$\LaTeX$ command declarations here.

In [1]:

```python
from __future__ import division

# scientific
%matplotlib inline
from matplotlib import pyplot as plt;
import numpy as np;

# ipython
import IPython;

# python
import os;

#####################################################

# image processing
import PIL;

# trim and scale images
def trim(im, percent=100):
    print("trim:", percent);
    bg = PIL.Image.new(im.mode, im.size, im.getpixel((0,0)))
    diff = PIL.ImageChops.difference(im, bg)
    diff = PIL.ImageChops.add(diff, diff, 2.0, -100)
    bbox = diff.getbbox()
    if bbox:
        x = im.crop(bbox)
        return x.resize(((x.size[0]*percent)//100, (x.size[1]*pe
rcent)//100), PIL.Image.ANTIALIAS);


#####################################################

# daft (rendering PGMs)
import daft;

# set to FALSE to load PGMs from static images
RENDER_PGMS = False;

# decorator for pgm rendering
def pgm_render(pgm_func):
    def render_func(path, percent=100, render=None, *args, **kwa
rgs):
        print("render_func:", percent);
        # render
        render = render if (render is not None) else RENDER_PGM
S;

        if render:
            print("rendering");
            # render
            pgm = pgm_func(*args, **kwargs);
            pgm.render();
```

```
            pgm.figure.savefig(path, dpi=300);

            # trim
            img = trim(PIL.Image.open(path), percent);
            img.save(path, 'PNG');
        else:
            print("not rendering");

        # error
        if not os.path.isfile(path):
            raise "Error:  Graphical model image %s not found.
You may need to set RENDER_PGMS=True.";

        # display
        return IPython.display.Image(filename=path);#trim(PIL.Im
age.open(path), percent);

    return render_func;

######################################################
```

EECS 545: Machine Learning

Lecture 14: Exponential Families & Bayesian Networks

- Instructor: **Jacob Abernethy**
- Date: March 9, 2016

Lecture Exposition Credit: Benjamin Bray & Valliappa Chockalingam

References

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- **[Koller & Friedman 2009]** Koller, Daphne and Nir Friedman. *Probabilistic Graphical Models* (https://mitpress.mit.edu/books/probabilistic-graphical-models). 2009.
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- **[Blei 2011]** Blei, David. *Notes on Exponential Families* (https://www.cs.princeton.edu/courses/archive/fall11/cos597C/lectures/exponential-families.pdf). 2011.
- **[Jordan 2010]** Jordan, Michael I.. *The Exponential Family: Basics* (http://www.cs.berkeley.edu/~jordan/courses/260-spring10/other-readings/chapter8.pdf). 2008.

Outline

- Exponential Families
  - Sufficient Statistics & Pitman-Koopman-Darmois Theorem
  - Mean and natural parameters
  - Maximum Likelihood estimation
- Probabilistic Graphical Models
  - Directed Models (Bayesian Networks)
  - Conditional Independence & Factorization
  - Examples

Exponential Families

> Uses material from **[MLAPP]** §9.2 and **[Hero 2008]** §3.5, §4.4.2

Exponential Family: Introduction

We have seen many distributions.

- Bernoulli
- Gaussian
- Exponential
- Gamma

Many of these belong to a more general class called the **exponential family**.

Exponential Family: Introduction

Why do we care?

- only family of distributions with finite-dimensional **sufficient statistics**
- only family of distributions for which **conjugate priors** exist
- makes the least set of assumptions subject to some user-chosen constraints (**Maximum Entropy**)
- core of generalized linear models and **variational inference**

Sufficient Statistics

Recall: A **statistic** $T(\mathcal{D})$ is a function of the observed data $\mathcal{D}$.

- Mean, $T(x_1, \ldots, x_n) = \frac{1}{n} \sum_{k=1}^n x_k$
- Variance, maximum, mode, etc.

Sufficient Statistics: Definition

Suppose we have a model P with parameters θ. Then,

> A statistic $T(\mathcal{D})$ is **sufficient** for θ if no other statistic calculated from the same sample provides any additional information about the parameter.

That is, if $T(\mathcal{D}_1) = T(\mathcal{D}_2)$, our estimate of θ given $\mathcal{D}_1$ or $\mathcal{D}_2$ will be the same.

- Mathematically, $P(\theta | T(\mathcal{D}), \mathcal{D}) = P(\theta | T(\mathcal{D}))$ independently of $\mathcal{D}$

Sufficient Statistics: Example

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and we observe $\mathcal{D} = (x_1, \ldots, x_n)$. Let

- $\hat{\mu}$ be the sample mean
- $\hat{\sigma}^2$ be the sample variance

Then $T(\mathcal{D}) = (\hat{\mu}, \hat{\sigma}^2)$ is sufficient for $\theta = (\mu, \sigma^2)$.

- Two samples $\mathcal{D}_1$ and $\mathcal{D}_2$ with the same mean and variance give the same estimate of θ

(we are sweeping some details under the rug)

Exponential Family: Definition

$p(x|\theta)$ has **exponential family form** if:

$$\begin{aligned} p(x|\theta) &= \frac{1}{Z(\theta)} h(x) \exp\left[\eta(\theta)^T \phi(x)\right] \\ &= h(x) \exp\left[\eta(\theta)^T \phi(x) - A(\theta)\right] \end{aligned}$$

- $Z(\theta)$ is the **partition function** for normalization
- $A(\theta) = \log Z(\theta)$ is the **log partition function**
- $\phi(x) \in \mathbb{R}^d$ is a vector of **sufficient statistics**
- $\eta(\theta)$ maps θ to a set of **natural parameters**
- $h(x)$ is a scaling constant, usually $h(x) = 1$

