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The term structure of credit spreads with jump risk

Chunsheng Zhou *

Guanghua School of Management, Beijing University, Beijing 100871, People's Republic of China
The Anderson Graduate School of Management, The University of California, Riverside, CA 92521, USA

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Abstract

Default risk analysis is important for valuing corporate bonds, swaps, and credit derivatives and plays a critical role in managing the credit risk of bank loan portfolios. This paper offers a theory to explain the observed empirical regularities on default probabilities, recovery rates, and credit spreads. It incorporates jump risk into the default process. With the jump risk, a firm can default instantaneously because of a sudden drop in its value. As a result, a credit model with the jump risk is able to match the size of credit spreads on corporate bonds and can generate various shapes of yield spread curves and marginal default rate curves, including upward-sloping, downward-sloping, flat, and hump-shaped, even if the firm is currently in a good financial standing. The model also links recovery rates to the firm value at default so that the variation in recovery rates is endogenously generated and the correlation between recovery rates and credit ratings before default reported in Altman [J. Finance 44 (1989) 909] can be justified. © 2001 Elsevier Science B.V. All rights reserved.

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* Tel.: +1-909-787-6448; fax: +1-909-787-3970.
E-mail address: chunsheng.zhou@ucr.edu (C. Zhou).

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1. The term structure of credit spreads with jump risk

There are two basic approaches to modeling corporate default risks. The structural approach, pioneered by Black and Scholes (1973) and Merton (1974) and extended by Black and Cox (1976), Longstaff and Schwartz (1995), Leland (1998), Zhou (2001), and others, explicitly models the evolution of the firm value. A firm defaults when its market value falls below certain exogenously given threshold level or the value of its debt. One critical common assumption of the Merton–Black–Cox–Longstaff–Schwartz approach is that the evolution of firm value follows a diffusion process. Under a diffusion process, because a sudden drop in the firm value is impossible, firms never default by surprise. Thus, the large credit spreads of corporate bonds, especially those with short maturities, are unexplained in the context (see, e.g., Jones et al., 1984; Sarig and Warga, 1989; Fons, 1994).

The reduced-form approach, adopted by Duffie and Singleton (1999), Jarrow et al. (1997), Madan and Unal (1994), and others, does not consider the relation between default and the firm value in an explicit (or structural) way. It treats default as an unpredictable Poisson event. It is not clear from this approach what the link is between the structure of a firm's assets and liabilities and the firm's default risk. For example, since the hazard rate of default in the reduced-form approach is modeled as an exogenous process, it is unknown what economic mechanism is behind the default process. Therefore, even though the reduced-form approach generates the rich dynamics of the term structure of credit spreads, it provides few theoretical insights on the causes of these dynamics. Moreover, according to Duffie (1999), the parameters of the reduced-firm models are unstable when the models are applied to fit observed yield spreads.

The main objective of this paper is to build a model that possesses the advantages of both the reduced-from approach and the traditional diffusion approach. That is, on one hand, the new model should be flexible to capture both short-term and long-term yield spreads and default rates; on the other hand, the new model should also provide sufficient conceptual insights on the economic mechanism of default risk. To fulfill this objective, the paper develops a structural approach to valuing risky debt by modeling the evolution of firm value as a jump-diffusion process. The jump-diffusion model has many nice features, including:

- The jump-diffusion model is consistent with the fact that bond prices often drop in a surprising manner at or around the time of default (Beneish and Press, 1995; Duffie and Lando, 1997). Duffie and Lando (1997) attribute this phenomenon to incomplete accounting information. That is, around the time of default, substantial accounting information about the issuer will be revealed to the market. Because of a jump in market information, bond prices jump accordingly. The jump caused by incomplete information is just a special case of many possible kinds of jumps, such as lawsuits and sudden financial turmoils.

- Jump risk can substantially raise credit spreads, especially the spreads of short-term bonds. A jump-diffusion model has the flexibility to generate a wide variety of the term structure of credit spreads, including upward sloping, flat, hump-shaped or downward sloping. [1] In a diffusion model, some of these shapes (flat and downward-sloping) do not exist.
- In a jump-diffusion model, the remaining value of a firm at default is an endogenous random variable that is not necessarily equal to the default boundary. Thus, the model is able to endogenously generate random variations in recovery rates that are linked to a firm's capital structure and asset value at default. In a Longstaff–Schwartz-type diffusion model, however, the value of a firm at default should be the same as the prespecified default boundary.
- For a bond with a relatively short time-to-maturity remaining, its recovery rate at default is positively correlated with the credit quality of the bond before default. This implication is consistent with a number of stylized empirical regularities detailed in Altman (1989).

The importance of jump processes in pricing risky bonds was also noticed by Mason and Bhattacharya (1981). In Mason and Bhattacharya's model, the evolution of firm value follows a pure jump process with jump amplitude following a binomial distribution. Our jump-diffusion model is more flexible and more general. It is also more realistic. In our model, the dynamics of firm value have two random components: a continuous diffusion component and a discontinuous jump component. The jump amplitude follows a log-normal distribution rather than a binomial distribution.

The remainder of this paper is structured as follows. Section 1 presents the basic economic framework. Section 2 solves the general model. The implications of the model are illustrated in Section 3. Section 4 extends our economic framework to allow for stochastic interest rates. Section 5 concludes.

2. The model

This section builds a continuous-time valuation framework for risky debt. The basic assumptions are listed and discussed below. Some of them parallel those of Merton (1974), Black and Cox (1976), and Longstaff and Schwartz (1995).

[1] For the empirical documentation of these shapes of credit spread curves, see, e.g., Fons (1994) and Sarig and Warga (1989).

Assumption 1. Let V denote the total market value of the assets of the firm. The dynamics of V are given by the following jump-diffusion process:

$$\frac{\mathrm{d}V_t}{V_t} = (\mu - \lambda v)\,\mathrm{d}t + \sigma\,\mathrm{d}Z_1 + (\Pi - 1)\,\mathrm{d}Y, \tag{1}$$

where μ represents the expected return on the firm's assets; v, λ, and σ are positive constants; Z_1 is a standard Brownian motion; $\mathrm{d}Y$ is a Poisson process with intensity parameter λ; $\Pi > 0$ is the jump amplitude with expected value equal to $v+1$; $\mathrm{d}Z_1$, $\mathrm{d}Y$, and Π are mutually independent.

We assume that Π follows an i.i.d. log-normal distribution, such that

$$\ln(\Pi) \sim N\left(\mu_\pi, \sigma_\pi^2\right). \tag{2}$$

This assumption implies that

$$v := E[\Pi - 1] = \exp\left(\mu_\pi + \sigma_\pi^2/2\right) - 1.$$

The diffusion process in Eq. (1) characterizes the 'normal' fluctuation in the firm value, due to gradual changes in economic conditions or the arrival of new information which causes marginal changes in the firm's value. The jump component describes the sudden changes in the firm value due to the arrival of important new information. For a detailed discussion of jump-diffusion processes, see, for example, Kushner (1967), Merton (1974) and Zhou (2001). Empirical evidence on jumps in asset values is abundant (see, e.g., Bates, 1996; Jorion, 1988; Kon, 1984).

Assumption 2. There exists a time-dependent positive threshold value K_t for the firm at which financial distress occurs. The firm continues to operate and to be able to meet its contractual obligations as long as $V_t > K_t$. However, if its value V_t falls to or below the threshold level K_t, it defaults on all of its obligations immediately and some form of corporate restructuring takes place.

Following Black and Cox (1976), we assume that the time dependence of K_t takes an exponential form: $K_t = \mathrm{e}^{\phi t} K_0$. In a related work, Longstaff and Schwartz (1995) assume that the default boundary K_t is constant through time, i.e., $\phi = 0$. Their assumption is a special case of ours.

