DERIVATIVES

CALI 2025

DERIVATIVES

- Derivatives
- How to calculate derivatives
- L’Hôpital’s rule

GALILEO AND THE TOWER OF PISA

How far did the stones fall?

$$x = \frac{1}{2} g t^2$$

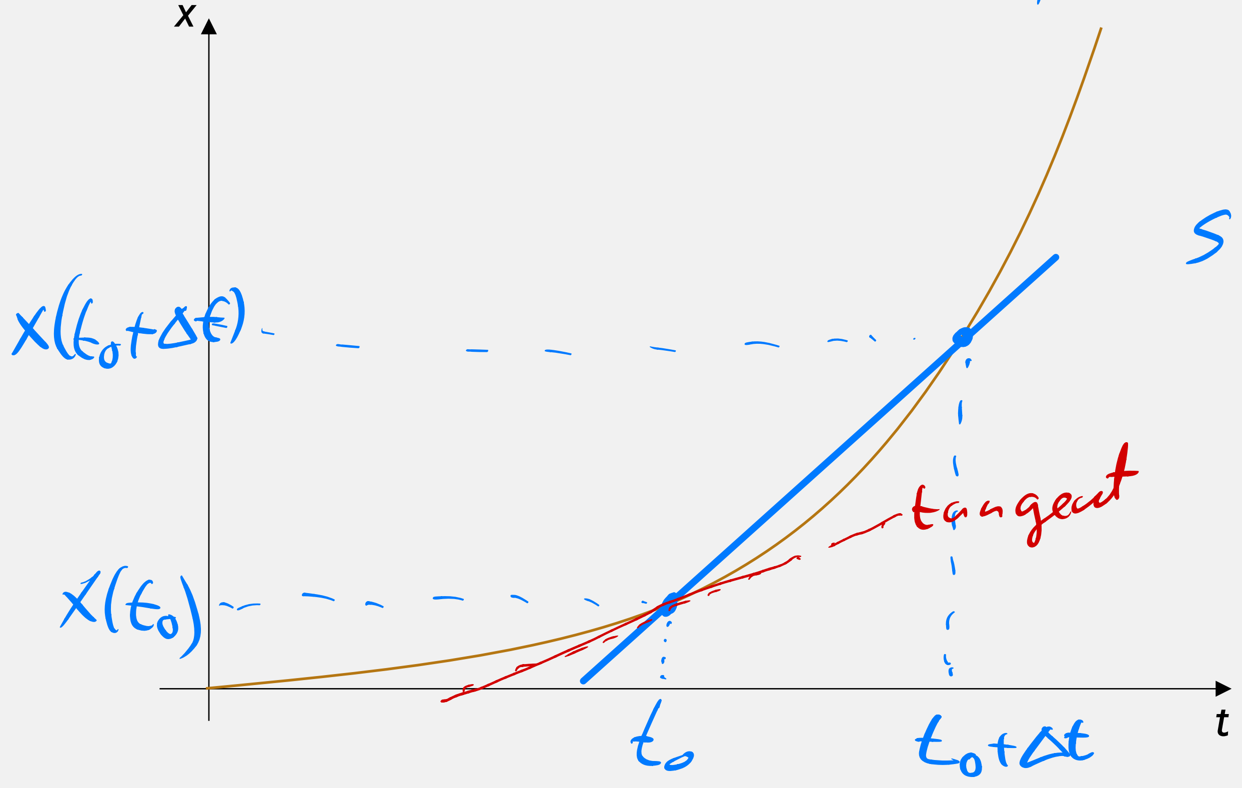
gravitational acceleration

in DK $g = 9.82 \frac{m}{s^2}$

GALILEO AND THE TOWER OF PISA

what is the velocity ?

$$x(t) = \frac{1}{2} g t^2$$



$$\text{slope} = \frac{x(t_0+\Delta t) - x(t_0)}{\Delta t}$$

GALILEO AND THE TOWER OF PISA

$(a+b)^2 = a^2 + b^2 + 2ab$

$$v(t_0) = \lim_{\Delta t \to 0} \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{1}{2} g (t_0 + \Delta t)^2 - \frac{1}{2} g t_0^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{1}{2} g (t_0^2 + \Delta t^2 + 2 t_0 \Delta t) - \frac{1}{2} g t_0^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\frac{1}{2} g t_0^2} + \frac{1}{2} g \Delta t^{\cancel{2}} + g t_0 \cancel{\Delta t} - \cancel{\frac{1}{2} g t_0^2}}{\cancel{\Delta t}}$$