Example: Bernoulli

The Bernoulli distribution can be written as

$$\begin{aligned} \mathrm{Ber}(x|\mu) &= \mu^x (1-\mu)^{1-x} \\ &= \exp[x \log \mu + (1-x)\log(1-\mu)] \\ &= \exp\left[\eta(\mu)^T \phi(x)\right] \end{aligned}$$

where $\eta(\mu) = (\log \mu, \log(1-\mu))$ and $\phi(x) = (x, 1-x)$

- There is a linear dependence between features $\phi(x)$
- This representation is **overcomplete**
- η is not uniquely determined

Example: Bernoulli

Instead, we can find a **minimal** parameterization:

$$\mathrm{Ber}(x|\mu) = (1-\mu) \exp\left[x \log \frac{\mu}{1-\mu}\right]$$

This gives **natural parameters** $\eta = \log \frac{\mu}{1-\mu}$.

- Now, η is unique

Other Examples

Exponential Family Distributions:

- Multivariate normal
- Exponential
- Dirichlet

Non-examples:

- Student t-distribution can't be written in exponential form
- Uniform distribution support depends on the parameters θ

Log-Partition Function

Derivatives of the **log-partition function** $A(\theta)$ yield **cumulants** of the sufficient statistics *(Exercise!)*

- $\nabla_\theta A(\theta) = E[\phi(x)]$
- $\nabla_\theta^2 A(\theta) = \mathrm{Cov}[\phi(x)]$

This guarantees that $A(\theta)$ is convex!

- Its Hessian is the covariance matrix of X, which is positive-definite.
- Later, this will guarantee a unique global maximum of the likelihood!

Proof of Convexity: First Derivative

$$\begin{aligned}
\frac{dA}{d\theta} &= \frac{d}{d\theta}\left[\log \int exp(\theta\phi(x))h(x)dx\right] \\
&= \frac{\frac{d}{d\theta}\int exp(\theta\phi(x))h(x)dx)}{\int exp(\theta\phi(x))h(x)dx)} \\
&= \frac{\int \phi(x) exp(\theta\phi(x))h(x)dx}{exp(A(\theta))} \\
&= \int \phi(x)\exp[\theta\phi(x) - A(\theta)]h(x)dx \\
&= \int \phi(x)p(x)dx \\
&= E[\phi(x)]
\end{aligned}$$

Proof of Convexity: Second Derivative

$$\begin{aligned}\frac{d^2A}{d\theta^2} &= \int \phi(x)\exp[\theta\phi(x) - A(\theta)]h(x)(\phi(x) - A'(\theta))dx \\ &= \int \phi(x)p(x)(\phi(x) - A'(\theta))dx \\ &= \int \phi^2(x)p(X)dx - A'(\theta)\int \phi(x)p(x)dx \\ &= E[\phi^2(x)] - E[\phi(x)]^2 \qquad (\because A'(\theta) = E[\phi(x)]) \\ &= Var[\phi(x)]\end{aligned}$$

Proof of Convexity: Second Derivative

For multi-variate case, we have

$$\frac{\partial^2 A}{\partial\theta_i \partial\theta_j} = E[\phi_i(x)\phi_j(x)] - E[\phi_i(x)]E[\phi_j(x)]$$

and hence,

$$\nabla^2 A(\theta) = Cov[\phi(x)]$$

Since covariance is positive definite, we have $A(\theta)$ convex as required.

Exponential Family: Likelihood

For data $\mathcal{D} = (x_1, \ldots, x_N)$, the likelihood is

$$p(\mathcal{D}|\theta) = \left[\prod_{k=1}^N h(x_k)\right] Z(\theta)^{-N} \exp\left[\eta(\theta)^T \left(\sum_{k=1}^N \phi(x_k)\right)\right]$$

The sufficient statistics are now $\phi(\mathcal{D}) = \sum_{k=1}^N \phi(x)$.

- **Bernoulli:** $\phi = \#Heads$
- **Normal:** $\phi = [\sum_k x_k, \sum_k x_k^2]$

Pitman-Koopman-Darmois Theorem

> Among families of distributions $P(x|\theta)$ whose support does not vary with the parameter θ, only in exponential families is there a sufficient statistic $T(x_1, \ldots, x_N)$ whose dimension remains bounded as the sample size N increases.

Exponential Family: MLE

For natural parameters θ and data $\mathcal{D} = (x_1, \ldots, x_N)$,

$$\log p(\mathcal{D}|\theta) = \theta^T \phi(\mathcal{D}) - NA(\theta)$$

Since $-A(\theta)$ is concave and $\theta^T \phi(\mathcal{D})$ linear,

- the log-likelihood is concave
- there is a unique global maximum!

Exponential Family: MLE

To find the maximum, recall $\nabla_\theta A(\theta) = E_\theta[\phi(x)]$, so \begin{align} *\nabla\theta \log p(\D | \theta) & = \nabla\theta(\theta^T \phi(\D) - N A(\theta)) \ & = \phi(\D) - N E_\theta[\phi(X)] = 0 \end{align}* Which gives

$$E_\theta[\phi(X)] = \frac{\phi(\mathcal{D})}{N} = \frac{1}{N}\sum_{k=1}^{N} \phi(x_k)$$

At the MLE $\hat{\theta}_{MLE}$, the empirical average of sufficient statistics equals their expected value.

- this is called **moment matching**

Exponential Family: MLE

As an example, consider the Bernoulli distribution

- Sufficient statistic N, $\phi(\mathcal{D}) = \#Heads$

$$\hat{\mu}_{MLE} = \frac{\#Heads}{N}$$

Bayes for Exponential Family

Exact Bayesian analysis is considerably simplified if the prior is **conjugate** to the likelihood.

