Autoencoders

Jie Tang

Tsinghua University

Overview

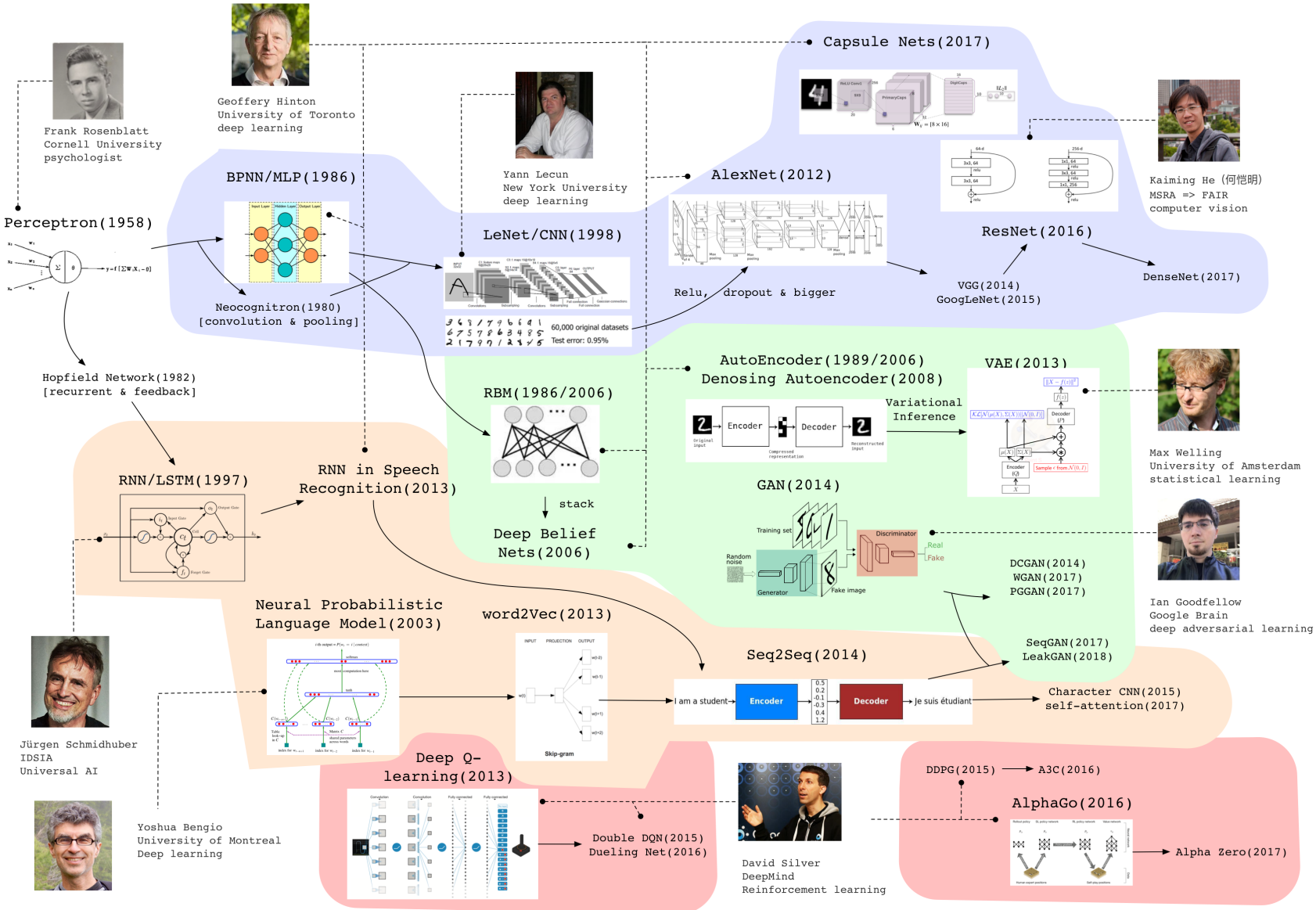
Overview

Autoencoder

- An autoencoder neural network is an unsupervised learning algorithm that applies back-propagation, setting the target values to be equal to the inputs;
- was first introduced by Dana Ballard as a way of conducting pretraining in ANNs[1];
- typically for the purpose of dimensionality reduction;
- the learned hidden layers are called the representation (encoding) of the input data.[2]

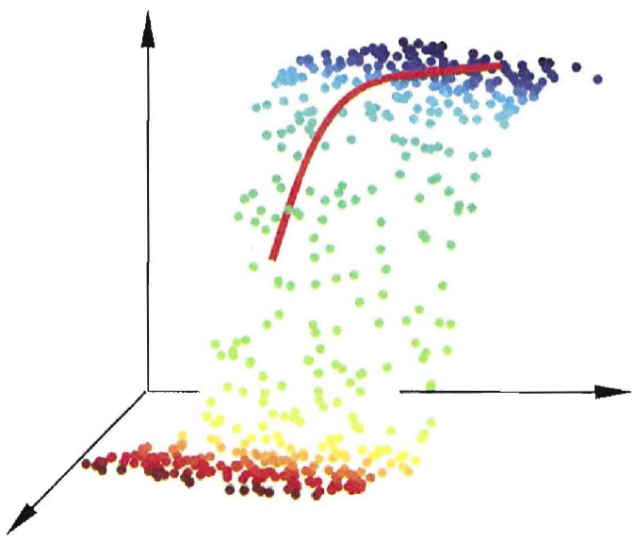
[1] Dana H. Ballard Modular learning in neural networks. AAAI'87, pp. 279-284

[2] Yoshua Bengio. Learning Deep Architectures for AI. Foundations and Trends in Machine Learning, 2, 2009.

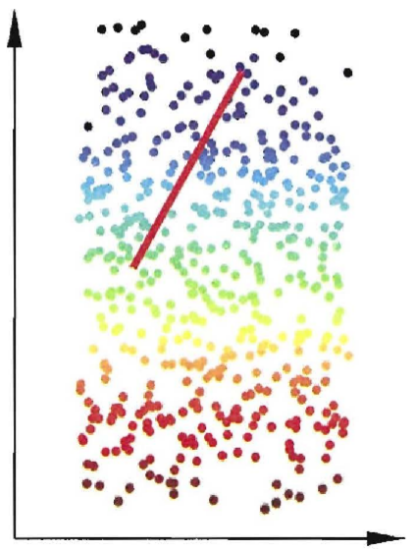

Frank Rosenblatt
Cornell University
psychologist
Geoffery Hinton
University of Toronto
deep learning
Yann Lecun
New York University
deep learning
Capsule Nets(2017)
Kaiming He (何恺明)
MSRA => FAIR
computer vision
BPNN/MLP(1986)
Perceptron(1958)
AlexNet(2012)
LeNet/CNN(1998)
ResNet(2016)
Neocognitron(1980)
[convolution & pooling]
60,000 original datasets
Test error: 0.95%
Relu, dropout & bigger
VGG(2014)
GoogLeNet(2015)
DenseNet(2017)
Hopfield Network(1982)
[recurrent & feedback]
AutoEncoder(1989/2006)
Denosing Autoencoder(2008)
VAE(2013)
Variational
Inference
Encoder
Decoder
RBM(1986/2006)
stack
Deep Belief
Nets(2006)
Max Welling
University of Amsterdam
statistical learning
RNN/LSTM(1997)
RNN in Speech
Recognition(2013)
GAN(2014)
Training set
Random
noise
Generator
Discriminator
Real
Fake
Fake image
DCGAN(2014)
WGAN(2017)
PGGAN(2017)
Ian Goodfellow
Google Brain
deep adversarial learning
Neural Probabilistic
Language Model(2003)
word2Vec(2013)
Skip-gram
Seq2Seq(2014)
I am a student
Encoder
Decoder
Je suis étudiant
SeqGAN(2017)
LeakGAN(2018)
Character CNN(2015)
self-attention(2017)
Jürgen Schmidhuber
IDSIA
Universal AI
Yoshua Bengio
University of Montreal
Deep learning
Deep Q-
learning(2013)
Double DQN(2015)
Dueling Net(2016)
David Silver
DeepMind
Reinforcement learning
DDPG(2015)
A3C(2016)
AlphaGo(2016)
Alpha Zero(2017)

Why Autoencoder: Curse of dimensionality

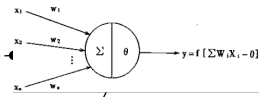
- Data in *high-dimensional spaces* (often with hundreds or thousands of dimensions) proves a serious obstacle.
  - train data is too sparse
  - computation and memory cost is high
- One important solution is dimension reduction.
  - autoencoder is used to learn a representation ($\mathbf{x} \in \mathbb{R}^n \rightarrow \mathbf{y} \in \mathbb{R}^d$, $d << n$) for data, typically for the purpose of dimensionality reduction.



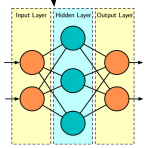
Observation data in 3D space



Projection in 2D space



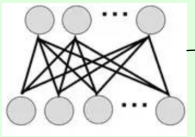
Perceptron(1958)



BPNN/MLP(1986)

Andrew Ng
Stanford University
Machine learning

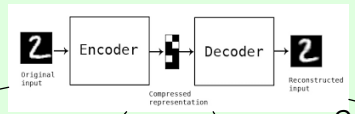
RBM(1986/2006)



Geoffery Hinton
University of Toronto
deep learning

AutoEncoder(1986)

Deep Belief Nets(2006)



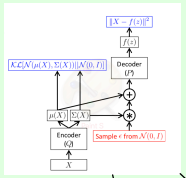
Max Welling
University of Amsterdam
statistical learning

Ian Goodfellow
Google Brain
deep adversarial learning

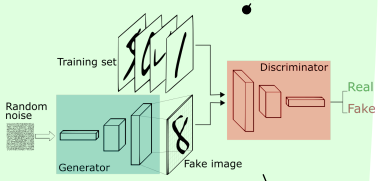
Regularization

Generative model

Variational AutoEncoder(2013)



GAN(2014)



Noise tolerance

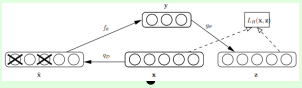
Specific domain

Sparse AutoEncoder(2007)

Dennoising AutoEncoder(2011)

Convolutional AutoEncoder(2011)

Contractive AutoEncoder(2011)



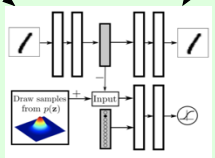
LSTM AutoEncoder(2015)

Robust AutoEncoder(2014)

beta-VAE(2017)

info-VAE(2017)

factorVAE(2018)



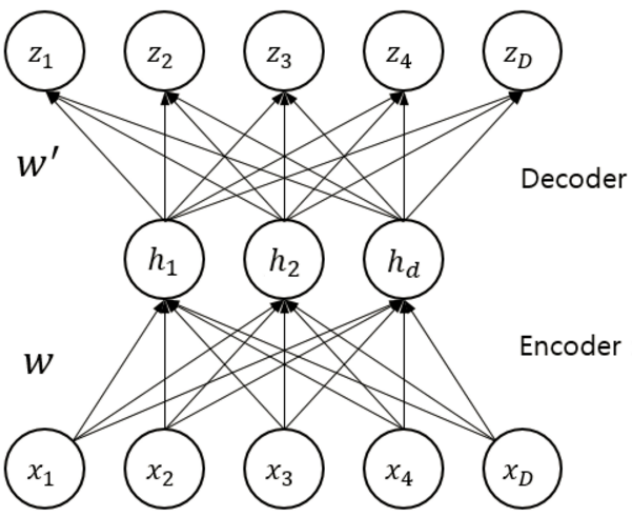
adversarial AutoEncoder(2016)

Yoshua Bengio
University of Montreal
Deep learning

Jürgen Schmidhuber
IDSIA
Universal AI

Autoencoder

- Unsupervised learning: only use the inputs $X = \{x^{(1)}, x^{(2)}, \cdots, x^{(n)}\}$ for learning.
  - automatically extract meaningful features for your data
  - leverage the availability of unlabeled data
  - add a data-dependent regularizer to trainings
- Autoencoder is a feedforward neural network trained to reproduce its input at the output layer: $\mathbf{z} \approx \mathbf{x}$



