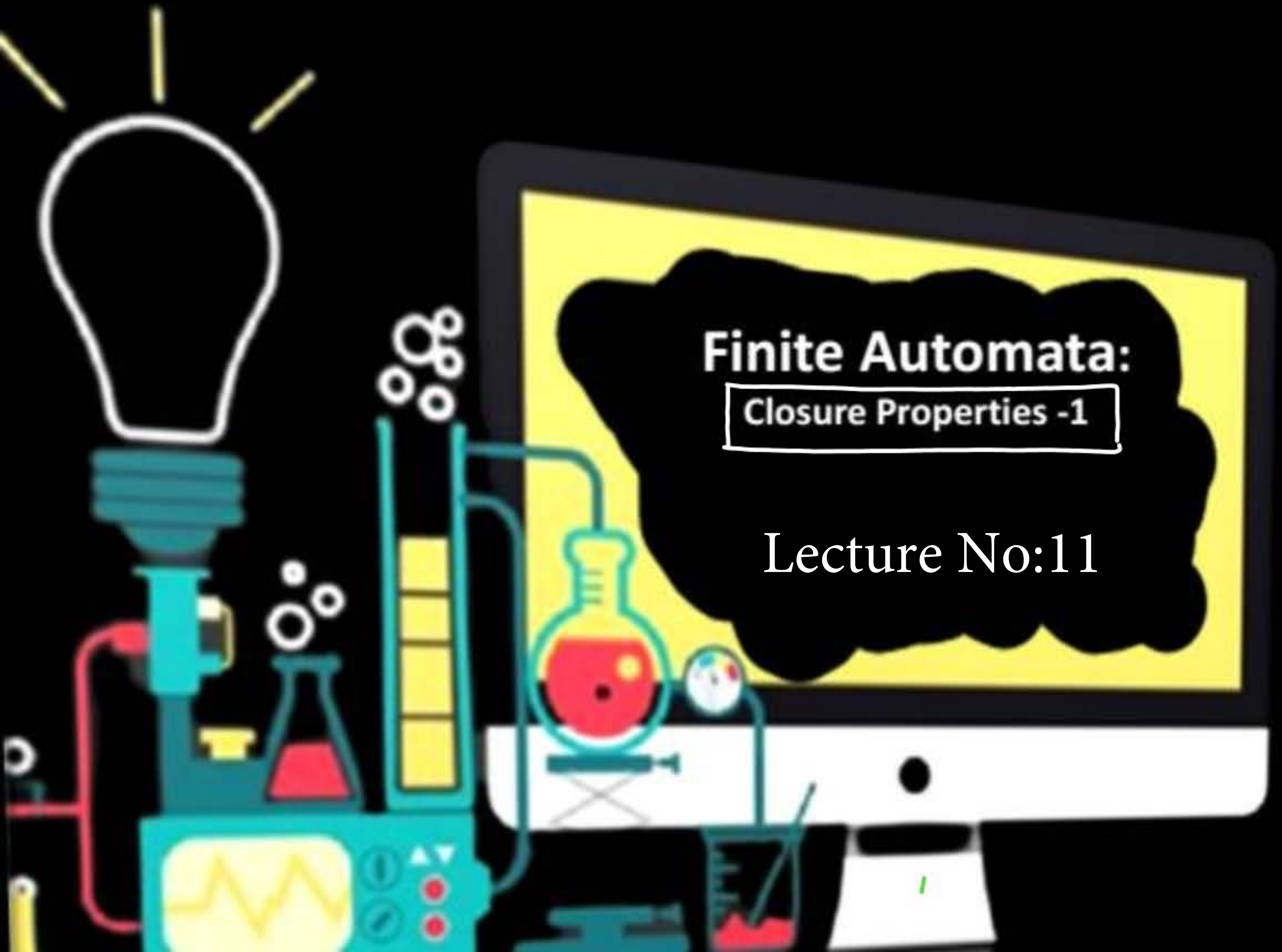
# CS & IT Engineering



## Finite Automata:

### Closure Properties -1

Lecture No:11

Deva sir

## Previous Class Summary:

↳ Identification of regular languages

# Topics to be covered Today:

- closure properties
  - closure
  - Types of operations
  - Domains
    - for finite languages
    - for Infinite languages
    - * for regular languages
    - for nonregular languages

closure (operation)

$(\underset{\text{Domain}}{N}, \underset{\text{operation}}{+})$ is $\underbrace{\text{closed}}$

for every 2 elements of $\mathbb{N}$ $x, y$

$x+y \in N$

$(N, -)$ is not closed

Some 2 elements of $\mathbb{N}$ $a, b$

$a+b \notin N$

1, 2, 3, 4, 5, 6, 7, . . .