- Simply, this means that prior $p(\theta)$ has the same form as the posterior $p(\theta|\mathcal{D})$.

This requires likelihood to have finite sufficient statistics

- Exponential family to the rescue!

Note: We will release some notes on cojugate priors + exponential families. It's hard to learn from slides and needs a bit more description.

Likelihood for exponential family

Likelihood:

$$p(\mathcal{D}|\theta) \propto g(\theta)^N \exp[\eta(\theta)^T s_N]$$
$$s_N = \sum_{i=1}^N \phi(x_i)$$

In terms of canonical parameters:

$$p(\mathcal{D}|\eta) \propto \exp[N\eta^T \bar{s} - NA(\eta)]$$
$$\bar{s} = \frac{1}{N} s_N$$

Conjugate prior for exponential family

- The prior and posterior for an exponential family involve two parameters, τ and ν, initially set to τ_0, ν_0

$$p(\theta|\nu_0, \tau_0) \propto g(\theta)^{\nu_0} \exp[\eta(\theta)^T \tau_0]$$

- Denote $\tau_0 = \nu_0 \bar{\tau}_0$ to separate out the size of the **prior pseudo-data**, ν_0 , from the mean of the sufficient statistics on this pseudo-data, τ_0 . Hence,

$$p(\theta|\nu_0, \bar{\tau}_0) \propto \exp[\nu_0 \eta^T \bar{\tau}_0 - \nu_0 A(\eta)]$$

- Think of τ_0 as a "guess" of the future sufficient statistics, and ν_0 as the strength of this guess

Prior: Example

$$p(\theta|\nu_0, \tau_0) \propto (1-\theta)^{\nu_0} \exp[\tau_0 \log(\frac{\theta}{1-\theta})]$$
$$= \theta^{\tau_0}(1-\theta)^{\nu_0 - \tau_0}$$

Define $\alpha = \tau_0 + 1$ and $\beta = \nu_0 - \tau_0 + 1$ to see that this is a **beta distribution**.

Posterior

Posterior:

$$p(\theta|\mathcal{D}) = p(\theta|\nu_N, \tau_N) = p(\theta|\nu_0 + N, \tau_0 + s_N)$$

Note that we obtain **hyper-parameters** by adding. Hence,

$$p(\eta|\mathcal{D}) \propto \exp[\eta^T(\nu_0\bar{\tau}_0 + N\bar{s}) - (\nu_0 + N)A(\eta)]$$
$$= p(\eta|\nu_0 + N, \frac{\nu_0\bar{\tau}_0 + N\bar{s}}{\nu_0 + N})$$

where $\bar{s} = \frac{1}{N}\sum_{i=1}^{N}\phi(x_i)$.

- *posterior hyper-parameters are a convex combination of the prior mean hyper-parameters and the average of the sufficient statistics.*

Break time!

Probabilistic Graphical Models

> Uses material from **[MLAPP]** §10.1, 10.2 and **[Koller & Friedman 2009]**.

> "I basically know of two principles for treating complicated systems in simple ways: the first is the principle of modularity and the second is the principle of abstraction. I am an apologist for computational probability in machine learning because I believe that probability theory implements these two principles in deep and intriguing ways — nameley through factorization and through averaging. Exploiting these two mechanisms as fully as possible seems to me to be the way forward in machine learning" – Michael Jordan (qtd. in MLAPP)

Graphical Models: Motivation

Suppose we observe multiple correlated variables $x = (x_1, \ldots, x_n)$.

- Words in a document
- Pixels in an image

How can we compactly represent the **joint distribution** $p(x|\theta)$?

- How can we tractably *infer* one set of variables given another?
- How can we efficiently *learn* the parameters?

Joint Probability Tables

One (bad) choice is to write down a **Joint Probability Table**.

- For n binary variables, we must specify $2^n - 1$ probabilities!
- Expensive to store and manipulate
- Impossible to learn so many parameters
- Very hard to interpret!

> Can we be more concise?

Motivating Example: Coin Flips

What is the joint distribution of three independent coin flips?

- Explicitly specifying the JPT requires $2^3 - 1 = 7$ parameters.

Assuming independence, $P(X_1, X_2, X_3) = P(X_1)P(X_2)P(X_3)$

- Each marginal $P(X_k)$ only requires one parameter, the bias
- This gives a total of 3 parameters, compared to 8.

> Exploiting the **independence structure** of a joint distribution leads to more concise representations.

Motivating Example: Naive Bayes

In Naive Bayes, we assumed the features $X_1, \dots, X_N$ were independent given the class label C:

$$P(x_1, \dots, x_N, c) = P(c) \prod_{k=1}^{N} P(x_k | c)$$

This greatly simplified the learning procedure:

- Allowed us to look at each feature individually
- Only need to learn $O(CN)$ probabilities, for C classes and N features

Conditional Independence

The key to efficiently representing large joint distributions is to make **conditional independence** assumptions of the form

$$X \perp Y \mid Z \iff p(X, Y | Z) = p(X|Z)p(Y|Z)$$

> Once z is known, information about x does not tell us any information about y and vice versa.

An effective way to represent these assumptions is with a **graph**.

Bayesian Networks: Definition

A **Bayesian Network** $\mathcal{G}$ is a directed acyclic graph whose nodes represent random variables $X_1, \dots, X_n$.

- Let $\text{Parents}_{\mathcal{G}}(X_k)$ denote the parents of X_k in $\mathcal{G}$
- Let $\text{NonDesc}_{\mathcal{G}}(X_k)$ denote the variables in $\mathcal{G}$ who are not descendants of X_k.

> Examples will come shortly...

