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Structural models of corporate bond pricing with maximum likelihood estimation ☆

Ka Leung Li [1], Hoi Ying Wong *

Department of Statistics, The Chinese University of Hong Kong, Hong Kong

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Abstract

This paper empirically examines the proxy, volatility-restriction (VR) and maximum likelihood (ML) approaches to implementing structural corporate bond pricing models, and documents that ML estimation is the best among the three implementation methods. Empirical studies using either the proxy approach or the VR method conclude that barrier-independent models significantly underestimate corporate bond yields. Although barrier-dependent models tend to overestimate the yield on average, they generate a sizable degree of underestimation. The present paper shows that the proxy approach is an upwardly biased estimator of the corporate assets and makes the empirical framework work systematically against structural models of corporate bond pricing. The VR approach may generate inconsistent corporate bond prices or may fail to give a positive corporate bond price for some structural models. When the Merton, LS, BD and LT models are implemented with ML estimation, we find substantial improvement in their performances. Our empirical analysis shows that the LT model is very accurate for predicting short-term bond yields, whereas the LS and BD models are good predictors for medium-term and long-term bonds. The Merton model however significantly overestimates short-term bond yields and underestimates long-term bond yields. Unlike empirical studies in the past, the Merton model implemented with ML estimation does not consistently underestimate corporate bond yields.

JEL classification: G32; G33
Keywords: Corporate bond pricing; Credit risk; Maximum likelihood estimation; Structural models

1. Introduction

Structural models of corporate bond pricing originated with the seminal work of Black and Scholes (1973) and Merton (1974; henceforth the Merton model). By considering the capital structures of firms, the Merton model views

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* Corresponding author. Fax: +852 2603 5188.
E-mail addresses: jeffery@alumni.cuhk.net (K.L. Li), hywong@cuhk.edu.hk (H.Y. Wong).
[1] Ka Leung Li is in HSBC. This work was finished when he was in CUHK.

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equity as a call option on corporate assets, and views corporate debt as a default-free debt less a put option. However, this simple construction is inadequate to describe actual situations because it excludes the possibility of default before maturity, the effect of a stochastic interest rate, and the valuation of coupon-bearing bonds.

Extensions and refinements of the structural models have been continually made since the work of Merton. Black and Cox (1976) modeled the early default feature by introducing a default barrier. Geske (1977) viewed a corporate coupon bond as a portfolio of compound options. Longstaff and Schwartz (1995, LS) developed a simple framework that incorporates default barrier and stochastic interest rate to price corporate coupon bonds. Leland and Toft (1996, LT) derived the optimal capital structure to determine corporate bond value. Briys and de Varenne (1997, BD) considered the default barrier to be a fixed quantity discounted at the riskless rate up to the maturity date of the risky corporate bond. Collin-Dufresne and Goldstein (2001, CDG) proposed a floating default barrier approach to model the target leverage ratio.

Each model claims to be able to theoretically capture certain market phenomena, but it is important to contain empirical evidence with actual data. Jones, Mason and Rosenfeld (1984, JMR) were the first to test the Merton model empirically. Based on a sample of firms with simple capital structures and bond prices in the secondary market in 1977–1981, they showed that the predicted prices from the Merton model were too high, by 4.52% on average, and that the errors were more severe for non-investment grade bonds. Ogden (1987) conducted a similar empirical study with newly issued bonds and obtained a similar result. Lyden and Saraniti (2000) compared the performance of the Merton and LS models, and found that yield spreads were underestimated with the Merton model and that the LS model made no significant improvement.

Eom, Helwege and Huang (2004, EHH) recently conducted a comprehensive empirical study of structural models. They tested five structural models, including the Merton, Geske, LT, LS, and CDG models, taking into account the asset payout ratio and stochastic interest rate. After carefully examining the capital structures of firms and characteristics of bonds, EHH obtained a sample of bonds from 1986–1997. To implement the structural models, they proxied the market value of corporate assets by the sum of the market value of equities and the book value of total liabilities and used different ways to estimate the other model parameters. Within this setting, both the Merton and Geske models (barrier-independent models) underestimated yield spreads on average, whereas the LT, LS and CDG models (barrier-dependent models) tended to overestimate yield spreads on average, although there was also a sizable number of underestimations. Despite their empirical results, EHH proposed an extended Merton model for coupon bonds, and set some criteria for bond selection.

There are many different ways to implement structural credit risk models. The major concern is the estimation of the value and risk of a firm's assets, neither of which are directly observable. Ronn and Verma (1986) proposed a volatility-restriction (VR) method to estimate firm value and volatility by simultaneously solving a pair of equations that match the observed stock prices and estimated stock volatility with model outputs. The VR method was adopted in the empirical works of JMR and Ogden (1987). EHH used the proxy for firm value but estimated the firm asset volatility by matching the estimated stock volatility with the model output. Therefore, it is a mixed approach. Duan (1994, 2000) pointed out several theoretical problems of the VR method and derived a maximum likelihood (ML) approach. For commercial purposes, Moody's KMV uses the VR method to generate an initial guess of volatility and then puts it into an iteration process to obtain constant volatility in the stable stage of the process.

The implementation of structural models has gained attention in the literature recently because it affects the performance and testing of structural models. Ericsson and Reneby (2005) showed, by a series of simulations, that the ML approach obviously outperforms the VR method in terms of both lack of bias and efficiency. They claimed that, no matter how satisfactory the theoretical feature of a model, its empirical use may have been limited by the chosen implementation method. Duan et al. (2004) showed that the iteration process in the KMV approach is actually an EM algorithm for obtaining the maximum likelihood estimator, which means that it is equivalent to the ML approach of Duan (1994). However, to our knowledge, no empirical study of the structural models of corporate bond pricing has used the ML approach, and no academic work has been devoted to a comparison of the proxy, VR, and ML approaches empirically.

This paper shows that structural models perform much better if the ML approach is used to estimate the model parameters, and, more importantly, that the proxy for firm value has many potential problems. When the proxy is carried forward to valuation, a significant pricing error is associated with all of the structural models of corporate bond pricing. In contrast, the pricing error is reduced significantly when the ML estimation is employed in an empirical study on the same set of market data. As the proxy for firm value essentially replaces the market value of debt with the book value of debt, our study is related to the paper of Sweeny, Warga and Winters (1997), which points out that empirical studies in which the market value of debt is replaced by the book value of debt distort the debt–equity ratio and cost of capital calculations.

The VR approach has a deficiency compared to ML estimation. In particular, when the VR approach is applied to barrier-dependent models, the system of equations often produces multiple solutions for different initial values. In this situation, we have to select the most appropriate estimator from the set of possible solutions manually. For the LT model, we even face a situation in which the system of equations has no solution and cannot calculate the corporate bond prices. This deficiency would not occur in the ML approach because there must be a maximum value for the likelihood function, and the likelihood function is bounded above.

The empirical data support our claim that the ML approach is the best implementation method among the three methods considered in this paper. We base our empirical study on the construction of EHH, including their criteria for the selection of corporate bonds. We then apply the proxy, VR, and ML approaches to the extended Merton, LS, BD, and LT models. The CDG model is excluded from our analysis because it is often regarded as a reduced form model.[2] Our aim is to examine the quality of each of the implementation methods, given a structural model. The main empirical evidence is that structural models perform much better with the ML estimation than do alternative implementation methods. As a byproduct, we also compare the predictive power of the four structural corporate bond models when the MLE approach is applied.

The rest of the paper is organized as follows. Section 2 reviews the possible implementations of structural models and discusses potential strengths and weaknesses of each approach. Section 3 describes the ML estimation and verifies the estimation quality with a simulation. Section 4 reports our empirical framework and the results for the extended Merton, BD, LS, and LT models. Concluding remarks are presented in Section 5.

2. Implementation of structural models

A difficulty that arises when implementing structural models is the estimation of hidden variables, such as the value and risk of a firm's assets, the asset payout ratio, and default barriers. In this section, we focus on different approaches to the estimation of the market value and volatility of a firm, and discuss the potential strengths and weaknesses of each approach.

2.1. The pure proxy approach

The simplest approach uses a proxy to measure the market value of a firm and then estimates the volatility through a time series of the proxy firm values. We call this estimation the "pure proxy approach", which is actually the "Method I" used by JMR. In accounting principles, the market value of a firm's assets must be equal to the market value of equities plus the market value of debts. As the latter is not observable in the market, the proxy approach approximates it by using the book value of debts. Therefore, the proxy firm value, which is the sum of market value of equities and the book value of liabilities, changes over time through the fluctuation of equity values alone. This means that firm asset volatility is estimated as the standard deviation of the returns of the proxy firm values. This approach does not depend on the particular features of a structural model and is typically easy to implement. The pure proxy approach is also adopted by Barclay and Smith (1995a,b), Brockman and Turtle (2003), and others.

We summarize the implementation steps of the proxy approach. Suppose we have a time series of observed equity prices: $S=\{S_1, S_2,\dots, S_{N+1}\}$ and the book value of total liability, X, extracted from the accounting book of a firm. The proxy approach runs as follows.

1. Calculate the proxy firm values as $V_i^{\mathrm{p}}=S_i+X$ for $i=1, 2, \dots, N+1$.
2. Calculate the standard deviation, $\sigma_{\mathrm{v}}^{\mathrm{p}}$, of the sample:
$$\left\{\ln\left(V_{i+1}^{\mathrm{p}}/V_i^{\mathrm{p}}\right)|i=1,2,\dots,N\right\}.$$
3. Substitute V_i^{p} and $\sigma_{\mathrm{v}}^{\mathrm{p}}$ into a corporate bond pricing formula to obtain the corporate bond price at time t_i.

[2] An anonymous referee suggested that we remove the CDG model but add the BD and LT models. For those who are interested in the empirical performance of CDG with ML estimation, please read the working paper version downloadable from http://www.sta.cuhk.edu.hk/hywong.

It is obvious that replacing the market value of debt with its higher book value leads to an upward bias in the estimate of the asset value. For a theoretical and complete justification, please see Appendices A.1 and A.2.

2.2. The volatility-restriction approach

To respect the features of structural models, Ronn and Verma (1986) proposed a volatility-restriction (VR) method that obtains the firm value and volatility by solving a system of two equations. Specifically, for the Merton model, the two equations are

$$S = C(V, X; \sigma_{\mathrm{v}}) \text{ and } \sigma_{\mathrm{e}} = \sigma_{\mathrm{v}} \frac{V}{S} \frac{\partial S}{\partial V},$$

where V and σ_{v} are the value and volatility of a firm, S and σ_{e} are the value and volatility of the equity, and $C(V; \sigma_{\mathrm{v}})$ is the call option pricing formula. In general, the first equation matches the observed equity prices with the prices of the model under investigation. The second equation restricts the estimated equity volatility to match the volatility that is generated by applying the Ito lemma to the equity pricing formula used in the first equation. Although the implementation is slightly more tedious than for the pure proxy approach, the speed is very fast given modern computing power. At each point in time, this method produces a pair of estimates of firm value and volatility. Although the VR method violates the constant volatility assumption of most structural models, it is the most popular way of implementing structural models. Empirical studies using the VR approach include JMR, Huang and Huang (2002), Lyden and Saraniti (2000), and Ogden (1987). Apart from academic research, Moody's KMV uses this approach as one part of its estimation process.

We summarize the implementation steps of the VR approach. A structural model should be specified in advance and the corresponding equity pricing formula, $h(V, X, H; \sigma_{\mathrm{v}})$, where X is the book value of liabilities and H is the default barrier level, should be coded. Then, run the following steps.

1. Calculate the standard deviation, σ_{e}, of the log-return of equities

   $$\{\ln S_{i+1}/S_i | i = 1, 2, \ldots, N\},$$

   where S_i is an element of S.
2. Calculate the asset value V_i^{r} and the asset volatility $\sigma_{\mathrm{v}}^{\mathrm{r}}(t_i)$ at time t_i by solving the following system of equations.

   $$S_i = h\left(V_i^{\mathrm{r}}, X, H; \sigma_{\mathrm{v}}^{\mathrm{r}}(t_i)\right) \text{ and } \sigma_{\mathrm{e}} = \sigma_{\mathrm{v}}^{\mathrm{r}}(t_i) \frac{V_i^{\mathrm{r}}}{S} \frac{\partial h}{\partial V_i^{\mathrm{r}}}. \tag{1}$$

3. Substitute V_i^{r} and $\sigma_{\mathrm{v}}^{\mathrm{r}}(t_i)$ into a corporate bond pricing formula to obtain the corporate bond price at time t_i.

