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Source: *The Journal of Finance*, Vol. 56, No. 5 (Oct., 2001), pp. 1929-1957
Published by: Wiley for the American Finance Association
Stable URL: https://www.jstor.org/stable/2697744
Accessed: 18-01-2019 14:54 UTC

THE JOURNAL OF FINANCE • VOL. LVI, NO. 5 • OCT. 2001

Do Credit Spreads Reflect Stationary Leverage Ratios?

PIERRE COLLIN-DUFRESNE and ROBERT S. GOLDSTEIN*

ABSTRACT

Most structural models of default preclude the firm from altering its capital structure. In practice, firms adjust outstanding debt levels in response to changes in firm value, thus generating mean-reverting leverage ratios. We propose a structural model of default with stochastic interest rates that captures this mean reversion. Our model generates credit spreads that are larger for low-leverage firms, and less sensitive to changes in firm value, both of which are more consistent with empirical findings than predictions of extant models. Further, the term structure of credit spreads can be upward sloping for speculative-grade debt, consistent with recent empirical findings.

CLAIM DILUTION, THE ABILITY OF FIRMS to issue additional debt of equal or greater priority in the future, has been identified by Fama and Miller (1972) and Smith and Warner (1979) as a method by which equity can expropriate wealth from current bondholders. Brick and Fisher (1987) report that prior to 1950 covenants were in place that precluded firms from issuing *pari-passu*, or equal-priority debt. By 1976, however, such covenants had mostly disappeared. Analyzing 414 long-term public debentures issued during 1960–1992 by industrial firms, Malitz (1994) finds that covenants in place typically allow for a significant amount of equal-priority debt to be issued in the future.[1] Further, most indentures even permit a small amount of secured debt to be issued in the future.[2] Hence, although many debentures have covenants in place to protect bondholders against major changes in capital structure such as leveraged buyouts, in general, firms with sufficient sol-

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[1] On average, she finds future debt can be issued to a point where the leverage ratio can rise 21.3 percent higher than the leverage ratio that exists at the date of the prior debt's issuance. Here, leverage ratio is defined as the ratio of future funded debt (basically, long-term debt) to consolidated net-tangible assets.

[2] See Stulz and Johnson (1985). Limiting future secured debt has been coined a negative pledge. Secured debt is effectively of higher priority than senior unsecured debt, having a significantly higher recovery rate than unsecured debt. See, for example, Franks and Torous (1989) and Altman (1992).

vency ratios have the legal right to issue additional equal-priority debt. Clearly, firms take advantage of this freedom. For example, Longstaff and Schwartz (1995) note that GMAC had 53 outstanding long-term debt issues listed in the 1992 Moody's Bank and Finance Manual, and that each would likely obtain the same recovery rate in the event of bankruptcy. Hence, in determining the appropriate credit spread for a corporate bond, it seems necessary to account for both the firm's current liability structure, and its right to alter this structure in the future. Below, we propose a structural model of credit spreads that accounts for this feature.

Theoretical models of credit spreads can be categorized as either structural or reduced-form models of default. Reduced-form models abstract from the firm-value process and effectively model default as a pure jump process (Jarrow, Lando, and Turnbull (1997), Madan and Unal (1998), Duffie and Singleton (1999)). In contrast, structural models of default use contingent-claims analysis to determine the appropriate credit spread for a given risky bond. A structural model proposed by Merton (1974) specifies the firm-value process and assumes that default is triggered on the maturity date if firm value is less than the value promised to bondholders. We note that this model precludes the firm from issuing additional debt at intermediate dates. Below, we demonstrate that accounting for the firm's option to issue additional debt in the future significantly increases the predicted credit spreads for previously issued debt. Further, we demonstrate that precluding this option generates a downward-sloping term structure of credit spreads for speculative-grade debt, in conflict with the empirical findings of Helwege and Turner (1999).

Besides this unrealistic restriction on future debt issues, Merton's model is difficult to implement for firms that have complicated capital structures. Longstaff and Schwartz (LS; 1995) circumvent this problem by exogenously specifying a default boundary, which they assume remains constant over time.[3] Note, though, that if this default boundary is presumed to be a monotonic function of the level of outstanding debt, then their model predicts that expected leverage ratios will decline exponentially over time.[4] In practice, however, leverage ratios appear to be stationary. At an aggregate (industry) level, leverage ratios have remained within a fairly narrow band even as equity indices have increased 10-fold over the past 20 years. At the firm level, Opler and Titman (1997) provide empirical support for the existence of target leverage ratios within an industry. Further, dynamic models of optimal capital structure by Fischer, Heinkel, and Zechner (1989), and Goldstein, Ju, and Leland (2001) find that firm value is maximized when a firm acts to keep its leverage ratio within a certain band.

[3] Leland (1994) finds that if the firm is restricted to issuing a perpetuity and maintaining that static capital structure decision, then such a time-invariant default threshold is optimal for the case of constant interest rates. In his framework the log-default threshold is linear in the log-debt level.

[4] LS assume the firm value process follows geometric Brownian motion, so that expected firm value increases exponentially over time. Thus, if the debt level is assumed to be a monotonic function of the default boundary, it follows that the debt level also remains constant over time, leading to expected leverage ratios that decline exponentially over time.

In this paper, we develop a structural model of default with stochastic interest rates that generates stationary leverage ratios. Compared to previously proposed structural models, our model generates larger credit spreads for firms with low initial leverage ratios, more in line with empirical observation.[5] Further, we demonstrate that our model generates term structures of credit spreads for speculative-grade debt that are consistent with the empirical findings of Helwege and Turner (1999).

In contrast to reduced-form models of default, the major disadvantage of the structural models is the mathematical complexity associated with modeling default as a first passage time event. Few one-factor models possess closed-form expressions for the first-passage density. When dealing with two-factor models, analytic results are even more scarce. Hence, most investigations have been restricted to one-factor, constant interest rate frameworks. Longstaff and Schwartz (1995) is a notable exception. They develop a structural model of default in the presence of stochastic interest rates and firm value. However, their proposed solution uses a formula due to Fortet (1943) that is valid only for one-dimensional Markov processes. As such, their proposed solution serves only as an approximation to the true solution of their model. Generalizing the formula of Fortet, and building on the approach of LS, we develop an efficient method for pricing corporate debt within a multifactor framework that is applicable to both our model and the original LS model.

Other papers have considered stochastic default boundaries, such as Nielsen, Saá-Requejo, and Santa-Clara (1993), Briys and de Varenne (1997) and Schöbel (1999). However, the source of the randomness of the boundary in these models is tied to the interest rate process, and will not lead to stationary leverage ratios. These models assume that the default boundary possesses a drift that increases linearly with the spot rate. This assumption causes the probability of default, and in turn the credit spreads, to be independent of the level of the spot rate. This theoretical prediction is in contrast to the empirical findings of Longstaff and Schwartz (1995), and Duffee (1998, 1999), who find credit spreads to be a decreasing function of interest rates.

Upon completion of this paper, we became aware of a related approach proposed by Taurén (1999). However, Taurén restricts his investigation to a constant interest rate economy. Further, his model predicts that credit spreads are independent of the level of the interest rate, contrary to the empirical findings discussed previously, as well as to predictions of our model.

The rest of the paper is organized as follows. In Section I, we present a simple extension of Merton's (1974) model to illustrate that a firm's ability to issue equal-priority debt in the future can have a significant impact on

[5] Jones, Mason, and Rosenfeld (1984) and others report predicted credit spreads from structural models of default are well below those observed in practice. Recently other improvements to the traditional structural model have improved structural model predictions. These improvements include equity's ability to force concessions (Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997)), and accounting for creditor protection via Chapter 11 (Francois and Morellec (1999)).

the credit spreads of previously issued debt. In Section II, we propose a model of mean-reverting leverage ratios and use it to price corporate debt in a constant interest rate framework. In Section III, we generalize the model to account for stochastic interest rates, and offer an efficient scheme for computing the first passage time density of a two-factor Markov system. We conclude in Section IV. Proofs of propositions are placed in Appendices A and B.

I. An Illustrative Example: Generalizing Merton (1974)

The Merton (1974) model assumes that a firm with initial value V_0 issues a zero-coupon bond with face value D and maturity T. The firm-value process is assumed to follow a geometric Brownian motion (under the risk-neutral measure):

$$\frac{dV_t}{V_t} = (r - \delta)dt + \sigma\, dz^{\mathcal{Q}}(t), \tag{1}$$

where the risk-free rate r, payout rate δ, and volatility σ are all assumed constant. Default occurs at date-T if $V_T < D$. Assuming bankruptcy costs are a constant proportion α of remaining firm value, the risky zero-coupon bond price is given by

$$\begin{aligned} P_M &= e^{-rT}\mathrm{E}^{\mathcal{Q}}[D\mathbf{1}_{\{V_T>D\}} + (1-\alpha)V_T\mathbf{1}_{\{V_T<D\}}] \\ &= De^{-rT}(N(d^{\mathcal{Q}}_{(\Gamma,T)}) + (1-\alpha)\Gamma e^{(r-\delta)T}N(-d^{\mathcal{R}}_{(\Gamma,T)})), \end{aligned} \tag{2}$$

where $\Gamma = V_0/D$ is the inverse-leverage ratio, $\mathbf{1}_{\{A\}}$ is the indicator function of event A, and we have defined

$$d^{\mathcal{Q}}_{(x,T)} \equiv \frac{\ln x + \left(r - \delta - \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d^{\mathcal{R}}_{(x,T)} \equiv \frac{\ln x + \left(r - \delta + \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}. \tag{3}$$

Now consider an otherwise equivalent firm that has the option to increase leverage at some intermediate date $\tau \in (0,T)$. Assume that if the firm exercises this option, it does so by issuing a zero coupon bond with the same maturity T as the previously issued debt. Further, assume that the face value of the newly issued debt, N_τ, is chosen to reset firm leverage back to its initial "target" value. In other words, N_τ is obtained implicitly from $V_\tau/(D + N_\tau) = \Gamma$. Finally, assume the proceeds of the new debt issuance are used to repurchase existing equity, leaving firm value unchanged. With these assumptions, the initial price of the original risky bond is determined via