There are many interpretations of the default boundary K_t. Black and Cox (1976) interpret K_t as the minimum firm value required by the safety covenant of the debt contract for the firm to continue its operation. If the value of the firm falls to K_t, bondholders are entitled to force the firm into bankruptcy and obtain the ownership of the firm's assets. According to Black and Cox, K_t takes an exponential form in t because the expected debt value usually takes this form. In many practical applications, K_t is set to a weighted average of the firm's long-term and short-term liabilities. For instance, in KMV's Credit Monitor,[2] K_t is set to 100% of the firm's short-term liabilities plus 50% of the firm's long-term liabilities. In this case, ϕ can be easily interpreted as the growth rate of firm i's liabilities. In a recent paper, Collin-Dufresne and Goldstein (2000) consider a Longstaff and Schwartz-type model in which the ratio of the firm value to the default boundary follows a stationary, mean-reverting process. This, they argue, reflects a company's tendency to adjust its capital structure to a target leverage ratio. This mean-reversion feature in the leverage ratio can significantly increase the credit spreads of long-term debt, but has little impact on the short-term credit spreads since the change in the default boundary in the short term is negligible.

In Black and Cox (1976) and Longstaff and Schwartz (1995), because the firm's value V has a continuous path, V_t is always equal to K_t in the event of default. In our model, because V has a jump component, V_t is a random number at default time t that is not necessary to be equal to K_t. It is quite natural in our model to randomize the recovery rate of debt issues and to link this rate to the firm's value if a default occurs.

Assumption 3. The firm issues both equity and debt (bonds). If it defaults during the life of a bond, the bond holder receives $1 - w(X_s)$ times the face value of the security at maturity T. Here $s = \min(\tau, T)$ with τ being the time of default and $X_t := V_t/K_t$ is the ratio of firm's value V_t to the threshold level K_t.

This assumption follows Longstaff and Schwartz (1995). The assumption that bondholders will be paid at the maturity time even though a default may have occurred before that time is made for expositional convenience. One can easily relax this assumption by assuming that bondholders get paid immediately at default time without affecting the basic structure of the model.

In practice, w is usually a non-increasing function of X, that is, the inequality $w'(X) \leqslant 0$ holds. The factor w represents the percentage write-down on a bond if there is a reorganization of the firm. When $w = 0$, there is no write-down and bondholders are not affected by the firm's reorganization. When $w = 1$, bondholders receive nothing in a reorganization.

In general, w will differ across various bond issues in the firm's capital structure. For example, Altman and Bencivenga (1995) find that the average recovery rate $(1 - w)$ for secured, senior, senior subordinated, cash-pay subordinated, and non-cash-pay subordinated debt for a sample of defaulted bond issues during the 1985–1994 period is 0.593, 0.508, 0.365, 0.306, and 0.187,

[2] KMV is a financial consulting company.

respectively. Similar results are also found by Altman (1992) and Franks and Torous (1994). It is important to note that even for the same class of bond issues, the write-down w differs significantly across different time periods and different firms. Altman and Bencivenga (1995) report, for example, the average recovery rate for the defaulted issues of secured debt is 0.827 in 1989 but is only 0.120 in 1987. A number of factors may have contributed to this large disparity, but the firm's value in the event of default is certainly important. Most valuation models do not explain the variation in write-down ratios for the same kind of bonds. A primary advantage of our model is that the model generates such variation endogenously.

Assumption 4. The short-term risk-free interest rate r is constant over time.

This assumption is made for convenience only and will be relaxed in Section 4. We also make an additional assumption, stated below, regarding the market premium of jump risk: investors have time-separable preferences.

3. The solution to the model

Based on the above model specification, the default event is characterized by the first time τ such that $V_\tau \leqslant K_\tau$, or equivalently, $X_\tau = V_\tau / K_\tau \leqslant 1$. Mathematically,

$$\tau := \inf\{t \,|\, X_t \leqslant 1,\ t \geqslant 0\}.$$

Explicit solutions for first passage times are not known, except for some very special diffusion processes. [3]

We now turn to the pricing issue. Because our basic results about the impact of jump risk on defaults are not sensitive to coupon structure, we will focus our attention on zero-coupon bonds.

If a default occurs before or at the maturity of a bond, the bondholder will receive, at the maturity time, a value of $1 - w(X_\tau)$ times the face value. Using the well-known results in the derivative pricing literature, the price of a discount bond promising to pay \$1 at time T can be expressed as

$$B(X, T) = \exp(-rT) - \exp(-rT)E^Q[w(X_\tau)I_{\tau \leqslant T}], \tag{3}$$

where Q is the risk-adjusted probability measure and I is an indicator function such that

$$I_{\tau \leqslant T} = \begin{cases} 1 & \text{if } \tau \leqslant T, \\ 0 & \text{otherwise.} \end{cases}$$

[3] See Abrahams (1986) for a survey on the first passage time problem.

According to Bates (1991, 1996), under the assumption that investors have time-separable power utility and other regularity assumptions, X_t follows the following jump-diffusion process under the risk-adjusted probability measure:

$$\mathrm{d}\ln(X) = \left(r - \phi - \sigma^2/2 - \lambda_q \cdot \nu_q\right)\mathrm{d}t + \sigma\,\mathrm{d}Z_1 + \ln t(\Pi_q)\,\mathrm{d}Y_q, \tag{4}$$

where $\lambda_q = \lambda \cdot E[1 + (\Delta J_W)/J_W]$ is the jump frequency under Q, with J_W being the investor's marginal utility of wealth and ΔJ_W being the jump in marginal utility of wealth conditional on a jump in V occurring; $\mathrm{d}Y_q$ is a Poisson process with intensity parameter λ_q; $\Pi_q > 0$ is the jump amplitude: $\ln(\Pi_q) \sim N(\mu_{\pi q}, \sigma_\pi^2)$; $\mu_{\pi q} = \mu_\pi + \mathrm{Cov}[\Pi, (\Delta J_W)/J_W]/E[1 + (\Delta J_W)/J_W]$; $\nu_q = E[\Pi_q - 1] = \exp(\mu_{\pi q} + \sigma_\pi^2/2) -1$.

Eq. (4) suggests that the impact of r and ϕ on bond price can be summarized in one term, $r - \phi$. The differences between the risk-adjusted parameters λ_q and $\mu_{\pi q}$ and the true parameters λ and μ_π depend on the risk aversion of investors and how 'systematic' jumps are. The more idiosyncratic is jump risk, the smaller are risk adjustments in the jump parameters, that is, the smaller are $|\lambda_q - \lambda|$ and $|\mu_{\pi q} - \mu_\pi|$. If the jump risk is diversifiable, then according to Merton (1976), $\lambda_q = \lambda$ and $\mu_{\pi q} = \mu_\pi$. Bates (1991) shows that λ_q and $\mu_{\pi q}$ do not differ either qualitatively or quantitatively from λ and μ_π, that is, $\lambda_q \approx \lambda$ and $\mu_{\pi q} \approx \mu_\pi$, even if the jump risk is completely systematic.

The following theorem provides a tractable approach to valuing the bond in the general framework where a default can occur at any time.

Theorem 1. *Assume $X > 1$. The price of a risky discount bond, $B(X, T)$, given in Eq. (3), can be expressed as*

$$B(X, T) = \exp(-rT) - \exp(-rT) \lim_{n\to\infty} \sum_{i=1}^{n} E^Q t\left[w\left(X_{t_i}^*\right) \mid \Omega_i\right] Q_i, \tag{5}$$

where

$$t_i = \frac{i}{n} T,$$
$$\Omega_i = \left\{X_{t_i}^* \leqslant 1 \text{ and } X_{t_j}^* > 1 \ \forall j < i\right\},$$
$$Q_i = Q(\Omega_i),$$

and moreover, $X_{t_i}^*$ is defined recursively as

$$X_{t_0}^* = X,$$
$$\ln(X_{t_i}^*) - \ln(X_{t_{i-1}}^*) = x_i + y_i \cdot \pi_i, \quad i = 1, 2, \ldots, n.$$

Here x_i, y_i, and π_i are mutually and serially independent random variables drawn from

$$x_i \sim N\left[\left(r - \phi - \sigma^2/2 - \lambda_q \cdot v_q\right) \cdot \frac{T}{n}, \sigma^2 \cdot \frac{T}{n}\right],$$

$$\pi_i \sim N\left(\mu_{\pi q}, \sigma_\pi^2\right),$$

and

$$y_i = \begin{cases} 0 & \textit{with prob. } 1 - \lambda \cdot T/n, \\ 1 & \textit{with prob. } \lambda \cdot T/n. \end{cases}$$

Briefly speaking, the theorem holds because in a very short-time period, no more than one jump can occur and the diffusion process cannot move a large distance almost surely. The proof of the theorem is outlined in Appendix A.