Duan (1994) pointed out several theoretical inconsistencies of the VR approach. The major weakness is that it forces the Ito lemma to hold at each time point. This may be too restrictive. When the equity price abruptly changes at a certain time point, the Ito lemma is hardly satisfied. Consequently, the system of equations may have no solution. In our empirical analysis, we do find such examples.

2.3. The mixed proxy approach

In between the two aforementioned approaches is the mixed proxy approach that is used in the empirical study of EHH (2004). The market value of a firm is estimated as the proxy firm value, whereas a firm's volatility is calibrated to the second equation of the VR method. In this way, the estimation procedure is simpler than that of the VR approach but respects the model features through the second equation of (1).

We summarize the implementation steps of the mixed proxy approach. A structural model should be specified in advance and the equity pricing formula, $h(V, X, H; \sigma_{\mathrm{v}})$, should be coded. The implementation goes as follows.

1. Calculate the proxy firm values as $V_i^{\mathrm{p}} = S_i + X$ for $i = 1, 2, \ldots, N+1$.
2. Calculate the standard deviation, σ_{e}, of the log-return of equities.

3. Calculate the asset volatility, $\sigma_{\mathrm{v}}^{\mathrm{m}}(t_i)$ by solving the equation

$$\sigma_{\mathrm{e}} = \sigma_{\mathrm{v}}^{\mathrm{m}}(t_i)\frac{V_i^{\mathrm{p}}}{S}\frac{\partial h}{\partial V_i^{\mathrm{p}}}.$$

4. Substitute V_i^{p} and $\sigma_{\mathrm{v}}^{\mathrm{m}}(t_i)$ into a corporate bond pricing formula to obtain the corporate bond price at time t_i.

Appendix A.3 proves that the mixed proxy approach produces an asset volatility that is larger than that of the pure proxy approach. The higher the asset volatility the higher the corporate bond yield is. As it may be argued that the higher volatility generated by the mixed proxy approach can offset the downward bias of the pure proxy approach in estimating corporate bond yields, we carry out a series of simulations to check the bias contained in the mixed proxy approach in Section 3.

2.4. Maximum likelihood estimation

The idea of the ML estimation proposed by Duan (1994) is to derive the likelihood function for the equity returns based on the assumptions that the firm value is log-normally distributed and the equity value is an option on the firm. By maximizing the likelihood function, parameters, such as the drift and volatility of a firm, are obtained. The firm asset value is then extracted by equating the pricing formula to the observed equity price. This approach is theoretically sound as it is proven to be asymptotically unbiased and allows the confidence interval for the parameter estimates to be derived. The drawback of the ML approach is that it is a tedious and relatively time consuming approach, usually taking some 10 s or longer to complete the estimation of one sample path. However, most empirical studies involve the estimation of several thousand firms, and hence several thousand paths. One possible solution is to use several computers at once. Although this approach relies heavily on the distribution of the asset value, it does not force the Ito lemmas to hold for the equity price process. To our knowledge, there is no empirical work on using ML estimation to examine corporate bond pricing models.

We summarize the implementation steps of the MLE approach. A structural model should be specified in advance. Not only the equity pricing formula, $h(V, X, H; \sigma_{\mathrm{v}})$, but also the corresponding likelihood function, $L(\mu, \sigma_{\mathrm{v}})$, should be coded, where μ is the asset drift and σ_{v} is the asset volatility. The implementation is performed as follows.

1. Obtain parameters μ and σ_{v} by maximizing the likelihood function, $L(\mu, \sigma_{\mathrm{v}})$, subject to the constraint that

$$S_i = h(V_i, X, H; \sigma_{\mathrm{v}}) \text{ for all } i = 1, 2, \ldots, N+1.$$

2. Calculate the firm's asset value by solving the above equation.
3. Substitute V_i and σ_{v} into a corporate bond pricing formula to obtain the corporate bond price at time t_i.

However, the likelihood function can change across models. The following provides likelihood functions for the Merton and barrier-dependent models.

2.4.1. The likelihood function with the Merton model

The parameters are the asset drift (μ) and asset volatility (c_{v}). For the Merton model, Duan (1994) showed that the likelihood function for the equity return is

$$L(\mu, \sigma) = \sum_{i=2}^{n} \{\ln g(v_i|v_{i-1}) - \ln[V_i \cdot N(d_i)|v = v_i]\}, \tag{2}$$

where $N(\cdot)$ is the cumulative distribution function for a standard normal random variable, $g(\cdot)$ is the density function of a normal random variable, and V_i and v_i denote the asset price and the log of the asset price at time i, respectively. The explicit expressions of $g(\cdot)$ and d_1 are given in Appendix B, where we also present the detailed formulation.

ML estimators are parameters that maximize the likelihood function (2), subject to the constraints that the market values of equities are equal to the call option pricing formula, that is,

$$\max_{\mu, \sigma_{\mathrm{v}}} L(\mu, \sigma_{\mathrm{v}}) \quad \text{s.t.} \quad S(t_i) = C(t_i, V(t_i), \sigma_{\mathrm{v}}), \forall i = 1, 2, \ldots, n.$$

2.4.2. The likelihood function with barrier-dependent models

For barrier-dependent models, the market value of equity is viewed as a barrier option, rather than a standard call option. As the asset price should not go below the default barrier before bankruptcy occurs, the density function of the log-asset-price becomes [see Rubinstein and Reiner (1991)],

$$g^{\mathrm{B}}(v_i|v_{i-1};\mu,\sigma) = \varphi(v_i - v_{i-1}) - e^{2(\eta-1)(b-v_{i-1})}\varphi(v_i + v_{i-1} - 2b), \tag{3}$$

where

$$b = \log H, \quad \eta = \frac{\mu}{\sigma_{\mathrm{v}}^2} + \frac{1}{2},$$
$$\varphi(x) = \frac{1}{\sigma_{\mathrm{v}}\sqrt{2\pi(t_i - t_{i-1})}} \exp\left\{ -\frac{\left[x - \left(\mu - \sigma_{\mathrm{v}}^2/2\right)\cdot(t_i - t_{i-1})\right]^2}{2\sigma_{\mathrm{v}}^2(t_i - t_{i-1})} \right\}.$$

In our estimation process, the function $g^{\mathrm{B}}(\cdot|\cdot)$ takes the form of Eq. (3) if the underlying asset value is larger than the barrier, and zero otherwise. Given the explicit formula of $h(V, X, H; \sigma_{\mathrm{v}})$, the option delta ($\Delta(V)$) is calculated by differentiating the pricing formula with respect to V. Note that we use the DOC pricing formula provided in Appendix C for the LS model, the equity pricing formula (D.8) for the BD model and (D.13) for the LT model. Following a similar procedure to that of the Merton model, we obtain the log-likelihood function as follows.

$$L^{\mathrm{B}}(\mu,\sigma_{\mathrm{v}}) = \sum_{i=2}^{n} \left\{ \ln g^{\mathrm{B}}(v_i|v_{i-1}) - \ln\left[V_i \cdot \Delta(V_i)|_{V=V_i} \right] \right\}. \tag{4}$$

We then estimate the parameters by solving the following optimisation problem.

$$\max_{\mu,\sigma_{\mathrm{v}}} L^{\mathrm{B}}(\mu,\sigma_{\mathrm{v}}) \quad \text{s.t.} \quad S(t_i) = h(V(t_i), X, H; \sigma_{\mathrm{v}}), \forall i = 1, 2, \ldots, n.$$

2.4.3. Survivorship consideration

In our empirical study, the sample is drawn from surviving companies, which may lead to a survivorship bias in the estimations. We recognize that maximum likelihood estimation with survivorship has been considered by Duan et al. (2003), who found that the original approach of Duan (1994) leads to an upward bias in the asset drift, but that the other parameters are obtained with a high quality. However, the survivorship bias has no impact on the testing of corporate bond pricing models. Structural models value corporate bonds in the risk-neutral world in which asset drift is replaced by the risk-free interest rate. Thus, the biased drift value plays no role in corporate bond pricing formulas. The inclusion of the drift in the estimation procedure aims to enhance the estimation quality of the volatility.

3. Simulation tests

Ericsson and Reneby (2005) use a series of simulations to show that the ML approach of Duan (1994) clearly outperforms the VR method in parameter estimation for both barrier-independent and barrier-dependent models. They find that the VR approach underprices corporate bonds when stock price increases, whereas it overprices bonds when stock price decreases. As we avoid repeating their works, this section is devoted to comparing the mixed proxy approach with the ML approach. Interestingly, our simulation shows that the mixed proxy approach significantly underestimates corporate bond yields and the results resemble those of EHH (2004). Therefore, the empirical results of EHH (2004) may be mainly driven by the mixed proxy approach.

Our simulation concentrates on the extended Merton and LS models because these two models are commonly considered in the empirical literature. In this simulation exercise, we use r=6.5%, μ=8%, σ=0.25, and an initial firm value of 1. One-year (260-day) sample paths are generated according to the Black–Scholes dynamics. Consider debt maturities T of 2, 5, 10 and 20 years. The face value of debt X takes three possible values, 0.3, 0.5 and 0.7, which represent the different leverage levels (or creditworthiness) of a company. To test the Merton model, we compute the market values of corporate equities by the standard call option formula. However, we use the DOC option pricing formula to calculate the market values of equities under the LS model.

Suppose that the extended Merton model introduced by EHH (see Appendix D) and the LS model are correct models for two different economies. This simulation, on the one hand, attempts to show that the proxy for firm value leads to an underestimation of corporate bond yields and, on the other hand, is used to check the performance of the ML estimation. We directly compare the ML approach with the mixed proxy approach, which produces less underestimation in corporate bond yields than the pure proxy approach does. We first simulate equally time-spaced market values of the firm based on specified parameters and then generate equity values and corporate bond prices using both models. These generated data are then regarded as market observable values. The detailed procedures of the MLE approach and the proxy approach are summarized as follows.

1. The extended Merton model with $K=X$ and $\omega=0$.
   (a) ML approach. The approach of Duan (1994) is employed to estimate the asset volatility and the market value of assets. By plugging the estimates back into the extended Merton model of corporate bond pricing, the predicted credit yield spreads are obtained.
   (b) Proxy approach. We estimate the market value of a firm's assets by the proxy for firm value. The asset value volatility, σ_{v}, is solved by the equation $\sigma_{\mathrm{e}} = \sigma_{\mathrm{v}} \frac{V_t}{S_t} \frac{\partial S_t}{\partial V_t}$, where σ_{e} is the equity volatility, and S_t and V_t denote the market value of equity and the proxy asset value at time t, respectively. These estimates are substituted into the extended Merton model to estimate credit yield spreads.