- Encoder:

$$\mathbf{h} = f_\theta(\mathbf{x}) = a(\mathbf{W}\mathbf{x} + \mathbf{b})$$

- Decoder:

$$\mathbf{z} = g_{\theta'}(\mathbf{h}) = a(\mathbf{W}'\mathbf{h} + \mathbf{b}')$$

$$\theta, \theta' : \{\mathbf{W}, \mathbf{b}\}, \{\mathbf{W}', \mathbf{b}'\}$$

Autoencoder—Parameter Estimation

- Parameter estimation:

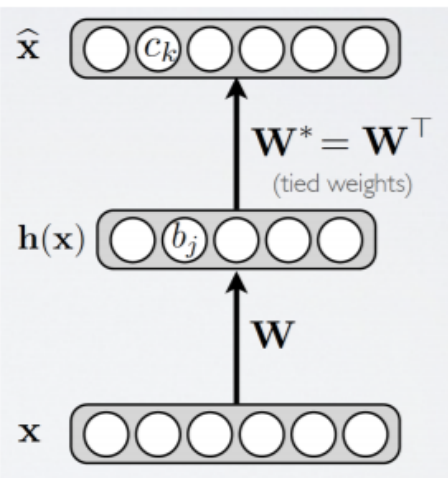
$$\begin{aligned} \theta^*, \theta'^* &= \underset{\theta,\theta'}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^{n} L\left(\mathbf{x}^{(i)}, \mathbf{z}^{(i)}\right) \\ &= \underset{\theta,\theta'}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^{n} L\left(\mathbf{x}^{(i)}, g_{\theta'}(f_\theta(\mathbf{x}^{(i)}))\right) \end{aligned} \tag{1}$$

  where L is a loss function such as *squared error* $L(\mathbf{x}, \mathbf{z}) = ||\mathbf{x} - \mathbf{z}||^2$.

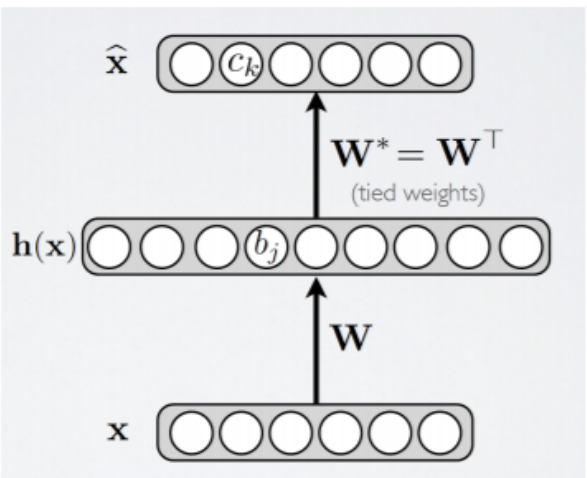
- If $\mathbf{x}$ and $\mathbf{z}$ are vectors of bit probabilities (Bernoullis), L is also chosen to be the *cross-entropy*: $L_H(\mathbf{x}, \mathbf{z}) = H(\beta_{\mathbf{x}}||\beta_{\mathbf{z}}) = -\sum_{k=1}^{d}[\mathbf{x}_k \log \mathbf{z}_k + (1-\mathbf{x}_k)\log(1-\mathbf{z}_k)]$

Undercomplete Vs Overcomplete

- Hidden layer is undercomplete if its dimension is smaller than that of the input layer:
  - hidden layer "compress" the data
  - hidden layer will be good features for the train distribution
- Hidden layer is overcomplete if its dimension is greater than that of the input layer:
  - no compression in hidden layer
  - no guarantee that the hidden units will extract meaningful structure



(a) undercomplete autoencoder.



(b) overcomplete autoencoder.

*Optionally, the reverse mapping W^* may be constrained to be the transpose of the forward mapping: $W^* = W^T$. This is referred to as *tied weights*.

Linear Autoencoder Corresponds to PCA

To do the proof, we need the following theorem:

- let $\mathbf{A}$ be any matrix, with singular value decomposition $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$
  - Σ is a diagonal matrix
  - $\mathbf{U}, \mathbf{V}$ are orthonormal matrices (columns/rows are orthonormal vectors)
- the matrix $\mathbf{B}$ of rank k that is closest to $\mathbf{A}$:

$$\mathbf{B}^* = \underset{\mathbf{B}\ s.t.\ rank(\mathbf{B})=k}{\operatorname{argmin}} ||\mathbf{A} - \mathbf{B}||_F = \mathbf{U}_{.,\leq k}\Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k} \quad (2)$$

  where $\mathbf{U}_{.,\leq k}\Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k}$ is the decomposition where we keep only the k largest singular values.

- the sum of square error in training data:

$$\min_{\theta} \sum_t \frac{1}{2}(\mathbf{x}_i^{(t)} - \mathbf{z}_i^{(t)})^2 \geq \min_{\mathbf{W}', h(\mathbf{X})} \frac{1}{2}||\mathbf{X} - \mathbf{W}' h(\mathbf{X})||_F^2 \quad (3)$$

  here $h(\mathbf{X})$ could be any encoder.

Linear autoencoder corresponds to PCA

Based on previous theorem, where $\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^T$ and k is hidden layer size:

$$\underset{\mathbf{W}', h(\mathbf{X})}{\text{argmin}} \frac{1}{2}||\mathbf{X} - \mathbf{W}' h(\mathbf{X})||_F^2 = (\mathbf{W}' \leftarrow \mathbf{U}_{.,\leq k}, h(\mathbf{X}) \leftarrow \Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k})$$

$$\begin{aligned}
h(\mathbf{X}) &= \Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k} \\
&= \Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k}(\mathbf{X}^T\mathbf{X})^{-1}(\mathbf{X}^T\mathbf{X}) && \leftarrow \textit{multiplying by identity} \\
&= \Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k}(\mathbf{V}\Sigma^T\mathbf{U}^T\mathbf{U}\Sigma\mathbf{V}^T)^{-1}(\mathbf{V}\Sigma^T\mathbf{U}^T\mathbf{X}) && \leftarrow \textit{replace with SVD} \\
&= \Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k}(\mathbf{V}\Sigma^T\Sigma\mathbf{V}^T)^{-1}\mathbf{V}\Sigma^T\mathbf{U}^T\mathbf{X} && \leftarrow \mathbf{U}^T\mathbf{U} = \mathbf{I} \\
&= \Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k}\mathbf{V}(\Sigma^T\Sigma)^{-1}\mathbf{V}^T\mathbf{V}\Sigma^T\mathbf{U}^T\mathbf{X} && \leftarrow \mathbf{V}(\Sigma^T\Sigma)^{-1}\mathbf{V}^T\mathbf{V}\Sigma^T\Sigma\mathbf{V}^T = \\
&= \Sigma_{\leq k,\leq k}\mathbf{V}^T_{.,\leq k}(\Sigma\mathbf{V}^T)^{-1}\mathbf{U}^T\mathbf{X} \\
&= \Sigma_{\leq k,\leq k}\mathbf{I}_{\leq k,.}\Sigma^{-1}\mathbf{U}^T\mathbf{X} && \leftarrow \textit{multiplying by } \mathbf{I}_{\leq k,.} \\
&= \mathbf{U}^T_{.,\leq k}\mathbf{X}
\end{aligned}$$

Linear autoencoder corresponds to PCA

- So an optimal pair of encoder and decoder is

$$h(\mathbf{x}) = \mathbf{U}_{.,\leq k}^{T}\mathbf{x}$$
$$\mathbf{z} = \mathbf{U}_{.,\leq k}h(\mathbf{x})$$

  - for the sum of squared difference error
  - for an autoencoder with a linear decoder
  - where optimality means "has the lowest reconstruction error"
- If inputs are normalized as follows:
  $\mathbf{x}^{(t)} \leftarrow \frac{1}{\sqrt{T}}(\mathbf{x}^{(t)} - \frac{1}{T}\sum_{i=1}^{T}\mathbf{x}^{(i)})$
  - encoder corresponds to Principal Component Analysis
  - singular values and vectors = eigenvalues/vectors of covariance matrix

Example

- Suppose that inputs $\mathbf{x}$ are the pixel values from a 10×10 image, $\mathbf{x} \in \mathbb{R}^{100}$.
- There are 50 units in hidden layers, $\mathbf{y} \in \mathbb{R}^{50}$
- The network must try to reconstruct the 100-dimensional input $\mathbf{x}$ by hidden representation $\mathbf{y}$
- Since there are only 50 hidden units, it is forced to learn a *compressed representation* of the input.
- If the input were completely random (e.g., each $\mathbf{x}_i$ comes from an IID Gaussian independent of other features) then this compression task would be very difficult.
- If there is structure in data (e.g., some of features are correlated), then this algorithm often ends up learning a low-dimensional representation very similar to PCAs.

Overview

Regular autoencoder is not enough

- What is a good representation?
  - Sparsity? $\Rightarrow$ Sparse AE
  - Denoise? $\Rightarrow$ Denoising AE
  - Robust to disturbance? $\Rightarrow$ contractive AE
  - High level? $\Rightarrow$ Stacked AE
  - Specific data (image) structure? Convolutional AE
  - Generate new data? $\Rightarrow$ Variational AE

Overview

Sparse Autoencoder: Overcomplete Autoencoder

- When the number of hidden units is greater than the number of input units, we can still discover interesting structure, by imposing a sparsity constraint on the hidden units.
- A neuron will be thought as "active" if its output value is close to 1, or as "inactive" if its output value is close to 0.[3]
- Let ρ_j be the average activation of hidden unit j

$$\hat{\rho}_j = \frac{1}{m} \sum_{i=1}^{m} [a_j^{(2)}(x^{(i)})] \tag{4}$$

- We'd like to constrain the neurons to be inactive most of the time:

$$\hat{\rho}_j = \rho \tag{5}$$

where ρ is a **sparsity parameter**, typically a small value close to zero, which forces most hidden units' activations to be near 0.

[3] This assumes a sigmoid activation function. If you are using a tanh activation function, we think of a neuron as being inactive when its output is close to -1.

Sparsity Penalty

- We add an extra penalty term to our optimization objective that penalizes $\hat{\rho}_j$ deviating significantly from ρ. For instance:

$$\sum_{j=1}^{s_2} \rho \log \frac{\rho}{\hat{\rho}_j} + (1-\rho) \log \frac{1-\rho}{1-\hat{\rho}_j} \tag{6}$$

  where s_2 is the number of neurons in the hidden layer

- This sparsity penalty is based on Kullback-Leibler (KL) divergence:

$$\sum_{j=1}^{s_2} KL(\beta_\rho || \beta_{\hat{\rho}_j}) \tag{7}$$

  where $KL(\beta_\rho || \beta_{\hat{\rho}_j}) = \rho \log \frac{\rho}{\hat{\rho}_j} + (1-\rho) \log \frac{1-\rho}{1-\hat{\rho}_j}$ is the KL divergence between two Bernoulli random variables with mean ρ and $\hat{\rho}_j$.

Sparsity penalty

Sparsity penalty has the property that $KL(\beta_\rho||\beta_{\hat{\rho}_j}) = 0$ if $\hat{\rho}_j = \rho$, and otherwise it increases monotonically as $\hat{\rho}_j$ diverges from ρ.