One feature of Theorem 1 is that the write-down $w(X)$ in the event of default can be any continuous function. Another feature is that the movement of X_t is governed by two simple probability distributions: normal distributions and binomial distributions. The following is a Monte Carlo approach to valuing $B(X, T)$.

- *Step 1*. Divide the time interval $[0, T]$ into n equal sub-periods for sufficiently large n, say $n = 100$ or $n = 500$. Denote $t_i \equiv T \cdot i/n$.
- *Step 2*. Do Monte Carlo simulations by repeating the following sub-procedures for M ($j = 1, 2, \ldots, M$) times. Typically, one can choose M between 10,000 and 100,000.
  (a) For each j, generate a series of mutually and serially independent random vectors (x_i, π_i, y_i) for $i = 1, 2, \ldots, n$ according to distributions described in Theorem 1.
  (b) Let $X^*_{t_0} = X$ and calculate $\ln(X^*_{t_i})$ or $X^*_{t_i}$ according to the formula

$$\ln\left(X^*_{t_i}\right) = \ln\left(X^*_{t_{i-1}}\right) + x_i + y_i \cdot \pi_i \quad \text{for } i = 1, \ldots, n.$$

  (c) Find the smallest integer $i \leqslant n$ such that $\ln(X^*_{t_i}) \leqslant 0$.[4] If such an i exists, let $W_j = w(X^*_{t_i})$. Otherwise, $W_j = 0$.
- *Step 3*. Let $B(X, T) = \exp(-rT)(1 - \sum_{j=1}^{M} W_j/M)$. $B(X, T)$ will be a numerical solution to the bond price.

[4] Mathematically, t_i obtained in this way is the first passage time of X_t to the lower bound 1 in a discretized model.

4. Empirical implications

The following numerical examples illustrate some of the rich implications of the jump-diffusion model. For simplification, we assume that the write-down function has a linear form

$$w(X) = w_0 - w_1 X,$$

where w_0 and w_1 are non-negative constants. [5]

First, we demonstrate the effect of the jump component on the pricing of default-risky bonds. To do this, we keep the instantaneous volatility [6] of X ($\mathrm{Var}[\mathrm{d}\ln(X)]/\mathrm{d}t$), constant as we change the parameter values which govern the random components of $\mathrm{d}X$, so that the variations in bond prices are truly caused by the relative importance of the jump component rather than by the changes in the overall volatility of the firm's value. Also, by fixing the total volatility of the firm value, we can clearly show the effect of misspecifying a jump-diffusion process as a pure diffusion process in credit analysis.

Eq. (4) implies that under the risk-adjusted measure Q, the volatility of X_t satisfies

$$\sigma_X^2 \equiv \frac{\mathrm{Var}(\mathrm{d}\ln(X))}{\mathrm{d}t} = \sigma^2 + \lambda_q \cdot \sigma_\pi^2 \tag{6}$$

if $\mu_{\pi q} = 0$. We will keep $\mu_{\pi q} = 0$ and $\sigma_X^2 = 0.035$ in our numerical simulations [7] so that the results shown here are really driven by the change in the extent of 'discontinuity' of the firm's value process rather than by the variation in the volatility of the firm's value.

As is well known, a diffusion process has a continuous sample path and cannot cross a boundary from somewhere else instantaneously. Therefore, under a diffusion process, if a firm is not currently in financial distress ($X > 1$), its probability of defaulting on very short-term debt is nearly 0 and therefore

[5] Because the recovery $1 - w(X)$ can be negative under this assumption, the limited liability rule is not satisfied here. To impose the limited liability restriction, one may assume that $w(X) = \min(1, w_0 - w_1 X)$. However, we find in our numerical simulations that the probability of a negative recovery rate (i.e., $w_0 - w_1 X > 1$) is very small and that the limited liability restriction has little impact on credit spreads. For simplicity, the limited liability restriction will not be imposed.

[6] Because $\ln X_t = \ln V_t - \ln K_t$, when K_t is a deterministic function of time t, we have $\mathrm{Var}(\mathrm{d}\ln X_t)/\mathrm{d}t = \mathrm{Var}(\mathrm{d}\ln V_t)/\mathrm{d}t$.

[7] Denote E as the equity value of a firm. According to Ingersoll (1987), the instantaneous volatility of E can be given as $\sigma_E = (X/(X-1)) \cdot (\mathrm{d}E/\mathrm{d}V) \cdot \sigma_X$, which is about 30% a year when $\sigma_X^2 = 0.035$. This is a slightly conservative choice of the volatility of individual stock prices. Obviously, we would obtain higher credit spreads if we choose higher volatility for X.

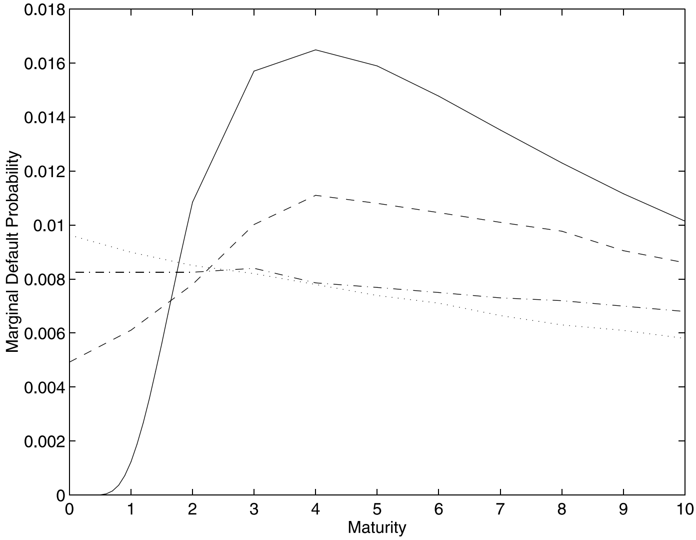

Fig. 1. The relationship between marginal default probabilities and jump size volatility σ_π^2. The parameter values used are $X = 2.0$, $r = 0.05$, $\lambda_q = 0.05$, $\mu_{\pi q} = 0.0$, $\sigma_X^2 = 0.035$, $w_0 = 1.4$, $w_1 = 1.0$, $\phi = 0$, and $\sigma^2 = \sigma_X^2 - \lambda_q \cdot \sigma_\pi^2$. (—): $\sigma_\pi^2 = 0.00$; (- -): $\sigma_\pi^2 = 0.25$; (–·): $\sigma_\pi^2 = 0.50$; (···): $\sigma_\pi^2 = 0.65$.

the marginal default probability curve [8] of the firm is upward-sloping at the beginning, as shown in Fig. 1 with $\sigma_\pi^2 = 0$. In the real world, the default probabilities of short-term bonds are often much larger than 0. If the evolution of the firm value follows a jump-diffusion process, however, the story will be different. Under a jump-diffusion process, a default can happen instantaneously because of a sudden drop in the firm value. As a result, a jump-diffusion model can generate many different shapes of marginal default probability curves, including upward-sloping, downward-sloping, flat, and hump-shaped. This is interesting since all the shapes of default rate curves shown in the figure have been identified in empirical studies. For instance, Fons (1994) found that different bonds have very different shapes of default rate curves. Only highly rated bonds have upward-sloping term structures of marginal default rates with initial default probabilities close to 0. Low-grade bonds often have non-zero initial default rates. Under certain circumstances, the marginal default rates of low-grade bonds could be flat or even downward-sloping. Without jumps, it would be very hard to understand this kind of default rate curves.

[8] Denote $P(t)$ as the cumulative default probability of a firm over time period $[0, t]$. The marginal default probability at time t is defined as $[\mathrm{d}P(t)]/\mathrm{d}t$, which is the default probability at a given instant.