Bayesian Networks: Local Independencies

Every Bayesian Network $\mathcal{G}$ encodes a set $\mathcal{I}_\ell(\mathcal{G})$ of **local independence assumptions**:

> For each variable X_k, we have $(X_k \perp \text{NonDesc}_{\mathcal{G}}(X_k) \mid \text{Parents}_{\mathcal{G}}(X_k))$

Every node X_k is conditionally independent of its nondescendants given its parents.

Example: Naive Bayes

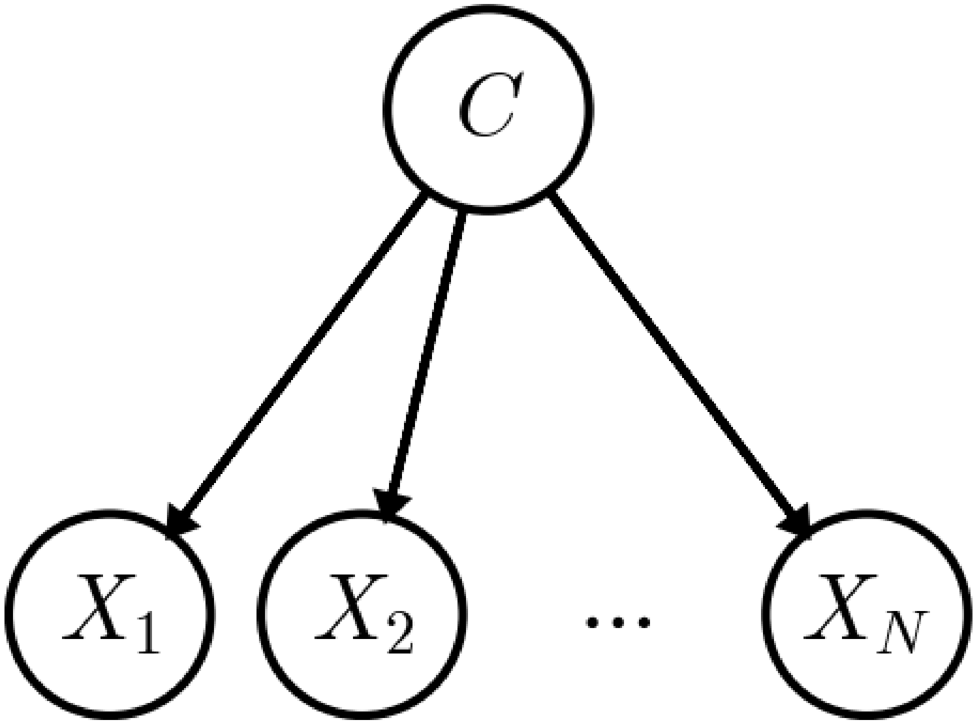
The graphical model for Naive Bayes is shown below:

- $\text{Parents}_{\mathcal{G}}(X_k) = \{C\}$, $\text{NonDesc}_{\mathcal{G}}(X_k) = \{X_j\}_{j\neq k}$
- Therefore $X_j \perp X_k \mid C$ for any $j \neq k$

In [2]:
```
@pgm_render
def pgm_naive_bayes():
    pgm = daft.PGM([4,3], origin=[-2,0], node_unit=0.8, grid_uni
t=2.0);
    # nodes
    pgm.add_node(daft.Node("c", r"$C$", -0.25, 2));
    pgm.add_node(daft.Node("x1", r"$X_1$", -1, 1));
    pgm.add_node(daft.Node("x2", r"$X_2$", -0.5, 1));
    pgm.add_node(daft.Node("dots", r"$\cdots$",  0, 1, plot_para
ms={ 'ec' : 'none' }));
    pgm.add_node(daft.Node("xN", r"$X_N$",  0.5, 1));

    # edges
    pgm.add_edge("c", "x1", head_length=0.08);
    pgm.add_edge("c", "x2", head_length=0.08);
    pgm.add_edge("c", "xN", head_length=0.08);

    return pgm;
```

In [3]:
```
%%capture
pgm_naive_bayes("images/naive-bayes.png");
```

Out[3]:



Subtle Point: Graphs & Distributions

A Bayesian network $\mathcal{G}$ over variables $X_1, \dots, X_N$ encodes a set of **conditional independencies**.

- Shows independence structure, nothing more.
- Does **not** tell us how to assign probabilities to a configuration $(x_1, \dots x_N)$ of the variables.

There are **many** distributions P satisfying the independencies in $\mathcal{G}$.

- Many joint distributions share a common structure, which we exploit in algorithms.
- The distribution P may satisfy other independencies **not** encoded in $\mathcal{G}$.

Subtle Point: Graphs & Distributions

If P satisfies the independence assertions made by $\mathcal{G}$, we say that

- $\mathcal{G}$ is an **I-Map** for P
- or that P **satisfies** $\mathcal{G}$.

Any distribution satisfying $\mathcal{G}$ shares common structure.

- We will exploit this structure in our algorithms
- This is what makes graphical models so **powerful**!

Review: Chain Rule for Probability

We can factorize any joint distribution via the **Chain Rule for Probability**:

$$\begin{aligned} P(X_1, \dots, X_N) &= P(X_1)P(X_2, \dots, X_N | X_1) \\ &= P(X_1)P(X_2|X_1)P(X_3, \dots, X_N | X_1, X_2) \\ &= \prod_{k=1}^{N} P(X_k | X_1, \dots, X_{k-1}) \end{aligned}$$

> Here, the ordering of variables is arbitrary. This works for any permutation.