$x+y$

$\exists a$ $\exists b$

$\forall x$ $\forall y$

$\mathbb{N}$

$a-b \notin \mathbb{N}$

$(D, *)$ is closed

iff

$\forall L_1 \in D, \forall L_2 \in D \ni L_1 * L_2 \in D$

$D \rightarrow$ set of languages

$* \rightarrow$ any operation

Example:

(Set of finite languages, $\cup$)

$\hookrightarrow$ closed

$\boxed{\text{Finite} \cup \text{Finite} \Rightarrow \text{Finite}}$

Set of finite languages

- $\{a^n b^n \mid n < 100\}$
- $\{\varepsilon, a, abb\}$
- $\{w \mid w \in \{a,b\}^*, |w| \leq 10\}$
- . . . . . . .

$(D, *)$ is not closed

iff

$\exists L_1 \in D, \exists L_2 \in D \ni L_1 * L_2 \notin D$

(Set of Finite sets, Kleene star)

↳ is not closed

$L = \{a\} \Rightarrow L^* = \underbrace{a^*}_{\notin \text{ set of finite sets}}$

finite

$a^*$

set of finite languages

- $F_1 = \phi$
- $F_2 = \{\varepsilon\}$
- $F_3 = \{a\}$
- $F_4$
- $F_5$
- $F_6$

⋮

Closed

↓

We require proof

(Algorithm)

Not closed

↓

We require example

Domain : Set of finite languages

$\overline{L} = \Sigma^* - L$

$\overline{Fin} = \underbrace{\Sigma^*}_{Inf} - Fin \Rightarrow Inf$

| | Closed ✓ / Not closed ✗ | |
|---|---|---|
| ① Union | ✓ | $Fin \cup Fin \Rightarrow Fin$ |
| ② Intersection | ✓ | $Fin \cap Fin \Rightarrow Fin$ |
| *** ③ complement | ✗ | $\overline{Fin} \Rightarrow$ Always Infinite |
| ④ Difference | ✓ | $Fin - Fin \Rightarrow Fin$ |
| ⑤ Concatenation | ✓ | $Fin . Fin \Rightarrow Fin$ |
| ⑥ Kleene star | ✗ | $(Fin)^* \Rightarrow$ may or may not finite |
| ⑦ Subset | ✓ | Subset of finite set $\Rightarrow$ finite set |
| ⑧ Reversal | ✓ | Reversal of finite set $\Rightarrow$ finite set |

Closure Properties

Domain : Set of Infinite languages

$a^* \cap b^* = \{\varepsilon\}$

| | ✓ closed / ✗ Not closed | |
|---|---|---|
| ① Union | ✓ | Inf ∪ Inf ⟹ Infinite |
| ② Intersection | ✗ | Inf ∩ Inf ⟹ Need not be Inf |
| ③ complement | ✗ | $\overline{\text{Inf}}$ ⟹ Need not be Inf |
| ④ Difference | ✗ | Inf − Inf ⟹ Need not be Inf |
| ⑤ Concatenation | ✓ | Inf . Inf ⟹ Inf |
| ⑥ Kleene star | ✓ | $(\text{Inf})^*$ ⟹ Inf |
| ⑦ Subset | ✗ | Subset of Inf set ⟹ Need not be Inf |
| ⑧ Reversal | ✓ | Reversal of Inf ⟹ Inf |

# Closure Properties for regular languages :

1. Union
2. Intersection
3. Complement
4. Difference
5. Concatenation
6. Reversal
7. Kleene star
8. Kleene plus
9. ~~Subset~~
10. Symmetric Difference
11. Substitution
12. Homomorphism
13. $\epsilon$-free Homomorphism
14. Inverse Homomorphism
15. prefix(L)
16. Suffix
17. Substring
18. Quotient
19. $Half(L) = \frac{1}{2}(L)$
20. Second Half(L)
21. one-third(L)
22. Middle $\frac{1}{3}(L)$
23. Last $\frac{1}{3}(L)$
24. Finite Union
25. Finite Intersection
26. Finite Difference
27. Finite Concatenation
28. Finite Subset
29. Finite Substitution
30. ~~Infinite Union~~
31. ~~Infinite Intersection~~
32. ~~Infinite Difference~~
33. ~~Infinite Concatenation~~
34. ~~Infinite Subset~~
35. ~~Infinite Substitution~~

Subset, Inf ....

