553.633/433

Homework #11

Due Monday 11/26/18

Four problems:

6.2. (textbook) Use a plot to determine when the process seems to achieve stationarity. Start with $X_0 = 0$, which is not a very good starting point, but one that allows the M-H algorithm to demonstrate its utility with even a poorly chosen initial condition.

6.6 (textbook) Assume that "plot the data" in part (b) of the problem statement means plot the data in three histograms of the marginal density functions, one for each ρ. Assume no burn-in period. Consolidate the values of X and the values of Y into one histogram for each ρ since the marginal density of X and Y is the same. So, each histogram represents the approximation of f_X (same as f_Y) at the given ρ. Given the three histograms, offer a brief comment about the relative performance of the Gibbs sampler for the three values of ρ. Which value of ρ provides the best performance and, briefly, why?

A. Gibbs sampling with a Bernoulli distribution. The sampling need not be performed according to continuous random variables and associated probability *density* functions. Consider the following example associated with a Bernoulli distribution.

Suppose that a matrix of joint probabilities for two random variables, X and Y, is

$$\begin{bmatrix} P(X=0,Y=0) & P(X=0,Y=1) \\ P(X=1,Y=0) & P(X=1,Y=1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.2 \end{bmatrix}.$$

Given the above as a target distribution, there are four "full conditionals" that we need to determine, corresponding to the entries in a Markov transition matrix. We are interested in the 2×2 transition matrix $\boldsymbol{P}$ governing the probability for the variable X of going from X_k to X_{k+1}. Show that this transition matrix is:

$$\boldsymbol{P} = \begin{bmatrix} 0.3889 & 0.6111 \\ 0.2619 & 0.7381 \end{bmatrix}.$$

B. Consider a simple state-space model with scalar state and measurement: $x_{k+1} = Fx_k$ and $z_k = x_k + v_k$, where the v_k are i.i.d. with mean 0 and variance σ^2. Answer the following based on use of the Kalman filter for the state estimation:

(a) Give the recursion (a difference equation) for the error-variance P_{k+1} in terms of P_k and the model parameters F and σ^2. Solve the difference equation in terms of F, σ^2, and the initial state variance P_0. Present the solution in a closed form that does *not* include a cumulative summation such as $\sum_{i=1}^{K}(\cdot)$, where K depends on k.

(b) From the results in part (a), summarize what is known about the accuracy of the Kalman filter estimate for large k in the cases $|F| < 1$, $|F| = 1$, and $|F| > 1$.