Study of the 2D elastica in a narrow tube

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THREAD
EUROPEAN TRAINING NETWORK

1 Euler's elastica under contact

This section delves into the problem of an elastica undergoing contact. We will explore the intricacies of this problem and its formulation as a constrained second order Lagrangian problem with augmented Lagrangian, while also discussing the implementation of the associated Python code. The link to the GitHub repository of this code is `https://github.com/THREAD-2-3/elastica_narrow_environment`.

1.1 Euler's elastica

The elastica problem was one of the earliest uses of the calculus of variations, first posed by Jacob Bernoulli and solved by Leonhard Euler, after being introduced to it by his nephew and pupil Daniel Bernoulli [1]. It was precisely Daniel who reduced the problem as that of finding a curve which minimizes the functional

$$\int_0^l \kappa(s)^2 ds$$

where $\kappa(s)$ denotes the curvature of a curve, and $s, l \in \mathbb{R}$ are the arc-length parameter and total length of the curve respectively. If $q : [0, l] \to \mathbb{R}^2$ denotes a curve parametrized by arc-length, then $\kappa(s) = q''(s)$, where $' = \frac{d}{ds}$.

The requirement of arc-length on its changing rate, i.e. $\|q'(s)\| = 1$, can be introduced as a constraint in the system. Thus, the problem can be stated as a constrained second order Lagrangian problem with augmented Lagrangian $L : T^{(2)}Q \times \mathbb{R} \to \mathbb{R}$ [2].

$$L(q, q', q'', \Lambda) = \frac{1}{2}EI\|q''\|^2 + \Lambda(\|q'\|^2 - 1) \tag{1}$$

Here, Λ is the Lagrange multiplier enforcing the constraint, and the stiffness of the elastica has been parametrized by E, its Young's modulus, and I, the second moment of area of the beam's cross-section. The continuous action integral S is defined as:

$$S = \int_0^l L(q, q', q'', \Lambda)\, ds \tag{2}$$

Applying Hamilton's principle of stationary action $\delta S = 0$ with fixed boundary conditions $(q(0), q'(0)) = (q_0, q'_0)$ and $(q(l), q'(l)) = (q_N, q'_N)$, yields the Euler-Lagrange equations.

$$\frac{d^2}{ds^2}\frac{\partial L}{\partial q''} - \frac{d}{ds}\frac{\partial L}{\partial q'} + \frac{\partial L}{\partial q} = 0 \tag{3}$$

The Legendre transformation map $\mathbb{F}L : T^{(3)}Q \to T^*TQ$ associated with $L(q, q', q'', \Lambda)$ is given by [3]

$$\mathbb{F}L(q, q', q'', q^{(3)}) = \left(q, q', p = \frac{\partial L}{\partial q'} - \frac{d}{ds}\frac{\partial L}{\partial q''}, \tilde{p} = \frac{\partial L}{\partial q''}\right) \tag{4}$$

where p and $\tilde{p}$ are the conjugate momenta w.r.t. coordinates q and slopes q' respectively. For the elastica, we have

$$\begin{aligned} p &= -q^{(3)} + 2\Lambda q' \\ \tilde{p} &= q'' \end{aligned} \tag{5}$$

The former is related to axial and transverse forces, while the latter is related to bending moments.

The action can be discretised over the length of the beam with space steps h, resulting in a discrete Lagrangian formulation. A natural discrete Lagrangian for an augmented second order Lagrangian theory is of the form $L_d : TQ \times TQ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ [3]. In particular, we apply the discretisation proposed in [4]

$$L_d\left(q_i, q'_i, q_{i+1}, q'_{i+1}, \Lambda_i, \Lambda_{i+1}\right) = \frac{h}{2}\left[L(q_i, q'_i, (q''_i)^\alpha, \Lambda_i) + L(q_{i+1}, q'_{i+1}, (q''_{i+1})^\alpha, \Lambda_{i+1})\right] \tag{6}$$

where

$$(q_i'')^\alpha = \frac{\left[(1-3\alpha)q_{i+1}' - (1+3\alpha)q_i'\right]h + 6\alpha(q_{i+1} - q_i)}{h^2} \tag{7}$$

$$(q_{i+1}'')^\alpha = \frac{\left[(1+3\alpha)q_{i+1}' - (1-3\alpha)q_i'\right]h - 6\alpha(q_{i+1} - q_i)}{h^2} \tag{8}$$