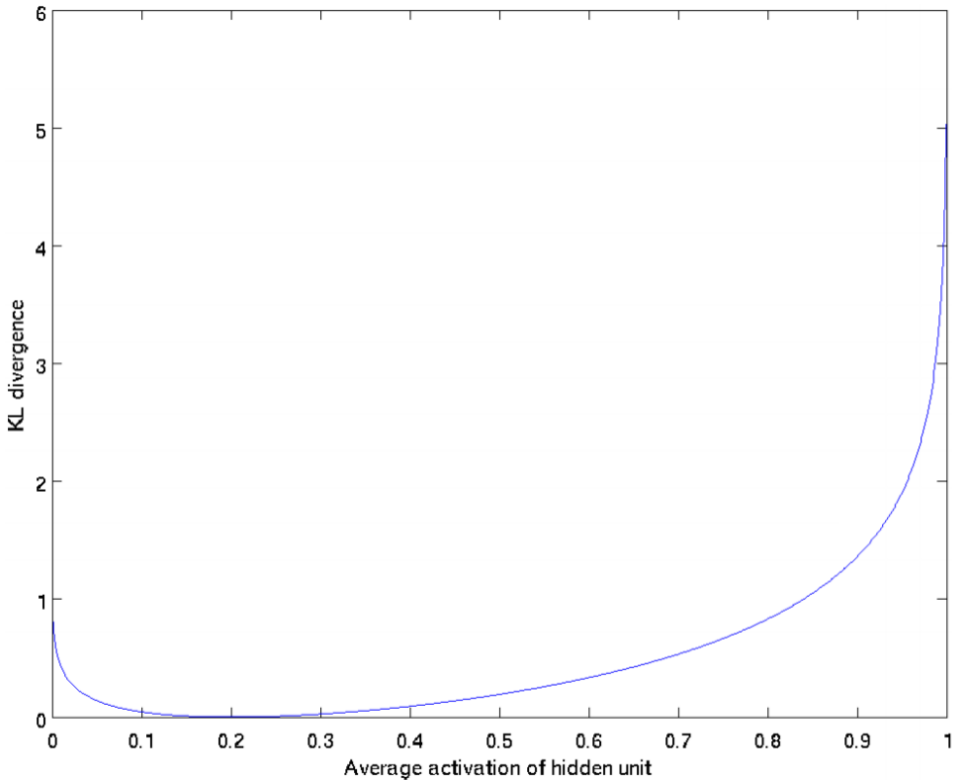


Abbildung 1: Example: $KL(\beta_\rho||\beta_{\hat{\rho}_j})$ is plotted in a range of values of $\hat{\rho}_j$ while ρ is set to be 0.2

Learning with Back Propagation

- Loss function can be written

$$L_{sparse}(W, b) = L(W, b) + \alpha \sum_{j=1}^{s_2} KL(\beta_\rho || \beta_{\hat{\rho}_j}) \tag{8}$$

- Previously for the second layer ($l = 2$), during back-propagation we would have computed

$$\delta_i^{(2)} = (\sum_{j=1}^{s_2} W_{ji}^{(2)} \delta_j^{(3)}) f'(z_i^{(2)}) \tag{9}$$

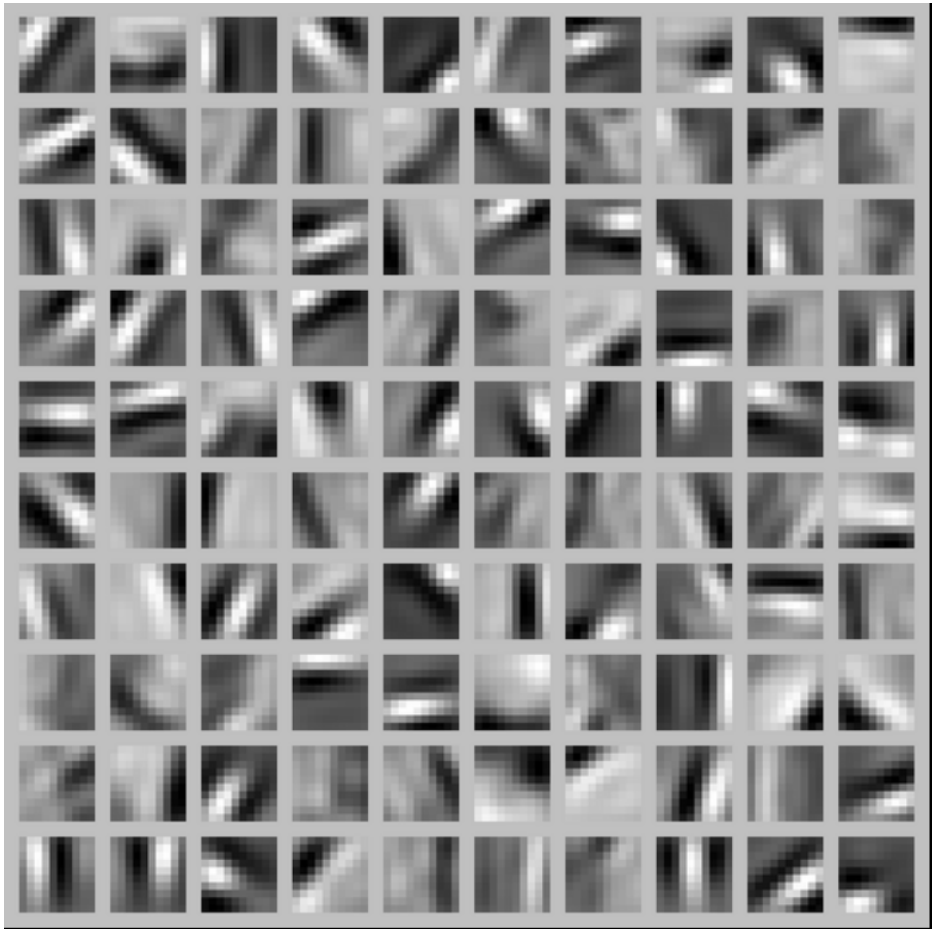
- By adding sparsity penalty, we have

$$\delta_i^{(2)} = ((\sum_{j=1}^{s_2} W_{ji}^{(2)} \delta_j^{(3)}) + \alpha(-\frac{\rho}{\hat{\rho}_i} + \frac{1-\rho}{1-\hat{\rho}_i})) f'(z_i^{(2)}) \tag{10}$$

  One subtlety is that you need to know $\hat{\rho}_i$.

Sparse autoencoder

- When we train a sparse autoencoder with 100 hidden units on 10×10 pixel inputs we get the following result:



Sparse Coding vs Sparse AE

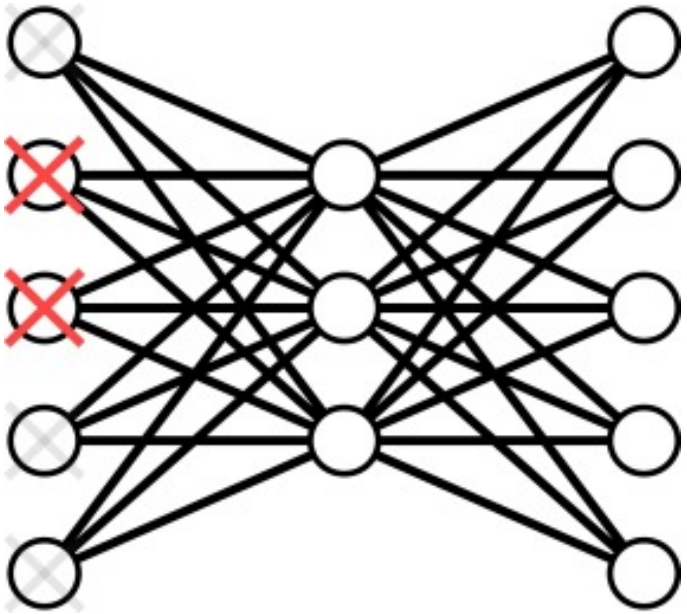
- Sparse coding also aims to learn a sparse representation for data.
- The corresponding objective function is:

$$L_{sc} = \underbrace{||WH - X||_2^2}_{reconstructionterm} + \underbrace{\lambda||H||_1}_{sparsityterm} \tag{11}$$

  Where W is a matrix of bases, H is a matrix of representation of data and X is a matrix of the data we wish to represent.
- In sparse AE, the sparse representation is obtained through an encoder layer: $H = h(X)$. While sprase coding learns H directly by optimizing the loss function.

