H1 - Introduction

H2 - The Dirac Equation

L1 - Setting Stuff Up

Units

- In principle, use natural units to measure mass in GeV/c^2, length in $\hbar c/GeV$, time in $\hbar/GeV$ and charge in $(\epsilon_0 \hbar c)^{1/2}$.
  - In practice, adopt Heaviside-Lorentz convention with $c = \hbar = \epsilon_0 = 1 (= \mu_0)$ meaning mass if measured in GeV, length in $1/GeV$, time in $1/GeV$ and charge is dimensionless.
    - These can be remembered by considering that energies and momenta should be high, but lengths and times should be small.
  - Can convert back to S.I. using $1 = \hbar c = 197 \text{ MeVfm}$.
    - To find the factors of c and $\hbar$, use the following

Property	Relation	Conclusion
Mass	$E \sim mc^2$	Energy $\propto GeV/c^2$
Momentum	$E \sim (mc)(c) \sim pc$	Momentum $\propto GeV/c$
Time	$\Delta E \Delta t \sim \hbar$	Time $\propto \hbar/GeV$
Length	$1 = \hbar c \approx 197 \text{Mev} \cdot \text{fm}$	Length $\propto \hbar c/GeV$

Dirac δ-functions

- Note that

$$\int_x g(x)\delta(u(x))dx = \int g(x(u)) \left|\frac{dx}{du}\right| du = \sum_{x \in X \text{ s.t. } u(x)=0} \frac{g(x)}{\left|\frac{du}{dx}\right|}$$

  - In the general case with n delta functions, the derivative in the denominator becomes the jacobian of the delta function dependencies.
- Note that this means

$$\int_{-\infty}^{\infty} e^x \delta(x^2 - a^2)dx = \sum_{x=\pm a} \frac{e^x}{|2x|} = \frac{1}{|a|}\cosh(a)$$

Standard Model

Matter

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	-1	0.0005	d	$-1/3$	~ 0.3
	ν_1	0	≈ 0	u	$+2/3$	~ 0.3
Second Generation	μ^-	-1	0.106	s	$-1/3$	~ 0.5
	ν_2	0	≈ 0	c	$+2/3$	~ 1.5
Third Generation	τ^-	-1	1.77	b	$-1/3$	~ 4.5
	ν_3	0	≈ 0	t	$+2/3$	173

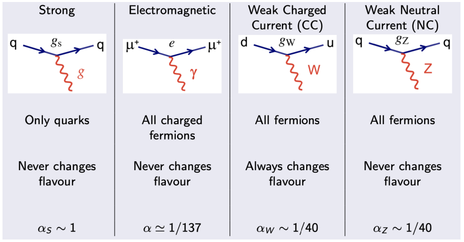
Forces

- Mediated by spin-1 gauge bosons

Force	Boson(s)	J^P	m/GeV	Charge "g"	coupling constant "α"	Notes
EM (QED)	Photon γ	1^-	0	e	α	Handout 3
Weak	$W^\pm/Z$	1^-	80/91	g_W, g_Z	α_W, α_Z	Handout 13
Strong (QCD)	8 Gluons g	1^-	0	g_s	α_s	Handout 8
Gravity	Graviton?	2^+ ?	0 ?	?	?	Not in SM!

- Coupling constants α are always dimensionless and set the intrinsic strength of each force.
  - Within the Heaviside-Lorentz convention, the charge is then always $g = \sqrt{4\pi\alpha}$ for all forces.
  - Electron charge is also related to the dimensionless quantities by $e = \sqrt{4\pi\alpha\epsilon_0\hbar c}$

Standard Model Vertices



Feynman Diagrams

- Usually show initial state on the LHS and final state on the RHS, with the centre showing how the process could have happened.
- Antiparticles indicated by arrows in the 'negative' time direction.

SR and 4-Vectors

- In this course, use 4-vector notation with t as the time-like component, such that

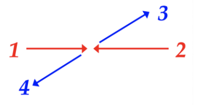
$$\begin{aligned} p^\mu &= \{E, \vec{p}\} \quad \text{(contravariant)} \\ p_\mu &= \{E, -\vec{p}\} \quad \text{(covariant) with} \\ g_{\mu\nu} = g^{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

- In particle physics, all calculations should be LI, so for example, use;

$$\begin{aligned} p^\mu p_\mu &= E^2 - p^2 = m^2 \quad \text{(invariant mass)} \\ x^\mu p_\mu &= Et - \vec{p} \cdot \vec{r} \quad \text{(Phase)} \end{aligned}$$

- On notation, write four-vectors as p^μ or p and three vectors as $\vec{p}$.
  - Quantities in the COM frame are written as $\vec{p}^*$ or p^*.

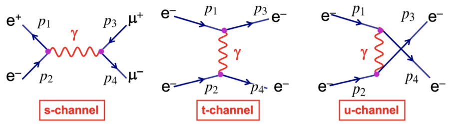
Mandelstam Variables



- A scattering process like $1 + 2 \to 3 + 4$, can be characterised by three LI mandelstam variables s, t and u;

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

- This corresponds to the naming of s, t and u-channel diagrams;



  - These variables are useful, as when the four external particle masses are known, only two of the variables are independent. Indeed it can be shown (example sheet) that

$$s + t + u \equiv m_1^2 + m_2^2 + m_3^2 + m_4^2$$

    - This is particularly useful, as there is only a fixed set of particles with known masses. Therefore can reduce the dimension of a collision problem from 16 to 12 dimensions. We then get time and energy conservation from Noether's theorem, further reducing dimensionality to 8. Also implementing the equivalence principle - ie moving into the correct frame - we can reduce the dimensionality further by 3 to 5. Furthermore, the isotropic nature of the universe is encoded by rotations, giving a further 3 dimensional reduction, making the collision problem 2 dimensional.
    - Essentially we therefore try to find $P(t|s)$ only.
  - s is LI, so $p_1^* = (E_1, \vec{p}^*)$ and $p_2^* = (E_2, -\vec{p}^*)$ giving $s = (E_1 + E_2, \vec{0})^2 = (\text{c.o.m energy})^2$.
    - Therefore, centre-of-mass energy is $\sqrt{s}$.

L2 - Particle Decay Rates

Particle Decay Rates

Fermi's Golden Rule

- Allows for calculation of decay rates and scattering probabilities. The following gives the rate (average number of transitions per unit time) at which an initial state $|i\rangle$ transitions to a final state $|f\rangle$ subject to the below assumptions.

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- All basis states normalised (note that $\mathcal{V}$ is a box of volume V containing the states and that $|i\rangle$ are eigenstates of $\hat{H}_{\text{free}}$.

$$\langle i|j\rangle = \int_{\mathcal{V}} \psi_i^* \psi_j dV = \delta_{ij}$$

- T_{fi} is the transition matrix element;

$$T_{fi} = \langle f|\hat{H}_{if}|i\rangle + \sum_{j \neq i} \frac{\langle f|\hat{H}_{int}|j\rangle \langle j|\hat{H}_{int}|i\rangle}{E_i - E_j} + \dots$$

- $\hat{H}_{int}$ is the interaction part of the full Hamiltonian written $\hat{H}_{full} = \hat{H}_{free} + \hat{H}_{int}$
- $\rho(E_f)$ is the density of final states defined below, where dN is the number of final states with energy between E_f and $E_f + dE$.

$$\rho(E_f) = \left|\frac{dN}{dE}\right|_{E=E_f}$$

  - Note that E_f = final state energy = initial state energy by energy conservation.
- Therefore, reaction rates are dictated by:
  - Matrix element - contains quantum physics, particle content, SM, SYSY, etc.
  - Phase space - relativity, momentum conservation, energy conservation.

Phase Space

- Within a box $\mathcal{V}$ of side length a, and imposing periodic boundary conditions, states must have quantised momenta;

$$\psi \propto e^{\pm i(\vec{p}\cdot\vec{x} - \omega t)}$$
$$\text{where} \quad \vec{p} = (p_x, p_y, p_z) = \frac{2\pi}{a}(n_x, n_y, n_z)$$

  - Therefore, single particle states live on a lattice, and in momentum state, each state has volume $(2\pi/a)^3$.
  - Hence, the number of states dN in an element $d^3\vec{p}$ is

$$dN = \frac{d^3\vec{p}}{\frac{(2\pi)^3}{V}} = V\frac{d^3\vec{p}}{(2\pi)^3}$$

  - We could then find $\rho(E_f)$ in terms of the velocity, but this is awful if we want to form a generalised, Lorentz invariant expression.

Writing the Golden Rule in a Nicer Form

- Note that the density of states may be rewritten in the form

$$\rho(E_f) = \left|\frac{dN}{dE}\right|_{E_f} = \int \frac{dN}{dE}\delta(E - E_i)dE$$

  - This is advantageous because the integral scans all energies, which is a much nicer set under Lorentz transformation. When substituted, this allows all final states of any energy;

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \frac{dN}{dE}\delta(E - E_i)dE$$
$$= 2\pi \int |T_{fi}|^2 \delta(E - E_i)dN$$

- Considering the special case where $i \to 1 + 2$ in the rest frame, including momentum conservation and noting that $E \equiv E_f \equiv E_1 + E_2$ we find from the above that

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_1 + E_2 - E_i) V \frac{d^3\vec{p}}{(2\pi)^3} \tag{1}$$
$$= \underbrace{(2\pi)^4 V}_{\text{maths}} \int \underbrace{|T_{fi}|^2}_{\text{physics}} \underbrace{\delta(E_1 + E_2 - E_i)}_{\text{Energy cons.}} \underbrace{\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_i)}_{\text{mom. cons.}} \underbrace{\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}}_{\text{density of states}} \tag{2}$$

- The previous normalisation condition isn't very good because it is not manifestly Lorentz covariant.
  - Regardless of particle energy or momentum, if they are within the same box $\mathcal{V}$ ie the universe, then locally each look like a bit of $1/V$-particles-per-unit-volume soup.
  - Consider two states, $|a\rangle$ and $|b\rangle$; if travelling past $|a\rangle$ with a velocity such as to transform $E_a \to E_a' = E_b$ and $\vec{p}_a \to \vec{p}_a' = \vec{p}_b$.
    - In this case, we find that $|a\rangle' = \sqrt{\gamma}\,|b\rangle$ as length contraction increases the local particle density by a factor γ.
  - Therefore, we adopt the normalisation convention that is consistent for all observers by defining

$$\psi_a'' \equiv \sqrt{\gamma_a}\psi_a$$

    - However, for historical and practical reasons we instead use

$$\psi_a' \equiv \sqrt{2E_a}\sqrt{V}\psi_a$$

- This leads to the conclusion that the number of particles per unit volume is;

$$\frac{\int_V \psi_a'^* \psi_a dV}{V} = 2E_a$$

- This new normalisation then allows us to define a Lorentz invariant matrix element as

$$M_{fi} \equiv \frac{1}{V} \langle \psi_1', \psi_2' \ldots | \hat{H}_{int} | \ldots \psi_{N-1}', \psi_N' \rangle + \ldots$$

  - This is related to T_{fi} by;

$$\boxed{\begin{aligned} M_{fi} &= \frac{1}{V}\sqrt{2E_1}\sqrt{V}\ldots\sqrt{2E_N}\sqrt{V}\langle \psi_1, \psi_2 \ldots |\hat{H}_{int}| \ldots \psi_{N-1}, \psi_N\rangle && (3) \\ &= V^{N/2-1}(\sqrt{2E_1}\ldots\sqrt{2E_N})T_{fi} && (4) \end{aligned}}$$

- Considering the $N = 3$ process $i \to 1 + 2$ therefore gives

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_1 + E_2 - E_i)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_i)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

  - Notice that here the first factor gives time dilation by a factor γ via $E_i = \gamma m_i$ as expected.
  - It can also be shown that the δ-function terms form a Lorentz invariant object and the density of states terms are also each Lorentz invariant.
  - Therefore, we conclude that $|M_{fi}|^2$ must also be Lorentz invariant.

Decay Rate Calculations

- As the integral above is Lorentz invariant, we can chose to evaluate it in the CoM frame.
  - Therefore, $E_i = m_i$ and $\vec{p}_i = 0$, so

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1(p_1) - E_2(p_2))\delta^3(\vec{p}_1 + \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1(p_1)}\frac{d^3\vec{p}_2}{2E_2(p_2)} \qquad (5)$$

  - Integrating over $\vec{p}_2$ and using the δ-function then gives

$$\begin{aligned} \Gamma_{fi} &= \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1(p_1) - E_2(p_1))\frac{d^3\vec{p}_1}{4E_1(p_1)E_2(p_1)} \\ &= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right)\frac{p_1^2 dp_1 d\Omega}{E_1(p_1)E_2(p_1)} \end{aligned}$$

  - Letting $p_1 = p^*$ where the argument of the δ-function goes to zero and evaluating the integral using the property of δ-functions discussed in lecture 1 and noting that $E_1 + E_2 = m_i$ by energy conservation then eventually gives

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 d\Omega$$

  - Finally, then in the particle's rest frame we have $E_i = m_i$, giving the result

$$\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 d\Omega$$

    - Solving for where the δ-function argument is zero also gives

$$p^* = \frac{1}{2m_i}\sqrt{[m_i^2 - (m_1 + m_2)^2][m_i^2 - (m_1 - m_2)^2]}$$

L3 - Cross Sections

Definitions and Fermi's Golden Rule (again)

- We define the cross section as

$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

  - This is a property of the pair fo things interacting, rather than either one singly.
- Define the differential cross section as

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$
$$\text{where} \quad d\Omega = d(\cos\theta)d\phi$$

- Within a cylinder containing particles travelling in opposite directions, the expected interaction rate is

$$(v_1 + v_2)\nu_2 \sigma AL\nu_1$$

  - The form of Fermi's golden rule as derived above is only valid for the case where $\nu_1 = \nu_2 = \frac{1}{V}$ and $AL = V$, so this simplifies to

$$\begin{aligned} \Gamma_{if} &= (v_1 + v_2)\frac{1}{V}\sigma V\frac{1}{V} \\ \implies \sigma &= \frac{\Gamma_{if} V}{v_1 + v_2} \end{aligned}$$

$2 \to 2$ Body Scattering

- Taking the formula from before for the decay $i \to 1 + 2$ [using (2) and (4)], noting that here $N = 4$ and suitably relabelling, such as for $E_i \to E_1 + E_2$, we find that

$$\Gamma_{if} = \frac{(2\pi)^4}{V(2E_1)(2E_2)}\int |M_{fi}|^2 \delta(E_3 + E_4 - E_1 - E_2)\delta^3(\vec{p}_3 + \vec{p}_4 - \vec{p}_1 - \vec{p}_2)\frac{d^3 p_3}{(2\pi)^3 2E_3}\frac{d^3 p_4}{(2\pi)^3 2E_4}$$

  - Therefore, combining this with the expression for σ above, we find

$$\sigma = \underbrace{\frac{(2\pi)^{-2}}{(2E_1)(2E_2)(v_1 + v_2)}}_{A}\int \underbrace{|M_{fi}|^2}_{\text{physics}}\underbrace{\delta(E_3 + E_4 - E_1 - E_2)}_{\text{energy cons.}}\underbrace{\delta^3(\vec{p}_3 + \vec{p}_4 - \vec{p}_1 - \vec{p}_2)}_{\text{mom cons.}}\underbrace{\frac{d^3 p_3}{2E_3}\frac{d^3 p_4}{2E_4}}_{\text{density of states}} \qquad (6)$$

  - It can be shown with considerable effort that A is LI provided $\vec{p}_1$ and $\vec{p}_2$ are colinear. Therefore, this whole expression is invariant with respect to longitudinal Lorentz boosts. In fact A may be rewritten using the Lorentz Invariant Flux F;

$$(2E_1)(2E_2)(v_1 + v_2) = F \equiv 4\sqrt{(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2}$$

    - This can also be used to reaffirm that $|M_{fi}|^2$ is longitudinally Lorentz invariant, although it is actually fully Lorentz invariant anyway.
    - We will also find it helpful to note that in the CoM frame, this reduces to

$$\begin{aligned} F &= 4E_1^* E_2^*\left(\frac{|\vec{p}_1^*|}{E_1^*} + \frac{|\vec{p}_2^*|}{E_2^*}\right) \\ &= 4(E_2^* + E_1^*)|\vec{p}^*| \quad \text{using} \quad |\vec{p}_1^*| = |\vec{p}_2^*| = |\vec{p}^*| \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

    - Furthermore, in a fixed target frame, this reduces to

$$F = 4E_1 m_2 \frac{|\vec{p}_1|}{E_1} = 4m_2|\vec{p}_1|$$

In the CoM frame

- Here, we have that $\vec{p}_1 + \vec{p}_2 = 0$ and $E_1 + E_2 = \sqrt{s}$, so using (6) we find that

$$\sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s} - E_3 - E_4)\delta^3(\vec{p}_3 + \vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

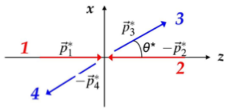
- As this integral is exactly the same as in (4), we simply reuse the result to find

$$\sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}^2 d\Omega^*$$

$$\implies \sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^* \tag{7}$$

  - This reduces even further in the case of elastic scattering where $|\vec{p}_i^*| = |\vec{p}_f^*|$.
  - This is only really useful in the CoM frame and cannot easily be used to calculate the differential cross-section in other frames, as the angles in $d\Omega^*$ refer to the CoM frame.

In Any Frame



- The above gives us $\frac{d\sigma}{d\Omega^*}$, so we could find $\frac{d\Omega^*}{d\Omega}$ to convert to any frame, but this turns out to be very difficult due to the aforementioned angle converting. Instead we try to calculate $\frac{d\sigma}{dt}$ which can then be used with any $\frac{dt}{d\Omega}$ to achieve the result in any frame.
- Using Mandelstram $t = (p_1 - p_3)^2$ we note that in the CoM frame with the setup above we find

$$\begin{aligned} p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|)\,, \\ p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*| \sin\theta^*, 0, |\vec{p}_3^*| \cos\theta^*) \\ \implies p_1^\mu p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*| \cos\theta^* \end{aligned}$$

  - Therefore we get

$$\begin{aligned} t &= m_1^2 + m_3^2 - 2p_1 \cdot p_3 \\ &= m_1^2 + m_3^2 - 2E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*| \cos\theta^* \\ \implies dt &= 2|\vec{p}_1^*||\vec{p}_3^*| d(\cos\theta^*) \end{aligned}$$

  - Combining this with $d\Omega^* = d(\cos\theta^*)d\phi^*$ we then find

$$d\Omega^* = \frac{dt d\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$$

  - Hence substituting this into (7) then gives

$$d\sigma = \frac{1}{2 \cdot 64\pi^2 s|\vec{p}_1^*|^2} |M_{fi}|^2 d\phi^* dt$$

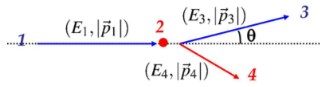
  - Finally, assuming there is no ϕ^* dependence of $|M_{fi}|^2$ gives the lorentz invariant differential cross section as

$$\boxed{\frac{d\sigma}{dq^2} = \frac{d\sigma}{dt} = \frac{1}{64\pi s|\vec{p}_i|^2}|M_{fi}|^2}$$

    - This is valid in all frames with

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$$

In the Lab Frame



- Consider the case of elastic scattering at high energy where the mass of the incoming particles can be neglected; $m_1 = m_3 = 0$ and $m_2 = m_4 = M$. We must express the cross section in terms of the scattering angle, so

$$\begin{aligned} d\Omega &= 2\pi d(\cos\theta) \\ \implies \frac{d\sigma}{d\Omega} &= \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{d\sigma}{dt}\frac{dt}{d(\cos\theta)} \end{aligned}$$

  - Using $t = (p_1 - p_3)^2$ and writing each p_j in terms of $\cos\theta$, it can be shown that

$$\begin{aligned} t &= -2M(E_1 - E_3) \\ \implies \frac{dt}{d(\cos\theta)} &= 2M\frac{dE_3}{d(\cos\theta)} \\ E_3 &= \frac{E_1 M}{M + E_1 - E_1\cos\theta} \\ \implies \frac{dE_3}{d(\cos\theta)} &= \frac{E_3^2}{M} \end{aligned}$$

  - Combining all of the above and noting that here $|\vec{p}_i^*|^2 = \frac{(s-M^2)^2}{4s}$ and $s = M^2 + 2ME_1$, we find

$$\frac{d\sigma}{d\Omega} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s - M^2)^2}|M_{fi}|^2$$

$$\begin{aligned} \implies \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \quad \text{where} \quad m_1 \to 0 \qquad (8) \\ &= \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad (9) \end{aligned}$$

- This can also be done in the lab frame where $m_1 \neq 0$, giving

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2} \cdot \frac{1}{|m_2||\vec{p}_1|} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |\mathcal{M}_{fi}|^2 \\ \text{with} \quad E_1 + m_2 &= \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2}. \end{aligned}$$

H2 - The Dirac Equation

H1 - Introduction H3 - Interaction by Particle Exchange and QED

Non-Relativistic QM

- The Schrödinger equation is obviously not LI. Nevertheless, it can be written (with $V=0$) as;

$$-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t} \tag{1}$$
$$-\frac{1}{2m}\vec{\nabla}^2\psi^* = -i\frac{\partial\psi^*}{\partial t} \tag{2}$$

  - Then taking $\psi^*\times(1)-\psi\times(2)$ gives

$$-\frac{1}{2m}\vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*) = \frac{\partial}{\partial t}(\psi^*\psi)$$

  - Then compare this with the continuity equation

$$\vec{\nabla}\cdot\vec{j}+\frac{\partial\rho}{\partial t}=0$$

    - We find a probability density and current;

$$\rho = |\psi|^2$$
$$\vec{j} = \frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*\right)$$

The Klein-Gordon Equation

- We apply the quantum operators to the relativistic energy equation to obtain the KG equation;

$$\begin{aligned} E^2 &= |\vec{p}|^2+m^2 \\ \implies \frac{\partial^2\psi}{\partial t^2} &= \vec{\nabla}^2\psi - m^2\psi \\ \implies 0 &= (\partial^\mu\partial_\mu+m^2)\psi \end{aligned}$$

  - For plane wave solutions $\psi = Ne^{i(\vec{p}\cdot\vec{r}-Et)}$ this has solutions $E=\pm\sqrt{|\vec{p}|^2+m^2}$. Historically, the negative solutions were viewed as problematic.
  - Furthermore, if we continue as before to find probability and current densities then we obtain;

$$\begin{aligned} \frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right) &= \vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*) \\ \implies \rho &= i\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right) \\ \text{and}\quad \vec{j} &= i(\psi^*\vec{\psi}-\psi\vec{\psi}^*) \end{aligned}$$

    - For a plane wave $\psi = Ne^{i(\vec{p}\cdot\vec{r}-Et)}$ we therefore find

$$\rho = 2E|N|^2 \quad \text{and} \quad \vec{j} = 2\vec{p}|N|^2$$

    - This incorporates length contraction as $E\propto\gamma$. However, due to the negative energy solutions, also implies the existence of negative particle densities?!?!?!?

L4 - The Dirac Equation

The Dirac Equation

- To solve this issue, Dirac proposes a Hamiltonian with only linear terms

$$\hat{H}\psi = (\vec{\alpha}\cdot\vec{p}+\beta m)\psi = i\frac{\partial\psi}{\partial t} \tag{3}$$

  - Applying this on itself gives

$$\begin{aligned} \left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\psi &= -\frac{\partial^2\psi}{\partial t^2} \\ \implies -\frac{\partial^2\psi}{\partial t^2} = -\alpha_x^2\frac{\partial^2\psi}{\partial x^2}-\alpha_y^2\frac{\partial^2\psi}{\partial y^2}-\alpha_z^2\frac{\partial^2\psi}{\partial z^2}+\beta^2m^2\psi & \\ -(\alpha_x\alpha_y+\alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y}-(\alpha_y\alpha_z+\alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z}-(\alpha_z\alpha_x+\alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} & \\ -(\alpha_x\beta+\beta\alpha_x)m\frac{\partial\psi}{\partial x}-(\alpha_y\beta+\beta\alpha_y)m\frac{\partial\psi}{\partial y}-(\alpha_z\beta+\beta\alpha_z)m\frac{\partial\psi}{\partial z}. & \end{aligned}$$

  - A free particle must also obey $E^2=\vec{p}^2+m^2$, and therefore also the KG equation;

$$-\frac{\partial^2\psi}{\partial t^2} = -\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}+m^2\psi$$

    - For this compatibility, we therefore require

$$\alpha_x^2=\alpha_y^2=\alpha_z^2=\beta^2=1, \tag{4}$$
$$\alpha_j\beta+\beta\alpha_j=0, \tag{5}$$
$$\alpha_j\alpha_k+\alpha_k\alpha_j=0 \quad (j\neq k). \tag{6}$$

    - This implies α_j and β must be four mutually anti-commuting (4×4) matrices and $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)$. Therefore, the wave equation has extra degrees of freedom!
    - Further imposing that the Hamiltonian must be Hermitian requires each of these matrices to be Hermitian.
  - Although everything could be derived from (4-6), we introduce explicit representations for $\vec{\alpha}$ and β for convenience such that

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{with}$$
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Probability Density and Current

- Now just for the fun of it, where D is the Dirac equation (3) we compute $\psi^\dagger\times D - D^\dagger\times\psi$, which eventually (using identity $\frac{\partial(\psi^\dagger\alpha_x\psi)}{\partial x} = \psi^\dagger\alpha_x\frac{\partial\psi}{\partial x}+\frac{\partial\psi^\dagger}{\partial x}\alpha_x\psi$) gives the continuity equation;

$$\vec{\nabla}\cdot(\psi^\dagger\vec{\alpha}\psi)+\frac{\partial\psi^\dagger\psi}{\partial t}=0$$

  - We can then identify probability density and current with $\rho=\psi^\dagger\psi>0$ and $\vec{j}=\psi^\dagger\vec{\alpha}\psi$.
    - This solves the negative probability density issue from the KG equation and also introduces the four-component Dirac Spinors, which naturally give rise to intrinsic spin and antiparticles with magnetic moment $\vec{\mu}=\frac{1}{m}\vec{S}$.

Covariant Form

- Introducing the Dirac gamma matrices such that $\gamma^0=\beta$, $\gamma^1=\beta\alpha_x$, $\gamma^2=\beta\alpha_y$ and $\gamma^3=\beta\alpha_z$, then premultiplying D by β, it may be rewritten as

$$\gamma^1\frac{\partial\psi}{\partial x}+i\gamma^2\frac{\partial\psi}{\partial y}+i\gamma^3\frac{\partial\psi}{\partial z}-m\psi = -i\gamma^0\frac{\partial\psi}{\partial t}$$

$$\implies (i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{7}$$

  - However, also note that the gamma matrices are not four-vectors and are constant matrices which do not change under Lorentz transform. Despite this, the Dirac equation is itself Lorentz covariant with ψ and ∂_μ changing.

- It can also be found that the gamma matrices obey the anti-commutation relations below which define an algebra;

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

- Furthermore, γ^0 is Hermitian, whilst γ^1, γ^2 and γ^3 are anti-Hermitian;

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{1\dagger} = -\gamma^1, \quad \gamma^{2\dagger} = -\gamma^2, \quad \gamma^{3\dagger} = -\gamma^3$$

- The γ matrices may also be written in Pauli-Dirac representation as;

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

  - Using these, we can rewrite ρ and $\vec{j}$ using $j^\mu = (\rho, \vec{j}) = \psi^\dagger\gamma^0\gamma^\mu\psi$ which is a four-vector current, simplifying the continuity equation to $\partial_\mu j^\mu = 0$.