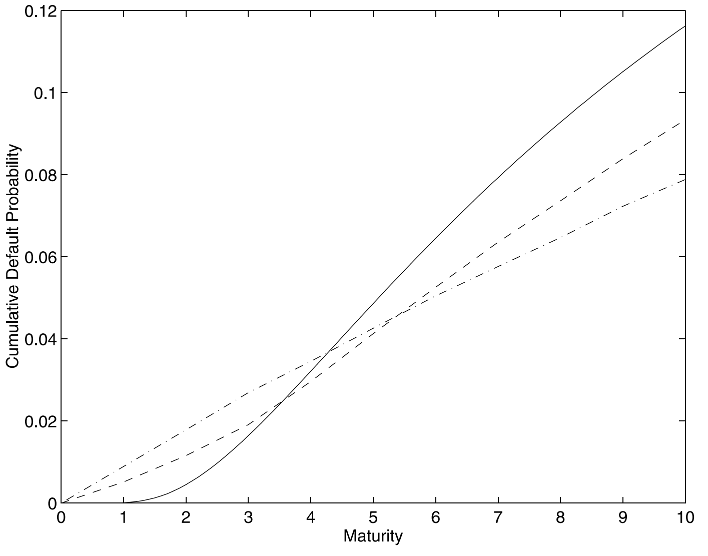

Fig. 2. The relationship between cumulative default probabilities and jump size volatility σ_π^2. The parameter values used are $X = 2.0$, $r = 0.05$, $\lambda_q = 0.05$, $\mu_{\pi q} = 0.0$, $\sigma_X^2 = 0.035$, $w_0 = 1.4$, $w_1 = 1.0$, $\phi = 0$, and $\sigma^2 = \sigma_X^2 - \lambda_q \cdot \sigma_\pi^2$. (—): $\sigma_\pi^2 = 0.00$; (- -): $\sigma_\pi^2 = 0.25$; (-·): $\sigma_\pi^2 = 0.50$.

As for cumulative default probabilities, [9] Fig. 2 shows that holding constant the volatility of X and the jump intensity λ_q, a firm with a more volatile jump component (i.e., a larger σ_π^2) is more likely to default on its short-maturity bonds than is a firm with a more volatile diffusion component. Interestingly, Fig. 2 also illustrates the reverse relation at longer maturities. A firm with a more volatile diffusion component is more likely to default on its long-maturity bonds than is a firm with a more volatile jump component. Because this is a surprising result, we now outline its intuition.

For a given $T > 0$ which is not very small in magnitude, an increase in the volatility of a diffusion process can substantially increase the probability of default during period $[0, T]$. However, for a jump process, the effect on default probability of the jump size volatility σ_π^2 is largely limited by the jump intensity λ_q. If λ_q is very small such that $\lambda_q T$ is also a small number, then the probability that there is at least one jump in period $[0, T]$ is approximately $\lambda_q T$. As a result, no matter how large the jump size volatility σ_π^2 is, the probability of default in

[9] Cumulative default probabilities reported in this paper are not annualized. They are calculated under risk-adjusted probability measure Q. They often look higher than default probabilities of real bonds under the physical probability measure. This is because the drift of a firm's value under the risk-adjusted measure is smaller than the corresponding real drift under the physical measure.

period $[0, T]$ caused by the jump process is always smaller than $\lambda_q T$, even though $\lambda_q T$ is already small. In this case, an increase in σ_π^2 mainly affects the remaining value of the firm upon default but has a very small effect on the default probability. This intuition is demonstrated more rigorously in the following concrete example.

Consider two extreme X processes for illustration. The first one is a pure diffusion process with the volatility σ^2 and the second one is a pure jump process with a small jump intensity λ_q and a large volatility of jump amplitude $\sigma_\pi^2 \equiv \mathrm{Var}(\ln(\Pi))$. We assume that $\sigma^2 = \lambda_q \cdot \sigma_\pi^2 = 0.035$, the same as the volatility of $\ln(X)$ used in Fig. 2.

Denote $F(T)$ as the cumulative distribution function of the first passage time to default for the pure diffusion process and $J(T)$ as the cumulative distribution function of the first passage time to default for the pure jump process, where T is the maturity time and default occurs whenever X falls to or below 1. Using the result of Harrison (1990), we have

$$F(T) = N\left(-\frac{\ln(X) + (r - \phi - \sigma^2/2)T}{\sigma\sqrt{T}} \right) + X^{1-2r/\sigma^2} N\left(-\frac{\ln(X) - (r - \phi - \sigma^2/2)T}{\sigma\sqrt{T}} \right). \tag{7}$$

Assuming that $X = 2$ at time 0 [10] and $r = 0.05$ as in Fig. 2, we obtain immediately that $F(1) = 0.0001$ and $F(10) = 0.116$.

There is no explicit expression for $J(T)$. However, in a pure jump process with a positive drift, default must be caused by jumps. Assume that $\lambda_q = 0.01$ and that $\sigma_\pi^2 = 0.035/0.01 = 3.50$. If $T = 1$, then the probability of one jump in $[0, T]$ is about $\lambda_q T = 0.01$ and the probability of two or more jumps in $[0, T]$ is small enough to ignore. If a jump occurs at time $t < 1$ and there are no other jumps before t, the probability that X_t falls to or below 1 is

$$N(-\ln(X)/\sigma_\pi) = N\left(-\ln(2)/\sqrt{3.5} \right) = 0.36.$$

As a result, we have $J(1) \approx 0.01 \times 0.36 = 0.0036$.

Now let us consider $T = 10$. The probability that there is no jump in the time interval $[0, T]$ is

$$\exp(-\lambda_q T) = \exp(-0.01 \times 10) = 0.90.$$

[10] We assume that $X = 2$ in the most of our numerical simulation exercises. Using the leverage data reported in Wigmore (1990) and Blume et al. (1998), $X = 2$ approximately corresponds to BB+ or BB rating category. As mentioned earlier, according to KMV's model, the face value of a long-term bond should be multiplies by some number less than 1 in calculating K.

That is, the probability that there are one or more jumps in $[0, T]$ is $1 - 0.90 = 0.10$. Denote d as the conditional probability of a default if there are jumps. Then we have

$$J(10) = 0.10 \cdot d \ll 0.10.$$

As a matter of fact, no matter how one increases the volatility σ_π^2, $J(10)$ is always much smaller than 0.10.

Obviously, $J(1) \gg F(1)$ and $J(10) \ll F(10)$. That is, a jump process is more likely to cause a default over a short horizon but less likely to cause a default over a long horizon than a diffusion process. This explains why when we fix the total volatility of the mixed jump-diffusion firm value process, σ_X^2, a firm with a relatively larger jump component is more likely to default over a short horizon but less likely to default over a long horizon.

Straightforwardly, under a jump-diffusion process, a firm's value V can jump below the boundary K without hitting it. This implies that the remaining value of the firm upon default is random and is possibly less than K. If V is stochastic, it is very natural that the recovery rate of a defaulted bond is also stochastic because what bondholders recover upon default depends on the remaining value (V) of the firm. It is intuitive that the volatility of write-down increases with the volatility of the jump component, σ_π^2, which provides a reasonable explanation for why the recovery rates of bonds are volatile and unpredictable.

Generally, the larger the jump size volatility σ_π^2 is the farther X is below 1 on average upon default. (If X follows a pure diffusion process then upon default, X always equals to 1.) This implies that average write-downs of bonds are larger when the jump volatility is larger. (Recall that a lower X at default means a lower recovery rate or a higher write-down.) In the examples shown in Fig. 3 ($w_0 = 1.4$ and $w_1 = 1.0$), the average write-down is 0.40 when $\sigma_\pi^2 = 0.00$ or there is no jump component. It rises to 0.50–0.55 when $\sigma_\pi^2 = 0.25$ and increases further to about 0.65 when $\sigma_\pi^2 = 0.50$. Fig. 3 also shows that under a jump-diffusion process, not only the ex post recovery of a bond is not a constant, the ex ante recovery rate of a bond is not a constant either. Because a diffusion process is almost unlikely to cause a default in a short period of time, the defaults of short-term bonds are usually caused by the jump component of the firm value. As the maturity gets longer, the probability that a default is caused by the diffusion process becomes larger. In other words, a three-year bond's default is more likely caused by the diffusion process than a two-year bond's default does, conditional on that the defaults of bonds have occurred. If a default is caused by the diffusion component, then $X = 1$ upon default; while if a default is caused by the jump component, upon default, $X < 1$ with probability 1. As a result, under a jump-diffusion process, short-maturity bonds are usually have lower expected recovery rates (higher expected write-downs) than are long-maturity bonds.