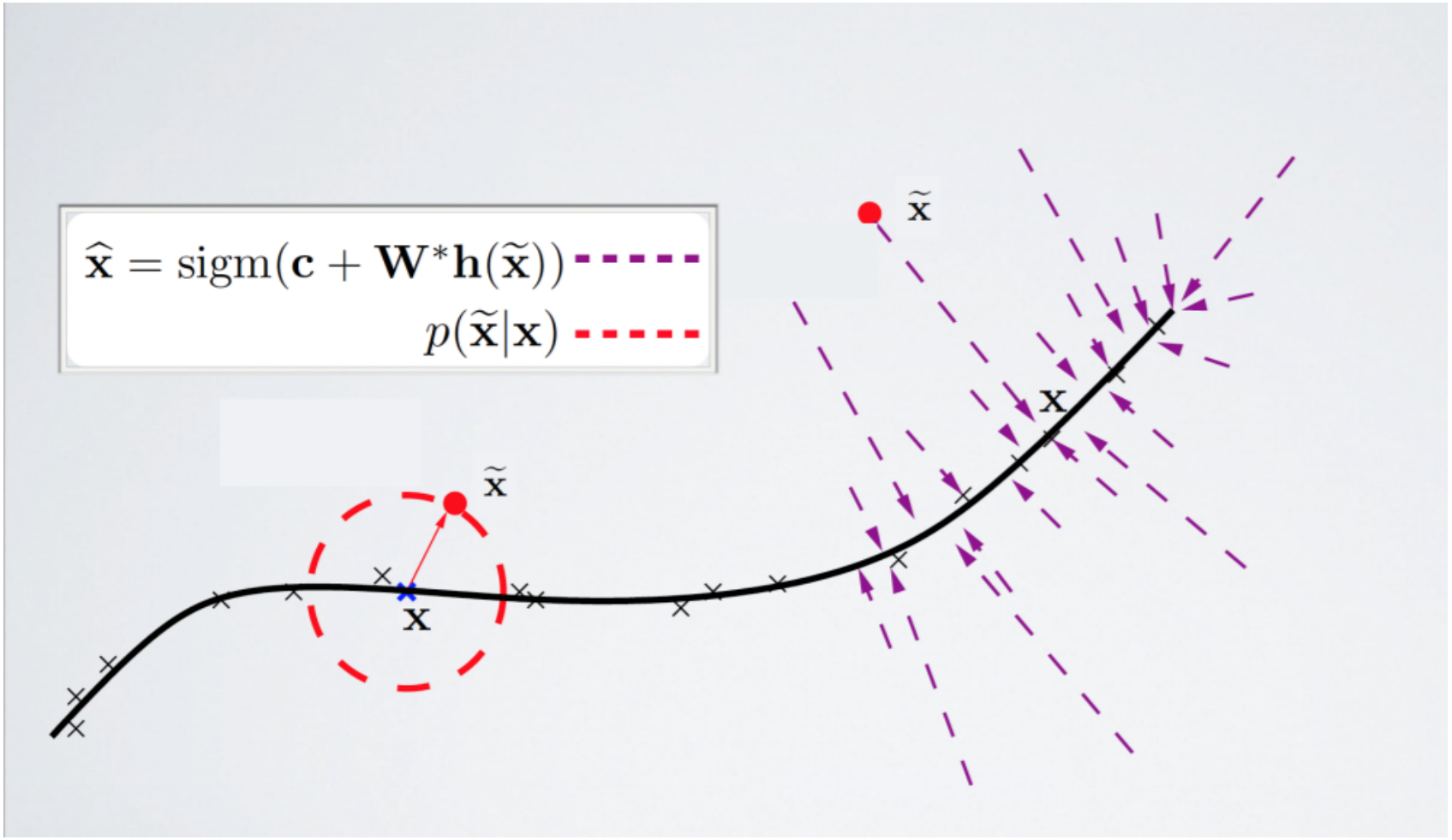
Overview

Denoising Autoencoder

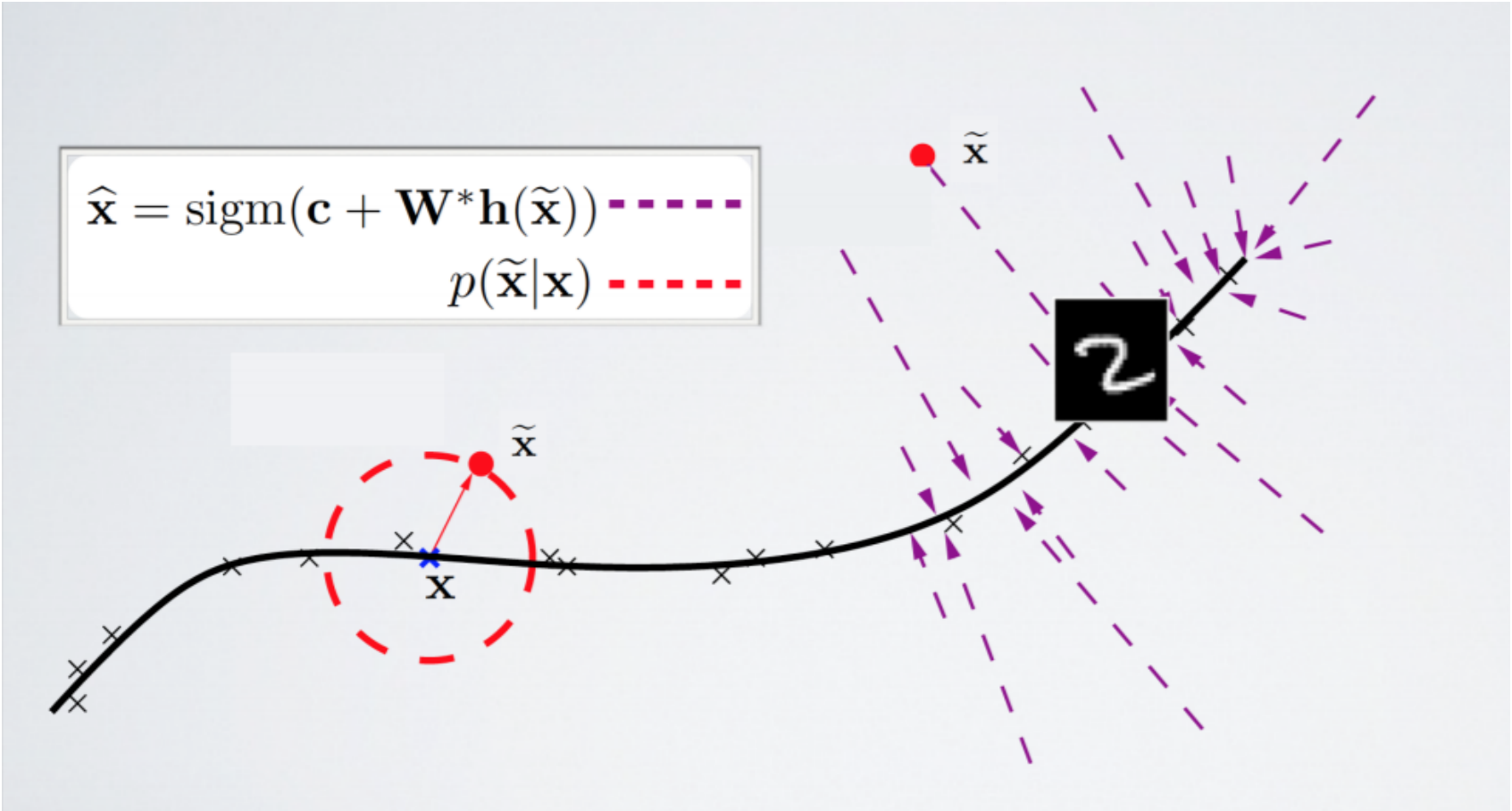


- Idea: representation should be robust to the introduction of noise:
  - random assignment of inputs to 0, with probability $q_{\mathcal{D}}$
  - additive isotropic Gaussian noise $\mathcal{N}(0, \sigma^2 \mathbf{I})$
- Hidden representation $\mathbf{y} = f_\theta(\widetilde{\mathbf{x}}) = f(\mathbf{W}\widetilde{\mathbf{x}} + \mathbf{b})$ is mapped from corrupted input
- Loss function compares reconstruction $\mathbf{z} = g_{\theta'}(\mathbf{y}) = f(\mathbf{W}'\mathbf{y} + \mathbf{b}')$ with the original raw input $\mathbf{x}$

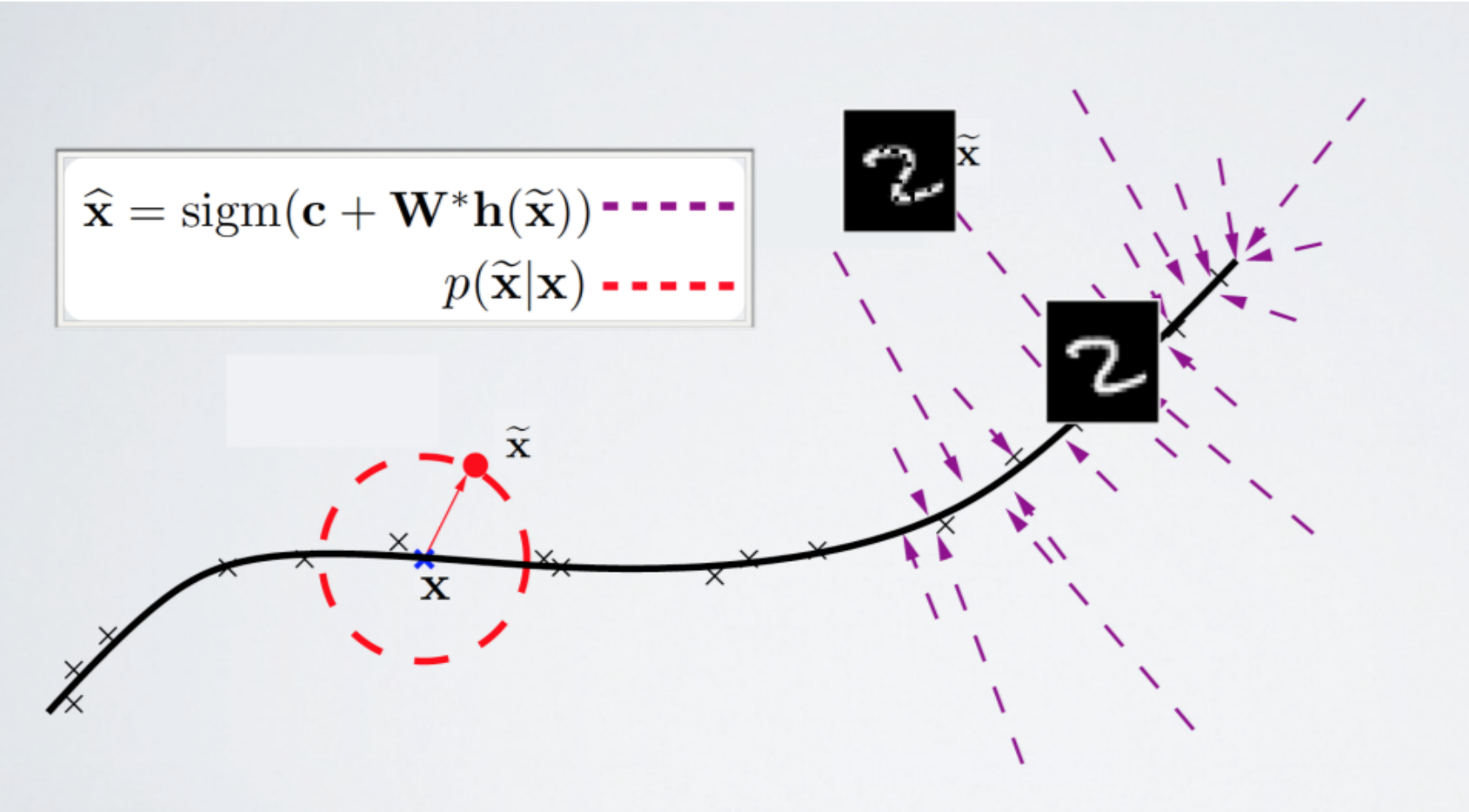
Manifold Learning Perspective



Manifold Learning Perspective



Manifold Learning Perspective



Overview

Contractive Autoencoder

- Idea: representation should be robust to small changes of the *training input*.
- Add the Jacobian of the encoder to penalize its sensitivity to training input:

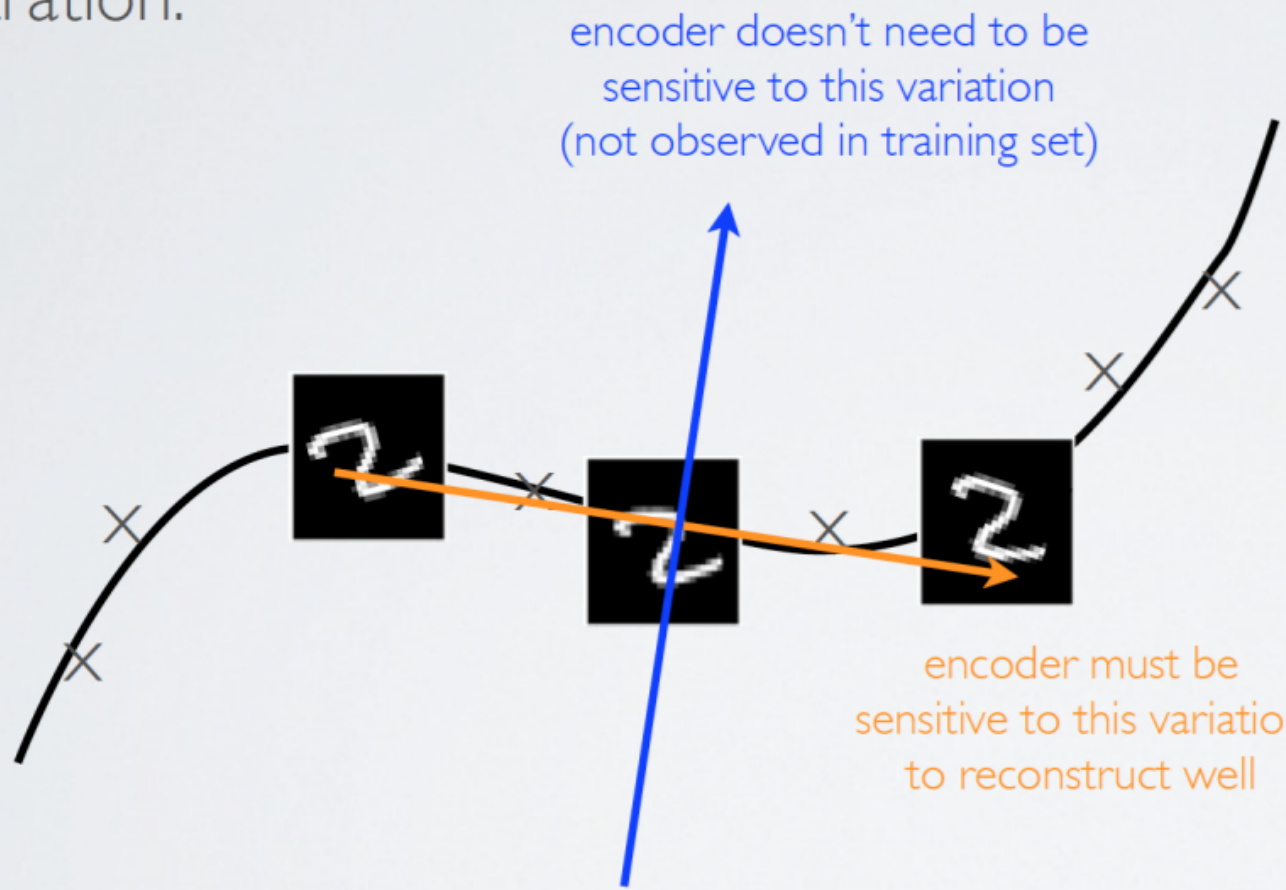
$$||\mathcal{J}_f(\mathbf{x})||_F^2 = \sum_{ij} \left(\frac{\partial \mathbf{h}_j}{\partial \mathbf{x}_i} \right)^2 \tag{12}$$

- The penalty $||\mathcal{J}_f||_F^2$ encourages the mapping to the feature space to be *contractive* in the neighborhood of the training data.
- New loss function:

$$\mathcal{L}(\mathbf{x}, g(f(\mathbf{x}))) + \lambda ||\mathcal{J}_f(\mathbf{x})||_F^2 \tag{13}$$

Manifold Learning Perspective

- Illustration:



Which Autoencoder?