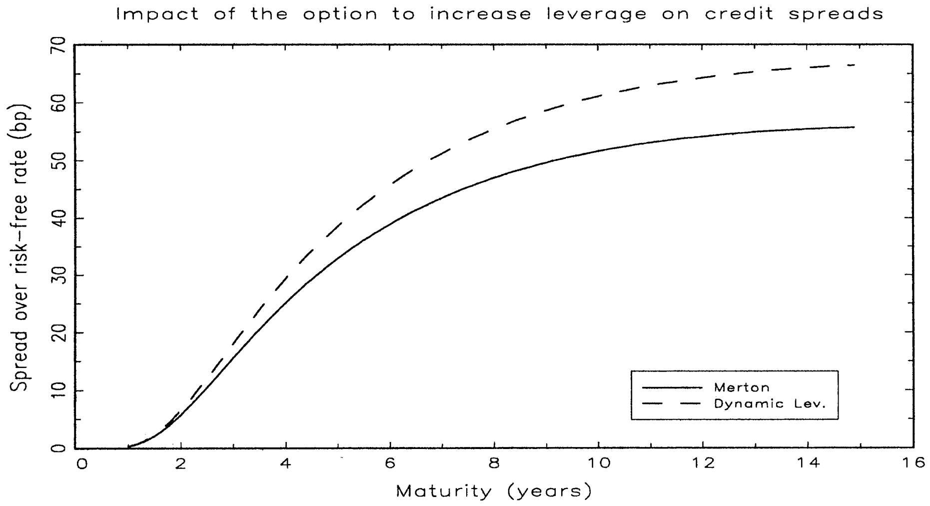

Figure 1. Impact of the option to increase leverage on credit spreads. Parameter values are: $r = 0.06$, $\delta = 0.03$, $\sigma = 0.2$ $L \equiv 1/\Gamma = 0.40$ $\alpha = .15$. For simplicity, it is assumed that τ, the time at which the firm decides whether or not to issue additional debt, is precommitted to at date-0. Further, it is chosen so as to expropriate as much value as possible from the initial debt-holders.

$$
\begin{aligned}
P_L &= e^{-rT}\mathrm{E}^{Q}\Bigg[D\mathbf{1}_{\{V_T>D, V_\tau<V_0\}} + (1-\alpha)V_T\mathbf{1}_{\{V_T<D, V_\tau<V_0\}} + D\mathbf{1}_{\{V_T>D+N_\tau, V_\tau>V_0\}} \\
&\qquad + (1-\alpha)V_T\frac{D}{D+N_\tau}\mathbf{1}_{\{V_T<D+N_\tau, V_\tau>V_0\}}\Bigg] \\
&= De^{-rT}N_2\left(d^{Q}_{(\Gamma,T)}, -d^{Q}_{(1,\tau)}, -\sqrt{\frac{\tau}{T}}\right) \\
&\quad + (1-\alpha)V_0e^{-\delta T}N_2\left(-d^{\mathcal{R}}_{(\Gamma,T)}, -d^{\mathcal{R}}_{(1,\tau)}, \sqrt{\frac{\tau}{T}}\right) \\
&\quad + DN(d^{Q}_{(\Gamma,T-\tau)})N(d^{Q}_{(1,\tau)}) + (1-\alpha)\Gamma e^{-\delta(T-\tau)}N(-d^{\mathcal{R}}_{(\Gamma,T-\tau)})N(d^{Q}_{(1,\tau)}),
\end{aligned}
\tag{4}
$$

where $N(x)$ and $N_2(x, y, \rho)$ are the univariate and bivariate standard-normal cumulative distribution functions, respectively. The proof of this formula is available upon request. The last term of this formula captures the assumption that the newly issued debt is of equal priority to the previously issued debt, so that in bankruptcy each claim receives the same recovery rate.

Figure 1 demonstrates the significant increase in credit spreads generated by equity's right to a one-time increase in debt levels. Permitting the firm to issue debt several times in the future would increase credit spreads even further. Of course, firms may also choose to decrease leverage to reduce default-risk as firm value declines. In the next section we empirically investigate how interest rates and a firm's current leverage ratio affects its decision to change outstanding debt levels.

II. A Model of Credit Spreads with Stationary Leverage Ratios

Assume firm-value dynamics follows geometric Brownian motion:

$$\frac{dV_t}{V_t} = (\mu - \delta)\,dt + \sigma dz_t, \tag{5}$$

where μ is the expected drift under the original measure, δ is the payout ratio, and σ is the volatility. It is convenient to define $y = \log V$. Hence

$$dy_t = \left(\mu - \delta - \frac{\sigma^2}{2}\right)dt + \sigma dz_t. \tag{6}$$

As in Black and Cox (1976) and Longstaff and Schwartz (1995) we assume default is triggered the first time firm value reaches some exogenously specified threshold. This approach captures the idea of Merton's (1974) original model, but is more flexible. In particular, it accommodates complicated liability structures and payoffs as well as stochastic interest rates. In contrast to these papers, however, we assume that the default threshold changes dynamically over time. In particular, we model the dynamics of the log-default threshold, k_t, as

$$dk_t = \lambda(y_t - \nu - k_t)\,dt. \tag{7}$$

The interpretation of this model is straightforward: When k_t is less than $(y_t - \nu)$, the firm acts to increase k_t, and vice-versa. This model captures in a parsimonious way the notion that firms tend to issue debt when their leverage ratio falls below some target, and are more hesitant to replace maturing debt when their leverage ratio is above that target.[6] More generally, even though the default threshold need not be equal to the outstanding book value of debt, it seems reasonable to assume both are related. In fact, in Merton's (1974) model the two are identical. In Leland's (1994) model, the log-default threshold is linear in the log of outstanding debt level. Our model could also be interpreted with a more general definition of leverage, namely the log-ratio of firm value to a default threshold that reflects the market value of total liabilities of the firm.

[6] A simple way to interpret this model is to assume that, as in Merton's model, default is triggered when firm value hits the book value of debt, $K(t)$. Modeling the latter as a multiple of past average firm values, that is, $K(t) = e^{-\nu+Q_t}$ where $Q_t = \lambda\int_{-\infty}^{t} ds\, e^{-\lambda(t-s)} y_s$, we obtain equation (7) above.

Define the log-leverage $\ell_t = k_t - y_t$. From Ito's lemma, ℓ_t follows the (one-factor Markov) dynamics:

$$d\ell_t = \lambda(\bar{\ell} - \ell_t)\,dt - \sigma\,dz_t, \tag{8}$$

where

$$\bar{\ell} \equiv \frac{-\mu + \delta + \dfrac{\sigma^2}{2}}{\lambda} - \nu. \tag{9}$$

As a base case, we take $\bar{\ell} = -1$, $\mu = .122$, $\delta = .03$, $\sigma = .2$, $\lambda = .18$.[7] This implies $\nu = 0.6$.

A. Pricing Corporate Debt

To obtain reasonable parameter values, we investigated the system's dynamics under the true measure. However, to price corporate debt, we need the system's dynamics under the risk-neutral measure. In particular, firm-value dynamics are specified by

$$\frac{dV_t}{V_t} = (r - \delta)\,dt + \sigma\,dz_t^Q. \tag{10}$$

The dynamics of the other relevant variables under the risk-neutral measure are

$$dy_t = \left(r - \delta - \frac{\sigma^2}{2}\right)dt + \sigma\,dz_t^Q \tag{11}$$

$$dk_t = \lambda(y_t - \nu - k_t)\,dt \tag{12}$$

$$d\ell_t = (dk - dy) = \lambda(\bar{\ell}^Q - \ell_t)\,dt - \sigma\,dz_t^Q, \tag{13}$$

where, if we take $r = .06$, we find for the base-case $\bar{\ell}^Q$ is

$$\bar{\ell}^Q \equiv \frac{-r + \delta + \dfrac{\sigma^2}{2}}{\lambda} - \nu \approx -.6556. \tag{14}$$

[7] Setting $\lambda = .18$ falls closer to the estimate of Fama and French (1999), ($\lambda \approx .1$), who investigate the universe of firms, than Shyam-Sunder and Myers (1999), ($\lambda \approx .4$), whose sample is weighted towards large, financially conservative firms.

Note that the estimate of the level towards which log-leverage reverts to under the risk-neutral measure is significantly higher than its counterpart under the original measure. This will be important when estimating credit spreads.

Define $\tilde{\tau}$ as the random time at which $\ell(t)$ reaches zero for the first time, triggering default. Assume that a risky discount bond with maturity T receives one dollar at T if $\tilde{\tau} > T$, or $(1-\omega)$ at time T if $\tilde{\tau} \leq T$. An alternative interpretation is that the risky bond pays a fraction $(1-\omega)$ of the T-maturity risk-free bond at the time of default. The price of this risky discount bond can be written as

$$P^T(\ell_0) = e^{-rT}\mathrm{E}^Q[\mathbf{1}_{(\tilde{\tau}>T)} + (1-\omega)\mathbf{1}_{(\tilde{\tau}<T)}] = e^{-rT}(1-\omega Q(\ell_0,T)). \tag{15}$$

Here, $Q(\ell_0,T)$ is the risk-neutral probability that default occurs before time T given that the leverage ratio is ℓ_0 at time 0.