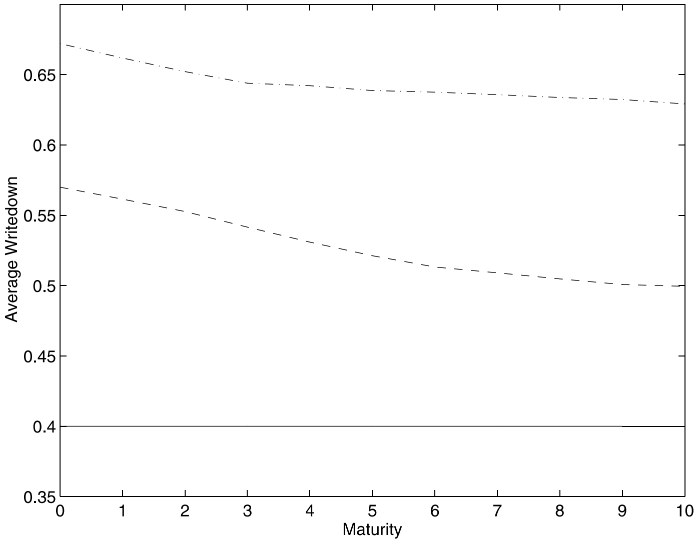

Fig. 3. The relationship between jump size volatility σ_π^2 and expected write-down. The parameter values used are $X = 2.0$, $r = 0.05$, $\lambda_q = 0.05$, $\mu_{\pi q} = 0.0$, $\sigma_X^2 = 0.035$, $w_0 = 1.4$, $w_1 = 1.0$, $\phi = 0$, and $\sigma^2 = \sigma_X^2 - \lambda_q \cdot \sigma_\pi^2$. (—): $\sigma_\pi^2 = 0.00$; (- -): $\sigma_\pi^2 = 0.25$; (–·): $\sigma_\pi^2 = 0.50$.

Theoretically, the risk-neutral default probability and expected recovery rate upon default determine the credit spread of a bond. According to Fig. 1, under a diffusion process, if a firm is not currently in financial distress ($X > 1$), its probability of defaulting on very short-term debt is nearly 0 and therefore, its short-term debt should have virtually 0 credit spreads, as shown in Fig. 4 with $\sigma_\pi^2 = 0$. This strong implication of diffusion models for credit spreads is not valid in the real world. The credit spreads of typical short-term bonds are significantly above 0. As mentioned earlier, Fons (1994) and Sarig and Warga (1989) even find that the yield spread curves of certain kind of bonds (BB-rated or B-rated) are relatively flat or downward sloping. [11] As illustrated in Fig. 4, these yield spread curves are captured by a jump-diffusion model with a non-trivial jump component.

[11] Helwege and Turner (1999) argue that credit spread curves of many B-rated bonds are still upward-sloping based on a particular data sample, but they cannot reject that some B-rated bonds really have downward-sloping credit spread curves. Moreover, no one suspects that the credit spreads of most short-term bonds are non-zero. Merton (1974) model which is based on a diffusion approach can generate a downward-sloping credit spread curve only if the firm is exceptionally highly leveraged, that is, if the firm's debt-ratio is greater than 1 or in terms of my modeling assumptions, the current X is smaller than 1. However, according to Helwege and Turner (1999), the data from Standard and Poor's on median book values of debt-to-capitalization ratios by rating indicate that B-rated and even many CCC-rated firms do not have debt ratios greater than 1.

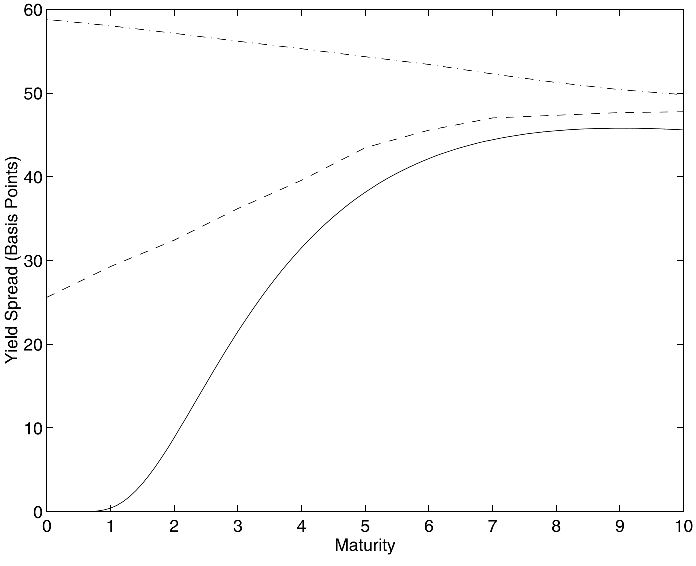

Fig. 4. The relationship between credit spreads and jump size volatility σ_π^2. The parameter values used are $X = 2.0$, $r = 0.05$, $\lambda_q = 0.05$, $\mu_{\pi q} = 0.0$, $\sigma_X^2 = 0.035$, $w_0 = 1.4$, $w_1 = 1.0$, $\phi = 0$, and $\sigma^2 = \sigma_X^2 - \lambda_q \cdot \sigma_\pi^2$. (—): $\sigma_\pi^2 = 0.00$; (- -): $\sigma_\pi^2 = 0.25$; (–·): $\sigma_\pi^2 = 0.50$.

Fig. 4 shows that the jump risk significantly raises the credit spreads of bonds with short- to middle-maturities, even holding constant the total volatility of the firm value. For example, for a two-year discount bond, the annualized credit spread shown in the figure is only seven basis points when the jump component does not exist ($\sigma_\pi^2 = 0$). The spread rises to 32 basis points as σ_π^2 becomes 0.25 and rises further to 57 basis points as σ_π^2 reaches 0.50. This result suggests that a misspecification of stochastic processes governing the dynamics of firm value, i.e., falsely specifying a jump-diffusion process as a continuous Brownian motion process, can substantially understate the credit spreads of corporate bonds. The results here may shed some light on the failure of traditional diffusion models in matching actual yield spread curves. More specifically, compared with the observed yield spreads of corporate bonds, the spreads generated by the traditional diffusion models are much too small (Jones et al., 1984).