- Both denoising and contractive autoencoders perform well
  - robust to the input data with some noise
  - DAE= $(g \circ f)(\mathbf{x})$ vs. CAE= $f(\mathbf{x})$
- Advantage of denoising autoencoder: simpler to implement
  - requires adding one or two lines of code to regular autoencoder
  - no need to compute Jacobian of hidden layer
- Advantage of contractive autoencoder:
  - can use second order optimizers (conjugate gradient, LBFGS, etc.)
  - robustness of encoder $f(\mathbf{x})$ is more important than robustness of reconstruction ($(g \circ f)(\mathbf{x})$ in DAEs) for classification

Overview

Stacked Autoencoder

- Stacked autoencoder is a neural network consisting of multiple layers of sparse autoencoders.
- The encoding step for the stacked autoencoder is:

$$\mathbf{z}^{(l+1)} = a(\mathbf{W}^{(l)}\mathbf{z}^{(l)}) \tag{14}$$

- Then the decoding step is:

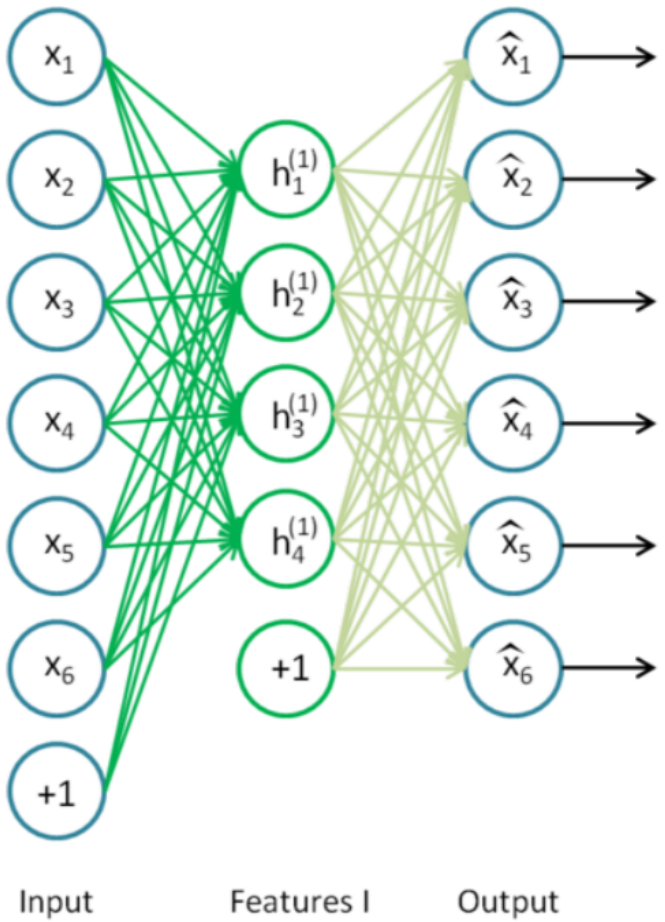
$$\mathbf{z}^{(n+l+1)} = a(\mathbf{W}^{T^{(n-l)}}\mathbf{z}^{(n+l)}) \tag{15}$$

where $\mathbf{W}^{'} = \mathbf{W}^{T}$, referred to as tiled weight.

Greedy Layer-Wise Training

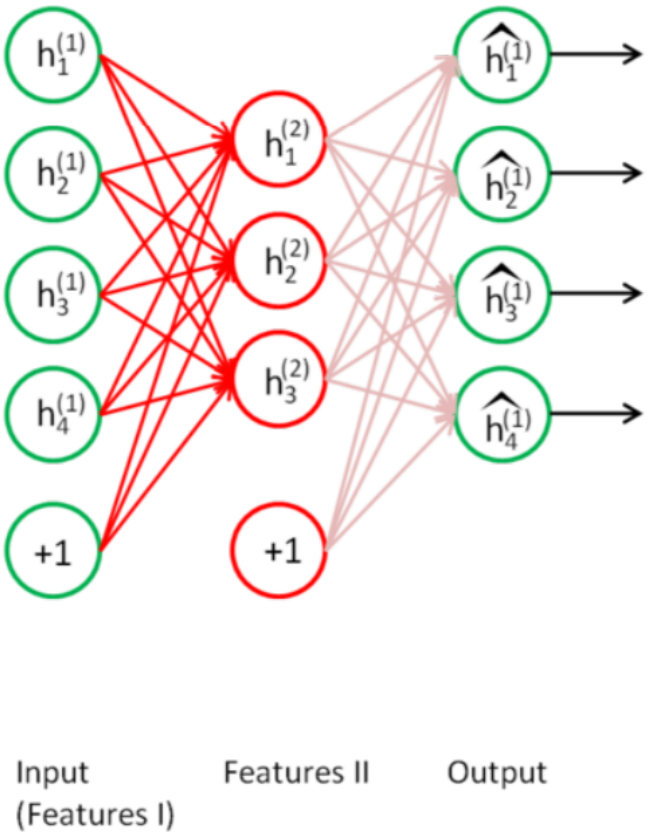
- Step 1: *greedy layer-wise training*
  - We first train a network with 1 hidden layer, and only after that is done, train a network with 2 hidden layers, and so on.
  - At each step, we take the old network as the $k-1$ hidden layers, and add an additional k-th hidden layer.
  - Training is usually unsupervised, but can be also supervised (e.g. with classification error as loss function on each step).
  - The weights from training the layers individually are then used to initialize the weights in the deep network.
- Step 2: *fune-tuning*
  - example: add a softmax layer of the output of Stacked AE and train the whole structure to fune-tune the parameters.

Greedy Layer-Wise Training



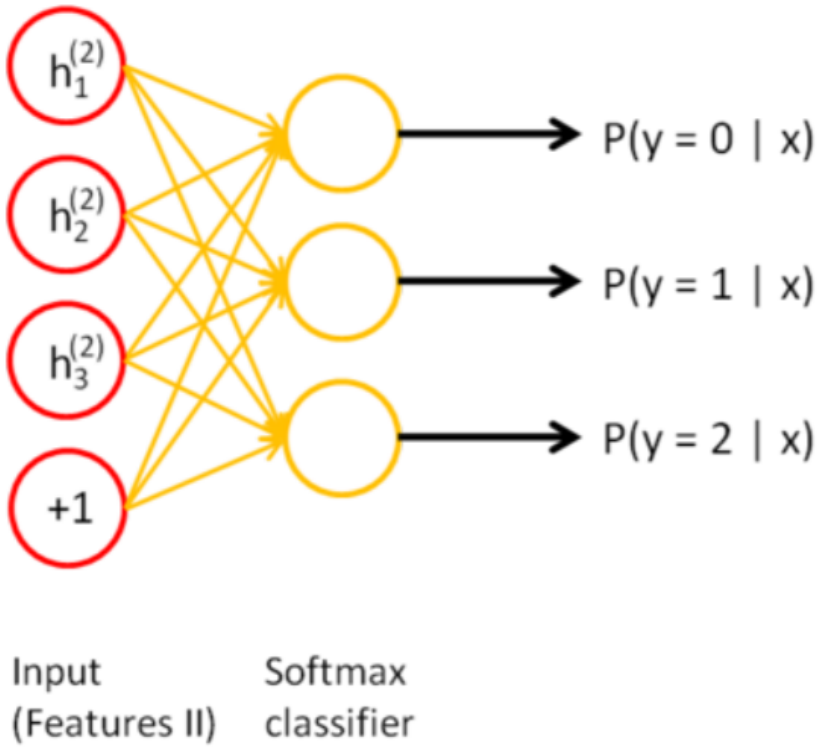
First, we train a sparse autoencoder on the raw inputs $\mathbf{x}$ to learn primary features $\mathbf{h}^{(1)}$.

Greedy Layer-Wise Training



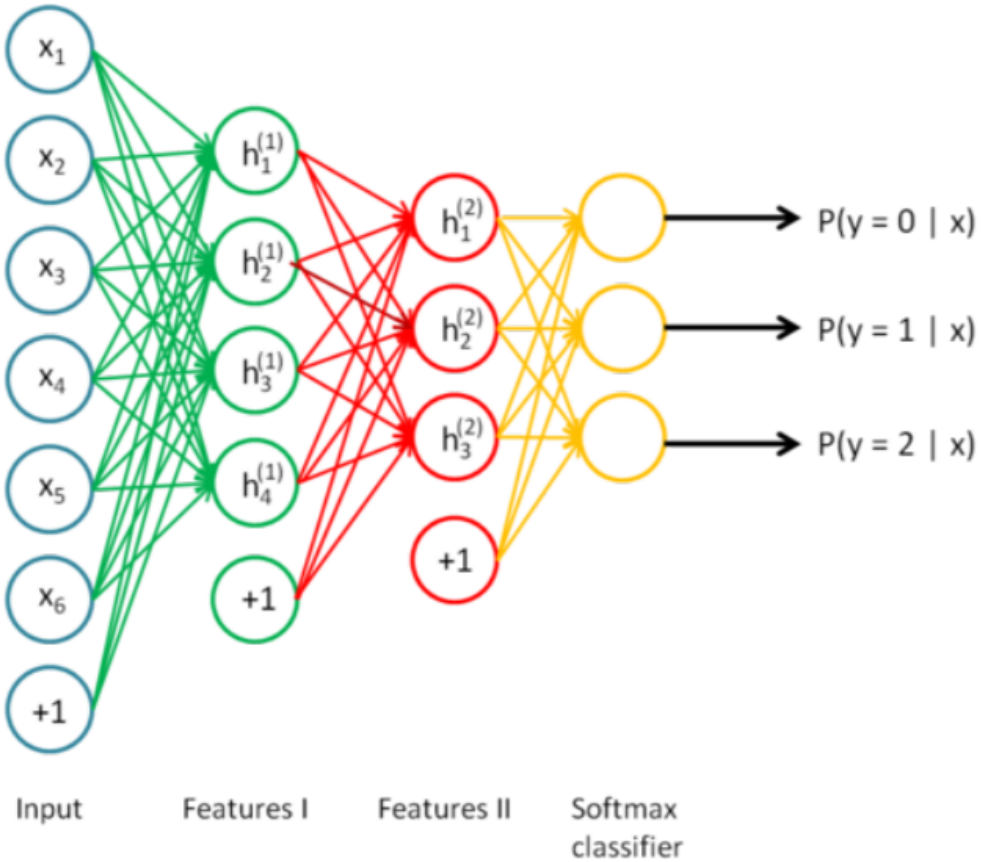
Next, we use these primary feature activations $\mathbf{h}^{(1)}$ as the "raw input" to another sparse to learn secondary features $\mathbf{h}^{(2)}$.

Greedy Layer-Wise Training



We then treat these secondary features as "raw input" to a softmax classifier, training it to map secondary features to digit labels.

Fine-tuning



Finally, we would combine all three layers together to form a stacked autoencoder with 2 hidden layers and a final softmax classifier layer.

Why does it work?

- Availability of data
  - The algorithm is able to learn and discover patterns from massively more amounts of data than purely supervised approaches.
- Better local optima
  - After having trained the network on the unlabeled data, the weights are now starting at a better location in parameter space than if they had been randomly initialized.
  - We can then further fine-tune the weights starting from this location.

Overview

Convolutional AE

- When processing image data, standard AEs do not explicitly consider the 2-dimensional structure.
- Convolutional AE solves this by substituting fully connected layers with convolutional layers.
- The encoding step of Convolutional AE is:

$$h^k = \sigma(x * W^k + b^k) \tag{16}$$

where h^k refers to the k^{th} feature map.