The bond price formula (equation (15)) depends on the first-passage density of an Ornstein–Uhlenbeck process. This process possesses a closed-form solution (Ricciardi and Sato (1988)). However, to motivate the approach used in the next section, here we use an integral equation due to Fortet (1943), which provides an implicit formula for the first hitting-time density:

$$\Pi^f(T|\ell_0,0) = \int_0^T dt\, g(\ell_t = 0, t|\ell_0,0)\Pi^f(T|\ell_t = 0, t). \tag{16}$$

Here, $g(\ell_t = 0, t|\ell_0,0)$ is the probability density that the first hitting time is at time t, and $\Pi^f(T|\ell_s,s)$ is the date-s conditional probability that $\ell_T > 0$, where we assume that $\{\ell_t\}$ follows the "free" (unabsorbed) process of equation (13). This equation has a straightforward interpretation: The only way that the process can start below the boundary ($\ell_0 < 0$) and end up above the boundary ($\ell_T > 0$) is that at some intermediate time t, it must pass through the boundary for the first time. This equation is valid for one-dimensional Markov processes, and has been used previously by Longstaff and Schwartz (1995) to estimate a hitting-time density.

Proposition 1. *Discretize time into n equal intervals, and define date $t_j = jT/n \equiv j\Delta t$ for $j \in (1,2,\ldots,n)$. The price of a risky discount bond is given by equation (15), where*

$$Q(\ell_0, t_j) = \sum_{i=1}^{j} q_i \qquad j = 2,3,\ldots,n \tag{17}$$

$$q_1 = \frac{N(a_1)}{N(b_{(1/2)})} \tag{18}$$

$$q_i = \left(\frac{1}{N(b_{(1/2)})}\right)\left[N(a_i) - \sum_{j=1}^{i-1} q_j N(b_{i-j+\frac{1}{2}})\right] \qquad i = 2,3,\ldots,n \tag{19}$$

$$a_i = \frac{M(i\Delta t)}{S(i\Delta t)} \tag{20}$$

$$b_i = \frac{L(i\Delta t)}{S(i\Delta t)} \tag{21}$$

$$M(t) = \ell_0 e^{-\lambda t} + \bar{\ell}^Q(1 - e^{-\lambda t}) \tag{22}$$

$$L(t) = \bar{\ell}^Q(1 - e^{-\lambda t}) \tag{23}$$

$$S^2(t) = \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t}). \tag{24}$$

Proof: See Appendix A.

B. *Credit Spreads*

Consider a coupon bond with promised coupon payments C at dates t_j, $j \in (1,N)$, where $t_N \equiv T$. The price of this coupon bond can be written as

$$\begin{aligned} P^T(\ell_0) &= \sum_{j=1}^{N} Ce^{-rt_j}\mathrm{E}^Q[\mathbf{1}_{(\tilde{\tau}>t_j)} + (1 - \omega_{coup})\mathbf{1}_{(\tilde{\tau}<t_j)}] \\ &\quad + e^{-rT}[\mathbf{1}_{(\tilde{\tau}>T)} + (1 - \omega)\mathbf{1}_{(\tilde{\tau}<T)}] \\ &\equiv \sum_{j=1}^{N} Ce^{-rt_j}(1 - \omega_{coup}Q(\ell_0, t_j)) + e^{-rT}(1 - \omega Q(\ell_0, T)). \end{aligned} \tag{25}$$

Comparing equations (15) and (25), we see that a coupon bond can be thought of as a portfolio of discount bonds. Because of this fact, most theoretical models of risky debt limit their investigation to discount bonds. However, our model predicts that the term structure of credit spreads generated by discount bonds is qualitatively different than those generated by coupon bonds. Since we wish to compare our results to empirical studies, and since almost all studies investigate coupon bonds, here we examine the term structure of credit spreads of coupon bonds. In practice, claims to future coupon payments are of the lowest priority, and rarely receive any compensation in bankruptcy (Helwege and Turner (1999)). We thus set $\omega_{coup} = 1$, that is, only future principal payments receive compensation in bankruptcy.

The yield to maturity for this coupon bond Y^T is defined implicitly through the equation

$$P_c^T(\ell_0) = e^{-Y^T T} + C\sum_{j=1}^{N} e^{-Y^T t_j}. \tag{26}$$

Finally, the credit spread $CS(T)$ is defined via

$$CS(T) = Y^T - r. \tag{27}$$

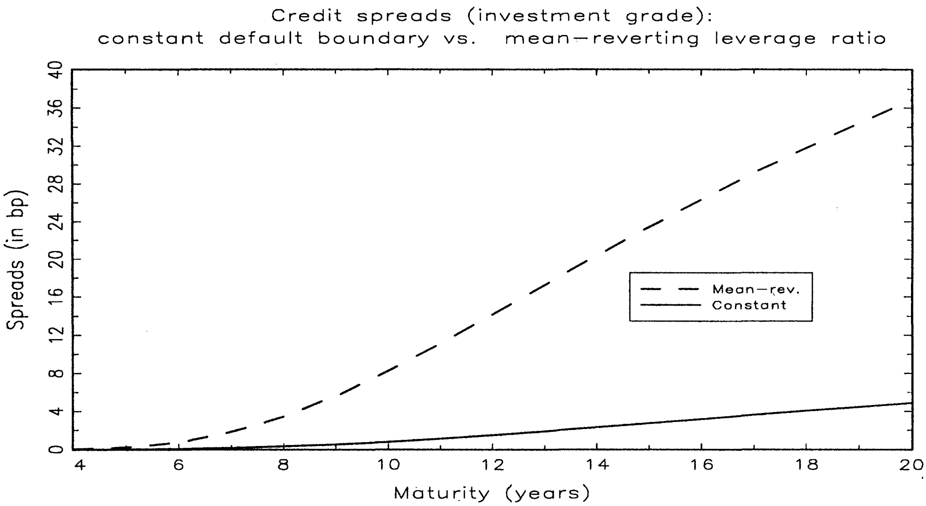

Figure 2. Credit spreads as a function of maturity for the standard structural model and the stationary leverage model for investment-grade (low-leverage) firms. In both models we set $r = .06$, $C = .075$, $\delta = .03$, $\sigma = .2$, $\omega = .56$, $\omega_{coup} = 1$. The initial leverage is set to 15 percent. For the stationary leverage ratio model, $\nu = 0.6$ and $\lambda = .18$ (which implies a long-term leverage of about 36 percent, or $\bar{\ell} = -1$), consistent with estimates obtained in our empirical investigation, which is available on request.

In Figure 2 we compare the credit spread predictions of the constant boundary model with the predictions of the stationary leverage model when the initial leverage ratio is 15 percent. This example is intended to represent a AAA "investment-grade" bond. For the stationary leverage model, we set the long-term leverage ratio under the true measure to about 36 percent (i.e., a log-leverage $\bar{\ell} = -1$), which corresponds closely to the average log-leverage observed in our own (unreported) empirical investigation. The corresponding long-term leverage under the Q measure is approximately 52 percent (log-leverage of $-.656$, see equation (14)). It is well documented that structural models predict negligible credit spreads for very short maturities, but theory can be made compliant with empirical findings either by accounting for imprecisely measured firm value (Duffie and Lando (1997)), jumps in the firm-value process, or by introducing a liquidity premium into the model. More relevant to the issue at hand is the term structure of credit spreads at longer maturities. We find that the traditional structural models of default predict negligible credit spreads for maturities up to 30 years. In contrast, our model performs considerably better for longer maturities. Further, it will be seen in the next section that when stochastic interest rates are accounted for, our model's prediction improves even further.

In Figure 3, we compare the credit spread predictions of the two models for speculative-grade debt, where we assume the initial leverage ratio is 65 percent. The constant boundary model predicts that credit spreads peak at about five years for speculative-grade debt, and drop after that. In contrast,

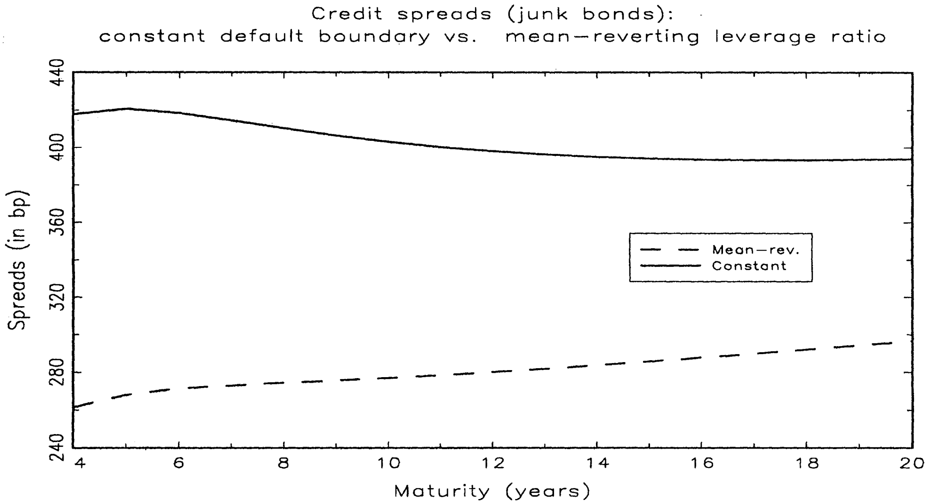

Figure 3. Credit spreads for junk bonds as a function of maturity for the standard structural model and the stationary leverage model. In both models we set $r = .06$, $C = .075$, $\delta = .03$, $\sigma = .2$, $\omega = .56$, $\omega_{coup} = 1$. The initial leverage is set to 65 percent. For the stationary case, we set $\lambda = .18$, and the long-run leverage ratio to 40 percent, implying $\nu = 0.5$.

for the parameter values chosen, our model predicts an upward-sloping yield curve.[8] Hence, in contrast to the traditional constant default-boundary model, for reasonable parameter choice, our model is consistent with the empirical findings of Helwege and Turner (1999). Also notable is that our model predicts credit spreads should be much less sensitive to changes in leverage than does the constant default boundary model. Indeed, as is apparent from comparing Figures 2 and 3, the traditional model predicts for all maturities counter-factually low credit spreads for low leverage firms and high credit spreads for speculative grade debt. In contrast, modeling the leverage ratio dynamics as mean reverting improves the predictions of structural models. The intuition is straightforward: With constant default boundary, a low leverage firm almost never defaults and a high leverage firm almost certainly defaults over a short period of time. On the other hand, with mean reversion in leverage ratios, default depends on both the initial leverage ratio and the long-term mean. If the latter is assumed to take on "moderate" values for all firms, then default probabilities will be less variable across firms.