The frequency of jump occurrences, λ_q, is another important parameter in characterizing the jump component. Given the volatility of the jump component, $\lambda_q\sigma_\pi^2$, the jump frequency λ_q determines how 'discontinuous' the jump process $\Pi_q \cdot \mathrm{d}Y_q$ is. A larger λ_q and a smaller σ_π^2 mean that jumps occur more frequently but each jump may cause a smaller movement. In other words, the path of the process with a larger λ_q and a smaller σ_π^2 looks more 'continuous' than that of a process with a smaller λ_q but a larger σ_π^2. Fig. 5 plots the relation between credit spreads and the jump intensity parameter λ_q based on a pure

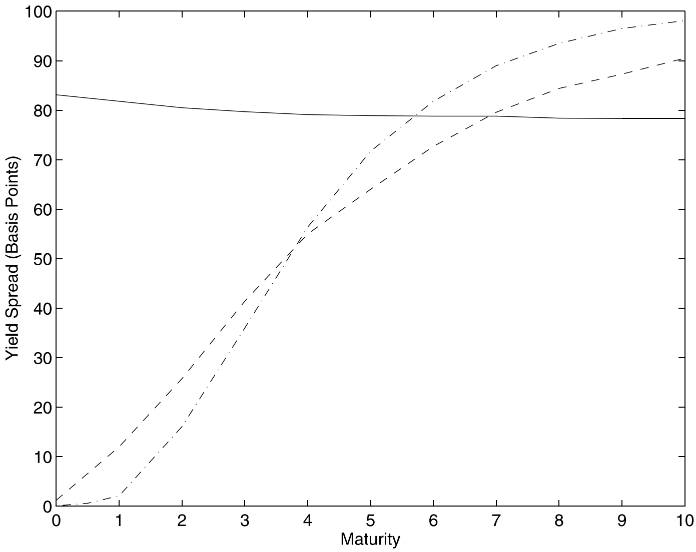

Fig. 5. The relationship between credit spreads and jump intensity λ_q. The parameter values used are $X = 2.0$, $r = 0.05$, $\mu_{\pi q} = 0.0$, $\sigma^2 = 0$, $w_0 = 1.4$, $w_1 = 1.0$, $\phi = 0$, and $\sigma_X^2 = \sigma^2 + \lambda_q \cdot \sigma_\pi^2 = 0.035$. (—): $\lambda_q = 0.10$, $\sigma_\pi^2 = 0.35$; (- -): $\lambda_q = 1$, $\sigma_\pi^2 = 0.035$; (–·): $\lambda_q = 10.0$, $\sigma_\pi^2 = 0.0035$.

jump process in which the instantaneous volatility of $d\ln(X)$ remains constant, i.e., $\sigma_X^2 = \lambda_q \cdot \sigma_\pi^2 = 0.035$. The figure shows an interesting relation between credit spreads and the parameters of the jump process. That is, a large λ_q and a small σ_π^2 are generally associated with low credit spreads of short-term bonds but high credit spreads of long-term bonds. The relationship between the term structure of credit spreads and σ_π^2 (with given $\lambda_q\sigma_\pi^2$) looks similar to Fig. 2.

Figs. 6–8 illustrate the relations between a firm's X and credit spreads, default probabilities, and expected write-downs, respectively. It is not surprising to see from Figs. 6 and 7 that credit spreads and default probabilities decrease with X. The farther is the firm's value V from the threshold level K, the smaller is the likelihood of a default. What is interesting here is the non-monotonic relation between X and the expected write-downs as shown in Fig. 8.

For bonds with very short maturities, a lower initial X generally implies a higher expected write-down [12] or a lower expected recovery rate. This is because a quick default is generally caused by a jump in X. The greater is the initial X, the higher is the expected value of X before a jump, and therefore the higher is the expected value of X after a jump. For bonds with middle matu-

[12] Throughout this paper, we define the expected write-down as the average value of a bond's write-down if the bond defaults before or at its maturity.

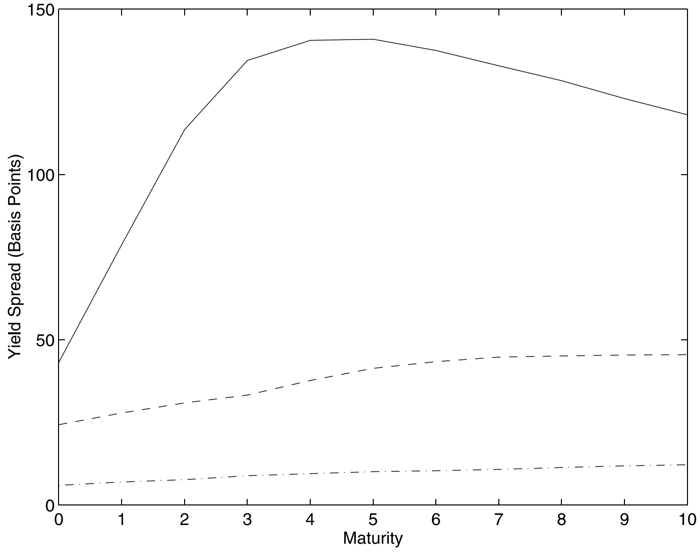

Fig. 6. The relationship between credit spreads and current X. The parameter values used are $r = 0.05$, $\lambda_q = 0.05$, $\sigma^2 = 0.0225$, $\mu_{\pi q} = 0.0$, $\sigma_\pi^2 = 0.25$, $w_0 = 1.4$, $\phi = 0$, and $w_1 = 1.0$. (—): $X = 1.5$; (- -): $X = 2.0$; (–·): $X = 3.0$.

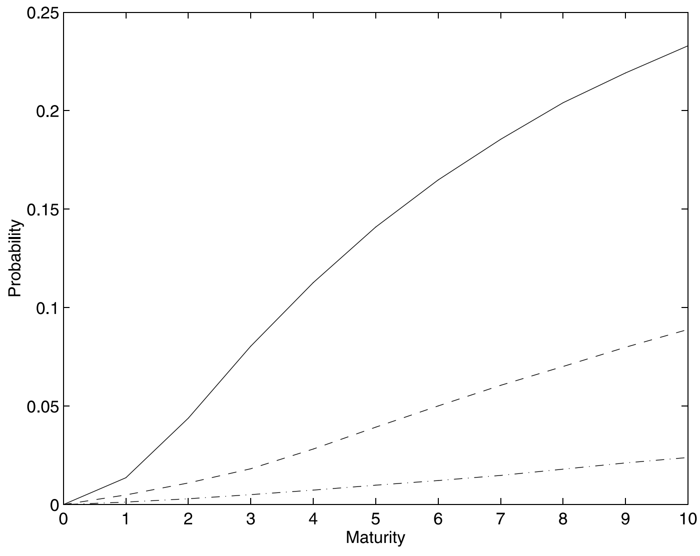

Fig. 7. The relationship between cumulative default probabilities and current X. The parameter values used are $r = 0.05$, $\lambda_q = 0.05$, $\sigma^2 = 0.0225$, $\mu_{\pi q} = 0.0$, $\sigma_\pi^2 = 0.25$, $w_0 = 1.4$, $\phi = 0$, and $w_1 = 1.0$. (—): $X = 1.5$; (- -): $X = 2.0$; (–·): $X = 3.0$.

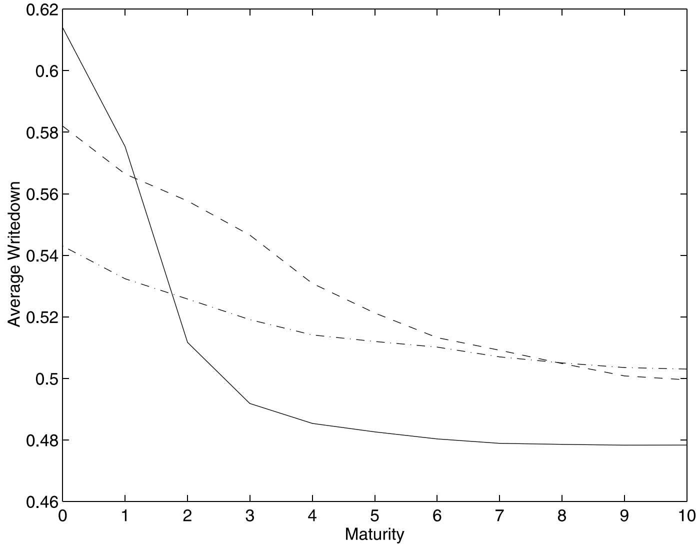

Fig. 8. The relationship between average write-down and current X. The parameter values used are $r = 0.05$, $\lambda_q = 0.05$, $\sigma^2 = 0.0225$, $\mu_{\pi q} = 0.0$, $\sigma_\pi^2 = 0.25$, $w_0 = 1.4$, $\phi = 0$, and $w_1 = 1.0$. (—): $X = 1.5$; (- -): $X = 2.0$; (–·): $X = 3.0$.

rities, if the initial X is close to the default threshold value, 1, the expected write-down is low. This is because a default in this case is very likely caused by the diffusion part of X_t process and there is a good chance that $X = 1$ (the highest value of X upon default) at default. If the initial X is sufficiently far away from 1, expected recovery rates will be positively correlated with the initial X. This is because when the initial X is sufficiently large, the default of the firm will be mainly caused by the jump component of the X process over the middle horizon. As just mentioned, the greater is the initial X, the higher is the expected value of X before a jump, and the higher is the expected value of X after a jump.