- We therefore introduce the adjoint spinor defined as $\bar{\psi}$, which also allows j^μ to be rewritten;

$$\begin{aligned} \bar{\psi} &= \psi^\dagger\gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) \\ \implies j^\mu &= \bar{\psi}\gamma^\mu\psi \end{aligned}$$

Plane Wave Solutions

- Looking for plane wave solutions to the Dirac Equation of the form (note $(u_+, u_-) \leftrightarrow (u, v)$ sometimes)

$$\psi = e^{\pm i(\vec{p}\cdot\vec{r} - Et)} u_\pm(E, \vec{p})$$

- We therefore find from (7) that

$$\begin{aligned} (i\gamma^\mu(\mp ip_\mu) - m)e^{\pm i(\vec{p}\cdot\vec{r}-Et)}u_\pm(E,\vec{p}) &= 0 \\ \implies (\gamma^\mu p_\mu \mp m)u_\pm &= 0 \end{aligned}$$

- Writing this in the Dirac Algebra gives

$$\begin{aligned} \left(E\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - \vec{p}\cdot\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \mp m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right)u_\pm &= 0 \\ \implies \begin{pmatrix} E \mp m & -\vec{\sigma}\cdot\vec{p} \\ +\vec{\sigma}\cdot\vec{p} & -E \mp m \end{pmatrix}\begin{pmatrix} A_\pm \\ B_\pm \end{pmatrix} &= 0 \end{aligned}$$

  - Now note that the two simultaneous equations encoded here are really the same. This can be shown by first noting that

$$(\vec{\sigma}\cdot\vec{p})^2 = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}^2 = (p_x^2 + p_y^2 + p_z^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\vec{p}|^2 I = (E^2 - m^2)I = (E-m)(E+m)I$$

    - Then where $E^2 \neq m^2$ (note this is fine as nothing special happens as $p \to 0$ so we may assume $E^2 \neq m^2$ wherever required);

$$\begin{aligned} &[(E \mp m)A_\pm = (\vec{\sigma}\cdot\vec{p})B_\pm] && (8) \\ \implies &[(E \mp m)(\vec{\sigma}\cdot\vec{p})A_\pm = (\vec{\sigma}\cdot\vec{p})^2 B_\pm] \\ \implies &[(E \mp m)(\vec{\sigma}\cdot\vec{p})A_\pm = (E-m)(E+m)B_\pm] \\ \implies &[(\vec{\sigma}\cdot\vec{p})A_\pm = (E \pm m)B_\pm] && (9) \end{aligned}$$

- Therefore, we regard one of $A_\pm$ and $B_\pm$ free and the other fixed. When considering u_+, we chose to use (8) to fix B_+ in terms of A_+ and when considering u_-, we use (9) to fix A_- in terms of B_-, such that;

$$\begin{aligned} u_+ = u &\in \left\{ \begin{pmatrix} A_+ \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}A_+ \end{pmatrix} \middle| \forall A_+ \right\}, \\ u_- = v &\in \left\{ \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}B_- \\ B_- \end{pmatrix} \middle| \forall B_- \right\}. \end{aligned}$$

  - Equivalently we could also say $u_+ = u \in \text{span}\{u_1, u_2\}$ and $u_- = v \in \text{span}\{v_1, v_2\}$ with

$$\begin{aligned} u_1 &= N\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}, \quad u_2 = N\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}, \\ v_1 &= N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}, \quad v_2 = N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}. \end{aligned}$$

    - Note that these unpack with matrix multiplication as $\vec{\sigma}\cdot\vec{p}$ is a 2×2 matrix.
    - Also, for $\psi^\dagger(u_1)\psi(u_1) = \psi^\dagger(u_2)\psi(u_2) = \psi^\dagger(v_1)\psi(v_1) = \psi^\dagger(v_2)\psi(v_2) = 2E$ as required for the 2E-particles-per-unit-volume normalisation, then we also need

$$N = \sqrt{E+m}$$

- Real observations show that Dirac particles have positive energy, but using the plane wave form we find

$$\hat{E}\psi(u_\pm) = \left(i\frac{\partial}{\partial t}\right)\left(e^{\pm(\vec{p}\cdot\vec{r}-Et)}u_\pm(E,\vec{p})\right) = \pm E\psi(u_\pm)$$

  - Dirac tried to account for these negative energies using a sea / hole model where the vacuum contains all -ve energy states which are filled by Pauli exclusion, and particles in these states correspond to -ve energy particles.

Charge Conjugation

- In relativistic electrodynamics, we have the substitution $\partial_\mu \to \partial_\mu + ieA_\mu$ for $A^\mu = (\phi, \vec{A})$. Appling this to the Dirac equation gives;

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0 \tag{10}$$

  - Now taking the complex conjugate and pre-multiplying by $-i\gamma^2$ gives;

$$-i\gamma^2\gamma^{\mu*}(\partial_\mu - ieA_\mu)\psi^* - m\gamma^2\psi^* = 0 \tag{11}$$

  - Then using the fact that in the form used so far,

$$\begin{aligned} \gamma^{0*} = \gamma^0; \quad \gamma^{1*} = \gamma^1; \quad \gamma^{2*} = -\gamma^2; \quad \gamma^{3*} = \gamma^3 \\ \underbrace{\implies}_{\text{with some expanding}} \gamma^2\gamma^{\mu*} = -\gamma^\mu\gamma^2 \end{aligned}$$

    - We may therefore rewrite (11) using $i^2 = -1$ and the Charge Conservation Operator $\hat{C}$:

$$\text{Definition:} \quad \psi' = \hat{C}\psi = i\gamma^2\psi^*$$

$$\begin{aligned} \implies \gamma^\mu(\partial_\mu - ieA_\mu)i\gamma^2\psi^* + imi\gamma^2\psi^* &= 0 \\ \implies \gamma^\mu(\partial_\mu - ieA_\mu)\psi' + im\psi' &= 0 \end{aligned}$$

    - Finally, comparing this with (10), we see that the spinor ψ' describes a particle with the same mass, but opposite charge - an anti-particle!
      - Hence, $\hat{C}$ converts a particle spinor to an anti-particle spinor, so every Fermion has an anti-fermionic partner!

- Applying $\hat{C}$ on the free particle wavefunction, we find that

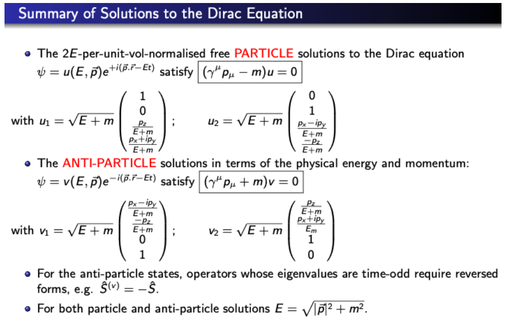
$$\psi' = \hat{C}\psi = \hat{C}\left(u_j e^{i(\vec{p}\cdot\vec{r}-Et)}\right) = v_j e^{-i(\vec{p}\cdot\vec{r}-Et)}$$

  - Therefore, we can call v_2 the anti-partner of u_1 and v_2 the anti-partner of u_2.

Special Operators for Anti-Particle Solutions

- As this course is somewhere between QFT and experiment, we define different operators to act on anti-particles such as to fix the signs;

$$\hat{H}^{(\nu)} = -i\frac{\partial}{\partial t}$$
$$\hat{p}^{(\nu)} = i\vec{\nabla}$$
$$\hat{S}^{(\nu)} = -\hat{S}$$



L5 - Spin, Helicity and Parity

Spin

- Consider Ehrenfest's Theorem;

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle [H,A]\rangle + \langle\frac{\partial A}{\partial t}\rangle$$

  - In the Dirac formulation, we want to check whether orbital angular momentum L is a good quantum number. For this, $L = \vec{r}\times\vec{p}$ must commute with the Hamiltonian; $[H,\vec{L}] = [\vec{\alpha}\cdot\vec{p} + \beta m, \vec{r}\times\vec{p}] = [\vec{\alpha}\cdot\vec{p}, \vec{r}\times\vec{p}]$.
  - Continuing in the normal way, we find that $[H, L_x] = -i(\alpha_y p_z - \alpha_z p_y) = -i(\vec{\alpha}\times\vec{p})_x$.
    - Eventually, we find that L is not a good quantum number, as;

$$[H,\vec{L}] = -i\vec{\alpha}\times\vec{p}$$

    - Therefore we have found that orbital angular momentum is not a constant of motion!
- Now introduce a 4×4 operator

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

  - We then wish to evaluate the commutator $[H,\vec{\Sigma}] = [\vec{\alpha}\cdot\vec{p} + \beta m, \vec{\Sigma}]$. First analysing the β part;

$$[\beta,\vec{\Sigma}] = \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} - \begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} = 0$$

    - Therefore, $[H,\vec{\Sigma}] = [\vec{\alpha}\cdot\vec{p},\vec{\Sigma}]$. Analysing the x-component of this commutator, we find

$$[\alpha_x,\Sigma_x] = 0$$
$$[\alpha_y,\Sigma_x] = -2i\alpha_z$$
$$[\alpha_z,\Sigma_x] = 2i\alpha_y$$
$$\implies [H,S_x] = \frac{1}{2}(0 - 2ip_y\alpha_x + 2ip_z\alpha_y) = i(\vec{\alpha}\times\vec{p})_x$$

    - In general it is then found that $\vec{S}$ is not a constant of motion;

$$[H,\vec{S}] = i\vec{\alpha}\times\vec{p}$$

- Combining the above two observations we then find an amazing result, $L+S$ is conserved!

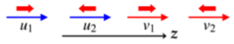
$$\frac{d}{dt}(L+S) \propto [H,\vec{L}+\vec{S}] = 0$$

  - Furthermore, as $\vec{S} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$, the commutation relations for $\vec{S}$ are the same as for $\vec{\sigma}$.
  - Even-furthermore, S^2 and S_z are diagonal;

$$S^2 = \frac{1}{4}(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) = \frac{3}{4}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}; \quad S_z = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

    - This also gives $S^2\psi = S(S+1)\psi = \frac{3}{4}$ and for a particle travelling along z, $S_z\psi = \pm\frac{1}{2}\psi$, so Dirac particles are spin-$\frac{1}{2}$.
  - Note that general spinors u_1, u_2, v_1, v_2 are not Eigenstates of $\hat{S}_z$, but particles/ anti-particles travelling in the z-direction are;

$$\hat{S}_z u_2 = -\frac{1}{2}u_2, \qquad \hat{S}_z u_1 = +\frac{1}{2}u_1$$
$$\hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\frac{1}{2}v_1, \qquad \hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\frac{1}{2}v_2$$



Helicity

- As $[H,\vec{S}\cdot\vec{p}] = 0$, the component of a particle's spin along its direction of flight is a good quantum number. Therefore, we define the helicity operator

$$h = \vec{\Sigma}\cdot\hat{\vec{p}} = \vec{\Sigma}\cdot\left(\frac{\vec{p}}{|\vec{p}|}\right)$$

  - Due to this definition, helicity of a particle is independent of boosts along the particle's momentum direction and helicity will be either ± 1.
  - Conventionally $h = +1$ is called right-handed and $h = -1$, left-handed. Don't confuse with chiral eigenstates.
- We now wish to make a better spinor basis - removing the arbitrary choice of $A_+ = (1,0)^T$ from before.
  - This can be achieved by noting that since $\vec{\Sigma}\cdot\hat{p} \propto \vec{\sigma}\cdot\vec{p}$ and all the spinor-halves here are already proportional to some power of $\vec{\sigma}\cdot\vec{p}$, we may find the new basis $\{u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow\}$ by replacing the arbitrary choices of A_+ or B_- with the eigenvectors of $\vec{\sigma}\cdot\vec{p}$.
  - Assuming momentum points in the (θ,ϕ) direction, then

$$\vec{p} = |\vec{p}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\implies \vec{\sigma}\cdot\hat{\vec{p}} = \begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$$

    - Doing the maths we then find the normalised eigenvalues of this as

$$\vec{e}_+ = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} \quad \text{and} \quad \vec{e}_- = \begin{pmatrix}-\sin\frac{\theta}{2}\\ e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix}$$

- Accordingly, we change to a basis

$$u_\uparrow = N\begin{pmatrix}\vec{e}_+ \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\vec{e}_+\end{pmatrix}, \quad u_\downarrow = N\begin{pmatrix}\vec{e}_- \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\vec{e}_-\end{pmatrix},$$
$$v_\uparrow = N\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\vec{e}_- \\ \vec{e}_-\end{pmatrix}, \quad v_\downarrow = N\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\vec{e}_+ \\ \vec{e}_+\end{pmatrix}.$$

  - Fully expanded (excluding $N = \sqrt{E+m}$ pre-factors);

$$u_\uparrow \propto \begin{pmatrix}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}\cos\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}, u_\downarrow \propto \begin{pmatrix}-\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}\sin\frac{\theta}{2} \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix}, v_\uparrow \propto \begin{pmatrix}\frac{|\vec{p}|}{E+m}\sin\frac{\theta}{2} \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix}, v_\downarrow \propto \begin{pmatrix}\frac{|\vec{p}|}{E+m}\cos\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$$

Intrinsic Parity

- Parity is defined by spatial inversion through the origin;

$$x' \equiv -x, \quad y' \equiv -y, \quad z' \equiv -z, \quad t' \equiv t$$

  - Consider premultiplying the Dirac equation (7) by γ^0 and substituting from unprimed to primed coordinates and then simplifying (note the use of $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I$ in the second step);

$$\begin{aligned} -i\gamma^0\gamma^1\frac{\partial\psi}{\partial x'} - i\gamma^0\gamma^2\frac{\partial\psi}{\partial y'} - i\gamma^0\gamma^3\frac{\partial\psi}{\partial z'} - m\gamma^0\psi &= -i\gamma^0\gamma^0\frac{\partial\psi}{\partial t'} \\ \implies i\gamma^1\frac{\partial(\gamma^0\psi)}{\partial x'} + i\gamma^2\frac{\partial(\gamma^0\psi)}{\partial y'} + i\gamma^3\frac{\partial(\gamma^0\psi)}{\partial z'} - m\gamma^0\psi &= -i\gamma^0\frac{\partial\gamma^0\psi}{\partial t'} \\ \implies i\gamma^1\frac{\partial\psi'}{\partial x'} + i\gamma^2\frac{\partial\psi'}{\partial y'} + i\gamma^3\frac{\partial\psi'}{\partial z'} - m\psi' &= -i\gamma^0\frac{\partial\psi'}{\partial t'} \end{aligned}$$

  - In the final step we introduce $\psi'(x', y', z', t')$ defined by $\psi'(x', y', z', t') = \gamma^0\psi(x, y, z, t)$.
    - Therefore, we have found that the parity operator to transform spinors to the parity-conjugates has the form $\hat{P} = \lambda\gamma^0$ for some λ.
    - Further constraining $\hat{P}$ to be Hermitian and to have $(\hat{P})^2 = 1$ required $\lambda = \pm 1$, but this is most often taken as $\lambda = 1$, so the Parity Operator;

$$\hat{P} = \gamma^0$$

  - Taking the $\vec{p} \to 0$ limit of this basis () gives;

$$u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad v_1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$
$$\text{and} \quad \gamma^0 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

    - Therefore, the fermions (u_1 and u_2) has positive parity and the anti-fermions (v_1 and v_2) have negative parity when at rest.

H9 - the Weak Interaction and V-A

H8 - Quantum Chromodynamics

H10 - Leptonic Weak Interactions and Neutrino Deep Inelastic Scattering

Parity

- The parity operator performs spatial inversion through the origin;

$$\psi'(\vec{x}, t) = \hat{P}\psi(\vec{x}, t) = \psi(-\vec{x}, t)$$

  - Furthermore, to preserve wave-function normalisation,

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\psi|\hat{P}^\dagger\hat{P}|\psi\rangle$$

  - Hence, $\hat{P}^\dagger\hat{P} = I$ and noting that $\hat{P}\hat{P} = I$, we see $\hat{P}^{-1} = \hat{P}^\dagger = \hat{P}$, so $\hat{P}$ is unitary and Hermitian.
    - We therefore expect parity to be an observable which is conserved if $[\hat{H}, \hat{P}] = 0$.
    - Furthermore, if λ is the eigenvalue for an eigenvector $\vec{x}$ of $\hat{P}$, then $\hat{P}^2\vec{x} = \lambda^2\vec{x} = \vec{x}$, so $\lambda^2 = 1$.
- Gauge Field Theory eventually shows that gauge bosons have $P = P_\gamma = P_g = P_{W^+} = P_{W^-} = P_Z = -1$.
- As shown (here), we conventionally take spin-1/2 particles to have parity +1 and spin-1/2 anti-particles to have spin -1. We also previously saw that for Dirac spinors, the parity operator;

$$\hat{P} = \gamma^0 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

  - As an ordinary and adjoint spinors therefore transform as;

$$u \xrightarrow{\hat{P}} u' \equiv \hat{P}u = \gamma^0 u$$
$$\bar{u} \xrightarrow{\hat{P}} \bar{u}' \equiv \overline{\hat{P}u} = \bar{u}\hat{P} = \bar{u}\gamma^0$$

    - Note that the adjoint results follow from;

$$\overline{\hat{P}u} = (\hat{P}u)^\dagger\gamma^0 = (\gamma^0 u)^\dagger\hat{P} = u^\dagger\gamma^{0\dagger}\hat{P} = u^\dagger\gamma^0\hat{P} = \bar{u}\hat{P}$$

- For convenience, we can also define vectors such that $\vec{v} \xrightarrow{\hat{P}} -\vec{v}$ and axial vectors such that $\vec{a} \xrightarrow{\hat{P}} \vec{a}$.
  - For example, $\vec{L} = \vec{r} \times \vec{p}$ is an axial vector, as both $\vec{r}$ and $\vec{p}$ acquire minus signs which cancel.

Currents

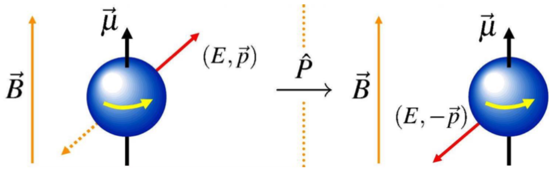
- All currents transform as Lorentz four-vectors under continuous transformations - however, Parity transformations are not continuous. Applying the parity operator to a 4-current gives;

$$j^\mu = \bar{\phi}\gamma^\mu\phi \xrightarrow{\hat{P}} (j')^\mu = (\bar{\phi}\gamma^0)\gamma^\mu(\gamma^0\psi) = \bar{\phi}(\gamma^0\gamma^\mu\gamma^0)\psi$$
$$\implies \begin{cases} j^0 \xrightarrow{\hat{P}} \bar{\phi}\gamma^0\psi = j^0 \\ j^i \xrightarrow{\hat{P}} -\bar{\phi}\gamma^i\psi = -j^i \end{cases}$$

  - Hence, currently behave like normal 4-vectors under parity, so scalar products between currents are invariant under parity.
  - As QED and QCD matrix elements involve terms like $j_e \cdot j_q$, they are also invariant under parity.
    - Therefore, QED and QCD conserve parity and predictions are invariant under parity inversion.

Parity Violation - β-Decay

- Observing β decays in an applied magnetic field, it is seen that electrons are emitted preferentially in a direction opposite to the applied B-field.
    - Considering that $\vec{B}(\vec{r}) \propto I d\vec{l} \times \vec{r}'$ according to the Biot-Savart law, we see that B is an axial vector, whereas the momentum of emitted electrons, p is a normal vector.



    - Therefore, applying parity, on a β-decay, we (surprisingly) see that in a coordinate inverted system, B is unchanged, but $\vec{p}$ is inverted relative to what they were in a non-inverted coordinate system.
        - Therefore, parity must be violated by the weak interaction and the weak vertex is **not** of the form $\bar{u}_e \gamma^\mu u_v$.

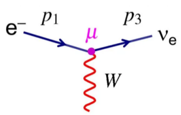
The Weak Interaction

- The requirement for Lorentz invariance severely restricts possible forms of an interaction vertex. There are only 5 possible combinations of two spinors and the gamma matrices that form Lorentz covariant currents, the "bilinear covariants";

Type	Form	Components	Boson Spin
Scalar	$\bar{\psi}\phi$	1	0
Pseudoscalar	$\bar{\psi}\gamma^5\phi$	1	0
Vector	$\bar{\psi}\gamma^\mu\phi$	4	1
Axial Vector	$\bar{\psi}\gamma^\mu\gamma^5\phi$	4	1
Tensor	$\bar{\psi}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\phi$	6	2

    - These 16 components are linearly independent, forming a basis for a general 4×4 matrix - a linear combination can therefore represent the interaction between any fermion and boson.

The V-A Structure of the Weak Interaction



- For interactions involving spin-1 bosons, the most general vertex element is a linear combination of vector and axial-vector currents. This is observed to take the form vector - axial-vector, also known as $V - A$. Hence, for the process above,

$$j^\mu \propto \bar{u}_{\nu_e}(\gamma^\mu - \gamma^\mu\gamma^5)u_e$$

    - Applying parity on an axial-vector gives (note the difference of the vector case)

$$j_A = \bar{\psi}\gamma^\mu\gamma^5\phi \xrightarrow{\hat{P}} \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\phi = -\bar{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^5\phi$$
$$\implies \begin{cases} j_A^0 \xrightarrow{\hat{P}} -\bar{\psi}\gamma^0\gamma^5\phi = -j_A^0 \\ j_A^i \xrightarrow{\hat{P}} \bar{\psi}\gamma^k\gamma^5\phi = j_A^k \end{cases}$$

    - Considering any matrix element $M \propto g_{\mu\nu} j_a^\mu j_2^\nu = j_1^0 j_2^0 - \vec{j}_1 \cdot \vec{j}_2$, we see that if both currents are axial-vectors, then M is invariant under parity (the minuses cancel). Therefore, parity is also conserved for pure axial-vector interactions.
    - However, if one of the currents is a vector and the other an axial vector, we have

$$j_{v1} \cdot j_{A2} \xrightarrow{\hat{P}} \left(\left(j_{v1}^0\right)\left(-j_{A2}^0\right) - \left(-\vec{j}_{v1}\right) \cdot \left(\vec{j}_{A2}\right)\right) = -j_{v1} \cdot j_{A2}$$

        - This changes sign under parity, so can interfere with terms in M to give parity violating cross-sections.
    - Considering matrix elements of the form $g_v\gamma^\mu + g_A\gamma^\mu\gamma^5$, it becomes apparent (with some work) that parity is conserved if either g_A or g_v is zero and the relative strength of a parity violating part is proportional to

$$\frac{g_v g_A}{g_v^2 + g_A^2}$$

Chirality and Helicity in the Weak Interaction

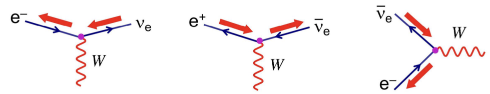
- The $W^\pm$ weak vertex is;

$$-\frac{ig_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1 - \gamma^5)$$

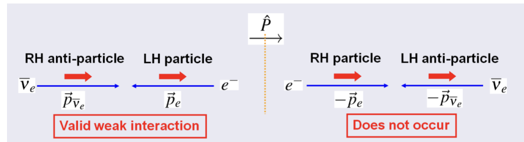
    - Hence, as $\frac{1}{2}(1 - \gamma^5)$ projects the left-handed chiral particle states, we see that where $\bar{\psi} = \bar{\psi}_R + \bar{\psi}_L$,

$$-\bar{\psi}\frac{ig_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1 - \gamma^5)\phi = -\frac{ig_W}{\sqrt{2}}\bar{\psi}_L\gamma^\mu\phi_L$$

    - **Therefore, only the left chiral components of spinors participate in weak charged current interactions.**
        - **In the relativistic limit, this implies that only left-helicity particles and right-helicity antiparticles interact by the weak charged current.** The only possible electron-neutrino interactions (in the relativistic limit) are then;



            - Note here, that the arrows on a Feynman diagram don't denote the direction in which a particle is moving, so where the spins are labelled by red arrows, and if we assume a very fast incoming particles, this all makes sense.
        - Note that this means some interactions, such as $\bar{\nu}_e + e^- \to W^-$ are parity-violating;



- Also note, this implies all neutrinos are left helicity and all anti-neutrinos are right helicity.

Helicity in Pion Decay

- Somewhat surprisingly, the decay of a pion into electrons is suppressed greatly relative to the decay into muons;

$$\frac{\Gamma(\pi^- \to e^-\bar{\nu}_e)}{\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)} = 1.23 \times 10^{-4}$$

    - Considering this decay in the frame of the (spin-0) pion, we see that the lepton and anti-neutrino must have spins in opposite directions and they must travel in opposite directions.
        - Then noting that the anti-neutrino is almost massless, it must be in a right-helicity state, **so the lepton must also be in a right helicity state**.
            - Recalling that the weak interaction only produces left chiral states, and noting that an electron is much lighter than a muon, so is always in a state very similar to that of chirality, we expect the electron decay rate to be suppressed relative to that of a muon decay. This is also found if we calculate the matrix element explicitly.
                - Note, this is due to the fact that the right-helicity contribution to a left-chirality state is not always zero.

- Sometimes, decay to a right-helicity particle state is the only possibility.

The Weak Current Propagator

- Unlike QED and QCD, the weak interaction is mediated by massive W-bosons, resulting in a different form for the propagator (where μ and ν denote the ends of the W-boson connection);

$$\frac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m_W^2\right]}{q^2 - m_W^2}$$

  - Note however that where $q^2 \ll m_W^2$ this simplifies to

$$\frac{i g_{\mu\nu}}{m_W^2}$$

Fermi Theory

- Before parity violation was known, Fermi proposed a matrix element for β-decay of the form $M_{fi} = G_F g_{\mu\nu}[\bar{\psi}\gamma^\mu\psi][\bar{\psi}\gamma^\mu\psi]$. Upon the discovery of parity violation, this was updated to;

$$M_{fi} = \frac{G_F}{\sqrt{2}} g_{\mu\nu}\left[\bar{\psi}\gamma^\mu\left(1-\gamma^5\right)\psi\right]\left[\bar{\psi}\gamma^\nu\left(1-\gamma^5\right)\psi\right]$$

  - The modern version is;

$$M_{fi} = \left[\frac{g_W}{\sqrt{2}}\bar{\psi}\frac{1}{2}\gamma^\mu\left(1-\gamma^5\right)\psi\right]\frac{g_{\mu\nu} - q_\mu q_\nu / m_W^2}{q^2 - m_W^2}\left[\frac{g_W}{\sqrt{2}}\bar{\psi}\frac{1}{2}\gamma^\nu\left(1-\gamma^5\right)\psi\right]$$

  - In the $q^2 \ll m_W^2$ limit, this reduces to

$$M_{fi} = \frac{g_W^2}{8m_W^2} g_{\mu\nu}\left[\bar{\psi}\gamma^\mu\left(1-\gamma^5\right)\psi\right]\left[\bar{\psi}\gamma^\nu\left(1-\gamma^5\right)\psi\right]$$

    - Hence, for consistency we require

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}$$

- Defining the strength of a weak interaction with

$$\alpha_W = \frac{g_W^2}{4\pi} \approx \frac{1}{30}$$

  - Where the final equality comes from above substituting experimental values.
  - The weak interaction is therefore intrinsically slightly stronger than the EM interaction, but is made to appear weak due to the high W-boson mass in the propagator.
    - For $q^2 \gg m_W^2$, weak interactions are more likely than EM interactions.

H4 - Electron-Positron Annihilation

H3 - Interaction by Particle Exchange and QED H5 - Electron-Proton Elastic Scattering

QED Calculations

- To calculate a cross section using QED:
  1. Draw all possible (unique) Feynman diagrams - including those of high orders.
  2. For each diagram, calculate the matrix element.
  3. Sum the individual matrix elements (note this ruins any notion of knowing which virtual particle is involved - much like you don't know which slit a photon passes through in Young's slits);

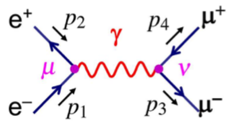
$$M_{fi} = M_1 + M_2 + M_3 + \dots$$

  4. Find the square of the modulus

$$|M_{fi}|^2 = (M_1 + M_2 + M_3 + \dots)(M_1^* + M_2^* + M_3^* + \dots)$$

     - For QED, we have $e^2 \propto \alpha_{em} \sim \frac{1}{137}$, so it is often sufficient to neglect higher order diagrams.
  5. Apply decay rate/ cross section formulae from .