III. Credit Spreads with Stochastic Interest Rates

Most structural models of default restrict their attention to a constant interest rate framework, implicitly assuming that firm value is the only state variable that affects credit spreads (Merton (1974), Black and Cox (1976),

[8] For the example displayed, we assumed the long-term leverage ratio was 40 percent ($\bar{\ell} = -0.9$) under the true measure, leading to $\nu = 0.5$ and $\theta^Q = 57$ percent.

Leland (1994), Leland and Toft (1996)). However, Longstaff and Schwartz (1995) and Duffee (1998) find that credit spreads are a decreasing function of interest rates. Further, as documented by Malitz (1994), firms tend to issue less debt when interest rates are high. Hence, stochastic interest rates appear to be a relevant factor in determining credit spreads. In this section, we generalize our model to investigate credit spreads in a stochastic interest rate environment, taking the LS model as a benchmark for comparison.

A. The Benchmark: Constant Default Boundary

LS assume the spot rate $\{r_t\}$ follows Vasicek (1977) dynamics, and that firm-value $\{V_t\}$ follows geometric Brownian motion. It is convenient to introduce the log-firm value $y_t \equiv \log V_t$. The LS model is specified under the risk-neutral measure, and can be characterized by the two-factor Markov system:

$$dy_t = \left(r_t - \delta - \frac{\sigma^2}{2} \right) dt + \sigma \, dz_1^Q(t) \tag{28}$$

$$dr_t = \kappa(\theta - r_t)\,dt + \eta \, dz_2^Q(t), \tag{29}$$

with $dz_1^Q dz_2^Q = \rho \, dt$. Default is triggered when asset value falls below a fixed boundary K. Since there is no known closed-form solution for the first-passage time density of y_t at $\log K$, LS propose a numerical solution based on a formula due to Fortet (1943), and investigated further by Buonocore, Nobile, and Ricciardi (1987). However, the Fortet formula is only valid for one-factor Markov processes. Hence, the LS formula serves only as an approximation to the true solution of their model. However, using LS's insight, and extending Fortet's (1943) equation to a two-factor environment, we derive in Appendix B an efficient algorithm for computing the exact solution to the LS model. We also demonstrate in Appendix B that the difference between the LS approximation and the exact solution to their model can be significant for typical parameter values. We thus use the exact solution to their model as the benchmark (denoted LS) for comparison to our model proposed below.

B. Stationary Leverage Ratio

Here we incorporate stochastic interest rates into our stationary leverage model of default. As in LS, we assume that log-firm value follows equation (28), and that interest rates follow equation (29). However, rather than assuming a constant default threshold, we assume that the log-default threshold follows the process

$$dk_t = \lambda[y_t - \nu - \phi(r_t - \theta) - k_t]\,dt, \tag{30}$$

where it is assumed that the parameter $\phi \geq 0$. Equation (30) generalizes equation (7) in that the drift of the log-default threshold is a decreasing function of the spot rate, consistent with the findings of Malitz (1994), who finds that debt issuances dropped dramatically during the high interest rate period of the early 1980s.

Defining as before the log-leverage ratio

$$\ell_t \equiv k_t - y_t,$$

and applying Itô's lemma, we obtain

$$d\ell_t = \lambda(\bar{\ell}^Q(r_t) - \ell_t)\,dt - \sigma\, dz_1^Q(t), \tag{31}$$

where we have defined

$$\bar{\ell}^Q(r) \equiv \frac{\delta + \dfrac{\sigma^2}{2}}{\lambda} - \nu + \phi\theta - r\left(\frac{1}{\lambda} + \phi\right). \tag{32}$$

Note that equation (32) implies the risk-neutral target leverage ratio is a decreasing function of the current interest rate.

Equations (29) and (31) together generate a framework which is two-factor Markov in $\{\ell, r\}$. This model reduces back to the one-factor model proposed in the previous section for the special case $\phi = 0$, $\kappa = 0$, $\eta = 0$.

C. Pricing Corporate Debt

As in the previous section we assume that a risky discount bond pays a proportion $(1 - \omega)$ of the principal at the maturity of the bond if default occurs prior to maturity. We define $Q^T(r_0, \ell_0, T) \equiv \mathrm{E}_0^T[\mathbf{1}_{(\tilde{\tau}<T)}]$ as the time-0 probability under the T-forward measure that default occurs before the bond maturity T. The risky discount bond is thus given by:

$$P^T(r_0, \ell_0) = \mathrm{E}^Q\left[e^{-\int_0^T ds\, r_s}(1 - \omega\mathbf{1}_{(\tilde{\tau}<T)})\right] = D^T(r_0)[1 - \omega Q^T(r_0, \ell_0, T)], \tag{33}$$

where

$$D^T(r_0) = e^{A^{(T)} - r_0 B_\kappa^{(T)}} \tag{34}$$

is the Vasicek (1977) risk-free bond price, with standard deterministic functions $A^{(\cdot)}$ and $B_\kappa^{(\cdot)}$.

As before, $\tilde{\tau}$ is the first passage time of the firm value reaching the default boundary or, equivalently, of ℓ_t reaching zero. We claim the following proposition.

PROPOSITION 2. *Discretize time into n_T equal intervals, and define date $t_j = jT/n_T \equiv j\Delta t$ for $j \in (1,2,\ldots,n_T)$. Similarly discretize r-space into n_r equal intervals between some chosen minimum $\underline{r}$ and maximum $\bar{r}$, and define*

$r_i = \underline{r} + i * \Delta r$ for $i \in (1,2,\ldots,n_r)$, where $\Delta r = (\bar{r} - \underline{r})/n_r$. The price of a risky discount bond is given by equation (33), where

$$Q^T(r_0,\ell_0,T) \equiv \sum_{j=1}^{n_T}\sum_{i=1}^{n_r} q(r_i,t_j) \tag{35}$$

$$q(r_i,t_1) = \Delta r \Psi(r_i,t_1) \quad \forall i \in (1,2,\ldots,n_r) \tag{36}$$

$$q(r_i,t_j) = \Delta r\left[\Psi(r_i,t_j) - \sum_{v=1}^{j-1}\sum_{u=1}^{n_r} q(r_u,t_v)\psi(r_i,t_j|r_u,t_v)\right] \qquad \forall i \in (1,..,n_r), \quad \forall j \in (2,..,n_T) \tag{37}$$

$$\Psi(r,t) \equiv \pi(r_t,t|r_0,0)N\left(\frac{\mu(r_t,t|\ell_0,r_0,0)}{\Sigma(r_t,t|\ell_0,r_0,0)}\right) \tag{38}$$

$$\psi(r_t,t|r_s,s) \equiv \pi(r_t,t|r_s,s)N\left(\frac{\mu(r_t,t|\ell_s=0,r_s,s)}{\Sigma(r_t,t|\ell_s=0,r_s,s)}\right) \quad \forall(t>s). \tag{39}$$

Here, $\pi(r_t,t|r_s,s)$ is the transition density for the interest rate, and $\mu(r_t,t|\ell_s,r_s,s)$ and $\Sigma(r_t,t|\ell_s,r_s,s)$ are the expected value and variance of the date-t log-leverage, respectively, conditional upon the values $\{r_t,\ell_s,r_s\}$.

Proof: Proof and expressions for N and Σ given in Appendix B.

We emphasize that the methodology used to price the risky bond is not specific to our model. In particular, it should prove useful for other applications (e.g., in exotic option pricing). For the remainder of this section, we focus on the implications of our model for credit spreads, and, in particular, stress the differences between our model and the constant boundary (LS) benchmark.

For reasons explained previously, we price the coupon bond as a portfolio of discount bonds, but assume that coupons have a 100 percent write-down: $\omega_{coup} = 1$. Define $P_c^T(r_0,\ell_0)$ as the date-0 price of a coupon bond with maturity T and coupon payments of C at dates $\{t_j\}$, $j = 1,\ldots,N$. The yield to maturity Y^T of this bond is obtained implicitly through

$$P_c^T(r_0,\ell_0) = \sum_{j=1}^{N} Ce^{-Y^T t_j} + e^{-Y^T T}. \tag{40}$$

Finally, the credit spread is defined via

$$CS^T(r_0,\ell_0) = Y^T - R^T, \tag{41}$$

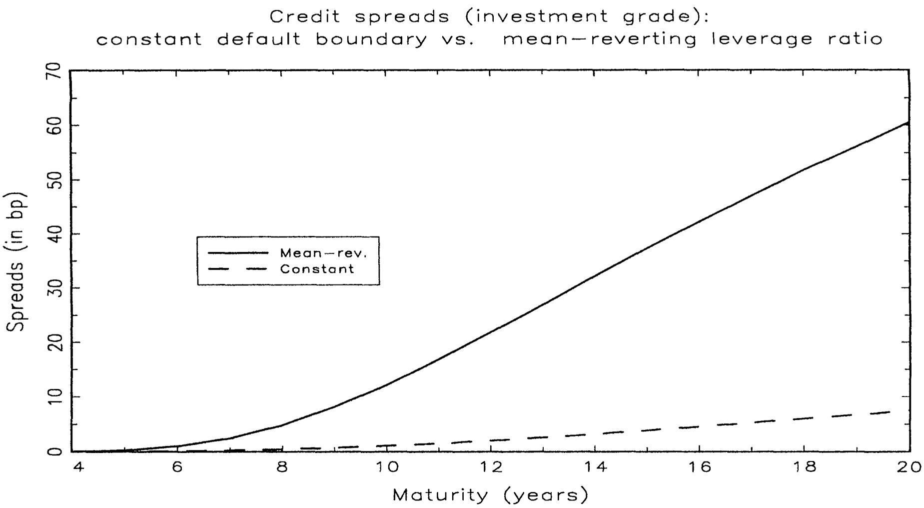

Figure 4. Investment-grade credit spreads for constant boundary versus mean-reverting leverage ratio. In both models we set C = .075, r_0 = .06, θ = .06, κ = .1, η = .015, δ = .03, σ = .2, ρ = −.2. For the stationary leverage model λ = .18, ν = 0.6, ϕ = 2.8. As before ω = .56, ω_{coup} = 1. The initial leverage is set to 15 percent.

where the risk-free T-spot rate R^T is defined implicitly via

$$D_C^T(r_0, \ell_0) = \sum_{j=1}^{N} Ce^{-R^T t_j} + e^{-R^T T}, \tag{42}$$

where $D_C^T(r_0, \ell_0)$ is the price of a portfolio of riskless discount bonds with the same promised cash-flows.