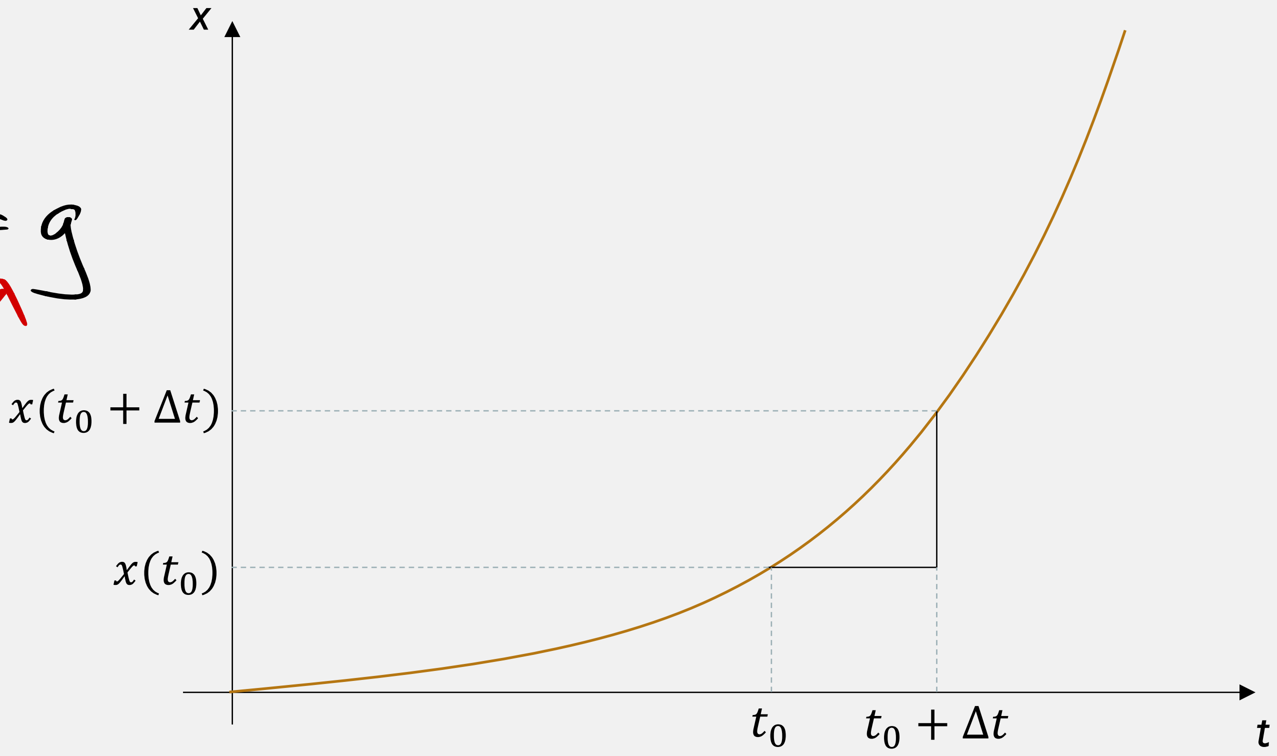
$$= \lim_{\Delta t \to 0} \left(\frac{1}{2} g \Delta t + g t_0 \right) = g t_0$$

GALILEO AND THE TOWER OF PISA

$$v(t_0) = \lim_{\Delta t \to 0} \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} = g t_0$$

$$a(t_0) = \lim_{\Delta t \to 0} \frac{v(t_0 + \Delta t) - v(t_0)}{\Delta t} = g$$

constant acceleration



THE DERIVATIVE AT A POINT

The derivative of function f at point x_0

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

[given that the limit exist]

is

- Slope of the tangent at $x = x_0$
- rate of change at $x = x_0$

THE DERIVATIVE AS A FUNCTION

$f'(x_0)$ is a number

$f'(x)$ is a function

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $f'(x_0)$ exists at every point in the domain

$\Rightarrow f$ is differentiable

DERIVATIVES OF COMMON FUNCTIONS

- Constant $f(x) = c$, $f'(x) = 0$
- Linear $f(x) = ax + b$, $f'(x) = a$

  Specifically $f(x) = x$, $f'(x) = 1$
- Polynomial $f(x) = x^n$, $f'(x) = n x^{n-1}$
- Exponential $f(x) = e^x$, $f'(x) = e^x$
- Logarithmic $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$
- Trigonometric $f(x) = \sin(x)$, $f'(x) = \cos(x)$

  $f(x) = \cos(x)$, $f'(x) = -\sin(x)$

  $f(x) = \tan(x)$ $f'(x) = 1 + \tan^2(x) = \frac{1}{\cos^2(x)}$

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DIFFERENTIATION

→ Finding the derivative

$$f'(x) = \frac{d}{dx} f(x)$$

differentiation operator

DIFFERENTIATION RULES

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$ [sum rule]

$$\frac{d}{dx}(cf) = c\,\frac{df}{dx}$$ [constant multiple rule]

$$\frac{d}{dx}(fg) = f\,\frac{dg}{dx} + \frac{df}{dx}\,g$$ [product rule]

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$ [chain rule]

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$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

$$\frac{d}{dx}\left(x^3 + \sin(x)\right)$$

$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(\sin(x))$$

$$= 3x^2 + \cos(x)$$

DIFFERENTIATION RULES

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$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

$$\frac{d}{dx}(4x^2) = 4\cdot\frac{d}{dx}(x^2)$$

$$= 4\cdot 2x$$

$$= 8x$$

DIFFERENTIATION RULES

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g \quad \text{[product rule]}$$