Remember "Not closed operations" for regulars

① Union

↳ It is closed for regular languages

$$Reg_1 \cup Reg_2 \Rightarrow \text{Always Regular}$$

① $L_1 = a^*$, $L_2 = b^*$ $\Rightarrow L_1 \cup L_2 = a^* + b^*$

$= a^+ + b^+ + \varepsilon$

$= a^+ + b^*$

$= a^* + b^+$

proofs:

1) Use Reg Exps

2) Use RGs

3) Use FAs ($FA_1 \times FA_2$)

4) Use ε-NFA

⋮

② $L_1 = a^*$, $L_2 = (a+b)^*$ $\Rightarrow L_1 \cup L_2 = L_2 = (a+b)^*$

③ If $L_1 \subseteq L_2$ then $L_1 \cup L_2 = \underline{\underline{L_2}}$

*** ④ If $L_1$ = Regular, and
$L_2$ = Non-regular then $L_1 \cup L_2$ is Need not be regular

⑤ If $L_1 = \phi$, $L_2 = a^n b^n$ then $L_1 \cup L_2$ is Non reg

⑥ If $L_1 = a^* b^*$, $L_2 = a^n b^n$ then $L_1 \cup L_2$ is reg

PYQ GATE ★★★

(7) If $L_1 \cup L_2$ is regular then $L_1$ is may or may not regular

(8) If $L_1 \cup L_2$ is non regular then $L_1$ is may or may not regular

(9) If $L_1$ is reg, and $L_2$ is reg $\Rightarrow$ $L_1 \cup L_2$ is always regular

(10) If $L_1$ is nonregular, and $L_2$ is nonreg $\Rightarrow$ $L_1 \cup L_2$ is may or may not regular

i) $\underbrace{\{a^n b^n\}}_{\text{nonreg}} \cup \underbrace{\overline{\{a^n b^n\}}}_{\text{nonreg}} \Rightarrow \text{reg } (\Sigma^*)$

ii) $\{a^n b^n\} \cup \{a^n b^n\} \Rightarrow \text{nonreg } (\{a^n b^n\})$

(11) If $L_1$ is finite and $L_2$ is regular then

$L_1 \cup L_2$ is Regular language
(may or may not be finite)

(12) If $L_1$ is Infinite and $L_2$ is regular then

$L_1 \cup L_2$ is Always Infinite (need not be regular)

## 2. Intersection

→ closed for regular languages

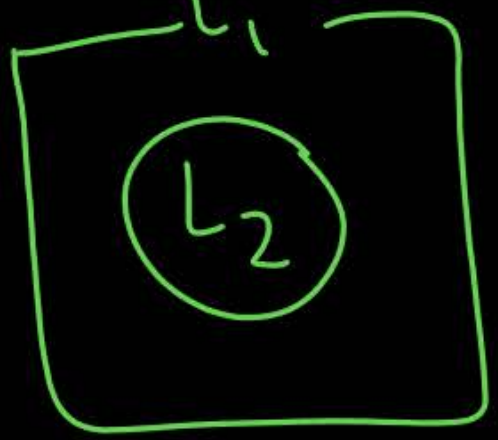
$$Reg_1 \cap Reg_2 \Rightarrow \text{Always Regular}$$

proof : $\underbrace{FA_1 \times FA_2}_{\text{composition}}$

1) $L_1 = \phi$, $L_2 = \Sigma^*$ $\Rightarrow L_1 \cap L_2 = \phi$

2) $L_1 = a^* b^*$, $L_2 = a^* + b^*$ $\Rightarrow L_1 \cap L_2 = L_2$

$a^* b^*$ → $a^*$, $b^*$, $a^+ b^+$




$L_2 \subseteq L_1$

$L_1 \cap L_2 = L_2$

$L_1 \cup L_2 = L_1$

3) $\left.\begin{array}{l} L_1 = a^* \\ L_2 = b^* \end{array}\right\} \Rightarrow L_1 \cap L_2$ is $\underline{\{\epsilon\}}$