Bayesian Networks: Topological Ordering

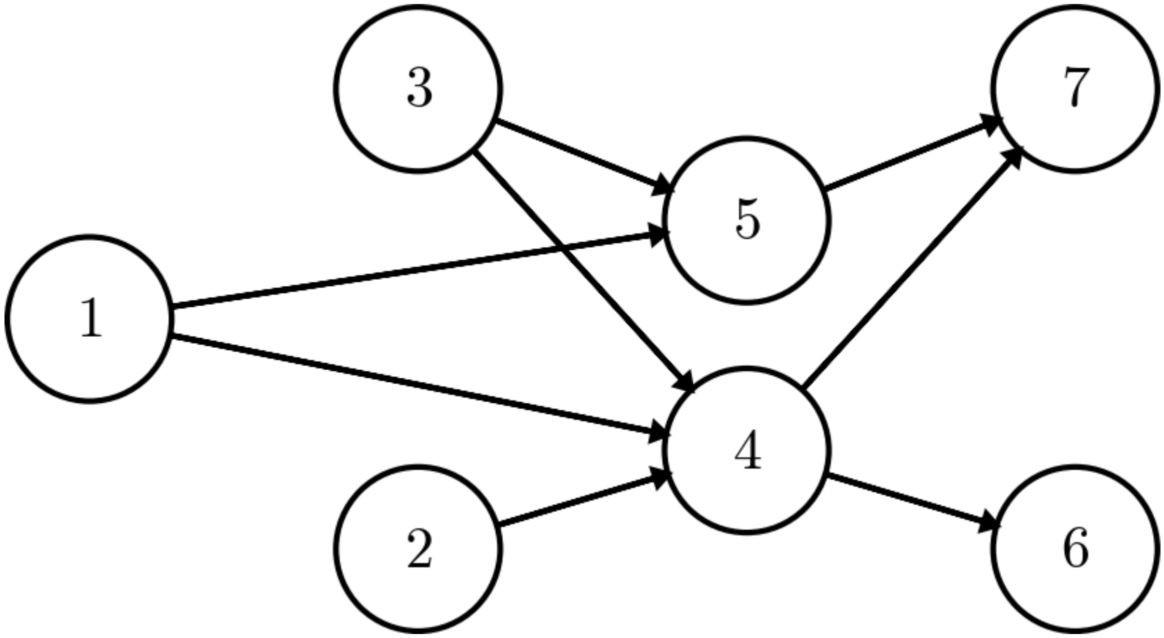
Every network $\mathcal{G}$ induces a **topological (partial) ordering** on its nodes:

> Parents assigned a lower index than their children

In [4]:
```
@pgm_render
def pgm_topological_order():
    pgm = daft.PGM([4, 4], origin=[-4, 0])

    # Nodes
    pgm.add_node(daft.Node("x1", r"$1$", -3.5, 2))
    pgm.add_node(daft.Node("x2", r"$2$", -2.5, 1.3))
    pgm.add_node(daft.Node("x3", r"$3$", -2.5, 2.7))
    pgm.add_node(daft.Node("x4", r"$4$", -1.5, 1.6))
    pgm.add_node(daft.Node("x5", r"$5$", -1.5, 2.3))
    pgm.add_node(daft.Node("x6", r"$6$", -0.5, 1.3))
    pgm.add_node(daft.Node("x7", r"$7$", -0.5, 2.7))

    # Add in the edges.
    pgm.add_edge("x1", "x4", head_length=0.08)
    pgm.add_edge("x1", "x5", head_length=0.08)
    pgm.add_edge("x2", "x4", head_length=0.08)
    pgm.add_edge("x3", "x4", head_length=0.08)
    pgm.add_edge("x3", "x5", head_length=0.08)
    pgm.add_edge("x4", "x6", head_length=0.08)
    pgm.add_edge("x4", "x7", head_length=0.08)
    pgm.add_edge("x5", "x7", head_length=0.08)

    return pgm;
```

In [5]:
```
%%capture
pgm_topological_order("images/topological-order.png")
```

Out[5]:



Factorization Theorem: Statement

Theorem: *(Koller & Friedman 3.1)* If $\mathcal{G}$ is an I-map for P, then P **factorizes** as follows:

$$P(X_1, \dots, X_N) = \prod_{k=1}^{N} P(X_k \mid \mathrm{Parents}_{\mathcal{G}}(X_k))$$

> Let's prove it together!

Factorization Theorem: Proof

First, apply the chain rule to any topological ordering:

$$P(X_1, \dots, X_N) = \prod_{k=1}^{N} P(X_k \mid X_1, \dots, X_{k-1})$$

Consider one of the factors $P(X_k \mid X_1, \dots, X_{k-1})$.

Factorization Theorem: Proof

Since our variables $X_1, \ldots, X_N$ are in topological order,

- $\mathbf{Parents}_\mathcal{G}(X_k) \subseteq \{X_1, \ldots, X_{k-1}\}$
- None of X_k's descendants can possibly lie in $\{X_1, \ldots, X_{k-1}\}$

Therefore, $\{X_1, \ldots, X_{k-1}\} = \mathbf{Parents}_\mathcal{G}(X_k) \cup \mathcal{Z}$

- for some $\mathcal{Z} \subseteq \mathrm{NonDesc}_\mathcal{G}(X_k)$.