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

$$\frac{d}{dx}(\overset{f}{x^2} \cdot \overset{g}{e^x}) = x^2 \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}(x^2) \cdot e^x$$

$$= x^2 e^x + 2x\, e^x$$

DIFFERENTIATION RULES

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$ [chain rule]

$$\frac{d}{dx}\left(\overset{f}{\ln}\left(\overset{g}{x^2}\right)\right) = \frac{1}{x^2}\cdot 2x$$

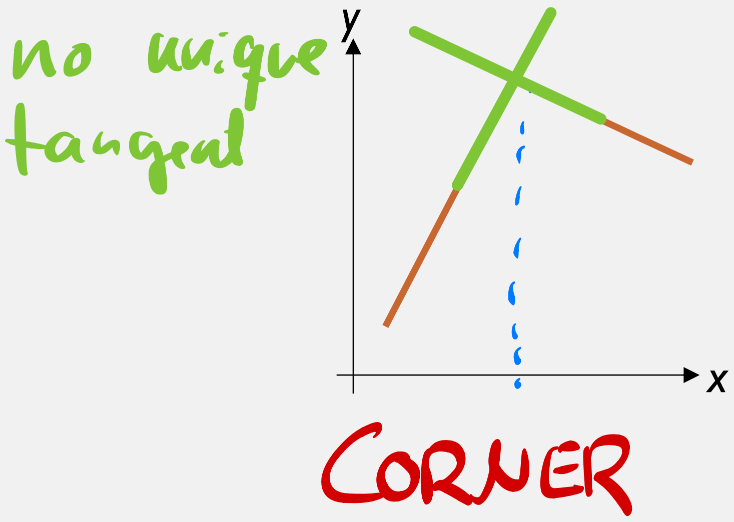
$$= \frac{2}{x}$$

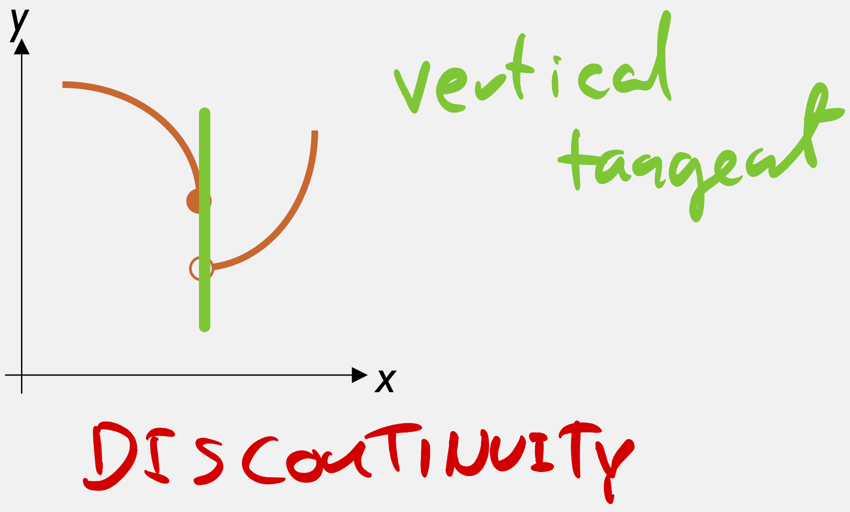
$$\frac{d}{dx}\tan(x^3+1) = \frac{1}{\cos^2(x^3+1)}\cdot 3x^2$$

DIFFERENTIATION RULES

If you know the derivatives of common functions, and you know the basic rules, you can (almost) always calculate a derivative …

DIFFERENTIABILITY





… unless it doesn't exist

DERIVATIVES

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- L’Hôpital’s rule

INDETERMINATE FORMS

$\hookrightarrow$ meaningless statement

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty - \infty \qquad \infty \cdot 0 \qquad 0^0 \qquad 1^\infty$$

L'HÔPITAL'S RULE

Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

assume that f and g are differentiable

assume that $g'(x) \neq 0$ when $x \neq a$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

"$\frac{0}{0}$"

EXAMPLES

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = \lim_{x\to 0} \cos(x) = 1$$

$$\text{"}\frac{0}{0}\text{"}$$

$$\lim_{x\to 0} \frac{\sin x - x}{x^2 + x} = \lim_{x\to 0} \frac{\cos x - 1}{2x + 1} = \frac{1 - 1}{0 + 1} = \frac{1-1}{1} = 0$$

$$\text{"}\frac{0}{0}\text{"}$$

∞ / ∞

- L'Hôpital's rule also applies when $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

EXAMPLES

$\sqrt{x} = x^{1/2}$

$(x^{1/2})' = \frac{1}{2} \cdot x^{-1/2} = \frac{1}{2\sqrt{x}}$

"$\frac{\infty}{\infty}$"

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{2\sqrt{x}}{1} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

"$0 \cdot \infty$"

$$\lim_{x\to 0^+} x \cdot \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} (-x) = 0$$

"$\frac{\infty}{\infty}$"

EXAMPLES

A NOTE OF CAUTION

- L'Hôpital's rule only applies to indeterminate forms!

$$2 = \lim_{x \to 1} \frac{x^2+1}{x^3} \neq \lim_{x \to 1} \frac{2x}{3x^2} = \frac{2}{3}$$