553.633/433

Homework #6

Due Monday 10/15/18

Four problems:

A. Suppose in a statistical experiment that $\overline{X} = 1.75$, $s^2 = 2.13$ (note: upper case S is usually used in the textbook; s and S are used interchangeably in 553.633), and $n = 25$. Compute a symmetric 95 percent confidence interval using the standard t-distribution-based method. Without knowing more about the nature of the data, mention at least two reasons why this interval could be flawed. (Note: This problem uses the basic statistical material from slides 23 – 25 in Chap1_633_handout.pdf; you should be familiar with that material from course prerequisites. The material will also be briefly reviewed in class on Monday, 10/8/17.)

B. Consider Example 2.14 in the textbook, including the settings for the coefficients (initial condition $X(0) = 1$, etc.). Do the following:

(a) Perform a statistical test of the accuracy of $\Delta t = 10^{-4}$ (same $h = \Delta t$ as in the textbook) versus $\Delta t = 10^{-1}$ in the E-M method. That is, for each Δt, compute 100 independent runs of the E-M process, each starting from $X(0) = 1$. Then, for each Δt, compute the two-sided P-value from a t-test of the sample mean of the 100 runs of the E-M process at the terminal value ($t = 1$) versus the true mean at the same (terminal) time. How do the two P-values compare? Does this experiment correspond to a test of the weak or strong measure of accuracy?

(b) For each Δt in (a), give a plot that is similar to Figure 2.12 in the text, except show the first *three* (of the 100) runs in part (a) versus the underlying true solution. (You should provide two plots, one for each Δt.)

C. Consider the Ornstein–Uhlenbeck process:

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t),$$

where $\theta = 1$, $\mu = 20$, and $\sigma = 10$, and $W(t)$ denotes the (standard) Wiener process. Simulate the stochastic process $X(t)$ using the Euler–Maruyama method described in the slide "Numerical Solution of Itô SDE" from the class notes (see also Sect. 4 of Higham, 2001). (That is, use the *full* E-M form in the class notes as if no solution to the SDE were available, rather than the solution of the O-U process discussed on the slide "Example 2 of Itô SDE: Ornstein–Uhlenbeck Process" in the class notes.) Run the E-M process 50 separate (statistically independent) times over the time interval [0, 5] with a $\Delta t = 0.01$ and $X(0) = 0$ (i.e., carry out the E-M process 50 independent times, each beginning at the same initial condition $X(0) = 0$). From these 50 runs, do the following:

(a) Show the first 5 (of 50) solution paths on one plot (one solution path is the sequence of X_j from the E-M process over all j, representing time from 0 to 5).

(b) Produce a separate line that represents the sample mean of the 50 paths and comment on how the sample mean differs from its limiting value as a function of t (this line may be on the plot in part (a) or on a separate plot). It is not necessary to run any formal statistical (or other) tests for analyzing the difference; a brief "words-only" discussion is sufficient.

(c) Perform a statistical t-test on whether the (unknown) true mean of the value of X_j (from the E-M process) that represents $X(2)$ is μ. That is, report a two-sided P-value and provide some brief interpretation. Do the same for $X(5)$.

D. Consider the formula for S^2 based on i.i.d. data, as given in eqn. (4.2) of the textbook.

(a) Show that S^2 is an unbiased estimator of σ^2 (i.e., $E(S^2) = \sigma^2$).

(b) Show that S is a biased estimator of σ.