Electron and Muon Currents



- Applying this process to the electron-muon interaction above, we find (to first order)

$$M = -\frac{e^2}{s} g_{\mu\nu}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma^\nu v(p_4)\right]$$

  - We therefore chose to define electron and muon currents as (noting that $j_e \cdot j_\mu = (j_e)^\mu (j_\mu)^\nu g_{\mu\nu}$);

$$(j_e)^\mu = \bar{v}(p_2)\gamma^\mu u(p_1) \quad \text{and} \quad (j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$$
$$\implies M = -\frac{e^2}{s} j_e \cdot j_\mu$$

- In general, we find the current $\bar{\psi}\gamma^\mu\phi$ expands as;

$$\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4, && (1)\\
\overline{\psi}\gamma^1\phi &= \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1, && (2)\\
\overline{\psi}\gamma^2\phi &= \psi^\dagger\gamma^0\gamma^2\phi = -i\left(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1\right), && (3)\\
\overline{\psi}\gamma^3\phi &= \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2. && (4)
\end{aligned}$$

- Note that it can sometimes save time to relate $\bar{\psi}\gamma^\mu\phi$ to $\bar{\phi}\gamma^\mu\psi$ using the Hermitian conjugate and the fact that $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$;

$$\begin{aligned}
\left[\overline{u}(p_3)\gamma^\mu v(p_4)\right]^\dagger &= \left[u(p_3)^\dagger\gamma^0\gamma^\mu v(p_4)\right]^\dagger\\
&= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_3)\\
&= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^0 u(p_3) && (5)\\
&= v(p_4)^\dagger\gamma^0\gamma^\mu u(p_3)\\
&= \overline{v}(p_4)\gamma^\mu u(p_3).
\end{aligned}$$

Spin Considerations

- In a general e^+e^- annihilation event, each electron could have either right or left handed helicity. The same is true for the muons. Hence, there are 16 possible helicity combinations. Therefore,

$$\langle |M|^2 \rangle = \frac{1}{4}\sum_{\text{spins}} |M_i|^2 = \frac{1}{4}\left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \dots\right)$$

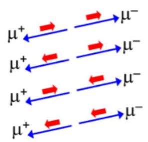
  - This is difficult to work with, so take the high energy $E \gg m$ limit such that where $c = \cos\theta/2$ and $s = \sin\theta/2$, the left ($\downarrow$) and right ($\uparrow$)-handed particles / antiparticle spinors reduce to;

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}, \quad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};$$
$$v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}, \quad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}.$$

Electron-Positron Annihilation

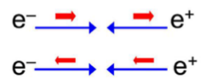
- Again considering the electron-positron annihilation case shown above, but now assuming the electron and positron are incident along the x-axis ($\theta = 0$) in the CoM frame gives different terms ($u_\uparrow(p_1)$ or $u_\downarrow(p_1)$, $v_\uparrow(p_2)$ or $v_\downarrow(p_2)$, $u_\uparrow(p_3)$ or $u_\downarrow(p_3)$, $v_\uparrow(p_4)$ or $v_\downarrow(p_4)$) depending on the helicity states.
  - For the decay products (muons), we may arbitrarily say one is directed with $(\theta,\phi) = (\theta, 0)$ and the other $(\theta,\phi) = (\pi - \theta, \pi)$.
  - Using these states and equations (1-4) we find that for the two muons produced in the above e^+e^- annihilation event, the current terms are

$$\overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta), \tag{6}$$
$$\overline{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) = (0,0,0,0),$$
$$\overline{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) = (0,0,0,0),$$
$$\overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta). \tag{7}$$



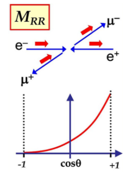
- **Only two of these helicity combinations are non-zero. This is a general feature of QED which also applies to QCD. In the Weak interaction, only one helicity combination contributes.**
  - Therefore, applying similar reasoning to the initial particles, we find that only 4 of 16 helicity combinations contribute.
  - **This essentially shows that in such an interaction, the helicity of a particle doesn't change along its 'tube line'.** (Not sure if this is completely true, check)
  - We can therefore label events based on the helicity of one muon and one electron, so the surviving matrix elements are $M_{RR}, M_{LR}, M_{RL}, M_{LL}$.
- Finding the electron currents using relation (5) to convert (6-7) and using $\theta = 0$ to give the appropriate case yields;

$$e_R^- e_L^+: \quad \overline{v}_\downarrow(p_2)\gamma^\nu u_\uparrow(p_1) = 2E(0, -1, -i, 0),$$
$$e_L^- e_R^+: \quad \overline{v}_\uparrow(p_2)\gamma^\nu u_\downarrow(p_1) = 2E(0, -1, i, 0).$$



Helicity Considerations

- Using the above it is then possible to calculate M for each of the four possible helicity combinations.
  - Calculating $M_{RR} = 4\pi\alpha(1+\cos\theta)$ gives an interesting θ dependence - the chance of producing a muon leaving in the opposite direction to the incoming electron goes to zero, corresponding to the fact that this makes conservation of angular momentum from the spins impossible.



$e^2(1+\cos\theta)^2$

  - Assuming incoming electrons and positrons are unpolarised, all states are equally likely, so the differential cross section is found by averaging over the initial states and summing over the final spin states;

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \times \underbrace{\frac{1}{4}\left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2\right)}_{\langle |M_{fi}|^2\rangle}$$
$$= \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$
$$\implies \sigma = \frac{4\pi\alpha^2}{3s}$$

  - This prediction for σ agrees with prediction to around 1%.
- Note that above, we have written $\langle |M_{fi}|^2\rangle$ in terms of the muon angle in the CoM frame. This may be made Lorentz invariant by combining momentum dot products such as to form an expression from mandelstam variables. Eventually, we find

$$\langle |M_{fi}|^2\rangle = 2e^4 \frac{(p_1\cdot p_3)^2 + (p_1\cdot p_4)^2}{(p_1\cdot p_2)^2} = 2e^4\left(\frac{t^2+u^2}{s^2}\right)$$

Classical Explanation

- Note that the angular dependence for M_{RR} etc. can be understood in terms of normal QM spin states with a spin-1 state aligned with the z axis and a spin-1 state at an angle θ to that axis.
  - We may represent such states as $|1,1\rangle$ and

$$\psi = |1,1\rangle_\theta = \frac{1}{2}(1-\cos\theta)\,|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta\,|1,0\rangle + \frac{1}{2}(1+\cos\theta)\,|1,1\rangle$$

  - Therefore evaluating the inner products, we recover the results found by QED;

$$|M_{RR}|^2 = |\langle\psi|1,1\rangle|^2 = \frac{1}{4}(1+\cos\theta)^2$$
$$\text{and} \quad |M_{LR}|^2 = |\langle\psi|1,-1\rangle|^2 = \frac{1}{4}(1-\cos\theta)^2$$

Chirality

- Chirality is an intrinsic property of particles, unlike helicity which can vary depending on your frame of reference (run sufficiently fast past a particle and it starts going the other way in your frame).
    - We find the components of chirality using chiral projection operators;

$$P_R = \frac{1}{2}(1+\gamma^5) \quad P_L = \frac{1}{2}(1-\gamma^5)$$
$$\text{where} \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

        - These essentially select either the symmetric or antisymmetric parts, allowing for decomposition, similarly to how we can write any matrix as the sum of its symmetric + antisymmetric parts.
    - We can then use P_R and P_L to write any spinor as the sum of its right and left chiral parts. Hence, defining

$$\psi_R = P_R\psi, \quad \overline{\psi_R} = \overline{P_r\psi}, \quad \psi_L = P_L\psi, \quad \text{and} \quad \overline{\psi_L} = \overline{P_L\psi}$$

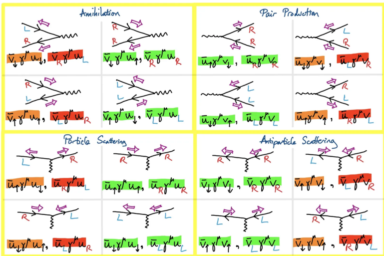
        - We find that the chiral parts of each spinor are eigenstates of γ^5 with opposite chirality;

$$\gamma^5\psi = \psi_R \quad \gamma^5\psi_L = -\psi_L$$

        - And also that a current may always be written in the form

$$\overline{\phi}\gamma^\mu\psi = \overline{\phi}_R\gamma^\mu\psi_R + \overline{\phi_L}\gamma^\mu\psi_L$$

            - Notably, as $\bar{\phi}\gamma^5\psi = \bar{\phi}_\downarrow\gamma^5\psi_\uparrow + \bar{\phi}_\uparrow\gamma^5\psi_\downarrow + \bar{\phi}_\uparrow\gamma^5\psi_\uparrow + \bar{\phi}_\downarrow\gamma^5\psi_\downarrow = \overline{\phi}_R\gamma^\mu\psi_R + \overline{\phi_L}\gamma^\mu\psi_L$ and in the relativistic limit, $\bar{\phi}_\downarrow = \bar{\phi}_R$, $\psi_\uparrow = \phi_R$, $\bar{\phi}_\uparrow = \bar{\phi}_L$ and $\psi_\downarrow = \psi_L$, we require that $\bar{\phi}_\uparrow\gamma^5\psi_\uparrow = \bar{\phi}_\downarrow\gamma^5\psi_\downarrow = 0$.
- **In the high energy limit $E \gg m$, we find helicity and chirality eigenstates correspond;**
    - **Right helicity particles and left helicity antiparticles are right chiral.**
    - **Left helicity particles and right helicity antiparticles are left chiral.**
- Check can predict all the colours in the summary slide (below) - red indicates a process that never occurs, orange is a process that can occur but is disfavoured and green is a very allowed process.



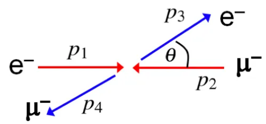
H5 - Electron-Proton Elastic Scattering

H4 - Electron-Positron Annihilation H6 - Deep Inelastic Scattering

Point-like t-channel scattering

- Considering scattering from a point-like particle in the CoM frame following a t-channel form, we find the appropriate matrix element (note the similarity to the s-channel form);

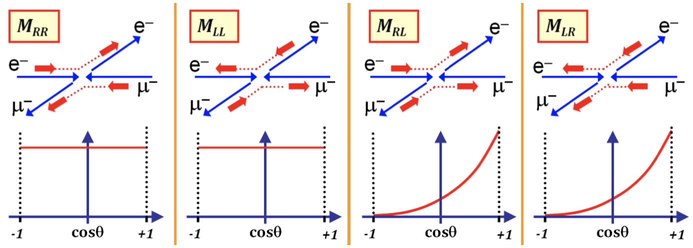
$$\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1\cdot p_4)^2 + (p_1\cdot p_2)^2}{(p_1\cdot p_3)^2} = 2e^4\left(\frac{s^2+u^2}{t^2}\right)$$



    - Then considering the arrangement above, we find

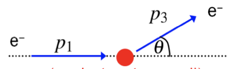
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle|M_{fi}|^2\rangle = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$

        - Here, the denominator comes from the propagator $-ig_{\mu\nu}/q^2$, so as $q^2 = (p_1-p_3)^2 = E^2(1-\cos\theta) \to 0$, the cross-section tends to infinity.
            - This makes sense, all particles interact with everything else in the universe a little bit, we can set a limit on the minimum amount of deflection we care about to recover finite values.
        - The numerator reflect the chiral structure of QED. There are four non-zero helicity contributions - two generate the 1 term, as they have $S_z = 0$ and two generate the $\frac{1}{4}(1+\cos\theta)^2$ term as they have $S_z = \pm 1$.



Probing Proton Structure

- The nature of an electron-proton interaction varies significantly with the electron energy (wavelength);
    - $\lambda \gg r_p \implies$ equivalent to scattering from a point-like spin-less object.
    - $\lambda \sim r_p \implies$ equivalent to scattering from an extended charge object.
    - $\lambda < r_p \implies$ scattering from constituent quarks.
    - $\lambda \ll r_p \implies$ scattering from a sea of quarks and gluons.



(neglect proton recoil)

Point-Like Protons Without Recoil

- Considering Rutherford scattering with low energy and neglecting proton recoil, we use the RH and LH Helicity particle spinors in terms of $\alpha = \frac{|\vec{p}|}{E+m_e}$ in the above situation to get;

$$u_\uparrow(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix}; \quad u_\downarrow(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix}$$
$$u_\uparrow(p_3) = N_e \begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix}; \quad u_\downarrow(p_3) = N_e \begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

  - Then considering the four possible electron currents (helicity combinations);

$$\begin{aligned} \overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right], \\ \overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right], \\ \overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right], \\ \overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right]. \end{aligned}$$

    - In the relativistic limit, the lower two go to zero. However, in the non-relativistic limit $|\vec{p}| \ll E$ we have $\alpha = 0$, so all four helicity combinations have non-zero matrix elements (the helicity eigenstates $\neq$ chirality eigenstates).
    - Repeating the same analysis with proton spinors (at rest, before and after) gives $j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p(1,0,0,0)$ and $j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$.
  - There are therefore 8 allowed helicity combinations; summing over them and finding the initial spin average (divide by the four initial spin states) in the non-relativistic limit gives (note that now $E = m_e$);

$$\langle |M_{fi}^2| \rangle = \frac{1}{4}\frac{e^4}{q^4}\left((2M_p)^2(2m_e)^2\right)(4c^2+4s^2) = \frac{16M_p^2 m_e^2 e^4}{q^4}$$

    - Here, $q^2 = (p_1 - p_3)^2 = -2|\vec{p}|(1-\cos\theta) = -4|\vec{p}|^2\sin^2(\theta/2)$.
  - Now using the formula for differential cross-section (equation 9) from the first handout and noting that $E \sim m_e \ll M_p$, we find

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^2}{64\pi^2|\vec{p}|^4\sin^4(\theta/2)}$$

    - Hence, writing this in terms of $e^2 = 4\pi\alpha$ and electron kinetic energy gives the Rutherford differential cross-section;

$$\implies \left.\frac{d\sigma}{d\Omega}\right|_{\text{Rutherford}} = \frac{\alpha^2}{16E_K^2\sin^4(\frac{\theta}{2})}$$

    - As this form could be derived using only the static Coulomb potential, we conclude that in this non-relativistic limit, only the interactions between **electric charges** matter (not magnetic moments).
  - Repeating the above in the relativistic limit and using a static potential such that we only have the 0 components contributing gives the Mott differential cross section;

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2} \cdot \underbrace{\cos^2\frac{\theta}{2}}_{\text{Initial / final state overlap}}$$

With Recoil

- To allow for recoil, we allow exiting particles to have different energies and momenta; $p_3 = (E_3, 0, E_3\sin\theta, E_3\cos\theta)$ and $p_4 = (E_4, \vec{p}_4)$ and use the form for the matrix element for a t-channel diagram where masses are not negligible;

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)M^2 - (p_1\cdot p_4)m^2 + 2m^2M^2\right]$$

  - Now neglecting the electron mass $m = m_e = 0$, eliminating p_4 by momentum conservation and evaluating the dot products gives

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1-p_3)^4}2ME_1E_3\left[(E_1-E_3)\sin^2(\theta/2) + M\cos^2(\theta/2)\right]$$

    - To progress further, note that $q^2 = (p_1-p_3)^2 \approx -2E_1E_3(1-\cos\theta) < 0$ and $q\cdot p_2 = (p_1-p_3)\cdot p_2 = M(E_1 - E_3)$.
      - Therefore, using

$$\begin{aligned} (q+p_2)^2 &= p_4^2 \\ \implies q^2 + M^2 + 2q\cdot p_2 &= M^2 \\ \implies E_1 - E_3 &= -\frac{q^2}{2M} \end{aligned} \tag{1}$$

    - Combining the above we find

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\frac{E_3}{E_1}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)$$

      - Here, the term E_3/E_1 is due to proton recoil and the term $\propto \sin^2\frac{\theta}{2}$ encodes the magnetic interaction due to the spin-spin interaction.
    - Using the expressions for $E_1 - E_3$ and q^2 above, we can also find

$$q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M + E_1(1-\cos\theta)}$$

L9

Scattering From an Extended Proton

Form Factors

- Considering the potential from an extended charge distribution, we find

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}d^3\vec{r}' \quad \text{where} \quad \int \rho(\vec{r})d^3]\vec{r} = 1$$

  - Then evaluating the first order perturbation theory matrix element by rewriting the exponential term as $e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}e^{i\vec{q}\cdot\vec{r}'}$ with $\vec{q} = \vec{p}_1 - \vec{p}_3$ and $\vec{R} = \vec{r} - \vec{r}'$

$$M_{fi} = \underbrace{\int e^{i\vec{q}\cdot\vec{R}}\frac{Q}{4\pi|\vec{R}|}d^3\vec{R}}_{(M_{fi})_{\text{point}}}\underbrace{\int \rho(\vec{r}')e^{i\vec{q}\cdot\vec{r}'}d^3\vec{r}'}_{F(\vec{q}^2)}$$

    - This is equivalent to the matrix element for scattering from a point source multiplied by the form factor $F(\vec{q}^2)$.

- We can then find the charge density by taking the inverse Fourier transform of the form factor.

General Form (Rosenbluth Formula)

- Conducting this analysis accounting for QFT eventually gives the Rosenbluth Formula ($G(q^2)$ are structure functions, one electric and one magnetic);

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4(\theta/2)}\frac{E_3}{E_1}\left(\cos^2(\theta/2) - \frac{q^2}{2M^2}\sin^2(\theta/2)\right)$$

Rutherford Proton recoil Electric/ Magnetic scattering Magnetic term due to spin.

$$\text{where} \quad \tau = -\frac{q^2}{4M^2} > 0$$

  - This is a function of q^2 rather than $\vec{q}^2$. However, using (1), we see that

$$q^2 = (E_1 - E_3)^2 - \vec{q}^2$$
$$\implies -\vec{q}^2 = q^2\left[1 - \left(\frac{q}{2M}\right)^2\right]$$

    - Therefore, where $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$, and $G(q^2) \approx G(\vec{q}^2)$, allowing us to interpret the structure functions as the Fourier transforms of the charge and magnetic moment distributions;

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}\cdot\vec{r}}\,\rho(\vec{r})\,d^3\vec{r}$$
$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}\cdot\vec{r}}\,\mu(\vec{r})\,d^3\vec{r}$$

- The Rosenbluth formula can be rewritten in the form;

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0\left(\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2\tan^2\frac{\theta}{2}\right)$$
$$\text{with} \left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

  - Hence, varying $\tan^2\frac{\theta}{2}$ allows for a straight line to be plotted, with the y-intercept and gradient combined to find G_E and G_M.
    - Performing the experiment, it is found that $G_M(q^2) = 2.79G_E(q^2)$ as expected based on the magnetic moment of a proton.
      - This is highly suggestive that the same thing is the source of the magnetic and electric parts.
    - We also find that the form of these distributions follow.

$$G_E(q^2) \approx \frac{G_M}{2.79} \approx \frac{1}{(1 + q^2/0.71\text{GeV}^2)^2}$$

    - Therefore, taking the FT gives the charge and magnetic moment spatial distributions;

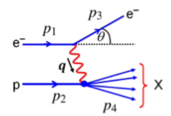
$$\rho(r) \approx \rho_0 e^{-r/a}$$

    - Hence, we see that the proton is an extended object with rms charge radius 0.8 fm.

H6 - Deep Inelastic Scattering

H5 - Electron-Proton Elastic Scattering H7 - Symmetries and the Quark Model

Non-Elastic Scattering Processes



- At high $Q^2 = -q^2$, substituting the experimentally discovered form of $G_M(q^2)$ into the Rosenbluth formula, we find that

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic and high } q^2} \propto q^{-6}$$

  - Hence, in this regime, elastic scattering is unlikely and inelastic reactions involving proton breakup dominate.
  - In a regime involving breakup, (s, t) are no longer sufficient to describe the whole situation - we require a third variable to describe the variability in the mass of p_4.

Kinematic Variables

- Defining the following kinematic variables (x is Bjorken x and y is Bjorken y), then for fixed s, at most two of them are independent, so there are many relations between them;

$$Q^2 = -q^2, \quad x = \frac{Q^2}{2p_2\cdot q}, \quad y = \frac{p_2\cdot q}{p_2\cdot p_1} \quad \text{and} \quad \nu = \frac{p_2\cdot q}{M}$$

  - Properties of x and Q^2:
    - Since $Q^2 = -(p_1 - p_3)^2 = 2m_e(E_3 - m_e) \geq 0$ (using LI of dot products), we have $Q^2 \geq 0$.
    - Since $2p_2\cdot q = (p_2 + q)^2 - p_2^2 - q^2 = M_X^2 - M^2 + Q^2$ we have $2p_2\cdot q \geq Q^2$.
    - Hence, $0 < x < 1$ for an inelastic and $x = 1$ for elastic scattering.
  - Properties of y:
    - Evaluating y in the lab frame with $p_1 = (E_1, 0, 0, E_1)$, $p_2 = (M, 0, 0, 0)$ and $q = (E_1 - E_3, \vec{p}_1 - \vec{p}_3)$ gives;

$$y = 1 - \frac{E_3}{E_1}$$

      - Hence, y is the fractional energy loss of the incoming particle in the lab frame.
    - Doing the same in the CoM frame using $p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$ ($E \gg M$) gives;

$$y = \frac{1}{2}(1 - \cos\theta^*)$$

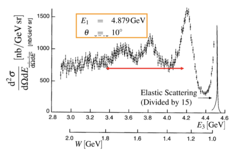
    - Combining these also gives $0 < y < 1$.
  - Properties of ν:
    - Evaluating ν in the lab frame gives;

$$\nu = E_1 - E_3$$

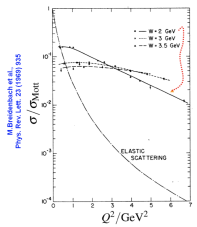
      - Hence, ν is the energy lost by the incoming particle in the lab frame.
  - Using $s = (p_1 + p_2)^2 \approx 2p_1\cdot p_2 + M^2$ it can also be shown that

$$
\begin{aligned}
x &= \frac{Q^2}{2M\nu} \quad \text{and} \quad y = \frac{2M}{s - M^2}\nu \\
xy &= \frac{Q^2}{s - M^2} \quad \text{and} \quad Q^2 = (s - M^2)xy
\end{aligned}
\tag{1}
$$

Inelastic Scattering



- Considering the cross-sections measured above, we see (note, $W = M_X$):
  - Elastic scattering - a large peak where the proton remains intact ($W = M$).
  - Inelastic scattering - excited states (red arrow) ($W = M_\Delta$).
  - Deep inelastic scattering - many particle final states corresponding to total proton breakup ($W \gg M$).
- To probe these, repeat the experiments done with inelastic scattering, but now record all of s, the angle of deflection θ and the energy of the scattered electron E_3.
  - We can then plot the inelastic cross sections for different values of W (below)



  - Here, we see that the inelastic and deep inelastic cross sections only weakly depend on Q^2, so their form factors are ~ 1, which implies the scattering is from point-like objects within the proton!

Deep Inelastic Scattering

- Note that in inelastic scattering, it is possible to rewrite the Rosenbluth formula in terms of only Q^2 as;

$$
\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[f_2(Q^2)\left(1 - y - \frac{M^2y^2}{Q^2}\right) + \frac{1}{2}y^2 f_1(Q^2)\right]
$$

- For deep inelastic scattering, there are two independent variables for the final state, so we require a double differential cross section.
  - It can be shown that the most general LI expression for $e^- p \to e^- X$ inelastic scattering is;

$$
\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2y^2}{Q^2}\right)\frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2)\right] \tag{2}
$$

    - This therefore replaces the form factors $f_i(Q^2)$ with structure functions $F_i(x, Q^2)$, which may (see below) be interpreted as the momentum distribution of quarks within a proton.
  - In the lab frame, this can then be expressed in terms of the scattering angle and energy E_3 as;

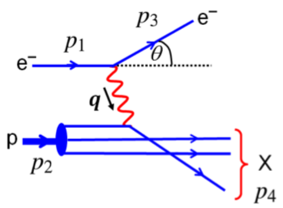
$$
\frac{d^2\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)}\left[\frac{1}{\nu}F_2(x, Q^2)\cos^2\left(\frac{\theta}{2}\right) + \frac{2}{M}F_1(x, Q^2)\sin^2\left(\frac{\theta}{2}\right)\right]
$$

    - Here, F_1 and F_2 are the magnetic and electromagnetic structure functions respectively.
      - These can be determined by finding the differential cross-section at different scattering angles and beam energies. Both are found to be (almost) independent of Q^2!
- The (almost) independence of the structure functions on Q^2 is called Bjorken scaling and strongly suggests scattering from point-like constituents within the proton.
  - We also observe that $F_1(x)$ and $F_2(x)$ obey the Callan-Gross relation;

$$
F_2(x) = 2xF_1(x)
$$

L10

The Quark-Parton Model



- In the parton model, we assume elastic scattering from a "quasi-free" spin-1/2 quark within the proton.
  - Considering the proton in the infinite momentum frame such that the proton mass is negligible and $p_2 = (E_2, 0, 0, E_2)$ allows us to also neglect any momentum transverse to the proton direction (like a quark may well have).
  - If we assume a quark carries a fraction ξ of the proton's four-momentum ($\xi^2 p_2^2 = m_q^2 \approx 0$), then after the interaction, as the quark is still a quark we have

$$
m_q^2 = (\xi p_2 + q)^2 = (\xi p_2)^2 + 2\xi p_2 \cdot q + q^2 = m_q^2 + 2\xi p_2 \cdot q + q^2
$$

$$
\implies \xi = -\frac{q^2}{2p_2 \cdot q} = x
$$

    - Hence, Bjorken x is the fraction of proton momentum carried by the struck quark in the infinite momentum frame.
  - Evaluating expressions with the proton momentum g

$$
s = (p_1 + p_2)^2 \approx 2p_1 \cdot p_2, \quad y = \frac{p_2 \cdot q}{p_2 \cdot p_1}, \quad x = \frac{Q^2}{2p_2 \cdot q}
$$

  - Now ooking for equivalent expressions in terms of the quark interaction we find;

$$
\begin{aligned}
s^q &= (p_1 + xp_2)^2 \approx 2xp_1 \cdot p_2 = xs, \\
y_q &= \frac{p_q \cdot q}{p_q \cdot p_1} = \frac{xp_2 \cdot q}{xp_2 \cdot p_1} = y \\
x_q &= \frac{Q^2}{2q_q \cdot q} = \frac{Q^2}{2xp_2 \cdot q} = \frac{x}{x} = 1
\end{aligned}
$$

    - Hence, the quark collision is elastic.
  - Applying the result from Ex. Q11 (add some more here once done) we find the differential cross-section for this elastic scattering as;

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2 e_q^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s_q}\right)^2\right]$$

- Then using $Q^2 = (s_q - m^2)x_q y_q$ from (1) and assuming the massless limit gives $\frac{q^2}{s_q} = -y_q = -y$, so

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1 + (1-y)^2\right] \tag{3}$$

- We also wish to account for the distribution of quark momenta within the proton, so introduce the parton distribution functions.
  - We define $q^p(x)dx$ as the number of quarks of type q within a proton p with momenta between $x \to x + dx$ - it is essentially a probability distribution function in x. Hence,

$$p_q(x)dx = q^p(x)dx$$

  - Using (3) and summing over all the quarks in a proton gives

$$\frac{d^2\sigma^{\text{ep}}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]\sum_q e_q^2 q^p(x)$$

  - Hence, comparing with (2) in the high energy ($Q^2 \gg M^2y^2$) recovers the Callan-Gross relation and Bjorken scaling (as the RHS is only a function of x);

$$F_2^p(x, Q^2) = 2xF_1^p(x, Q^2) = x\sum_q e_q^2 q^p(x)$$

- Therefore, measurements of the structure functions allows for determination of the parton distribution functions!

Predictions of the Parton Model

- For electron-proton (ep) and electron-neutron (en) scattering, we therefore find (assuming up quarks have charge $2/3$ and down $1/3$);

$$F_2^{\text{ep}}(x) = x\left(\frac{4}{9}u^p(x) + \frac{1}{9}d^p(x) + \frac{4}{9}\bar{u}^p(x) + \frac{1}{9}\bar{d}^p(x)\right) \tag{4}$$
$$F_2^{\text{en}}(x) = x\left(\frac{4}{9}u^n(x) + \frac{1}{9}d^n(x) + \frac{4}{9}\bar{u}^n(x) + \frac{1}{9}\bar{d}^n(x)\right) \tag{5}$$

  - Note, a proton contains anti-up and anti-down quarks due to higher orders.
  - Assuming that a neutron (ddu) is the same as a proton (uud) but with up and down quarks interchanged (isospin symmetry) implies;

$$d^n(x) = u^p(x) = u(x) \quad \text{and} \quad u^n(x) = d^p(x) = d(x)$$

  - Rewriting (4-5) in terms of $u(x)$ and $d(x)$ defined above and integrating gives

$$\int_0^1 F_2^{\text{ep}}(x)dx = \frac{4}{9}f_u + \frac{1}{9}f_d$$
$$\int_0^1 F_2^{\text{en}}(x)dx = \frac{4}{9}f_d + \frac{1}{9}f_u$$
$$\text{where} \quad f_u = \int_0^1 xu(x) + x\bar{u}(x)dx \quad \text{and} \quad f_d = \int_0^1 xd(x) + x\bar{d}(x)dx$$

- Experimentally we then find $\int F_2^{\text{ep}}(x)dx \approx 0.18$ and $\int F_2^{\text{en}}(x)dx \approx 0.12$, which gives $f_u \approx 0.36$ and $f_d \approx 0.18$.
  - This implies up quarks carry twice the momentum of the down quarks and that the quarks carry just over 50% of the total proton momentum. The rest is carried by gluons, which don't contribute to electron-nucleon scattering because they are neutral.

