In [854]:

```
%matplotlib inline

import imp

try:
    imp.find_module('daft')
except ImportError:
    !pip install --user --upgrade daft

import daft
```

In [855]:

```
from PIL import Image, ImageChops

def trim(im, percent=36):
    bg = Image.new(im.mode, im.size, im.getpixel((0,0)))
    diff = ImageChops.difference(im, bg)
    diff = ImageChops.add(diff, diff, 2.0, -100)
    bbox = diff.getbbox()
    if bbox:
        x = im.crop(bbox)
        return x.resize(((x.size[0]*percent)/100, (x.size[1]*percent)/100))
```

EECS 545: Machine Learning

Lecture 13: Bayesian Networks

- Instructor: **Jacob Abernethy**
- Date: March 9, 2016

Lecture Exposition Credit: Benjamin Bray, Valliappa Chockalingam

Problem Setting

- Many problems in Machine Learning can be described with some number random variables.
- How do we model interactions between them?

Joint Probability Table

- Given a set of random variables $\{x_1, \ldots, x_N\}$, the Joint Probability Table (JPT) $p(x_1, \ldots, x_N)$ lets you answer any probabilistic question that can be asked.
- Can we express a JPT any more concisely?

Decomposing the JPT using the Chain Rule

By the chain rule of probability, we can also represent a joint distribution as follows:

$$\begin{aligned} p(\mathbf{x}_{1:N}) &= p(x_1) \cdot p(x_2 \mid x_1) \cdot p(x_3 \mid x_2, x_1) \cdot \ldots \cdot p(x_N \mid \mathbf{x}_{1:N-1}) \\ &= \prod_{n=1}^{N} p(x_n \mid \mathbf{x}_{1:n-1}) \end{aligned}$$

where N is the number of variables and $\mathbf{x} = \{x_1, \ldots, x_N\}$.

Problem?

Even with decomposition, **The expression can become very large!**

- As n gets large, $p(x_n \mid \mathbf{x}_{1:n-1})$ becomes more and more complicated to represent.
- Suppose all the variables can take K values, then
  - $p(x_1)$ has $O(K)$ parameters,
  - $p(x_2 \mid x_1)$ has $O(K^2)$ parameters, and so on giving,
  - $O(K^N)$ parameters in the model.

A Solution: Use a graphical representation

Before explaining the solution, a brief review of graph theory and related terms is presented.

- Definition: A graph $G = (\nu, \epsilon)$ consists of
  - A set of nodes or vertices, $V = \{\nu_1, \cdots, \nu_n\}$, and,
  - A set of edges or links, $\epsilon = \{\epsilon_1, \ldots, \epsilon_m\} \subseteq V \times V$ where each ϵ_i is thus an ordered pair.

Edge Related Terminology

- A node u is said to be adjacent to another node v if there is an edge from u to v or from v to u.
- If there is an edge *from u to v* then u is often called the tail, while v is called the head.

Sparsity

- A fully connected graph of n nodes contains edges between every pair of nodes for a total of $n + (n-1) + (n-2) + \ldots + 1 = \frac{n(n-1)}{2}$ edges.
- A graph is said to be sparse if the number of edges $m \ll \frac{n(n-1)}{2}$.
- A graph is said to be dense if the number of edges $m \approx \frac{n(n-1)}{2}$.

Directedness

- A graph is said to be undirected if $\forall s, t, G(s,t) = 1 \leftrightarrow G(t,s) = 1$.
- Otherwise it is said to be directed.

Representations: Adjacency Matrices

- We often use an adjacency **matrix** to represent graphs: $G(t,s) = 1$ denotes $(s,t) \in \epsilon$.
  - Space Complexity of Adjacency Matrices: $O(n^2)$ where n is the number of nodes.

Representations: Adjacency List

- For sparse graphs, more compact representations exist such as Adjacency lists: Mapping of nodes to nodes they have an edge to (or from depending on the convention being used).
  - Space Complexity of Adjacency Lists: $O(m+n)$ where m is the number of edges and n is the number of nodes.

Cyclicity

- A graph is said to be cyclic if there exists a sequence of vertices that can be traversed using edges of the graph such that the sequence's first and last nodes are the same.
- If a graph is not cyclic, it is said to be acyclic.
  - Note: By definition, an acyclic graph is not reflexive (has no edges from nodes to themselves).

The Graphical Representation

- Let each node represent a random variable.
- Link each variable to those it is conditionally dependent on.
  - ***Directed Acyclic Graph*** : Directed Edges, No Loops

Review: Independence

- Random Variables x and y are said to be ***independent*** if $\boxed{p(x, y) = p(x) \cdot p(y)}$
- Knowing x doesn't tell us any information about y and vice versa.

Review: Conditional Independence

- Random Variables x and y are said to be ***conditionally independent*** given z if $\boxed{p(x, y \mid z) = p(x \mid z) \cdot p(y \mid z)}$
- Once z is known, information about x does not tell us any information about y and vice versa.

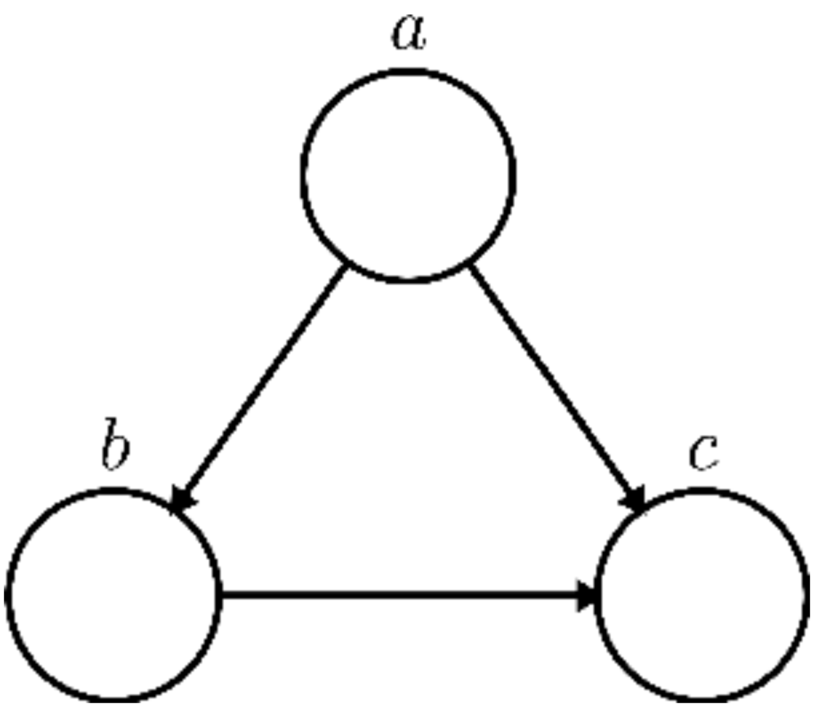
In [856]:

```
%%capture
### Example

pgm = daft.PGM([4, 4], origin=[0, 0])



# Nodes
pgm.add_node(daft.Node("a", r"$a$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("c", r"$c$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("a", "b", plot_params={'head_length' : 0.08})
pgm.add_edge("a", "c", plot_params={'head_length' : 0.08})
pgm.add_edge("b", "c", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("Example.png", dpi=1000)
```

In [857]:

```
### An Example

trim(Image.open('Example.png'), 20)

# p(a, b, c) = p(c | a, b) x p(a, b) = p(c | a, b) x p(b | a) x p(a)
```

Out[857]:



Missing Links

- A ***Fully Connected Graph*** represents the fully general Joint Probability Table.
- In most domains, some variables will be independent or conditionally independent of others. So, some links will be missing.
  - The main advantage of the graphical representation.

Building a Bayesian Network

- Specify the variables $\mathbf{x} = \{x_1, \ldots, x_N\}$.
- Identify what each variable depends on:
  - x_n depends on it's parent variables $pa_n \subseteq \{x_1, \ldots, x_{n-1}\}$
  - Observation: Given pa_n, x_n and $\{x_1, \ldots, x_{n-1}\} \setminus pa_n$ are conditionally independent.

Building a Bayesian Network

- Add links to x_n from its parent.
  - These are often causal relations in the domain.
- Annotate x_n with $p(x_n \mid pa_n)$.
  - $p(x_n \mid pa_n)$ is called a Conditional Probability Table (CPT).
- Finally, the Joint Probability Table (JPT) is then given by $p(\mathbf{x}) = \prod_{n=1}^{N} p(x_n \mid pa_n)$. This is sometimes refered to as the General Factorization for Bayesian Networks.

Exercise

Construct a Bayesian Network to represent the Joint Probability Distribution:

$$
\begin{aligned}
p(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \; & p(x_1) \cdot p(x_2) \cdot p(x_3) \cdot \\
& p(x_4 \mid x_1, x_2, x_3) \cdot p(x_5 \mid x_1, x_3) \cdot \\
& p(x_6 \mid x_4) \cdot p(x_7 \mid x_4, x_5)
\end{aligned}
$$

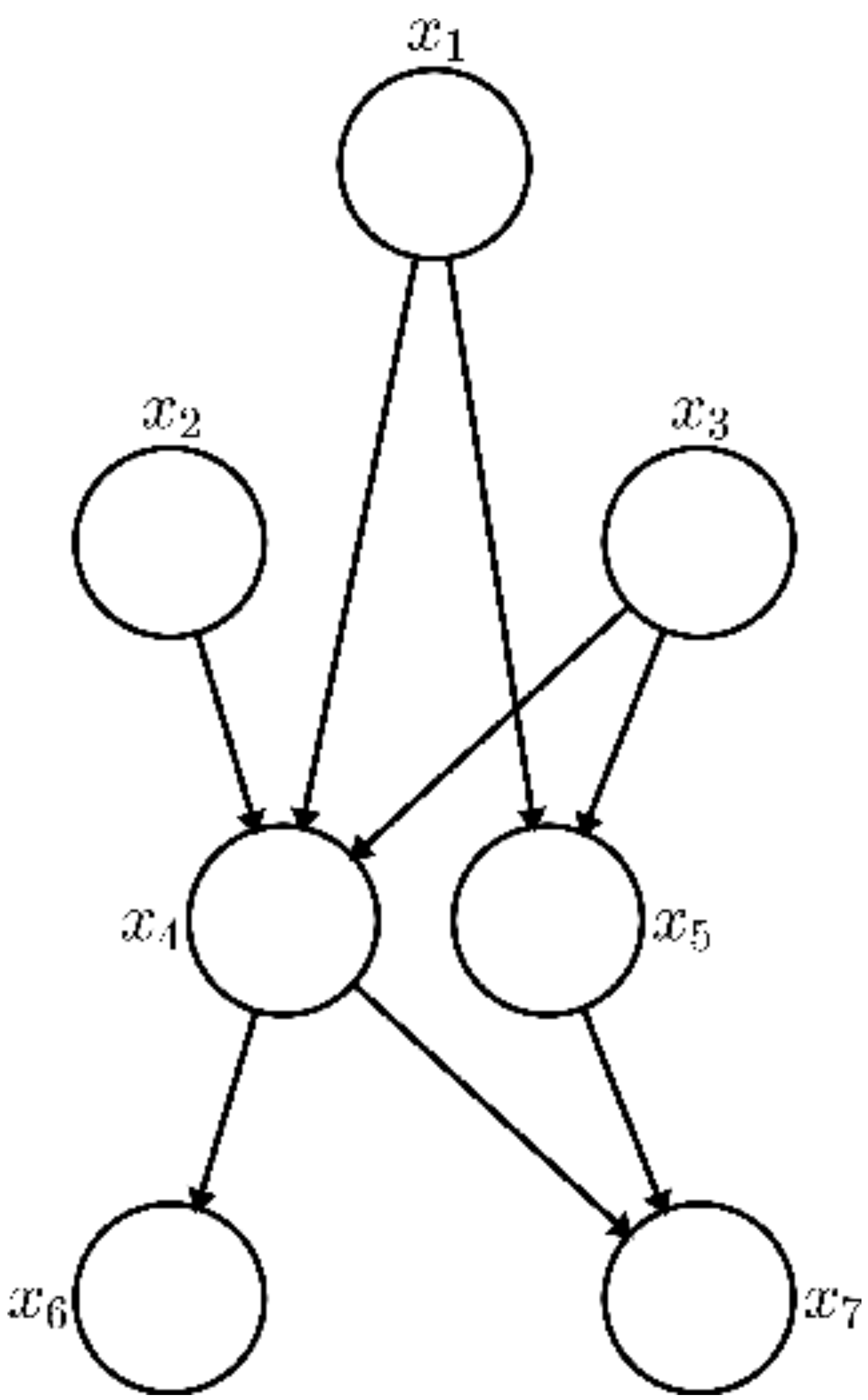
In [858]:

```
%%capture
# Answer to Exercise

pgm = daft.PGM([4, 4], origin=[0, 0])


# Nodes
pgm.add_node(daft.Node("x1", r"$x_1$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("x2", r"$x_2$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("x3", r"$x_3$", 2.7, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("x4", r"$x_4$", 1.6, 1.5, offset=(-20, 0)))
pgm.add_node(daft.Node("x5", r"$x_5$", 2.3, 1.5, offset=(20, 0)))
pgm.add_node(daft.Node("x6", r"$x_6$", 1.3, 0.5, offset=(-20, 0)))
pgm.add_node(daft.Node("x7", r"$x_7$", 2.7, 0.5, offset=(20, 0)))

# Add in the edges.
pgm.add_edge("x1", "x4", plot_params={'head_length' : 0.08})
pgm.add_edge("x1", "x5", plot_params={'head_length' : 0.08})
pgm.add_edge("x2", "x4", plot_params={'head_length' : 0.08})
pgm.add_edge("x3", "x4", plot_params={'head_length' : 0.08})
pgm.add_edge("x3", "x5", plot_params={'head_length' : 0.08})
pgm.add_edge("x4", "x6", plot_params={'head_length' : 0.08})
pgm.add_edge("x4", "x7", plot_params={'head_length' : 0.08})
pgm.add_edge("x5", "x7", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("Exercise.png", dpi=1000)
```

In [859]:

```
# Answer
trim(Image.open('Exercise.png'), 18)
```

Out[859]:



Bayesian Curve-Fitting

- Bayesian Regression problem.
- Random variables are the:
  - Vector of polynimal coefficients, $\mathbf{w}$, and the
  - Vector of observed data, $\mathbf{t} = (t_1, \ldots, t_N)^T$.
- Polynomial: $y(x, \mathbf{w}) = \sum_{j=0}^{M} w_j x^j$.
- Bayes Theorem Formulation: $p(\mathbf{t}, \mathbf{w}) = p(\mathbf{w}) \prod_{n=1}^{N} p(t_n \mid y(\mathbf{w}, x_n))$

Plate Models

- We can use random variables t_i and a weight vector $\mathbf{w}$.
- $\mathbf{w}$ is a parent of each of the t_i variables.
- This is an ideal scenario for using plate models.
    - **Purpose of plate models:** Compactly represent multiple iterated nodes.