4) $\left.\begin{array}{l} L_1 = (a+b)^* = \Sigma^* \\ L_2 = \text{Any} \end{array}\right\} \Rightarrow L_1 \cap L_2 = \underline{L_2}$

(3) Complement:

$\hookrightarrow$ closed for Regulars

$\overline{Reg} \Longrightarrow$ Always Regular

Proof:

$L$
$\Downarrow$
DFA
$\Downarrow$ $f \longleftrightarrow nf$
DFA
$\Downarrow$
$\overline{L}$

(1) $L = \phi \Longrightarrow \overline{L} = \Sigma^*$

(2) $L = \Sigma^* \Longrightarrow \overline{L} = \phi$

(3) $L = \Sigma^+ \Longrightarrow \overline{L} = \{\varepsilon\}$

(4) $L = a(a+b)^* \Longrightarrow \overline{L} = \varepsilon + b(a+b)^*$

4 Difference

→ closed for regulars

$$\boxed{Reg_1 - Reg_2 \Rightarrow Regular}$$

$$\boxed{L_1 - L_2 = L_1 \cap \overline{L_2}}$$

$$Reg_1 - Reg_2 = Reg_1 \cap \overline{Reg_2}$$

1. $L_1 = \phi,\ L_2 = \text{Any} \Rightarrow L_1 - L_2 = \phi$
   $L_2 - L_1 = L_2$
2. $L_1 = \Sigma^*,\ L_2 = \text{Any} \Rightarrow L_1 - L_2 = \overline{L_2}$
   $L_2 - L_1 = \phi$
3. $L_1 = a^*,\ L_2 = b^* \Rightarrow L_1 - L_2 = a^+$
   $L_2 - L_1 = b^+$

⑤ Concatenation:

↳ closed for regular language

$$\boxed{Reg_1 \cdot Reg_2 \Rightarrow Regular}$$

proof : ε-NFA

1) $L_1 = \phi,\ L_2 = \text{Any} \Rightarrow L_1 \cdot L_2 = \phi$
$L_2 \cdot L_1 = \phi$

2) $L_1 = a^*,\ L_2 = b^* \Rightarrow L_1 L_2 = a^* b^*$
$L_2 L_1 = b^* a^*$

3) $L_1 = a^*,\ L_2 = (a+b)^* \Rightarrow L_1 L_2 = a^* \cdot (a+b)^* = (a+b)^* = L_2$
$L_2 L_1 = (a+b)^* \cdot a^* = (a+b)^* = L_2$

⑥ Reversal

↳ closed for regulars

$$(\text{Regular})^{\text{Reversal}} \Rightarrow \text{Always Regular}$$

1) $L = \phi \Rightarrow L^{Rev} = \phi$

2) $L = \Sigma^* \Rightarrow L^{Rev} = \Sigma^*$

3) $L = a\Sigma^* \Rightarrow L^{Rev} = \Sigma^* a$

4) $L = \Sigma^* a \Rightarrow L^{Rev} = a\Sigma^*$

5) $L = a^* b^* \Rightarrow L^{Rev} = b^* a^*$

Proof:

1) Use Reg Exp

2) use ε-NFA

$$\boxed{\begin{array}{l} \text{I)}\ \overline{\left(\overline{L}\right)} = L \\ \text{II)}\ \left(L^{Rev}\right)^{Rev} = L \end{array}}$$

⑦ Kleene star
↳ closed for regular

$$(\text{Reg})^* \Rightarrow \text{Regular}$$

1) $L = \phi \Rightarrow L^* = \{\epsilon\}$

2) $L = \{\epsilon\} \Rightarrow L^* = \{\epsilon\}$

3) $L = \Sigma^* \Rightarrow L^* = \Sigma^*$

4) $L = a^*b^* \Rightarrow L^* = (a+b)^*$

5) $L = a^* + b^* \Rightarrow L^* = (a+b)^*$

⑧ Kleene plus
↳ closed for regular

I) If $L$ is regular $\Rightarrow L^*$ is also regular

II) If $L^*$ is regular $\Rightarrow L$ may or may not be regular

# Summary

$\cup$ ✓

$\cap$ ✓

$\overline{L}$ ✓

$L_1 - L_2$ ✓

$L_1 . L_2$ ✓

$L^{Rev}$ ✓

$L^*$ ✓

$L^+$ ✓

# Thank you