Valence and Sea Quarks

- The proton parton distribution function has contributions from both the **v**alence quarks and the virtual quarks produced by gluons (the "**s**ea"). Hence, we can resolve the contributions;

$$u(x) = u_V(x) + u_S(x) \quad \bar{d}(x) = \bar{d}_S(x)$$
$$\bar{u}(x) = \bar{u}_S(x) \quad d(x) = d_V(x) + d_S(x)$$

  - As these sea quarks arise from pair production, we expect $m_u = m_d$, so say

$$u_S(x) = d_S(x) = \bar{u}_S(x) = \bar{d}_S(x) = S(x)$$

  - We can then rewrite (4-5) as

$$F_2^{\text{ep}}(x) = x\left(\frac{4}{9}u_V(x) + \frac{1}{9}d_V(x) + \frac{10}{9}S(x)\right)$$
$$F_2^{\text{en}}(x) = x\left(\frac{4}{9}d_V(x) + \frac{1}{9}u_V(x) + \frac{10}{9}S(x)\right)$$

- We can then investigate the ratio $\frac{F_2^{\text{en}}(x)}{F_2^{\text{ep}}(x)}$. As the gluon propagator has $1/q^2$ dependence, it is much more likely to produce low energy gluons, so we expect the sea to comprise of low energy q and $\bar{q}$.
  - Hence, at low x, we expect the sea to dominate - this is experimentally observed.
  - At high x we expect the sea contribution to be small, so using $u_V = 2d_V$ gives the ratio as $2/3$ as $x \to 1$.
    - This is not seen; the ratio actually tends to $1/4$ - not fully understood.

Experiments

- Non-point like scattering becomes apparent when $\lambda_\gamma \sim$ the size of the scattering centre ($\lambda_\gamma = h/|\vec{q}|$).
  - If quarks were not point-like then at high Q^2 we could see a rapid increase in the cross-section with increasing Q^2.
  - Hence, we try to probe quark sub-structure using high Q^2 (HERA collider).
    - At HERA, no evidence was found for a rapid decrease of cross-section at high Q^2, so $R_{\text{quark}} < 10^{-18}$ m.
    - However, at low x we find that $F_2(x, Q^2) \neq F_2(x)$, giving deviations from exact Bjorken scaling.

H7 - Symmetries and the Quark Model

H6 - Deep Inelastic Scattering H8 - Quantum Chromodynamics

L11

Symmetries and Conservation Laws

- If physics is invariant under a transformation $\hat{U}$, such that $\psi \to \psi' = \hat{U}\psi$, then in order to preserve state normalisations, we require

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle \implies \boxed{\hat{U}^\dagger\hat{U} = 1; \quad \hat{U} \text{ is unitary}}$$

  - For the transformation to not affect predictions, we also require

$$\begin{aligned}\langle\psi|\hat{H}|\psi\rangle = \langle\psi'|\hat{H}|\psi'\rangle &\implies \hat{U}^\dagger\hat{H}\hat{U} = \hat{H} \\ &\implies \boxed{[\hat{H},\hat{U}] = 0}\end{aligned}$$

  - If the transformation is small then it can be written in terms of infinitesimal transformations with generator $\hat{G}$;

$$\hat{U} = 1 + i\epsilon\hat{G}$$

    - Requiring $\hat{U}\hat{U}^\dagger = 1$ then requires $\hat{G} = \hat{G}^\dagger$, so is Hermitian with a corresponding observable.
    - Hence, $[\hat{H},\hat{U}] = 0 \implies [\hat{H},\hat{G}] = 0$, so Ehrenfest's Theorem states that

$$\text{Symmetry} \Longleftrightarrow \text{Conservation Law}$$

      - E.g. spatial translation corresponds to conservation of momentum.
    - In general, the symmetry could depend on multiple parameters; $\hat{U} = 1 + i\vec{\epsilon}\cdot\vec{G}$.
  - Expressing a finite transformation in terms of a series of infinitesimal transformations gives

$$\hat{U}(\vec{\epsilon}) = \lim_{n\to\infty}\left(1 + i\frac{\vec{\epsilon}}{n}\cdot\vec{G}\right)^n = e^{i\vec{\epsilon}\cdot\vec{G}}$$

Isospin

- If we assume that protons and neutrons are indistinguishable if we ignore the electric charge of the proton, then we could consider them as two states of the nucleon;

$$p = \begin{pmatrix}1\\0\end{pmatrix} \quad n = \begin{pmatrix}0\\1\end{pmatrix}$$

  - We expect physics to be invariant under continuous "rotations" in isospin space.
- If we now extend this idea to quarks and assume the strong interaction treats all quark flavours equally, then we can chose the basis

$$u = \begin{pmatrix}1\\0\end{pmatrix} \quad d = \begin{pmatrix}0\\1\end{pmatrix}$$

  - Expressing the "rotation" in the isospin space with $\hat{U}$ below where each component is complex, we have 8 degrees of freedom. However, requiring $\hat{U}$ be unitary reduces this to 4 real degrees of freedom, so there are four independent $\hat{U}$ s.

$$\hat{U} = \begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}$$

    - One of these degrees of freedom $(\hat{U}_1)$ corresponds to multiplication by a phase factor - this is irrelevant as it is not a flavour transformation.

$$\hat{U}_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}e^{i\phi}$$

    - The other **three** degrees of freedom parameterise the $SU(2)$ group of 'special' ($\det U = 1$) unitary matrices. If we put these d.o.f in ϵ_i, then

$$\hat{U}(\vec{\epsilon}) = 1 + i\sum_{i=1}^{3}\epsilon_i\hat{G}_i$$

      - The requirement for $\det U = 1$ then also imposes that $\text{Tr}(\hat{G}_i) = 0$ for infinitesimal transformations. **(Exercise to check)**
      - We therefore require $\hat{G}_i$ to be linearly independent, traceless, Hermitian matrices. These requirements are fulfilled by the Pauli Matrices

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

      - Traditionally, the $\hat{G}_i$ with $SU(2)$ symmetry are defined as

$$\hat{T}_i = \frac{1}{2}\sigma_i \implies U(\vec{\epsilon}) = e^{i\vec{\epsilon}\cdot\vec{T}}$$

  - We can then notice that isospin generators have the same properties as spin;

$$\begin{gathered}[T_1,T_2] = iT_3, \quad [T_2,T_3] = iT_1, \quad [T_3,T_1] = iT_2, \\ [T^2,T_3] = 0, \quad T^2 = T_1^2 + T_2^2 + T_3^2.\end{gathered}$$

    - We therefore label states with total isospin I and the third component of isospin I_3, such that $T^2|I,I_3\rangle = I(I+1)|I,I_3\rangle$ and $T_3|I,I_3\rangle = I_3|I,I_3\rangle$. Therefore, we can switch to notation;

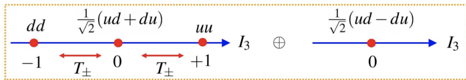
$$u = \begin{pmatrix}1\\0\end{pmatrix} = |\frac{1}{2},+\frac{1}{2}\rangle \quad d = \begin{pmatrix}0\\1\end{pmatrix} = |\frac{1}{2},-\frac{1}{2}\rangle$$

      - In general, $I_3 = \frac{1}{2}(N_u - N_d)$.
    - Defining isospin ladder operators as normal with $T_- = T_1 - iT_2$, $T_+ = T_1 + iT_2$ (and below) allows for transitioning between these states.

$$\begin{aligned}T_+|I,I_3\rangle &= \sqrt{I(I+1) - I_3(I_3+1)}\,|I,I_3+1\rangle, \\ T_-|I,I_3\rangle &= \sqrt{I(I+1) - I_3(I_3-1)}\,|I,I_3-1\rangle.\end{aligned}$$

Combining Quarks

- We can combine isospin the same as for combining angular momentum. For a two quark system, we see that one combined state is $|I,I_3\rangle = |1,+1\rangle = uu$.
  - Applying T_- on this twice gives two more basis states. A fourth can then be found by orthogonality with $|1,0\rangle$, giving;



- We can now add an additional up or down quark to each of these states to generate a sextet and a doublet. The doublet states are conveniently then iso-doublets, but of the sextet, only the uuu and ddd states have well defined isospin.

$$2\otimes 2\otimes 2 = 2\otimes(3\oplus 1) = (2\otimes 3)\oplus(2\otimes 1) = 4\oplus 2\oplus 2$$

  - To generate an 8-dimensional basis, we therefore repeatedly act on the uuu state with T_- and use orthogonality, giving

$$
\left.\begin{aligned}
\left|\tfrac{3}{2},+\tfrac{3}{2}\right\rangle &= uuu \\
\left|\tfrac{3}{2},+\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(uud+udu+duu) \\
\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{3}}(ddu+dud+udd) \\
\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle &= ddd
\end{aligned}\right\} S
$$

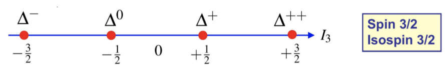
$$
\left.\begin{aligned}
\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle &= +\tfrac{1}{\sqrt{6}}(2uud-udu-duu) \\
\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle &= -\tfrac{1}{\sqrt{6}}(2ddu-udd-dud)
\end{aligned}\right\} M_S
$$

$$
\left.\begin{aligned}
\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(udu-duu) \\
\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle &= \tfrac{1}{\sqrt{2}}(udd-dud)
\end{aligned}\right\} M_A
$$

- S is a quadruplet of states which are **S**ymmetric under the interchange of any two quarks.
- M_S is **M**ixed, **S**ymmetric under the interchange of quarks $1 \leftrightarrow 2$.
- M_A is **M**ixed, **A**ntisymmetric under the interchange of quarks $1 \leftrightarrow 2$.
  - The M_S and M_A states have no definite symmetry under interchange of the third quark with any others.

Baryon Wave-Functions

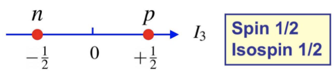
- As quarks are fermions, their total wavefunction must be anti-symmetric under interchange of any two quarks.
  - Factorising the wave-function as $\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$ and taking ξ_{colour} as antisymmetric for bound qqq states and η_{space} as symmetric by only considering ground state ($L=0$) baryons, we deduce that $\phi_{\text{flavour}}\xi_{\text{spin}}$ **is symmetric under interchange of any two quarks.**
  - We can therefore produce totally symmetric wavefunctions either by;
    - Combining totally symmetric isospin and spin wavefunctions giving;



    - Combining mixed symmetry spin and mixed symmetry isospin states, we find a linear combination (below) to be totally symmetric (including with swapping q_3);

$$
\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)
$$

      - Therefore, the possible combinations are;



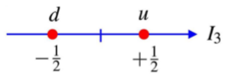
      - This gives the spin-up proton wave-function as;

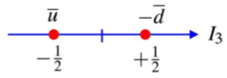
$$
|p\uparrow\rangle = \frac{1}{6\sqrt{2}}\Big((2uud-udu-duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{1}{2\sqrt{2}}(udu-duu)(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)\Big)
$$

Anti-Quarks and Light (ud) Mesons

- In order for anti-quarks and quarks to transform the same way (and hence make predictions invariant under $u \leftrightarrow d$ and $\bar{u} \leftrightarrow \bar{d}$) we define

$$
\bar{u} = \begin{pmatrix}0\\1\end{pmatrix}; \quad \bar{d} = -\begin{pmatrix}1\\0\end{pmatrix}
$$





  - We therefore find that applying ladder operators gives;

$$
T_-\bar{u} = 0; \quad T_+\bar{u} = -\bar{d}; \quad T_-\bar{d} = -\bar{u}; \quad T_+\bar{d} = 0
$$

  - Considering $q\bar{q}$ combinations, we can immediately construct states

$$
\begin{aligned}
|1,+1\rangle &= -u\bar{d} \\
|1,-1\rangle &= d\bar{u}
\end{aligned}
$$

    - Then applying ladder operators and orthogonality, we find

$$
\begin{aligned}
|1,0\rangle &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\
|0,0\rangle &= \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})
\end{aligned}
$$

    - We have therefore generated a triplet of $I=1$ states and singlet $I=0$ state. This can be written $2 \otimes \bar{2} = 3 \oplus 1$.
      - Note, a singlet state is a 'dead end' in terms of ladder operators.

L12

SU(3)-Flavour

The Strange Quark

- As $m_s \approx m_u \approx m_d$, the strong interaction acts as if it is approximately symmetric under $u \leftrightarrow d \leftrightarrow s$. We therefore apply SU(3) symmetry rules.
  - We represent the three quarks as

$$
u = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad d = \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad s = \begin{pmatrix}0\\0\\1\end{pmatrix}
$$

  - For SU(n) there are n^2-1 degrees of freedom, so in a SU(3) representation, we have 8 degrees of freedom and therefore must find 8 LI traceless hermitian operator matrices labelled λ_i - the Gell-Mann matrices.
    - The typically used generators are defined as $\vec{G} = \frac{1}{2}\vec{\lambda}$ giving a general element of the group as $\hat{U}(\vec{\epsilon}) = e^{i\vec{\epsilon}\cdot\vec{G}} = e^{\frac{i}{2}\vec{\epsilon}\cdot\vec{\lambda}}$.
    - Distributing the Pauli matrices over 3×3 matrices forms 9 traceless, hermitian matrices including three traceless diagonal matrices.
      - In SU(3), at most two diagonal matrices can be independent, so we create a linear combination of two of them, λ_8.

$$
\lambda_8 = \frac{1}{\sqrt{3}}\left(\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix} + \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\right) = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}
$$

      - The other diagonal matrix is λ_3;

$$
\lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}
$$

        - As these matrices are diagonal, their eigenvalues do not change with time. We therefore identify;

- Eigenvalues of λ_3 again as I_3 - counting the number of up quarks minus the number of down quarks in a state.
- Eigenvalues of λ_8 as hypercharge (Y).

- We can then use these two quantum numbers to specify a state in the 2D plane (I_3, Y).

- The other λ_i are;

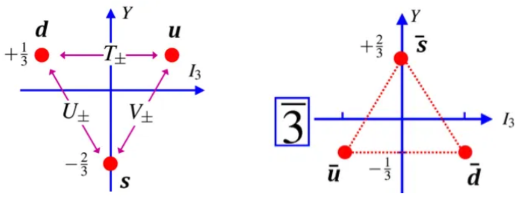
$$u \leftrightarrow d \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$u \leftrightarrow s \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$d \leftrightarrow s \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

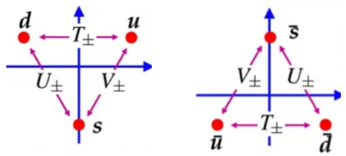
$$\text{other} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- In this representation, anti-quarks have the opposite quantum numbers, so we can represent quarks and antiquarks as;



- Using the above, it is also possible to define a new set of ladder operators to step between states:

$$T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$$

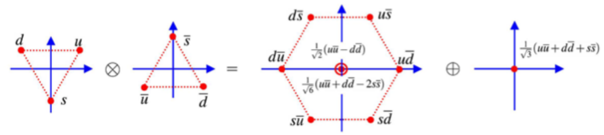


- For antiparticles, pushing downwards diagonally generates an extra minus;

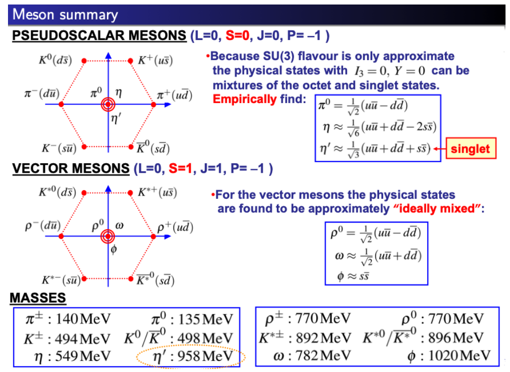
$$\begin{aligned} T_+d &= u, & T_-u &= d; & T_+\bar{u} &= -\bar{d}, & T_-\bar{d} &= -\bar{u}; \\ V_+s &= u, & V_-u &= s; & V_+\bar{u} &= -\bar{s}, & V_-\bar{s} &= -\bar{u}; \\ U_+s &= d, & U_-d &= s; & U_+\bar{d} &= -\bar{s}, & U_-\bar{s} &= -\bar{d}; \end{aligned}$$

Light (uds) Mesons

- Combining one quark with one antiquark schematically as described in the last part of L12 tells us that;

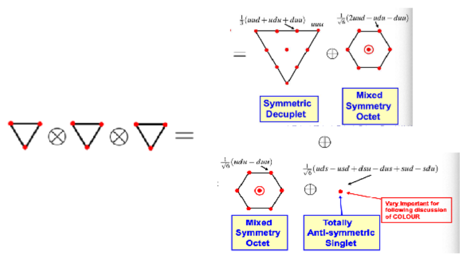


- Here, the **group theory method (MAKE SURE YOU KNOW THIS)** gives us that $3 \otimes \bar{3} = 8 \oplus 1$ and ladder operators indicate the two states in the middle of the hexagon should be linear combinations of $|u\bar{u}\rangle - |d\bar{d}\rangle$, $|u\bar{u}\rangle - |s\bar{s}\rangle$ and $|d\bar{d}\rangle - |s\bar{s}\rangle$. Hence, chose to reuse the π^0 isospin triplet as $\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$.
  - The second central octet state can then be found as a linear combination of the other states $\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |+s\bar{s}\rangle)$, which combined with $\langle\psi_1|\psi_2\rangle = 0$ and $\langle\psi_2|\psi_2\rangle = 1$ gives $\psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$.
  - The singlet state is then found as whatever is orthogonal to ψ_1 and ψ_2; $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$.
    - This state has no angular momentum, so in a sense is flavourless.
- As the SU(3) flavour is only approximate, the physical states with $I_3 = 0, Y = 0$ can be mixtures of the octet and singlet states.

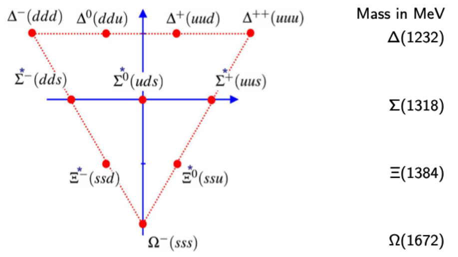


Baryons

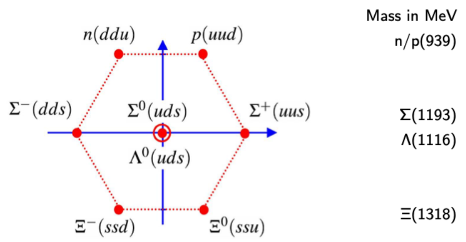
- Combining three sets of quarks as above, we find;



- In other words, $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$.
- We can then form a spin-$\frac{3}{2}$ decuplet using the symmetric flavour and symmetric spin wave-functions $\psi_{\text{flavour}}(S)\chi_{\text{spin}}(S)$ $(L = 0, S = 3/2, J = 3/2, P = +1)$;



- A spin-$\frac{1}{2}$ octet can be formed with mixed symmetry flavour and mixed symmetry spin wavefunctions as $\alpha\phi(M_S)\chi(M_S) + \beta\phi(M_A)\chi(M_A)$;



- It isn't possible to form a totally symmetric wavefunction from the anti-symmetric flavour singlet, as there is no totally anti-symmetric spin wavefunction for three quarks.
  - Note that none of the masses are the same indicating that SU(3)-flavour is a broken symmetry.

H8 - Quantum Chromodynamics

H7 - Symmetries and the Quark Model H9 - the Weak Interaction and V-A

Gauge Field Theories

- When deriving QED, we require physics to be invariant under the local phase transformation $\psi \to \psi' = e^{iq\chi(x)}\psi$.
  - Unfortunately, this transforms the Dirac equation as;

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad \longrightarrow \quad i\gamma^\mu(\partial_\mu + iq\partial_\mu \chi)\psi - m\psi = 0$$

  - In order to restore invariance, we must introduce a massless gauge boson A_μ and modify the Dirac equation to include this such that;

$$\begin{gathered} i\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\psi = 0 \\ \text{with} \quad A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi \end{gathered}$$

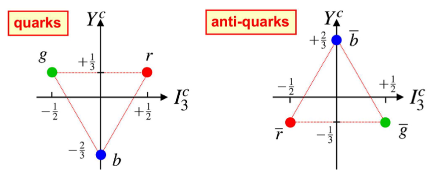
    - This results in the introduction of the $ie\gamma^\mu$ vertex factor in the QED Feynman rules.
    - Note this local phase transformation is a unitary $U(1)$ transformation.

QCD

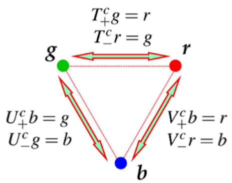
- Now assume a fundamental symmetry of the universe to be: invariance under SU(3) local phase transformations $\psi \to \psi' = e^{ig\vec{\lambda}\cdot\vec{\theta}(x)}\psi$ where $\vec{\lambda}$ are the Gell-Mann matrices.
  - This requires the wavefunction to be a vector in colour space $\psi = (\psi_1, \psi_2, \psi_3)$.
  - Requiring invariance under local SU(3) transformations allows for creation of the QCD Lagrangian.
    - Analysis of this Lagrangian find the QCD interaction vertex to be $-\frac{1}{2}ig_s\lambda^a\gamma^\mu$.
    - We also find that there are 8 massless gauge bosons (gluons) - one per λ, and that there can be 3 and 4 gluon vertices, but no others.
- This is all very similar to QED. However, now the strong interaction is fully invariant under rotation in colour space with $r \leftrightarrow b$, $r \leftrightarrow g$ and $b \leftrightarrow g$, as the SU(3)-colour symmetry is exact. If we represent states as;

$$r = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

  - If we then label such states using I_3^c - colour isospin and Y^c - colour hypercharge, then the permitted eigenstates for each quark (or antiquark) are;

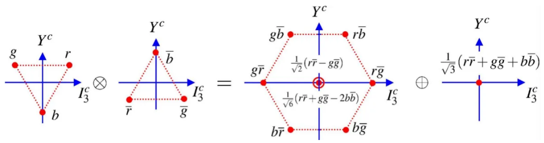


    - We can therefore reuse the SU(3) flavour results with $(u, d, s) \to (r, g, b)$ with ladder operators;



- The **Colour Confinement Hypothesis** states that: all free particles have colour singlet wavefunctions.
  - Suspected that this is derivable, but has not yet been done.
  - Note that a colour singlet is not just where $I_3^c = 0$ and $Y^c = 0$, but also require the singlet to be invariant under SU(3) colour transformations, and for all ladder operators to yield zero when applied on the singlet.

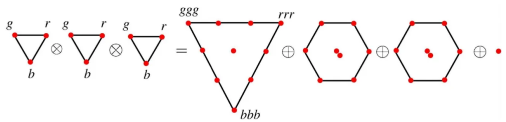
Mesons



- Considering $q\bar{q}$ combinations, we find a coloured octet and a colourless singlet.
  - Using the colour confinement hypothesis we therefore find the colour part of the wavefunction for mesons as;

$$\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

Baryons



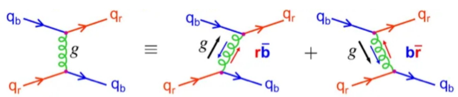
- Similar analysis with qq and $qq\bar{q}$ states doesn't produce colour singlets. However, qqq combinations do (above).
  - Hence, the colour singlet wavefunction for qqq baryons is

$$\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)$$

  - It is also possible to construct singlet states from $q\bar{q}q\bar{q}$ and $qqqq\bar{q}$ - tetra- and penta-quarks.

Gluons

- Quarks interact by exchanging virtual massless gluons which carry colour and anti-colour;



  - Gluons have the same colour + anti-colour wavefunctions as those which are obtained for mesons.

- Note that this would introduce a colourless gluon which would propagate like a photon giving infinite-ranged strong forcing. This is not observed.
  - Therefore, we see nature uses an SU(3) symmetry and not S(3), which would give 9 gluons.

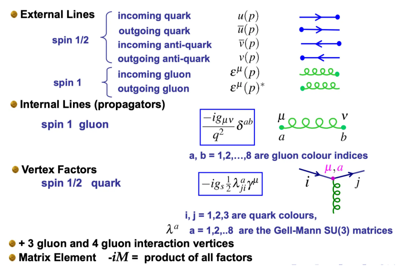
- Also note that unlike photons in QED, gluons carry the charge of QCD (colour), allowing self-interactions.
  - This therefore gives rise to triple and quadruple gluon vertices.
  - We therefore have both quark-quark and gluon-gluon scattering.
  - These self-interactions are believed to give rise to colour confinement.
    - Separating two coloured objects forms a flux tube of interacting gluons of energy density $\lambda \sim 1\ \mathrm{GeV/fm}$, so $V(r) \sim \lambda r$.
      - It would therefore require infinite energy to separate two coloured objects to infinity.
- The high energy density of quark separation gives rise to hadronisation and jets;
  - A quark- anti-quark pair produced from an annihilation event travel apart rapidly, forming a flux tube between them.
  - Once the energy stored in the flux tube is sufficiently high, another quark - anti-quark pair forms.
  - This repeats until the quarks pair up into jets of colourless hadrons (hadronisation).

Matrix Elements

- Rewriting the colour indices as $r = c_1$, $g = c_2$ and $b = c_3$ allows us to define the colour part of an interaction as;

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix} \lambda^a_{1i} \\ \lambda^a_{2i} \\ \lambda^a_{3i} \end{pmatrix} = \lambda^a_{ij}$$

  - This is the part of the matrix element responsible for the quark-gluon interaction.
- We can then write the Feynman rules for QCD as;



  - These are remarkably similar to those of QED (here), just replacing $e \to g_s$ and adding $\frac{1}{2}\lambda^a_{ji}$ at an index and δ^{ab} at an internal line. In all cases, the indices are still given by the piccadilly line rule.
  - Furthermore, when calculating QED and QCD matrix elements for isomorphic (ish) cases like $e^-\mu^- \to e^-\mu^-$ and $ud \to ud$, the QCD Matrix element is found to only differ by $e \to g_s$ and an additional Colour factor;

$$C(ik \to jl) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}$$

    - For example, if a process is $rr \to rr$, we only sum over Gell-Mann matrices with non-zero λ_{11} entries $\implies C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}$.
    - Note that this colour factor vanishes if we have any configuration where colour is not 'piped' around. Hence, **colour must be conserved**.

Experimental Evidence

Electron-Positron Collisions

- Electron-positron collisions produce all quark flavours for which $\sqrt{s} > 2m_q$ - generally resulting in jets.
    - As we have shown (here), the differential cross-section for $e^+e^- \to \mu^+\mu^-$ is

$$\sigma = \frac{4\pi\alpha^2}{3s} \quad \text{and}$$
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

    - The angular distribution of the jets produced is observed to be $\propto (1+\cos^2\theta)$, implying that quarks are also spin-$\frac{1}{2}$.
    - For a single quark flavour and a single colour, we expect (since in a Feynman diagram, the QED quark tube line would contribute $(Q_q e)^2$ and the QED electron-positron tube line contribute $(e)^2$);

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{4\pi\alpha^2}{3s}Q_q^2$$

        - Hence, for a jet of hadrons (where the factor of 3 arises from colour), we have

$$\sigma(e^+e^- \to \text{hadrons}) = 3\sum_{u,d,s,\dots}\frac{4\pi\alpha^2}{3s}Q_q^2$$

        - Hence, we find the ratio

$$R_\mu = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+ mu^-)} = 3\sum_{u,d,s,\dots}Q_q^2$$

            - Experimentally, we see R_μ increasing in steps $2 \to \frac{10}{3} \to \frac{11}{3}$ corresponding to situations with u,d,s; u,d,s,c; and u,d,s,c,b quarks.
                - This exemplifies the need for the factor of 3 from colour.
- Similarly, colour contributions accurately predict the relative rates at which two, three and four-jet events occur.

Quark-Quark Scattering

- Considering a $qq \to qq$ scattering process, there are 9 possible, equally likely, colour combinations for the colliding quarks.
    - Hence, the average matrix element $\langle |M_{fi}|^2\rangle$ contains the average colour factor

$$\langle |C|^2\rangle = \frac{1}{9}\sum_{i,j,k,l}^{3}|C(ij \to kl)|^2$$

        - For $qq \to qq$, we therefore get

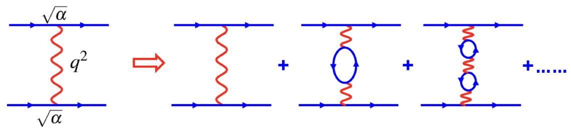
$$\langle |C|^2\rangle = \frac{1}{9}\left[\underbrace{3\times\left(\frac{1}{3}\right)^2}_{rr\to rr,\dots} + \underbrace{6\times\left(-\frac{1}{6}\right)^2}_{rb\to rb,\dots} + \underbrace{6\times\left(\frac{1}{2}\right)^2}_{rb\to br,\dots}\right] = \frac{2}{9}$$

        - Hence, the QCD differential cross-section for $ud \to ud$ is the same as that for $e^-\mu^- \to e^-\mu^-$ but replacing $\alpha \to \alpha_s$ and multiplying by $\langle |C|^2\rangle$.
            - Calculating hadron-hadron scattering is very difficult, as must include parton structure factors and account for all interactions, including those with gluons, but measurements do eventually match very well with predictions.
            - Again, at low E_T the cross-sections are dominated by low x partons (gluon-gluon scattering) and at high E_T they are dominated by high x (quark-antiquark) scattering.

Running Coupling Constants

Running of α - QED

- In QED, we see that the bare charge of an electron is screened by virtual e^+e^- pairs.



    - Assuming we can represent an e^+e^- interaction as $M = M_1 + rM_1 + r^2M_1 + \dots$, then computing the geometric sum we find

$$M = \frac{M_1}{1-r}$$

    - This gives some motivation for the (probably QED) result that this interaction is equivalent to a single diagram with a "running" coupling constant

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - \frac{\alpha(Q_0^2)}{3\pi}\ln\left(\frac{Q^2}{Q_0^2}\right)}$$

        - This is potentially problematic, as it becomes infinite for $\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$ - however, this is above where quantum gravity effect are likely to contribute, so QED probably doesn't apply.

Running of α_S - QCD

- In QCD, we have similar processes to QED, but with extra contributions from gluon loops.
    - As we sum the amplitudes of each process, we can have negative interference, giving smaller contributions. Eventually, we find (from QFT)

$$\alpha_S(Q^2) = \frac{\alpha_S(Q_0^2)}{1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)} \quad \text{with} \quad B = \frac{11N_c - 2N_f}{12\pi}$$

        - Here, N_c is the number of colours in the universe and N_f the number of quark flavours. With $N_c = 3$ and $N_f = 6$, we find $B > 0$, so α_S decreases with Q^2!
            - Therefore, at low Q^2, α_S is large, so perturbation theory cannot be used for low energy calculations in QCD.
            - At high Q^2, α_S is relatively small, allowing for perturbation theory approaches and explains why quarks have quasi-free behaviour in high energy deep inelastic scattering.

H9 - the Weak Interaction and V-A

H8 - Quantum Chromodynamics   H10 - Leptonic Weak Interactions and Neutrino Deep Inelastic Scattering

Parity

- The parity operator performs spatial inversion through the origin;

$$\psi'(\vec{x},t)=\hat{P}\psi(\vec{x},t)=\psi(-\vec{x},t)$$

  - Furthermore, to preserve wave-function normalisation,

$$\langle\psi|\psi\rangle=\langle\psi'|\psi'\rangle=\langle\psi|\hat{P}^\dagger\hat{P}|\psi\rangle$$

  - Hence, $\hat{P}^\dagger\hat{P}=I$ and noting that $\hat{P}\hat{P}=I$, we see $\hat{P}^{-1}=\hat{P}^\dagger=\hat{P}$, so $\hat{P}$ is unitary and Hermitian.
    - We therefore expect parity to be an observable which is conserved if $[\hat{H},\hat{P}]=0$.
    - Furthermore, if λ is the eigenvalue for an eigenvector $\vec{x}$ of $\hat{P}$, then $\hat{P}^2\vec{x}=\lambda^2\vec{x}=\vec{x}$, so $\lambda^2=1$.
- Gauge Field Theory eventually shows that gauge bosons have $P=P_\gamma=P_g=P_{W^+}=P_{W^-}=P_Z=-1$.
- As shown (here), we conventionally take spin-1/2 particles to have parity +1 and spin-1/2 anti-particles to have spin -1. We also previously saw that for Dirac spinors, the parity operator;

$$\hat{P}=\gamma^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

  - As an ordinary and adjoint spinors therefore transform as;

$$u\xrightarrow{\hat{P}}u'\equiv\hat{P}u=\gamma^0u$$
$$\bar{u}\xrightarrow{\hat{P}}\bar{u}'\equiv\overline{\hat{P}u}=\bar{u}\hat{P}=\bar{u}\gamma^0$$

    - Note that the adjoint results follow from;

$$\overline{\hat{P}u}=(\hat{P}u)^\dagger\gamma^0=(\gamma^0u)^\dagger\hat{P}=u^\dagger\gamma^{0\dagger}\hat{P}=u^\dagger\gamma^0\hat{P}=\bar{u}\hat{P}$$

- For convenience, we can also define vectors such that $\vec{v}\xrightarrow{\hat{P}}-\vec{v}$ and axial vectors such that $\vec{a}\xrightarrow{\hat{P}}\vec{a}$.
  - For example, $\vec{L}=\vec{r}\times\vec{p}$ is an axial vector, as both $\vec{r}$ and $\vec{p}$ acquire minus signs which cancel.

Currents

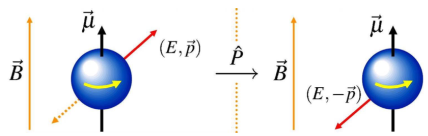
- All currents transform as Lorentz four-vectors under continuous transformations - however, Parity transformations are not continuous. Applying the parity operator to a 4-current gives;

$$j^\mu=\bar{\phi}\gamma^\mu\phi\xrightarrow{\hat{P}}(j')^\mu=(\bar{\phi}\gamma^0)\gamma^\mu(\gamma^0\psi)=\bar{\phi}(\gamma^0\gamma^\mu\gamma^0)\psi$$
$$\implies\begin{cases}j^0\xrightarrow{\hat{P}}\bar{\phi}\gamma^0\psi=j^0\\ j^i\xrightarrow{\hat{P}}-\bar{\phi}\gamma^i\psi=-j^i\end{cases}$$

  - Hence, currently behave like normal 4-vectors under parity, so scalar products between currents are invariant under parity.
  - As QED and QCD matrix elements involve terms like $j_e\cdot j_q$, they are also invariant under parity.
    - Therefore, QED and QCD conserve parity and predictions are invariant under parity inversion.

Parity Violation - β-Decay

- Observing β decays in an applied magnetic field, it is seen that electrons are emitted preferentially in a direction opposite to the applied B-field.
  - Considering that $\vec{B}(\vec{r})\propto Id\vec{l}\times\vec{r}$ according to the Biot-Savart law, we see that B is an axial vector, whereas the momentum of emitted electrons, p is a normal vector.



  - Therefore, applying parity, on a β-decay, we (surprisingly) see that in a coordinate inverted system, B is unchanged, but $\vec{p}$ is inverted relative to what they were in a non-inverted coordinate system.
    - Therefore, parity must be violated by the weak interaction and the weak vertex is **not** of the form $\bar{u}_e\gamma^\mu u_v$.

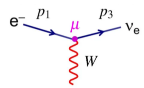
The Weak Interaction

- The requirement for Lorentz invariance severely restricts possible forms of an interaction vertex. There are only 5 possible combinations of two spinors and the gamma matrices that form Lorentz covariant currents, the "bilinear covariants";

Type	Form	Components	Boson Spin
Scalar	$\bar{\psi}\phi$	1	0
Pseudoscalar	$\bar{\psi}\gamma^5\phi$	1	0
Vector	$\bar{\psi}\gamma^\mu\phi$	4	1
Axial Vector	$\bar{\psi}\gamma^\mu\gamma^5\phi$	4	1
Tensor	$\bar{\psi}\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right)\phi$	6	2

  - These 16 components are linearly independent, forming a basis for a general 4×4 matrix - a linear combination can therefore represent the interaction between any fermion and boson.

The V-A Structure of the Weak Interaction



- For interactions involving spin-1 bosons, the most general vertex element is a linear combination of vector and axial-vector currents. This is observed to take the form vector - axial-vector, also known as $V-A$. Hence, for the process above,

$$j^\mu\propto\bar{u}_{\nu_e}(\gamma^\mu-\gamma^\mu\gamma^5)u_e$$

  - Applying parity on an axial-vector gives (note the difference of the vector case)

$$j_A=\bar{\psi}\gamma^\mu\gamma^5\phi\xrightarrow{\hat{P}}\bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\phi=-\bar{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^5\phi$$
$$\implies\begin{cases}j_A^0\xrightarrow{\hat{P}}-\bar{\psi}\gamma^0\gamma^5\phi=-j_A^0\\ j_A^i\xrightarrow{\hat{P}}\bar{\psi}\gamma^k\gamma^5\phi=j_A^k\end{cases}$$

  - Considering any matrix element $M\propto g_{\mu\nu}j_a^\mu j_2^\nu=j_1^0j_2^0-\vec{j}_1\cdot\vec{j}_2$, we see that if both currents are axial-vectors, then M is invariant under parity (the minuses cancel). Therefore, parity is also conserved for pure axial-vector interactions.
  - However, if one of the currents is a vector and the other an axial vector, we have

$$j_{v1}\cdot j_{A2}\xrightarrow{\hat{P}}\left(\left(j_{v1}^0\right)\left(-j_{A2}^0\right)-\left(-\vec{j}_{v1}\right)\cdot\left(\vec{j}_{A2}\right)\right)=-j_{v1}\cdot j_{A2}$$

    - This changes sign under parity, so can interfere with terms in M to give parity violating cross-sections.
  - Considering matrix elements of the form $g_v\gamma^\mu+g_A\gamma^\mu\gamma^5$, it becomes apparent (with some work) that parity is conserved if either g_A or g_v is zero and the relative strength of a parity violating part is proportional to

$$\frac{g_vg_A}{g_v^2+g_A^2}$$

Chirality and Helicity in the Weak Interaction

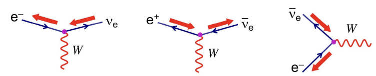
- The $W^\pm$ weak vertex is;

$$-\frac{ig_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)$$

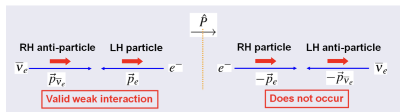
  - Hence, as $\frac{1}{2}(1-\gamma^5)$ projects the left-handed chiral particle states, we see that where $\bar{\psi}=\bar{\psi}_R+\bar{\psi}_L$,

$$-\bar{\psi}\frac{ig_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)\phi=-\frac{ig_W}{\sqrt{2}}\bar{\psi}_L\gamma^\mu\phi_L$$

  - **Therefore, only the left chiral components of spinors participate in weak charged current interactions.**
    - **In the relativistic limit, this implies that only left-helicity particles and right-helicity antiparticles interact by the weak charged current.** The only possible electron-neutrino interactions (in the relativistic limit) are then;



      - Note here, that the arrows on a Feynman diagram don't denote the direction in which a particle is moving, so where the spins are labelled by red arrows, and if we assume a very fast incoming particles, this all makes sense.
    - Note that this means some interactions, such as $\bar{\nu}_e+e^-\to W^-$ are parity-violating;



- Also note, this implies all neutrinos are left helicity and all anti-neutrinos are right helicity.

Helicity in Pion Decay

- Somewhat surprisingly, the decay of a pion into electrons is suppressed greatly relative to the decay into muons;

$$\frac{\Gamma(\pi^- \to e^-\bar{\nu}_e)}{\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)} = 1.23 \times 10^{-4}$$

  - Considering this decay in the frame of the (spin-0) pion, we see that the lepton and anti-neutrino must have spins in opposite directions and they must travel in opposite directions.
    - Then noting that the anti-neutrino is almost massless, it must be in a right-helicity state, **so the lepton must also be in a right helicity state**.
      - Recalling that the weak interaction only produces left chiral states, and noting that an electron is much lighter than a muon, so is always in a state very similar to that of chirality, we expect the electron decay rate to be suppressed relative to that of a muon decay. This is also found if we calculate the matrix element explicitly.
        - Note, this is due to the fact that the right-helicity contribution to a left-chirality state is not always zero.
        - Sometimes, decay to a right-helicity particle state is the only possibility.

The Weak Current Propagator

- Unlike QED and QCD, the weak interaction is mediated by massive W-bosons, resulting in a different form for the propagator (where μ and ν denote the ends of the W-boson connection);

$$\frac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m_W^2\right]}{q^2 - m_W^2}$$

  - Note however that where $q^2 \ll m_W^2$ this simplifies to

$$\frac{i g_{\mu\nu}}{m_W^2}$$

Fermi Theory

- Before parity violation was known, Fermi proposed a matrix element for β-decay of the form $M_{fi} = G_F g_{\mu\nu}[\bar{\psi}\gamma^\mu\psi][\bar{\psi}\gamma^\mu\psi]$. Upon the discovery of parity violation, this was updated to;

$$M_{fi} = \frac{G_F}{\sqrt{2}} g_{\mu\nu}\left[\bar{\psi}\gamma^\mu\left(1-\gamma^5\right)\psi\right]\left[\bar{\psi}\gamma^\nu\left(1-\gamma^5\right)\psi\right]$$

  - The modern version is;

$$M_{fi} = \left[\frac{g_W}{\sqrt{2}}\bar{\psi}\frac{1}{2}\gamma^\mu\left(1-\gamma^5\right)\psi\right]\frac{g_{\mu\nu} - q_\mu q_\nu/m_W^2}{q^2 - m_W^2}\left[\frac{g_W}{\sqrt{2}}\bar{\psi}\frac{1}{2}\gamma^\nu\left(1-\gamma^5\right)\psi\right]$$

  - In the $q^2 \ll m_W^2$ limit, this reduces to

$$M_{fi} = \frac{g_W^2}{8m_W^2} g_{\mu\nu}\left[\bar{\psi}\gamma^\mu\left(1-\gamma^5\right)\psi\right]\left[\bar{\psi}\gamma^\nu\left(1-\gamma^5\right)\psi\right]$$

    - Hence, for consistency we require

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}$$

- Defining the strength of a weak interaction with

$$\alpha_W = \frac{g_W^2}{4\pi} \approx \frac{1}{30}$$

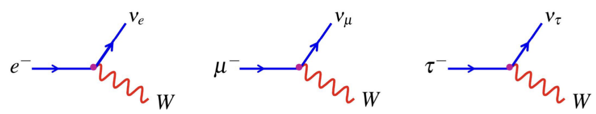
  - Where the final equality comes from above substituting experimental values.
  - The weak interaction is therefore intrinsically slightly stronger than the EM interaction, but is made to appear weak due to the high W-boson mass in the propagator.
    - For $q^2 \gg m_W^2$, weak interactions are more likely than EM interactions.

H10 - Leptonic Weak Interactions and Neutrino Deep Inelastic Scattering

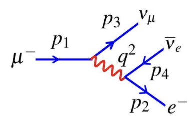
H9 - the Weak Interaction and V-A H11 - Neutrino Oscillations

Lepton Universality

- As we can never directly observe neutrinos, we define ν_l to be the neutrino state produced along with lepton l.
  - Note that the flavour states ν_e, ν_μ and ν_τ are not the fundamental particles - those are ν_1, ν_2 and ν_3, but this distinction doesn't matter much on short length scales.
  - In this case, possible $W^\pm$ interaction vertices are;



    - Note that for lepton decays, $q^2 \ll m_W^2$, so the propagator is constant (see here)



- With the above in mind, we can find the matrix element for a muon decay producing an electron (above) as;

$$M_{fi} = \frac{g_W^{(e)} g_W^{(\mu)}}{8m_W^2}\left[\bar{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)\right] g_{\mu\nu}\left[\bar{u}(p_2)\gamma^\nu(1-\gamma^5)v(p_4)\right]$$

  - $1 \to 3$ decays like the above result in Sargent's Rule - that the decay rate is proportional to the fifth power of the energy available from the decay (here pretty much the mass of the decaying lepton). Hence,

$$\Gamma(\mu \to e\nu\bar{\nu}) = \frac{G_F^e G_F^\mu m_\mu^5}{192\pi^3} = \frac{1}{\tau_\mu}$$

  - The tau lepton can also decay to an electron with a decay rate of the same form. Note, however that the tau lepton can also decay to other products less massive than itself. Taking the ratio of the muon and tauon decay rates gives;

$$\frac{\Gamma(\tau \to e\nu\bar{\nu})}{\Gamma(\mu \to e\nu\bar{\nu})} = \frac{G_F^e G_F^\tau m_\tau^5}{G_F^e G_F^\mu m_\mu^5}$$

    - Therefore, where Γ_l is the decay rate of the lepton into anything and BR denotes a branching ratio,

$$\frac{G_F^\tau}{G_F^\mu} = \frac{m_\mu^5}{m_\tau^5}\frac{\Gamma_\tau BR(\tau \to e\nu\bar{\nu})}{\Gamma_\mu} = \frac{m_\mu^5}{m_\tau^5}\frac{\tau_\mu}{\tau_\tau} \cdot BR(\tau \to e\nu\bar{\nu})$$

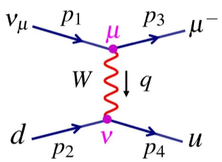
    - Experimentally, it is then observed that $\frac{G_F^\tau}{G_F^\mu} = 1.000 \pm 0.004$. Hence, the weak charged current is the same form all leptonic vertices - Charged Current Lepton Universality.

Neutrino Scattering

- By producing a sufficiently collimated beam of π^+ or π^-, and allowing them to decay whilst collimated, we can produce a narrow neutrino beam. Note, $\pi^+ \to \nu_\mu$ and $\pi^- \to \bar{\nu}_\mu$.

Neutrino-Quark Scattering

- Considering a ν_μ-proton deep inelastic scattering process, the neutrino must undergo $\nu_\mu \to \mu^- + W^+$ because all neutrinos are left helicity, and charge conservation must be obeyed.
  - As this can only produce W^+ bosons, we conclude that this interaction must then convert a quark from $d \to u$ within the proton (as to interact with a u would need to create a charge $5/2$ quark). Therefore, we have the process;



    - Note, the W^+ could also interact with a $\bar{u}$ present in the quark sea.
  - This has matrix element in the $q^2 \ll m_W^2$ limit of;

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu} \left[\bar{u}(p_3)\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right) u(p_1)\right] \left[\bar{u}(p_4)\frac{1}{2}\gamma^\nu \left(1-\gamma^5\right) u(p_2)\right]$$

    - Conducting this interaction in the extreme relativistic limit, where we still have $q^2 \ll m_W^2$ (note this is a compromise), we can approximate the helicity states as chiral states. Since the weak interaction 'conserves helicity', we therefore only have a contribution from;

$$M_{fi} = \frac{g_W^2}{2m_W^2} g_{\mu\nu} \left[\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)\right] \left[\bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2)\right].$$

    - Considering this interaction in the CoM frame and performing the calculations, we find;

$$\begin{aligned} \bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2E(c, s, -is, c) \\ \bar{u}_\downarrow(p_4)\gamma^\nu u_\downarrow(p_2) &= 2E(c, -s, -is, -c) \\ \implies & \boxed{M_{fi} = \frac{g_W^2 \hat{s}}{m_W^2} \quad \text{with} \quad \hat{s} = (2E)^2} \end{aligned}$$

      - Note, $\hat{s}$ is Mandelstram s.
      - Since this helicity combination has spins anti-parallel, this matrix element matches our expectation for there to be no preferred polar angle.
- Using the above result, and noting that only half of the quarks will be in a left helicity (and therefore able to participate in the charged current Weak interaction), we find;

$$\langle |M_{fi}|^2 \rangle = \frac{1}{2}\left|\frac{g_W^2}{m_W^2}\hat{s}\right|^2$$

$$\implies \frac{d\sigma}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s} \implies \sigma_{\nu q} = \frac{G_F^2 \hat{s}}{\pi}(1)$$

  - Note that since this cross section is longitudinally Lorentz invariant, it is also valid in the lab frame.

Antineutrino-Quark Scattering

- In this case, we repeat the above, but note that the interaction now occurs in a total angular momentum $J = 1$ state, so we expect angular dependence. Eventually, we obtain;

$$\frac{d\sigma_{\bar{\nu}q}}{d\Omega^*} = \frac{d\sigma_{\nu q}}{d\Omega^*}\frac{1}{4}(1+\cos\theta^*)^2 \hat{s}$$

  - This angular part can be understood in terms of the overlap of initial and final angular momentum wave-functions.
  - Integrating over the solid angle, we find;

$$\boxed{\sigma_{\bar{\nu}q} = \frac{G_F^2 \hat{s}}{3\pi}} \implies \frac{\sigma_{\bar{\nu}q}}{\sigma_{\nu q}} = \frac{1}{3}$$

(Anti)neutrino-(Anti)quark Scattering

- We can then immediately find by analogy the cross-sections for the other two allowed interactions ($\nu_\mu \bar{u} \to \mu^- \bar{d}$) and ($\bar{\nu}_\mu \bar{d} \to \mu^+ \bar{u}$) which are present due to the sea quarks in the proton. These give;

$$\begin{aligned} \frac{d\sigma_{\nu\bar{q}}}{d\Omega^*} = \frac{G_F^2}{16\pi^2}(1+\cos\theta^*)^2 \hat{s} \quad &\text{and} \quad \frac{d\sigma_{\bar{\nu}\bar{q}}}{d\Omega^*} = \frac{G_F^2}{4\pi^2}\hat{s}; \\ \sigma_{\nu\bar{q}} = \frac{G_F^2 \hat{s}}{3\pi} \quad &\text{and} \quad \sigma_{\bar{\nu}\bar{q}} = \frac{G_F^2 \hat{s}}{\pi} \end{aligned}$$

Lorentz Invariant Differential Cross Sections

- In the relativistic limit where $y = \frac{1}{2}(1-\cos\theta^*)$, transforming the scattering cross section as a Jacobian, we see that;

$$\frac{d\sigma}{dy} = 4\pi \frac{d\sigma}{d\Omega^*}$$

  - Hence, using (1), we see

$$\frac{d\sigma_{\nu q}}{dy} = \frac{d\sigma_{\bar{\nu}\bar{q}}}{dy} = \frac{G_F^2}{\pi}\hat{s}$$

  - Similarly, noting that $1 - y = \frac{1}{2}(1+\cos\theta^*)$, we see

$$\frac{d\sigma_{\bar{\nu}q}}{dy} = \frac{d\sigma_{\nu\bar{q}}}{dy} = \frac{G_F^2}{\pi}(1-y)^2 \hat{s}$$

    - Interestingly, the QED cross-section can also be written in a similar (but different) form as (from here);

$$\frac{d\sigma_{e^\pm q}}{dy} = \frac{2\pi\alpha^2}{q^4} e_q^2 \left[1 + (1-y)^2\right] \hat{s}$$

The Parton Model for Neutrino Deep Inelastic Scattering

- Defining the parton distribution functions the same as before, we see that neutrino-proton scattering has contributions from down and anti-up quarks, such that, where the CoM energy of the $\nu_\mu d$ for a specific x is, $\hat{s} = xs$;

$$\frac{d^2\sigma^{\nu p}}{dxdy} = \frac{G_F^2}{\pi} sx \left[d^p(x) + (1-y)^2 \bar{u}^p(x)\right]$$

  - Similarly for anti-neutrino proton scattering, we have

$$\frac{d^2\sigma^{\bar{\nu} p}}{dxdy} = \frac{G_F^2}{\pi} sx \left[(1-y)^2 u^p(x) + \bar{d}^p(x)\right]$$

  - We can then simply relabel to find the differential cross-section for (anti)neutrino-neutron scattering;

$$\begin{aligned} \frac{d^2\sigma^{\nu n}}{dxdy} &= \frac{G_F^2}{\pi} sx \left[d^n(x) + (1-y)^2 \bar{u}^n(x)\right] \\ \frac{d^2\sigma^{\bar{\nu} n}}{dxdy} &= \frac{G_F^2}{\pi} sx \left[(1-y)^2 u^n(x) + \bar{d}^n(x)\right] \end{aligned}$$

  - We then again assume isospin symmetry;

$$d^n(x) = u^p(x) = u(x) \quad \text{and} \quad u^n(x) = d^p(x) = d(x)$$

- This then allows us to write the mean cross-section per nucleon N for an isoscalar target (same number of protons and neutrons) as;

$$\frac{d^2\sigma^{\nu N}}{dxdy} = \frac{1}{2}\left(\frac{d^2\sigma^{\nu p}}{dxdy} + \frac{d^2\sigma^{\nu n}}{dxdy}\right)$$
$$\implies \frac{d^2\sigma^{\nu N}}{dxdy} = \frac{G_F^2}{2\pi} sx\left[u(x) + d(x) + (1-y)^2(\bar{u}(x) + \bar{d}(x))\right]$$

- Integrating over x therefore gives (where the second is for antineutrino-Nucleon interactions);

$$\boxed{\begin{aligned}\frac{d\sigma^{\nu N}}{dy} &= \frac{G_F^2}{2\pi} s\left[f_q + (1-y)^2 f_{\bar{q}}\right] (2)\\ \frac{d\sigma^{\bar{\nu} N}}{dy} &= \frac{G_F^2}{2\pi} s\left[(1-y)^2 f_q + f_{\bar{q}}\right] (3)\end{aligned}}$$
$$\text{with} \quad \begin{cases} f_q = f_d + f_u = \int_0^1 x[u(x) + d(x)]dx \\ f_{\bar{q}} = f_{\bar{d}} + f_{\bar{u}} = \int_0^1 x[\bar{u}(x) + \bar{d}(x)]dx \end{cases}$$

Experimental Evidence

- To measure these functions, collide high energy neutrinos with metal plates within a magnetic field which will cause any emitted leptons to follow a helical shape.
  - We can then measure three variables; the energy from X, E_X; the momentum of an emitted lepton using its curvature in the B-field, E_μ; and the initial angle with which the lepton is emitted. This gives

$$E_\nu = E_X + E_\mu$$

  - Furthermore, as we are in the lab frame, we can use this, giving

$$y = 1 - \frac{E_\mu}{E_\nu} \implies E_\mu = (1-y)E_\nu$$

  - Integrating (2-3), we find

$$\sigma^{\nu N} = \frac{G_F^2 s}{2\pi}\left[f_q + \frac{1}{3}f_{\bar{q}}\right] \quad \text{and} \quad \sigma^{\bar{\nu} N} = \frac{G_F^2 s}{2\pi}\left[\frac{1}{3}f_q + f_{\bar{q}}\right]$$

    - Note that in the lab frame with $p^\mu(\nu) = (E_\nu, 0, 0, E_\nu)$ and $p^\mu(p) = (m_p, 0, 0, 0)$, we have $s = (E_\nu + m_p)^2 - E_\nu^2 \approx 2E_\nu m_p$, so Neutrino DIS cross section is approximately proportional to the lab frame neutrino energy!
    - Measuring these cross-sections, we find $f_q \approx 0.41$ and $f_{\bar{q}} \approx 0.08$, so around 50% of the momentum within the nucleons is carried by gluons.
    - This also provides evidence for anti-quarks, as with $f_{\bar{q}} = 0$, we expect $\sigma^{\nu N}/\sigma^{\bar{\nu} N} = 3$. The measured value is ≈ 2.
- Note that neutrinos also interact via the neutral current with the Z-boson.

H11 - Neutrino Oscillations

H10 - Leptonic Weak Interactions and Neutrino Deep Inelastic Scattering
H12 - The CKM Matrix and CP Violation

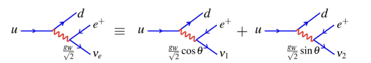
Neutrino Mass vs Weak Eigenstates

- Neutrinos must be made by the weak interaction, as they don't have colour or charge, so don't interact by QED or QCD.
- Previously, we defined the neutrino weak eigenstates as those produced or absorbed by their respective lepton.
  - We now define the neutrino mass eigenstates ν_1, ν_2, ν_3 as the states with a well defined mass and which are therefore eigenstates of the free Hamiltonian.
  - It is observed that neutrinos change flavour on long length-scales, so the weak eigenstates must be a combination of the mass eigenstates - much like the double slit experiment. The mass eigenstates interfere across a long gap.

Neutrino Oscillations for Two Flavours

- In the two neutrino flavour case, we write the weak eigenstates ν_e and ν_μ as coherent linear combinations of the mass eigenstates;

$$\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} \implies \begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} \quad (1)$$



  - Since the mass eigenstates are free-particle solutions to the wave equation, they evolve as

$$|\nu_1(t)\rangle = |\nu_1\rangle\, e^{i\vec{p}_1\cdot\vec{x} - iE_1 t}; \quad |\nu_2(t)\rangle = |\nu_1\rangle\, e^{i\vec{p}_2\cdot\vec{x} - iE_2 t}$$

  - If at $t = 0$ a neutrino is produced in a ν_e state, then

$$|\psi(0)\rangle = |\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$
$$\implies |\psi(t)\rangle = \cos\theta\,|\nu_1\rangle\, e^{-ip_1\cdot x} + \sin\theta\,|\nu_2\rangle\, e^{-ip_2\cdot x}$$

  - If the neutrino interacts with a detector a distance L away at time T, then defining $\phi_i = p_i \cdot x = E_i T - |\vec{p}_i| L$ gives;

$$\boxed{|\psi(L,T)\rangle = \cos\theta\,|\nu_1\rangle\, e^{-i\phi_1} + \sin\theta\,|\nu_2\rangle\, e^{-i\phi_2}}$$

    - Using (1) this is re-expressed as;

$$|\psi(L,T)\rangle = |\nu_e\rangle\left(\cos^2\theta e^{-i\phi_1} + \sin^2\theta e^{-i\phi_2}\right) + |\nu_\mu\rangle \sin\theta\cos\theta\left(-e^{-i\phi_1} + e^{-i\phi_2}\right)$$

    - Continuing with this, we find that

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |\langle\nu_\mu|\psi(L,T)\rangle|^2\\ &= \cos^2\theta\sin^2\theta(-e^{-i\phi_1} + e^{-i\phi_2})(-e^{i\phi_1} + e^{i\phi_2})\\ \implies & \boxed{P(\nu_e \to \nu_\mu) = \sin^2 2\theta \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right)} (2)\end{aligned}$$

      - Hence, for neutrino oscillations to occur, we require BOTH $\sin(2\theta) \neq 0$ and $\sin\left(\frac{\phi_1-\phi_2}{2}\right) \neq 0$.
- Using (2), if the masses of $|\nu_1\rangle$ and $|\nu_2\rangle$ are the same such that $\phi_1 = \phi_2$, then a neutrino produced by an electron would also produce an electron when it interacts.
  - Evaluating $\Delta\phi_{12} = \phi_1 - \phi_2 = (E_1 - E_2)T - (|p_1| - |p_2|)L$ using $|p_1| = |p_2| = p$, we find (also using $L \approx (c)T$ - but taking $c = 1$);

$$\Delta\phi_{12} = p\left[\left(1+\frac{m_1^2}{p^2}\right)^{1/2} - \left(1+\frac{m_2^2}{p^2}\right)^{1/2}\right]L \approx \frac{m_1^2-m_2^2}{2p}L$$

- A full analysis accounting for how mass changes momentum recovers the same result.

- This can also be justified by writing the phase difference as

$$\Delta\phi_{12} = (E_1-E_2)T - \left(\frac{|p_1|^2-|p_2|^2}{|p_1|+|p_2|}\right)L$$
$$\implies \Delta\phi_{12} = (E_1-E_2)\left[T - \left(\frac{E_1+E_2}{|p_1|+|p_2|}\right)L\right] + \left(\frac{m_1^2-m_2^2}{|p_1|+|p_2|}\right)L$$

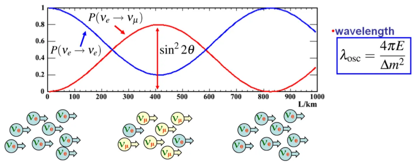
- The first RHS term then vanishes if $E_1 = E_2$ or $\beta_1 = \beta_2$.

- In all of these cases, we have

$$\Delta\phi_{12} \approx \frac{m_1^2-m_2^2}{2p}L \approx \frac{\Delta m^2}{2E}L$$

- Therefore, (2) reduces to

$$P(\nu_e\to\nu_\mu) = \sin^2 2\theta \sin^2\left(\frac{\Delta m_{21}^2 L}{4E}\right)$$



Neutrino Oscillations for Three Flavours

- Repeating a similar process for three flavours, we write the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix relation (where unitarity $\implies U^{-1} = U^\dagger$ gives the second line);

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}$$
$$\implies \begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = \begin{pmatrix}U_{e1}^* & U_{\mu1}^* & U_{\tau1}^*\\ U_{e2}^* & U_{\mu2}^* & U_{\tau2}^*\\ U_{e3}^* & U_{\mu3}^* & U_{\tau3}^*\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$

- Furthermore, as $UU^\dagger = I$, we have a set of six unique relations given by;

$$\begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}\begin{pmatrix}U_{e1}^* & U_{\mu1}^* & U_{\tau1}^*\\ U_{e2}^* & U_{\mu2}^* & U_{\tau2}^*\\ U_{e3}^* & U_{\mu3}^* & U_{\tau3}^*\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{3}$$

- Defining the state at $t=0$ similarly to here, after travelling a distance L in the z-direction, it takes the form

$$|\psi(L)\rangle = U_{e1}|\nu_1\rangle e^{-\phi_1} + U_{e2}|\nu_2\rangle e^{-i\phi_2} + U_{e3}|\nu_3\rangle e^{-i\phi_3}$$
$$\text{with} \quad \phi_i \approx \frac{m_i^2}{2E_i}L$$

- Note, the ϕ_i value again matches the result from a full analysis.

- Substituting for the mass eigenstates in terms of the weak eigenstates eventually gives

$$\begin{aligned}|\psi(L)\rangle =& \left(U_{e1}U_{e1}^*e^{-i\phi_1} + U_{e2}U_{e2}^*e^{-i\phi_2} + U_{e3}U_{e3}^*e^{-i\phi_3}\right)|\nu_e\rangle\\ &+ \left(U_{e1}U_{\mu1}^*e^{-i\phi_1} + U_{e2}U_{\mu2}^*e^{-i\phi_2} + U_{e3}U_{\mu3}^*e^{-i\phi_3}\right)|\nu_\mu\rangle\\ &+ \left(U_{e1}U_{\tau1}^*e^{-i\phi_1} + U_{e2}U_{\tau2}^*e^{-i\phi_2} + U_{e3}U_{\tau3}^*e^{-i\phi_3}\right)|\nu_\tau\rangle.\end{aligned}$$

- Therefore, we see that

$$\begin{aligned}P(\nu_e\to\nu_\mu) &= |\langle\nu_\mu|\psi(L)\rangle|^2\\ &= \left|U_{e1}U_{\mu1}^*e^{-i\phi_1} + U_{e2}U_{\mu2}^*e^{-i\phi_2} + U_{e3}U_{\mu3}^*e^{-i\phi_3}\right|^2\end{aligned} \tag{4}$$

- From (3), we require

$$U_{e1}U_{\mu1}^* + U_{e2}U_{\mu2}^* + U_{e3}U_{\mu3}^* = 0 \tag{5}$$

- Therefore, for neutrino oscillations to occur, the ϕ_i in (4) must be different, so the neutrinos must have different masses.

$$\boxed{|z_1+z_2+z_3|^2 \equiv |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,\mathrm{Re}\left(z_1z_2^* + z_1z_3^* + z_2z_3^*\right)}$$

- Now applying the identity above to (4) and (5), we see that (4) may be rewritten as;

$$\begin{aligned}P(\nu_e\to\nu_\mu) =& 2\,\mathrm{Re}\left\{U_{e1}U_{\mu1}^*U_{e2}^*U_{\mu2}\left[e^{-i(\phi_1-\phi_2)}-1\right]\right\}\\ &+2\,\mathrm{Re}\left\{U_{e1}U_{\mu1}^*U_{e3}^*U_{\mu3}\left[e^{-i(\phi_1-\phi_3)}-1\right]\right\}\\ &+2\,\mathrm{Re}\left\{U_{e2}U_{\mu2}^*U_{e3}^*U_{\mu3}\left[e^{-i(\phi_2-\phi_3)}-1\right]\right\}.\end{aligned} \tag{6}$$

- Similarly to above, but defining Δ_{ij} as

$$\Delta_{ij} = \frac{\Delta m_{ij}^2 L}{4E} \quad \text{with} \quad \Delta m_{21}^2 = m_2^2 - m_1^2$$

- And noticing that only two of the mass-squared differences and therefore Δ_{ij} are independent;

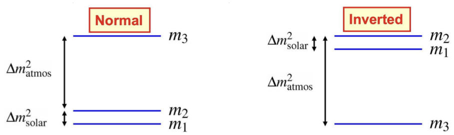
$$m_3^2 - m_1^2 = \left(m_3^2 - m_2^2\right) + \left(m_2^2 - m_1^2\right)$$

- We find the ν_e survival probability as

$$\begin{aligned}P(\nu_e\to\nu_e) = 1 &- 4|U_{e1}|^2|U_{e2}|^2\sin^2\Delta_{21}\\ &- 4|U_{e1}|^2|U_{e3}|^2\sin^2\Delta_{31}\\ &- 4|U_{e2}|^2|U_{e3}|^2\sin^2\Delta_{32}\end{aligned} \tag{7}$$

Neutrino Mass Hierarchy

- Current neutrino oscillation results have only been able to determine $|\Delta m_{ji}^2|$ and detect two very different mass scales, one "atmospheric" - produced primarily by cosmic ray interactions in the upper atmosphere - with $|\Delta m^2|_{\text{atm}} \sim 2.5\times 10^{-3}\ \text{eV}^2$ and one "solar" - produced primarily in the sun - with $|\Delta m^2|_{\text{sol}} \sim 8\times 10^{-5}\ \text{eV}^2$. This gives two possibilities;



- We can therefore approximate $|\Delta m_{31}^2| \approx |\Delta m_{32}^2|$ and therefore, $|\Delta_{31}| \approx |\Delta_{32}|$.
  - Now assuming the PMNS matrix is real (neglecting CP violation, see below), using the above, using and (5), we can simplify the oscillation probability (6) to;

$$P(\nu_e\to\nu_\mu) \approx -4U_{e1}U_{\mu1}U_{e2}U_{\mu2}\sin^2\Delta_{21} + 4U_{e3}^2U_{\mu3}^2\sin^2\Delta_{32}$$

- Similarly using the relation $U_{e1}^2 + U_{e2}^2 + U_{e3}^2 = 1$ from (3), we find the survival probability (7) simplifies to;

$$P(\nu_e \to \nu_e) \approx 1 - 4U_{e1}^2 U_{e2}^2 \sin^2 \Delta_{21} - 4\left(1 - U_{e3}^2\right) U_{e3}^2 \sin^2 \Delta_{32} \tag{8}$$

- The same can be repeated for each interaction, giving;

Summary of approx three flavour oscillation probabilities neglecting *CP*-violation and using $|\Delta m_{31}^2| \approx |\Delta m_{32}^2|$

Survival probabilities

$$P(\nu_e \to \nu_e) \approx 1 - 4U_{e1}^2 U_{e2}^2 \sin^2 \Delta_{21} - 4\left(1 - U_{e3}^2\right) U_{e3}^2 \sin^2 \Delta_{32} \tag{133}$$

$$P(\nu_\mu \to \nu_\mu) \approx 1 - 4U_{\mu1}^2 U_{\mu2}^2 \sin^2 \Delta_{21} - 4\left(1 - U_{\mu3}^2\right) U_{\mu3}^2 \sin^2 \Delta_{32} \tag{134}$$

$$P(\nu_\tau \to \nu_\tau) \approx 1 - 4U_{\tau1}^2 U_{\tau2}^2 \sin^2 \Delta_{21} - 4\left(1 - U_{\tau3}^2\right) U_{\tau3}^2 \sin^2 \Delta_{32} \tag{135}$$

Oscillation probabilities

$$P(\nu_e \to \nu_\mu) = P(\nu_\mu \to \nu_e) \approx -4U_{e1}U_{\mu1}U_{e2}U_{\mu2} \sin^2 \Delta_{21} + 4U_{e3}^2 U_{\mu3}^2 \sin^2 \Delta_{32} \tag{136}$$

$$P(\nu_e \to \nu_\tau) = P(\nu_\tau \to \nu_e) \approx -4U_{e1}U_{\tau1}U_{e2}U_{\tau2} \sin^2 \Delta_{21} + 4U_{e3}^2 U_{\tau3}^2 \sin^2 \Delta_{32} \tag{137}$$

$$P(\nu_\mu \to \nu_\tau) = P(\nu_\tau \to \nu_\mu) \approx -4U_{\mu1}U_{\tau1}U_{\mu2}U_{\tau2} \sin^2 \Delta_{21} + 4U_{\mu3}^2 U_{\tau3}^2 \sin^2 \Delta_{32} \tag{138}$$

The wavelengths associated with $\sin^2 \Delta_{21}$ and $\sin^2 \Delta_{32}$ are:

"ATMOSPHERIC" $\lambda_{32} = \frac{4\pi E}{\Delta m_{32}^2}$ "Short"-Wavelength and "SOLAR" $\lambda_{21} = \frac{4\pi E}{\Delta m_{21}^2}$ "Long"-Wavelength.

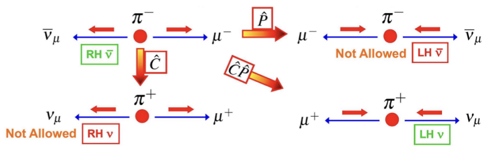
- The PMNS matrix can also be expressed in terms of 3 rotation angles θ_{12}, θ_{23}, θ_{13} and a complex phase δ. Where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$, this is expressed as

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{"Atmospheric"}} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{"Solar"}}$$

  - There are therefore six SM parameters to measure with ν oscillation experiments; $|\Delta m_{21}|^2$, $|\Delta m_{32}|^2$, θ_{12}, θ_{23}, θ_{13} and δ.

CP and CPT in the Weak Interaction

- As addition to parity ($\hat{P}$) time reversal ($\hat{T}$) and charge conjugation ($\hat{C}$) are discrete symmetries.
  - It is easily seen that the weak interaction violates $\hat{C}$, but is invariant under the combined effect of $\hat{C}$ and $\hat{P}$;



  - This can also be understood as under CP transformation,

$$\text{RH Particles} \leftrightarrow \text{LH Anti-Particles}$$
$$\text{LH Particles} \leftrightarrow \text{RH Anti-Particles}$$

    - If the weak interaction is invariant under CP, we expect

$$\Gamma(\pi^+ \to \mu^+\nu_\mu) = \Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)$$

  - The CPT-Theorem states that all Lorentz invariant QFTs are invariant under CPT, and therefore that particles/ anti-particles have identical mass, lifetimes and magnetic moments.
    - Therefore, T-violation must imply CP-violation, which could explain the excess of matter over anti-matter in the universe.
    - CP violation can occur in the weak interaction, see here.

CP and T Violation in Neutrino Oscillations

- Considering (6), we can find the oscillation probability $P(\nu_\mu \to \nu_e)$ by exchanging labels $e \leftrightarrow \mu$, therefore giving pre-factors within each real part of z and z^* in the $\nu_e \to \nu_\mu$ and $\nu_\mu \to \nu_e$ decays respectively.
  - Therefore, unless elements of the PMNS matrix are real,

$$P(\nu_e \to \nu_\mu) \neq P(\nu_\mu \to \nu_e) \tag{9}$$

    - This would imply that three-flavour neutrino oscillations are not invariant under time reversal.
  - Applying CPT on a neutrino oscillation,

$$\nu_\mu \to \nu_e \quad \xleftrightarrow{\hat{C}\hat{P}\hat{T}} \quad \bar{\nu}_e \to \bar{\nu}_\mu$$

    - Therefore, if weak interactions are invariant under CPT, we expect;

$$P(\nu_\mu \to \nu_e) = P(\bar{\nu}_e \to \bar{\nu}_\mu)$$

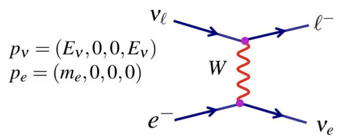
      - Combining this with (9), we find that if the PMNS matrix is not purely real,

$$\boxed{P(\nu_e \to \nu_\mu) \neq P(\bar{\nu}_e \to \bar{\nu}_\mu)}$$

      - Therefore, unless the PMNS matrix is real, CP will be violated in neutrino oscillations, although detection of this is currently well below experimental sensitivity.

Neutrino Experiments

- Neutrinos either interact with atomic electrons or the nucleons within the nucleus - either by the charge current (CC) or neutral current (NC) interactions.
  - Reactor and solar neutrinos typically have $E_\nu \sim 1 \text{ MeV}$, wheras atmospheric neutrinos typically have $E_\nu \sim 1 \text{ GeV}$. These are all relatively low energies, kinematically restricting feasible reactions.



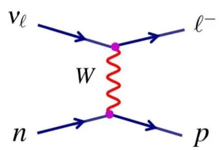
  - For interactions with atomic electrons, we require $s > m_l^2$. Therefore, for the above interaction, using $s = (E_\nu + m_e)^2 - E_\nu^2$, we require

$$E_\nu > \left[\left(\frac{m_l}{m_e}\right)^2 - 1\right]\frac{m_e}{2}$$

    - Therefore, for interactions with atomic electrons, we require

$$E_{\nu_e} > 0 \text{ GeV} \quad E_{\nu_\mu} > 11 \text{ GeV} \quad E_{\nu_\tau} > 3090 \text{ GeV}$$

    - Electron neutrinos from the sun which oscillate into ν_μ and ν_τ therefore seem to disappear, as they cannot interact.



  - For interactions with nucleons (above), we require $s > (m_l + m_p)^2$, so must have

$$E_\nu > \frac{(m_p^2 - m_n^2) + m_l^2 + 2m_p m_l}{2m_n}$$

- These interactions therefore require neutrinos to have

$$E_{\nu_e} > 0 \text{ GeV} \quad E_{\nu_\mu} > 110 \text{ MeV} \quad E_{\nu_\tau} > 3.5 \text{ GeV}$$

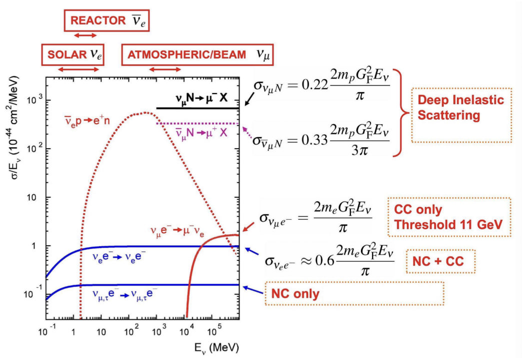
- Atmospheric muon neutrinos which oscillate into ν_τ therefore appear to disappear, as they cannot interact.

- To observe neutrino interactions, they must also have sufficiently large cross sections. Using this result shows that for muon neutrinos, the cross-section increases linearly with lab frame neutrino energy;

$$\sigma_{\nu_\mu e^-} = \frac{G_F^2 s}{\pi} \approx \frac{2m_e G_F^2 E_\nu}{\pi}$$

  - For electron neutrinos, a NC interaction is also possible, which negatively interferes with the CC interaction such that;

$$\sigma_{\nu_e e^-} \approx 0.6\sigma_{\nu_e e^-}^{CC}$$

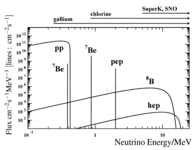


Long Baseline Experiments (Beam Neutrinos $E_\nu > 1 \text{ GeV}$)

- Generally work by creating an intense neutrino beam and building one near and one far detector.
  - Can then measure the ratio of the neutrino energy spectrum in the far detector (where the neutrinos may have oscillated) to that in the near detector (where much less oscillation has occurred).
  - Note that the ratio detected here is therefore expected to be well related to the predictions here, potentially allowing $\sin 2\theta^2$ and Δm^2 to be found.
- MINOS is an example of this technology, deployed at Fermilab to produce high energy neutrinos with $E_\nu > 1 \text{ GeV}$.
  - Neutrinos here interact with nucleons and can then use $E_\nu = E_\mu + E_X$ to investigate further. Can determine muon energy from range / curvature in the B-field and can determine hadronic energy from the amount of light observed.

Solar Neutrino Experiments ($E_\nu > 5 \text{ MeV}$)

- The solar neutrino emission spectrum is;

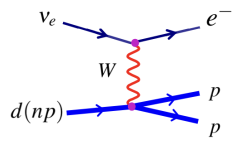


$p + p \to d + e^+ + \nu_e$	$E_\nu < 0.5 \text{ MeV}$
$^8\text{B} \to {}^8\text{Be}^* + e^+ + \nu_e$	$E_\nu \sim 5 \text{ MeV}$
$p + e^- + p \to d + \nu_e$	
$^7\text{Be} + e^- \to {}^7\text{Li} + \nu_e$	
$^3\text{He} + p \to {}^4\text{He} + e^+ + \nu_e$	

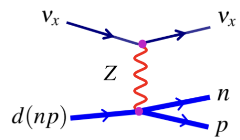
  - The number of electron neutrinos observed is often lower than expected from this - the solar neutrino problem.
- Super Kamiokande is a water Cerenkov detector.
  - Is possible to distinguish electron and muon interactions from the different light patterns - muons produce clean rings, whereas electrons produce fuzzy rings.
  - Is sensitive to solar neutrinos with $E_\nu > 5 \text{ MeV}$, so mostly detect B^8 decays.
  - Showed that the number of neutrinos detected was around 45% those that are expected from the standard solar model prediction. Measured;

$$\Delta m_{32}^2 \approx 2.4 \times 10^{-3} \text{ eV}^2$$

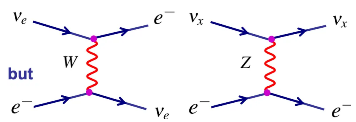
- SNO is another Cerenkov detector using D_2O surrounded by PMTs.
  - The use of D_2O allows this experiment to be sensitive to three different interactions;
    - A charged current interaction only sensitive to ν_e, giving $\text{CC Rate} \propto \phi(\nu_e)$;



    - A neutral current interaction which liberates a neutron which is then thermalised in the heavy water and observed following capture on another nuclei. This gives a measure of total neutrino flux, so $\text{NC Rate} \propto \phi(\nu_e) + \phi(\nu_\mu) + \phi(\nu_\tau)$;



    - Elastic scattering interactions with a larger cross-section for ν_e with $\text{ES Rate} \propto \phi(\nu_e) + 0.154(\phi(\nu_\mu) + \phi(\nu_\tau))$;



  - It is also possible to find information about the type of neutrino by its direction relative to the sun, and to measure background interactions as their contributions vary with distance from the edge of the detector.
    - Therefore, uniquely, SNO is able to measure electron neutrino flux and the total flux.
    - Find that measured $\phi(\nu_e) + \phi(\nu_\mu) + \phi(\nu_\tau)$ match the standard solar model (SMM) prediction for $\phi(\nu_e)$, providing evidence of $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_\tau$ transition.
- Note that when considering solar neutrino data, matter effects should be accounted for, noting that the coherent forward scattering process for an electron neutrino is different to that for a muon or tau neutrino.
  - This can enhance oscillations. Accounting for this, we find

$$\Delta m_{\text{solar}}^2 \approx 8 \times 10^{-5} \text{ eV}^2 \quad \sin^2 2\theta_{\text{solar}} \approx 0.85$$

Atmospheric Neutrino Experiments ($E_\nu \sim 1 \text{ GeV}$)

- When cosmic rays interact with the atmosphere, they mostly produce pions, as they are the lightest hadrons.

- Due to helicity suppression, these pions preferentially decay following;

$$\pi^+ \to \mu^+\nu_\mu \to e^+\nu_e\bar{\nu}_\mu\nu_\mu$$
$$\pi^- \to \mu^-\bar{\nu}_\mu \to e^-\bar{\nu}_e\nu_\mu\bar{\nu}_\mu$$

- We therefore expect

$$\frac{N(\nu_\mu+\bar{\nu}_\mu)}{N(\nu_e+\bar{\nu}_e)} \approx 2$$

  - We however see a lower ratio, with a deficit of $\nu_\mu/\bar{\nu}_\mu$ coming from below the horizon due to their production on the other side of the Earth (a long distance away) and ability to then oscillate

- In Super Kamiokande, the direction a neutrino is detected to have come from relative to the vertical can therefore be used to deduce how far it has travelled.
  - Experimental data is seen to match predictions for ν_e rates, but is below expectation for ν_μ, providing evidence for $\nu_\mu \to \nu_\tau$ oscillations.
  - The experimental resolution is insufficient to directly see neutrino oscillations, so see a 'smeared out' signal tending to a value of $P(\nu_\mu \to \nu_\mu) = 1/2$ for $\cos\theta \to -1$. We can then fit for this smeared out signal, finding $|\Delta m^2_{\text{atmos}}| \approx 0.0025\ \text{eV}^2$ and $\sin^2 2\theta_{\text{atmos}} \approx 1$.

Reactor Experiments

- Using (8) and substituting the PMNS matrix elements, we find

$$P(\nu_e \to \nu_e) = 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2\Delta_{32} \tag{10}$$

  - This therefore has contributions from the short wavelength and long wavelength mass differences.
  - Can place a detector to detect beta decay $\bar{\nu}_e + p \to e^+ + n$ using a liquid scintillator loaded with Gadolinium (large n capture cross section) to also detect photons from neutron capture on Gadolinium;

$$e^+ + e^- \to \gamma + \gamma$$
$$\text{and} \quad n + \text{Gd} \to \text{Gd}^* \to \text{Gd} + \gamma + \gamma + \ldots$$

    - The data from CHOOZ agree with unoscillated predictions for rate and energy spectrum, so $\sin^2 2\theta_{13}$ is small (<0.2).

- A similar method is used by KamLAND but utilising the nuclear reactors all around Japan - accounting for when each is turned on / off.
  - Using (10) and averaging over rapid oscillations $\langle\sin^2\Delta_{32}\rangle = 0.5$ gives (neglecting $\sin^4\theta_{13}$ in the final line);

$$\begin{aligned} P(\nu_e \to \nu_e) &\approx 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \frac{1}{2}\sin^2 2\theta_{13} \\ &= \cos^4\theta_{13} + \sin^4\theta_{13} - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} \\ &\approx \cos^4\theta_{13}\left(1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}\right) \end{aligned}$$

    - CHOOZ then constrains $\cos^4\theta_{13} > 0.9$.
    - In KamLand, the oscillatory structure is clearly visible and combined with SNO we find $|\Delta m^2_{21}| = (7.59 \pm 0.21) \times 10^{-5}\ \text{eV}^2$ and $\tan^2\theta_{12} = 0.47^{+0.06}_{-0.05}$.

Experiment	Dominant	Important
Solar Experiments	θ_{12}	Δm^2_{21} , θ_{13}
Reactor LBL (KamLAND)	Δm^2_{21}	θ_{12} , θ_{13}
Reactor MBL (Daya-Bay, Reno, D-Chooz)	θ_{13}, $\lvert\Delta m^2_{31,32}\rvert$	
Atmospheric Experiments (SK, IC-DC)		θ_{23},$\lvert\Delta m^2_{31,32}\rvert$, θ_{13},δ_{CP}
Accel LBL ν_μ,$\bar{\nu}_\mu$, Disapp (K2K, MINOS, T2K, NOνA)	$\lvert\Delta m^2_{31,32}\rvert$, θ_{23}	
Accel LBL ν_e,$\bar{\nu}_e$ App (MINOS, T2K, NOνA)	δ_{CP}	θ_{13} , θ_{23}

H12 - The CKM Matrix and CP Violation

H11 - Neutrino Oscillations H13 - Electroweak Unification and the W and Z Bosons

CP Violation

- Modern day universe is matter dominated with matter/anti-matter asymmetry at

$$\xi = \frac{n_B - n_{\bar{B}}}{n_\gamma} \approx \frac{n_B}{n_\gamma} \approx 10^{-9}$$

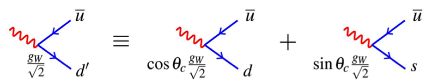
  - One possible explanation is that for every 10^9 anti-baryons in the early universe there were $10^9 + 1$ baryons, which annihilated to 1 baryon $+ \sim 10^9$ photons and no anti-baryons. For such an asymmetry from a symmetric initial state, we require:
    - Baryon number violation
    - C and CP violation - if CP were conserved in a reaction preferentially generating baryons, there would be a CP conjugate reaction preferentially generating anti-baryons.
    - Departure from thermal equilibrium - else any baryon number violating process will be balanced by the inverse reaction.
  - Note, we could argue that the matter distribution in the early universe was not symmetric, but controversial.

Weak Interaction of Quarks

- Quark flavour states appear to be distinct from their mass eigenstates (which we know). We therefore introduce the Cabibbo hypothesis (in 2D);

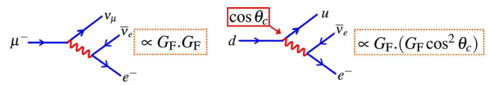
$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

  - We define the d' and s' states as the states which are made with the $\bar{u}$ and $\bar{c}$ respectively (below and similar for s');



    - Note, we could equivalently choose to define things in terms of u and c.

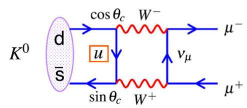
- This idea is backed experimental evidence, as measurements of G_F for μ and β decays give slightly different values. We also observe that Kaon decay is suppressed by a factor of ~ 20 compared to weak coupling assumptions (and therefore the pion decay).
  - Using the above, however, we expect $G_F^\beta = G_F^\mu \cos\theta_c$, as



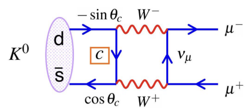
  - We also find $\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu) \propto |M|^2 \propto \cos^2\theta_c$ and $\Gamma(K^- \to \mu^-\bar{\nu}_\mu) \propto |M|^2 \propto \sin^2\theta_c$, as;

    - Hence, where $\theta_c = 13.1^\circ$, these predictions match the observations.

- As the weak interaction couples to both ud and us, neutral mesons can decay via box diagrams;



  - This would predict $M_1 \propto g_W^4 \cos\theta_c \sin\theta_c$. However, a much smaller branching is observed. Including the Charm quark (or supposing its existence) allows for another box diagram for $K^0 \to \mu^+\mu^-$ to be drawn;



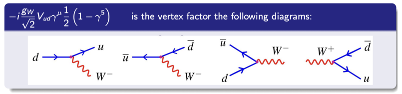
- This gives a matrix element $M_2 \propto -g_W^4 \cos\theta_c \sin\theta_c$.
  - Therefore, the combined amplitude $|M|^2 = |M_1 + M_2|^2 \approx 0$, however the cancelation is not exact because $m_u \neq m_c$.
  - This is the Glashow, Illiopoulos and Maiani (GIS) Mechanism.

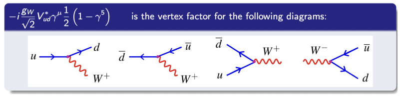
The CKM Matrix (Cabibbo-Kobayashi-Maskawa)

- Extending the above to three quark flavours we get the CKM matrix;

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

  - When writing matrix elements in Feynman rules, the happy quark index on V always comes first and when following the Piccadilly line rule, no conjugation is needed if a walk backwards up the fermion line also encounters the happy quark first.

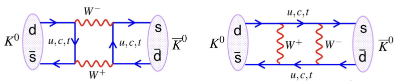




  - Note that the weak charged current $W^\pm$ is the only SM interaction that changes flavour.
    - Experimentally, we find the unitary CKM matrix is fairly diagonal, so flavour changes are discouraged.
    - Furthermore, weak interactions are largest between quarks of the same generation - that between the first and third generations is very small.
    - The CKM matrix also allows for CP violation in the SM.

Neutral Kaon Decays

- A neutral Kaon is defined to contain either $K^0 = d\bar{s}$ or $\bar{K}^0 = s\bar{d}$. These are able to interact by weak box diagrams;



  - Neutral Kaons are therefore seen to propagate as eigenstates of the overall strong + weak interaction - as a linear combination of K^0 and $\bar{K}^0$.
    - There are two such eigenstates - short and long, both with approximately the same mass, but lifetimes differing by three orders of magnitude.
- Noting that K^0 and $\bar{K}^0$ have $J^P = 0^-$, we see that

$$\hat{P}\,|K^0\rangle = -\,|K^0\rangle\,, \quad \hat{P}\,|\bar{K}^0\rangle = -\,|\bar{K}^0\rangle$$

  - Acting on these states with the charge conjugation operator we also see;

$$\hat{C}\,|K^0\rangle = \hat{C}\,|d\bar{s}\rangle = |s\bar{d}\rangle = |\bar{K}^0\rangle \quad \text{and} \quad \hat{C}\,|\bar{K}^0\rangle = |K^0\rangle$$

  - Therefore, $\hat{C}\hat{P}\,|K^0\rangle = -\,|\bar{K}^0\rangle$ and $\hat{C}\hat{P}\,|\bar{K}^0\rangle = -\,|K^0\rangle$ - neither K^0 nor $\bar{K}^0$ are eigenstates of CP.
  - We can however form CP eigenstates from linear combinations;

$$|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right) \implies \hat{C}\hat{P}\,|K_1\rangle = |K_1\rangle \tag{1}$$
$$|K_2\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right) \implies \hat{C}\hat{P}\,|K_2\rangle = -\,|K_2\rangle \tag{2}$$

Decay of Neutral Kaons to Pions (Assuming CP Conservation)

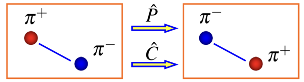
- Neutral kaons often decay to pions because they are the lightest hadrons. As the Kaon mass is $498\ \mathrm{MeV}$and pion mass around $140\ \mathrm{MeV}$, there is just enough energy to produce three pions.
  - Note, this is relevant, as the CPLEAR experiment produces low-energy particles to observe the below effects.
- There are two possible ways for neutral kaons to decay to two pions;
  - $K^0 \to \pi^0\pi^0$ with J^P: $0^- \to 0^- + 0^-$. As $J = L + S$, conservation of momentum also requires the $\pi^0\pi^0$ state to have $L = 0$, and hence

$$\hat{P}(\pi^0\pi^0) = -1 \cdot -1 \cdot (-1)^L = +1$$

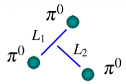
    - Furthermore, as $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ is an eigenstate of $\hat{C}$,

$$\hat{C}(\pi^0 \otimes \pi^0) = (\hat{C}\pi^0) \otimes (\hat{C}\pi^0) = 1 \cdot 1 = 1$$

      - Therefore, $CP(\pi^0\pi^0) = 1$.
  - $K^0 \to \pi^+\pi^-$ can be thought of differently;



    - Acting on a $\pi^+\pi^-$ state in space as in the above LHS, with a $\hat{C}$ or $\hat{P}$ operator, gives the RHS above. Therefore, acting with both $\hat{C}$ and $\hat{P}$ gives no change.
      - Therefore, $CP(\pi^+\pi^-) = 1$.
  - Neutral kaon decays to two pions therefore occur in CP-even ($CP = 1$) eigenstates. If CP is perfectly conserved, then $\pi\pi$ systems can only be formed by the decay of $|K_1\rangle$.
- Neutral Kaons can also decay to three pions in two ways;



  - $K^0 \to \pi^0\pi^0\pi^0$. Taking WLOG the angular momenta of the system as depicted above and again considering that J^P is $0^- \to 0^- + 0^- + 0^-$, we require $L_1 \oplus L_2 = 0 \implies L_1 = L_2$.
    - Therefore, $P(\pi^0\pi^0\pi^0) = -1 \cdot -1 \cdot -1 \cdot (-1)^{L_1} \cdot (-1)^{L_2} = -1$ and $C(\pi^0 \otimes \pi^0 \otimes \pi^0) = 1$, so $CP(\pi^0\pi^0\pi^0) = -1$.
  - $K^0 \to \pi^+\pi^-\pi^0$ again requires $L_1 = L_2$ for the same reason as above, so again $P(\pi^+\pi^-\pi^0) = -1$.
    - However, now $C(\pi^+\pi^-\pi^0) = 1 \cdot C(\pi^+\pi^-) = P(\pi^+\pi^-) = (-1)^{L_1}$, giving $CP(\pi^+\pi^-\pi^0) = -1(-1)^{L_1}$.
    - Given that there is little energy left after creating three pions, we see higher energy states with $L_1 > 0$ are suppressed by the angular momentum barrier effects.
  - Therefore, neutral kaon decays to three pions mostly occur in CP-odd ($CP = -1$) eigenstates. If CP is perfectly conserved, then $\pi\pi\pi$ systems are almost entirely formed by the decay of $|K_2\rangle$.
- As the decay rate for a process is directly proportional to the density of states for the final state energy, and the pions produced from a decay to only two pions will have higher energy than those produced from the decay to three pions, we expect the states which decay to only two pions to have a shorter lifetime.
  - We can therefore identify the short-lived state as $|K_S\rangle = |K_1\rangle$, decaying mostly to $\pi\pi$ and occasionally to $\pi\pi\pi$ where L_1 is odd; and the long-lived state as $|K_L\rangle = |K_2\rangle$, decaying only to $\pi\pi\pi$ states.
  - By forming $|K_0\rangle$ states from $p\pi^-$ collisions (which cannot produce $|\bar{K}_0\rangle$ states) and watching how they decay, we can observe the superposition of these decays.

$$\boxed{|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|K_S\rangle + |K_L\rangle)}$$

    - Suppose a pure K^0 beam is initially in the form $|\psi(t=0)\rangle$ above - which follows from combining (1-2), and evolves with time dependence $|K_S(t)\rangle = |K_S\rangle\, e^{-im_S t - \Gamma_S t/2} = \theta_S(t)\,|K_S\rangle$, such as to give an exponential decay $|\psi_S|^2 = e^{-\Gamma_S t}$ (and similarly for $|K_L(t)\rangle$). Note that here the $e^{-i(p \cdot x)} = e^{-imt}$ because we produce very low energy kaons.
    - Then note that the $|K^0\rangle$ and $|\bar{K}^0\rangle$ states can also decay to leptons; with$|K_L\rangle$ states having a much higher branching fraction to leptons than $|K_S\rangle$ states do due to the $|K_L\rangle$ having lower pion decay rates.
      - The only mechanisms for these decays are $K^0 \to \pi^- e^+ \nu_e$ and $\bar{K}^0 \to \pi^+ e^- \bar{\nu}_e$ with $\bar{K}^0 \not\to \pi^- e^+ \nu_e$ and $K^0 \not\to \pi^+ e^- \bar{\nu}_e$, because charge cannot be conserved at tree level.
      - We can therefore tag decays by the charge of the lepton emitted;

$$K^0_{t=0} \to K^0 \to \pi^- e^+ \nu_e$$
$$K^0_{t=0} \to \bar{K}^0 \to \pi^+ e^- \bar{\nu}_e$$

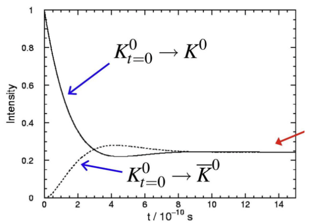
    - Therefore, finding the inner product of a time evolved $|\psi(t)\rangle = |K_0(t)\rangle$ state with $|K_0\rangle$ gives the probability of a decay from $|K_0\rangle$ occurring at time t which is therefore directly proportional to the decay rate;

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(\theta_S(t)\,|K_S\rangle + \theta_L(t)\,|K_L\rangle) = \frac{1}{2}(\theta_S + \theta_L)\,|K^0\rangle + \frac{1}{2}(\theta_L - \theta_S)\,|\bar{K}^0\rangle$$
$$\implies \begin{aligned} P(K^0_{t=0} \to K^0) &= \left|\langle K^0|\psi(t)\rangle\right|^2 = \frac{1}{4}|\theta_S + \theta_L|^2 \quad (3) \\ P(K^0_{t=0} \to \bar{K}^0) &= \left|\langle \bar{K}^0|\psi(t)\rangle\right|^2 = \frac{1}{4}|\theta_S - \theta_L|^2 \quad (4) \end{aligned}$$

- Using the identity $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\Re(z_1 z_2^*)$, and using $\Delta m = m(K_L) - m(K_S)$, we find (3-4) expand to;

$$P\left(K^0_{t=0} \to K^0\right) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$
$$P\left(K^0_{t=0} \to \bar{K}^0\right) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

- It is then measured that this oscillation period is $T_{\text{osc}} = \frac{2\pi\hbar}{\Delta m} \approx 1.2 \times 10^{-9}$ s - quite long compared to the K_S lifetime with $\tau(K_S) = 0.9 \times 10^{-10}$ s, so only one wiggle is seen in the intensity vs time graphs.



Note that the "flat" section here is itself exponentially decaying just with period $\tau(K_L) = 0.5 \times 10^{-7}$ s, so is not discernible.

- Using the above method, the decay rates for all possibilities can be found;

$$R_+ \equiv \Gamma\left(K^0_{t=0} \to \pi^- e^+ \nu_e\right) = N_{\pi e \nu}\frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$
$$R_- \equiv \Gamma\left(K^0_{t=0} \to \pi^+ e^- \bar{\nu}_e\right) = N_{\pi e \nu}\frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$
$$\bar{R}_- \equiv \Gamma\left(\bar{K}^0_{t=0} \to \pi^+ e^- \bar{\nu}_e\right) = N_{\pi e \nu}\frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$
$$\bar{R}_+ \equiv \Gamma\left(\bar{K}^0_{t=0} \to \pi^- e^+ \nu_e\right) = N_{\pi e \nu}\frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

- When performing experiments, we measure the asymmetry of these terms to remove dependence on $N_{\pi e \nu}$, finding

$$A_{\Delta m} = \frac{(R_+ + \bar{R}_-) - (R_- + \bar{R}_+)}{(R_+ + \bar{R}_-) + (R_- + \bar{R}_+)} = \frac{2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t}{e^{-\Gamma_S t} + e^{-\Gamma_L t}}.$$

CP-Violation in the Kaon System

- The above assumes CP is conserved, so at a long distance from the production of a Kaon source, we expect to see only $|K_L\rangle$ - the $|K_S\rangle$ component should have entirely decayed away. Therefore, we should only observe $\pi\pi\pi$ decay products. This is measured not to be the case to 2 parts in 1000.
- There are two possible explanations for this;
  - It could be that K_S and K_L do not correspond exactly to the CP-eigenstates K_1 and K_2. This can be accounted for by supposing, where $\varepsilon = 2 \times 10^{-3}$;

$$|K_S\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}[|K_1\rangle + \varepsilon|K_2\rangle] \quad \text{and} \quad |K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}[|K_2\rangle + \varepsilon|K_1\rangle] \tag{5}$$

    - This then still allows $|K_1\rangle \to \pi\pi$ decays at long distances.
  - And / or CP is violated in the decay, parameterised by ϵ';

$$\underbrace{|K_L\rangle = |K_2\rangle}_{\text{CP}=-1} \to \underbrace{\pi\pi\pi}_{\text{CP}=-1} \text{ and } \underbrace{\pi\pi}_{\text{CP}=+1}$$

  - Experiments show that both contribute, but the first dominates with $\epsilon'/\varepsilon = (1.7 \pm 0.3) \times 10^{-3}$.
- Using the forms in (1-2) allows us to rewrite (5) in terms of $|K^0\rangle$ and $|\bar{K}^0\rangle$;

$$|K_L\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)|K^0\rangle + (1-\varepsilon)|\bar{K}^0\rangle\right]$$

  - We can do the same for $|K_L\rangle$ and combine these forms to find an adjusted value for $|K_0\rangle$ in terms of $|K_L\rangle$ and $|K_R\rangle$, then allowing for inner products to again be taken to find decay rates.
    - Notably, this requires calculating components like $|1-\varepsilon|^2 \approx 1 - 2\Re(\varepsilon)$ as ε is small.
    - Conducting this calculation finds that the K_L decay to $\pi^- e^+ \nu_e$ is 0.7% more likely than the K_L decay to $\pi^+ e^- \bar{\nu}_e$.
      - This absolute difference between matter and anti-matter could enable determination of matter / antimatter incompatibility with aliens!
      - The electrons in our atoms have the same charge as those emitted least often in the decays of the long-lived neutral kaon.
      - Note however that this matter-antimatter difference is insufficient to explain the matter-antimatter asymmetry in the universe, so either there is another CP violating mechanism, or there was just more matter than antimatter to start with.

H13 - Electroweak Unification and the W and Z Bosons

H12 - The CKM Matrix and CP Violation H14 - Precision Tests of the Standard Model

The W Boson

Boson Polarisation Vectors

- Considering the electroweak interaction as a gauge field theory requires writing a boson wave-function in terms of its polarisation four-vectors ε^μ;

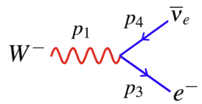
$$B^\mu = \varepsilon^\mu e^{-ip\cdot x}$$

  - A massless spin-1 boson may only exist in two transverse polarisations c.f. left and right circularly polarised, whereas a massive spin-1 boson may also be longitudinally polarised, with each polarisation corresponding to $S_z = -1, 0, 1$ as below;

$$\varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0); \quad \varepsilon_L = \frac{1}{m}(p_z, 0, 0, E); \quad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0, 1, i, 0)$$

    - Note that these may be easier to remember by noting that $\epsilon^\mu p_\mu = 0$ and $\epsilon^\mu \epsilon^*_\mu = -1$.

W-Boson Decays



- Considering the above, Feynman rules now give the matrix element for the above process as;

$$-iM_{fi} = \epsilon_\mu(p_1)\cdot \bar{u}(p_3)\cdot -i\frac{g_W}{\sqrt{2}}\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right)\cdot v(p_4)$$
$$\implies M_{fi} = \frac{1}{\sqrt{2}}g_w \varepsilon_\mu(p_1)\cdot j^\mu$$

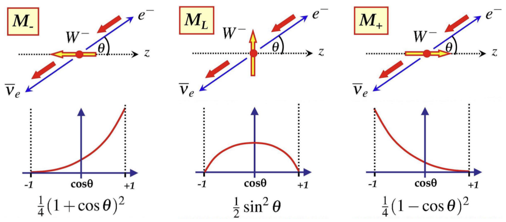
  - In the ultra-relativistic limit with $m_W \gg m_e$, only the left-particle and - right anti-particle helicity current will be non-zero. Using the result from before, we find that where $E = m_W/2$,

$$j^\mu_{\uparrow\downarrow} = m_W(0, -\cos\theta, -i, \sin\theta)$$

  - Combining this with the polarisation vectors in the W-boson rest frame gives

$$M_- = \frac{g_W}{\sqrt{2}} \times \frac{1}{\sqrt{2}}(0, 1, -i, 0)\cdot m_W(0, -\cos\theta, -i, \sin\theta) \implies |M_-|^2 = g_W^2 m_W^2 \frac{1}{4}(1+\cos\theta)^2, \tag{1}$$
$$M_L = \frac{g_W}{\sqrt{2}} \times (0, 0, 0, 1)\cdot m_W(0, -\cos\theta, -i, \sin\theta) \implies |M_L|^2 = g_W^2 m_W^2 \frac{1}{2}\sin^2\theta, \tag{2}$$
$$M_+ = \frac{g_W}{\sqrt{2}} \times \frac{-1}{\sqrt{2}}(0, 1, i, 0)\cdot m_W(0, -\cos\theta, -i, \sin\theta) \implies |M_+|^2 = g_W^2 m_W^2 \frac{1}{4}(1-\cos\theta)^2. \tag{3}$$

    - Graphically, this then gives the angular distribution, as expected for each spin as;



- The differential decay rate for each process is given by the equation below (originally here), where $p^* = m_W/2$ is the COM momentum of the final state particles;

$$\frac{d\Gamma}{d\Omega} = \frac{|p^*|}{32\pi^2 m_W^2}|M|^2$$

  - Integrating these differential decay rates over all angles for each polarisation matrix element, we see that the decay rate for a W-boson is independent of its polarisation;

$$\Gamma_- = \Gamma_L = \Gamma_+ = \frac{g_W^2 m_W}{48\pi}$$

    - This is unsurprising, as the decay rate should not depend on the arbitrary definition of the z-axis direction.

- For unpolarised W-decays, we also find the average matrix element using (1-3) as;

$$\langle |M_{fi}|^2 \rangle = \frac{1}{3}\left(|M_-|^2 + |M_L|^2 + |M_+|^2\right) = \frac{1}{3} g_W^2 m_W^2$$

  - We therefore see that W-bosons decay isotropically with

$$\frac{d\Gamma}{d\Omega} = \frac{g_W^2 m_W^2}{192\pi^2}$$

  - Therefore, the total unpolarised decay rate to a flavour of lepton is (multiplying the above by 4π);

$$\Gamma(W^- \to e^- \bar{\nu}) = \frac{g_W^2 m_W}{48\pi}$$

- Neglecting the final state particle masses, and therefore assuming no change in the phase space terms, we expect the same results for decay to μ or τ.
  - Furthermore, considering the rate at which $W^- \to d\bar{u}$, we get a factor of $|V_{ud}|^2$ due to CKM mixing and a factor of 3, as the $d\bar{u}$ may both be either red, green or blue. In this way we can get decays to any $\bar{u}$ or $\bar{c}$ products, but not $\bar{t}$, as the top quark mass (173 GeV) > W-boson mass (80 GeV). Defining $X = \frac{g_W^2 m_W}{48\pi} = \Gamma(W^- \to e^- \bar{\nu}_e)$, we find;

$$\begin{aligned}
&\Gamma(W^- \to e^- \bar{\nu}_e) = X \quad && \Gamma(W^- \to d\bar{u}) = 3|V_{ud}|^2 X \quad && \Gamma(W^- \to d\bar{c}) = 3|V_{cd}|^2 X \\
&\Gamma(W^- \to \mu^- \bar{\nu}_\mu) = X \quad && \Gamma(W^- \to s\bar{u}) = 3|V_{us}|^2 X \quad && \Gamma(W^- \to s\bar{c}) = 3|V_{cs}|^2 X \\
&\Gamma(W^- \to \tau^- \bar{\nu}_\tau) = X \quad && \Gamma(W^- \to b\bar{u}) = 3|V_{ub}|^2 X \quad && \Gamma(W^- \to b\bar{c}) = 3|V_{cb}|^2 X
\end{aligned}$$

  - Unitarity of the CKM matrix then gives $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$, so we expect and $\mathrm{BR}(W^- \to \text{hadrons}) = 2/3$.

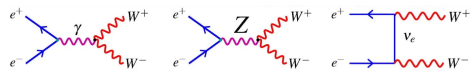
$$\begin{aligned}
\mathrm{BR}(W^- \to e\bar{\nu}_e) &= 1/9 \\
\mathrm{BR}(W^- \to \text{hadrons}) &= 2/3 \\
\implies \mathrm{BR}(W^- \to \text{anything}) &= 9X = \frac{3 g_W^2 m_W}{16\pi} \approx 2.07 \text{ GeV}
\end{aligned}$$

    - This matches (perhaps too well) the experimental value of $\Gamma_W = 2.085 \pm 0.042$ GeV.

Electroweak Unification

- It is seen that with only γ and ν_e supporting the production of W^+W^- from e^+e^- annihilation, the cross section for this interaction increases with COM energy, eventually violating unitarity - the probability of the electrons interacting is > 1!
  - This is overcome by introducing the Z-boson to negatively interfere with these two processes, such that;

$$|M_{\gamma WW} + M_{ZWW} + M_{\nu WW}|^2 < |M_{\gamma WW} + M_{\nu WW}|^2$$



- Electroweak unification arrises from the local gauge transformation below with the three $\vec{\sigma}$ generators;

$$\psi \to \psi' = e^{i\vec{\alpha}(x)\cdot\vec{\sigma}/2}\psi$$

  - Three generators implies we need three gauge bosons, call them $W_1^\mu, W_2^\mu, W_3^\mu$.
  - We then also place the left-chiral fermion (and anti-fermion) components in weak isospin doublets ($I_W = 1/2$) - note nothing to do with isospin - such that the left chiral component of any neutrino is indistinguishable from the left chiral component of its respective lepton.

$$\begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix} \to \begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}' = e^{i\vec{\alpha}(x)\cdot\vec{\sigma}/2}\begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}$$

  - Furthermore, the right-chiral fermion (and anti-fermion) components are placed in weak isospin singlets ($I_W = 0$) which are unmodified by gauge transformations;

Summary of fermion weak isospin assignments (anti-fermion assignments are opposite) :

$$\begin{array}{lll}
\text{DOUBLETs} \quad I_W = \dfrac{1}{2}: & \begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L^- \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L^- \end{pmatrix}, \begin{pmatrix} u_L \\ d_L' \end{pmatrix}, \begin{pmatrix} c_L \\ s_L' \end{pmatrix}, \begin{pmatrix} t_L \\ b_L' \end{pmatrix} & \begin{array}{l} \leftarrow I_W^3 = +\frac{1}{2} \\ \leftarrow I_W^3 = -\frac{1}{2} \end{array} \\
\text{SINGLETs} \quad I_W = 0 : & (\nu_{eR}), (e_R^-), \ldots\ldots\ldots\ldots\ldots\ldots, (t_R), (b_R) & \leftarrow I_W^3 = \quad 0.
\end{array}$$

  - Associating a current with each gauge boson with $j_i^\mu = g_W \bar{\chi}_L \gamma^\mu \frac{1}{2}\sigma_i \chi_L$ and then defining a linear combination of them;

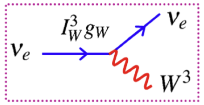
$$j_\pm^\mu = \frac{1}{\sqrt{2}}(j_1^\mu \mp i j_2^\mu) \implies j_\pm^\mu = \frac{g_W}{\sqrt{2}}\bar{\chi}_L \gamma^\mu \frac{1}{2}(\sigma_1 \mp i\sigma_2)\chi_L \quad \text{with} \quad \chi_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$$

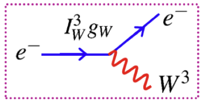
$$\implies \begin{aligned} j_+^\mu &= \frac{g_W}{\sqrt{2}}\bar{e}_L \gamma^\mu \nu_{eL} = \frac{g_W}{\sqrt{2}}\bar{e}\gamma^\mu \frac{1}{2}(1-\gamma^5)\nu_e \\ j_-^\mu &= \frac{g_W}{\sqrt{2}}\bar{\nu}_{eL}\gamma^\mu e_L = \frac{g_W}{\sqrt{2}}\bar{\nu}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)e \end{aligned}$$

    - We can therefore identify j^+ as the W^+ current and j^- as the W^- current. Analysing j_3^μ, we then find;

$$\begin{aligned}
j_3^\mu &= g_W \begin{pmatrix} \bar{\nu}_{eL} & \bar{e}_L \end{pmatrix} \gamma^\mu \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \\
&= g_W \left[\frac{1}{2}\bar{\nu}_{eL}\gamma^\mu \nu_{eL} - \frac{1}{2}\bar{e}_L \gamma^\mu e_L\right] \\
&= g_W \left[\left(+\frac{1}{2}\right)\cdot \bar{\nu}_{eL}\gamma^\mu \nu_{eL} + \left(-\frac{1}{2}\right)\cdot \bar{e}_L \gamma^\mu e_L + (0)\cdot \bar{e}_R \gamma^\mu e_R + (0)\cdot \bar{\nu}_{eR}\gamma^\mu \nu_{eR}\right] \\
&= g_W \left[I_W^3(\nu_{eL})\cdot \bar{\nu}_{eL}\gamma^\mu \nu_{eL} + I_W^3(e_L)\cdot \bar{e}_L \gamma^\mu e_L + I_W^3(e_R)\cdot \bar{e}_R \gamma^\mu e_R + I_W^3(\nu_{eR})\cdot \bar{\nu}_{eR}\gamma^\mu \nu_{eR}\right] \qquad (4)
\end{aligned}$$

      - This is a **flavour-conserving neutral current interaction** where the coupling strength for each interaction is $g_W I_W^3$ for the appropriate third component of weak isospin (note this component $I_W^3 = 0$ for any weak isospin singlet interaction);

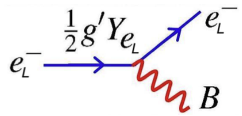




The Z-Boson

- We must not identify W_3 as the Z boson, as any particles within the SM with the same quantum numbers can mix, so we must account for mixing between the photon (A) and the Z.
  - This is potentially catastrophic, because we now don't know what a photon is and QED is all wobbly! We therefore define the particle we previously thought of as the photon as the B-boson (remember as it's as close to A as possible in the alphabet, so is almost a photon).
    - This B-boson is associated with a $U(1)_Y$ gauge symmetry called weak hypercharge, Y. Within this, we define the coupling strength of any fermion to the B-boson as in (5), where Y is the weak hypercharge of the fermion in question.

$$\frac{1}{2}g'Y \tag{5}$$



    - It can be shown (see below) that $Y = 2Q - 2I_W^3$, with I_W^3 allowing Y to be different for left/right chiral fermions.
  - We therefore chose to parameterise the photon (A) and Z-boson mass eigenstates in terms of a mixture (mixing angle θ_W) of the B and W^3 gauge bosons;

$$A_\mu = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W$$
$$Z_\mu = -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W$$

    - We can now realise the $U(1)_{em}$ gauge symmetry previously explored in QED is an emergent rather than fundamenetal gauge symmetry of the SM which actually lives in;

$$U(1)_Y \times SU(2)_L \times SU(3)_C$$

      - Here, Y denotes weak hypercharge, L denotes that this gauge symmetry acts on left-handed fermions and C denotes colour. Both the apparent $U(1)_{em}$ and Z-boson emerge from $U(1)_Y \times SU(2)_L$.
  - Previously when considering QED we constructed the photon (A) to have a neutral-current for electrons of $j_\mu^{em} = eQ_e \cdot \bar{\psi}\gamma_\mu\psi$, coupling equally to the left and right helicity components. Using this, the $U(1)_Y$ current and (4), we find;

$$\begin{aligned}
A\,(\gamma):&\quad j_\mu^{em} = eQ_e \cdot \bar{\psi}\gamma_\mu\psi = \quad eQ_e \cdot \bar{e}_L\gamma_\mu e_L + eQ_e \cdot \bar{e}_R\gamma_\mu e_R\\
B\quad:&\quad j_\mu^{Y} = \frac{g'}{2}Y_e\bar{\psi}\gamma_\mu\psi = \quad \frac{g'}{2}Y_{e_L} \cdot \bar{e}_L\gamma_\mu e_L + \frac{g'}{2}Y_{e_R} \cdot \bar{e}_R\gamma_\mu e_R\\
W^3\quad:&\quad j_\mu^{W^3} = \quad = g_W I_W^3(e_L) \cdot \bar{e}_L\gamma_\mu e_L + g_W I_W^3(e_R) \cdot \bar{e}_R\gamma_\mu e_R
\end{aligned}$$

    - Then noting that $A_\mu = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W \implies j_\mu^{em} = j_\mu^Y \cos\theta_W + j_\mu^{W_3} \sin\theta_W$, we can achieve consistency with the QED results iff for $x \in \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, s, c, b, t\}$ and $\odot \in \{L, R\}$,

$$eQ_x = \cos\theta_W \frac{g'}{2}Y_{x_\odot} + \sin\theta_W g_W I_W^3(x_\odot)$$

- The only way to make this work is by requiring the three new quantities $\{g', \theta_W, Y_{X_\odot}\}$ take values which solve

$$\begin{aligned}
e &= g' \cos\theta\\
e &= g_W \sin\theta_W\\
Q_x &= \frac{1}{2}Y_{X_\odot} + I_W^3(x_\odot)\\
&\implies Y_{x_\odot} = 2Q_x - 2I_W^3(x_\odot)
\end{aligned}$$

  - The last of these backs up the value of hypercharge declared earlier. Therefore, everything we found in QED holds as a superposition of B and Z.
- We can now substitute in to find the Z current in the same way with and find

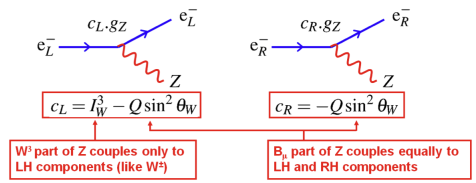
$$j_\mu^Z(x_\odot) = \left[g' \frac{(I_W^3 - Q\sin^2\theta_W)}{\sin\theta_W}\right] \cdot \bar{x}_\odot\gamma_\mu x_\odot$$

  - By defining $e = g_z \cos\theta_W \sin\theta_W \implies g_Z = \frac{g_W}{\cos\theta_W}$ this may be rewritten as;

$$j_\mu^Z(x_\odot) = g_Z\left(I_W^3(x_\odot) - Q_x \sin^2\theta_W\right) \cdot \bar{x}_\odot\gamma_\mu x_\odot$$

  - Note that this means the Z-boson couples to both left and right components, but not equally. This is clearer when written in terms of $c_\odot(x)$, as right chiral fermions (and anti-fermions) have $I_W^3 = 0$;

$$\begin{aligned}
c_\odot(x) &= (I_W^3(x_\odot) - Q_x \sin^2\theta_W)\\
\implies \quad c_L(x) &= (I_W^3(x_L) - Q_x \sin^2\theta_W)\\
c_R(x) &= (\quad\quad - Q_x \sin^2\theta_W)\\
\implies j_\mu^Z(x_\odot) &= g_z c_\odot(x) \cdot \bar{x}_\odot\gamma_\mu x_\odot
\end{aligned}$$



  - Using this form, we can then find the total current for j_μ^Z by summing both chirality contributions and write it neatly as;

$$\begin{aligned}
j_\mu^Z &= g_Z \bar{u}\gamma_\mu\left[c_L \frac{1}{2}(1-\gamma_5) + c_R \frac{1}{2}(1+\gamma_5)\right]u\\
&= \frac{g_Z}{2}\bar{u}\gamma_\mu\left[(c_L + c_R) + (c_R - c_L)\gamma_5\right]u\\
&= \boxed{\frac{g_Z}{2}\bar{u}\gamma_\mu\left[c_V - c_A\gamma_5\right]u}.\\
\text{with}\quad c_V(x) &= c_L(x) + c_R(x) = I_W^3(x_L) - 2Q_x\sin^2\theta_W,\\
c_A(x) &= c_L(x) - c_R(x) = I_W^3(x_L)
\end{aligned}$$

    - This therefore gives the Z-boson vertex factor as

$$-ig_Z\frac{1}{2}\gamma_\mu[c_V - c_A\gamma^5]$$

- Values for each of these coupling constants are often tabulated.
- Notably, $\sin^2\theta_2 \approx 1/4$, so for $x \in \{e, \mu, \tau\}$, $c_V(x) = I_W^3(x_L) - 2Q_x\sin^2\theta_W$ is almost zero, making interactions proportional to this difficult to measure.

Z-Boson Decay Rate

- By repeating a similar procedure as was done for the W-boson for the Z boson, but accounting for both left-left and right-right helicity contributions, we find the unpolarised average matrix element as

$$\langle|M_{fi}|^2\rangle = \frac{1}{3}\left[2c_L^2g_Z^2m_Z^2 + 2c_R^2g_Z^2m_Z^2\right] = \frac{2}{3}g_Z^2m_Z^2\left(c_L^2 + c_R^2\right)$$

  - Using $c_V^2 + c_A^2 = 2(c_L^2 + c_R^2)$, and integrating $\frac{d\Gamma}{d\Omega}$, we find

$$\Gamma(Z \to e^+e^-) = \frac{g_Z^2m_Z}{48\pi}\left((c_V^e)^2 + (c_A^e)^2\right)$$

  - Again neglecting Fermion masses, the same expression becomes valid for other decays, allowing the branching ratios to be calculated;

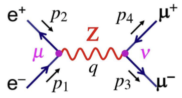
$$\Gamma(Z \to f\bar{f}) = \frac{g_Z^2m_Z}{48\pi}\left((c_V^f)^2 + (c_A^f)^2\right)$$

    - The most common decay is to hadrons, mostly due to the factor of 3 from colour.

H14 - Precision Tests of the Standard Model

H13 - Electroweak Unification and the W and Z Bosons

Z Resonance



- Calculating the cross-section for the above diagram, we apply Feynman rules to find the matrix element;

$$M_{fi} = \frac{-g_Z^2g_{\mu\nu}}{q^2 - m_Z^2}\left[\bar{v}(p_2)\frac{\gamma^\mu}{2}\left(c_V(e) - c_A(e)\gamma^5\right)u(p_1)\right]\cdot\left[\bar{u}(p_3)\frac{\gamma^\nu}{2}\left(c_V(\mu) - c_A(\mu)\gamma^5\right)v(p_4)\right]$$

  - Then rewriting this in terms of c_L and c_R with $\frac{1}{2}\left(c_V - c_A\gamma^5\right) = c_L\frac{1}{2}\left(1-\gamma^5\right) + c_R\frac{1}{2}\left(1+\gamma^5\right)$ gives;

$$\begin{aligned}M_{fi} = -\frac{g_Z^2g_{\mu\nu}}{q^2 - m_Z^2}&\left[c_L(e)\bar{v}(p_2)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(p_1) + c_R(e)\bar{v}(p_2)\gamma^\mu\frac{1}{2}(1+\gamma^5)u(p_1)\right]\\ &\times\left[c_L(\mu)\bar{u}(p_3)\gamma^\nu\frac{1}{2}(1-\gamma^5)v(p_4) + c_R(\mu)\bar{u}(p_3)\gamma^\nu\frac{1}{2}(1+\gamma^5)v(p_4)\right]\end{aligned}$$

    - Considering the relativistic limit, and recalling that in the relativistic limit, $\bar{v}_\uparrow\gamma^\nu u_\uparrow = 0$, this can be further simplified to (where L/R denote the helicities of the initial/final state particles (not anti-particles)).

$$\begin{aligned}M_{fi} &= -\frac{g_Z^2}{q^2 - m_Z^2}g_{\mu\nu}\left[c_L(e)\bar{v}_\uparrow(p_2)\gamma^\mu u_\downarrow(p_1) + c_R(e)\bar{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1)\right]\\ &\quad\times\left[c_L(\mu)\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) + c_R(\mu)\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)\right]\\ &\implies M_{fi} = M_{RR} + M_{RL} + M_{LR} + M_{LL}\quad\text{with}\\ M_{RR} &= -\frac{g_Z^2}{q^2 - m_Z^2}c_R(e)c_R(\mu)g_{\mu\nu}\left[\bar{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1)\right]\left[\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)\right]\quad\text{etc.}\end{aligned}$$

      - Calculating these $M_{L/R\ L/R}$ as before, noticing we only need replace $\frac{e^2}{q^2} \to \frac{g_Z^2}{q^2 - m_Z^2}c(e)c(\mu)$ where $q^2 = s = 4E_e^2$, we find (again in agreement with spin predictions),

$$|M_{RR}|^2 = s^2\left|\frac{g_Z^2}{s - m_Z^2}\right|^2\cdot c_R^2(e)\cdot c_R^2(\mu)\cdot(1+\cos\theta)^2,$$

$$|M_{RL}|^2 = s^2\left|\frac{g_Z^2}{s - m_Z^2}\right|^2\cdot c_R^2(e)\cdot c_L^2(\mu)\cdot(1-\cos\theta)^2,$$

$$|M_{LR}|^2 = s^2\left|\frac{g_Z^2}{s - m_Z^2}\right|^2\cdot c_L^2(e)\cdot c_R^2(\mu)\cdot(1-\cos\theta)^2,$$

$$|M_{LL}|^2 = s^2\left|\frac{g_Z^2}{s - m_Z^2}\right|^2\cdot c_L^2(e)\cdot c_L^2(\mu)\cdot(1+\cos\theta)^2.$$

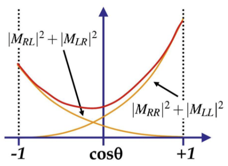
      - Note that here we have divergence where $s = m_Z^2$. This is an artifact from using only the simplest diagrams, with others accounting for Z-boson instability.
        - This can be corrected for (dubiously, theorists have better ways) by considering an unstable particle to decay as $\psi \sim e^{-imt}e^{-\Gamma t/2}$ and by making the Z mass slightly complex with $m_Z \to m_Z - i\Gamma_Z/2$ (would be required to stop exponential decay, so maybe accounts for the effects of higher order diagrams...).
          - Note that here Γ_Z is the total decay rate for a Z-Boson.
        - We can then modify the propagator with

$$
\begin{aligned}
(s-m_Z^2) &\to [s-(m_Z-i\Gamma_Z/2)^2] \approx s-m_Z^2+im_Z\Gamma_Z \\
\implies \left|\frac{1}{s-m_Z^2}\right|^2 &\to \frac{1}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2} \\
\implies |M_{RR}|^2 &= \frac{g_Z^4 s^2}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\cdot c_R^2(e)\cdot c_R^2(\mu)\cdot(1+\cos\theta)^2
\end{aligned}
$$

- This form gives Breit-Wigner resonances, which don't diverge, so are more physical.

$$
\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|M_{fi}|^2 \tag{1}
$$

- Analysing the differential cross sections using (1) demonstrates an asymmetry in the angular distribution as a consequence of $|M_{LL}|^2+|M_{RR}|^2 \neq |M_{LR}|^2+|M_{RL}|^2$ (more on this below);



- To find the total cross section, we must average over the four initial electron/ positron spin combinations, giving

$$
\begin{aligned}
\langle|M_{fi}|^2\rangle &= \frac{1}{2}\cdot\frac{1}{2}\cdot\left(|M_{RR}|^2+|M_{LL}|^2+|M_{LR}|^2+|M_{RL}|^2\right) \\
&= \frac{1}{2}\cdot\frac{1}{2}\frac{g_Z^4 s^2}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\times\left\{\left[c_R^2(e)c_R^2(\mu)+c_L^2(e)c_L^2(\mu)\right](1+\cos\theta)^2\right. \\
&\quad \left.+\left[c_L^2(e)c_R^2(\mu)+c_R^2(e)c_L^2(\mu)\right](1-\cos\theta)^2\right\}. \\
&= \frac{1}{4}\frac{g_Z^4 s^2}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\left\{\frac{1}{4}\left[c_V^2(e)+c_A^2(e)\right]\left[c_V^2(\mu)+c_A^2(\mu)\right]\left(1+\cos^2\theta\right)\right. \\
&\quad \left.+2c_V^{(e)}c_A^{(e)}c_V^{(\mu)}c_A^{(\mu)}\cos\theta\right\}
\end{aligned}
$$

  - Note that here the final line follows using $c_V^2+c_A^2=2(c_L^2+c_R^2)$ and $c_Vc_A=c_L^2-c_R^2$ and also that the final term is doubly small because $c_V(l)$ is small for any lepton l.

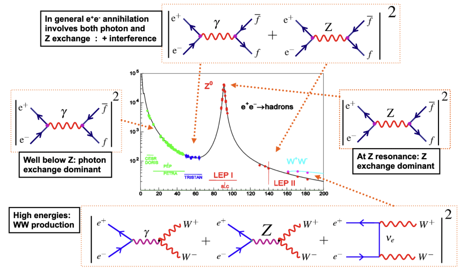
- Again using (1) to find the differential cross-section and then integrating over all angles we find

$$
\begin{aligned}
\sigma_{e^+e^-\to Z\to\mu^+\mu^-} &= \frac{1}{192\pi}\frac{g_Z^4 s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\left[c_V^2(e)+c_A^2(e)\right]\left[c_V^2(\mu)+c_A^2(\mu)\right] \\
\implies \sigma(e^+e^-\to Z\to f\bar{f}) &= \frac{12\pi}{m_Z^2}\frac{s}{(s-m_Z^2)^2+m_Z^2\Gamma_Z^2}\Gamma_{ee}\Gamma_{ff} \qquad (2)
\end{aligned}
$$

  - The is the **relativistic Breit-Wigner formula**, and the second line follows from before where we found the partial widths as;

$$
\Gamma(Z\to f\bar{f}) = \frac{g_Z^2 m_Z}{48\pi}\left((c_V^f)^2+(c_A^f)^2\right)
$$

e^+e^- Event Detection (LEP)



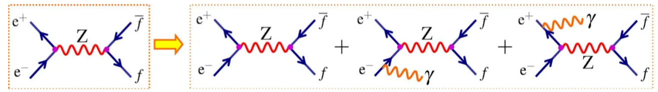
- The contributions to the e^+e^- decay rate at different energies are summarised in the above figure. Note that there is the constant background of γ events everywhere with the other processes activating primarily at specific energy ranges.
- We can introduce the integrated luminosity of the collider, L, such that for each type of event;

$$
\frac{N_{\text{events}}}{L} = \sigma
$$

  - The integrated luminosity is hard to calculate precisely, so instead we measure a process which is well understood (Bahbha Scattering) to determine its value.
  - Bahbha scattering is the QED electron / positron scattering process discussed, and may be treated classically because very little deflection is expected with $\frac{d\sigma}{d\Omega}\propto\frac{1}{\sin^4(\theta/2)} \implies \frac{d\sigma}{d\theta}\propto\frac{1}{\theta^3}$.
    - The large forward cross section then allows us to count the number of events where the electron is scattered in the very forward direction with $N_{\text{Bhabha}}=L\sigma_{\text{Bhabha}}$. Combined with the definition for L, we therefore find;

$$
\sigma_{Z\to f\bar{f}} = \frac{N_{Z\to f\bar{f}}}{N_{\text{Bhabha}}}\sigma_{\text{Bhabha}}
$$

- Using (2), we can find the maximum cross section occurs at $\sqrt{s}=m_Z$, giving a peak cross-section of $\sigma^0_{f\bar{f}}=\frac{12\pi}{m_Z^2}\frac{\Gamma_{ee}\Gamma_{ff}}{\Gamma_Z^2}$.
  - Furthermore, by equating $\sigma^0_{f\bar{f}}/2=\sigma_{f\bar{f}}$, making sensible approximations with $\Gamma_Z<m_Z$ and taking $E=\sqrt{s}=m_Z+\epsilon$ eventually gives half peaks at $\sqrt{s}=m_z\pm\frac{\Gamma_Z}{2}$, so $\Gamma_Z=\frac{\hbar}{\tau_Z}$ is the FWHM of the resonance.
  - In practice this is more difficult due to initial state radiation reducing the COM energy of the e^+e^- collision;



    - We can write the measured cross-section as

$$
\sigma_{\text{meas}}(\sqrt{s}) = \int_0^{\sqrt{s}}\sigma(\sqrt{s'})p(\sqrt{s'}|\sqrt{s})d\sqrt{s'}
$$

      - $p(\sqrt{s'}|\sqrt{s})$ can be calculated using QED. This form can then be used to compare measurements with predictions.

- As Γ_Z is the total decay rate for the Z-boson, it should include all particle widths, including those that are invisible.
  - When determining $\Gamma_Z=\Gamma_{\text{visible}}$ from the peak of σ, we are only actually counting the decays that are detected - any decays to neutrinos are missed!
  - When determining Γ_Z from the FWHM, we measure the full decay rate including those to decays we don't see! Call any contribution from a decay to neutrinos $\Gamma_{\nu_i\nu_i}=\Gamma_{\nu_1\nu_1}$ as they are the same for any generation.
  - We therefore find the true FWHM decay rate $\Gamma_Z=\Gamma_{\text{visible}}+N_\nu\Gamma_{\nu_1\nu_1}$, where N_ν is the number of neutrino generations. This gives $N_\nu=2.9840\pm0.0082$ - strong evidence for there only being three neutrino generations. Only other alternative is for additional generations to have $m_{\nu_4}>m_Z/2$.

Forward-Backward Asymmetry

- Again using the result for $\langle |M_{fi}|^2 \rangle$ from before to find the differential decay rate, we find (for a constant of proportionality κ);

$$\frac{d\sigma}{d\Omega} = \kappa \left[A \left(1 + \cos^2\theta \right) + 2B\cos\theta \right] \quad \text{where} \qquad (3)$$
$$A = \left[c_L^2(e) + c_R^2(e) \right] \left[c_L^2(\mu) + c_R^2(\mu) \right] \quad \text{and} \quad B = \left[c_L^2(e) - c_R^2(e) \right] \left[c_L^2(\mu) - c_R^2(\mu) \right]$$

  - Then defining the forward and backward cross sections as

$$\sigma_F = \int_0^1 \frac{d\sigma}{d\cos\theta} d\cos\theta \quad \text{and} \quad \sigma_B = \int_{-1}^0 \frac{d\sigma}{d\cos\theta} d\cos\theta$$

  - Allows us to define the Forward-Backward Asymmetry A_{FB} as

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}$$

    - Now noticing that $d\Omega = d\cos\theta d\phi$, integrating (3) gives;

$$\begin{aligned} \sigma_F &\propto \int_0^1 \left[A(1+x^2) + 2Bx \right] dx \propto 4A + 3B \\ \sigma_B &\propto \int_{-1}^0 \left[A(1+x^2) + 2Bx \right] \propto 4A - 3B \\ \implies A_{FB} &= \frac{3}{4} \left[\frac{c_L^2(e) - c_R^2(e)}{c_L^2(e) + c_R^2(e)} \right] \cdot \left[\frac{c_L^2(\mu) - c_R^2(\mu)}{c_L^2(\mu) + c_R^2(\mu)} \right] = \frac{3}{4} A_e A_\mu \quad \text{where} \\ A_x &= \frac{c_L^2(x) - c_R^2(x)}{c_L^2(x) + c_R^2(x)} = \frac{2c_V(x)c_A(x)}{c_V^2(x) + c_A^2(x)} = \frac{2 \cdot (c_V(x)/c_A(x))}{1 + (c_V(x)/c_A(x))^2} = f\left(\frac{c_V(x)}{c_A(x)} \right) \quad (4) \end{aligned}$$

    - We therefore see that $A_{FB} \neq 0$, as couplings to left and right chiral particles differ, meaning $A_e \neq 0 \neq A_\mu$.
  - A_{FB} can be seen in experimental data, especially just above or below the Z resonance peak due to interference with the γ-mediated process. By finding $A_{FB}^x = \frac{3}{4} A_e A_x$ and combining each $x \in \{e, \mu, \tau\}$ to eliminate A_e, we find values for A_e, A_μ, A_τ.
    - Using (4) we can then calculate each $c_V(x)/c_A(x)$, which then allows for calculation of the weak mixing angle as;

$$\begin{aligned} \frac{c_V(x)}{c_A(x)} &= \frac{I_W^3(x_L) - 2Q_x \sin^2\theta_W}{I_W^3(x_L)} \\ &= 1 - 4|Q_x| \sin^2\theta_W \quad (\text{for any Fermion}) \\ &= 1 - 4\sin^2\theta_W \quad (\text{for } x \in \{e, \mu, \tau\}) \end{aligned}$$

    - Combining the measurements from all x give the best measurements of $\sin^2\theta_W = 0.23154 \pm 0.00016$.

W^+W^- Production

- As previously noted, without the Z-boson, the cross-section for $e^+e^- \to W^+W^-$ explodes past the unitarity limit. Having now introduced the Z-boson, we wish to check this is fixed (using LEP)!
  - This is tricky, as the branching fractions to hadrons, as well as all lepton configurations are relatively high, and each decay to leptons removes information, as the corresponding neutrino cannot be detected.
    - However, this can be overcome because the incident electron and positron collide with net zero momentum. Hence, for a decay $W^+W^- \to q\bar{q}e^-\bar{\nu}_e$, we find the masses by solving;

$$\begin{aligned} m_{W1}^2 &= (p_e + p_{\nu_e})^\mu (p_e + p_{\nu_e})_\mu \\ m_{W2}^2 &= (p_{q1} + p_{q2})^\mu (p_{q1} + p_{q2})_\mu \\ (\sqrt{s}, 0)^\mu &= (p_e + p_{\nu_e} + p_{q1} + p_{q2})^\mu \end{aligned}$$

    - This gives $m_W = 80.376 \pm 0.033$ GeV and $\Gamma_W = 2.196 \pm 0.083$ GeV.

The Higgs Mechanism

- The Higgs field is much more important than the Higgs boson. It will be described first and its consequences for the Higgs Boson then explained.

The Higgs Field

- The Higgs Field gives mass to bosons and fermions in different ways.
  - Mass is given to bosons by spontaneous symmetry breaking and goldstone bosons (gauge field theory).
  - Mass is given to fermions by Yukawa interactions of the form;

$$\mathcal{L} = \left(\cdots + y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R + \ldots \right) + \text{c.c.}$$

    - Note that this is $SU(2)_L$ gauge invariant with

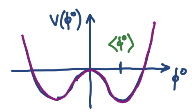
$$\begin{aligned} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L &\longrightarrow e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} &\longrightarrow e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \\ e_R &\longrightarrow e_R \\ \implies y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R &\longrightarrow y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} e^{i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R = y_e \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R. \end{aligned}$$

    - Furthermore, expanding this Lagrangian term gives

$$\begin{aligned} \text{Yukawa Mass Term} &= y_e \left(\overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R \right) + \text{c.c.} \\ &= (y_e\, \overline{e}_L\, \phi^0 e_R + y_e\, \overline{\nu_e}_L\, \phi^+ e_R) + \text{c.c.} \\ &= y_e\, \phi^0 (\overline{e}_L e_R) + y_e\, \phi^+ (\overline{\nu_e}_L e_R) + \text{c.c.} \\ &= (y_e \phi^0)(\overline{e}_L e_R + \overline{e_R} e_L) + (y_e \overline{\nu_e}_L \phi^+ e_R + \text{c.c.}) \\ &= (y_e \phi^0)\bar{e}e \; + \; \left(y_e \overline{\nu_e}_L \phi^+ e_R + y_e \overline{e_R}\, \phi^- \nu_{e_L} \right). \end{aligned}$$

      - The first term on the final line behaves like the Dirac mass term $m_e \bar{e}e$ so long as ϕ^0 doesn't vary too much and $\langle y_e \phi^0 \rangle = m_e$. Equivalently, we therefore require the Higgs vacuum expectation value $\langle \phi^0 \rangle = \frac{m_e}{y_e} \neq 0$.
        - This is explained by the Lagrangian having a Higgs potential like

$$V(\phi^0) = a(\phi^0)^4 - b(\phi^0)^2$$



          - This is then minimised at $\phi^0 = \langle \phi^0 \rangle_{VEV} \neq 0$. The measured value is $\langle \phi^0 \rangle = 246$ GeV.
          - Note that at sufficiently high temperatures $\sim 10^{15}$ K, the Higgs field has sufficient energy that it doesn't need to settle in these minima, causing particles to spread out and become less meaningful. This is what the universe was like before electroweak symmetry breaking - particles essentially didn't have mass.
    - Note, we cannot use the Dirac form;

$$\mathcal{L} = \cdots + m_e \bar{e}e + \ldots$$

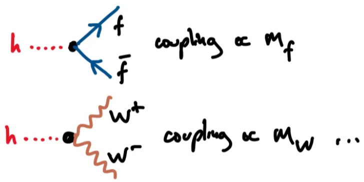
- Because it is not $SU(2)_L$ gauge invariant;

$$
\begin{aligned}
m_e\,\bar{e}e &= m_e\,\bar{e}(P_L + P_R)e \quad (P_L + P_R = 1)\\
&= m_e\,\bar{e}(P_L^2 + P_R^2)e \quad (P_L^2 = P_L,\ P_R^2 = P_R)\\
&= m_e\,\bar{e}P_L^2 e \ +\ m_e\,\bar{e}P_R^2 e\\
&= m_e\,\overline{e_R}e_L \ +\ m_e\,\overline{e_L}e_R \quad (\bar{\psi}P_{R/L} = \bar{\psi}_{L/R},\ P_{R/L}\psi = \psi_{R/L})
\end{aligned}
$$

$$\Rightarrow m_e\,\overline{e_R}(c\,\nu_{eL} + d\,e_L) \ +\ m_e(\overline{c\,\nu_{eL}} + \overline{d\,e_L})e_R \quad (\text{not } m_e\,\bar{e}e\text{! Not invariant!})$$

The Higgs Boson

- Higgs Bosons are excitations in the Higgs field.
- Must be a scalar (spin - 0), as the Yukawa interaction has no spare indices to contract with a spinor or vector field.
- Prefers to decay to the heaviest allowed pair of particles, since its coupling is $\propto$ the daughter mass.



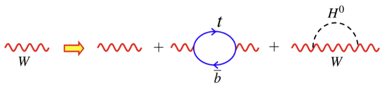
Precision Electroweak Tests

- Within the SM, electroweak unification and the Higgs mechanism give rise to the following relations;

$$m_W = \left(\frac{\pi\alpha_{em}}{\sqrt{2}G_F}\right)^{1/2}\frac{1}{\sin\theta_W} \quad \text{and} \quad m_Z = \frac{m_W}{\cos\theta_W}$$

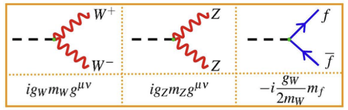
  - Therefore, we can find the other two given any three of $\alpha_{em}, G_F, m_W, m_Z, \sin\theta_W$.
  - This therefore predicts $m_W = 79.946 \pm 0.008$ GeV, but we measure $m_W = 80.376 \pm 0.033$ GeV. This is due to us only considering the lowest order diagrams.
    - More detailed calculations show that W mass also has contributions from the virtual loops below such that

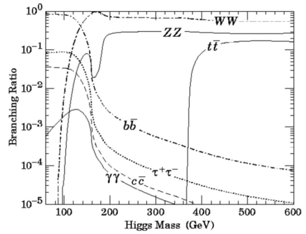
$$m_W = m_W^0 + a\cdot m_t^2 + b\cdot\ln\left(\frac{m_H}{m_W}\right)$$



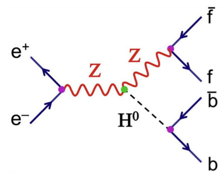
- Therefore, very precise measurements can give measurements of the masses inside the virtual loops.
- This was successfully used to approximate the top quark mass, which is difficult to directly calculate because it decays almost exclusively to bottom quarks as the Kabbibo elements $|V_{tb}|^2 \gg |V_{td}|^2 + |V_{ts}|^2$.
- It is more difficult to detect the Higgs mass using this method due to its logarithmic dependence, but LEP was able to determine $m_H < 200$ GeV.

Trying to Detect the Higgs Boson

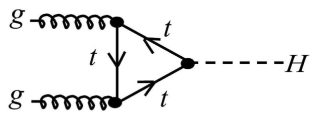




- Since the Higgs boson couples proportionally to the daughter mass, most Higgs decays are to the heaviest kinematically allowed particle.
  - For $m_H < 2m_W$, decays are mostly $H^0 \to b\bar{b}$ and 10% $H^0 \to \tau^+\tau^-$.
  - For $2m_W < m_H < 2m_t$, decays are almost entirely $H^0 \to W^+W^-$ and $H^0 \to Z^0Z^0$.
  - For $m_H > 2m_t$, decays are either $H^0 \to W^+W^-$, $H^0 \to Z^0Z^0$ or $H^0 \to t\bar{t}$.
- At LEP with COM energy 207 GeV, the Higgsstahlung process would dominate for a Higgs mass below 116 GeV;



  - Decays would therefore be expected to be to $\bar{b}b\bar{x}x$ for $x \in \{q, l, \nu\}$.
    - Due to the weak Cabbibo coupling of the b with $V_{cb} \approx 0.04$, the b-hadrons are relatively long-lived, travelling around 3 mm before decaying, allowing them to be effectively identified.
    - This has a very small cross section - LEP would only expect a few 10s of events with large backgrounds, primarily from another Z in place of the H^0.
      - These can only be distinguished by the invariant mass of the jets from boson decays.
- At the LHC, the dominant Higgs production mechanism is gluon fusion;



  - With the measured Higgs mass of around 125 GeV, we expect decays mostly to $b\bar{b}$.