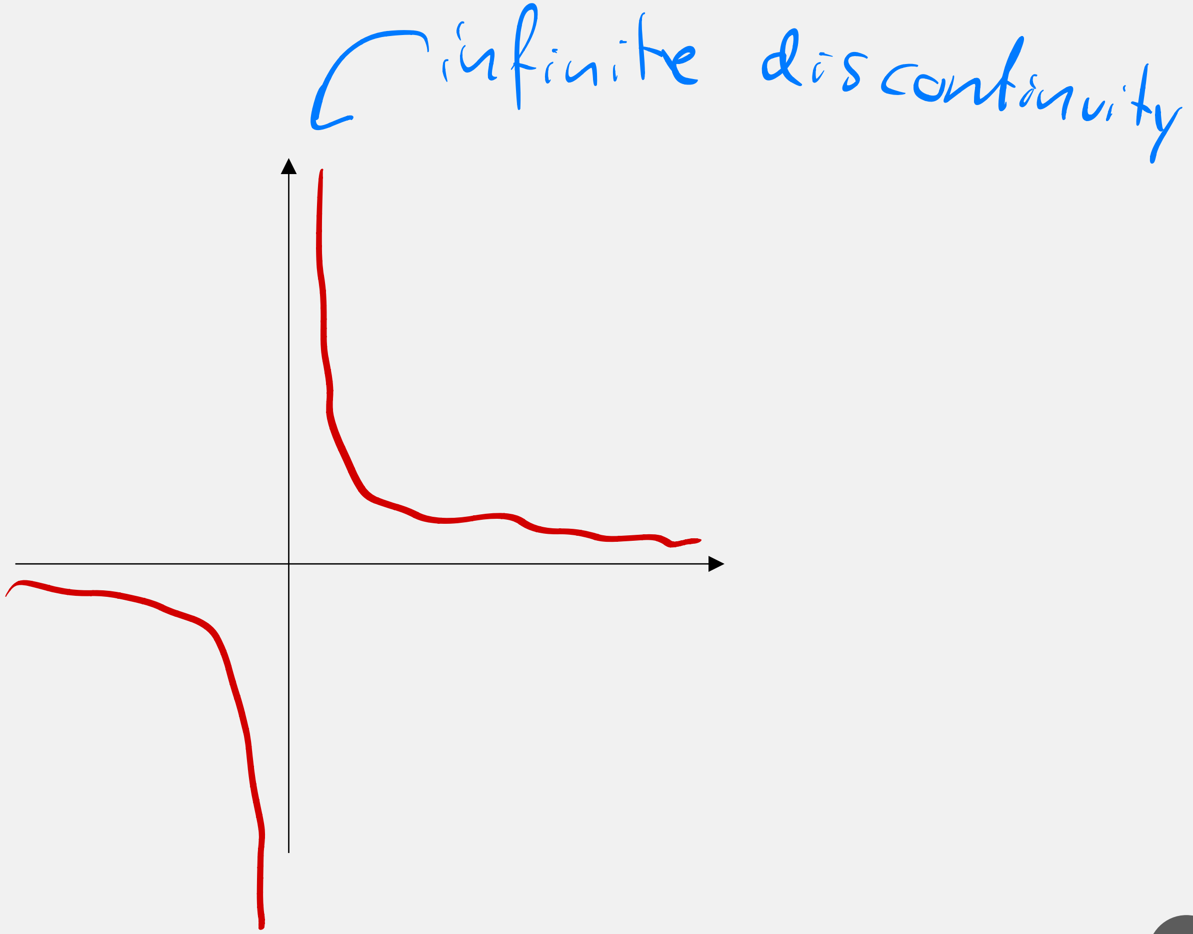
CONTINUITY



Continous function means "no jumps"

TYPES OF DISCONTINUITIES





CONTINUITY TEST

A function $f(x)$ is continuous at point $x = c$ if and only if:

1. $f(c)$ must exist
2. $\lim_{x \to c} f(x)$ must exist
3. $f(c) = \lim_{x \to c} f(x)$

CONTINUOUS FUNCTIONS

are functions that are continous at all points in the domain

Otherwise, it is discontinous