- The decoding step of Convolutional AE is:

$$y = \sigma(\sum_{k \in H} h^k * \tilde{W}^k + c) \tag{17}$$

where H is the group of latent feature maps, $\tilde{W} = W^T$

Max-Pooling

- Max-pooling layer introduces sparsity over the hidden representation by erasing all non-maximal values in non overlapping sub-regions.
- This forces feature detectors to become more broadly applicable.

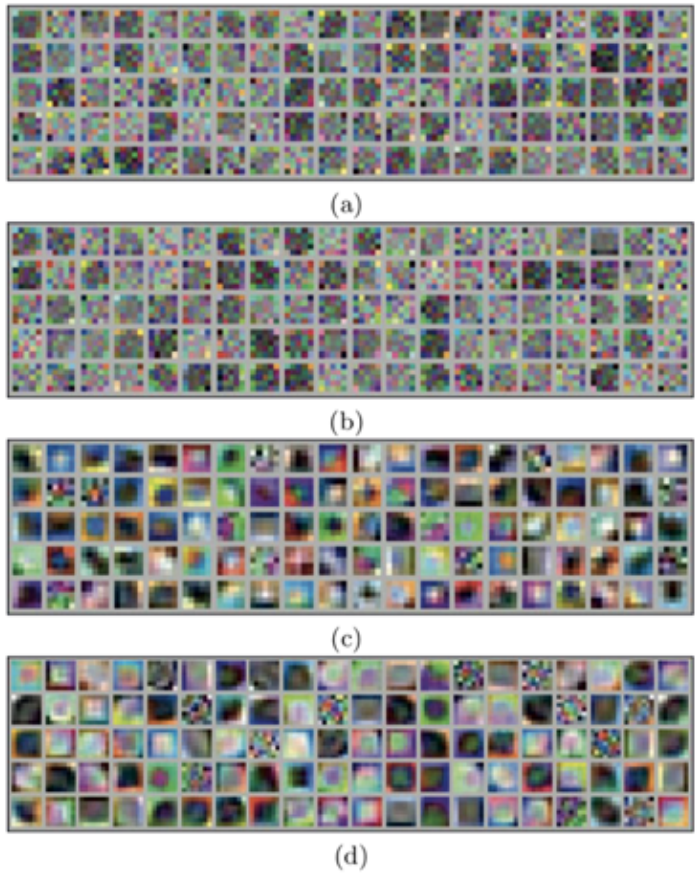
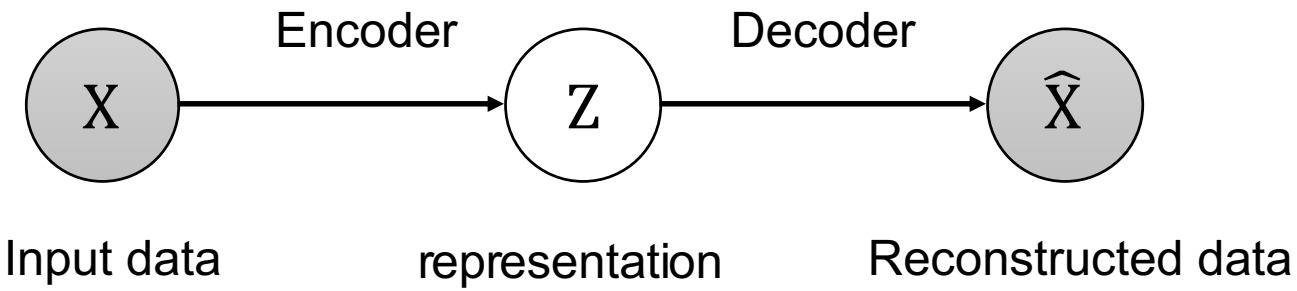


Fig. 2. A randomly selected subset of the first layer's filters learned on CIFAR10 to compare noise and pooling (best viewed in colours). (a) No pooling and 0% noise, (b) No pooling and 50% noise, (c) Pooling of 2x2 and 0% noise, (d) Pooling of 2x2 and 50% noise.

Overview

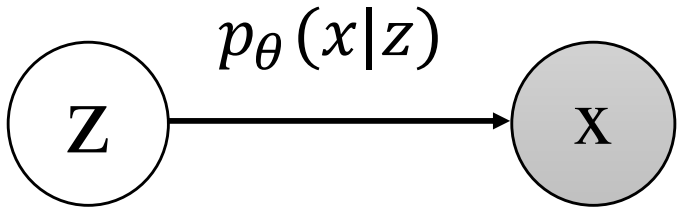
Recap Autoencoder



- Autoencoders can resonstruct data, and can learn features to initialize a supervised model.
- Features capture factors of variation in training data. Can we generate new data from an autoencoder?

Variational AE

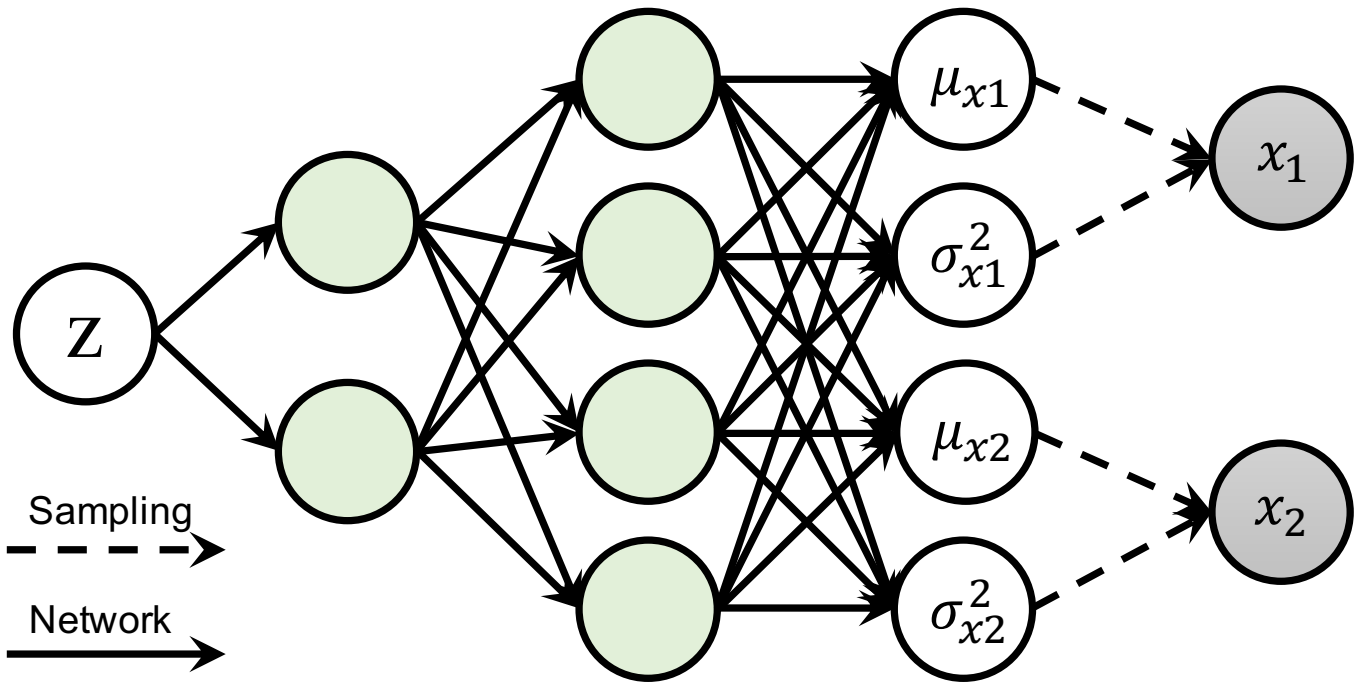
- Probabilistic spin on autoencoder - will let us sample from the model to generate data!
- Assume training data $X = \{x^{(1)}, x^{(2)}, \cdots, x^{(N)}\}$ is generated from underlying latent (unobserved) representation $z \sim p(z)$ (Gaussian)



- We wish to estimate the true parameters θ from training data x.
- Modeling the conditional distribution $p_\theta(x|z)$ by a neural neural network (Decoder network).

Decoder Network

- Idea: NN + Gaussian (or Bernoulli) with a diagonal covariance: $p_\theta(x|z) = N(\mu_x, \sigma_x^2)$



How to train the model?

Training as an Autoencoder

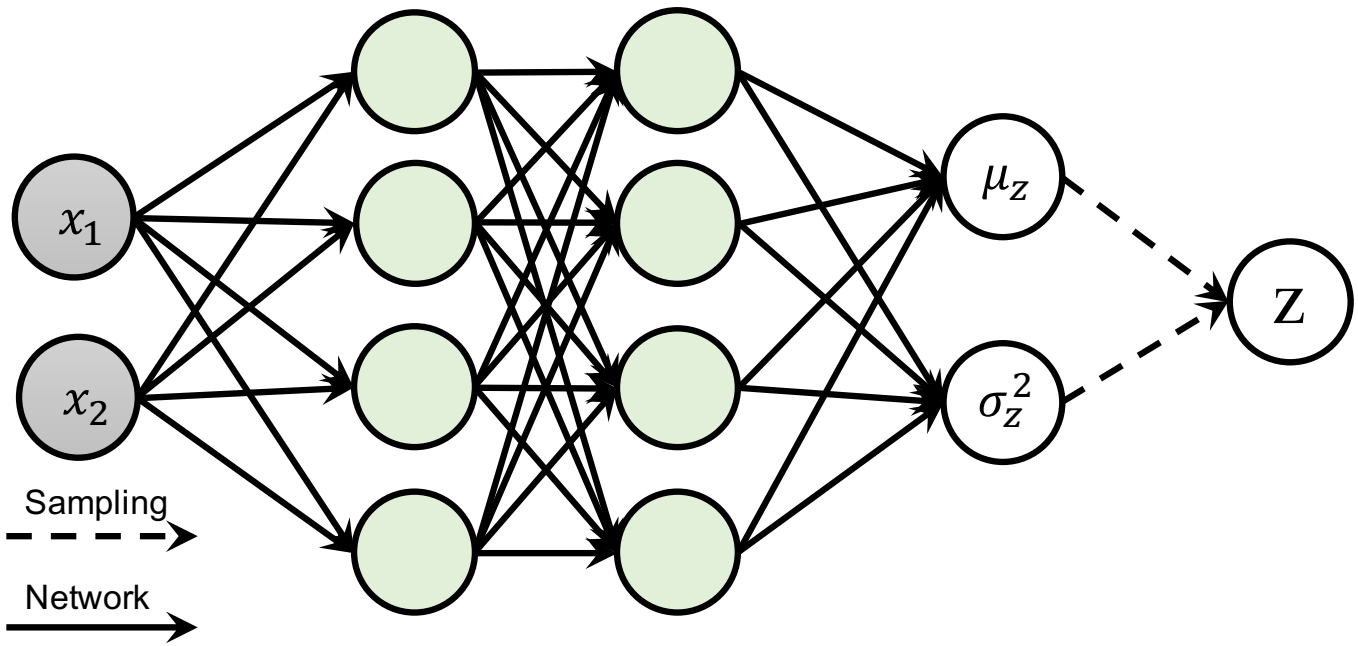
- For generative model, the training objective is to maximize likelihood of training data:

$$p_\theta(x) = \int p_\theta(z) p_\theta(x|z) dz \tag{18}$$

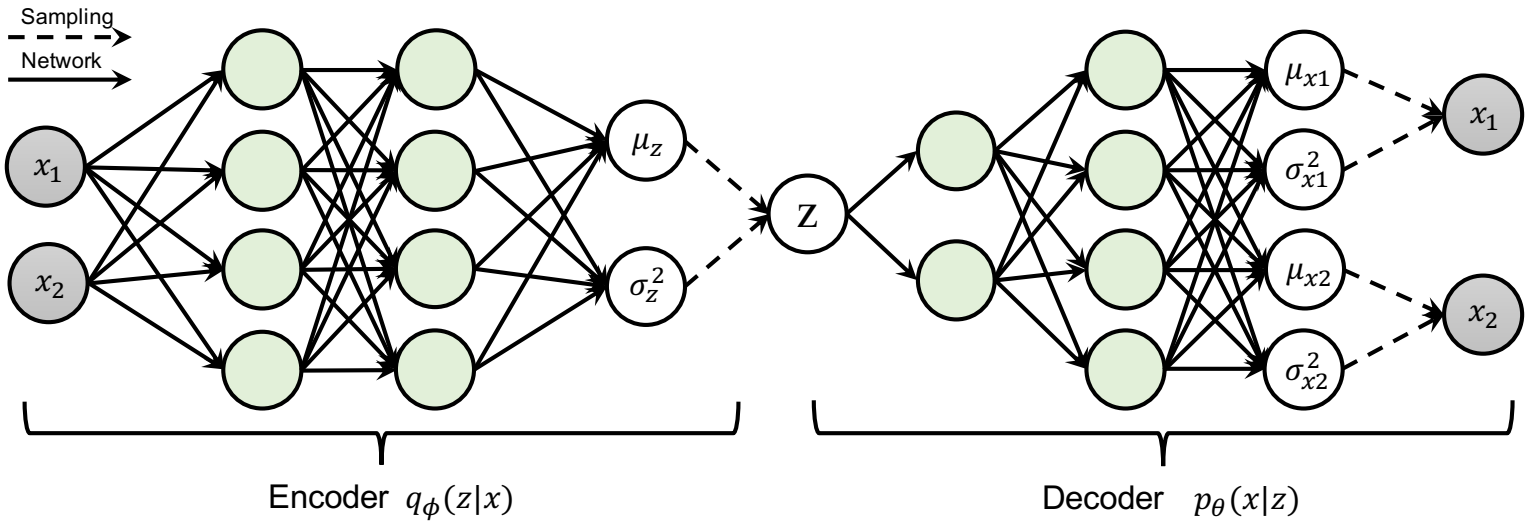
- Problem: It is intractable to compute $p(x|z)$ for every z. The posterior distribution $p_\theta(z|x)$ is also intractable.
- Solution: In addition to decoder network $p_\theta(x|z)$, define an additional encoder network $q_\phi(z|x)$ that approximates $p_\theta(z|x)$.
- We will see that this allow us to derive a lower bound on the data likelihood that is tractable, which we can optimize.

Encoder Network

- A feed forward NN + Gaussian. $q_\phi(z|x) = \mathcal{N}(\mu_z(x), \sigma_z(x))$



The Complete Auto-encoder



- Encoder and decoder networks also called "recognition" / "inference" and "generation" networks
- We aim to learn the parameters ϕ and θ via backpropagation.

Training Objective

- Equipped with our encoder and decoder networks, let's work out the data likelihood:

$$
\begin{aligned}
\log p_\theta(x) &= \mathbb{E}_{z\sim q_\phi(z|x)}\left[\log p_\theta(x)\right] \quad (p_\theta(x) \text{ Does not depend on } z) \\
&= \mathbb{E}_z\left[\log \frac{p_\theta(x,z)}{p_\theta(z|x)}\right] \quad \text{(Bayes's Rule)} \\
&= \mathbb{E}_z\left[\log \frac{p_\theta(z,x)}{q_\phi(z|x)}\frac{q_\phi(z|x)}{p_\theta(z|x)}\right] \quad \text{(Multiply by constant)} \\
&= \mathbb{E}_z\left[\log \frac{p_\theta(z,x)}{q_\phi(z|x)}\right] + \mathbb{E}_z\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right] \quad \text{(Logarithms)} \\
&= L(\theta,\phi,x) + \underbrace{D_{KL}(q_\phi(z|x)||p_\theta(z|x))}_{\geq 0} \\
&\geq \underbrace{L(\theta,\phi,x)}_{\text{variational lower bound}}
\end{aligned}
$$

Variational Lower Bound

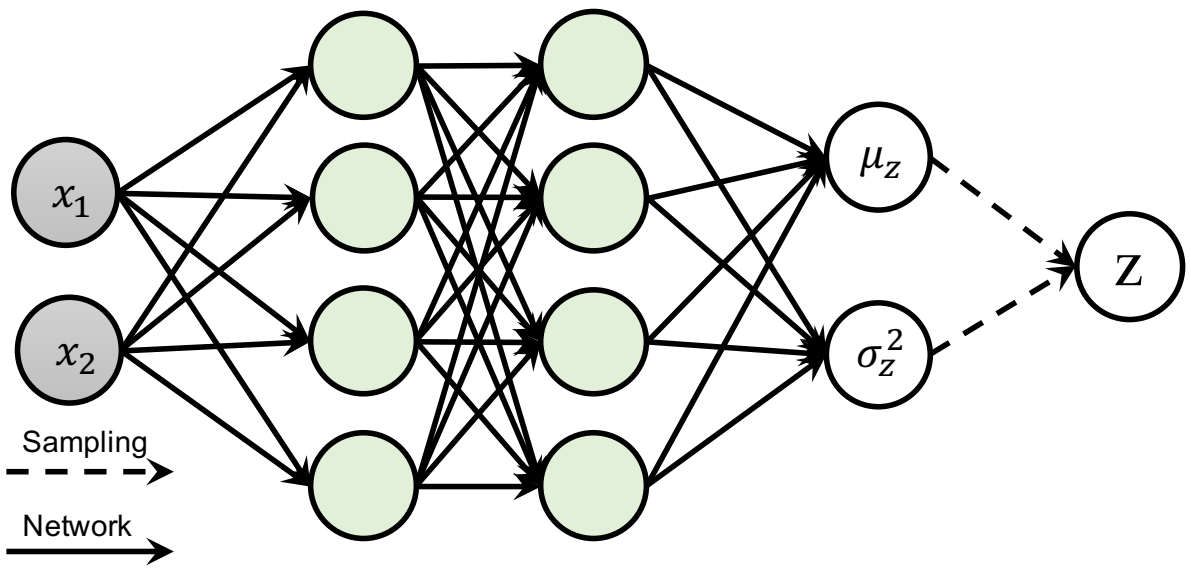
$$
\begin{aligned}
L(\theta, \phi, x) &= \mathbb{E}_z \left[\log \frac{p_\theta(z, x)}{q_\phi(z|x)} \right] \\
&= \mathbb{E}_z \left[\log \frac{p_\theta(x|z) p_\theta(z)}{q_\phi(z|x)} \right] \quad \text{(Bayes's Rule)} \\
&= \mathbb{E}_z \left[\log \frac{p_\theta(z)}{q_\phi(z|x)} \right] + \mathbb{E}_z \left[\log p_\theta(x|z) \right] \quad \text{(Logarithms)} \\
&= \underbrace{-D_{KL}(q_\phi(z|x)||p_\theta(z))}_{\text{Regularization}} + \underbrace{\mathbb{E}_z \left[\log p_\theta(x|z) \right]}_{\text{Reconstruction error}}
\end{aligned}
\tag{19}
$$

- The variational lower bound $L(\theta, \phi, x)$ is tractable.
- Our objective is to maximaze $L(\theta, \phi, x)$.

Example with Gaussian Distribution

- Use $\mathcal{N}(0,1)$ as prior for z; $q_\phi(z|x)$ is Gaussian distribution $\mathcal{N}(\mu_z(x;\phi), \sigma_z^2(x;\phi))$ determined by NN.
  - The KL-divergence:

$$-D_{KL}(q_\phi(z|x)||p_(z)) = \frac{1}{2}(1 + \log \sigma_z^2 - \mu_z^2 - \sigma_z^2) \qquad (20)$$

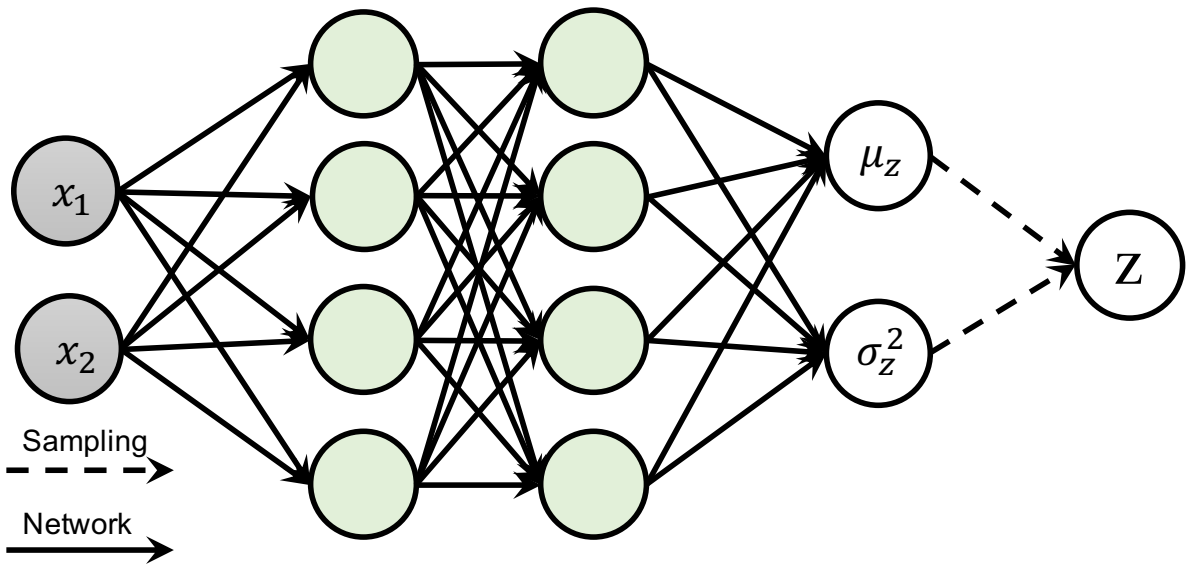


Example with Gaussian Distribution

- Use $\mathcal{N}(0,1)$ as prior for z; $q_\phi(z|x)$ is Gaussian distribution $\mathcal{N}(\mu_z(x;\phi), \sigma_z^2(x;\phi))$ determined by NN.
  - Approximate $\mathbb{E}_{q_\phi(z|x))}(\log(p_\theta(x|z)))$ via Monte Carlo methods:

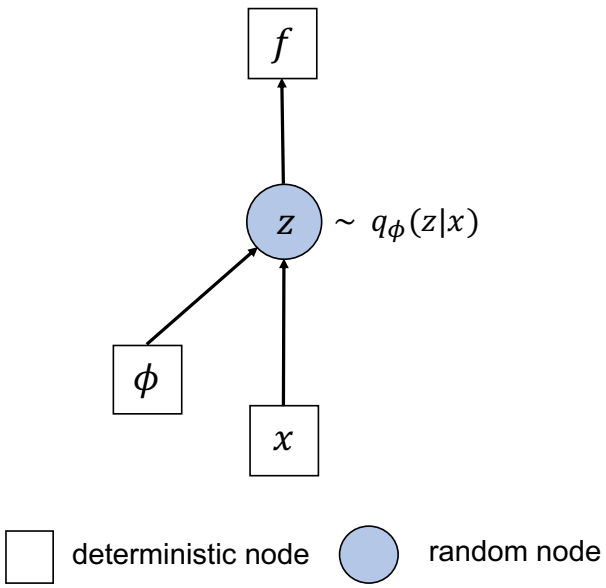
$$\mathbb{E}_z\left[\log p_\theta(x|z)\right] \approx \frac{1}{L}\sum_k^L \log p_\theta(x|z^{(k)}), \quad z^{(k)} \sim q_\phi(z|x) \quad (21)$$

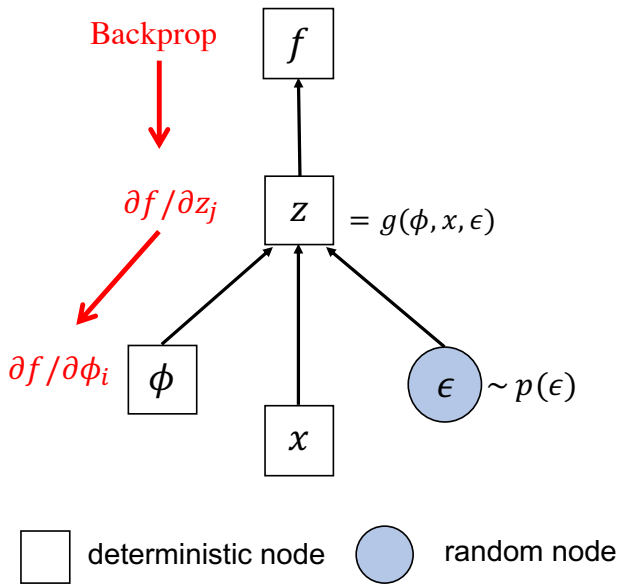
  - However, $z^{(k)}$ is a random variable, we can't take gradient over a random drawn number.



Reparameterization Trick

- Backpropagation not possible through random sampling.





$$z^{(k)} \sim \mathcal{N}(\mu_z(x;\phi), \sigma_z^2(x;\phi))$$

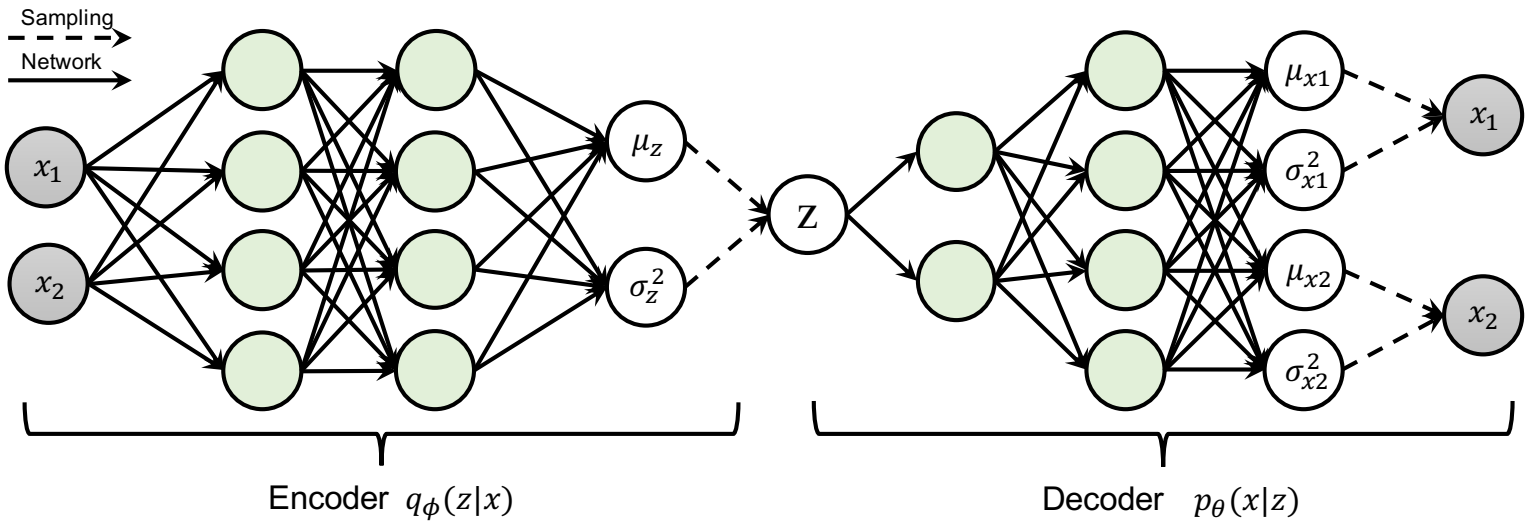
Cannot back-propagate through a randomly drawn number

$$\epsilon^{(k)} \sim \mathcal{N}(0,1)$$

$$z^{(k)} = \mu_z(x,\phi) + \sigma_z(x,\phi) \cdot \epsilon^{(k)}$$

z has the same distribution, but now can back-propagate.

VAE Summary



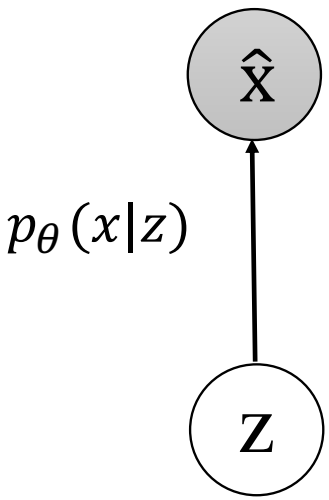
- Loss function:

$$L(\theta, \phi, x) \approx \frac{1}{2} \sum_{j}^{J} (1 + \log \sigma_{zj}^2 - \mu_{zj}^2 - \sigma_{zj}^2) + \frac{1}{L} \sum_{k}^{L} \log(p_\theta(x|z^{(k)})) \tag{22}$$

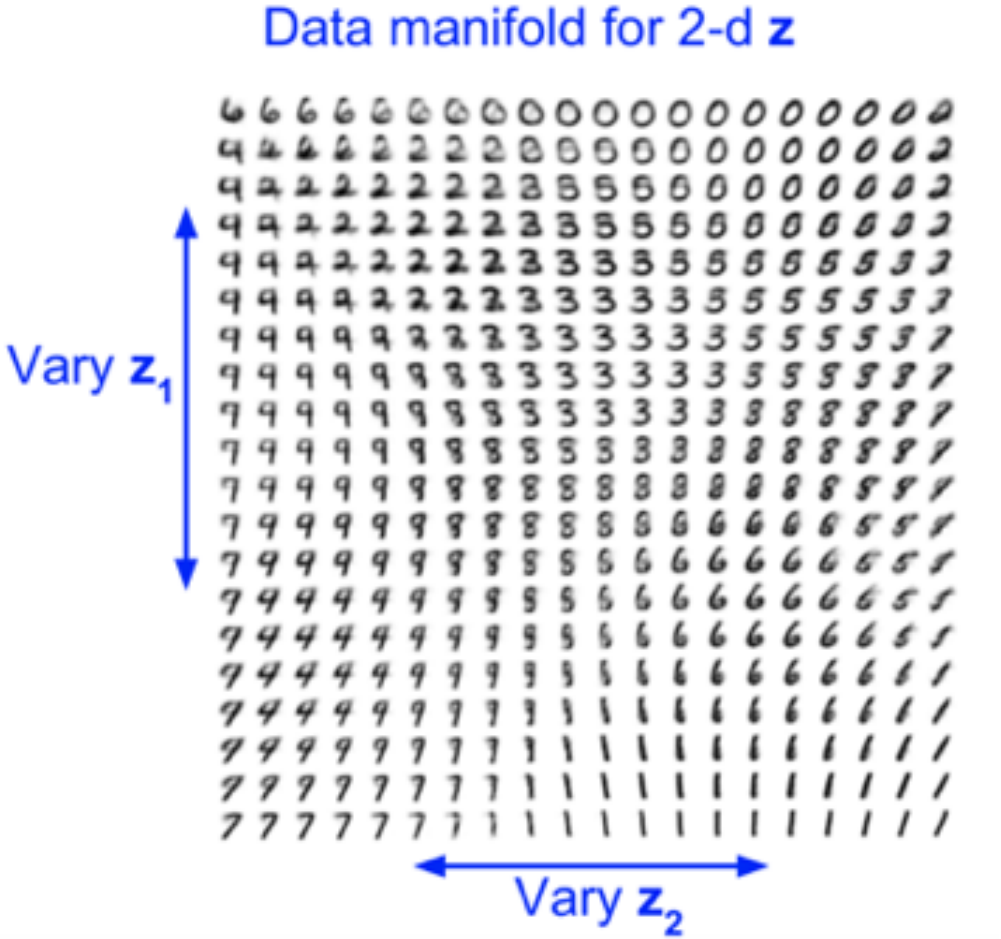
where $z^{(k)} = \mu_z(x, \phi) + \sigma_z(x, \phi) \cdot \epsilon^{(k)}$ and $\epsilon^{(k)} \sim \mathcal{N}(0, 1)$.

VAE: Generating New Data

- Use decoder network, sample z from prior!

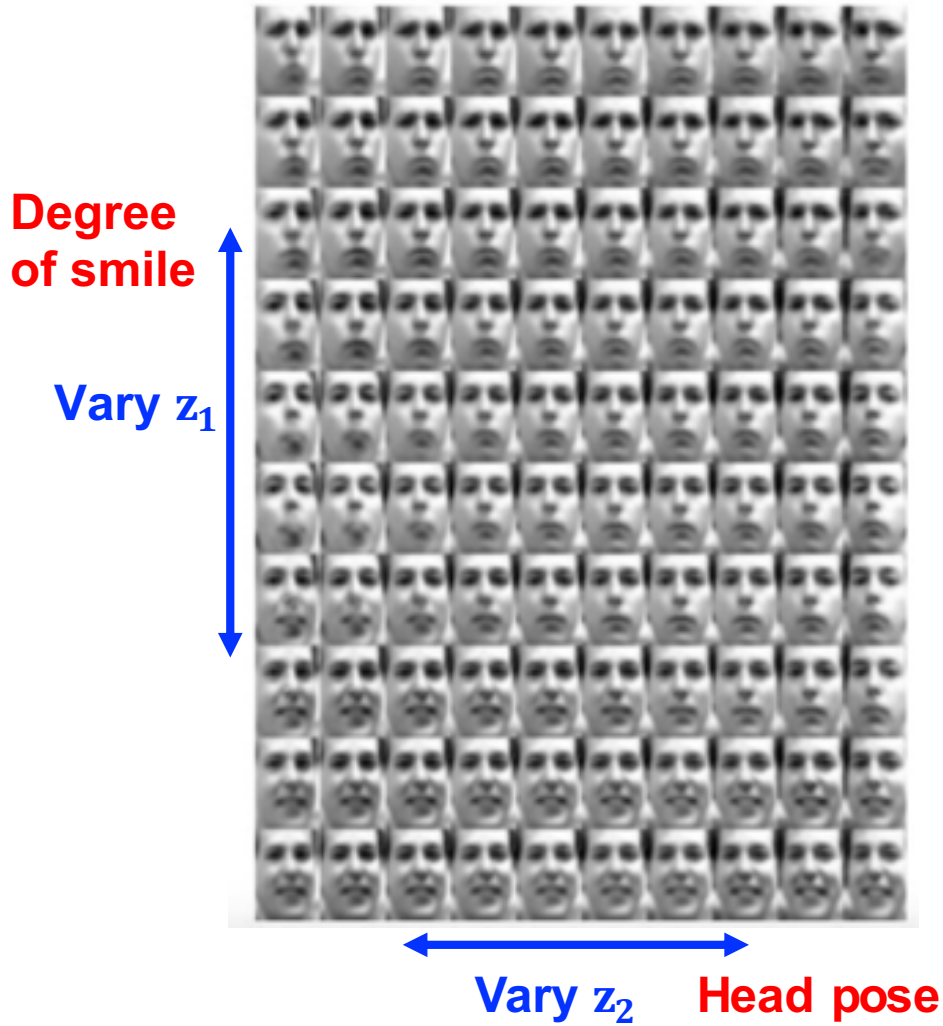


Sample z from $z \sim \mathcal{N}(0, I)$



VAE: Generating new data

- Diagonal prior on z $\Rightarrow$ independent latent variables
- Different dimensions of z encode interpretable factors of variation



Summary

- Successfully applied to dimensionality reduction and information retrieval tasks
- Add a data-dependent penalty to trainings
  - sparse autoencoder
  - denoising autoencoder
  - contractive autoencoder
- An efficient algorithm for initializing the deep neural network
  - stacked autoencoder
- Substituting fully connected layers with convolutional layers
  - convolutional autoencoder
- Add prior distribution assumption to representation
  - variational autoencoder

Thanks.

HP: http://keg.cs.tsinghua.edu.cn/jietang/
Email: jietang@tsinghua.edu.cn