Factorization Theorem: Proof

Recall the following property of conditional independence:

$$(X \perp Y, W \mid Z) \implies (X \perp Y \mid Z)$$

Since $\mathcal{G}$ is an I-map for P and $\mathcal{Z} \subseteq \mathrm{NonDesc}_\mathcal{G}(X_k)$, we have

$$\begin{aligned} &(X_k \perp \mathrm{NonDesc}_\mathcal{G}(X_k) \mid \mathbf{Parents}_\mathcal{G}(X_k)) \\ \implies &(X_k \perp \mathcal{Z} \mid \mathbf{Parents}_\mathcal{G}(X_k)) \end{aligned}$$

Factorization Theorem: Proof

We have just shown $(X_k \perp \mathcal{Z} \mid \mathbf{Parents}_\mathcal{G}(X_k))$, therefore

$$P(X_k \mid X_1, \ldots, X_{k-1}) = P(X_k \mid \mathbf{Parents}_\mathcal{G}(X_k))$$

- Recall $\{X_1, \ldots, X_N\} = \mathbf{Parents}_\mathcal{G}(X_k) \cup \mathcal{Z}$.

> **Remember:** X_k is conditionally independent of its nondescendants given its parents!

Factorization Theorem: End of Proof

Applying this to every factor, we see that

$$\begin{aligned} P(X_1, \ldots, X_N) &= \prod_{k=1}^{N} P(X_k \mid X_1, \ldots, X_{k-1}) \\ &= \prod_{k=1}^{N} P(X_k \mid \text{Parents}_{\mathcal{G}}(X_k)) \end{aligned}$$

Factorization Theorem: Consequences

We just proved that for any P satisfying $\mathcal{G}$,

$$P(X_1, \ldots, X_N) = \prod_{k=1}^{N} P(X_k \mid \text{Parents}_{\mathcal{G}}(X_k))$$

It suffices to store **conditional probability tables** $P(X_k | \text{Parents}_{\mathcal{G}}(X_k))$!

- Requires $O(N2^k)$ features if each node has $\leq k$ parents
- Substantially more compact than **JPTs** for N large, $\mathcal{G}$ sparse
- We can also specify that a CPD is Gaussian, Dirichlet, etc.

Example: Fully Connected Graph

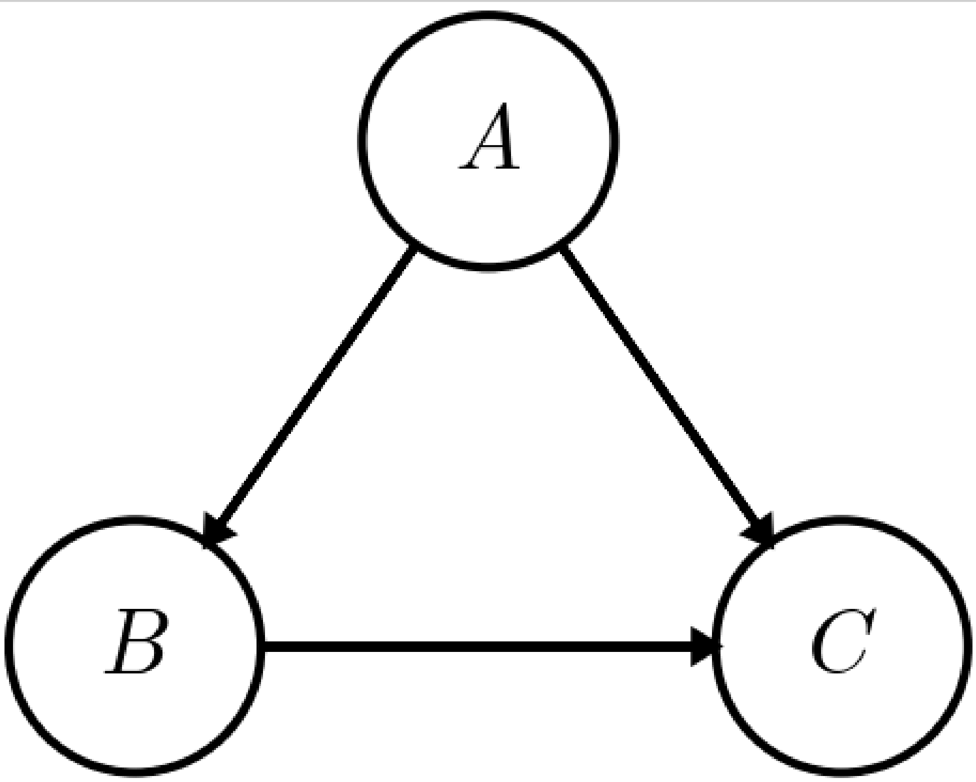
A **fully connected graph** makes no independence assumptions.

$$P(A, B, C) = P(A)P(B|A)P(C|A, B)$$

In [6]:
```
@pgm_render
def pgm_fully_connected_a():
    pgm = daft.PGM([4, 4], origin=[0, 0])

    # nodes
    pgm.add_node(daft.Node("a", r"$A$", 2, 3.5))
    pgm.add_node(daft.Node("b", r"$B$", 1.3, 2.5))
    pgm.add_node(daft.Node("c", r"$C$", 2.7, 2.5))

    # add in the edges
    pgm.add_edge("a", "b", head_length=0.08)
    pgm.add_edge("a", "c", head_length=0.08)
    pgm.add_edge("b", "c", head_length=0.08)

    return pgm;
```

In [7]:
```
%%capture
pgm_fully_connected_a("images/fully-connected-a.png")
```

Out[7]:



Example: Fully Connected Graph

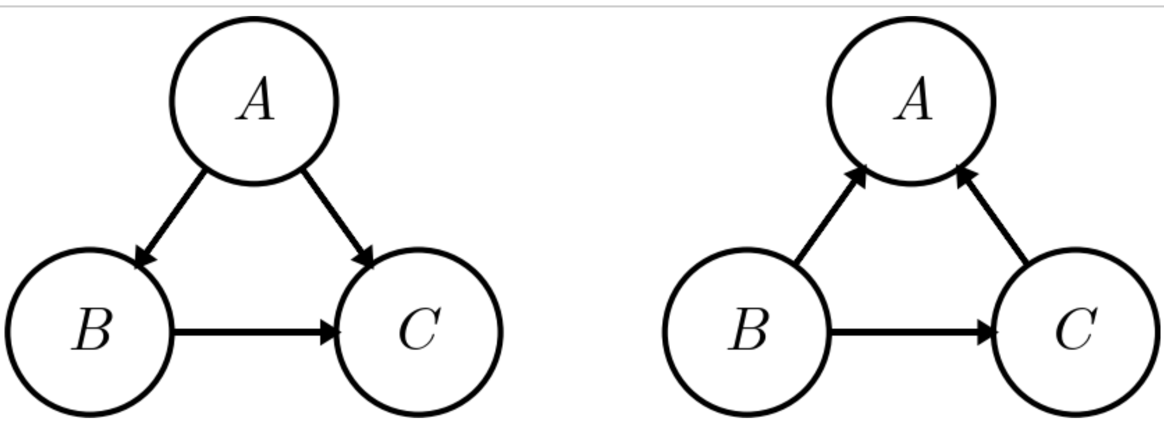
There are many possible fully connected graphs:

$$\begin{aligned} P(A, B, C) &= P(A)P(B|A)P(C|A, B) \\ &= P(B)P(C|B)P(A|B, C) \end{aligned}$$

In [8]:
```
@pgm_render
def pgm_fully_connected_b():
    pgm = daft.PGM([8, 4], origin=[0, 0])

    # nodes
    pgm.add_node(daft.Node("a1", r"$A$", 2, 3.5))
    pgm.add_node(daft.Node("b1", r"$B$", 1.5, 2.8))
    pgm.add_node(daft.Node("c1", r"$C$", 2.5, 2.8))

    # add in the edges
    pgm.add_edge("a1", "b1", head_length=0.08)
    pgm.add_edge("a1", "c1", head_length=0.08)
    pgm.add_edge("b1", "c1", head_length=0.08)

    # nodes
    pgm.add_node(daft.Node("a2", r"$A$", 4, 3.5))
    pgm.add_node(daft.Node("b2", r"$B$", 3.5, 2.8))
    pgm.add_node(daft.Node("c2", r"$C$", 4.5, 2.8))

    # add in the edges
    pgm.add_edge("b2", "c2", head_length=0.08)
    pgm.add_edge("b2", "a2", head_length=0.08)
    pgm.add_edge("c2", "a2", head_length=0.08)

    return pgm;
```

In [9]:
```
%%capture
pgm_fully_connected_b("images/fully-connected-b.png")
```

Out[9]:



Bayesian Networks & Causality

The fully-connected example brings up a crucial point:

> Directed edges do **not** necessarily represent causality.

Bayesian networks encode **independence assumptions** only.

- This representation is not unique.

Example: Markov Chain

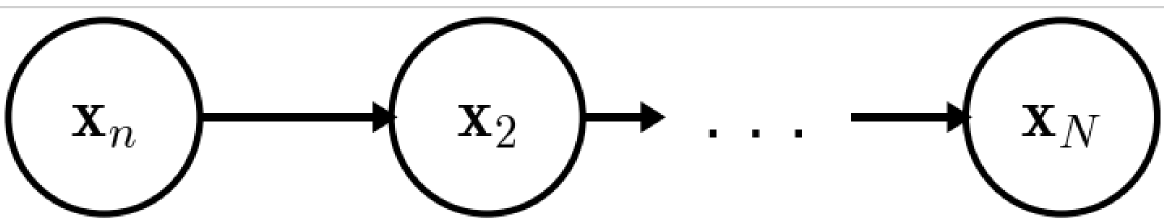
State at time t depends only on state at time $t-1$.

$$P(X_0, X_1, \ldots, X_N) = P(X_0) \prod_{t=1}^{N} P(X_t \mid X_{t-1})$$

In [10]:
```
@pgm_render
def pgm_markov_chain():
    pgm = daft.PGM([6, 6], origin=[0, 0])

    # Nodes
    pgm.add_node(daft.Node("x1", r"$\mathbf{x}_n$", 2, 2.5))
    pgm.add_node(daft.Node("x2", r"$\mathbf{x}_2$", 3, 2.5))
    pgm.add_node(daft.Node("ellipsis", r" . . . ", 3.7, 2.5, off
set=(0, 0), plot_params={"ec" : "none"}))
    pgm.add_node(daft.Node("ellipsis_end", r"", 3.7, 2.5, offset
=(0, 0), plot_params={"ec" : "none"}))
    pgm.add_node(daft.Node("xN", r"$\mathbf{x}_N$", 4.5, 2.5))

    # Add in the edges.
    pgm.add_edge("x1", "x2", head_length=0.08)
    pgm.add_edge("x2", "ellipsis", head_length=0.08)
    pgm.add_edge("ellipsis_end", "xN", head_length=0.08)

    return pgm;
```

In [11]:
```
%%capture
pgm_markov_chain("images/markov-chain.png")
```

Out[11]:



Example: Hidden Markov Model

Noisy observations X_k generated from hidden Markov chain Y_k.