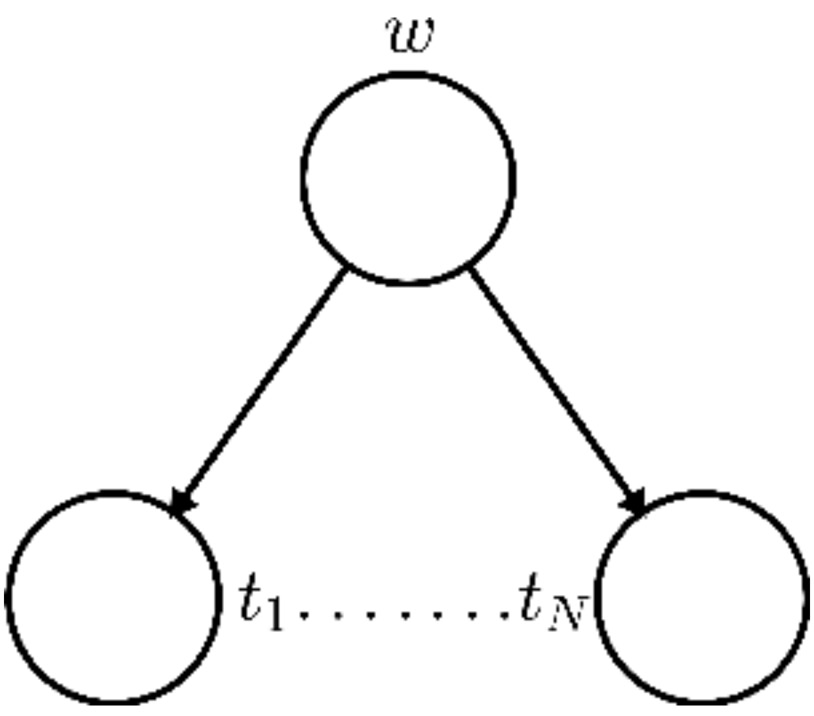
In [860]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("w", r"$w$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("t1", r"$t_1 . . . . . . . . $", 1.3, 2.5, offset=(35, 0)))
pgm.add_node(daft.Node("t2", r"$t_N$", 2.7, 2.5, offset=(-20, 0)))

# Add in the edges.
pgm.add_edge("w", "t1", plot_params={'head_length' : 0.08})
pgm.add_edge("w", "t2", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("Poly_ellipsis.png", dpi=1000)
```

In [861]:

```
trim(Image.open('Poly_ellipsis.png'), 20)
```

Out[861]:



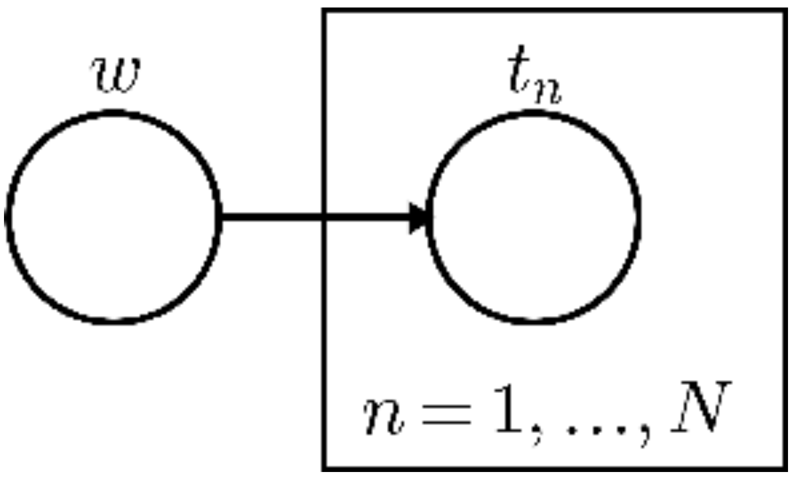
In [862]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("w", r"$w$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("t_n", r"$t_n$", 3, 3.5, offset=(0, 20)))

pgm.add_plate(daft.Plate([2.5, 3, 1.1, 1], label=r"$n = 1, \ldots, N$",
    shift=-0.1))

# Add in the edges.
pgm.add_edge("w", "t_n", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("Poly_plate.png", dpi=1000, frameon=False)
```

In [863]:

```
# Plate Example
trim(Image.open('Poly_plate.png'), 20)
```

Out[863]:



$$\text{Model: } p(\mathbf{T}, \mathbf{w}) = p(\mathbf{w}) \prod_{n=1}^{N} p(t_n \mid \mathbf{w})$$

Constant Parameters and Inputs?

- Open circles represent random variables.
- Small filled circles represent constant parameters such as input, variance, and hyperparameters.

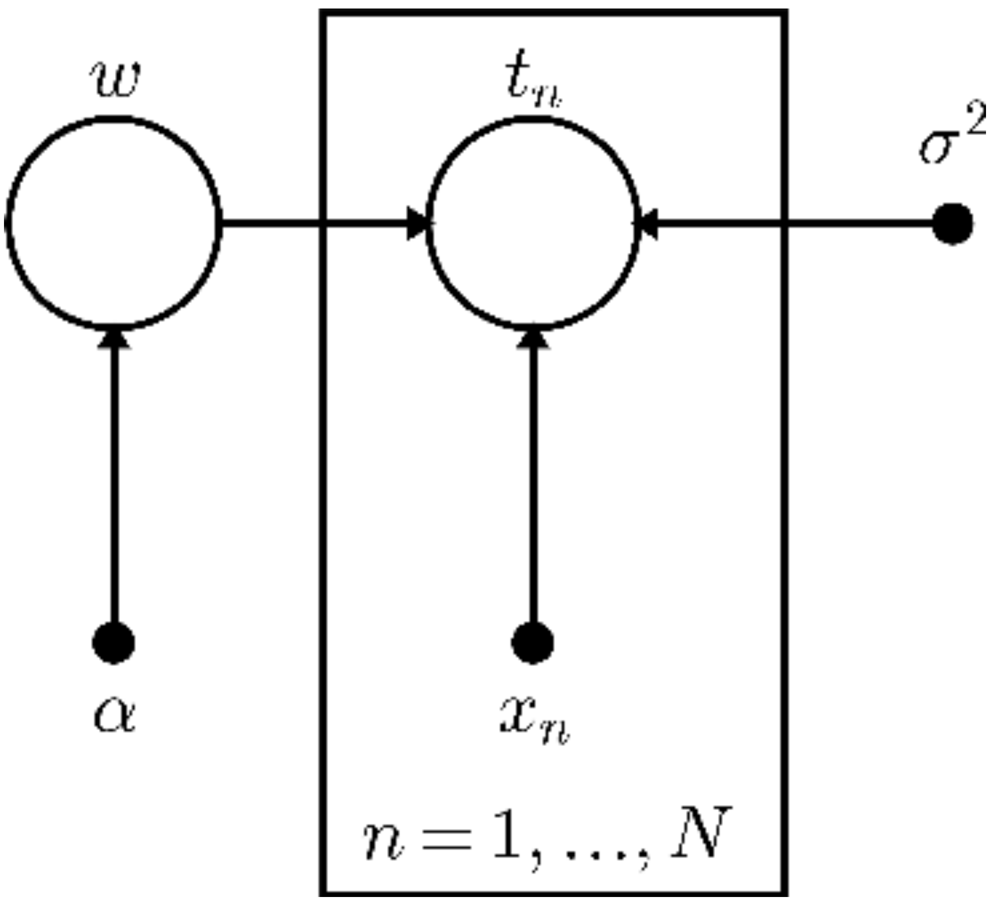
In [864]:

```
%%capture
pgm = daft.PGM([5, 5], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("w", r"$w$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("t_n", r"$t_n$", 3, 3.5, offset=(0, 20)))

pgm.add_node(daft.Node("x_n", r"$x_n$", 3, 2.5, offset=(0, -20), fixed=True))
pgm.add_node(daft.Node("var", r"$\sigma^2$", 4, 3.5, fixed=True))
pgm.add_node(daft.Node("alpha", r"$\alpha$", 2, 2.5, offset=(0, -20), fixed=True
))

pgm.add_plate(daft.Plate([2.5, 2, 1.1, 2], label=r"$n = 1, \ldots, N$",
    shift=-0.1))

# Add in the edges.
pgm.add_edge("w", "t_n", plot_params={'head_length' : 0.08})
pgm.add_edge("alpha", "w", plot_params={'head_length' : 0.08})
pgm.add_edge("x_n", "t_n", plot_params={'head_length' : 0.08})
pgm.add_edge("var", "t_n", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("Poly_with_parameters.png", dpi=1000)
```

In [865]:

```
trim(Image.open('Poly_with_parameters.png'), 20)
```

Out[865]:



$$\text{Model: } p(\mathbf{T}, \mathbf{w} \mid \mathbf{x}, \alpha, \sigma^2) = p(\mathbf{w} \mid \alpha) \prod_{n=1}^{N} p(t_n \mid \mathbf{w}, x_n, \sigma^2)$$

Learning

Shaded Circles represent random variables with observed values.

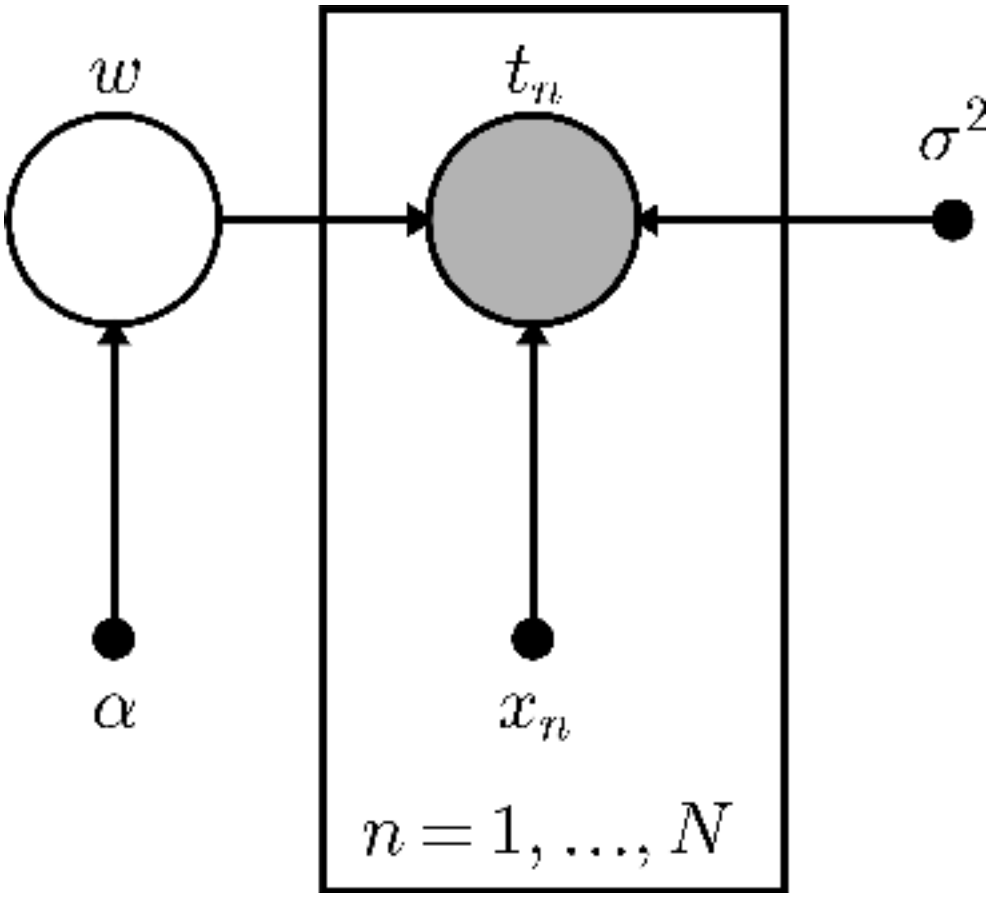
In [866]:

```
%%capture
pgm = daft.PGM([5, 5], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("w", r"$w$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("t_n", r"$t_n$", 3, 3.5, offset=(0, 20), observed=True))

pgm.add_node(daft.Node("x_n", r"$x_n$", 3, 2.5, offset=(0, -20), fixed=True))
pgm.add_node(daft.Node("var", r"$\sigma^2$", 4, 3.5, fixed=True))
pgm.add_node(daft.Node("alpha", r"$\alpha$", 2, 2.5, offset=(0, -20), fixed=True
))

pgm.add_plate(daft.Plate([2.5, 2, 1.1, 2], label=r"$n = 1, \ldots, N$",
    shift=-0.1))

# Add in the edges.
pgm.add_edge("w", "t_n", plot_params={'head_length' : 0.08})
pgm.add_edge("alpha", "w", plot_params={'head_length' : 0.08})
pgm.add_edge("x_n", "t_n", plot_params={'head_length' : 0.08})
pgm.add_edge("var", "t_n", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("Poly_after_observation.png", dpi=1000)
```

In [867]:

```
trim(Image.open('Poly_after_observation.png'), 20)
```

Out[867]:



$$\text{Note: } p(\mathbf{w} \mid \mathbf{T}) \propto p(\mathbf{w}) \prod_{n=1}^{N} p(t_n \mid \mathbf{w})$$

Prediction

- Add new variables to represent a query and the estimated target value.
- We are interested in the probability distribution $p(\hat{t})$ for a new input value $\hat{x}$ conditioned on the observed data.

Prediction

- The joint distribution can then be formulated as
$$p(\hat{t}, \mathbf{T}, \mathbf{w} \mid \hat{x}, \mathbf{x}, \alpha, \sigma^2) =$$
$$\left[\prod_{n=1}^{N} p(t_n \mid x_n, \mathbf{w}, \sigma^2)\right] p(\mathbf{w} \mid \alpha) p(\hat{t} \mid \hat{x}, \mathbf{w}, \sigma^2)$$
- So the predictive distribution for $\hat{x}$ is given by
$$p(\hat{t} \mid \hat{x}, \mathbf{x}, \mathbf{T}, \alpha, \sigma^2) \propto \int p(\hat{t}, \mathbf{T}, \mathbf{w} \mid \hat{x}, \mathbf{x}, \alpha, \sigma^2)\ d\mathbf{w}$$

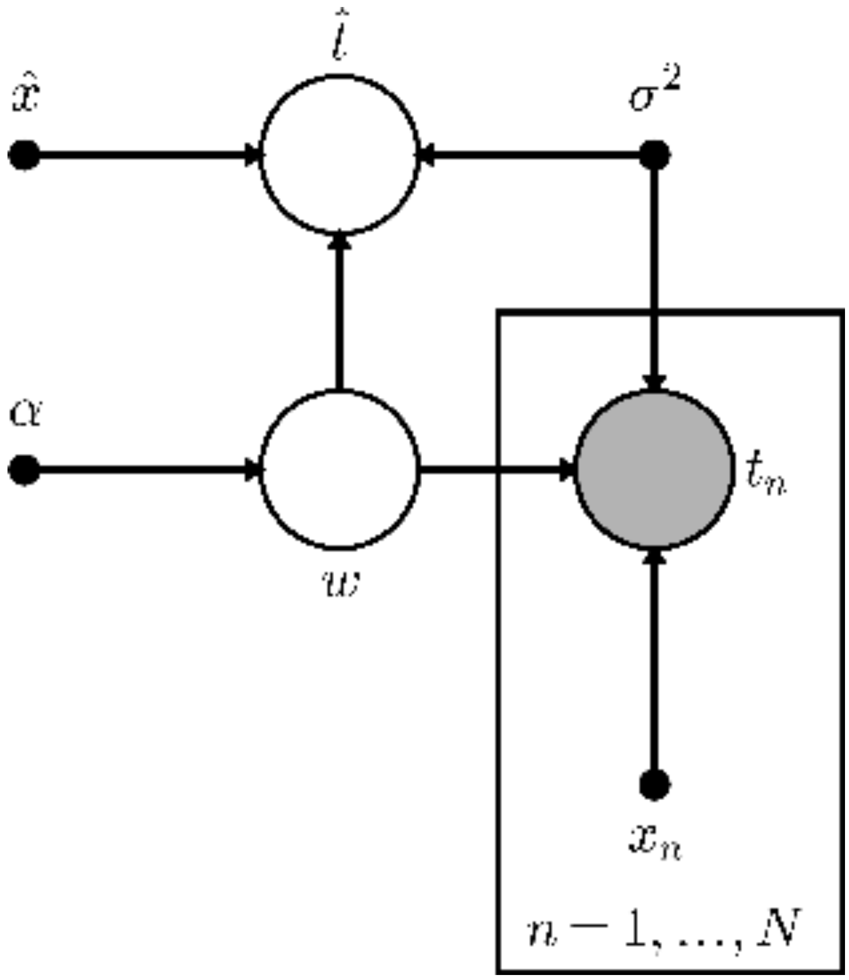
In [868]:

```
%%capture
pgm = daft.PGM([6, 6], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("w", r"$w$", 2, 3.5, offset=(0, -20)))
pgm.add_node(daft.Node("t_n", r"$t_n$", 3, 3.5, offset=(20, 0), observed=True))
pgm.add_node(daft.Node("that", r"$\hat{t}$", 2, 4.5, offset=(0, 20)))

pgm.add_node(daft.Node("x_n", r"$x_n$", 3, 2.5, offset=(0, -20), fixed=True))
pgm.add_node(daft.Node("var", r"$\sigma^2$", 3, 4.5, fixed=True))
pgm.add_node(daft.Node("alpha", r"$\alpha$", 1, 3.5, fixed=True))
pgm.add_node(daft.Node("xhat", r"$\hat{x}$", 1, 4.5, fixed=True))


pgm.add_plate(daft.Plate([2.5, 2, 1.1, 2], label=r"$n = 1, \ldots, N$",
    shift=-0.1))

# Add in the edges.
pgm.add_edge("w", "t_n", plot_params={'head_length' : 0.08})
pgm.add_edge("alpha", "w", plot_params={'head_length' : 0.08})
pgm.add_edge("x_n", "t_n", plot_params={'head_length' : 0.08})
pgm.add_edge("var", "t_n", plot_params={'head_length' : 0.08})
pgm.add_edge("var", "that", plot_params={'head_length' : 0.08})
pgm.add_edge("xhat", "that", plot_params={'head_length' : 0.08})
pgm.add_edge("w", "that", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("Poly_prediction.png", dpi=1000)
```

In [869]:

```
trim(Image.open('Poly_prediction.png'), 15)
```

Out[869]:



Directedness in Bayesian Networks and Sampling the Joint Distribution

- Question: How to sample the JPT $p(x_1, \ldots, x_k)$?
- In $p(x_n \mid pa_n)$, the variables in pa_n come before x_k in the ordering of variables, so we can sample in sequence.

Example

Given $p(a, b, c) = p(c \mid a, b)p(b \mid a)p(a)$

- Draw a from the distribution $p(a)$.
- Given that value of a, draw b from $p(b \mid a)$.
- Given a and b, draw c from $p(c \mid a, b)$.
- (a, b, c) is then drawn from $p(a, b, c)$.

Factorization: Joint Distribution of Discrete Variables

Single Variable Case

- The probability distribution $p(\mathbf{x} \mid \mu)$ for a single discrete variable $\mathbf{x}$ that can take K states is given by $p(\mathbf{x} \mid \mu) = \prod_{k=1}^{K} \mu_k^{x_k}$.
- With the constraint that $\sum_k \mu_k = 1$, only $K - 1$ free parameters are needed
  - Another way of thinking of this is with degrees of freedom.

Factorization: Joint Distribution of Discrete Variables

Two Variable Case

- Consider two discrete random variables $\mathbf{x}_1$ and $\mathbf{x}_2$, each of which has K states.
- Let the parameter μ_{kl} denote the probability $p(\mathbf{x}_1 = k, \mathbf{x}_2 = l)$.
- The joint distribution is then given by $p(\mathbf{x}_1, \mathbf{x}_2 \mid \mu) = \prod_{k=1}^{K} \prod_{l=1}^{K} \mu_{kl}^{x_{1k} x_{2l}}$
- With $\sum_k \sum_l \mu_{kl} = 1$, the distribution is then governed by $K^2 - 1$ parameters.

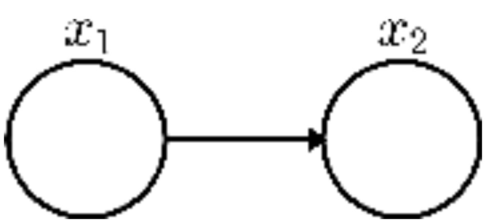
In [870]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("x1", r"$x_1$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("x2", r"$x_2$", 3, 3.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("x1", "x2", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("2-gen.png", dpi=1000)
```

In [871]:

```
trim(Image.open('2-gen.png'), 15)
```

Out[871]:



Factorization: Joint Distribution of Discrete Variables

General Case

Extending on the previous observations, we see that for a joint distribution over N discrete variables, there are $K^N - 1$ states which grows ***exponentially*** in N.

Factorization: Joint Distribution of Independent Discrete Variables - Two Variable Case

- Consider two random variables $\mathbf{x}_1$ and $\mathbf{x}_2$ that are independent and each of which can take K states.
- Each variable can then be described a separate multinomial distribution with $K - 1$ parameters for a total of $2(K - 1)$ parameters.

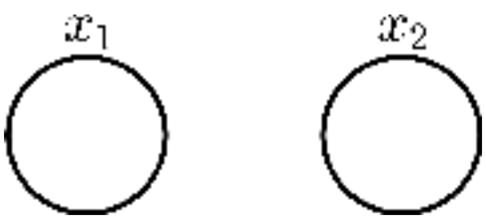
In [872]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("x1", r"$x_1$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("x2", r"$x_2$", 3, 3.5, offset=(0, 20)))

# Render and save.
pgm.render()
pgm.figure.savefig("2-ind.png", dpi=1000)
```

In [873]:

```
trim(Image.open('2-ind.png'), 15)
```

Out[873]:



Factorization: Joint Distribution of Independent Discrete Variables - General Case

- With N independent discrete random variables, we can describe each with a multinomial distribution consisting of $K - 1$ parameters.
- Total number of parameters: $N(K - 1)$
  - Grows ***linearly*** in N.

Observations

- A Fully Connected Graph with N nodes has $K^N - 1$ parameters.
- A Fully Disconected Graph (one without any edges) had $N(K-1)$ parameters.
- Graphs with intermediate levels of connectivity
  - Allow for more general distributions than the fully factorized one, and,
  - Require fewer parameters than the fully connected one.

Conjugate Priors

- Dirichlet Priors can be added to the multinomial random variables.
- Furthermore, a Diriclet prior can be shared by conditional distributions.

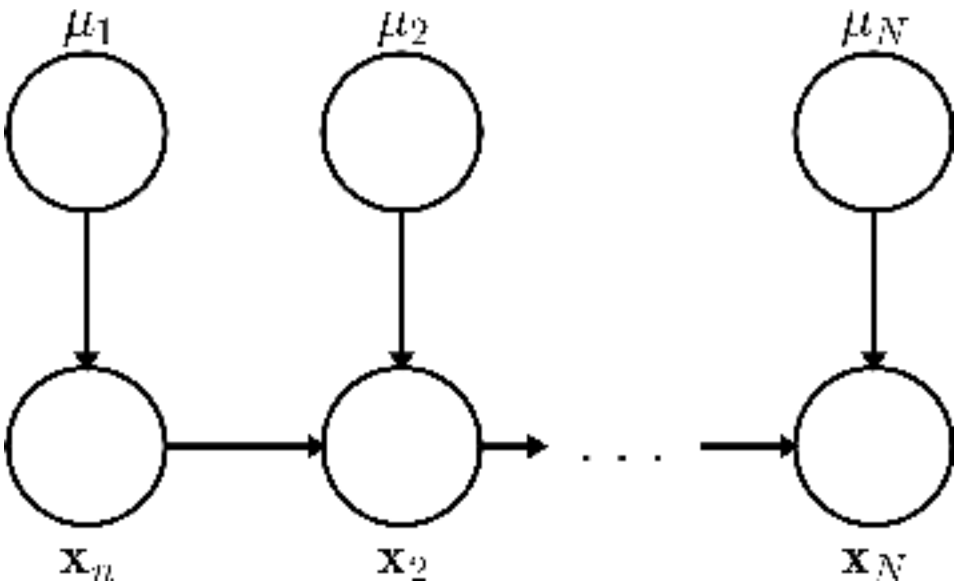
In [874]:

```
%%capture
pgm = daft.PGM([6, 6], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("mu1", r"$\mathbf{\mu}_1$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("mu2", r"$\mathbf{\mu}_2$", 3, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("muN", r"$\mathbf{\mu}_N$", 4.5, 3.5, offset=(0, 20)))

pgm.add_node(daft.Node("x1", r"$\mathbf{x}_n$", 2, 2.5, offset=(0, -20)))
pgm.add_node(daft.Node("x2", r"$\mathbf{x}_2$", 3, 2.5, offset=(0, -20)))
pgm.add_node(daft.Node("ellipsis", r" . . . ", 3.7, 2.5, offset=(0, 0), plot_params={"ec" : "none"}))
pgm.add_node(daft.Node("ellipsis_end", r"", 3.7, 2.5, offset=(0, 0), plot_params={"ec" : "none"}))
pgm.add_node(daft.Node("xN", r"$\mathbf{x}_N$", 4.5, 2.5, offset=(0, -20)))


# Add in the edges.
pgm.add_edge("mu1", "x1", plot_params={'head_length':0.08})
pgm.add_edge("mu2", "x2", plot_params={'head_length':0.08})
pgm.add_edge("muN", "xN", plot_params={'head_length':0.08})
pgm.add_edge("x1", "x2", plot_params={'head_length':0.08})
pgm.add_edge("x2", "ellipsis", plot_params={'head_length':0.08})
pgm.add_edge("ellipsis_end", "xN", plot_params={'head_length':0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("Priors.png", dpi=1000)
```

In [875]:

```
# Dirichlet Priors
trim(Image.open('Priors.png'), 15)
```

Out[875]:



$p(\{\mathbf{x}_n, \mu_n\}) = p(\mathbf{x}_1 \mid \mu_1)p(\mu_1) \prod_{n=2}^{N} p(\mathbf{x}_n \mid \mathbf{x}_{n-1}, \mu_n)p(\mu_n)$, where $p(\mu_n) = Dir(\mu_n \mid \alpha_n)$

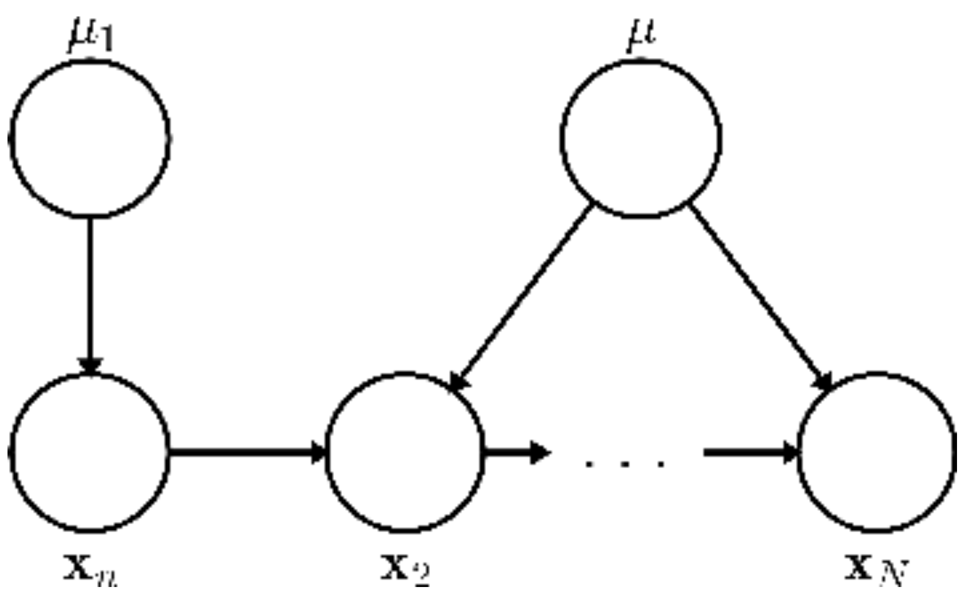
In [876]:

```
%%capture
pgm = daft.PGM([6, 6], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("mu1", r"$\mathbf{\mu}_1$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("mu", r"$\mathbf{\mu}$", 3.75, 3.5, offset=(0, 20)))

pgm.add_node(daft.Node("x1", r"$\mathbf{x}_n$", 2, 2.5, offset=(0, -20)))
pgm.add_node(daft.Node("x2", r"$\mathbf{x}_2$", 3, 2.5, offset=(0, -20)))
pgm.add_node(daft.Node("ellipsis", r" . . . ", 3.7, 2.5, offset=(0, 0), plot_par
ams={"ec" : "none"}))
pgm.add_node(daft.Node("ellipsis_end", r"", 3.7, 2.5, offset=(0, 0), plot_params
={"ec" : "none"}))
pgm.add_node(daft.Node("xN", r"$\mathbf{x}_N$", 4.5, 2.5, offset=(0, -20)))



# Add in the edges.
pgm.add_edge("mu1", "x1", plot_params={'head_length':0.08})
pgm.add_edge("mu", "x2", plot_params={'head_length':0.08})
pgm.add_edge("mu", "xN", plot_params={'head_length':0.08})
pgm.add_edge("x1", "x2", plot_params={'head_length':0.08})
pgm.add_edge("x2", "ellipsis", plot_params={'head_length':0.08})
pgm.add_edge("ellipsis_end", "xN", plot_params={'head_length':0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("SharedPrior.png", dpi=1000)
```

In [877]:

```
# Shared Prior
trim(Image.open('SharedPrior.png'), 15)
```

Out[877]:



$$p(\{\mathbf{x}_n\}, \mu_1, \mu) = p(\mathbf{x}_1 \mid \mu_1)p(\mu_1) \prod_{n=2}^{N} p(\mathbf{x}_n \mid \mathbf{x}_{n-1}, \mu)p(\mu)$$

Parametrization Example

- Consider N binary random variables, each governed by a single parameter μ_i representing the probability $p(x_i = 1)$.
  - Total number of parameters: N

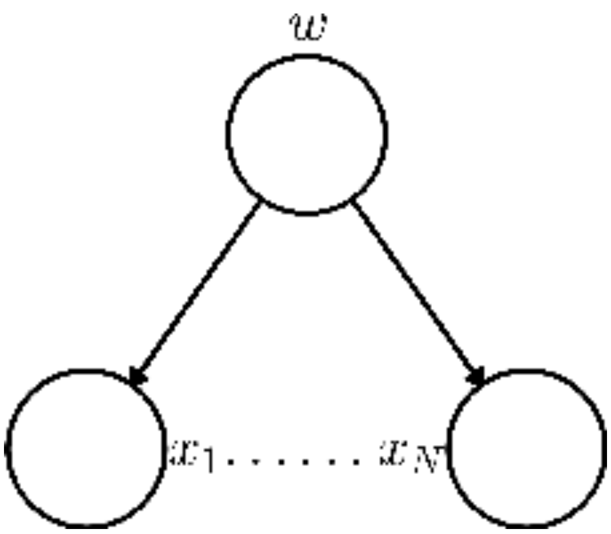
In [878]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("y", r"$w$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("x1", r"$x_1 . . . . . . .$", 1.3, 2.5, offset=(32, 0)))
pgm.add_node(daft.Node("xN", r"$x_N$", 2.7, 2.5, offset=(-21, 0)))

# Add in the edges.
pgm.add_edge("y", "x1", plot_params={'head_length' : 0.08})
pgm.add_edge("y", "xN", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("ParametrizationEx-1.png", dpi=1000)
```

In [879]:

```
trim(Image.open("ParametrizationEx-1.png"), 15)
```

Out[879]:



- However, note that the conditional distribution $p(y \mid x_1, \ldots, x_N)$ would require 2^N parameters.

Parametrization Example

One way to simplify the model is to use the ***logistic sigmoid function*** on a ***linear combination of the parent variables*** as follows

$$p(y=1 \mid x_1, \ldots, x_N) = \sigma\left(w_0 + \sum_{i=1}^{N} w_i x_i\right) = \sigma(\mathbf{W}^T \mathbf{x})$$

where $\sigma(t) = (1 + \exp(-t))^{-1}$ is the logistic sigmoid, $\mathbf{x} = (1, x_1, \ldots, x_N)^T$ is an $(N+1)$-dimensional vector and $\mathbf{w} = (w_0, \ldots, w_N)^T$ is a vector of $N+1$ parameters! ***Note: Linear in the number of variables as opposed to exponential!***

Data Complexity

- How many parameters specify the distribution of N discrete (say, binary) variables?
  - If the graph is disconnected (no edges at all), $O(N)$.
  - If the graph is fully connected, $O(2^N)$.
  - Partial structure gives intermediate complexity.
- For N Gaussian variables? (Verify these.)
  - If graph is disconnected (again, no edges at all), $O(N)$.
  - If graph is fully connected, $O(N^2)$.

Conditional Independence Revisited

- Recall, Random Variables x and y are said to be ***conditionally independent*** given z if
  $\boxed{p(x, y \mid z) = p(x \mid z) \cdot p(y \mid z)}$

- Notation:
  - $a \perp\!\!\!\perp b \mid c$, or,
  - $a \perp b \mid c$.

- How to infer such structure from a graph?

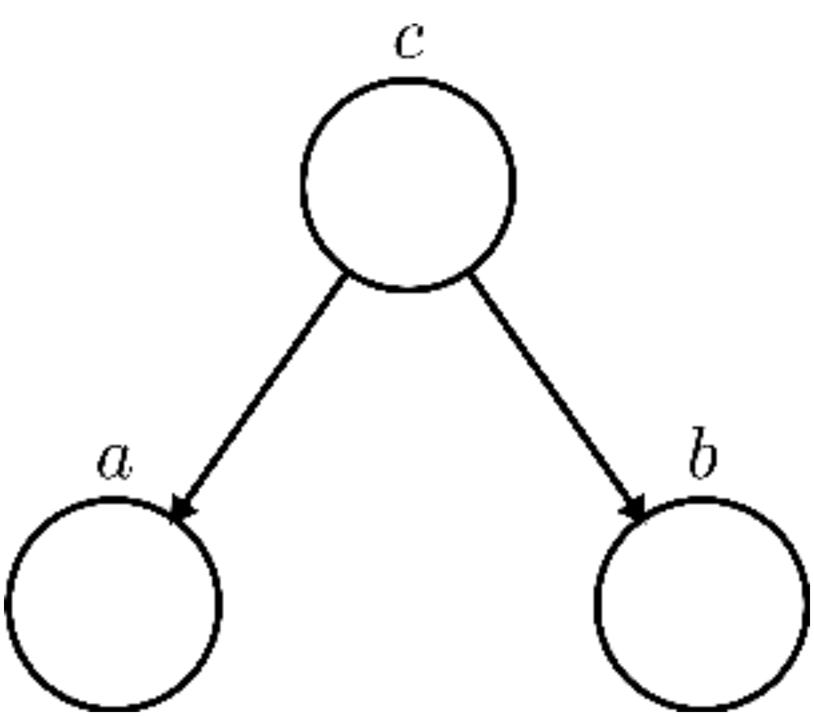
In [880]:

```
%%capture
### Example

pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("c", r"$c$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("a", r"$a$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("c", "a", plot_params={'head_length' : 0.08})
pgm.add_edge("c", "b", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("CondIndExample1.png", dpi=1000)
```

In [881]:

```
# Example 1
# Question: Are a and b conditionally independent
#           in the given scenario?
trim(Image.open('CondIndExample1.png'), 20)
```

Out[881]:



Answer

- No, a and b are not conditionally independent given the empty set.
- Note: c is ***not*** given! In particular, it is not shaded.
- Explanation:
  - The joint distribution is given by $p(a, b, c) = p(a \mid c)p(b \mid c)p(c)$
  - The marginal distribution is given by $p(a, b) = \sum_c p(a \mid c)p(b \mid c)p(c)$ which in general does not factorize into $p(a)p(b)$.
  - Therefore, $a \not\perp\!\!\!\perp b \mid \emptyset$.

A subtle point

- Following the notation of the book, expressions such as $P(E \mid \emptyset)$ and $a \not\perp\!\!\!\perp b \mid \emptyset$ are used here.
- However, technically, expressions such as $p(E \mid \emptyset)$ are undefined.
- The intuition is to think of the empty as denoteing "given no information" or rather the universal set in fact.
  - In this scenario, the conditional independence of a and b is really equivalent to a and b being independent. This is the reason we are looking for whether the marginal distribution $p(a, b)$ is equal to $p(a)p(b)$.

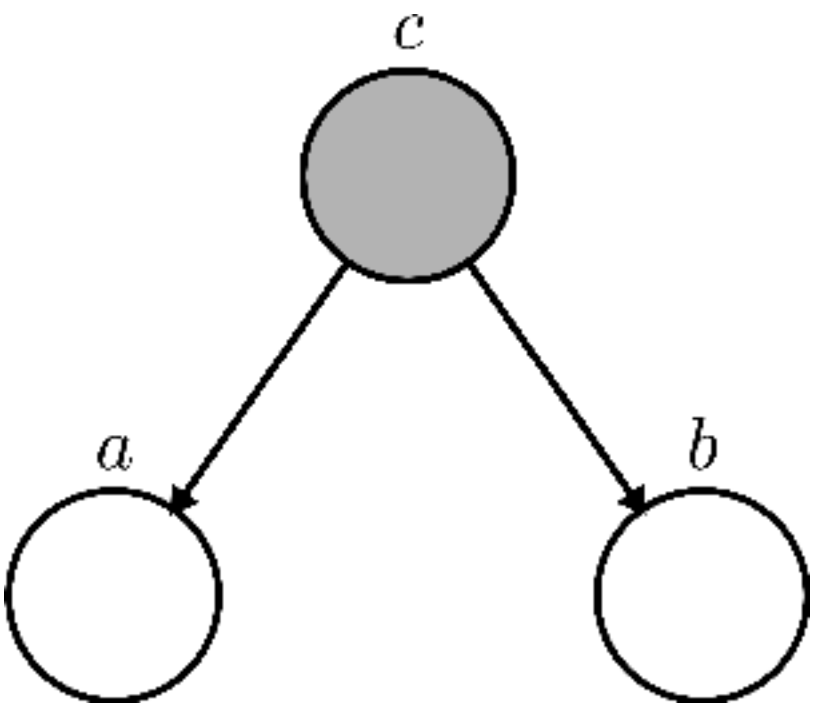
In [882]:

```
%%capture
### Example

pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("c", r"$c$", 2, 3.5, offset=(0, 20),
                       observed=True))
pgm.add_node(daft.Node("a", r"$a$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("c", "a", plot_params={'head_length' : 0.08})
pgm.add_edge("c", "b", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("CondIndExample1-observed.png", dpi=1000)
```

In [883]:

```
# Example 1 after Observation
# Question: Are a and b conditionally independent
#           in the given scenario?
trim(Image.open('CondIndExample1-observed.png'), 20)
```

Out[883]:



Answer

- Yes, a and b are conditionally independent given the c.
- Given c, $p(a, b \mid c) = \frac{p(a,b,c)}{p(c)} = p(a \mid c)p(b \mid c)$ (recall previous slide about the joint distribution.)
- Thus, we have $a \perp\!\!\!\perp b \mid c$.

Tail-to-Tail

- Note that there is an ***undirected*** path from node a to node b via c.
- Node c is tail-to-tail with respect to this path (node c is at the tail of both of the edges).
- Thus, for a tail-to-tail connections,
  - a and b are dependent, but,
  - the conditioned node, c, ***blocks*** the path from a to b and causes a and b to become independent.

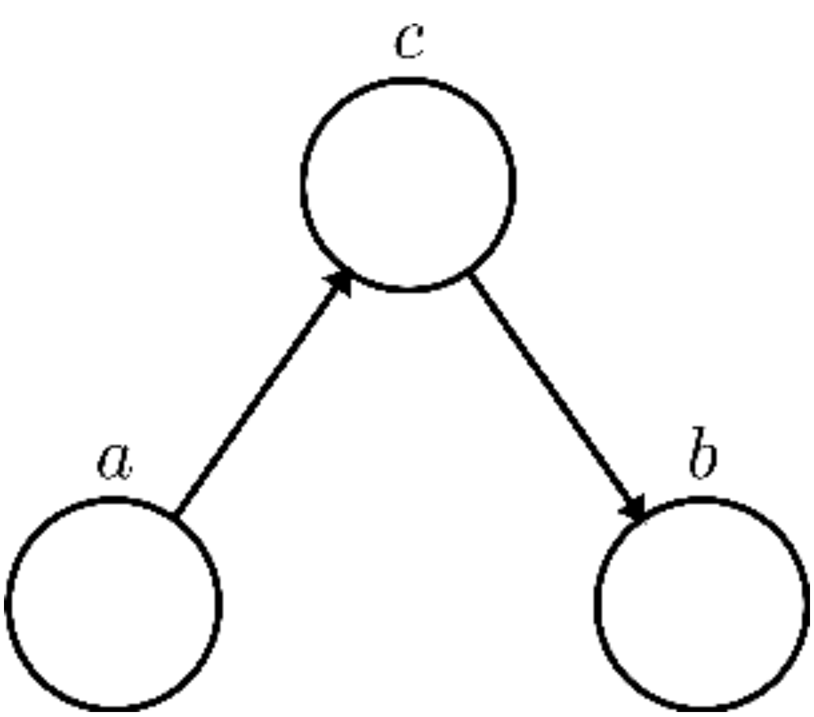
In [884]:

```
%%capture
### Example

pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("c", r"$c$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("a", r"$a$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("a", "c", plot_params={'head_length' : 0.08})
pgm.add_edge("c", "b", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("CondIndExample2.png", dpi=1000)
```

In [885]:

```
# Example 2
# Question: Are a and b conditionally independent
#           in the given scenario?
trim(Image.open('CondIndExample2.png'), 20)
```

Out[885]:



Answer

- No, a and b are not conditionally independent given the empty set.
- Note: Again, c is ***not*** given! In particular, it is not shaded.
- Explanation:
  - The joint distribution is given by $p(a, b, c) = p(a)p(c \mid a)p(b \mid c)$
  - $p(a, b)$ is given by $p(a) \sum_c p(c \mid a)p(b \mid c) = p(a)p(b \mid a)$ which in general does not factorize into $p(a)p(b)$.
  - Therefore, $a \not\perp\!\!\!\perp b \mid \emptyset$.

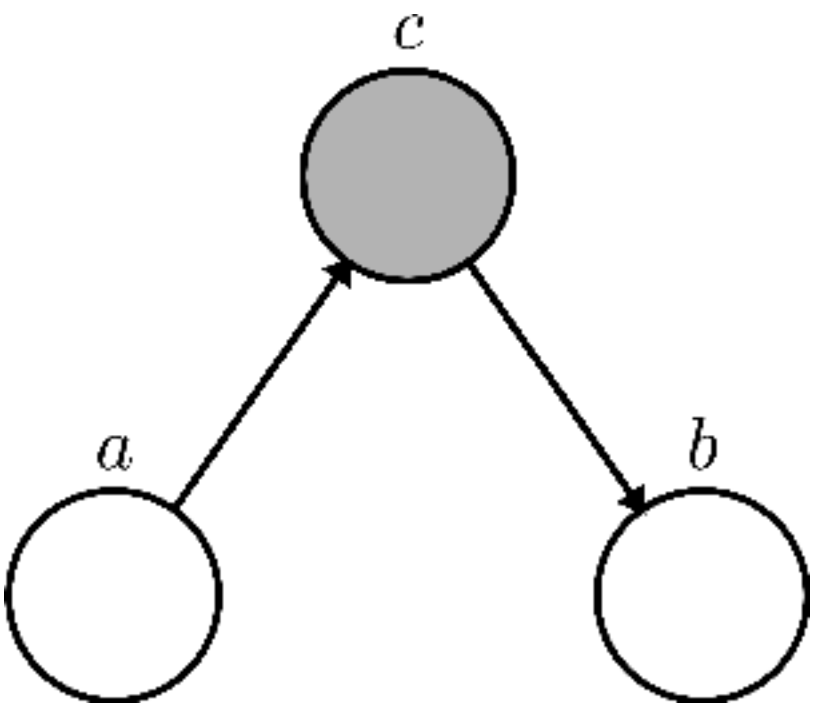
In [886]:

```
%%capture
### Example

pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("c", r"$c$", 2, 3.5, offset=(0, 20),
                       observed = True))
pgm.add_node(daft.Node("a", r"$a$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("a", "c", plot_params={'head_length' : 0.08})
pgm.add_edge("c", "b", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("CondIndExample2-observed.png", dpi=1000)
```

In [887]:

```
# Example 2 after observation
# Question: Are a and b conditionally independent
#           in the given scenario?
trim(Image.open('CondIndExample2-observed.png'), 20)
```

Out[887]:



Answer

- Yes, a and b are conditionally independent given the c.
- Given c, $p(a, b \mid c) = \frac{p(a,b,c)}{p(c)} = \frac{p(a)p(c|a)p(b|c)}{p(c)} = p(a \mid c)p(b \mid c)$
- Thus, we have $a \perp\!\!\!\perp b \mid c$.

Head-to-Tail

- Note that there is an ***undirected*** path from node a to node b via c.
- Node c is head-to-tail with respect to this path (node c is at the head of the edge from a and at the tail of the edge to b.)
- Thus, for a head-to-tail connection,
  - a and b are dependent, but,
  - the conditioned node, c, ***blocks*** the path from a to b and causes a and b to become independent.

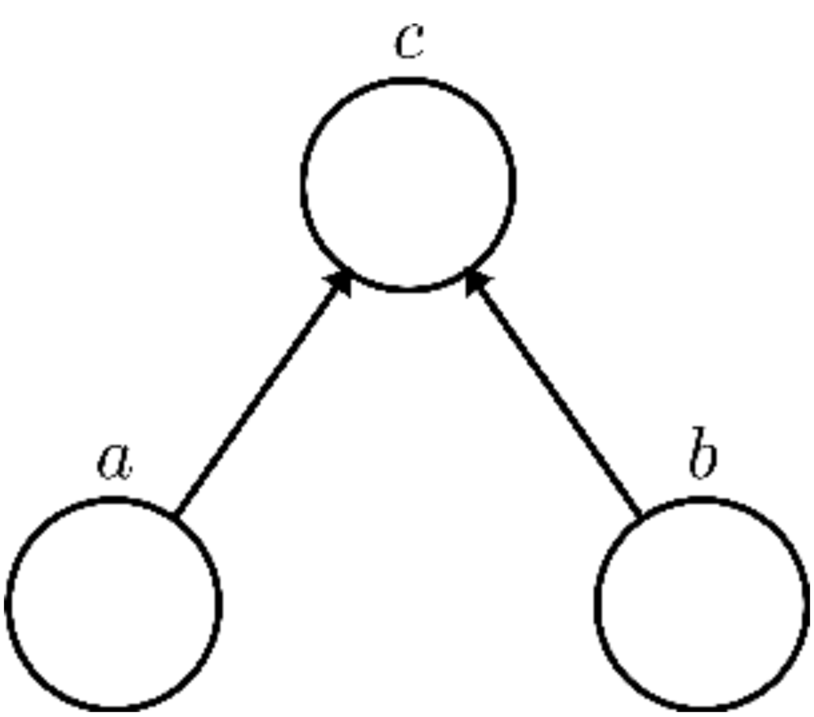
In [888]:

```
%%capture
### Example

pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("c", r"$c$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("a", r"$a$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("a", "c", plot_params={'head_length' : 0.08})
pgm.add_edge("b", "c", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("CondIndExample3.png", dpi=1000)
```

In [889]:

```
# Example 3
# Question: Are a and b conditionally independent
#           in the given scenario?
trim(Image.open('CondIndExample3.png'), 20)
```

Out[889]:



Answer

- Yes, a and b are conditionally independent given the empty set!
- Explanation:
  - The joint distribution is given by $p(a, b, c) = p(a)p(b)p(c \mid a, b)$.
  - When none of the variables are observed, marginalizing both sides leads to $p(a, b) = p(a)P(b)$.
  - Therefore, a and b are independent with no variables observed, in contrast to the previous two examples, $a \perp\!\!\!\perp b \mid \emptyset$

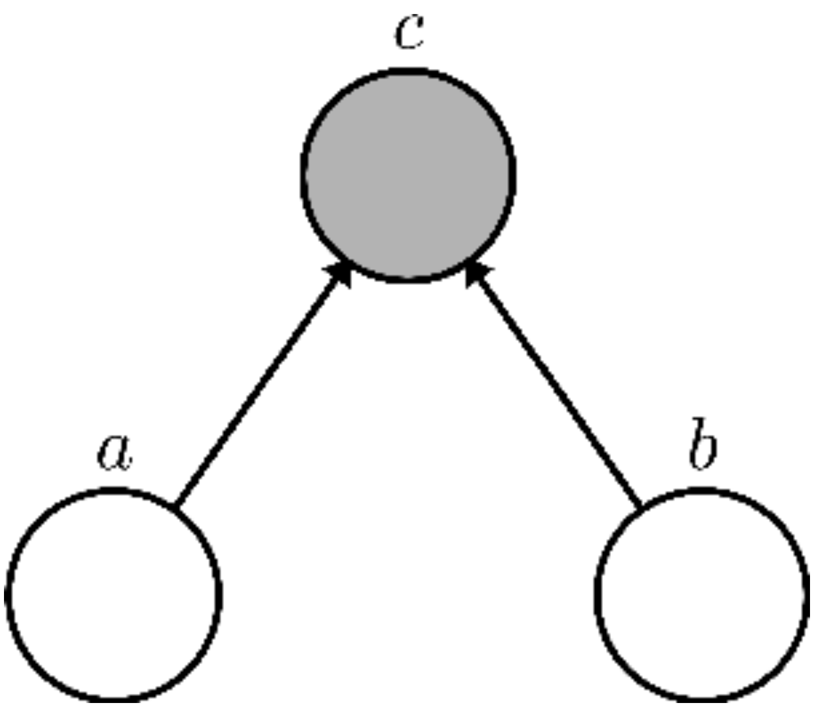
In [890]:

```
%%capture
### Example

pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("c", r"$c$", 2, 3.5, offset=(0, 20),
                        observed=True))
pgm.add_node(daft.Node("a", r"$a$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("a", "c", plot_params={'head_length' : 0.08})
pgm.add_edge("b", "c", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("CondIndExample3-observed.png", dpi=1000)
```

In [891]:

```
# Example 3 after observation
# Question: Are a and b conditionally independent
#           in the given scenario?
trim(Image.open('CondIndExample3-observed.png'), 20)
```

Out[891]:



Answer

- No, a and b are not conditionally independent given the c.
- Given c, $p(a, b \mid c) = \frac{p(a, b, c)}{p(c)} = \frac{p(a)p(b)p(c \mid a, b)}{p(c)}$.
  - This in general does not factorize to $p(a \mid c)p(b \mid c)$.
- Thus, we have $a \not\perp\!\!\!\perp b \mid c$.

Head-to-Head

- Note that there is an ***undirected*** path from node a to node b via c.
- Node c is head-to-head with respect to this path (node c is at the head of both the edges.)
- Thus, for a head-to-head connection,
  - a and b are ***independent***, but,
  - the conditioned node, c, ***unblocks*** the path from a to b and causes a and b to become ***dependent***.

Battery, Fuel and Gauge Example

- Consider three random variables:
  - B which represents the state of the battery that is either charged, $B = 1$, or flat, $B = 0$,
  - F which represents the state of the fuel tank that is either full, $F = 1$, or empty, $F = 0$, and,
  - G which represents the state of an electical fuel gauge which indicates either full, $G = 1$ or empty, $G = 0$.
- B and F are independent with priors:
  - $p(B = 1) = 0.9$
  - $p(F = 1) = 0.9$

Further Setup of the example

Given the state of the fuel tank and the battery, the fuel gauge reads full with probabilities:

- $p(G = 1 \mid B = 1, F = 1) = 0.8$
- $p(G = 1 \mid B = 1, F = 0) = 0.2$
- $p(G = 1 \mid B = 0, F = 1) = 0.2$
- $p(G = 1 \mid B = 0, F = 0) = 0.1$

Analysis

- Without any observations, the prior of the fuel tank being empty is given by $1 - P(F = 1) = 1 - 0.9 = 0.1$.

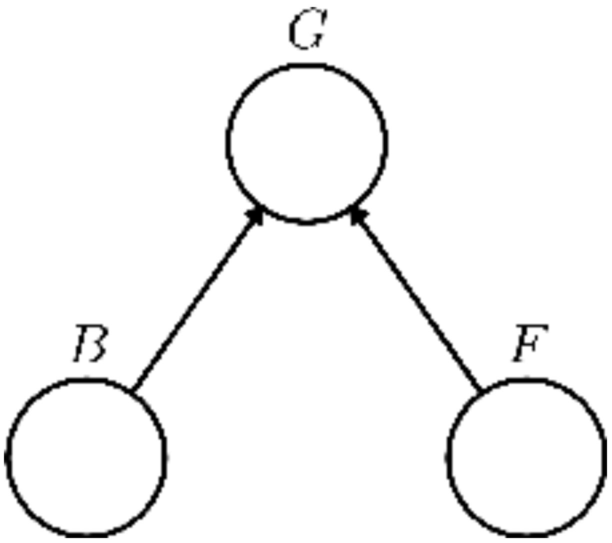
In [892]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("G", r"$G$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("B", r"$B$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("F", r"$F$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("B", "G", plot_params={'head_length' : 0.08})
pgm.add_edge("F", "G", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("OutOfFuel1.png", dpi=1000)
```

In [893]:

```
trim(Image.open("OutOfFuel1.png"), 15)
```

Out[893]:



Analysis

- Now, suppose we observe $G = 0$.
  - $p(G = 0) = \sum_{B \in \{0,1\}} \sum_{F \in \{0,1\}} p(G = 0 \mid B, F) p(B) p(F) = 0.315$

(Try to verify this.)

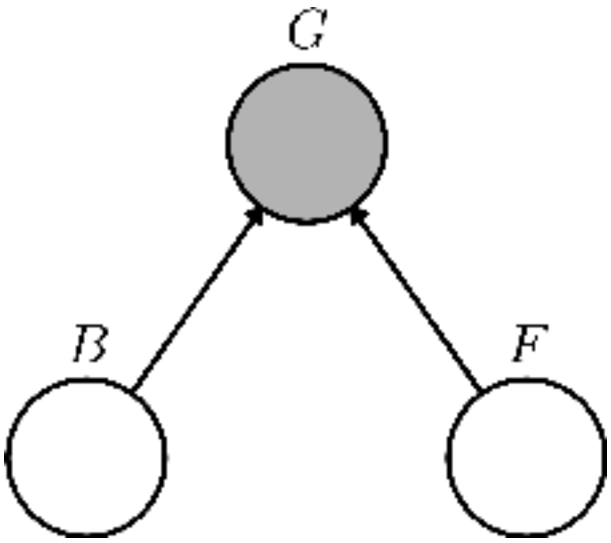
In [894]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("G", r"$G$", 2, 3.5, offset=(0, 20), observed=True))
pgm.add_node(daft.Node("B", r"$B$", 1.3, 2.5, offset=(0, 20)))
pgm.add_node(daft.Node("F", r"$F$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("B", "G", plot_params={'head_length' : 0.08})
pgm.add_edge("F", "G", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("OutOfFuel2.png", dpi=1000)
```

In [895]:

```
trim(Image.open("OutOfFuel2.png"), 15)
```

Out[895]:



Analysis

- Now, we also have $p(G = 0 \mid F = 0) =$
$$\sum_{B \in \{0,1\}} p(G = 0 \mid B, F = 0)p(B) = 0.81.$$
- So, by Bayes Rule, $p(F = 0 \mid G = 0) = \frac{p(G=0|F=0)p(F=0)}{p(G=0)}$
$$\approx 0.257 > p(F = 0) = 0.1$$
- Thus, observing the gauge reads empty makes it more likely that the tank is empty.

Analysis

- Now, suppose we also observe that the battery is flat, $B = 0$.
- The posterior that the fuel tank is empty given this new observation is then $p(F = 0 \mid G = 0, B = 0) = \frac{p(G=0|B=0,F=0)p(F=0)}{\sum_{F \in \{0,1\}} p(G=0|B=0,F)p(F)} \approx 0.111$

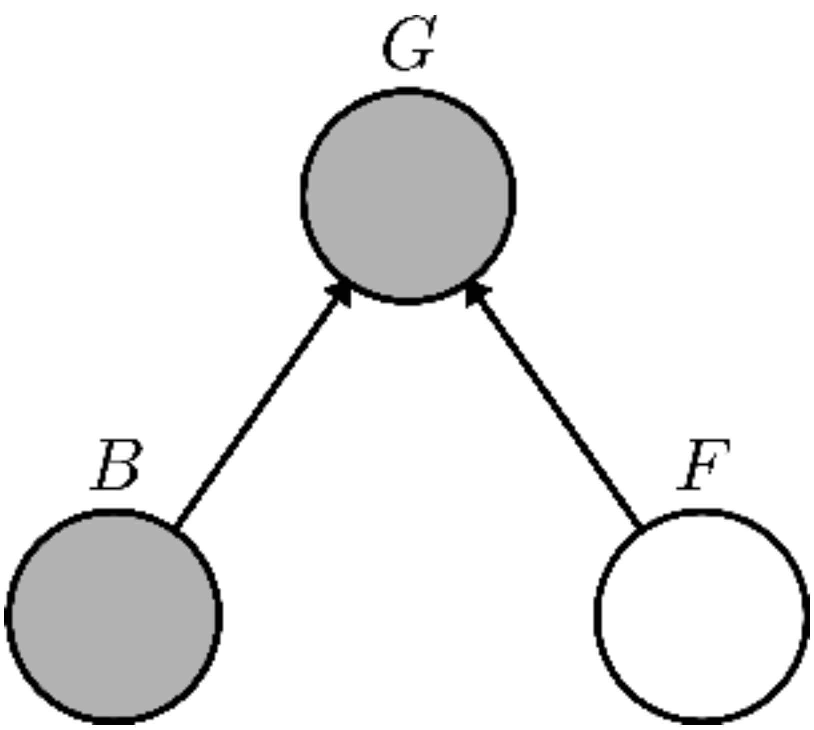
In [896]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("G", r"$G$", 2, 3.5, offset=(0, 20), observed=True))
pgm.add_node(daft.Node("B", r"$B$", 1.3, 2.5, offset=(0, 20), observed=True))
pgm.add_node(daft.Node("F", r"$F$", 2.7, 2.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("B", "G", plot_params={'head_length' : 0.08})
pgm.add_edge("F", "G", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("OutOfFuel3.png", dpi=1000)
```

In [897]:

```
trim(Image.open("OutOfFuel3.png"), 20)
```

Out[897]:



"Explanining Away"

- The probability that the tank is empty has now decreased from 0.257 to 0.111 as a result of oberving the state of the battery!
- On the other hand (try to verify this as well), on observing that the battery is full, the posterior, $p(F = 0 \mid G = 0, B = 1)$ increases to 0.308.
- Observing B = 0 (battery is flat) explains away F = 0 (the fuel tank is empty). Therefore, B and F become dependent on each other as a result of observing G = 0.

D-Separation

- Method to determine whether any pair of sets A and B of variables are conditionally independent, given knowledge of values of variables in C.
- Graphical model contains everything needed to infer conditional independence.

D-Separation

- Let A, B, and C be ***mutually exclusive*** subsets of nodes in a Bayesian Network.
- A path from A to B is blocked (when C is observed) if it contains a node such that either:
  - The arrows on the path meet head-to-tail or tail-to-tail at the node, and the node is in set C, or
  - The arrows meet head-to-head at the node, and neither the node, nor ***any*** of its descendants, are in set C.

D-Separation

- If all paths from A to B are blocked,
  - A is said to be d-separated from B by C, and,
  - The joint distribution over all variables in the graph satsifies $A \perp B \mid C$.

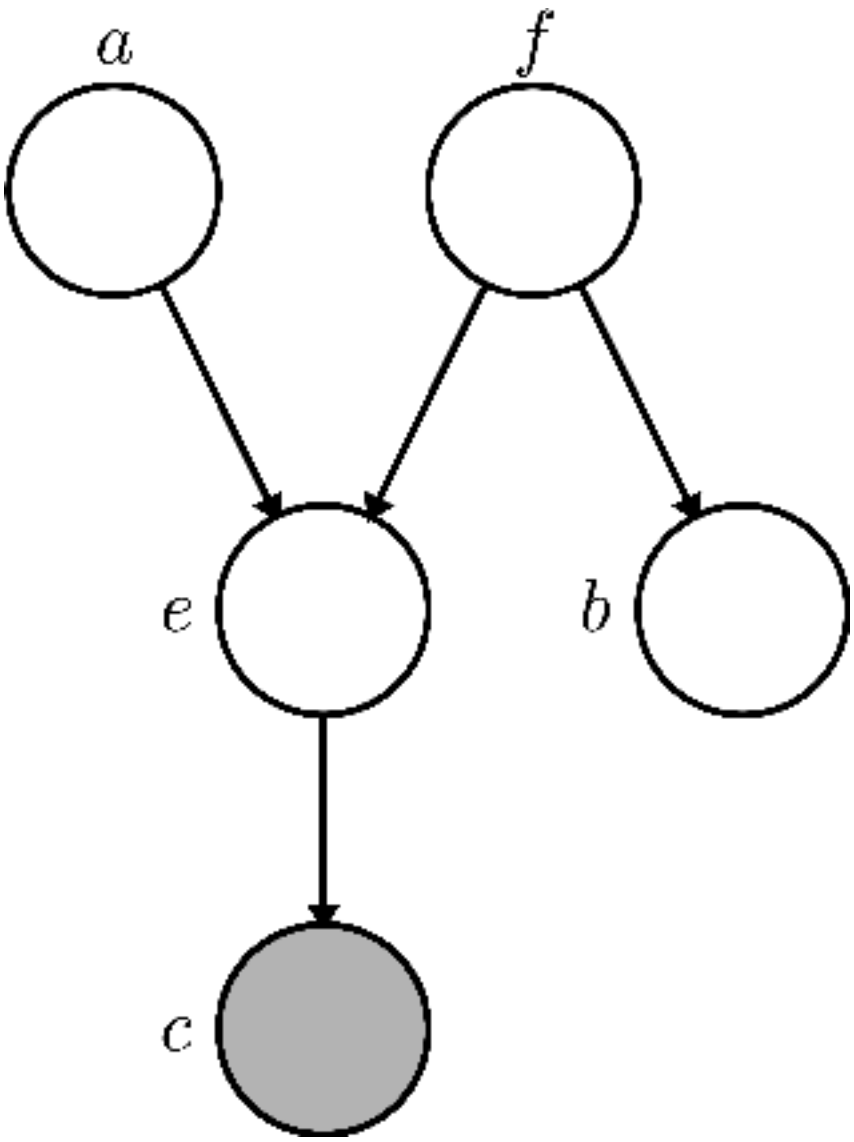
In [898]:

```
%%capture
pgm = daft.PGM([5, 5], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("a", r"$a$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 3.5, 2.5, offset=(-20, 0)))
pgm.add_node(daft.Node("c", r"$c$", 2.5, 1.5, offset=(-20, 0), observed=True))
pgm.add_node(daft.Node("e", r"$e$", 2.5, 2.5, offset=(-20, 0)))
pgm.add_node(daft.Node("f", r"$f$", 3, 3.5, offset=(0, 20)))

# Add in the edges.
pgm.add_edge("a", "e", plot_params={'head_length' : 0.08})
pgm.add_edge("f", "e", plot_params={'head_length' : 0.08})
pgm.add_edge("e", "c", plot_params={'head_length' : 0.08})
pgm.add_edge("f", "b", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("DSepExample-1.png", dpi=1000)
```

In [899]:

```
# Example
# Question: Given c, are a and b conditionally independent?
trim(Image.open("DSepExample-1.png"), 20)
```

Out[899]:



Answer

- The path from a to b is not blocked by node f as f is a tail-to-tail node and it is not observed.
- Furthermore, while e is a head-to-head node with respect to the undirected path from a to b, the path is not blocked by e since e has a descendant c in the conditioning set.
- Therefore, $a \not\perp\!\!\!\perp b \mid c$.

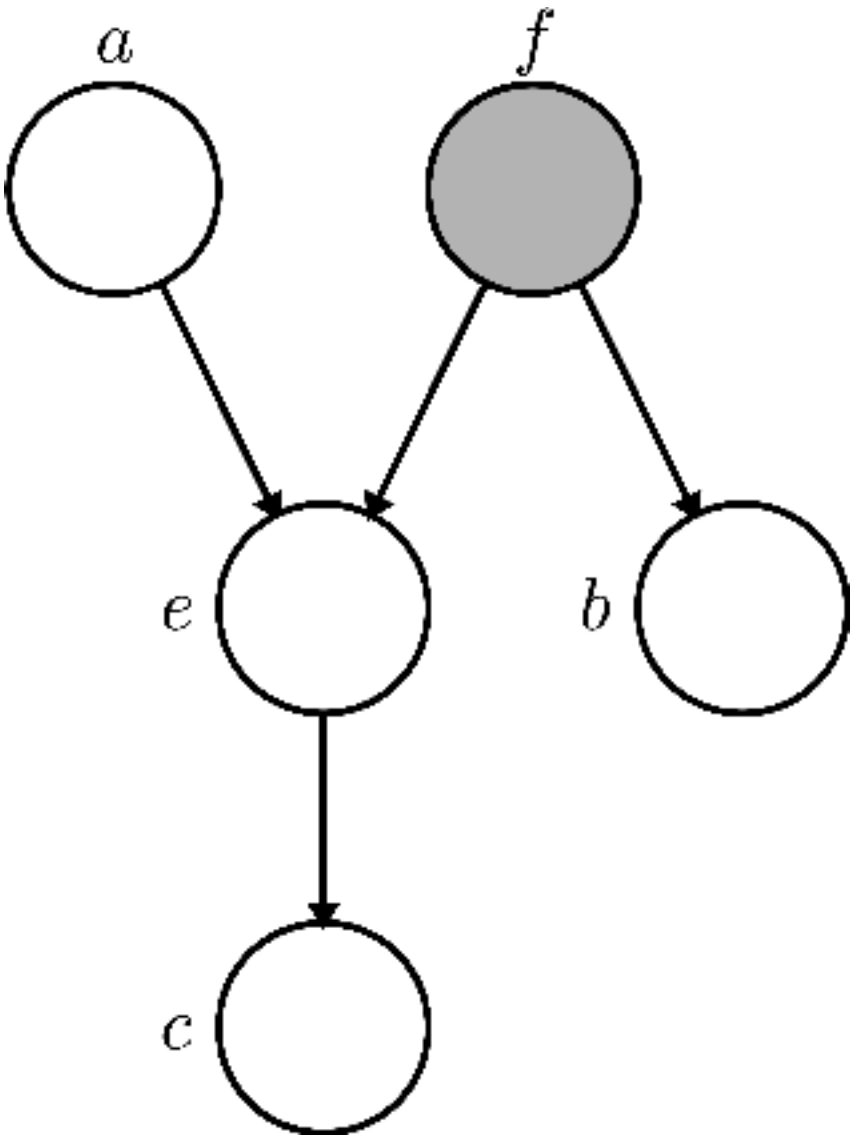
In [900]:

```
%%capture
pgm = daft.PGM([5, 5], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("a", r"$a$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("b", r"$b$", 3.5, 2.5, offset=(-20, 0)))
pgm.add_node(daft.Node("c", r"$c$", 2.5, 1.5, offset=(-20, 0)))
pgm.add_node(daft.Node("e", r"$e$", 2.5, 2.5, offset=(-20, 0)))
pgm.add_node(daft.Node("f", r"$f$", 3, 3.5, offset=(0, 20), observed=True))

# Add in the edges.
pgm.add_edge("a", "e", plot_params={'head_length' : 0.08})
pgm.add_edge("f", "e", plot_params={'head_length' : 0.08})
pgm.add_edge("e", "c", plot_params={'head_length' : 0.08})
pgm.add_edge("f", "b", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("DSepExample-2.png", dpi=1000)
```

In [901]:

```
# Example
# Question: Given f, are a and b conditionally independent?
trim(Image.open("DSepExample-2.png"), 20)
```

Out[901]:



Answer

- The path from a to b is blocked by node f as f is a tail-to-tail node and it is observed.
- Therefore, $a \perp b \mid f$.
- Furthermore, the undirected path from a to b is blocked by e because
  - e is a head-to-head node, and,
  - Neither it nor any of its descendants (in this case only c) are in the conditioning set.
- Thus, clearly, we again then have that $a \perp b \mid f$

Parameters in D-Separation?

- Recall that parameter nodes are the small filled nodes.
- In D-Sepration, they behave in a similar manner to observed nodes.
- There are no marginal distributions associated with such nodes, and therefore, parameter nodes never have parents and all paths through these will always be
  - Tail-to-tail, and hence,
  - Blocked.
- Hence, parameters play no role in D-Separation.

D-separation: I.I.D. Data

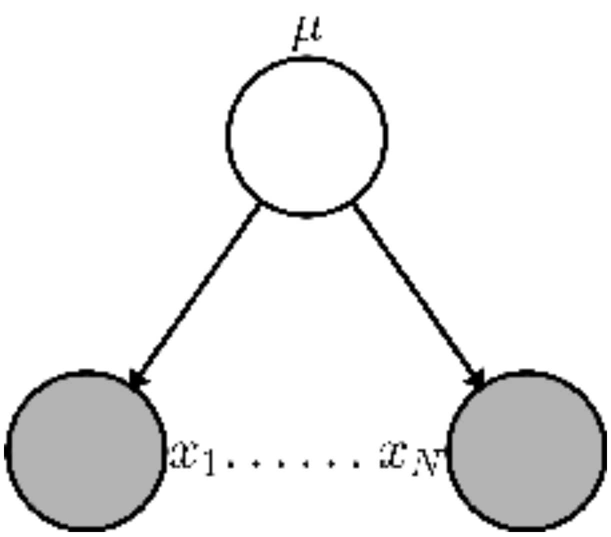
In [902]:

```
%%capture
pgm = daft.PGM([4, 4], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("mu", r"$\mu$", 2, 3.5, offset=(0, 20)))
pgm.add_node(daft.Node("x1", r"$x_1 . . . . . .$", 1.3, 2.5, offset=(32, 0), obs
erved=True))
pgm.add_node(daft.Node("xN", r"$x_N$", 2.7, 2.5, offset=(-21, 0), observed=True)
)

# Add in the edges.
pgm.add_edge("mu", "x1", plot_params={'head_length' : 0.08})
pgm.add_edge("mu", "xN", plot_params={'head_length' : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("IID.png", dpi=1000)
```

In [903]:

```
trim(Image.open("IID.png"), 15)
```

Out[903]:



- Let us condition on μ and consider the joint distribution of the observations, $\mathcal{D} = \{x_1, \cdots, x_N\}$.
- There is a unique path from any x_i to any other $x_{j \neq i}$.
  - This path is tail-to-tail w.r.t. μ.
  - Every such path is blocked and the observations are independent given μ:

$$p(\mathcal{D} \mid \mu) = \prod_{n=1}^{N} p(x_n \mid \mu)$$

D-Separation: I.I.D. Data

- Now, integrating over μ (taking μ to be unobserved), the observations ($\mathcal{D}$) are in general no longer independent:

$$p(\mathcal{D}) = \int_{-\infty}^{\infty} p(\mathcal{D} \mid \mu) p(\mu) d\mu \neq \prod_{n=1}^{N} p(x_n)$$

Markov Blanket

- Consider a joint distribution $p(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ represented by a directed graph having N nodes, and the conditional distribution of a node $\mathbf{x}_i$ conditioned on all of the remaining variables $\mathbf{x}_{j \neq i}$.
- Using factorization, we can express the conditional distribution in the form

$$p(\mathbf{x}_i \mid \mathbf{x}_{j \neq i}) = \frac{p(\mathbf{x}_1, \ldots, \mathbf{x}_N)}{\int p(\mathbf{x}_1, \ldots, \mathbf{x}_N) d\mathbf{x}_i} = \frac{\prod_k p(\mathbf{x}_k \mid pa_k)}{\int \prod_k p(\mathbf{x}_k \mid pa_k) d\mathbf{x}_i}$$

Markov Blanket

- Any factor $p(\mathbf{x}_k \mid pa_k)$ that does not have a function dependence on $\mathbf{x}_i$ can be taken outside the intergral over $\mathbf{x}_i$, and will cancel with the numerator.
- Remaining factors will then be:
  - The conditional distribution $p(\mathbf{x}_i \mid pa_i)$ for node $\mathbf{x}_i$ itself, and,
  - Conditional distributions for nodes $\mathbf{x}_k$ such that $\mathbf{x}_i \in pa_k$

Markov Blanket

- The conditional $p(\mathbf{x}_i \mid pa_i)$ depends on the parents of $\mathbf{x}_i$.
- The conditionals $p(\mathbf{x}_k \mid pa_k)$ depend on the children of $\mathbf{x}_i$ as well as the co-parents.
- Definition: The set of nodes comprising the parents, the children and the co-parents is called the Markov blanket
  - The minimal set of nodes that isolates $\mathbf{x}_i$ from the rest of the graph.

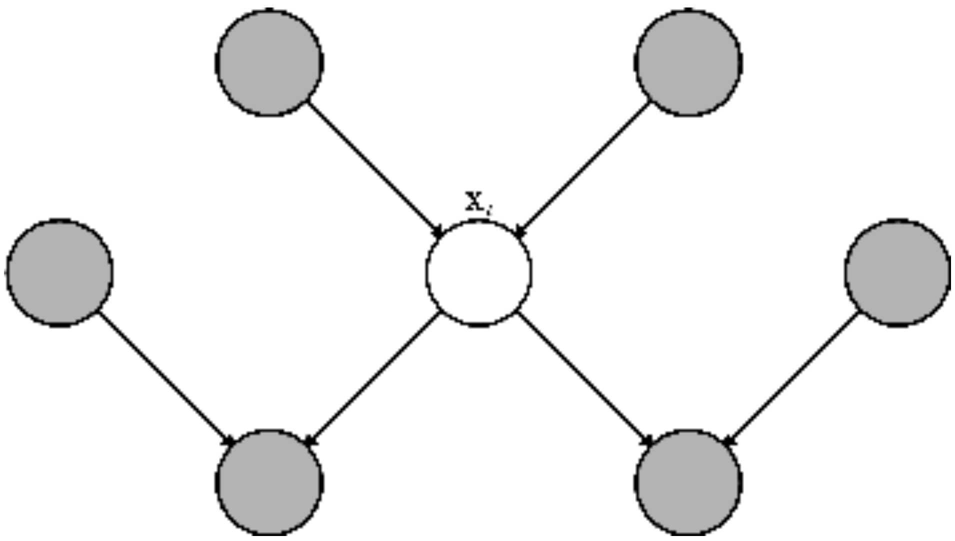
In [904]:

```
%%capture
# Instantiate the PGM.
pgm = daft.PGM([6, 6], origin=[-1, -1])

pgm.add_node(daft.Node("x2", "", 1, 0, observed=True))
pgm.add_node(daft.Node("x1", "", 0, 1, observed=True))
pgm.add_node(daft.Node("x3", "", 1, 2, observed=True))
pgm.add_node(daft.Node("xi", r"$\mathbf{x}_i$", 2, 1, offset=(0, 20)))
pgm.add_node(daft.Node("x4", "", 3, 0, observed=True))
pgm.add_node(daft.Node("x6", "", 4, 1, observed=True))
pgm.add_node(daft.Node("x5", "", 3, 2, observed=True))

pgm.add_edge("x1", "x2", plot_params={"head_length" : 0.08})
pgm.add_edge("xi", "x2", plot_params={"head_length" : 0.08})
pgm.add_edge("x3", "xi", plot_params={"head_length" : 0.08})
pgm.add_edge("x5", "xi", plot_params={"head_length" : 0.08})
pgm.add_edge("xi", "x4", plot_params={"head_length" : 0.08})
pgm.add_edge("x6", "x4", plot_params={"head_length" : 0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("MarkovBlanket.png", dpi=1000)
```

In [905]:

```
trim(Image.open("MarkovBlanket.png"), 10)
```

Out[905]:



Example Application of Bayesian Networks

Medical Diagnosis

- Findings: $\mathbf{f}$
- Diseases: $\mathbf{d}$

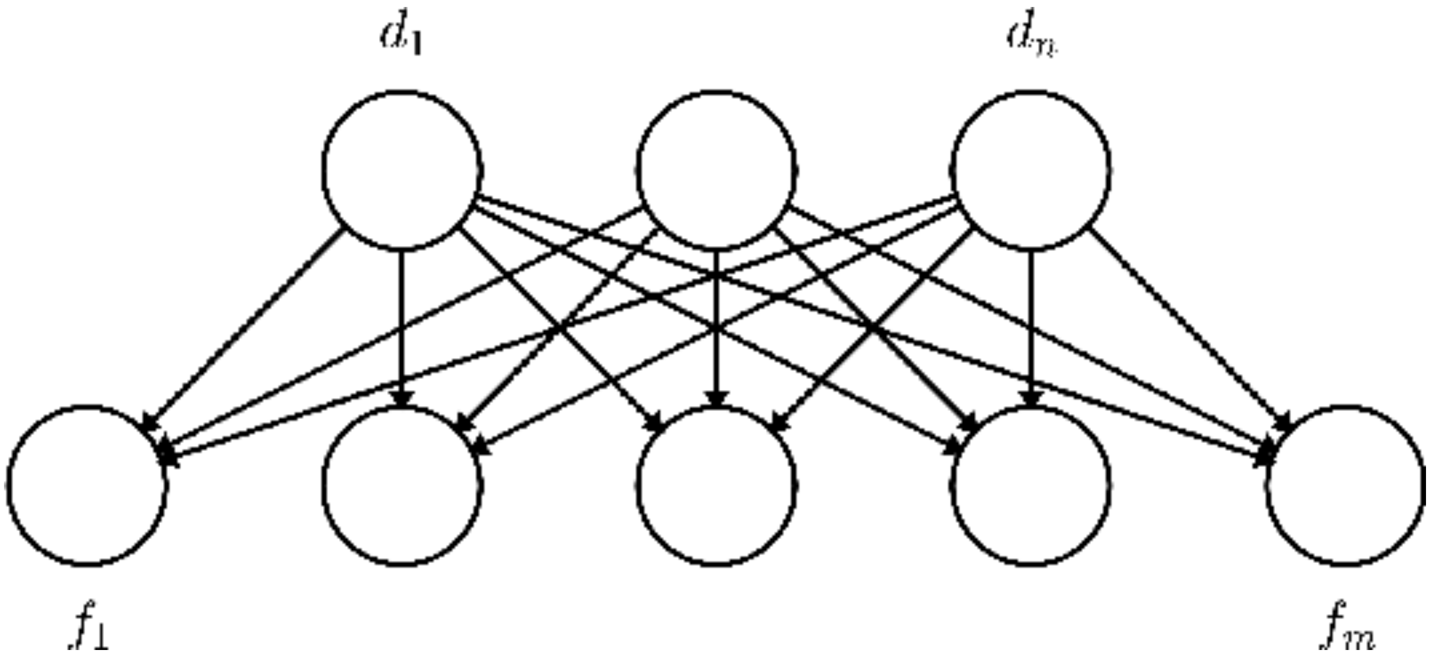
In [906]:

```
%%capture
pgm = daft.PGM([6, 6], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("d1", r"$d_1$", 2, 3.5, offset=(0, 25)))
pgm.add_node(daft.Node("di", r"", 3, 3.5))
pgm.add_node(daft.Node("dn", r"$d_n$", 4, 3.5, offset=(0, 25)))

pgm.add_node(daft.Node("f1", r"$f_1$", 1, 2.5, offset=(0, -25)))
pgm.add_node(daft.Node("fi-1", r"", 2, 2.5))
pgm.add_node(daft.Node("fi", r"", 3, 2.5))
pgm.add_node(daft.Node("fi+1", r"", 4, 2.5))
pgm.add_node(daft.Node("fm", r"$f_m$", 5, 2.5, offset=(0, -25)))


# Add in the edges.
pgm.add_edge("d1", "f1", plot_params={'head_length':0.08})
pgm.add_edge("d1", "fi-1", plot_params={'head_length':0.08})
pgm.add_edge("d1", "fi", plot_params={'head_length':0.08})
pgm.add_edge("d1", "fi+1", plot_params={'head_length':0.08})
pgm.add_edge("d1", "fm", plot_params={'head_length':0.08})

pgm.add_edge("di", "f1", plot_params={'head_length':0.08})
pgm.add_edge("di", "fi-1", plot_params={'head_length':0.08})
pgm.add_edge("di", "fi", plot_params={'head_length':0.08})
pgm.add_edge("di", "fi+1", plot_params={'head_length':0.08})
pgm.add_edge("di", "fm", plot_params={'head_length':0.08})

pgm.add_edge("dn", "f1", plot_params={'head_length':0.08})
pgm.add_edge("dn", "fi-1", plot_params={'head_length':0.08})
pgm.add_edge("dn", "fi", plot_params={'head_length':0.08})
pgm.add_edge("dn", "fi+1", plot_params={'head_length':0.08})
pgm.add_edge("dn", "fm", plot_params={'head_length':0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("DiagnosisExample.png", dpi=1000)
```

In [907]:

```
trim(Image.open("DiagnosisExample.png"), 15)
```

Out[907]:



Medical Diagnosis

- Joint distribution factors as: $p(\mathbf{f}, \mathbf{d}) = \prod_j p(d_j) \prod_i p(f_i \mid \mathbf{d})$
- Model assumes: $d_i \perp d_j$ and $f_i \perp f_j \mid \mathbf{d}$

Medical Diagnosis

- Problem: Common findings can be caused by many diseases.
  - Too many parameters needed to specify the Conditional Probability Distribution of $p(f_i \mid \mathbf{d})$.
- Parametrization: Use Noisy-OR
  - $p(f_i = 0 \mid \mathbf{d}) = (1 - q_{i0}) \prod_{j \in pa_i} (1 - q_{ij})^d_j$ where q_{ij} represents the "independent failure probability" of witnessing disease d_j without finding f_i.

Example Application of Bayesian Networks

Hidden Markov Model

- Frequently used for speech recognition and part-of-speech tagging.

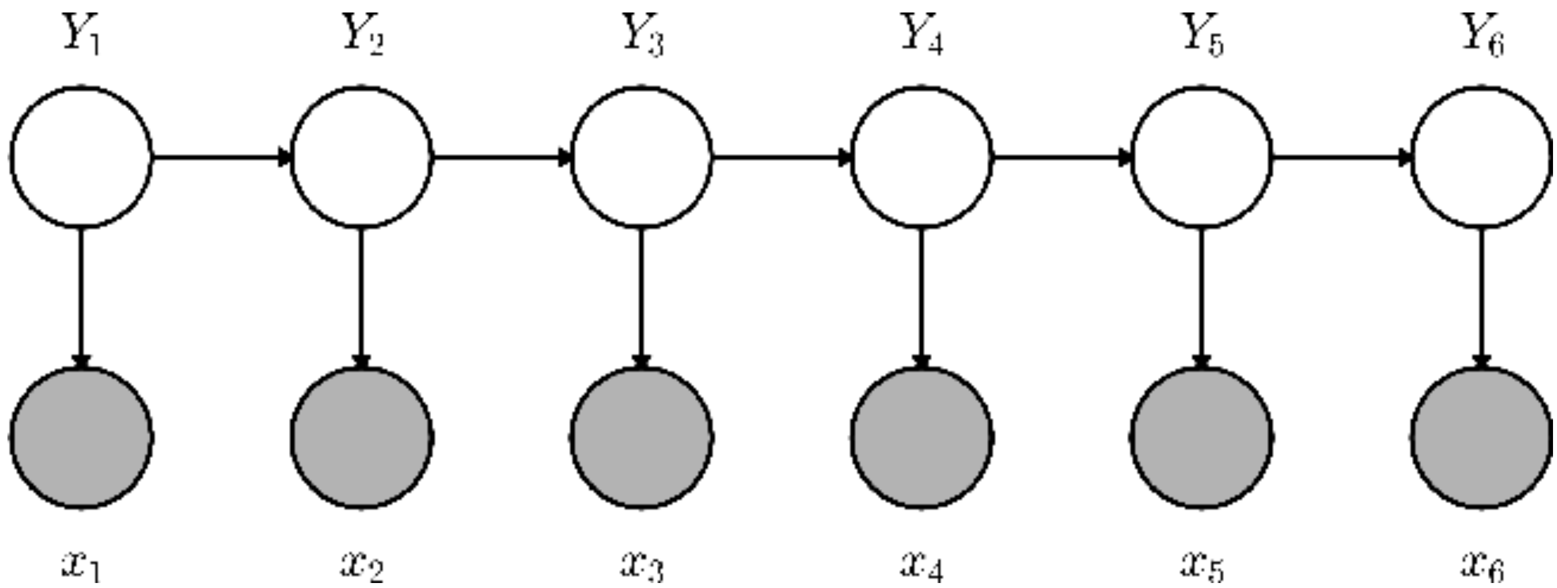
In [908]:

```
%%capture
pgm = daft.PGM([7, 7], origin=[0, 0])

# Nodes
pgm.add_node(daft.Node("Y1", r"$Y_1$", 1, 3.5, offset=(0, 25)))
pgm.add_node(daft.Node("Y2", r"$Y_2$", 2, 3.5, offset=(0, 25)))
pgm.add_node(daft.Node("Y3", r"$Y_3$", 3, 3.5, offset=(0, 25)))
pgm.add_node(daft.Node("Y4", r"$Y_4$", 4, 3.5, offset=(0, 25)))
pgm.add_node(daft.Node("Y5", r"$Y_5$", 5, 3.5, offset=(0, 25)))
pgm.add_node(daft.Node("Y6", r"$Y_6$", 6, 3.5, offset=(0, 25)))

pgm.add_node(daft.Node("x1", r"$x_1$", 1, 2.5, offset=(0, -25), observed=True))
pgm.add_node(daft.Node("x2", r"$x_2$", 2, 2.5, offset=(0, -25), observed=True))
pgm.add_node(daft.Node("x3", r"$x_3$", 3, 2.5, offset=(0, -25), observed=True))
pgm.add_node(daft.Node("x4", r"$x_4$", 4, 2.5, offset=(0, -25), observed=True))
pgm.add_node(daft.Node("x5", r"$x_5$", 5, 2.5, offset=(0, -25), observed=True))
pgm.add_node(daft.Node("x6", r"$x_6$", 6, 2.5, offset=(0, -25), observed=True))


# Add in the edges.
pgm.add_edge("Y1", "Y2", plot_params={'head_length':0.08})
pgm.add_edge("Y2", "Y3", plot_params={'head_length':0.08})
pgm.add_edge("Y3", "Y4", plot_params={'head_length':0.08})
pgm.add_edge("Y4", "Y5", plot_params={'head_length':0.08})
pgm.add_edge("Y5", "Y6", plot_params={'head_length':0.08})

pgm.add_edge("Y1", "x1", plot_params={'head_length':0.08})
pgm.add_edge("Y2", "x2", plot_params={'head_length':0.08})
pgm.add_edge("Y3", "x3", plot_params={'head_length':0.08})
pgm.add_edge("Y4", "x4", plot_params={'head_length':0.08})
pgm.add_edge("Y5", "x5", plot_params={'head_length':0.08})
pgm.add_edge("Y6", "x6", plot_params={'head_length':0.08})

# Render and save.
pgm.render()
pgm.figure.savefig("HMM.png", dpi=1000)
```

In [909]:

```
trim(Image.open("HMM.png"), 15)
```

Out[909]:



Hidden Markov Model

- Joint Distribution factors as: $p(\mathbf{y}, \mathbf{x}) = p(y_1)p(x_1 \mid y_1) \prod_{t=2}^{T} p(y_t \mid y_{t-1})p(x_t \mid y_t)$ where
  - $p(y_1)$ is the distribution of the starting state,
  - $p(y_t \mid y_{t-i})$ is the ***transition probability*** between states y_{t-1} and y_t, and,
  - $p(x_t \mid y_t)$ is the ***observation*** or ***emission*** probability.

Hidden Markov Model

- ***Markov Assumption***
  - $Y_{i+1} \perp \{Y_1, \ldots, Y_i - 1\} \mid Y_i$ or $Y_{i-1} \perp \{Y_{i+1}, \ldots, Y_T\} \mid Y_i$
  - "The future is independent of the past given on the present."