In this study case, $\alpha = 1$ is chosen. The discrete action S_d associated to this Lagrangian is

$$S_d = \sum_{i=0}^{N-1} L_d\left(q_i, q_i', q_{i+1}, q_{i+1}', \Lambda_i, \Lambda_{i+1}\right) \tag{9}$$

evaluated along the beam. By applying the discrete Hamilton's principle to S_d, one obtains the discrete Euler-Lagrange equations, which are the discrete equilibrium equations of the beam.

$$D_1 L\left(q_i, q_i', q_{i+1}, q_{i+1}', \Lambda_i, \Lambda_{i+1}\right) + D_3 L\left(q_{i-1}, q_{i-1}', q_i, q_i', \Lambda_{i-1}, \Lambda_i\right) = 0$$
$$D_2 L\left(q_i, q_i', q_{i+1}, q_{i+1}', \Lambda_i, \Lambda_{i+1}\right) + D_4 L\left(q_{i-1}, q_{i-1}', q_i, q_i', \Lambda_{i-1}, \Lambda_i\right) = 0 \tag{10}$$
$$D_5 L\left(q_i, q_i', q_{i+1}, q_{i+1}', \Lambda_i, \Lambda_{i+1}\right) + D_6 L\left(q_{i-1}, q_{i-1}', q_i, q_i', \Lambda_{i-1}, \Lambda_i\right) = 0$$

1.2 A formulation for contact

The problem of a beam in contact with a rigid wall can be described by a further augmented Lagrangian $\tilde{L}$, [5].

$$\tilde{L} = L - kg(q)\Lambda_c + \frac{p}{2}g(q)^2 - \frac{1}{2p}\left(dist\left(k\Lambda_c - pg(q), \mathbb{R}^+\right)\right)^2 \tag{11}$$

Here, k is a scaling factor, g is the gap function w.r.t. the wall, Λ_c is a Lagrange multiplier, and p is a positive penalty coefficient. One can recognise a Lagrange multiplier term, a penalty term and distance term, where $\xi = k\Lambda_c - pg$ is the augmented Lagrange multiplier. The latter can represent two possible scenarios, i.e. for $\xi < 0$ the contact forces are not active, while for $\xi > 0$ the contact is activated.

In this particular case study, we are interested in investigating the elastica in a narrow environment as in Figure 1 where two rigid straight walls are implemented.
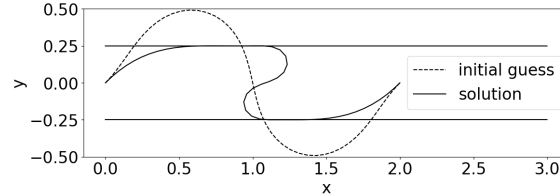

Figure 1: Case study: beam in a narrow straight tube

1.3 Running the code

The code is implemented in Python 3.9.12 using Spyder 5.1.5 and it is organised as follow:

- *elastica_Lagrangian_2walls*: main code containing the augmented Lagrangian formulation of the elastica in a straight tube, the boundary contidions (q_0, q_0') and (q_N, q_N'), the solution of the non-linear system and the study of the contact forces;

- *param*: definition of parameters such as bending stiffness of the beam EI, number of nodes n_nodes for space discretisation, parameters k and p for solving the contact problem, $r1$ and $r2$ for definition of the gap function w.r.t. the upper and down walls respectively;

- *rigid_wall_up* and *rigid_wall_down*: functions for contact problem description w.r.t the upper and down walls respectively;

- *mod_QVL*: functions useful to modify the structure of the matrix of q_i, q_i' and λ_i.

A Python environment, such as Spyder, is used to run the code.. Modifications to the code are possible by editing param.py.

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1.4 Conclusion and remarks

This contribution to deliverable D2.2 presents a variational formulation of the elastica in static equilibrium in a narrow straght tube. The open-source code related to this problem is the first step for the dynamic simulation of the elastica in a narrow environment with a more complex shape, i.e. L-shaped or U-shaped tube. This will help in modelling endoscopes undergoing certain load cases during surgeries, for better understanding of the device behaviour due to complex movements during endoscopies.

References

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[3] L. Colombo, S. Ferraro, and D. Martín de Diego, "Geometric integrators for higher-order variational systems and their application to optimal control," *Journal of Nonlinear Science*, vol. 26, pp. 1615–1650, 2016.

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