Because a firm with a high credit rating usually has a large X before default, the results of Fig. 8 may explain why among various investment grade bond issues, the recovery rates of defaulted bonds are positively correlated with bond ratings before defaults (Altman, 1989).

So far we have used $\phi = 0$ in our numerical simulation exercises. We now analyze the impact of ϕ on credit spreads when ϕ is not 0. Eq. (4) suggests that what really matters in determining credit spreads is $r - \phi$. All other conditions remaining the same, the credit spread curve with $r = 8\%$ and $\phi = 3\%$ would not be different from that with $r = 5\%$ and $\phi = 0$. Therefore, the simulation results shown in Fig. 4 will not change if one assumes $r = 8\%$ and $\phi = 3\%$ instead of $r = 5\%$ and $\phi = 0$. In other words, downward sloping credit spreads are still possible even if K_t is allowed to grow over time.

Of course, when the risk-free rate r is given, a positive drift in K_t (i.e., $\phi > 0$) would raise the default boundary over time. As a result, corporate bonds, especially those with longer maturities, would become riskier than they would be with $\phi = 0$. This may help to explain why the traditional Merton-type model with fixed default boundaries understates the credit spreads of long-term corporate bonds (Jones et al., 1984). Collin-Dufresne and Goldstein (2000) use a different approach to obtain higher yield spreads of long-term bonds. They endogenize K_t so that X_t is mean-reverting and is unlikely to get very large in the future. One could easily extend our model to incorporate such a mean-reverting process for X_t.

5. Stochastic interest rate model

We assumed constant risk-free interest rates earlier. We now relax this assumption by assuming that the instantaneous risk-free interest rates follow a diffusion process:

Assumption 4′. The dynamics of short-term risk-free rates r are given by

$$\mathrm{d}r = \beta(\gamma - r)\,\mathrm{d}t + \eta\,\mathrm{d}Z_2, \tag{8}$$

where γ, β, and η are constants and $\mathrm{d}Z_2$ is a standard Brownian motion. The instantaneous correlation between $\mathrm{d}Z_1$ in Assumption 1 and $\mathrm{d}Z_2$ is $\rho\mathrm{d}t$. $\mathrm{d}Z_2$ is independent of $\mathrm{d}Y$ and Π.

This assumption about the short-term interest rate dynamics is proposed by Vasicek (1977) in his well-known term structure model. It is a straightforward exercise to use other interest rate processes like Cox et al. (1985). Please refer to Appendix B for a technical treatment of bond pricing under the stochastic interest rate process specified in Assumption 4′.

The effect of the correlation between the interest rate movements and the changes in firm's value on credit spreads was first investigated by Longstaff and Schwartz (1995) in a diffusion model. Fig. 9 graphs the relation between credit spreads and the correlation coefficient between the diffusion component of the changes in firm's value and changes in short-term interest rate in our jump-diffusion model. The impact of the correlation coefficient ρ on credit spreads is significant. The reason why the credit spread increases with ρ is that the risk-neutral distribution of future values of X depends on the movements in r. Thus, the variance of changes in X depends on the correlation between changes in X and changes in short-term interest rate. When ρ is positive, the covariance term adds to the total variance of changes in X under the risk-adjusted probability measure, and therefore increases the probability of a default, as shown in Fig. 10. There is substantial empirical evidence that credit spreads are negatively

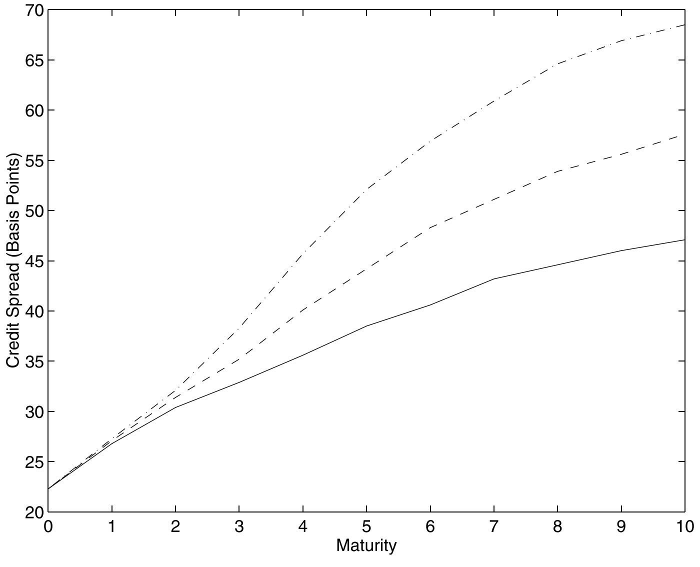

Fig. 9. The effect of correlation ρ on credit spreads. The parameter values used are $X = 2.0$, $r = 0.06$, $\sigma^2 = 0.0225$, $\lambda = 0.05$, $\mu_\pi = 0.0$, $\sigma_\pi^2 = 0.25$, $\sigma_X^2 = 0.035$, $w_0 = 1.4$, $w_1 = 1.0$, $\alpha = 0.05$, $\beta = 1.00$, and $\eta^2 = 0.001$. (—): $\rho = -0.50$; (- -): $\rho = 0.00$; (–·): $\rho = 0.50$.

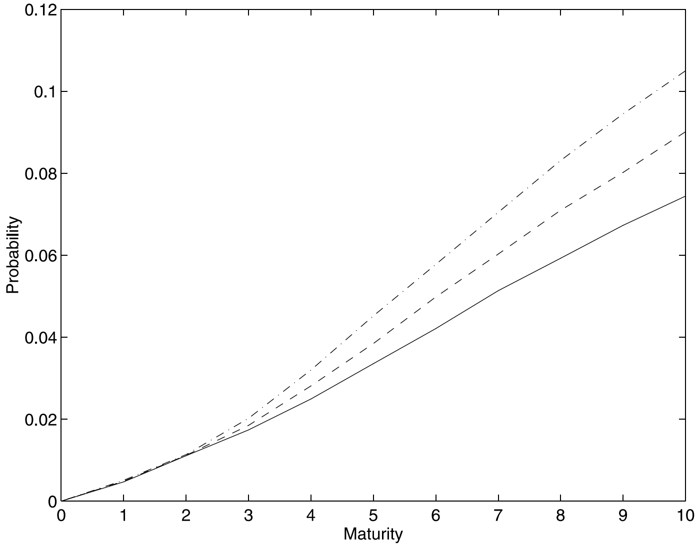

Fig. 10. The effect of correlation ρ on default probabilities. The parameter values used are $X = 2.0$, $r = 0.06$, $\sigma^2 = 0.0225$, $\lambda = 0.05$, $\mu_\pi = 0.0$, $\sigma_\pi^2 = 0.25$, $\sigma_X^2 = 0.035$, $w_0 = 1.4$, $w_1 = 1.0$, $\alpha = 0.05$, $\beta = 1.00$, and $\eta^2 = 0.001$. (—): $\rho = -0.50$; (- -): $\rho = 0.00$; (–·): $\rho = 0.50$.

correlated with risk-free interest rates. The said empirical finding can be easily explained with our model. As we just mentioned, the risk-neutral distribution of future values of X depends on the risk-free rate r. If r is high, the expected value of future X should also be high, and therefore, the default probability should be low under the risk-neutral probability measure.

6. Concluding remarks

This paper develops a tractable yet theoretically rigorous framework for valuing risky debt and credit derivatives that incorporates both default risk and interest rate risk and allows for both a continuous component and a jump component in the evolution of firm value.

The paper has a number of important implications. It shows that the structural pricing model with both a jump component and a continuous component is much more flexible in generating various shapes of the term structure of credit spreads than are other structural models and that a jump-diffusion model can explain a number of empirical regularities regarding default probabilities, recovery rates, and credit spreads.