D. Numerical Results

For our numerical illustrations we pick as base-case scenario the following parameters: C = .075, r_0 = .06, θ = .06, κ = .1, η = .015, δ = .03, σ = .2, ρ = −.2, ν = 0.6, λ = .18, ϕ = 2.8. Figure 4 compares the term structure of credit spreads for the constant boundary case (i.e., the LS model) and the mean-reverting leverage ratio case (MR) for investment-grade firms with relatively low initial leverage ratio (15 percent). Both models predict increasing term structures of credit spreads with counter-factually low credit spreads at the short end. At the long end of the term structure, however, the LS model predicts spreads of the order of eight basis points (bp) whereas the MR model predicts more realistic spreads, around 60 bp. Modeling mean reversion in leverage appears to solve the problem of structural models at fitting long-term credit spreads for low-leverage firms. Interestingly, the stochastic interest rate model is in better agreement with empirical observation than is the constant-r model in that, for all maturities, this model predicts

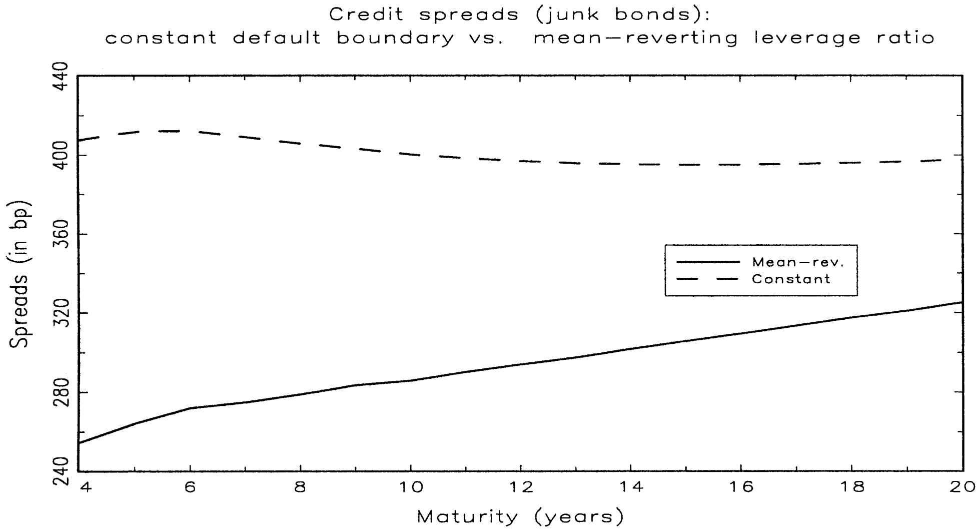

Figure 5. Speculative-grade credit spreads for constant default boundary versus mean-reverting leverage ratio case. In both models we set $C = .075$, $r_0 = .06$, $\theta = .06$, $\kappa = .1$, $\eta = .015$, $\delta = .03$, $\sigma = .2$, $\rho = -.2$. For the stationary leverage model $\lambda = .18$, $\nu = .5$, $\phi = 2.8$. As before $\omega = .56$, $\omega_{coup} = 1$. The initial leverage is set to 65 percent.

higher credit spreads (see Figure 2), with the effect being more dramatic at longer maturities. This is due to the fact that, for correlations not too negative, introducing stochastic interest rates increases the volatility of leverage, in turn increasing the probability that the first passage event occurs within a given time.[9] Moreover, the fact that interest rates affect the default boundary in our model magnifies this result.

We now investigate a few comparative-static results for the two models.

D.1. Credit Spreads for Junk Bonds

Figure 5 represents the term structure of credit spreads for junk bonds issued by a firm with initial leverage ratio equal to 65 percent. Note that, as in the constant interest rate case, the LS model exhibits a decreasing term structure of credit spreads (from 5 to 20 years), whereas the MR model exhibits an increasing term structure of credit spreads, more in line with the empirical results of Helwege and Turner (1999). Comparing Figures 3 and 5, we see that the stochastic model predicts a term structure of credit spreads that is even more upward sloping. The intuition is the following: Since the interest rate appears only in the drift of the leverage process, we expect the two models (with and without stochastic interest rates) to predict similar

[9] Since the instantaneous volatility of leverage is not affected by stochastic interest rates (interest rates appear only in the drift of leverage), it is intuitive that this result is more pronounced for larger maturities.

credit spreads for short maturities. For longer maturities, the difference in credit spreads depends on the correlation parameter ρ. When ρ is positive, the model with stochastic interest rates predicts a greater probability of default than the model with constant interest rates, because a decline in firm value on average decreases interest rates, which, in turn, lowers the expected future firm value (see Figure 7). On the other hand, when ρ is negative, there are two partially offsetting effects compared to the constant interest rate case: higher volatility in leverage rates due to stochastic interest rates, and lower volatility due to the natural hedge between movements in firm value and interest rates (which we discuss further in conjunction with Figure 7). Numerically we find that for $\rho = -.2$, a typical value for firms, the first effect dominates, generating larger credit spreads in the stochastic interest rate model for long maturities. Given that stochastic and constant interest rate framework produce similar credit spreads at the short end, we find the term structure of credit spreads is more upward sloping for the stochastic interest rate model than for its constant interest rate counterpart.[10]

D.2. Interest Rate Level

Figure 6 shows that an increase in the level of the short-term rate lowers credit spreads in our model. Qualitatively similar results hold in the standard model. Longstaff and Schwartz (1995) provide some intuition for this result: Equation (28) indicates the drift of firm value is an increasing function of the risk-free rate. Since our model incorporates this feature, and, in addition, it models the drift of the default boundary as a decreasing function of interest rates, this static prediction is amplified in our framework.

D.3. Correlation Between Changes in Firm Value and Changes in Interest Rates

While the comparative static for the effect of interest rate changes on credit spreads is unambiguous, the prediction is less clear from a dynamic standpoint. Empirical studies (Longstaff and Schwartz (1995), Duffee (1998, 1999), Collin-Dufresne and Solnik (2000)) find a negative correlation between innovations in credit spreads and changes in interest rates. As noted by Longstaff and Schwartz, the correlation coefficient between firm value and interest rates has an important impact on credit spreads. A negative correlation implies that a decrease in interest rates will have two countervailing effects on credit spreads. On one hand, credit spreads may increase due to the decrease in the drift of asset value. On the other hand, a decrease in interest rates will typically be associated with an increase in underlying asset value. Thus, credit spreads may decrease. In this sense, a negative correlation between firm value and interest rate provides a natural hedge

[10] Unreported results confirm that, for very large negative values of ρ, credit spreads are smaller in the model with stochastic interest rates.

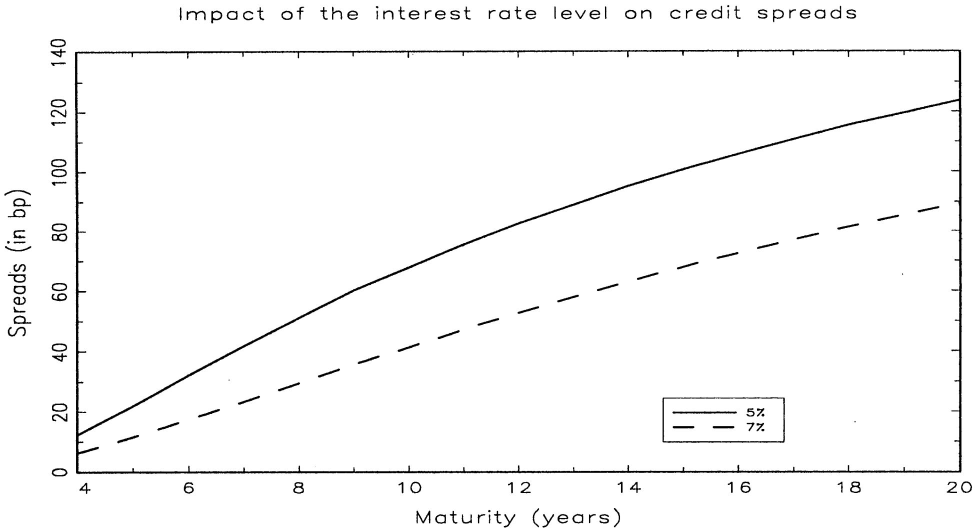

Figure 6. Impact of the level of the short term rate on credit spreads when leverage ratio is mean-reverting. Two cases are shown: $r_0 = .05$ and $r_0 = .07$. Remaining parameters are as before: $C = .075$, $\theta = .06$, $\kappa = .1$, $\eta = .015$, $\delta = .03$, $\sigma = .2$, $\rho = -.2$. For the stationary leverage model $\lambda = .18$, $\nu = 0.6$, $\phi = 2.8$. As before $\omega = .56$, $\omega_{coup} = 1$. The initial leverage is set to 35 percent (the average in the sample).

against credit risk. Conversely, when the correlation is positive, the two effects work in the same direction. Hence, one would expect higher credit spreads for positive correlations. Figure 7 verifies this intuition.