Table 1
Simulation results for Merton model

Characteristics	ML approach			Proxy approach		
	% error	% error	% error	% error	% error	% error
	In prices	In yields	In spreads	In prices	In yields	In spreads
	Mean	Mean	Mean	Mean	Mean	Mean
	(S.D.)	(S.D.)	(S.D.)	(S.D.)	(S.D.)	(S.D.)
Panel A: Different levels of coupon rate						
$c=0\%$	0.06%	−0.10%	−4.14%	1.37%	−2.53%	−93.22%
	(0.10%)	(0.18%)	(3.08%)	(2.04%)	(5.48%)	(12.81%)
$c=8\%$	0.03%	−0.09%	−4.09%	0.75%	−2.35%	−92.28%
	(0.05%)	(0.18%)	(3.96%)	(1.26%)	(5.64%)	(14.64%)
Panel B: Different levels of total liabilities						
$X=0.3$	0.00%	−0.01%	−2.92%	0.15%	−0.22%	−86.89%
	(0.01%)	(0.01%)	(3.84%)	(0.31%)	(0.48%)	(20.98%)
$X=0.5$	0.03%	−0.05%	−4.08%	0.77%	−1.64%	−94.68%
	(0.04%)	(0.07%)	(2.56%)	(1.05%)	(3.08%)	(7.12%)
$X=0.7$	0.10%	−0.22%	−5.35%	2.25%	−5.45%	−96.69%
	(0.11%)	(0.26%)	(3.69%)	(2.33%)	(8.28%)	(4.88%)
Panel C: Different levels of time to maturities						
$T=2$	0.03%	−0.15%	−4.50%	0.70%	−4.21%	−93.77%
	(0.05%)	(0.28%)	(5.08%)	(1.44%)	(8.95%)	(22.86%)
$T=5$	0.05%	−0.11%	−3.86%	1.18%	−3.06%	−95.34%
	(0.07%)	(0.18%)	(2.66%)	(1.86%)	(5.36%)	(5.70%)
$T=10$	0.05%	−0.07%	−3.83%	1.26%	−1.71%	−91.83%
	(0.09%)	(0.11%)	(2.71%)	(1.84%)	(2.69%)	(8.66%)
$T=20$	0.06%	−0.04%	−4.27%	1.10%	−0.77%	−90.08%
	(0.10%)	(0.06%)	(3.15%)	(1.69%)	(1.12%)	(10.67%)

Table 1 shows simulation results for Merton model, disaggregated by different values of coupon rates (c), total liabilities (X) and maturities (T). For each characteristic, means and standard deviations (in percentages) of percentage errors in prices, percentage errors in yields and percentage errors in spreads by using MLE and proxy approaches are presented. For percentage error in price of each observation, it is calculated as predicted price minus actual price, and then divided by actual price. Similar calculation is performed to obtain percentage error in yield and percentage error in spread for each observation.

2. The LS model with $H=X$ and $\omega=51.31\%$.
   (a) ML approach. We view the market value of equity as a DOC option and perform our proposed MLE approach to estimate the asset volatility and the market value of the firm. These estimated parameters are used to derive the corporate bond yield using the LS model.
   (b) Proxy approach: We use the same procedure as for the extended Merton model, except that the LS model is used this time.

Finally, we compare the credit yield spreads and bond prices that are obtained from the proxy and MLE approaches for each model.

The simulation results for the Merton and LS models are given in Tables 1 and 2, and the percentage errors in prices, yields, and yield spreads are reported. The percentage error in prices is the model prices minus the market price divided by the market price, where a positive number indicates an overestimation. The percentage errors in yields and yield spreads are calculated in the same manner.

3.1. Simulation results for the Merton model

Table 1 shows that the average percentage errors in prices and yields are all close to zero for the ML approach, whereas when using the proxy firm value, the average percentage errors in prices are significantly positive, and those in

Table 2
Simulation results for Longstaff & Schwartz model

Characteristics	ML approach			Proxy approach		
	% error	% error	% error	% error	% error	% error
	In prices	In yields	In spreads	In prices	In yields	In spreads
	Mean	Mean	Mean	Mean	Mean	Mean
	(S.D.)	(S.D.)	(S.D.)	(S.D.)	(S.D.)	(S.D.)
Panel A: Different levels of coupon rate						
$c=0\%$	0.07%	−0.09%	−0.73%	5.46%	−5.95%	−51.57%
	(0.13%)	(0.21%)	(2.76%)	(7.54%)	(9.79%)	(39.53%)
$c=8\%$	0.06%	−0.10%	−0.63%	4.32%	−6.58%	−55.01%
	(0.11%)	(0.22%)	(0.93%)	(6.90%)	(10.46%)	(39.50%)
Panel B: Different levels of total liabilities						
$X=0.3$	0.01%	−0.01%	−0.88%	2.29%	−2.53%	−69.45%
	(0.02%)	(0.02%)	(3.29%)	(2.76%)	(2.65%)	(25.52%)
$X=0.5$	0.04%	−0.06%	−0.50%	6.24%	−8.84%	−64.69%
	(0.05%)	(0.08%)	(0.90%)	(5.04%)	(7.57%)	(25.77%)
$X=0.7$	0.13%	−0.21%	−0.67%	6.14%	−7.42%	−25.74%
	(0.18%)	(0.33%)	(1.03%)	(10.72%)	(14.91%)	(47.22%)
Panel C: Different levels of time to maturities						
$T=2$	0.04%	−0.15%	−1.31%	2.88%	−8.19%	−74.88%
	(0.10%)	(0.34%)	(3.91%)	(6.90%)	(15.49%)	(45.48%)
$T=5$	0.06%	−0.11%	−0.69%	4.37%	−7.06%	−61.68%
	(0.13%)	(0.21%)	(0.83%)	(7.38%)	(9.99%)	(39.11%)
$T=10$	0.07%	−0.07%	−0.44%	5.77%	−5.64%	−45.96%
	(0.13%)	(0.13%)	(0.56%)	(7.34%)	(6.51%)	(31.79%)
$T=20$	0.07%	−0.05%	−0.28%	6.53%	−4.16%	−30.65%
	(0.12%)	(0.09%)	(0.40%)	(6.87%)	(4.56%)	(23.64%)

Table 2 shows simulation results for Longstaff & Schwartz model, disaggregated by different values of coupon rates (c), total liabilities (X) and maturities (T). For each characteristic, means and standard deviations (in percentages) of percentage errors in prices, percentage errors in yields and percentage errors in spreads by using MLE and proxy approaches are presented. For percentage error in price of each observation, it is calculated as predicted price minus actual price, and then divided by actual price. Similar calculation is performed to obtain percentage error in yield and percentage error in spread for each observation.

the yields and yield spreads are significantly negative. Panel A shows that the errors are more severe for zero coupon bonds, and Panel B implies that the errors are more pronounced for highly leveraged firms. The percentage errors in yield spreads, shown in Panel C, are consistently less than −90% for all maturities. The simulation suggests that underestimations of bond yields with the Merton model are probably due to the hidden bias of the proxy for firm value.

We further illustrate our simulation result by graphs. In Fig. 1, the circles represent the percentage errors in yields that are obtained from the ML approach, and the crosses represent those from the proxy approach. Fig. 1a contains the results for all credit qualities and Fig. 1b–d present the results for high, medium and low ratings respectively. The crosses are generally beneath the circles in all cases, which shows that the proxy for firm value leads to the underestimation of corporate bond yields. We would like to point out an interesting observation that Fig. 1 resembles Fig. 1 in EHH (2004) for testing the Merton model empirically.

3.2. Simulation results for the LS model

Table 2 summarizes the results of the LS model. It can be seen that the ML approach definitely outperforms the proxy approach, and that the errors induced by the proxy are less significant than those of the Merton model. However, the errors are not negligible.

Fig. 2 consists of four pictures. Fig. 2a plots the percentage errors in the yields against the debt maturities for all bonds. We can see that the error points of the ML approach are located around zero, whereas most of the points of the proxy approach are negative. Fig. 2b, c and d show the percentage errors in yields for high, medium, and low rating bonds in order. We recognize that the proxy always underestimates the yields of high and medium rating bonds. In Fig. 2d, we can see that there are some points with positive percentage errors in the yields with the proxy approach, and thus the errors that are generated by the proxy are partially offset by the imposition of a default barrier for low rating bonds. Fig. 2 resembles Fig. 4 of EHH (2004) for testing the LS model.

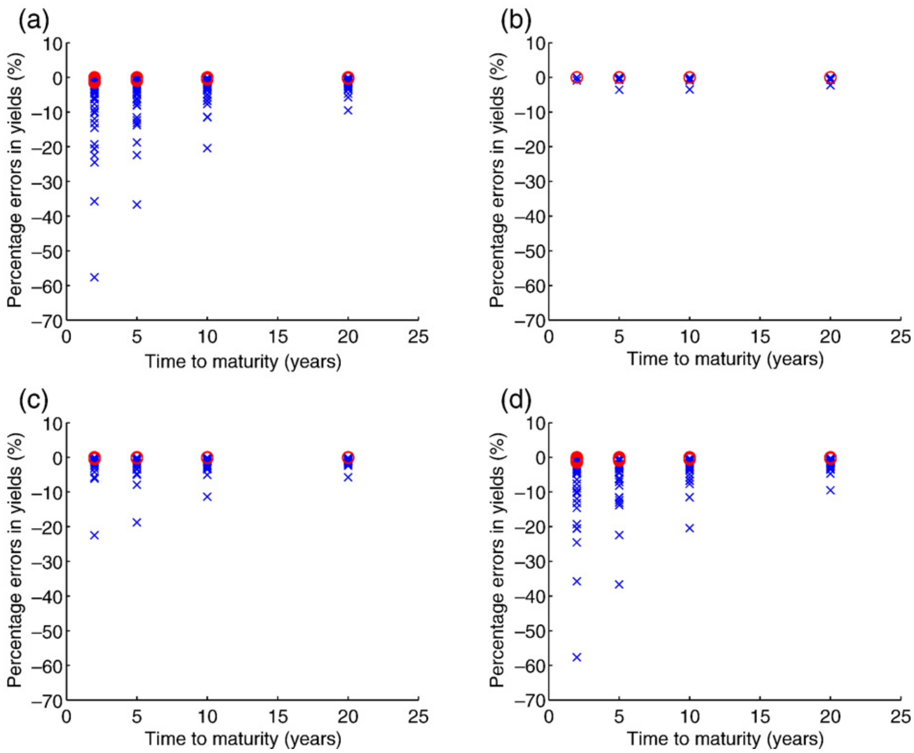

Fig. 1. These figures show simulation result on Merton model. (a) plots the results for all bonds. (b), (c) and (d) plot the results by high, medium and low credit qualities respectively. In all figures, '∘' and '×' indicate percentage error in yield by using ML and proxy approaches respectively. For percentage error in yield, it is calculated as predicted yield minus actual yield, and then divided by actual yield.

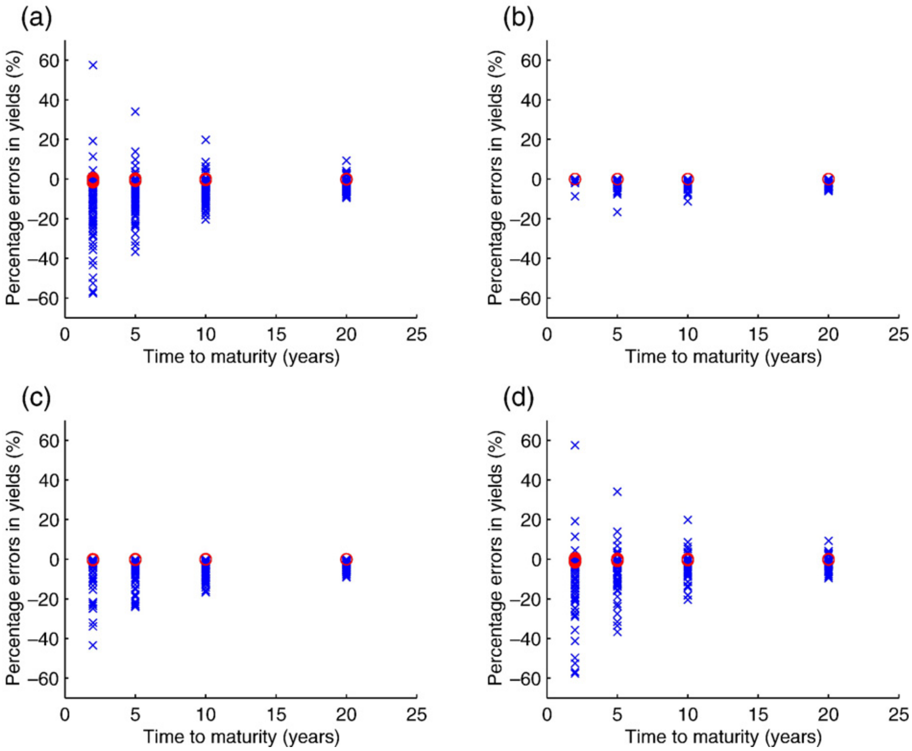

Fig. 2. These figures show simulation result on LS model. (a) plots the results for all bonds. (b), (c) and (d) plot the results by high, medium and low credit qualities respectively. In all figures, '○' and '×' indicate percentage error in yield by using ML and proxy approaches respectively. For percentage error in yield, it is calculated as predicted yield minus actual yield, and then divided by actual yield.

In summary, our simulation further indicates that the proxy of firm value is inappropriate and leads to the underestimation of bond yields when all other parameters are fixed. This bias occurs in both barrier-dependent and barrier-independent models.

4. Empirical study

An empirical study is conducted to check whether the performance of structural bond pricing models is improved when the ML approach is used, and whether the empirical results in the past were driven by alternative implementation methods. We empirically examine the performance of implementation methods using the Merton, BD, LS, and LT models.

4.1. Criteria of bond selection

Based on the criteria of EHH, we select bonds with simple capital structures and sufficient equity data. The bond prices on the last trading day of each December for the period 1986–1996 were obtained from the Fixed Income Database. We choose non-callable and non-putable bonds that are issued by industrial and transportation firms, and exclude bonds with matrix prices and those with maturities of less than one year. There are nearly 7000 bonds that meet these criteria.

To have simple capital structures, we consider firms with only one or two public bonds, and sinkable and subordinated bonds are excluded. We examine the characteristics of the firms with information that was provided by Rating Interactive of Moody's Investor Services. We regard firms with an organization type as the corporations and exclude those with a non-US domicile. Firms in broad industries such as finance, real estate finance, public utility, insurance, and banking, are also excluded from our sample. At this stage, our sample consists of 2033 bonds.

To measure the market value of corporate assets, we restrict ourselves to firms that have issued equity and provide regular financial statements. Therefore, we downloaded the market values of equities from Datastream and total liabilities and reported dividend yields from CompuStat for the period 1986 to 1996. By matching all of the available data and excluding some firms that were acquired, our sample ultimately contains 807 bonds issued by 171 firms.

The summary statistics of the data are exhibited in Table 3. Panel A shows that our sample contains bonds with maturities that range from 1 year to 50 years, with an average of 10 years. This wide range of maturities enables us to study the maturity effect of different structural bond pricing models. Our sample covers zero coupon bonds and bonds with high coupon rates, at a maximum of 15%. The range of yield-to-maturity is wide, from 4% to 22.5%. The bonds in the sample fall within a large credit spectrum. Most of the bonds belong to the investment grade according to Moody's and S&P, and some are junk bonds. These large discrepancies in ratings allow us to check the performance of the structural models for different credit qualities. Our sample includes different sizes of firms that carry at least US $231 million of market capitalization to a maximum of US$96 billion. The total liabilities of these firms range from US $114 million to a maximum of US$150 billion.

Panel B presents the mean of the time to maturity, coupon rates, yield-to-maturity, Moody's rating, S&P rating, market capitalizations, and total liabilities. The mean of the Moody's and S&P ratings are quite stable, but the values of time to maturity, coupon rate, and yield-to-maturity vary from 8 to 11.5, 7.55 to 9.75, and 6.4 to 10, respectively.

4.2. Parameters of the models

Firm-specific parameters include the market value of assets (V), asset volatility (σ), book value of liabilities (X), asset payout ratio (δ), and default barrier (H).

Table 3
Summary statistics of the bonds in empirical study

Panel A: Summary statistic of the bonds

Characteristics	Mean	S.D.	Minimum	Median	Maximum
Time to maturity (T)	9.91	8.03	1.04	7.87	49.95
Coupon rate (c)	8.20	1.52	0	8.5	15
Yield-to-maturity (y)	7.68	1.54	3.94	7.48	12.49
Moody's ratings	7.24	2.73	2	7	24
S&P ratings	6.99	2.67	2	7	16
Market capitalisation (MV)	7450.66	10733.12	230.55	3428.44	95983.1
Total liabilities (X)	5151.77	10728.75	113.6	2324.49	150424.59

Panel B: Summary statistic of the bonds

Observation year	Number of bonds	T	c	y	Moody's rating	S&P	MV	X
1986	20	11.47	9.75	8.17	6.95	6.65	4479.68	4622.74
1987	29	10.46	9.18	9.55	5.93	5.93	6309.20	5575.82
1988	47	8.08	9.07	10.02	6.45	6.26	5286.23	9584.63
1989	52	8.48	9.11	8.93	6.69	6.46	6355.56	8661.61
1990	49	9.26	9.16	9.14	6.31	6.27	8371.26	10086.37
1991	68	10.89	8.91	7.47	6.46	6.25	5573.71	5124.63
1992	77	10.19	8.43	7.38	7.25	6.77	6892.26	4050.76
1993	94	10.21	7.67	6.41	7.30	6.90	7572.97	4120.06
1994	99	9.62	7.75	8.72	7.55	7.17	7752.24	4518.75
1995	138	10.02	7.63	6.40	7.66	7.43	8107.86	3231.63
1996	134	10.23	7.55	7.05	8.07	7.95	7754.13	3231.63

Table 3 presents the summary statistics of the bonds in our empirical study. Panel A shows the mean, standard deviation (S.D.), minimum, median and maximum of time to maturity (measured in years), coupon rate (measured in %), yield-to-maturity (measured in %), Moody's ratings, S&P ratings, market capitalisation (measured in $ millions) and total liabilities (measured in $ millions). Panel B segregates the sample according to observation years. For the Moody's rating, 1 stands for Aaa+, 2 stands for Aaa and etc. For the S&P ratings, 1 stands for AAA+, 2 stands for AAA and etc. For both rating systems, 24 stands for NR, meaning the bond is not rated.

To make comparisons, we use the proxy, VR and ML approaches to estimate the market value of assets and the asset value volatility. Our proxy approach always refers to the mixed proxy approach discussed in Section 2. Although not reported, the pure proxy approach performs very poorly. In the ML framework, estimation is based on a one-year time series of market values of equities. We use the approach of Duan (1994) for the Merton model and the likelihood function of (4) for the barrier-dependent models, which include the BD, LS and LT models. We assume that a zero rebate is paid to equity holders upon default.

Both the proxy and VR approaches need the historical equity volatility as an input. We follow EHH (2004) to estimate the historical equity volatility over the most recent 150 trading days.

4.2.1. The default barrier

For the Merton model, the default barrier is set to zero. For the LT model, the default barrier is endogenously obtained through (D.14). The default barrier should be specified for exogenous bankruptcy models, such as LS and BD.

However, there is no consensus on the exact position of the default barrier for exogenous bankruptcy models. Empirical studies that use a prudential barrier setting to the debt level include the works of Ogden (1987) and EHH (2004). In the industry, Moody's KMV sets the default barrier to the default point, which is the short-term debts plus half of the long-term debts, and is less than the total debt value, see Crosbie and Bohn (1993). Wong and Choi (2007) show empirically that default barriers tend to be less than the book value of total liabilities and that the median default barrier is 73.8% of the total liability.

In our empirical study, we consider two cases for exogenous models. The first case sets H to $0.738X$ for both the LS and BD models, based on the empirical results of Wong and Choi (2007). The second case sets H to X.

4.2.2. The asset payout ratio

The asset payout ratio captures the payout that the firm makes in the form of dividend yields and share repurchases. The data of dividend yields and stock repurchases can be downloaded from Compustat.

Although EHH (2004) directly used the reported dividend yields downloaded from Compustat, we do not do this, but rather use a revised definition of dividend yield. The reported dividend yield from Compustat is calculated by dividing the annual dividends by the end-of-year stock price, which is an upwardly biased estimator. For firms that pay a large amount of dividends, the corresponding figures are usually in excess of 100%. For example, the reported dividend yields for the USG Corporation in 1988 and the Georgia Gulf Corporation in 1990 are 668% and 282%, respectively. These figures are misleading because the actual payouts could not be that high; otherwise, an arbitrage profit could be made by purchasing the stock to receive dividends, the total value of which is greater than the initial investment.

The asset payout ratio, δ, is the equity payout ratio times leverage. The equity payout ratio is the revised dividend yield for firms with no stock repurchases in the year. Let $\hat{q}$ be the reported dividend yield and D be the annual dividend. Then, the asset payout ratio for this case is given by

$$\delta = \frac{D}{S+D}\frac{S}{V} = \frac{\hat{q}}{1+\hat{q}}\frac{S}{V} = q\frac{S}{V}.$$

Otherwise, the asset payout ratio is calculated as

$$\delta = \frac{D+D_{\mathrm{r}}}{S+D+D_{\mathrm{r}}}\frac{S}{V},$$

where D_{r} is the total value of stock repurchases over a year.

4.2.3. Interest rate parameters

The Merton and LT model assume a constant interest rate, which we measure by the instantaneous interest rate fitted to the Nelson and Siegel (1987) model. The BD and LS models employ the stochastic interest rate model of Vasicek (1977). We calibrate the four parameters that are used in the Vasicek (1977) model to the yield data of constant maturity treasury bonds, which were obtained from the Federal Reserve Board's H15 release.

Specifically, the Nelson and Siegel (1987) model estimates the yield of default-free bonds, y_{NS}, as

$$\hat{y}_{\mathrm{NS}} = \beta_0 + \delta_1(\beta_1+\beta_2)\frac{1-e^{-\tau/\delta_1}}{\tau} - \beta_2 e^{-\tau/\delta_1},$$

where τ is the time to maturity. To calibrate the parameters, we search for the optimal values of β_0, β_1, β_2, and δ_1 such that the sum-of-squared-error between the model yields and the market yields is minimized.

The Vasicek model is calibrated in the same way as the Nelson–Siegel model. The BD and LS models require the correlation coefficient between the asset value returns and changes in the risk-free rate. As the market values of assets have been estimated either by the proxy, VR or MLE approach, we directly calculate the correlation between the asset returns and changes in the interest rate.

4.2.4. Bond specific parameters

The coupon rate (c) and maturity (T) of the bonds are obtained from the Fixed Income Database, which enables us to derive the remaining coupon paying days for each bond. For the Merton, LS and BD models, the coupon rate does not appear in the corresponding equity pricing formulas. However, the coupon rate does exist in that of the LT model. We calculate the weighted average of the coupon rate, weighted by the outstanding balance, for the LT model. We assume no bankruptcy cost for any of the models and a corporate tax rate of 35% for the LT model.

For the recovery rate (ω) of a bond, the paper by Altman and Kishore (1996) shows that the recovery rates for senior secured and senior unsecured debt are about 55% and 48%, respectively. Keenan, Shtogrin and Sobehart (1999) also find that the average bond recovery rate is around 51.31% of the face value of a bond. We follow EHH in taking a recovery rate of 51.31%. This recovery rate is applied to the extended Merton, BD, LS and LT models. In particular for the BD model, there are two recovery rates with one for early default and the other for default at the bond maturity. We however set these two values to be the same, that is, $f_1 = f_2 = 51.31\%$, where f_1 and f_2 are defined in Appendix D.3.

4.3. Empirical results

The empirical results for the Merton, BD, LS, and LT models are summarized in Tables 4–6, in which the percentage errors in prices, yields, and yield spreads are provided. The effects of agency ratings and bond maturities are reported in Tables 5 and 6, respectively. We regard bonds with an S&P rating of A or above as high rating bonds, those with a BBB-rating as medium rating bonds and others as junk bonds. We regard bonds with a maturity of less than or equal to 5 years as short-term bonds, of 5 to 15 years as medium-term bonds, and others as long-term bonds. Sections 4.3.1–4.3.4 report the impact of the implementation methods on each of the structural models. Section 4.3.5 compares the predictive powers of the structural models when they are implemented with the ML approach.

Table 4
Overall empirical results

Characteristics	ML approach			VR approach			Proxy approach		
	% error in prices	% error in yields	Yield differences	% error in prices	% error in yields	Yield differences	% error in prices	% error in yields	Yield differences
	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)	Mean (S.D.)
Merton	2.57% (8.55%)	−2.69% (33.74%)	−0.12% (2.93%)	3.03% (8.29%)	−3.80% (33.65%)	−0.22% (2.89%)	7.05 % (5.43%)	−14.84% (9.84%)	−1.21% (0.85%)
LS, Barrier = 0.738 × Total Liabilities	1.32% (6.84%)	−3.48% (20.42%)	−0.21% (1.84%)	2.21% (6.74%)	−5.06% (19.80%)	−0.35% (1.75%)	4.95% (4.84%)	−11.55% (11.43%)	−0.92% (0.95%)
LS, Barrier = Total Liabilities	−2.08% (9.92%)	4.63% (32.44%)	0.49% (2.90%)	−0.99% (10.12%)	2.63% (33.01%)	0.31% (2.93%)	3.26% (5.99%)	−8.28% (16.30%)	−0.65% (1.34%)
BD, Barrier = 0.738 × Total Liabilities	1.60% (5.97%)	−4.81% (16.88%)	−0.33% (1.45%)	1.94% (5.71%)	−5.74% (16.24%)	−0.39% (1.38%)	5.54% (5.07%)	−12.81% (11.74%)	−1.04% (0.98%)
BD, Barrier = Total Liabilities	0.47% (6.45%)	−2.66% (18.30%)	−0.16% (1.55%)	0.63% (6.41%)	−2.89% (18.34%)	−0.19% (1.55%)	4.62% (5.36%)	−11.27% (14.06%)	−0.91% (1.12%)
LT	2.02% (4.63%)	−1.99% (9.62%)	−0.22% (0.78%)	1.72% (4.99%)	−1.02% (12.36%)	−0.14% (0.97%)	2.16% (4.85%)	−2.28% (10.10%)	−0.25% (0.81%)

Table 4 shows the overall empirical performances of the proxy, VR and ML approaches for the Merton, LS, BD and LT models. For each model, means and standard deviations (in percentages) of percentage errors in prices, percentage errors in yields and yield differences by using ML, VR and proxy approaches are presented. For percentage error in price of each observation, it is calculated as predicted price minus actual price, and then divided by actual price. Similar calculation is performed to obtain percentage errors in yield for each observation. For yield difference, it is calculated by subtracting actual yield from predicted yield.

Table 5
Empirical results by ratings

Characteristics	ML approach			VR approach		
	A or above	BBB	BB or below	A or above	BBB	BB or below
	Mean	Mean	Mean	Mean	Mean	Mean
	(S.D.)	(S.D.)	(S.D.)	(S.D.)	(S.D.)	(S.D.)
Merton	−0.30%	−0.30%	3.66%	−0.37%	−0.38%	3.12%
	(2.42%)	(2.42%)	(7.39%)	(2.44%)	(2.34%)	(7.19%)
LS, H=0.738X	−0.24%	−0.26%	0.69%	−0.35%	−0.48%	0.24%
	(1.56%)	(1.56%)	(4.87%)	(1.54%)	(1.33%)	(4.58%)
LS, H=X	0.34%	0.50%	2.85%	0.23%	0.19%	2.36%
	(2.43%)	(2.55%)	(7.47%)	(2.55%)	(2.21%)	(7.64%)
BD, H=0.738X	−0.34%	−0.33%	−0.25%	−0.39%	−0.36%	−0.54%
	(1.31%)	(1.55%)	(2.76%)	(1.27%)	(1.48%)	(2.50%)
BD, H=X	−0.16%	−0.07%	−0.58%	−0.19%	−0.06%	−0.78%
	(1.41%)	(1.78%)	(2.47%)	(1.42%)	(1.76%)	(2.42%)
LT	−0.16%	−0.29%	−0.85%	−0.11%	−0.21%	−0.42%
	(0.73%)	(0.82%)	(1.12%)	(0.95%)	(0.90%)	(1.48%)

Table 5 shows the empirical performance of the VR and ML methods on the Merton, LS, BD and LT models disaggregated by ratings. For each model, means and standard deviations (in percentages) of yield differences by using ML and variance restriction approaches are presented. For yield difference of each observation, it is calculated by subtracting actual yield from predicted yield.

4.3.1. The Merton model

Table 4 shows the overall performance of the implementation methods for the Merton model. The average percentage errors in the prices and yields are 7.05% and −14.84% for the proxy approach, 3.03% and −3.8% for the VR approach, and 2.57% and −2.69% for the ML approach in order. The ML approach thus consistently improves the Merton model in predicting prices and yields over the alternative implementation methods.

A similar conclusion can be drawn for the yield spreads. In fact, the ML approach produces an average prediction error in the yields of −12 basis points, the VR approach −22 basis points, and the proxy approach −121 basis points. This offers empirical evidence that the proxy firm value makes the Merton model generate a sizable underestimation of yields. It can be seen that the standard deviations of the ML and VR approaches are greater than those of the proxy approach. We stress that a small standard deviation together with a wrong mean value indicates a serious bias. To select

Table 6
Empirical results by maturities

Characteristics	ML approach			VR approach		
	$T\leq 5$	$5<T\leq 15$	$T>15$	$T\leq 5$	$5<T\leq 15$	$T>15$
	Mean	Mean	Mean	Mean	Mean	Mean
	(S.D.)	(S.D.)	(S.D.)	(S.D.)	(S.D.)	(S.D.)
Merton	0.54%	−0.34%	−0.57%	0.43%	−0.44%	−0.64%
	(4.91%)	(1.42%)	(0.97%)	(4.88%)	(1.28%)	(0.98%)
LS, H=0.738X	−0.43%	−0.15%	−0.01%	−0.55%	−0.28%	−0.20%
	(2.75%)	(1.30%)	(1.07%)	(2.58%)	(1.30%)	(1.05%)
LS, H=X	0.41%	0.51%	0.54%	0.26%	0.34%	0.33%
	(4.44%)	(2.02%)	(1.57%)	(4.40%)	(2.12%)	(1.65%)
BD, H=0.738X	−0.59%	−0.34%	0.10%	−0.66%	−0.39%	0.15%
	(2.03%)	(1.15%)	(0.89%)	(1.93%)	(1.10%)	(0.84%)
BD, H=X	−0.50%	−0.15%	0.32%	−0.53%	−0.16%	0.33%
	(2.05%)	(1.35%)	(0.91%)	(2.08%)	(1.32%)	(0.92%)
LT	0.04%	−0.25%	−0.52%	0.17%	−0.20%	−0.48%
	(0.94%)	(0.69%)	(0.57%)	(1.33%)	(0.71%)	(0.75%)

Table 6 shows the empirical performances of the VR and ML approaches on the Merton, LS, BD and LT models disaggregated by maturities. For each model, means and standard deviations (in percentages) of yield differences by using ML and VR approaches are presented. For yield difference of each observation, it is calculated by subtracting actual yield from predicted yield.

a reasonably good estimator, the most important criterion is that it should be consistent or asymptotically unbiased, that is, the bias will be eliminated if the sample size tends to infinity. A wrong mean and a small standard deviation together imply that the estimator tends to a wrong value very quickly when the sample size is increased. Given a set of consistent estimators, the second criterion is to choose the one with the minimum standard deviation. Thus, unbiased estimators play a key role in testing models.

Fig. 3 plots the errors in the yields against the bond maturities. Fig. 3b shows the performance of the proxy approach, in which most points fall into the negative region. Fig. 3a shows the empirical results of the ML approach. We observe that most points are crowded near zero, which provides evidence that the proxy for firm value leads to the underestimation of bond yields. Moreover, the ML approach offers a better estimation of corporate bonds, with quite a number of outliers in the set of short-term bonds. Although the Merton model underestimates corporate bond yields on average, it does not consistently underestimate the yields, as Fig. 3a shows that there are many points in the positive region. Fig. 3c shows that the performance of the VR approach. Similar to the ML approach, the Merton model does not consistently underestimate corporate bond yields with the VR approach. The proxy approach is an obvious loser among these three implementation methods for its significant downward bias on bond yields.

We now concentrate on comparing the ML and VR approaches. Tables 5 and 6 display the performances of the VR and ML approaches by disaggregating the whole data set into different ratings and maturities, respectively. It can be seen that the ML approach outperforms the VR approach except for short-term bonds and lower rating bonds. This poor performance in predicting short-term and low rating corporate bond yields may be caused by the model itself rather than the implementation method. Merton (1974) also recognizes the weakness of his model that it should overestimate short-term and low rating bond yields.

4.3.2. The LS model

For the LS model, Table 4 shows that the proxy approach is the clear loser among the three implementation methods as it underestimates bond yields significantly. The proxy approach underestimates bond yields by 11.55% on average for $H=0.738X$ and 8.28% for $H=X$. The proxy approach considered in this paper uses the revised definition of the asset payout ratio. Therefore, the overestimation in the corporate yield of the LS model implemented with the proxy approach shown in EHH is mainly driven by the direct use of reported dividend yields from Compustat.

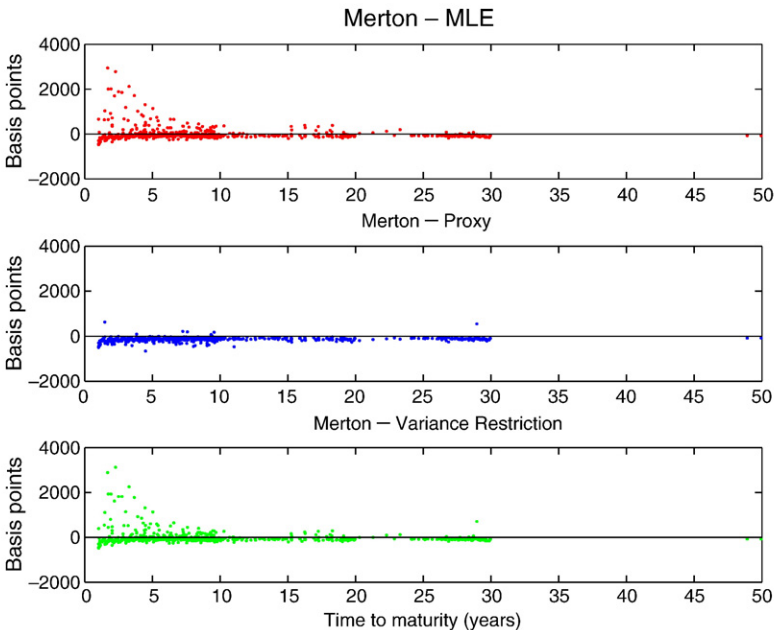

Fig. 3. These figures show the overall results of an empirical study on Merton model. The top figure shows the yield differences of all observations by using ML approach, while the middle and the bottom figures show those by using the proxy and VR approaches, respectively. For yield difference of each observation, it is calculated as predicted yield minus observed yield.

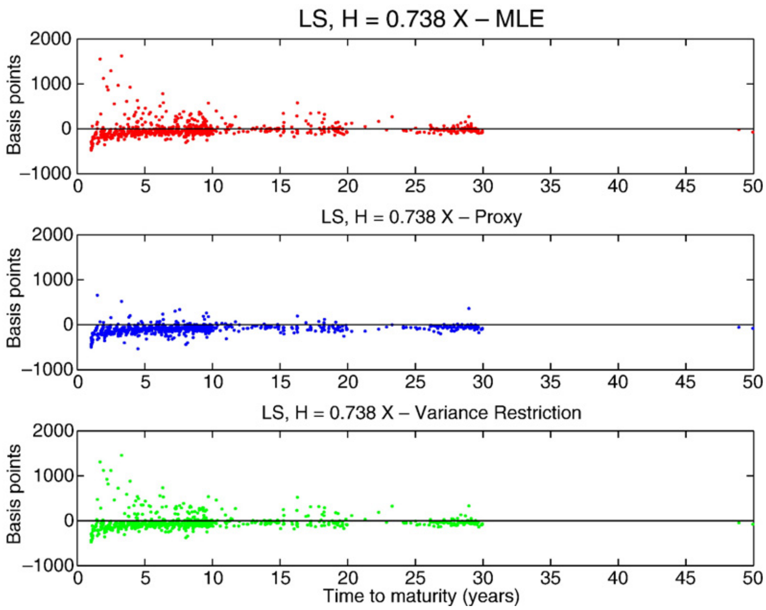

Fig. 4. These figures show the overall results of an empirical study on LS model, by setting barrier at 0.738 times total liabilities. The top figure shows the yield differences of all observations by using ML approach, while the middle and the bottom figures show those by using proxy and VR approaches respectively. For yield difference of each observation, it is calculated as predicted yield minus observed yield.

Fig. 4 plots the difference between the model and market yields against bond maturity for the ML, proxy and VR approaches when $H=0.738X$. It can be seen from Fig. 4b that the proxy approach underestimates the bond yields for almost all of the cases. In contrast, the ML and VR approaches have errors crowded near zero, indicating significant improvement in reducing the systematic underestimation. Fig. 5 shows a similar feature when $H=X$. In all of the cases, the empirical performance of the LS model can be improved by utilizing either the MLE or the VR approach. The proxy approach should be avoided.

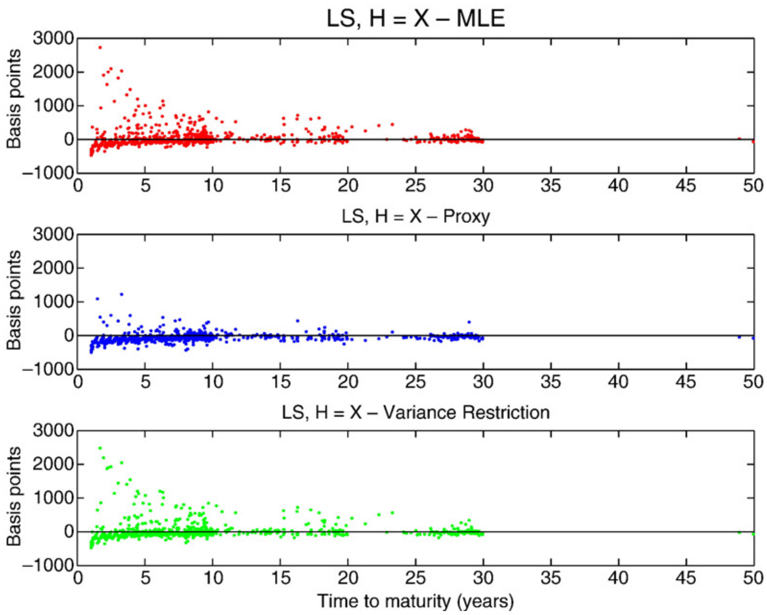

Fig. 5. These figures show the overall results of an empirical study on the LS model, by setting barrier at total liabilities. The top figure shows the yield differences of all observations by using ML approach, while the middle and the bottom figures show those by using the proxy and VR approaches respectively. For yield difference of each observation, it is calculated as predicted yield minus observed yield.

However, it is difficult to conclude whether the ML or the VR approach is the better implementation method for the LS model, based on the predicted errors of prices and yields. When the empirical result is disaggregated into different ratings, Table 5 shows that the LS model implemented with ML estimation gives a better prediction for high and medium rating bonds when H=0.738X , but the VR approach is better for the remaining cases. When the empirical result is displayed by time to maturity, Table 6 shows that the MLE approach gives a better prediction when H=0.738X, but the VR approach is better when $H=X$. It is important to note that the predictive power of the LS model depends not only on the implementation method, but also on the model structure itself.

In addition to their predictive power, the ML and VR approaches can be compared by considering their consistency and implementation robustness. Although we did not include a quantitative measure for these, we can share our experience of implementing the two methods. As the VR approach requires us to solve a system of two equations at each time point, we found that the system of equations sometimes produced more than one solution. In such a situation, we have to manually select one from all possible solutions, such that the predictive error is lowest. Therefore, the process is not completely automatic. This does not occur in the ML approach for the LS model. The main reason may be that there is only one maximum point for the likelihood function, because it is derived from the normal density, which has only one peak. The same difficulty in implementing the VR approach also appears in the BD model.

4.3.3. The BD model

Once again, Table 4 shows that the proxy approach is the worst among the three implementation methods. The underestimation of corporate bond yields with the proxy approach is more than 11%, regardless of the barrier level. Fig. 6 plots the errors in the yields against the bond maturities for the BD model with the barrier level equal to 0.738 of the total value of liabilities. From the figure, we can see that the proxy approach leads to underestimation for almost all corporate bonds, whereas the other two methods do not share the same bias.

The ML estimation is the best and outperforms the VR approach in predicting prices and yields for both $H=0.738X$ and $H=X$. Specifically, errors in yields in the ML approach are −4.81% and −2.66% for $H=0.738X$ and $H=X$, respectively; those in the VR approach are −5.47% and −2.89%, respectively. Given that the standard deviations of the ML and VR approaches are similar, we can conclude that the former outperforms the latter. An additional deficiency of the VR approach is that the system of equations may generate different solutions with different initial values for the numerical root finding algorithm. In such a situation, we manually choose the solution that is the best fit to the market bond yield. This makes the VR approach less useful in practice.

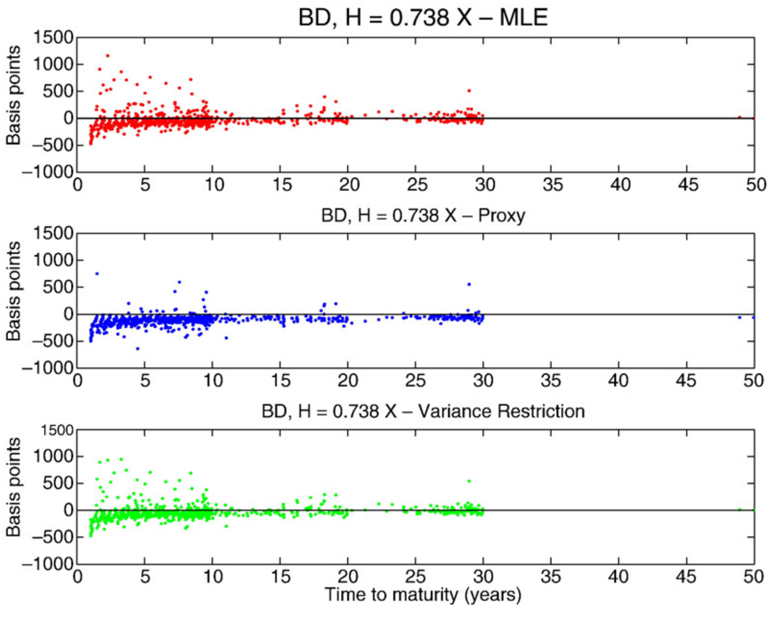

Fig. 6. These figures show the overall results of an empirical study on the BD model, by setting barrier at 0.738 times total liabilities. The top figure shows the yield differences of all observations by using ML approach, while the middle and bottom figures show those by using the proxy and VR approaches respectively. For yield difference of each observation, it is calculated as predicted yield minus observed yield.

No matter whether we display the empirical results by rating or by maturity, the ML approach consistently outperforms the VR approach in terms of predictive power in prices and yields, except for medium-rating bond when $H=X$. Tables 5 and 6 exhibit the corresponding results.

4.3.4. The LT model

The situation of the LT model is very interesting. We can see from Table 4 that, although the proxy approach is still the worst implementation method, the predictive power of the LT model using the proxy approach is not that bad compared to other models. The errors in prices and yields are 2.16% and −2.28%, respectively. The standard deviation for the error in yields is 10.10%, which is significantly less than those in the other structural models with any implementation method. As shown in Fig. 7, the proxy approach no longer consistently underestimates corporate bond yields for the LT model. In fact, the human eye cannot judge which method is the clear loser solely by looking at the three graphs in Fig. 7. A potential reason for this is that the endogenous barrier can offset some of the bias contained in the proxy approach. Although the parameter estimates and the firm values are biased, the endogenous barrier will be adjusted accordingly such that it optimally fits to the theoretical corporate bond price. Thus, the LT model is relatively more robust with this implementation method than are the other models considered in this paper.

However, an appropriate implementation method still improves the LT model. Let us compare the proxy and ML approaches numerically. From Table 4, we can see that the errors in prices and yields (2.02% and −1.99%, respectively) for the ML approach are smaller than those (2.16% and −2.28%, respectively) for the proxy approach. As the standard deviations of the ML approach are also smaller, we are comfortable to say that the ML approach outperforms the proxy approach.

Let us compare the ML and VR approaches. It seems that the VR approach outperforms the ML approach as shown in Table 4. When we disaggregate the empirical results by ratings, the VR approach still seems to be better than the MLE approach as shown in Table 5. When we look at the effect of bond maturities, Table 6 shows that the ML approach only outperforms the VR approach for short- term bonds. However, one important piece of information not included in the tables is that the VR approach gives no output for 10 companies, because there exists no solution to the system of equations derived from the LT model. The data reported in Tables 4–6 have removed these ill cases. For non-ill cases, the VR method may produce multiple solutions for the system of equations. We spent a lot of time manually selecting the best estimator with the VR approach. This deficiency does not occur in either the ML or the proxy approach. Thus, the ML approach is worth considering for implementing structural models in practice.

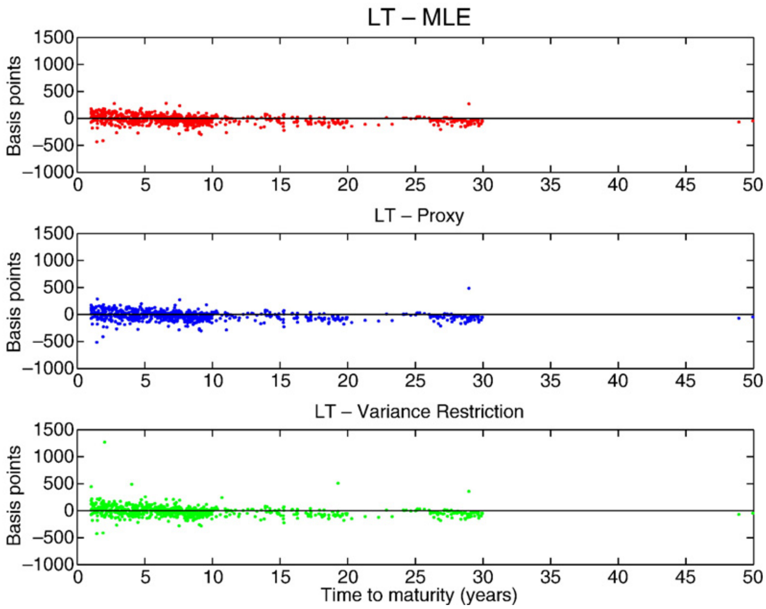

Fig. 7. These figures show the overall empirical result on the LT model. The top figure shows the yield differences of all observations by using ML approach, while the middle and the bottom figures show those by using the proxy and VR approaches respectively. For yield difference of each observation, it is calculated as predicted yield minus observed yield.

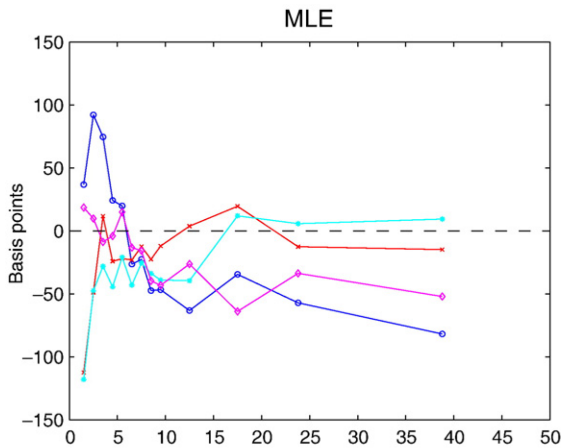

Fig. 8. These figures show the empirical result on the Merton, LS, BD, and LT models by using ML estimation. Observations are grouped into 13 segments, with cutting points of maturities equal 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20 and 27.5 years. In the figure, '○' indicates average yield differences for the Merton model, '×' for the LS model with $H=0.738X$, '∗' for the BD model with $H=0.738$X and '◇' for the LT model.

4.3.5. Comparing structural models implemented with MLE

Although there are many empirical studies on testing structural corporate bond pricing models, none of them performs the ML approach. This subsection is devoted to filling this vacuum. In fact, the performance of each structural model with ML estimation can be easily examined by reading the first few columns of Tables 4–6.

It may be more informative if we plot the average errors of yields for each model against the bond maturity. We can see from Fig. 8 that the Merton model overestimates short-term bond yields but underestimates medium-term and long-term bond yields. The errors are very significant compared with other models.

The LT model shows a similar feature to the Merton model, but it performs much better. In particular, the LT model has the smallest predictive error for short-term bond yields. However, it still suffers from quite significant underestimation for long-term bond yields.

The two exogenous barrier models have a similar performance. Note that we have used the barrier level of 0.738X suggested by Wong and Choi (2007). These two models both significantly underestimate short-term bond yields, but perform the best for long-term bonds. The LS model demonstrates that its predictive power for medium-term bonds is the best.

From this empirical result, we suggest the use of a hybrid structural model for corporate bonds. For instance, the LT model can be used for short-term bonds and the LS model for medium-term and long-term bonds. Of course, this hybrid model should be estimated using the MLE approach. The use of this potential model for credit derivatives remains for future research.

5. Conclusion

This paper tests the proxy, VR, and MLE approaches for implementing the structural models of corporate bond pricing. Specifically, we apply the aforementioned methods to the Merton, LS, BD and LT models. Our empirical results suggest that the proxy approach generates a systematic downward bias for corporate bond yields for all of the models considered in this paper. Although the VR approach sometimes performs very well, it may produce no output or, in contrast, many different outputs. This deficiency leads to implementation difficulties and inconvenience. The ML approach is more involved in terms of theoretical background. However, it consistently outperforms the alternative implementation methods in terms of consistency and efficiency.

Another important contribution of this paper is that it empirically examines the Merton, LS, BD, and LT models using maximum likelihood estimation. We find that the ML approach improves the performance of all of the structural models considered in this paper. The LT model performs the best for short-term bonds, the LS model is the best for medium-term bonds, and the LS and BD are the best for long-term bonds. However, none of the models consistently underestimates corporate bond yields, and this revises the empirical literature on the structural models of corporate bond pricing.

Appendix A. The bias of the proxy approach

A.1. The Merton model

In Merton (1974), there is no intermediate default, and thus the terminal payoff of zero coupon bond holders takes the minimum of the face value of the bond (X) and the market value of assets (V_T). The current bond price (B^{M} (V, X, T)) is valued as a risk-free bond minus a put option (P (V, X, T)) on the current market value of assets (V) with a strike price X and a maturity T. Specifically,

$$B^{\mathrm{M}}(V,X,T) = X \cdot D(T) - P(V,X,T), \tag{5}$$

where $D(T)$ denotes the default-free discount factor with a maturity T. However, the payoff for equity holders resembles the call option payoff with a strike price X. Denote S as the market value of equities. We then have

$$S = C(V,X,T),$$

where $C(V, X, T)$ is the standard call option pricing formula.

Let V_{proxy} be the proxy firm value. The definition of the proxy then asserts that

$$V_{\text{proxy}} = S + X \text{ or, equivalently, } S = V_{\text{proxy}} - X.$$

By a property of standard call options, a call option premium must be greater than the intrinsic value, which implies that

$$C(V,X,T) = S = V_{\text{proxy}} - X < C\left(V_{\text{proxy}}, X, T\right).$$

As a call option is an increasing function of the underlying asset price, the foregoing inequality implies that the proxy firm value is an upwardly biased estimator. This overstated asset value causes the bond price of Eq. (5) to be overestimated and hence the yield spreads to be underestimated, which explains the significant underestimation of corporate bond yields with the Merton model in many empirical studies.

A.2. Barrier-dependent models

Let V be the true market value of assets, V_{proxy} be the proxy for firm value and S be the market value of equity. The proxy for firm value then relates to the equity and liabilities by

$$V_{\text{proxy}} = S + X.$$

We view the market value of equity as a DOC option on the underlying asset V with a strike price X, default barrier H, and rebate R. Thus,

$$S = DOC(V,X,H,R,).$$

The no arbitrage pricing principle shows that the DOC price must be greater than the intrinsic value if the barrier is set to the book value of liabilities, that is,

$$DOC(V,X,H,R) > V - X, \tag{A.1}$$

where $H=X$. If this is not the case (that is, if $V-\mathrm{DOC}(V, X, X, R)-X\geq 0$), then an investor can make an arbitrage profit by selling the asset at V to purchase the DOC option. The remaining cash is put into a bank account. A profit can then be made by taking two different actions that correspond to two possible scenarios.

1. If the asset price V does not breach the barrier level X before maturity, then on the maturity day (T), the investor will exercise the option to purchase the asset for a value of X so that the investor's short position in the asset will be canceled. An arbitrage profit of

   $$[V - DOC(V,X,X)]e^{rT} - X$$

   is then made at time T.

2. If the asset value breaches the barrier level X at time $\tau<T$, then the investor will receive a rebate of R. The investor will purchase the asset from the market right away for an amount X to cancel the short position in the asset. An arbitrage profit of

$$[V - DOC(V, X, X)]e^{r\tau} - X + R$$

is then made at time τ.

This proves the inequality (Eq. (A.1)), which is a model-independent property of DOC options.

A consequence of the inequality is that

$$DOC\left(V_{\text{proxy}}, X, X, R\right) > V_{\text{proxy}} - X = S = DOC(V, X, X, R).$$

As the DOC option is an increasing function of the underlying asset price, the proxy firm value, V_{proxy}, is clearly larger than the true value, V, if the default barrier is set to X. This shows that the proxy firm value in the study of EHH is an upwardly biased estimator.

Actually, the inequality (Eq. (A.1)) holds for all $H<X$, because the DOC option is a decreasing function of the default barrier. Moreover, the difference DOC(V, X, H, R)−(V−X) can be widened by decreasing the value of H, which implies that the smaller the default barrier the more significant the upward bias that is induced by the proxy for firm value. Therefore, it is the most significant bias in the Merton model.

A.3. The pure proxy approach versus the mixed proxy approach

Both proxy approaches employ the same approximation of the firm value but differ in the estimation of a firm's volatility. We have just shown that both proxy approaches overestimate the firm value and hence underestimate the corporate bond yields. However, they may suffer from different degrees of underestimation due to the effect of the volatility estimate.

Consider a sample of n equally time-spaced observations of equity values $\{S_1, S_2, \ldots, S_n\}$ and a fixed book value of debt X over the period of observation. Both proxy approaches produce the same set of firm values $\{\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_n\}$, where $\hat{V}_j = S_j + X$. The pure proxy approach measures the asset volatility as the sample standard deviation of the asset returns. That is,

$$\sigma_{\text{pure}}^2 \Delta t|_{S_i} = \text{Var}\left(\frac{V_{i+1} - V_i}{V_i} |S_i\right) = \text{Var}\left(\frac{S_{i+1} - S_i}{S_i} \frac{S_i}{S_i + X} |S_i\right).$$

It is easy to see that the asset volatility obtained in this way is less than the stock volatility σ_e because $0 < S_i / (S_i + X) < 1$ for all $i = 1, 2, \ldots, n$. Moreover,

$$\sigma_{\text{pure}}|_{S_i} = \frac{S_i}{S_i + X} \times \sigma_e|_{S_i}.$$

For the mixed proxy approach, the volatility of the firm is estimated using the second equation of (1). By making the asset volatility the subject, we have

$$\sigma_{\text{mix}}|_{S_i} = \frac{S_i}{S_i + X} \left[\frac{\partial S}{\partial V}\right]_{V = S_i + X}^{-1} \times \sigma_e|_{S_i} = \left[\frac{\partial S}{\partial V}\right]_{V = S_i + X}^{-1} \times \sigma_{\text{pure}}|_{S_i}.$$

The quantity $\partial S / \partial V$, which is the delta of the standard call (down-and-out call) option for the Merton (LS) model, is always less than 1. This implies that the asset volatility of the mixed proxy approach is greater than that of the pure proxy approach.

A higher volatility leads to a high default risk of a firm and hence a higher credit yield spread. Therefore, the corporate bond yield predicted by the pure proxy approach is systematically less than that of the mixed proxy approach. In Section 3, we show by simulation that the mixed proxy approach underestimates corporate bond yield compared to the ML estimation. The underestimation should therefore be much more significant in the pure proxy approach.

Appendix B. Likelihood function of the Merton model

The underlying asset price evolves as the Black–Scholes dynamics,

$$d \ln V_t = \left(\mu - \sigma^2/2\right)dt + \sigma dZ_t,$$

where V_t is the market value of assets at time t, μ is the drift of the business, σ is the asset volatility, and Z_t is a standard Wiener process. Under the physical probability measure, the density function of ln V_t is given by

$$g(v_i|v_{i-1}) = \frac{1}{\sigma\sqrt{2\pi(t_i - t_{i-1})}} \times \exp\left\{-\frac{\left[v_i - v_{i-1} - (\mu - \sigma^2/2)(t_i - t_{i-1})\right]^2}{2\sigma^2(t_i - t_{i-1})}\right\}.$$

The Merton model views the market value of equity S as a standard call option on the market value of assets V such that

$$S = V \cdot N(d_1) - Xe^{-rT} \cdot N(d_2),$$

where X is the book value of corporate liabilities, r is the risk-free rate, T is maturity, $N(\cdot)$ is the cumulative distribution function for a standard normal random variable and

$$d_1 = \frac{\ln(V/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \text{ and } d_2 = \frac{\ln(V/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

As inference is made based on the observable market values of equities, we formulate the log-likelihood function of μ and σ by

$$L(\mu, \sigma) = \sum_{i=2}^{n} \ln f(S_i|S_{i-1}, \mu, \sigma), \quad S_i \equiv S(t_i),$$

where $f(\cdot)$ denotes the probability density function of S and $S(t_i)$ denotes the market value of equity at time t_i. After applying the standard change of variable technique, we obtain

$$f(S_i|S_{i-1}, \mu, \sigma) = g(v_i|v_{i-1}, \mu, \sigma) \times \left[V_i \cdot N(d_1)|_{V=V_i}\right]^{-1}.$$

Hence, the log-likelihood function reads

$$L(\mu, \sigma) = \sum_{i=2}^{n} \left\{\ln g(v_i|v_{i-1}) - \ln\left[V_i \cdot N(d_1)|_{V=V_i}\right]\right..$$

The MLE is the solution to the following optimisation problem.

$$\max_{\mu,\sigma} L(\mu, \sigma) \quad \text{s.t.} \quad S(t_i) = C(t_i, V(t_i), \sigma), \forall i = 1, 2, \ldots, n.$$

Appendix C. DOC option pricing formula

$$\begin{aligned} DOC(V, X, H, R) = {} & VN(a) - Xe^{-rT}N\left(a - \sigma\sqrt{T}\right) \\ & - V(H/V)^{2\eta}N(b) + Xe^{-rT}(H/V)^{2\eta-2}N\left(b - \sigma\sqrt{T}\right) \\ & + R(H/V)^{2\eta-1}N(c) + R(V/H)N\left(c - 2\eta\sigma\sqrt{T}\right), \end{aligned}$$

where V is the market value of firm assets, X is the future promised payment, H is the barrier level, σ is the asset volatility, r is the risk-free interest rate, T is the time to maturity, R is the rebate paid to the equity holders upon default (asset value breaches the barrier), $N(\cdot)$ is the cumulative distribution function for a standard normal random variable, and

$$a = \begin{cases} \dfrac{\ln(V/X) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \text{for } X \geq H, \\ \dfrac{\ln(V/H) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \text{for } X < H, \end{cases}$$

$$b = \begin{cases} \dfrac{\ln(H^2/VX) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \text{for } X \geq H, \\ \dfrac{\ln(H/V) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \text{for } X < H, \end{cases}$$

$$c = \frac{\ln(H/V) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad \eta \frac{r}{\sigma^2} + \frac{1}{2}.$$

Appendix D. Pricing formulas of structural models

D.1. The Merton model

The original Merton model considers a corporate zero coupon bond with a maturity T and face value X. The model assumes a constant interest rate r and market values of assets V_t follow a geometric Brownian motion, i.e.

$$dV_t = (\mu - \delta)V_t dt + \sigma V_t dW_{1t}, \tag{D.2}$$

where μ, δ and σ is the drift, payout ratio and volatility of market values of assets respectively and W_{1t} is a standard Brownian motion.