One thing on our future research agenda is to empirically test our model using statistical data. As shown in our numerical simulations, jumps that are important for bond pricing are those rare and big events which can substantially change the firm value instantaneously. A direct estimate of the parameters in Eq. (1) using the underlying time series data usually does not yield satisfactory results because of the so-called 'Peso problem'. The following two approaches can be useful in estimating our model or applying our model in practice. One approach is to recover the jump and the volatility parameters of the underlying stock from options data (Bakshi et al., 1997). As the stock value of a firm equals the total asset value of the firm minus the debt value, the relation between the bond price and the firm value derived in this paper also yields a relation between the stock price and the firm value. The parameters of the firm value process V then can be estimated from the parameters of the stock price process.

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Appendix A. Proof of Theorem 1

This appendix outlines the proof of Theorem 1.

Proof of Theorem 1. Dividing the time interval $[0, T]$ into n equal sub-periods and evaluating expected values on the right-hand side of Eq. (3) gives

$$E^Q[w(X_\tau)I_{\tau \leqslant T}] = \sum_{i=1}^{n} E^Q[w(X_\tau) \,|\, t_{i-1} < \tau \leqslant t_i] \cdot Q(t_{i-1} \leqslant \tau < t_i). \tag{A.1}$$

Denote

$$\hat{\Omega}_i := \left\{X_{t_i} \leqslant 1 \text{ and } X_{t_j} > 1,\ j < i\right\}$$

and

$$\hat{Q}_i := Q(\hat{\Omega}_i).$$

It is easy to show that

$$Q(t_{i-1} \leqslant \tau < t_i) = \hat{Q}_i + \mathrm{o}(T/n), \tag{A.2}$$

$$\begin{aligned} E^Q[w(X_\tau) \,|\, t_{i-1} < \tau \leqslant t_i] &= E^Q[w(X_{t_i}) \,|\, t_{i-1} \leqslant \tau < t_i] + \mathrm{O}(T/n) \\ &= E^Q[w(X_{t_i}) \,|\, \hat{\Omega}_i] + \mathrm{O}(T/n). \end{aligned} \tag{A.3}$$

On the other hand, Eq. (4) implies that

$$\ln(X_{t_i}) - \ln(X_{t_{i-1}}) = x_i + \sum_{j=0}^{\kappa_i} \pi_{ij}, \tag{A.4}$$

where

$$x_i \sim N\big((r - \phi - \sigma^2/2 - \lambda_q v_q)T/n, \sigma^2 \cdot T/n\big),$$
$$\pi_{ij} \sim N\big(\mu_{\pi q}, \sigma_\pi^2\big),$$

and

$$\kappa_i = k \quad \text{with prob. } \lambda^k \cdot \frac{\exp(-\lambda \cdot T/n)}{k!} \cdot (T/n)^k, \quad k = 0, 1, \ldots.$$

According to the definitions of $X_{t_i}^*$ and Q_i, we have

$$\widehat{Q_i} = Q_i + \mathrm{o}(T/n)$$

and

$$E^Q[w(X_{t_i}) \,|\, \hat{\Omega}_i] = E^Q\big[w\big(X_{t_i}^*\big) \,|\, \Omega_i\big] + \mathrm{o}(1),$$

where $X_{t_i}^*$, Ω_i and Q_i are defined as before.

As a result, we obtain from Eq. (A.1)

$$E^Q[w(X_\tau)I_{\tau\leqslant T}] = \lim_{n\to\infty}\sum_{i=1}^{n} E^Q\left[w\left(X^*_{t_i}\right) \mid \Omega_i\right]\cdot Q_i. \tag{A.5}$$

The theorem then establishes immediately. □

Appendix B. Bond pricing with stochastic interest rate

Under Assumption 4, the bond price $B(X,r,T)$ can be expressed as

$$B(X,r,T) = E^Q\left[\exp\left(-\int_0^T r\,\mathrm{d}t\right)\left(I_{\tau>T} + (1-w(X_\tau))I_{\tau\leqslant T}\right)\right]. \tag{B.1}$$

Here I is an indicator function as defined previously and Q is the risk-adjusted probability measure under which

$$\mathrm{d}\ln(X) = \left(r-\phi-\frac{\sigma^2}{2}-\lambda_q\cdot v_q\right)\mathrm{d}t + \sigma\,\mathrm{d}Z_1 + \ln(\Pi_q)\,\mathrm{d}Y_q, \tag{B.2}$$

$$\mathrm{d}r = \beta(\alpha-r)\,\mathrm{d}t + \eta\,\mathrm{d}Z_2, \tag{B.3}$$

where $\alpha \equiv \gamma + q\cdot(\eta/\beta)$ and q is the equilibrium market price of interest rate risk (Pennacchi, 1999).

Denote $D(r,T)$ as the price of the risk-free discount bond which pays \$1 at time T. In the above Vasicek term structure model, we have

$$D(r,T) = a(T)\cdot\exp(-b(T)r), \tag{B.4}$$

where

$$b(T) \equiv \frac{1-\exp(-\beta T)}{\beta},$$

$$a(T) \equiv \exp\left[(b(T)-T)\left(\alpha-\frac{\eta^2}{2\beta^2}\right)-\frac{\eta^2\cdot b(T)^2}{4\beta}\right].$$

Define $v_t \equiv [\ln(X_t), r_t]'$. Following the appendix of Pennacchi (1999) and the proof of Theorem 1, we obtain:

Theorem 2. *The bond price $B(X,r,T)$ given in Eq.* (B.1) *can then be expressed as*

$$B(X,r,T) = D(r,T) - \lim_{n\to\infty}\sum_{i=0}^{n} E^Q\left[\exp\left(-\sum_{j=0}^{n} r^*_{t_j}\cdot\tau\right)\cdot w\left(X^*_{t_i}\right) \mid \Omega_i\right]\cdot Q_i,$$

where

$$\tau \equiv \frac{T}{n},$$

$$t_i = \frac{i}{n}T,$$

$$\Omega_i = \left\{X^*_{t_i} \leqslant 1 \textit{ and } X_{t_j} > 1 \ \forall j < i\right\},$$

$$Q_i = Q(\Omega_i),$$

and moreover, $v^*_t \equiv [\ln(X^*_t), r^*_t]'$ *is defined recursively as*

$$v^*_{t_0} = [\ln(X), r]',$$

$$v^*_{t_i} = \left[\begin{array}{c} \left(\alpha - \phi - \frac{\sigma^2}{2} - \lambda_q \cdot v_q\right)\tau - \alpha \cdot b(\tau) \\ \alpha\beta \cdot b(\tau) \end{array}\right] + \begin{bmatrix} 1 & b(\tau) \\ 0 & \exp(-\beta\tau) \end{bmatrix} v^*_{t_{i-1}} + \begin{bmatrix} x_i \\ \delta_i \end{bmatrix} + \begin{bmatrix} y_i \cdot \pi_i \\ 0 \end{bmatrix}'.$$

Here x_i, y_i, δ_i, *and* π_i *are random variables drawn from*

$$\begin{bmatrix} x_i \\ \delta_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}\right),$$

$$\pi_i \sim N\left(\mu_{\pi q}, \sigma^2_\pi\right),$$

and

$$y_i = \begin{cases} 0 & \textit{with prob. } 1 - \lambda \cdot T/n, \\ 1 & \textit{with prob. } \lambda \cdot T/n. \end{cases}$$

The second-order moments are given by

$$s_{11} = \sigma^2\tau + 2\rho\sigma\eta[\tau - b(\tau)]/\beta + \eta^2\left[\tau - b(\tau) - \frac{\beta \cdot b(\tau)^2}{2}\right] \bigg/ \beta^2,$$

$$s_{12} = \rho\sigma\eta \cdot b(\tau) + \frac{\eta^2 \cdot b(\tau)^2}{2},$$

and

$$s_{22} = \frac{\eta^2 \cdot (1 - \exp(-2\beta\tau))}{2}.$$

The intuition of this theorem is similar to that of Theorem 1. One can follow procedures similar to those described after Theorem 1 to evaluate the bond price $B(X, r, T)$ numerically.

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