D.4. Correlation Between Credit Spreads and Firm Value

The standard model predicts a much higher impact of changes in firm value on credit spreads than our model with mean-reverting leverage ratios. This is related to our previous observation that accounting for mean reverting leverage ratios makes credit spreads less sensitive to cross-sectional variations in leverage levels. This prediction of our model is consistent with the empirical findings of both Hotchkiss and Ronen (1999), and Collin-Dufresne, Goldstein, and Martin (2001), who find small explanatory power of changes in leverage for changes in credit spreads.

D.5. Correlation Between Credit Spreads and Interest Rates

Even though the effect of the level of interest rates on credit spreads is unambiguous, that is, a higher interest rate leads to lower credit spreads (see Figure 6), determining the sign of the correlation between changes in interest rates and changes in credit spreads is less straightforward. As noted above, there are two partially offsetting effects of a change in interest rates on credit spreads when ρ is negative. For the LS model, the net effect may go either way, depending on the parameter values. In fact, numerical results

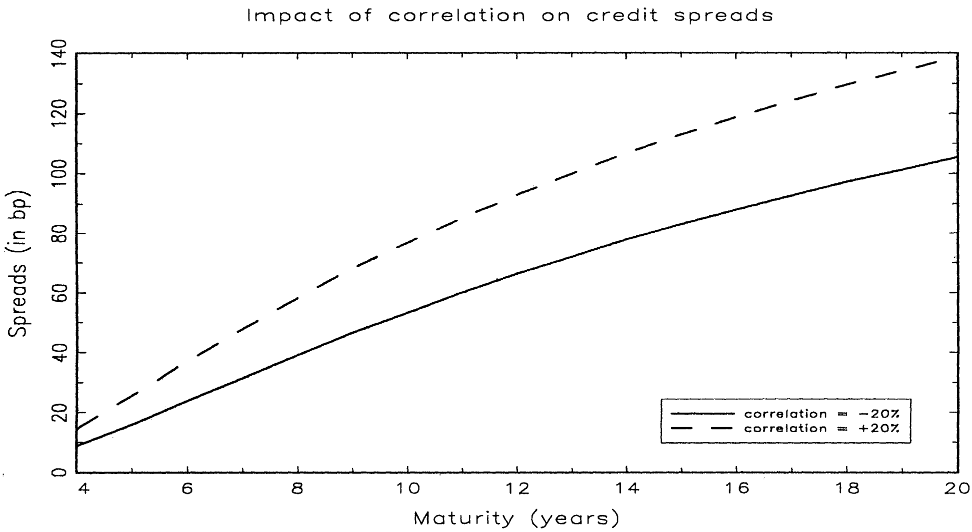

Figure 7. Impact of the correlation between interest rates and firm value on credit spreads when leverage ratio is mean-reverting. Two cases are shown: $\rho = -.2$ and $\rho = .2$. Remaining parameters are as before: $C = .075$, $r_0 = .06$, $\theta = .06$, $\kappa = .1$, $\eta = .015$, $\delta = .03$, $\sigma = .2$. For the stationary leverage model $\lambda = .18$, $\nu = 0.6$, $\phi = 2.8$. As before $\omega = .56$, $\omega_{coup} = 1$. The initial leverage is set to 35 percent.

show that in our base-case scenario, changing the interest rate mean-reversion parameter κ from 0.1 to 0.4 is sufficient to flip the sign of this correlation. In contrast, our model predicts a negative correlation between credit spreads and interest rates for all reasonable parameter choices.[11]

IV. Conclusion

If leverage ratios follow stationary processes, then it is natural to expect that market prices would account for this fact. In this paper, we develop a simple model where firms adjust their capital structure to reflect changes in asset value. This framework generates stationary (mean-reverting) leverage ratios and accounts for stochastic interest rates.

Compared to traditional structural models of default, credit spreads generated from our model are more consistent with recent empirical findings. First, our model predicts that the term structure of credit spreads of speculative-grade debt should in general be upward sloping. Second, it predicts that the sensitivity of credit spreads to changes in firm value is much lower. Finally, it predicts a negative correlation between credit spreads and interest rates for all reasonable parameter values.

[11] This is due to the fact that, in our model, the partial derivative $|-(\partial CS/\partial \ell)|$ is much smaller (due to the mean-reverting leverage ratio), and the partial derivative $|-(\partial CS/\partial r)|$ is much larger (in part due to the interest rate sensitivity of the default threshold) than in the standard model.

It thus appears that accounting for a firm's ability to control its level of outstanding debt has a significant impact on credit spread predictions, and that it helps reconcile many of the predictions of structural models of credit spreads with empirical observations. The framework we propose accounts for both stochastic interest rates and mean-reverting leverage ratios. Yet it remains tractable, and can be applied to pricing various forms of outstanding debt-payoff structures or derivatives.

Finally, we note that to solve our stochastic interest rate model, we generalize a formula due to Fortet (1943) so that it can be applied to multifactor models. Besides providing an efficient scheme to determine credit spreads in our model, the method also provides an exact solution to the model proposed by Longstaff and Schwartz (1995), and should prove useful in other applications, such as exotic option pricing.

Appendix A. First Passage Time for One-dimensional Markov Processes

Consider a one-factor continuous Markov process ℓ_t. Define $\pi(\ell_t, t|\ell_s, s)$ as the free transition density. Further, define $g(\ell_s = \underline{\ell}, s|\ell_0, 0)$ as the probability density that the first passage time through a constant boundary $\underline{\ell}$ occurs at date-s. An implicit formula for $g(\cdot)$ in terms of $\pi(\cdot)$ has been proposed by Fortet (1943):[12,13]

$$\pi(\ell_t, t|\ell_0, 0) = \int_0^t ds\, g(\ell_s = \underline{\ell}, s|\ell_0, 0)\pi(\ell_t, t|\ell_s = \underline{\ell}, s) \quad \forall(\ell_t > \underline{\ell} > \ell_0). \quad \text{(A1)}$$

In this equation it is assumed that ℓ_t and ℓ_0 are on opposite sides of the boundary $\ell = \underline{\ell}$. When the process ℓ is one-factor Markov, the above equation has a very intuitive interpretation: The only way that the process can start below the boundary ($\ell_0 < \underline{\ell}$) and end up above the boundary ($\ell_t > \underline{\ell}$) is that the process, at some intermediate time s, must pass through the boundary for the first time.

[12] More rigorously, we can write for arbitrary $\{\ell_t, \underline{\ell}, \ell_0\}$, where $\tilde{\tau}$ is the first passage time to $\underline{\ell}$:

$$\begin{aligned}\pi(\ell_t, t|\ell_0, 0) &= \int_0^t ds\, \pi(\ell_t, t; \tilde{\tau} = s|\ell_0, 0) + \pi(\ell_t, t; \tilde{\tau} > t|\ell_0, 0)\\ &= \int_0^t ds\, \pi(\ell_t, t|\tilde{\tau} = s; \ell_0, 0)\pi(\tilde{\tau} = s|\ell_0, 0) + \pi(\ell_t, t; \tilde{\tau} > t|\ell_0, 0)\\ &= \int_0^t ds\, \pi(\ell_t, t|\ell_s = \underline{\ell}, s)g(\ell_s = \underline{\ell}, s|\ell_0, 0) + \pi(\ell_t, t; \tilde{\tau} > t|\ell_0, 0),\end{aligned}$$

where we have used the Markov property in the last line. When $\ell_t > \underline{\ell} > \ell_0$, this last term vanishes, and we obtain equation (A1).

[13] We emphasize that imposing the constraint that the boundary be constant is not restrictive, since it is always possible to transform any time-dependent boundary into a static one by a suitable transformation of variables.

Here, we apply equation (A1) to the one-factor model of Section III. From equation (8), we see that ℓ_t is a Gaussian process with:

$$\mathrm{E}^Q[\ell_t|\ell_0] \equiv M(t) = \ell_0 e^{-\lambda t} + \bar{\ell}^Q(1 - e^{-\lambda t}) \tag{A2}$$

$$\mathrm{E}^Q[\ell_t|\ell_s = 0] \equiv L(t-s) = \bar{\ell}^Q(1 - e^{-\lambda(t-s)}) \tag{A3}$$

$$\mathrm{Var}_s^Q[\ell_t] \equiv S^2(t-s) = \left(\frac{\sigma^2}{2\lambda}\right)(1 - e^{-2\lambda(t-s)}). \tag{A4}$$

With the default boundary at $\underline{l} = 0$, we integrate both sides of equation (A1) by $\int_0^\infty d\ell_t$ to obtain

$$\mathrm{N}\left(\frac{M(t)}{S(t)}\right) = \int_0^t ds\, g(\ell_s = \underline{\ell}, s|\ell_0, 0)\,\mathrm{N}\left(\frac{L(t-s)}{S(t-s)}\right). \tag{A5}$$

To obtain an explicit function for the first passage density, we discretize time into n equal intervals, and define Δt via $t = n\Delta t$. Numerically, we find it most efficient to approximate the integral on the right hand side of equation (A5) by estimating values at the midpoint of the interval. Hence, defining $a_i = (M(i\Delta t)/S(i\Delta t))$ and $b_i = (L(i\Delta t)/S(i\Delta t))$, the first two terms from equation (A5) can be approximated as

$$\mathrm{N}(a_1) = \Delta t\, g(\ell_{(\Delta t/2)} = \underline{\ell}, (\Delta t/2)|\ell_0, 0)\mathrm{N}(b_{(1/2)}) \tag{A6}$$

$$\begin{aligned}\mathrm{N}(a_2) = {} & \Delta t\, g(\ell_{(\Delta t/2)} = \underline{\ell}, (\Delta t/2)|\ell_0, 0)\mathrm{N}(b_{(3/2)}) \\ & + \Delta t\, g(\ell_{(3\Delta t/2)} = \underline{\ell}, (3\Delta t/2)|\ell_0, 0)\mathrm{N}(b_{1/2}).\end{aligned} \tag{A7}$$

Continuing in this manner, we obtain n equations for the n unknowns $g(\ell_{(i-(1/2))\Delta t} = \underline{\ell}, (i - \frac{1}{2})\Delta t|\ell_0, 0)$, where $i \in (1, n)$. Defining $q_i = \Delta t\, g(\ell_{(i-(1/2))\Delta t} = \underline{\ell}, (i - \frac{1}{2})\Delta t|\ell_0, 0)$, we obtain the proposed solution.

Appendix B. First Passage Time Density for Two-dimensional Markov Processes

Consider a two-factor Markov process $\{r_t, \ell_t\}$ as in our model, or the one investigated in Longstaff and Schwartz (1995). In either case, the computation of the first passage time density of ℓ_t at $\underline{\ell}$ is required to determine credit spreads. Here we show that the solution proposed by Longstaff and Schwartz serves only as an approximation to the exact solution of their model. We then generalize their approach to derive an exact solution for the first passage time of a two-factor Gaussian Markov process.

A. Fortet's Equation for a Two-dimensional Markov Process

LS use equation (B1) to estimate the hitting-time density. However, their model is not one-factor Markov since the proposed firm value process is a function of the spot rate, which itself is stochastic. Hence, Fortet's equation must be generalized in order to obtain a first passage time density for this two-factor model.

Define $g[\ell_s = \underline{\ell}, r_s, s | \ell_0, r_0, 0]$ as the probability density that the first passage time is at time s, and that the random process r takes on the value r_s at that time. We claim that the two-dimensional generalization of the Fortet (1943) model is for $\ell_0 < \underline{\ell} < \ell_t$

$$\pi(\ell_t, r_t, t | \ell_0, r_0, 0) = \int_0^t ds \int_{-\infty}^{\infty} dr_s g[\ell_s = \underline{\ell}, r_s, s | \ell_0, r_0, 0]\, \pi(\ell_t, r_t, t | \ell_s = \underline{\ell}, r_s, s). \tag{B1}$$

The intuition of this equation is straightforward: The only way that the system can evolve from $\{\ell_0, r_0\}$, where $\ell_0 < \underline{\ell}$, and end up at $\{\ell_t, r_t\}$, where $\ell_t > \underline{\ell}$ is that for some intermediate time s, the ℓ-value must pass through the boundary $\underline{\ell}$ for the first time. Further, at that time, the r-process must take on some value r_s.

Integrating both sides of the equation by $\int_{-\infty}^{\infty} dr_t$ we obtain:

$$\pi(\ell_t, t | \ell_0, r_0, 0) = \int_0^t ds \int_{-\infty}^{\infty} dr_s g[\ell_s = \underline{\ell}, r_s, s | \ell_0, r_0, 0] \pi(\ell_t, t | \ell_s = \underline{\ell}, r_s, s). \tag{B2}$$

Note that if ℓ was actually one-factor Markov, then Bayes' rule implies

$$\pi(\ell_t, t | \ell_v, r_v, v) = \pi(\ell_t, t | \ell_v, v) \quad \forall \{v\} \tag{B3}$$

$$g[\ell_s = \underline{\ell}, r_s, s | \ell_0, r_0, 0] = g[\ell_s = \underline{\ell}, s | \ell_0, 0] \pi(r_s, s | \ell_0, r_0, 0), \tag{B4}$$

and equation (B2) reduces to Fortet's equation

$$\pi(\ell_t, t | \ell_0, 0) = \int_0^t ds \int_{-\infty}^{\infty} dr_s\, \pi(r_s, s | \ell_0, r_0, 0) g[\ell_s = \underline{\ell}, s | \ell_0, 0] \pi(\ell_t, t | \ell_s = \underline{\ell}, s) \tag{B5}$$

$$= \int_0^t ds\, g[\ell_s = \underline{\ell}, s | \ell_0, 0] \pi(\ell_t, t | \ell_s = \underline{\ell}, s). \tag{B6}$$

However, for multivariate Markov processes, this result does not obtain. Indeed, it is this assumption that is tacitly made in the proposed solution of LS.

B. Application of the Two-dimensional Fortet Equation to Gaussian Processes

To solve a two-factor model as proposed in our paper, or as in LS, we generalize the insightful approach taken in LS. It is convenient to introduce

$$\Psi(r_t, t) \equiv \int_{\underline{\ell}}^{\infty} d\ell_t \, \pi(\ell_t, r_t, t \,|\, \ell_0, r_0, 0) \tag{B7}$$

$$\psi(r_t, t, r_s, s) \equiv \int_{\underline{\ell}}^{\infty} d\ell_t \, \pi(\ell_t, r_t, t \,|\, \ell_s = \underline{\ell}, r_s, s) \tag{B8}$$

$$g(r_s, s) = g[\ell_s = \underline{\ell}, r_s, s \,|\, \ell_0, r_0, 0]. \tag{B9}$$

Integrating both sides of equation (B1) by $\int_{\underline{\ell}}^{\infty} d\ell_t$, we find that it can then be written in the form

$$\Psi(r_t, t) = \int_0^t ds \int_{-\infty}^{\infty} dr_s \, g(r_s, s) \psi(r_t, t, r_s, s). \tag{B10}$$

Generalizing the approach used in the one-dimensional case, we discretize equation (B10) to obtain an algorithm for estimating the first passage time of ℓ at $\underline{\ell} = 0$. Here we present an outline of the approach.

Discretize time into n_T equal intervals, and define date $t_j = jT/n_T \equiv j\Delta t$ for $j \in (1, 2, \ldots, n_T)$. Similarly discretize r-space into n_r equal intervals and define $r_i = \underline{r} + i * \Delta r$ for $i \in (1, 2, \ldots, n_r)$ and where $\Delta r = (\bar{r} - \underline{r})/n_r$. The discretized version of equation (B10) takes the form

$$\Psi(r_i, t_j) = \sum_{v=1}^{j} \sum_{u=1}^{n_r} q(r_u, t_v) \psi(r_i, t_j \,|\, r_u, t_v),$$

where

$$q(r_u, t_v) = \Delta t \Delta r \, g(r_u, t_v).$$

The probability under the T-forward measure that the first passage time is less than T is

$$Q^T(r_0, \ell_0, T) \equiv \sum_{j=1}^{n_T} \sum_{i=1}^{n_r} q(r_i, t_j). \tag{B11}$$

As in the one-dimensional case, the $q(r_i, t_j)$ can be found recursively (given Ψ and ψ) as follows:

$$q(r_i, t_1) = \Delta r \Psi(r_i, t_1) \quad \forall i \in (1,2,\ldots,n_r) \tag{B12}$$

$$q(r_i, t_j) = \Delta r \left[\Psi(r_i, t_j) - \sum_{v=1}^{j-1} \sum_{u=1}^{n_r} q(r_u, t_v) \psi(r_i, t_j | r_u, t_v) \right] \qquad \forall i \in (1,..,n_r) \quad \forall j \in (2,..,n_T). \tag{B13}$$

To compute the functions Ψ and ψ given in equations (B7) and (B8), we note that applying Bayes' rule to the bivariate density $\pi(\ell_t, r_t, t | \ell_v, r_v, v)$ gives

$$\pi(\ell_t, r_t, t | \ell_v, r_v, v) = \pi(r_t, t | \ell_v, r_v, v) \pi(\ell_t, t | r_t, \ell_v, r_v, v) \quad \forall \{v\}.$$

Using this in equations (B7) and (B8) we obtain

$$\Psi(r_t, t) = \pi(r_t, t | \ell_0, r_0, 0) \int_{\underline{\ell}}^{\infty} d\ell_t \, \pi(\ell_t, t | r_t, \ell_0, r_0, 0) \tag{B14}$$

$$\psi(r_t, t, r_s, s) = \pi(r_t, t | \ell_s = \underline{\ell}, r_s, s) \int_{\underline{\ell}}^{\infty} d\ell_t \, \pi(\ell_t, t | r_t, \ell_s = \underline{\ell}, r_s, s). \tag{B15}$$

Because (r, ℓ) follow a joint Gaussian process, we can apply the projection theorem to obtain

$$\mu(r_t, \ell_s, r_s) \equiv \mathrm{E}_s^T[\ell_t | r_t] = \mathrm{E}_s^T[\ell_t] + \frac{\mathrm{Cov}_s^T[\ell_t, r_t]}{\mathrm{Var}_s^T[r_t]} (r_t - \mathrm{E}_s^T[r_t]) \tag{B16}$$

$$\Sigma^2(r_t, \ell_s, r_s) \equiv \mathrm{Var}_s^T[\ell_t | r_t] = \mathrm{Var}_s^T[\ell_t] - \frac{\mathrm{Cov}_s^T[\ell_t, r_t]^2}{\mathrm{Var}_s^T[r_t]}. \tag{B17}$$

Since the interest rate process is one-factor Markov in both models, we may further simplify the expression and arrive at the final expressions:

$$\Psi(r_t, t) \equiv \pi(r_t, t | r_0, 0) N\left(\frac{\mu(r_t, t | \ell_0, r_0, 0)}{\Sigma(r_t, t | \ell_0, r_0, 0)} \right) \tag{B18}$$

$$\psi(r_t, t | r_s, s) \equiv \pi(r_t, t | r_s, s) N\left(\frac{\mu(r_t, t | \ell_s = \underline{\ell}, r_s, s)}{\Sigma(r_t, t | \ell_s = \underline{\ell}, r_s, s)} \right), \tag{B19}$$

where $N(\cdot)$ is the cumulative normal distribution and $\pi(r_t, t | r_s, s)$ is the well-known transition density for the one-factor Markov Gaussian interest rate process. Both are easy to program and fast computationally.

C. Implementation for LS

Hence, all that remains to be done in order to compute Q in equation (B11) is to determine the free and conditional transition densities of the model. For both our model and the LS model, the transition densities are bivariate normal. Hence, one only needs to determine the means, variances, and covariance. For the LS model, the dynamics of the state variables under the T-forward measure are

$$dy_t = \left[r_t - \delta - \frac{\sigma^2}{2} - \rho\sigma\eta B_\kappa^{(T-t)} \right] dt + \sigma\, dz_1^T \tag{B20}$$

$$dr_t = \kappa \left[\theta - r_t - \frac{\eta^2}{\kappa} B_\kappa^{(T-t)} \right] dt + \eta\, dz_2^T, \tag{B21}$$

where we have defined $B_\kappa^{(u)} \equiv (1/\kappa)(1 - e^{-\kappa u})$. To simplify the comparison of this model to ours, here we define $\ell_t = k - y_t$, where k is the constant log-default boundary. We find[14]

$$\begin{aligned} \mathrm{E}_u^T[\ell_t] = \ell_u &- \left(\theta - \frac{\eta^2}{\kappa^2} - \delta - \frac{\sigma^2}{2} - \frac{\rho\sigma\eta}{\kappa} \right)(t-u) \\ &- \left(r_u - \theta + \frac{\eta^2}{\kappa^2} + \frac{\rho\sigma\eta}{\kappa} e^{-\kappa(T-t)} \right) B_\kappa^{(t-u)} - \frac{\eta^2}{2\kappa} e^{-\kappa(T-t)} (B_\kappa^{(t-u)})^2 \end{aligned} \tag{B22}$$

$$\mathrm{E}_u^T[r_t] = r_u e^{-\kappa(t-u)} + \left(\theta\kappa - \frac{\eta^2}{\kappa} \right) B_\kappa^{(t-u)} + \frac{\eta^2}{\kappa} e^{-\kappa(T-t)} B_{2\kappa}^{(t-u)} \tag{B23}$$

$$\mathrm{Var}_u^T[\ell_t] = \left(\sigma^2 + 2\frac{\sigma\rho\eta}{\kappa} + \frac{\eta^2}{\kappa^2} \right)(t-u) - 2\left(\frac{\sigma\rho\eta}{\kappa} + \frac{\eta^2}{\kappa^2} \right) B_\kappa^{(t-u)} + \frac{\eta^2}{\kappa^2} B_{2\kappa}^{(t-u)} \tag{B24}$$

$$\mathrm{Var}_u^T[r_t] = \eta^2 B_{2\kappa}^{(t-u)} \tag{B25}$$

$$\mathrm{Cov}_u^T[\ell_t, r_t] = \frac{\eta^2}{\kappa} B_{2\kappa}^{(t-u)} - \left(\frac{\eta^2}{\kappa} + \rho\eta\sigma \right) B_\kappa^{(t-u)}. \tag{B26}$$

[14] We note that Longstaff and Schwartz (1995) incorrectly claim that the conditional expectation of y_T given y_t is equal to $y_t + \mathrm{E}_0^T[y_T] - \mathrm{E}_0^T[y_t]$, or in their notation, equal to $y_t + M(T,T) - M(t,T)$. This can be seen from a simple application of the projection theorem. Taurén (1999) makes a similar error.

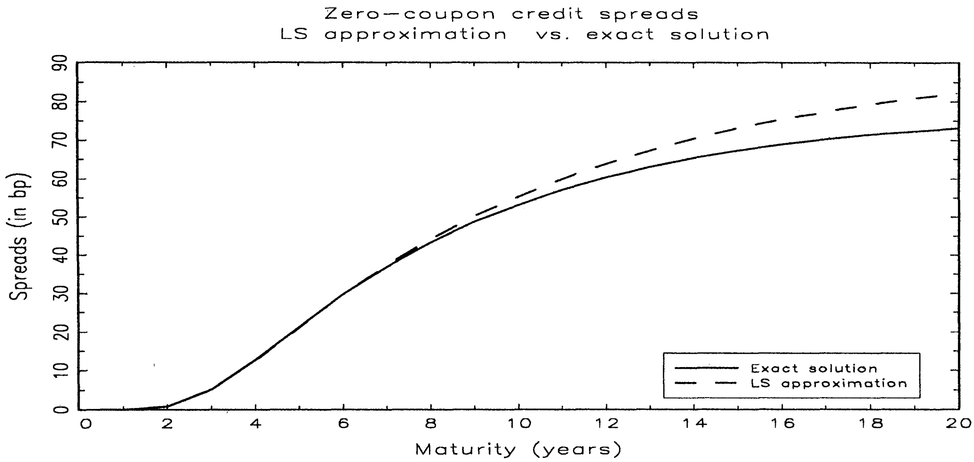

Figure 8. Credit spreads of zero coupon bonds for the Longstaff and Schwartz approximation and the true solution as a function of maturity. The parameter values chosen are $r_0 = .06$, $\theta = .06$, $\kappa = .1$, $\eta = .015$, $\delta = .03$, $\sigma = .2$ $\rho = -.2$. The initial leverage ratio is set to 35 percent.

The proposed numerical procedure is well behaved and converges rapidly. We find that choosing r_{min} and r_{max} to include three standard deviations about the long-run mean is sufficient for convergence when the initial interest rate is within one standard deviation of the long-term mean, and that taking $\Delta r = 0.2$ percent achieves convergence. In the time dimension, it is most efficient to use an extrapolation scheme (e.g., we used $\Delta t = 0.1, 0.3, 0.5, 0.7$ and the polint routine of Numerical Recipes).

In Figure 8, we plot both the exact credit spreads for the LS model, and the credit spreads using the LS approximation for reasonable parameter values. We find the discrepancy is economically significant, on the order of 15 bp at a maturity of 20 years. Unreported results show that the discrepancy widens for more volatile interest rates, and decreases for less volatile interest rates. This result is intuitive, for the LS approximation is exact when interest rate dynamics are deterministic. The difference also decreases with correlation. It seems difficult to give a precise description of the conditions under which it is reasonable to use the LS approximation instead of the true solution, especially with regard to dynamic implication of the model (e.g., hedge ratios).[15]

D. Implementation for the Model with a Mean-reverting Leverage Ratio

To determine the risky bond price in equation (33) we need to compute $Q^T(r_t, \ell_t, t)$ from equation (B11). Again, since $\{\ell_t, r_t\}$ is a bivariate normal

[15] Certainly, low volatility and high mean-reversion of interest rates provide such conditions, but, perhaps, it is more reasonable to use the constant interest rate model as an approximation in that case.

system, all that is needed are the conditional and unconditional moments of their distributions. Under the T-forward measure, the dynamics of the state variables $\{\ell_t, r_t\}$ for this model are:

$$d\ell_t = \lambda\left(\bar{\ell}^{\mathcal{Q}} - \frac{1+\lambda\phi}{\lambda} r_t - \ell_t + \frac{\eta\rho\sigma}{\lambda} B_\kappa^{(T-t)}\right) dt - \sigma\, dz_1^T(t) \tag{B27}$$

$$dr_t = \kappa\left[\theta - r_t - \frac{\eta^2}{\kappa} B_\kappa^{(T-t)}\right] dt + \eta\, dz_2^T(t). \tag{B28}$$

After some straightforward but tedious calculations we find

$$\begin{aligned}\mathrm{E}_u^T[\ell_t] = {} & \ell_u e^{-\lambda(t-u)} - (1+\lambda\phi)\left(r_u + \frac{\eta^2}{\kappa^2} - \theta\right) e^{-\kappa(t-u)} B_{(\lambda-\kappa)}^{(t-u)} \\ & - \left(\frac{\eta\rho\sigma}{\kappa} + (1+\lambda\phi)\frac{\eta^2}{2\kappa^2}\right) e^{-\kappa(T-t)} B_{(\lambda+\kappa)}^{(t-u)} \\ & + (1+\lambda\phi)\frac{\eta^2}{2\kappa^2} e^{-\kappa(T-t)} e^{-2\kappa(t-u)} B_{(\lambda-\kappa)}^{(t-u)} \\ & + \left(\frac{\eta\rho\sigma}{\kappa} + \lambda\bar{\ell}^{\mathcal{Q}} - (1+\lambda\phi)\left(\theta - \frac{\eta^2}{\kappa^2}\right)\right) B_\lambda^{(t-u)}\end{aligned} \tag{B29}$$

$$\mathrm{E}_u^T[r_t] = r_u e^{-\kappa(t-u)} + \left(\theta\kappa - \frac{\eta^2}{\kappa}\right) B_\kappa^{(t-u)} + \left(\frac{\eta^2}{\kappa}\right) e^{-\kappa(T-t)} B_{2\kappa}^{(t-u)} \tag{B30}$$

$$\begin{aligned}\mathrm{Var}_u^T[\ell_t] = {} & \left(\frac{(1+\lambda\phi)\eta}{\lambda-\kappa}\right)^2 B_{2\kappa}^{(t-u)} \\ & + \left[\sigma^2 + \left(\frac{(1+\lambda\phi)\eta}{\lambda-\kappa}\right)^2 - \left(2\,\frac{\rho\sigma(1+\lambda\phi)\eta}{\lambda-\kappa}\right)\right] B_{2\lambda}^{(t-u)} \\ & + 2\left[\left(\frac{\rho\sigma(1+\lambda\phi)\eta}{\lambda-\kappa}\right) - \left(\frac{(1+\lambda\phi)\eta}{\lambda-\kappa}\right)^2\right] B_{(\lambda+\kappa)}^{(t-u)}\end{aligned} \tag{B31}$$

$$\mathrm{Var}_u^T[r_t] = \eta^2 B_{2\kappa}^{(t-u)} \tag{B32}$$

$$\mathrm{Cov}_u^T[\ell_t, r_t] = -\frac{(1+\lambda\phi)\eta^2}{\lambda-\kappa} B_{2\kappa}^{(t-u)} - \left(\sigma\eta\rho - \frac{(1+\lambda\phi)\eta^2}{\lambda-\kappa}\right) B_{(\lambda+\kappa)}^{(t-u)}. \tag{B33}$$

In the above we have set $\bar{\ell}^{\mathcal{Q}} \equiv \bar{\ell}^{\mathcal{Q}}(0)$ where $\bar{\ell}^{\mathcal{Q}}(\cdot)$ is defined in equation (32).

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