Assuming no intermediate default, the terminal payoff of the bond is the minimum of the face amount of the bond and the market value of assets at maturity V_T. By discounting it under the risk-neutral measure, the corporate bond price is expressed as a risk-free bond minus a put option on the underlying assets V with a strike price of X and maturity T, that is,

$$BP_c^M(V_0, X, T) = Xe^{-rT} - P(V_0, X, T) = Xe^{-rT}N(d_2) + V_0 e^{-\delta T}N(-d_1), \tag{D.3}$$

where

$$d_1 = \frac{\ln(V_0/X) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

As the original Merton model only deals with a zero coupon bond, EHH propose the extended Merton model to treat a coupon-bearing bond as a portfolio of zero coupon bonds. Default is assumed to occur only at coupon paying dates when the market value of assets is less than a default barrier K. Upon default, bondholders receive a portion of market values of assets, the recovery rate ω. The pricing formula of the extended Merton model is found to be

$$\begin{aligned} BP_c^{EM}(V_0, X, T) = &\sum\nolimits_{i=1}^{n-1} e^{-rt_i} E^Q\left[(c/2)I_{\{V_{t_i} \geq K\}} + \min(wc/2, V_{t_i})I_{\{V_{t_i} < K\}}\right] \\ &+ e^{-rT}E^Q\left[(1+c/2)I_{\{V_T \geq K\}} + \min(w(1+c/2), V_T)I_{\{V_T < K\}}\right], \end{aligned} \tag{D.4}$$

where c is the coupon rate,

$$\begin{aligned} &E^Q\left[I_{\{V_t \geq K\}}\right] = N(d_2(K,t)), \\ &E^Q\left[I_{\{V_t < K\}} \min(\psi, V_t)\right] = V_0 e^{(r-\delta)t} N(-d_1(\psi,t)) + \psi[N(d_2(\psi,t)) - N(d_2(K,t))], \\ &d_1(x,t) = \frac{\ln(V_0/x) + (r - \delta + \sigma^2/2)t}{\sigma\sqrt{t}}, \\ &d_2(x,t) = d_1(x,t) - \sigma\sqrt{t}. \end{aligned}$$

In formula (D.4), we assume n coupon paying dates of $\{t_1, t_2, \ldots, t_n\}$, that $t_n = T$, and use $N(\cdot)$ to represent the cumulative distribution function of a standard normal random variable.

D.2. The LS model

For the LS model, asset prices are assumed to follow Eq. (D.2), and interest rates r_t are assumed to be stochastic with dynamics of

$$\mathrm{d}r_t = (\alpha - \beta r_t)dt + \eta dW_{2t},$$

or, equivalently,

$$dr_t = \kappa(\theta - r_t)dt + \eta dW_{2t}, \tag{D.5}$$

where α, β, η, κ and θ are some parameters and W_2 is another standard Brownian motion process. The underlying asset price and the interest rate are correlated processes with correlation coefficient ρ.

Under the LS framework, default occurs if the market value of assets at time t (V_t) reaches a threshold value K, or equivalent $L_t = V_t / K$ reaches one. Hence, the pricing formula for a corporate zero coupon bond can be calculated as

$$BP_{\mathrm{c}}^{\mathrm{LS}}(L_0, r_0, T) = D'(r_0, T)[1 - \omega Q(L_0, r_0, T)], \tag{D.6}$$

where

$$Q(L_0, r_0, T, n) = \sum_{i=1}^{n} q_i, q_1 = N(a_1), q_i = N(a_i) - \sum\nolimits_{j=1}^{i-1} q_j N(b_{ij}), \quad i = 2, 3, \ldots, n,$$
$$a_i = \frac{-\ln X - M(iT/n, T)}{\sqrt{S(iT/n)}}, b_{ij} = \frac{M(jT/n, T) - M(iT/n, T)}{\sqrt{S(iT/n) - S(jT/n)},}$$

and

$$M(t, T) = \left(\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2} - \delta\right)t + \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3}\right)\exp(-\beta T)[\exp(\beta t) - 1]$$
$$+ \left(\frac{r_0}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta^2}{\beta^3}\right)[1 - \exp(-\beta t)] - \left(\frac{\eta^2}{2\beta^3}\right)\exp(-\beta T)[1 - \exp(-\beta t)],$$
$$S(t) = \left(\frac{2\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2\right)t - \left(\frac{2\rho\sigma\eta}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)[1 - \exp(-\beta t)] + \left(\frac{\eta^2}{2\beta^3}\right)[1 - \exp(-2\beta t)],$$

where $D'(r_0, T)$ is the price of a zero coupon bond with a face value of \$1 and time to maturity T under interest rates that follow the Vasicek (1977) model, and $N(\cdot)$ is the cumulative density function of a standard normal distribution. When n tends to infinity, the term $Q(L_0, r_0, T)$ is the limit of $Q(L_0, r_0, T, n)$ and thus we can calculate the corporate bond price predicted by the LS model.

The pricing formula for a corporate coupon-bearing bond is simply the sum of all of the individual zero coupon bonds, that is,

$$BP_{\mathrm{c}}^{\mathrm{LS}}(L_0, r_0, T) = \sum_{i=1}^{n} D'(r_0, t_i) \cdot X_i \cdot [1 - \omega Q(L_0, r_0, t_i)], \tag{D.7}$$

and

$$X_1 = X_2 = \ldots = X_{n-1} = \frac{Xc}{2}, \quad X_n = X\left(1 + \frac{c}{2}\right).$$

D.3. The BD model

Briys and de Varenne (1997) consider a soft exogenous default barrier:

$$H(t) = H \cdot P(t, T),$$

where H is a constant less than or equal to the total book value of liability and $P\ (t,\ T)$ is the default-free zero coupon bond price. The default-free bond price is calculated using the Vasicek (1977) model as shown in Eq. (D.5).

Briys and de Varenne (1997) do not provide the corresponding model for the equity. As they claim that their work is an extension of Black and Cox (1976), we use the DOC option to model the equity with the constant default barrier replaced by the soft barrier, i.e.

$$S(t) = E\left\{ e^{-\int_t^T r(s)ds} \max\left(V_T - X, 0\right) 1_{\{\tau_H > T\}} \right\}, \quad \tau_H = \inf\{t | V_t = H(t)\}.$$

Simple calculation shows that

$$S(t) = DOC(V(t)/P(t,T), X, H, R; r = 0, \sigma = \Sigma(T)), \tag{D.8}$$

where DOC is defined in Appendix C, R is the rebate paid to equity holders and

$$\Sigma(t) = \int_0^T \left[(\rho\sigma_{\mathrm{v}} + \sigma_{\mathrm{P}}(\tau, T))^2 + (1 - \rho^2)\sigma_{\mathrm{v}}^2 \right] d\tau, \tag{D.9}$$

with $\sigma_{\mathrm{P}}\ (\tau,\ T)$ being the volatility of the default-free zero coupon bond, σ_{v} being the asset volatility, and ρ being the correlation coefficient between the asset value and the interest rate. In formula (D.8), we set $r=0$ because the interest rate has been absorbed by the default-free bond pricing formula, $P\ (t,\ T)$.

Briys and de Varenne (1997) derive the zero coupon corporate bond price as follows:

$$\begin{aligned} BP^{\mathrm{BD}}(0) = F \cdot P(0,T) \cdot \Bigg[& 1 - P_{\mathrm{E}}(\ell_0, 1) + P_{\mathrm{E}}\left(q_0, \frac{\ell_0}{q_0}\right) - (1 - f_1)\ell_0\left(N(-d_3) + \frac{N(-d_4)}{q_0}\right) \\ & -(1 - f_2)\ell_0\left(N(d_3) - N(d_1) + \frac{N(d_4) - N(d_6)}{q_0}\right)\Bigg], \end{aligned} \tag{D.10}$$

where F is the face value of the bond, f_1 is the recovery rate paid at the default time prior to the maturity and f_2 is the recovery rate paid at the bond maturity.

$$\begin{aligned} &\ell_0 = \frac{V_0}{FP(0,T)}, \quad q_0 = \frac{V_0}{HP(0,T)}, d_1 = \frac{\ln \ell_0 + \Sigma(T)/2}{\sqrt{\Sigma(T)}} = d_2 + \sqrt{\Sigma(T)}, \\ &d_3 = \frac{\ln q_0 + \Sigma(T)/2}{\sqrt{\Sigma(T)}} = d_4 + \sqrt{\Sigma(T)}, d_5 = \frac{\ln\left(q_0^2/\ell_0\right) + \Sigma(T)/2}{\sqrt{\Sigma(T)}} = d_6 + \sqrt{\Sigma(T)}, \\ &P_{\mathrm{E}}(\ell_0, 1) = -\ell_0 N(-d_1) + N(-d_2), P_{\mathrm{E}}\left(q_0 \cdot \frac{\ell_0}{q_0}\right) = -q_0 N(-d_5) + \frac{\ell_0}{q_0} N(-d_6). \end{aligned}$$

To value a corporate coupon bond, we use the approximation that the coupon bond can be expressed as a sum of zero coupon bonds and use formula (D.10) to calculate each zero coupon corporate bonds.

D.4. The LT model

Suppose that the default barrier, H, is exogenously given. Under the GBM for the asset price process, Leland and Toft (1996) determine the total value of all outstanding bonds:

$$\begin{aligned}
&B(V;H,T)=\frac{C}{r}+\left(K-\frac{C}{r}\right)\left(\frac{1-e^{-rT}}{rT}-I(T)\right)+\left[(1-\alpha)H-\frac{C}{r}\right]J(T), I(T)=\frac{1}{rT}\left[G(T)-e^{-rT}F(T)\right],\\
&J(T)=\frac{1}{z\sigma\sqrt{T}}\left[-\left(\frac{V}{H}\right)^{-a_1+z}N(q_1)q_1+\left(\frac{V}{H}\right)^{-a_1-z}N(q_2)q_2\right],\\
&F(T)=N(h_1)+\left(\frac{V}{H}\right)^{-2a_1-z}N(h_2)\\
&G(T)=\left(\frac{V}{H}\right)^{-a_1+z}N(q_1)+\left(\frac{V}{H}\right)^{-a_1-z}N(q_2),
\end{aligned}\tag{D.11}$$

where

$$\begin{aligned}
&q_1=\frac{\ln\dfrac{H}{V}-z\sigma^2T}{\sigma\sqrt{T}};\quad q_2=\frac{\ln\dfrac{H}{V}+z\sigma^2T}{\sigma\sqrt{T}}, h_1=\frac{\ln\dfrac{H}{V}-a_1\sigma^2T}{\sigma\sqrt{T}};\quad h_2=\frac{\ln\dfrac{H}{V}+a_1\sigma^2T}{\sigma\sqrt{T}},\\
&a_1=\frac{r-\delta-\sigma^2/2}{\sigma^2};\quad z=\frac{\left[\left(a_1\sigma^2\right)^2+2r\sigma^2\right]^{1/2}}{\sigma^2},
\end{aligned}$$

and δ is the asset payout ratio, K is the total principal value of all outstanding bonds, C is the total coupon paid by all outstanding bonds per year, and α is the fraction of firm asset value lost in bankruptcy; i.e. the remaining value $(1-\alpha)H$ is distributed to bond holders in bankruptcy.

The total market value of the firm, v, equals the asset value V plus the value of tax benefits, less the value of bankruptcy costs. Tax benefits accrue at rate τC per year as long as $V>H$, where τ is the corporate tax rate. The total firm value is given by

$$v(V;H)=V+\frac{\tau C}{r}\left[1-\left(\frac{V}{H}\right)^{a_1+z}\right]-\alpha H\left(\frac{V}{H}\right)^{-a_1-z}.\tag{D.12}$$

Hence, the market value of equity is given by

$$S(V;H,T)=v(V;H)-B(V;H,T).\tag{D.13}$$

To determine the equilibrium default barrier H endogenously, Leland and Toft (1996) specify that the equity valuation formula should satisfy the smooth-pasting condition, under which the value of equity and the value of the firm are maximised. Specifically, H must be solved from the following equation:

$$\frac{\partial S(V;H,T)}{\partial V}|_{V=H}=0.$$

The solution is given by

$$H-\frac{(C/r)(A/(rT)-B)-AK/(rT)-\tau C(a_1+z)/r}{1+\alpha(a_1+z)-(1-\alpha)B}\tag{D.14}$$

where

$$\begin{aligned}
&A=2a_1e^{rT}N(a_1\sigma\sqrt{T})-2zN(z\sigma\sqrt{T})+(z-a_1)-\frac{2}{\sigma\sqrt{T}}n(z\sigma\sqrt{T})+\frac{2e^{-rT}}{\sigma\sqrt{T}}n(a_1\sigma\sqrt{T});\\
&B=-\left(2z+\frac{2}{z\sigma^2T}\right)N(z\sigma\sqrt{T})-\frac{2}{\sigma\sqrt{T}}n(z\sigma\sqrt{T})+(z-a_1)+\frac{1}{z\sigma^2T},
\end{aligned}$$

and $n(\cdot)$ denotes the standard normal density function. Once the endogenous barrier is obtained, the equity and bond pricing formulas are respectively defined in Eqs. (D.13) and (D.11) with the endogenous barrier substituted into the formulas.

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