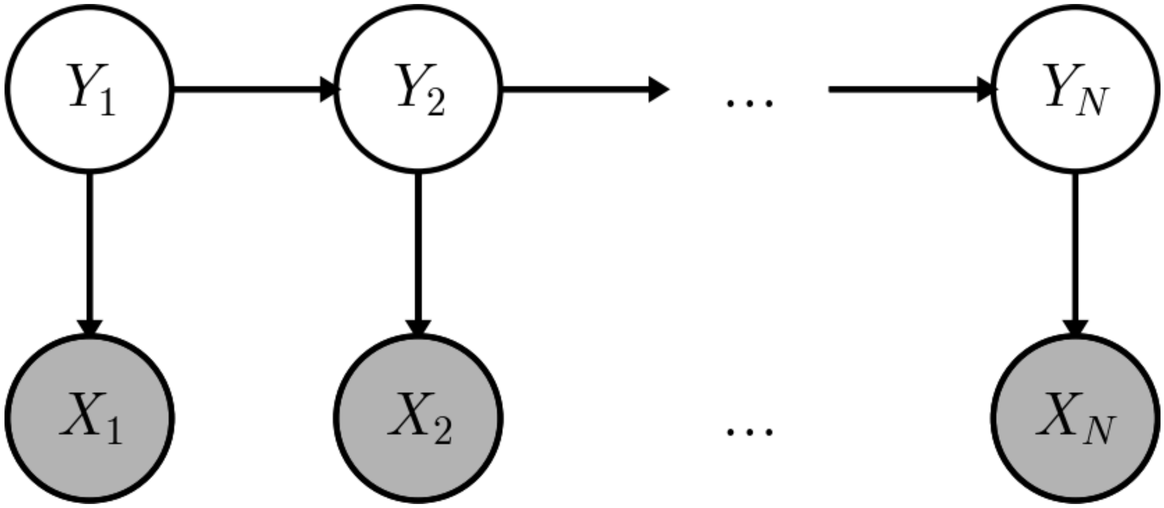
$$P(\mathbf{X}, \mathbf{Y}) = P(Y_1)P(X_1 \mid Y_1)\prod_{k=2}^{N}\left(P(Y_k \mid Y_{k-1})P(X_k \mid Y_k)\right)$$

In [12]:
```
@pgm_render
def pgm_hmm():
    pgm = daft.PGM([7, 7], origin=[0, 0])

    # Nodes
    pgm.add_node(daft.Node("Y1", r"$Y_1$", 1, 3.5))
    pgm.add_node(daft.Node("Y2", r"$Y_2$", 2, 3.5))
    pgm.add_node(daft.Node("Y3", r"$\dots$", 3, 3.5, plot_params
={'ec':'none'}))
    pgm.add_node(daft.Node("Y4", r"$Y_N$", 4, 3.5))

    pgm.add_node(daft.Node("x1", r"$X_1$", 1, 2.5, observed=Tru
e))
    pgm.add_node(daft.Node("x2", r"$X_2$", 2, 2.5, observed=Tru
e))
    pgm.add_node(daft.Node("x3", r"$\dots$", 3, 2.5, plot_params
={'ec':'none'}))
    pgm.add_node(daft.Node("x4", r"$X_N$", 4, 2.5, observed=Tru
e))


    # Add in the edges.
    pgm.add_edge("Y1", "Y2", head_length=0.08)
    pgm.add_edge("Y2", "Y3", head_length=0.08)
    pgm.add_edge("Y3", "Y4", head_length=0.08)

    pgm.add_edge("Y1", "x1", head_length=0.08)
    pgm.add_edge("Y2", "x2", head_length=0.08)
    pgm.add_edge("Y4", "x4", head_length=0.08)

    return pgm;
```

In [13]:

```
%%capture
pgm_hmm("images/hmm.png")
```

Out[13]:



Example: Plate Notation

We can represent (conditionally) iid variables using **plate notation**.

In [14]:
```
@pgm_render
def pgm_plate_example():
    pgm = daft.PGM([4,3], origin=[-2,0], node_unit=0.8, grid_uni
t=2.0);
    # nodes
    pgm.add_node(daft.Node("lambda", r"$\lambda$", -0.25, 2));

    pgm.add_node(daft.Node("t1", r"$\theta_1$", -1, 1.3));
    pgm.add_node(daft.Node("t2", r"$\theta_2$", -0.5, 1.3));
    pgm.add_node(daft.Node("dots1", r"$\cdots$",  0, 1.3, plot_p
arams={ 'ec' : 'none' }));
    pgm.add_node(daft.Node("tN", r"$\theta_N$",  0.5, 1.3));

    pgm.add_node(daft.Node("x1", r"$X_1$", -1, 0.6));
    pgm.add_node(daft.Node("x2", r"$X_2$", -0.5, 0.6));
    pgm.add_node(daft.Node("dots2", r"$\cdots$",  0, 0.6, plot_p
arams={ 'ec' : 'none' }));
    pgm.add_node(daft.Node("xN", r"$X_N$",  0.5, 0.6));


    pgm.add_node(daft.Node("LAMBDA", r"$\lambda$", 1.5, 2));
    pgm.add_node(daft.Node("THETA", r"$\theta_k$", 1.5,1.3));
    pgm.add_node(daft.Node("XX", r"$X_k$", 1.5,0.6));

    # edges
    pgm.add_edge("lambda", "t1", head_length=0.08);
    pgm.add_edge("lambda", "t2", head_length=0.08);
    pgm.add_edge("lambda", "tN", head_length=0.08);
    pgm.add_edge("t1", "x1", head_length=0.08);
    pgm.add_edge("t2", "x2", head_length=0.08);
    pgm.add_edge("tN", "xN", head_length=0.08);


    pgm.add_edge("LAMBDA", "THETA", head_length=0.08);
    pgm.add_edge("THETA", "XX", head_length=0.08);

    pgm.add_plate(daft.Plate([1.1,0.4,0.8,1.2], label=r"$\qquad
\quad\; K$",
    shift=-0.1))

    return pgm;
```

In [15]:
```
%%capture
pgm_plate_example("images/plate-example.png")
```

Out[15]: