L1 - Introduction

L2 - GR Recap

Einstein's Tower: The Incompatibility of Gravity and Special Relativity

- Consider dropping a weight from a tower onto a device which converts the weight into a photon which then travels back to the top of the tower, where it is again converted to a weight.
    - In this case, if the photon arrives at the top of the tower with the same energy it had at the bottom of the tower, then we are generating energy (mgh per cycle).
    - If instead we assume the energy of the photon arriving at the top is $E = mc^2$, then where $E' = E + mgh$ is the energy the photon had at the bottom, we see that where $E = 2\pi\hbar/\lambda$,

$$E_{\text{emisson}} = E' = mc^2 + mgh = E + mgh = E_{\text{recep}}\left(1 + \frac{gh}{c^2}\right)$$

    - Therefore where ϕ is the Newtonian potential per unit mass, redshift is found as

$$z = \frac{\lambda_{\text{recep}} - \lambda_{\text{emission}}}{\lambda_{\text{emission}}} = 1 + \frac{gh}{c^2} - 1 = \frac{gh}{c^2} = \frac{\Delta\phi}{c^2}$$

    - This is the gravitational redshift formula. The effect was observed in the 60s to within 1% by measuring the difference in redshift for γ photons moving up and down a tower.
        - The effect is also observed in spectral lines from stars. Since the processes which produce these lines are oscillatory, they can essentially be thought of as clocks and therefore this formula has implications for the rate at which time passes within a gravitational field.
- Consider if there are two observers in a Lorentz frame, one at the top and one at the bottom of the tower.
    - Applying the above logic, if the lower observer releases an electromagnetic wavetrain for Δt_1 at ν_1, then gravitational redshift causes the upper observer to observe the wavetrain with $\nu_2 < \nu_1$. The upper observer therefore sees $\Delta t_2 > \Delta t_1$.
    - According to a spacetime diagram, we must have $\Delta t_1 \equiv \Delta t_2$, so a contradiction.
    - Even if we don't assume the photons follow a straight path or are changed by gravity, those at the start and end of the wavetrain will still be congruent, so SR still gives $\Delta t_2 \equiv \Delta t_1$.
- The fundamental insight into relativity is that it's remarkable how heavier objects don't fall faster than lighter objects.
    - This leads to the weak equivalence principle - that gravitational and inertial masses are the same. This has been verified by a range of torsion balance experiments to 1 part in 10^{15}.
    - Notably, the masses in fictitious forces also cancel with the intertial mass, implying that perhaps gravity is just a fictitious force. They are listed below;

$$\begin{aligned}\text{Centrifugal force } F &= -m\omega^2 r\\ \text{Coriolis force } F &= -2m\omega \times \vec{v}\\ \text{Euler force } F &= -m\dot{\omega} \times \vec{x}\\ \text{G-force } F &= -ma\end{aligned}$$

    - Einstein built this into relativity by taking the inertial reference frame to be that of free-fall, therefore at least locally, removing the effect of gravity.

The Equivalence Principles

- **Weak Equivalence Principle (WEP) -** In a frame moving with the free fall acceleration at any point in a gravitational field, the laws of motion of free test particles have their usual special relativistic form, except from gravity, which disappears locally.

    - This can be viewed as analysing the situation in a frame without gravity, but moving at the free fall acceleration appropriate to the point.
- **Strong Equivalence Principle (SEP) -** In a frame moving with the free fall acceleration at any point in a gravitational field, all the laws of physics (dynamics, electromagnetism, quantum mechanics etc.) have their usual special relativistic form, except for gravity which disappears locally.
- Note that here, the equivalence in question is really that between acceleration and gravity - if you're on a spaceship, you can't tell whether it's the engines providing an acceleration or gravity.
    - You could think of being on the surface of earth as you accelerating upwards at $-\vec{g}$ relative to free fall.
    - All this is to say that the particle in question doesn't matter - gravity is based on geometry.

> Spacetime tells matter how to move
> Matter tells spacetime how to curve - John Wheeler

- Some views are that gravity could be treated as a gauge theory - possibly to be compatible with QFT. However, this is ongoing.

Takeaways

- Einstein's critical insight into unifying SR and gravity was how weird it is that heavier object don't fall faster
- Local gravity is a manifestation of an improper choice of reference frame.
- The true inertial frame (A Lorentz frame) is free fall.

L2 - GR Recap

L1 - Introduction L3 - Curvature

Einstein's Tower Revisited

- Consider a photon released from the base of a tower with frequency ω in the earth frame, ascending to height d to result in a new frequency $\omega_t = \omega(1 - gd/c^2)$.
  - Note, to show this frequency change, consider the bottom of the tower to emit a photon of energy $E = mc^2$ and a photon further up is detected with energy $E' = mc^2 - mgd$ corresponding to if it was converted to a mass and carried up. Then doing the same trick as normal gives;

$$E' = mc^2 \left(1 - \frac{gd}{c^2}\right) = E\left(1 - \frac{gd}{c^2}\right)$$

  - At the same time as the photon is released, allow an observer to fall from the top of the tower such that they have velocity $v = gd/c$ downwards (note d/c is time).
    - The photon frequency in the falling observer's frame is then

$$\begin{aligned}\omega' &= \omega_t \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} \\ &= \omega(1 + gd/c^2)^{1/2}(1 - gd/c^2)^{1/2} \\ &\approx \omega\end{aligned}$$

  - Hence, in the freely falling frame, the photon frequency doesn't change during the ascent, verifying the strong equivalence principle to around 1%.
    - However, this only works where the field can be taken as uniform - and even then, very precise measurements will begin to see neighbouring particles incur deviations - hinting at real gravity effects manifested by curvature.

True Gravity

- True gravity can be considered as the part which is not transformable away by an acceleration and has the same distance dependence as tidal forces.
  - It manifests as the separation or coming together of test particles initially on parallel tracks.
  - The universe can be understood to be locally euclidean at any point in space, but over sufficient distance and time, this does not hold. We therefore need curvature to link locally euclidean spaces.

Curvature as the Description of Gravity

- Einstein proposed that gravity should no longer be considered as a force in the conventional sense, but rather as a manifestation of the curvature of the spacetimes, where this curvature is induced by the presence of matter.
  - Hence, gravity doesn't cause time to speed up and slow down, gravity is the bending and stretching of time (and space) by matter.
  - Can understand a lot of GR with:
    - Metric - how distances relate to coordinates
    - Geodesics - how spacetime tells particles to move
    - Curvature - how spacetime responds to matter

Special Relativity

- We unify spacetime into four-vectors in Minkowski geometry with:

$$\begin{aligned}x &= (ct, \vec{x}) \\ u &= \dot{x} = \gamma(c, \vec{v}) \\ p &= mu = (E/c, \vec{p}) \\ \gamma &\equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\end{aligned}$$

  - Here, $\dot{x} = \frac{dx}{d\tau}$ is four-veclocity wrt proper time τ and $\vec{v} = \frac{d\vec{x}}{dt}$ is coordinate velocity.
- Invariant lengths of four-vectors:

$$\begin{aligned}w \cdot w &= w_t^2 - \vec{w}^2 \\ u \cdot u &= c^2 \text{ or } 0 \\ p \cdot p &= m^2c^2\end{aligned}$$

- Lorentz transform:

$$\begin{pmatrix} w_t' \\ w_x' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v/c \\ \gamma v/c & \gamma \end{pmatrix} \begin{pmatrix} w_t \\ w_x \end{pmatrix}$$

- Rapidity is defined by:

$$\begin{aligned} v &= c \tanh \psi \\ \implies \gamma &= \cosh \phi \\ \implies \gamma v &= c \sinh \phi \end{aligned}$$

- Can derive length contraction and time dilation by putting $(0, 1)$ and $(c, 0)$ into LT.
- Loss of simultaneity resolves most paradoxes.

GR Coordinates and Notation

- In GR, coordinates label physical events - they are not generally vectors.
  - Summation convention, whilst great, is terrible for doing specific calculations; instead of setting x^0, x^1 etc., set t, r, θ and ϕ when referring to specific coordinates.
- The dot product is generalised to a form involving a metric;

$$u \cdot u = c^2 = \left(\frac{cdt}{d\tau}\right)^2 - \left(\frac{d\vec{x}}{d\tau}\right)^2$$
$$\text{hence,}$$
$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2 = g_{\mu\nu} dx^\mu dx^\nu$$

  - At any point in space, there exist locally intertial coordinates where the metric takes Minkowski form. However, the metric does not need to take this form globally, either due to coordinate choice or curvature.
  - The Minkowski metric is defined as

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

  - The metric is the geometric encoding of angles, lengths, proper times and rapidities - ie it is the dot product.
    - In an arbitrary coordinate system, this is expressed as

$$a \cdot b = a^\mu g_{\mu\nu} b^\nu = a^\mu b_\mu = a_\nu g^{\nu\mu} b_\mu$$

  - The metric raises and lowers indices;

$$w^\mu g_{\mu\nu} \equiv w_\nu \quad \text{and} \quad A^{\mu\nu} g_{\nu\lambda} = A^\mu_\lambda$$

    - It can also be seen that $g^{\mu\nu}$ is the inverse of the metric;

$$g^\mu_\nu = \delta^\mu_\nu$$

Derivatives

- Taking derivatives of scalar fields and quantities is simple;

$$\nabla v = \partial v + \Gamma v \implies \begin{aligned} \nabla_a v^b &= \partial_a v^b + \Gamma^b_{ca} v^c \\ \nabla_a v_b &= \partial_a v_b - \Gamma^c_{ba} v_c \end{aligned}$$

  - Just remember that upper indices $\implies$ positive sign and the last index on Γ is that of the derivative. The connection coefficients encode how the metric changes throughout spacetime.
- Can also be generalised to tensors;

$$\nabla_a T^b_{cd} = \partial_a T^b_{cd} + \Gamma^b_{ea} T^e_{cd} - \Gamma^e_{ca} T^b_{ed} - \Gamma^e_{da} T^b_{ce}$$

- Parallel transport along a curve $x^\mu(\sigma)$ of a scalar/vector/tensor v is encoded by

$$\frac{D}{D\sigma}v = \dot{x}^\mu \nabla_\mu v$$

  - Also, note that $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$ and that coordinates are always upstairs, disambiguating $r \equiv x^1 \equiv x^r$ from x_1.

Curvature

- Geodesics are either the shortest path or the straightest line within a space.
  - The shortest path definition is global and is technically the "extremal" distance. The straightest line essentially means that the direction of the path doesn't change and the tangent vector is parallel transported.
  - These definitions are the same in torsionless manifolds (Einstein's GR assumes no torsion, but it can be included to try and include spin)
- Using the shortest path definition, we wish to minimise the length

$$L = \int_A^B ds = \int_A^B (g_{\mu\nu} dx^\mu dx^\nu)^{1/2} = \int_A^B (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2} ds/c$$

  - This is therefore Euler-Lagrange applied to $G = (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2}$, so solve

$$\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0$$

  - Noting, however that $G^2 = u \cdot u = c^2$ by the equivalence principle, $\dot{G} = 0$ and noting that

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial G^2}{\partial \dot{x}^\mu}\right) - \frac{\partial G^2}{\partial x^\mu} &= 2\dot{G}\frac{\partial G}{\partial \dot{x}^\mu} + 2G\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - 2G\frac{\partial G}{\partial x^\mu} \\ &= 2G\left(\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu}\right) \\ &= 0\end{aligned}$$

  - We see that we can in fact define a Lagrangian $\mathcal{L} = G^2$ and solve

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \quad \text{with} \quad \mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = (c^2 \text{ or } 0)$$

- (Derivation NE) Substituting gives

$$\frac{d}{ds}\left(\frac{\partial}{\partial \dot{x}^a} g_{bc}\dot{x}^b \dot{x}^c\right) - \partial_a g_{bc}\dot{x}^b \dot{x}^c = 0$$

  - Expanding derivatives in the first term gives

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial}{\partial \dot{x}^a} g_{bc}\dot{x}^b \dot{x}^c\right) &= \frac{d}{ds}\left(g_{ba}\dot{x}^b + g_{ac}\dot{x}^c\right) \\ &= \dot{g}_{ba}\dot{x}^b + \dot{g}_{ac}\dot{x}^c + g_{ba}\ddot{x}^b + g_{ac}\ddot{x}^c\end{aligned}$$

  - Using chain rule ($\dot{g} = \dot{x}^d \partial_d g$) and noting that the metric is symmetric

$$2g_{ba}\ddot{x}^b + \dot{x}^d \partial_d g_{ba}\dot{x}^b + \dot{x}^d \partial_d g_{ac}\dot{x}^c - \partial_a g_{bc}\dot{x}^b \dot{x}^c = 0$$

  - Finally contracting and relabelling gives the general solution;

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b \dot{x}^c = 0 \quad \text{where} \quad \Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}\right).$$

- This can be linked to the straight line definition of a geodesic via parallel transport;

$$\frac{D}{Ds}v^b \equiv \dot{x}^a \nabla_a v^b \equiv \dot{x}^a \partial_a v^b + \dot{x}^a \Gamma^b_{ca} v^c \equiv \dot{v}^b + \Gamma^b_{ca} v^c \dot{x}^a = 0$$

- Here, we have five equations and four unknowns (4 EL equations and the $\mathcal{L} = u \cdot u = c^2$ or 0 constraint), so drop the worst one.

  - Also note however that if $\frac{\partial \mathcal{L}}{\partial x^\mu} = 0$, equivalent to $\partial_\mu g_{ab}$, then no differential equations are needed at all (this is most cases);

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \propto g_{\mu\nu}\dot{x}^\nu = u_\mu = \text{const}$$

Simple Case

- Considering flat SR space, $\mathcal{L}$ doesn't depend explicitly on x^μ, so the geodesic equation reduces to

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = 0 \quad \Rightarrow \quad \ddot{t} = \ddot{x} = \ddot{y} = \ddot{z} = 0,$$

  - This represents a straight line in spacetime. This also shows that in Minkowski space, the extremal proper time between two points is joined by a straight line.
    - Notably, this is the maximum proper time - the minimum proper time would be that of a photon zig-zagging between the two points therefore taking zero proper time.
    - This is the solution to the twin paradox - the unaccelerated observer ages most. According to the equivalence principle, we therefore conclude that stronger gravity also slows ageing.

L3 - Curvature

L2 - GR Recap L4 - Stars

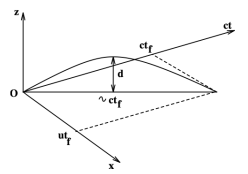
Motivations for Curvature in Spacetime

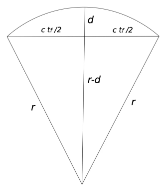
Mathematical

- SR introduces the link between time and space - by Lorentz boosting, an interval between events in time for one observer can be made to appear as an interval in space for another.
  - Therefore, as typical examples of curvature are surfaces embedded in 3D space, it would be weird for curvature to be applied only to space and not to time as well.

Geometrical

- To specify a geodesic through space, we require a magnitude and velocity, whereas in spacetime, we only need the direction; similar to how a geodesic over a 2D surface can be specified with only a direction.
  - Consider throwing a ball. The curvature of the path through space obviously varies. However, the curvature through spacetime is constant.
  - Making this problem 3D by assuming $y = 0$ gives the height of a thrown ball with velocities $(u, 0, v)$ as $z(t) = vt - \frac{1}{2}gt^2$ with a time of flight $t_f = 2v/g$.
    - Therefore the maximum height is $d = z(t_f/2) = \frac{gt_f^2}{8}$.
    - Assuming the velocities are non-relativistic, then the projection of the curve in the $x - t$ plane is $\sim ct_f$ (see left diagram).
    - Furthermore, approximating the curved path as the arc of a circle with radius of curvature $r = \left(\frac{ct_f}{2}\right)^2 + (r - d)^2$ (see right diagram), it is found that $r = \frac{c^2}{g} = 0.968 \text{ ly}$.
      - This is independent of u and v, so the curvature of world lines for freely falling bodies in spacetime is constant for a given gravitational field.
    - This is the embedding picture - could be better to think about this in terms of straight worldlines within a curved spacetime.





The Newtonian Limit - The Weak-Field Metric

- In absence of gravity, spacetime has Minkowski geometry; a weak gravitational field corresponds to only slightly curved spacetime.
  - There therefore exist coordinates x^μ such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, ie a small perturbation.

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \quad \text{where} \quad \Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}\right).$$

  - Using the geodesic equation (above) and noting the requirement for a static metric gives the equation of motion for a particle;

$$\frac{d^2\vec{x}}{dt^2} = -\frac{1}{2}c^2\vec{\nabla}h_{00}$$

  - This is therefore identical to the Newtonian equation of motion if $h_{00} = \frac{2\Phi}{c^2}$.
    - Hence, we recover Newtonian gravity if particles are slowly moving and the metric is the weak field metric;

$$g_{00} = \left(1 + \frac{2\Phi}{c^2}\right)$$

    - Even on the surface of a white dwarf star, $\frac{\Phi}{c^2} = -\frac{GM}{c^2 r} \sim -10^{-4}$, so this weak field limit is an excellent approximation.
- Consider the proper time interval $d\tau$ between two clocks of a clock at rest;

$$c^2 d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}c^2 dt^2$$

  - As $g_{00} \neq 1$, coordinate time t no longer measures proper time, so coordinate time t doesn't record the proper time interval;

$$d\tau = \left(1 + \frac{2\Phi}{c^2}\right)^{1/2} dt \tag{1}$$

  - Note that Φ is negative, so the proper time interval is shorter than the corresponding interval for a stationary observer a large distance from the object, so we get the equation for time dilation in a weak gravitational field.
  - To check this is consistent with the Einstein tower arguments, consider observers at two heights in Earth's field;

$$\frac{\nu_2}{\nu_1} = \left(\frac{1 + \frac{2\Phi(x_2)}{c^2}}{1 + \frac{2\Phi(x_1)}{c^2}}\right)^{-1/2} \approx 1 - \frac{\Phi(x_2) - \Phi(x_1)}{c^2}$$

    - This again gives $z \approx \frac{\Delta\Phi}{c^2}$ as expected.

Beyond the Weak Field

- For most purposes we only need to work on 2D sub-surfaces, as we have the time component around a spherically symmetric body (g_{00} above), and in these spherically symmetric cases, we can work in the equatorial plane with $\theta = \pi/2$ wlog, only leaving the (r, ϕ) surface.
  - In cosmology, there is also t dependence. However, as curvature is independent of spatial position, it is sufficient to work in a (t, r) subsurface.
  - The most general metric in 2D is

$$g_{\mu\nu} = \begin{pmatrix} g_{yy} & g_{yz} \\ g_{zy} & g_{zz} \end{pmatrix}$$

  - Noticing that we may always rewrite $ds^2 = g_{yy}dy^2 + g_{yz}dydz + g_{zy}dzdy + g_{zz}dz^2$ by amalgamating the middle two terms to give a new $g_{yz} = g_{zy}$ equal to half the old g_{yz} and g_{zy} terms allows us to also take this metric as symmetric.
    - This also means it is possible to find a coordinate system in which it is locally diagonal.
    - We may instead therefore at most consider

$$g_{\mu\nu} = \begin{pmatrix} g_{yy} & 0 \\ 0 & g_{zz} \end{pmatrix}$$

    - For example, on the surface of a sphere, $ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$ such that $g_{\mu\nu} = \text{diag}(a^2, a^2\sin^2\theta)$.

Gauss' Theorema Egregium

- States that no matter what coordinate transformations occur, in two dimensions $K(y, z)$ is invariant - measuring intrinsic curvature.

$$K(y,z) = \frac{1}{2g_{yy}g_{zz}}\left(-\partial_z^2 g_{yy} - \partial_y^2 g_{zz} + \frac{(\partial_y g_{zz})^2}{2g_{yy}} + \frac{(\partial_z g_{yy})^2}{2g_{zz}} + \frac{\partial_y g_{yy}\partial_y g_{zz} + \partial_z g_{yy}\partial_z g_{zz}}{2g_{yy}g_{zz}}\right)$$

- K is the Gaussian curvature. By setting $g_{\theta\theta} = a^2$ and $g_{\phi\phi} = a^2\sin^2\theta$, we find that a sphere has $K = 1/a^2$.
- The Riemann curvature tensor is the generalisation of this to 4D.

Meaning of K

- Any 2-d surface has two orthogonal directions in which it is changing most rapidly.
  - This can be quantified by k_1 and k_2 - the principal curvatures of the surface.
    - More properly, the principal curvature for a one-dimensional section is that of the circle which fits the section best at the point of contact.
  - These cannot be found by measurements intrinsic to the surface. However, $K = k_1 k_2$ can be.
    - Surprisingly, this gives the conclusion that the surface of a cylinder is not curved. However, this highlights the differences between the properties of a surface that depend on how they are embedded into a higher-dimensional space (extrinsic), and the properties that are intrinsic to a surface.
    - It can be shown that a bug wandering around on a sphere can find the intrinsic curvature for the surface as $1/a^2$ using, (where C is the circumference of a circle measured);

$$K = \frac{3}{\pi}\lim_{\epsilon\to 0}\left\{\frac{2\pi\epsilon - C}{\epsilon^3}\right\}$$

    - Again this shows that Gaussian curvature is intrinsic.
- Note that is is possible to measure a negative curvature, such as around a saddle, as the principal curvatures in the two directions have opposite signs. In fact, we can derive the form of a metric which has negative gaussian curvature everywhere;
  - Considering the metric in polar coordinates (r,ϕ) with

$$g_{\mu\nu} = \begin{pmatrix} f(r) & 0 \\ 0 & r^2 \end{pmatrix}$$

  - Substituting this into Gauss's Theorema Egregium (with $f' = df/dr$) gives

$$K(r,\phi) = \frac{f'}{2f^2 r}$$

  - This is independent of ϕ, so integrating gives

$$-\frac{1}{f} = r^2 K + \text{const}$$

  - To make the space flat for $K = 0$, we set the constant to -1. Therefore, the form of a metric which is negatively curved everywhere has the form

$$f(r) = \frac{1}{1 - Kr^2}$$

The Schwarzschild Metric

- In the empty space around a spherically symmetric body of mass M, we expect the spatial part of the metric to take the form

$$ds^2 = f(r)dr^2 + r^2\underbrace{(d\theta^2 + \sin^2\theta d\phi^2)}_{d\Omega^2}$$

  - This can be further simplified by working in the equatorial $\theta = \pi/2$ plane.

  - Using dimensional analysis under the assumption that K is a function of r, M, G and c and comparing to Newtonian results gives

$$K(r) = -\frac{GM}{c^2 r^3} = \frac{f'}{2f^2 r}$$

  - Integrating this gives

$$-\frac{1}{f} = \frac{2GM}{c^2 r} + \text{const 2}$$

  - Noting that space should be undistorted ($f = 1$) as $r \to \infty$ gives the constant as -1. Therefore, we find

$$f(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

  - Finally, we combine this with the Newtonian result for the time part in (1) with $\Phi = -GM/r$ (valid in the weak field) to find the full Schwarzschild metric;

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2.$$

  - Defining the Schwarzschild radius as $R_S = 2GM/c^2$ and using $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$ allows this to be rewritten as

$$ds^2 = \left(1 - \frac{R_S}{r}\right)c^2 dt^2 - \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 - r^2 d\Omega$$

    - An object which lies entirely within its Schwarzschild radius is a black hole.
    - Within R_S, an increment in dr causes a timelike increase in the interval ds and an increment of coordinate time causes a spacelike increase in ds - time and space swap their roles.
      - Note that despite this, the gravity as measured by the intrinsic curvature varies smoothly all the way to the origin.

Beyond Gaussian Curvature

- Riemann curvature tensor gives the curvature for every geodesic;

$$R^d_{abc} \equiv \partial_b \Gamma^d{}_{ac} - \partial_c \Gamma^d{}_{ab} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}$$

  - Note that Gaussian curvature K is related to the Riemann curvature tensor by

$$K = \frac{R_{1212}}{g} = \frac{R_{1212}}{\det[g_{\mu\nu}]}$$

- The Ricci tensor R_{ab} gives the average curvature for a volume;

$$R \equiv g^{ab}R_{ab} \equiv R^\lambda_{\mu\lambda\nu}$$

- The Einstein tensor describes the curvature of spacetime in the field equations of GR;

$$G^{ab} \equiv R^{ab} - \frac{1}{2}g^{ab}R$$

L4 - Stars

L3 - Curvature L5 - Motion and Energy in Schwarzschild Metrics

A Newtonian Treatment

- Considering momentum conservation for the fluid inside a star in a Lagrangian view we get

$$\rho\frac{D\vec{u}}{Dt} = -\vec{\nabla}p + \rho\vec{g} \quad \text{with} \quad \vec{u} = 0$$
$$\implies \frac{1}{\rho}\frac{dP}{dr} = -\frac{GM(r)}{r^2}$$

  - Assuming $\rho = \rho_0$ is uniform giving $M(r) = \frac{4}{3}\pi\rho_0 r^3$ then allows the above to be solved as, where P_0 is the central pressure;

$$P(r) = P_0 - \frac{2}{3}\pi G\rho_0^2 r^2$$

  - Also note that $P(r)$ should vanish at the surface of a star (at radius R), so

$$P(r) = \frac{2}{3}\pi G\rho_0^2\left(1 - \frac{r^2}{R^2}\right)$$

  - Furthermore, if $M = M(R)$, then we may write P_0 in terms of the Schwarzschild radius;

$$P_0 = \rho_0 c^2\frac{GM}{2c^2R} = \rho_0 c^2\frac{1}{4}\frac{R_S}{R}$$

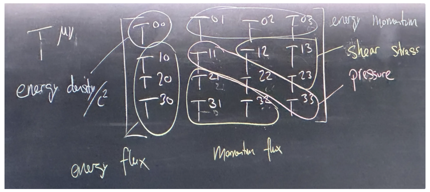
    - Therefore, pressure is tiny compared to energy density unless $R_S \to R$.

Stars in GR

Fluids and the Stress-Energy Tensor

- The relationship between geometry and matter is given by the Einstein field equations;

$$G_{\mu\nu} = -\frac{8\pi}{c^4}GT_{\mu\nu}$$



- $T_{\mu\nu}$ is the stress-energy tensor, which can be understood as follows;
  - We consider fluids to be made of electrically neutral, non-interacting particles of mass m_0. Furthermore, we can characterise a fluid at a point using its matter density ρ and velocity $\vec{u}$ in some intertial frame.
  - In the instantaneous rest frame (IRF), (S), of the fluid, it has proper density $\rho_0 = m_0 n_0$ where n_0 is the number of particles per unit volume.
  - Considering a frame (S') moving with speed v relative to (S) we see that a volume containing a fixed number of particles is Lorentz contracted along the direction of motion.
    - Therefore in (S'), the number density of particles is $n' = \gamma_v n_0$. The effective mass of each particle also increases to $m' = \gamma_v m_0$. Hence, matter density in (S') is;

$$\rho' = \gamma_v^2\rho_0$$

    - Note this is not a scalar, as it is not invariant under LT, but does transform as the tt-component of a rank-2 tensor, suggesting the source term for gravity is a rank-2 tensor.
  - As the time component for a 4-velocity is $\gamma_v c$, we therefore chose a form below for the energy-momentum tensor (stress-energy tensor);

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu$$

    - Here, $\rho_0(x)$ is the proper density of a fluid (measured by an observer co-moving with the local flow).

The Effect of Pressure

- As mass and energy are equivalent, we therefore expect both mass density and pressure to be sources of gravity. Consider;

$$T^{ij} = \rho u^i u^j = \gamma_u^2\rho v^i v^j$$

  - We see that T^{ij} is the rate of flow of the i-component of momentum per unit area in the j-direction. As rate of change of momentum is force, we therefore find that T^{ij} is the i-component of force per unit are perpendicular to the j-direction, so for a perfect fluid in the IRF;

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} = (\rho + P/c^2)u^\mu u^\nu - P\eta^{\mu\nu}$$

  - Replacing η by g we therefore find a fully covariant, symmetric expression for a perfect fluid;

$$T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\nu - Pg^{\mu\nu} \tag{1}$$

    - It is also possible to define symmetric energy-momentum tensor for imperfect fluids, charged fluids and the electromagnetic field.
- Usually pressure contributions to gravity are very small as pressure energy density / particle density is usually tiny. However, this is not the case for radiation, as particles with zero rest mass satisfy the relation below.

$$\text{pressure} = \frac{\text{energy density}}{3}$$

  - This is no longer small, so radiation pressure contributions to gravity must be accounted for.

Oppenheimer-Volkoff (OV) Equation

- In part II, we used a metric of the form below to derive the Schwarzschild vacuum solution;

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

- Now we want to solve the Einstein field equations more generally, so we wish to solve

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu}$$

  - Applying $g^{\mu\nu}$ generates

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = -\frac{8\pi G}{c^4}T^\mu_\nu$$

  - Contracting this with $\mu = \nu$ and using $T = T^\mu_\mu$ then gives;

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{2}$$

- In part II we showed that for the metric where $ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2$, the off-diagonal components of $R_{\mu\nu}$ vanish and the diagonal components are;

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \tag{3}$$
$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \tag{4}$$
$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \tag{5}$$
$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta. \tag{6}$$

  - Furthermore, here we are assuming the matter is a perfect fluid, so we can use equation (1) for $T_{\mu\nu}$ where $\rho(r)$ is the proper mass density and $P(r)$ is the isotropic pressure in the IRF of the fluid.
  - Multiplying (1) by $g_{\mu\nu}$ and using $u_\mu u^\mu = c^2$ then gives

$$T = \left(\rho + \frac{P}{c^2}\right)c^2 - P\delta^\mu_\mu = \rho c^2 - 3P$$

  - This therefore transforms (2) into

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left[\left(\rho + \frac{P}{c^2}\right)u_\mu u_\nu - \frac{1}{2}(\rho c^2 - P)g_{\mu\nu}\right]$$

    - First, solve for the off-diagonals, $R_{ti} = 0 \quad \forall\, i \in \{r, \theta, \phi\}$.
      - This requires $u_i u_t = 0$. However, as $u_\mu u^\mu = c^2 = (u_t)^2 - (u_i)^2$, we really require $u_i = 0$.
      - Therefore, the fluid 4-velocity is $[u_\mu] = c\sqrt{A}(1,0,0,0)$, implying the 3-velocity of the fluid vanishes everywhere.
        - Hence, the choice for the metric to be independent of t automatically ensures the matter distribution is static.
    - Now, using this u_μ, we evaluate for the diagonal elements, giving

$$\begin{aligned} R_{tt} &= -\frac{4\pi G}{c^4}\left(\rho c^2 + 3P\right)A, \\ R_{rr} &= -\frac{4\pi G}{c^4}\left(\rho c^2 - P\right)B, \\ R_{\theta\theta} &= -\frac{4\pi G}{c^4}\left(\rho c^2 - P\right)r^2, \\ R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta. \end{aligned}$$

    - Combining these gives

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} + \frac{2R_{\theta\theta}}{r^2} = -\frac{16\pi G}{c^2}$$

  - Substituting (3-6) gives

$$\begin{aligned} \left(1 - \frac{1}{B}\right) + \frac{rB'}{B^2} &= \frac{8\pi G}{c^2}r^2\rho \\ \implies \frac{d}{dr}\left[r\left(1 - \frac{1}{B}\right)\right] &= \frac{8\pi G}{c^2}r^2\rho \end{aligned}$$

  - Integrating then gives

$$B(r) = \left[1 - \frac{c_0}{r} - \frac{2Gm(r)}{c^2 r}\right]^{-1}$$

    - Then we note that it is only possible for $B(r)$ to be non-zero at $r = 0$ if the constant of integration $c_0 = 0$. Hence,

$$\begin{aligned} B(r) &= \left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1} \quad \text{where} \\ m(r) &= \int_0^r \rho(r)4\pi r^2 dr \end{aligned}$$

      - Note that $m(r)$ is not quite the mass contained within radius r.
- We can then solve for A in a similar way to obtain

$$\frac{r}{A}\frac{dA}{dr} = \frac{8\pi G}{c^4}r^2 BP - 1 + B$$

  - Finally then differentiating this wrt r and combining with A'', A' and B gives the Oppenheimer-Volkoff equation;

$$\frac{dP}{dr} = -\frac{1}{r^2}(\rho c^2 + P)\left[\frac{4\pi G}{c^4}Pr^3 + \frac{Gm(r)}{c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}$$

- Note this can actually be written as (in Planck units, $c = G = 1$);

$$\frac{dP}{dr} = -\frac{v^2}{r}(\rho + P) \quad \text{where (see L5)} \quad v^2 = \frac{rA'}{2A} = \frac{m(r) + 4\pi r^3 P}{r - 2m(r)} \tag{7}$$

- It is also possible to find $B(r)$ using Gauss' Theorema Egregium in the (r, ϕ) subspace of the A, B metric.
  - Note however, that the interpretation of $m(r)$ as the mass within the coordinate radius r is not quite correct, as the proper spatial volume element for the A,B metric is;

$$d^3V = \sqrt{g}d^3x = \sqrt{B(r)}r^2 \sin\theta dr d\theta d\phi$$

  - Therefore, the proper integrated mass within radius r is;

$$\tilde{m}(r) = \int_0^r \rho(r)\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1/2} 4\pi r^2 dr$$

  - For an object extending to $r = R$, the spacetime geometry outside of the object is that of the Schwarzschild metric with $M = m(R)$ rather than $\tilde{M} = \tilde{m}(R)$.
    - The difference $E = (\tilde{M} - M)c^2$ is the gravitational binding energy of the object; the energy required to disperse the material comprising the object to infinite spatial separation.

Physical Implications of the OV Equation

- Note that using equation (7), we may find $P(r)$ without finding A or B.
- Any time where we have $P = P(\rho)$, we have an equation of state.
  - Many astrophysical systems obey the polytropic equation of state $P = K\rho^\gamma$ where $\gamma = 1 + 1/n$ and n is the polytropic index.
    - Note a polytropic process obeys $PV^n = \text{const}$.
- Finally we therefore have three equations;

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r), \tag{8}$$

$$\frac{dP}{dr} = -\frac{1}{r^2}\left(\rho c^2 + P\right)\left[\frac{4\pi G}{c^4}Pr^3 + \frac{Gm(r)}{c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1} \tag{9}$$

$$P = P(\rho). \tag{10}$$

- We therefore require two boundary conditions for a unique solution.
  - One is obvious, must have $m(0) = 0$.
  - Second need to be specified, commonly use central pressure $P(0)$ or central density $\rho(0)$.
- We can compare this to the Newtonian limit by taking $P \ll \rho \implies 4\pi r^3 P \ll mc^2$ and assuming a roughly Minkowski metric such that $2Gm/c^2 r \ll 1$ - equivalently can just take $c \to \infty$.
  - Here the OV equation reduces to the Newtonian equation for hydrostatic equilibrium;

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

Schwarzschild Constant-Density Solution

- The simplest analytic solution for a relativistic star is using $\rho = \text{const}$ corresponding to an ultra-stiff equation of state for an incompressible fluid with infinite sound speed $(\frac{dP}{d\rho})^{1/2}$.
  - Working through the OV equation under this assumption produces; (may set $r = 0$ for central pressure)

$$P(r) = \rho c^2 \frac{\left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2} - \left(1 - \frac{2\mu}{R}\right)^{1/2}}{3\left(1 - \frac{2\mu}{R}\right)^{1/2} - \left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2}} \quad \text{where} \quad \boxed{\mu = \frac{GM}{c^2} = \frac{R_S}{2}}$$

  - The series expansion of this matches the Newtonian expression.
- Solving for A and B under the constant density assumption within R we find

$$A = \frac{c^2}{4}\left[3\left(1 - \frac{2\mu}{R}\right)^{1/2} - \left(1 - \frac{2\mu r^2}{R^3}\right)^{1/2}\right]^2,$$

$$B = \left(1 - \frac{2\mu r^2}{R^3}\right)^{-1}.$$

  - Outside of R we have the normal Schwarzschild solution with

$$A = \frac{1}{B} = \left(1 - \frac{2\mu}{r}\right)$$

Buchdahl's Theorem

- Taking the central pressure with $r = 0$ above we find

$$P_0 = \rho c^2 \frac{1 - \left(1 - \frac{2\mu}{R}\right)^{1/2}}{3\left(1 - \frac{2\mu}{R}\right)^{1/2} - 1}$$

  - This denominator vanishes where

$$3\left(1 - \frac{2\mu}{R}\right)^{1/2} = 1$$

  - Therefore, a constant density star where $\frac{2\mu}{R} = \frac{R_s}{R} \to \frac{8}{9}$ will have central pressure $\to \infty$.
  - As this pressure is a scalar, the infinity persists in any coordinate system, so we deduce that

$$\frac{2GM}{c^2 R} = \frac{2\mu}{R} = \frac{R_s}{R} < \frac{8}{9} \tag{11}$$

    - Buchdahl's theorem states that this constraint is valid for any equation of state, not just a constant density.
    - This also places an upper limit on a star's mass for a fixed radius.
- If attempting to pack more mass inside R than allowed by (11), then GR admits no static solution, hydrostatic equilibrium is broken by increased attraction and therefore the star must collapse inwards without stopping.
  - The whole time, the exterior geometry is described by the Schwarzschild metric, so we obtain a Schwarzschild black hole.

L5 - Motion and Energy in Schwarzschild Metrics

L4 - Stars L6 & 7 - Rotating Black Holes

The Schwarzschild Metric

Solving the Schwarzschild Geodesic Equations

- Working in $\theta = \pi/2$, we find the Schwarzschild Lagrangian as

$$\mathcal{L} = \frac{1}{2} g_{\mu\mu}\dot{x}^\mu \dot{x}^\nu = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$$

  - The EL equations then give

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$$

  - Then noting that $\mathcal{L}$ is independent of t and ϕ gives

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0 \implies u_t = \left(1 - \frac{2GM}{c^2 r}\right) c\dot{t} = \text{const} = kc \tag{1}$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0 \implies r^2 \dot{\phi} = \text{const} = h \tag{2}$$

    - These equations correspond to conservation of energy and momentum respectively.
  - Using $\mathcal{L} = c^2 \text{ or } 0$ gives a third equation, so substituting gives

$$\left(1 - \frac{2GM}{c^2 r}\right)^{-1} c^2 k^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - \frac{r^2 h^2}{r^4} = c^2 \tag{3}$$

$$\implies \dot{r}^2 = c^2(k^2 - 1) - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{2GM}{r} \tag{4}$$

  - Rearranging this and using (2), we find

$$\underbrace{\frac{1}{2} m\dot{r}^2 + \frac{1}{2} m (r\dot{\phi})^2 \left(1 - \frac{2GM}{c^2 r}\right)}_{\text{K.E. Like}} - \underbrace{\frac{GMm}{r}}_{\text{P.E. Like}} = \underbrace{\frac{1}{2} mc^2 (k^2 - 1)}_{\text{"Energy"}} \tag{5}$$

- Noting that we may write

$$\vec{p} = p_a e^a \quad \text{where} \quad p^a \equiv m \frac{dx^a}{d\tau}$$

  - We obtain an alternative form of the geodesic equations (derivation in appendix but NE) - essentially use $p = m\dot{x}$ and propagate

$$\dot{p}_a = \frac{1}{2m} (\partial_a g_{cd}) p^c p^d$$

  - We see that for any coordinate x^a on which the metric does not depend, there is a conserved quantity p_a. This is the conjugate momentum.

    - We therefore identify p_t as particle energy and p_ϕ as particle angular momentum, both of which are conserved if the metric is static (energy) and azimuthally symmetric (angular momentum).
- Applying the above point in the Schwarzschild metric, we see (noting that $x_b = g_{b\mu} x^\mu$),

$$p_t = m g_{t\mu} \dot{x}^\mu = m \left(1 - \frac{2GM}{rc^2}\right) c\dot{t} = kcm = E/c$$

  - Therefore, restoring c's gives particle energy as

$$E_{\text{part}} = kmc^2$$

  - We also find angular momentum as

$$p_\phi = m g_{\phi\mu} \dot{x}^\mu = -mr^2 \dot{\phi}$$

$$\implies L = -mh$$

Orbit Shapes

- Using (2) we find $\dot{r} = \frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{h}{r^2}\frac{dr}{d\phi}$, so then using (4) we get (using $u = 1/r$ in the second line);

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = c^2(k^2 - 1) + \frac{2GM}{r} + \frac{2GMh^2}{c^2 r^3}$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2}{h^2}(k^2 - 1) + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2}$$

  - Finally differentiating with respect to ϕ we get;

$$\frac{d^2 u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2} u^2 \tag{6}$$

    - In Newtonian gravity, the equation for planetary orbits is (below), so we note the $\frac{3GM}{c^2}u^2$ term gives all the relativistic effects. This implies that all orbits will be modified ellipses (causing precession)

$$\frac{d^2 u}{d\phi^2} + u = \frac{GM}{h^2}$$

    - Also, note that if $M \to 0$ then we get harmonic solutions - this is due to the fact that a straight line in polar coordinates has a trigonometric component.
    - Interestingly, differentiating (4) gives;

$$\frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} + \frac{h^2}{r^3} - \frac{3h^2 GM}{c^2 r^4}$$

      - This implies that for any orbit within $r = \frac{3GM}{c^2}$, centrifugal force changes direction such as to increase the rate of infall to a hole.
  - Bertrand's theorem states that only an inverse-square law or a harmonic oscillator potential result in stand, closed orbits for all bound particles. Orbits in GR and not closed.

Circular Orbits and Energy

- Here, r and therefore u are constant, so (5) and (6) respectively reduce to;

$$h^2 = \frac{GMr^2}{r - 3GM/c^2}$$
$$\frac{1}{2}mc^2(k^2-1) = \frac{1}{2}m(r\dot{\phi})^2\left(1 - \frac{2GM}{c^2r}\right) - \frac{GMm}{r}$$

- Combining the above with (2) then gives

$$E_{\text{circ}} = k = mc^2\left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{3\mu}{r}\right)^{-1/2}$$

  - This reduces to the Newtonian form on taking $c \to \infty$ and realising that the Newtonian energy is only a correction to the rest mass energy.
  - Where $m \to 0$, the singularity in the denominator at $r = \frac{3GM}{c^2}$ can be cancelled by the zero on top, giving a photon orbit radius (see below).
  - Particles with mass are also bound for $E_{\text{circ}} < mc^2$, so circular orbits in the range; $\frac{4GM}{c^2} < r < \infty$ are bound.

The Schwarzschild-de Sitter Metric

- Here, we use a metric of the form;

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

- If we assume that space could have a constant curvature $\Lambda/3$ - a cosmological constant, becomes;

$$K(r) = -\frac{GM}{c^2r^3} + \frac{\Lambda}{d} = \frac{f'}{2f^2r}$$

  - This has solution

$$f(r) = B(r) = \left(1 - \frac{2GM}{rc^2} - \frac{\Lambda r^2}{3}\right)^{-1}$$

  - It can also be found that $A(r) = \frac{1}{B(r)}$ where fluid pressure is non-zero, so we have found the Schwarzschild-de Sitter Metric;

$$ds^2 = \left(1 - \frac{2GM}{c^2r} - \frac{\Lambda r^2}{3}\right)c^2dt^2 - \left(1 - \frac{2GM}{c^2r} - \frac{\Lambda r^2}{3}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2.$$

    - Considering g_{00} as $\left(1 + \frac{2\Phi}{c^2}\right)$ gives;

$$\phi = -\frac{GM}{r} - \frac{\Lambda r^2c^2}{6}$$
$$\implies \frac{F_r}{m} = -\frac{d\Phi}{dr} = -\frac{GM}{r^2} + \frac{\Lambda c^2}{3}r$$

      - Therefore, Λ results in a repulsive force which increases with distance

General Circular Orbits

- Repeating the analysis at the start of this lecture, but for the general metric above, we find that

$$\mathcal{L} = Ac^2\dot{t}^2 - B\dot{r}^2 - r^2\dot{\phi}^2 = c^2,$$
$$k = A\dot{t} \quad \text{and} \quad h = r^2\dot{\phi}$$

  - We then substitute to find

$$\dot{r}^2 = \frac{1}{B}\left(\frac{c^2k^2}{A} - \frac{h^2}{r^2} - c^2\right)$$

  - Furthermore, note that

$$\frac{d}{dr}(\dot{r}^2) = \frac{d}{d\tau}(\dot{r}^2)\frac{d\tau}{dr} = \frac{2\ddot{r}\dot{r}}{\dot{r}} = 2\ddot{r}$$

  - For a circular orbit, we require $\ddot{r} = \dot{r} = 0$, so where $A' = dA/dr$, we find

$$-\frac{k^2c^2}{A^2}A' + \frac{2h^2}{r^3} = 0$$

    - Combining this with the $\dot{r} = 0$ condition we find

$$k^2 = \frac{2A^2}{2A - rA'} \quad \text{and} \quad h^2 = \frac{c^2r^3A'}{2A - rA'} \tag{7}$$

    - Note this scheme contains the Schwarzschild form and the Reissner-Nordström metric for charged black holes which replaces $-\frac{1}{3}\Lambda r^2$ in the SdS metric with $q^2G/4\pi\epsilon_0r^2$.

Velocity in a Circular Orbit

- Recalling the definition of rapidity such that

$$\frac{v}{c} = \tanh\alpha \implies \gamma = \cosh\alpha \quad \text{and} \quad \beta\gamma = \sinh\alpha$$

  - Defining a coordinate time t' at a radius r (not the Schwarzschild coordinate time t which is that of a stationary observer at infinity) gives $\frac{dt'}{d\tau} = \gamma = \cosh\alpha$. We then note that

$$v = r\frac{d\phi}{dt'} = r\frac{d\phi}{d\tau}\Big/\frac{dt'}{d\tau} = \frac{r\dot{\phi}}{\cosh\alpha}$$
$$\implies r\dot{\phi} = c\sinh\alpha$$

    - Combining this with the above result for $h^2 = r^4\dot{\phi}^2$ in (7) and a trig identity gives (as used in L4),

$$\frac{v^2}{c^2} = \frac{rA'}{2A}$$

Stability

- We could also rewrite $\dot{r}^2$ in the form

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \text{const.} \quad \text{with} \quad V_{\text{eff}}(r) = \frac{c^2}{2B(r)} + \frac{h^2}{2r^2B(r)} - \frac{c^2k^2}{2A(r)B(r)}$$

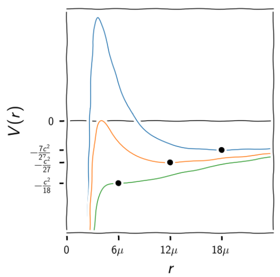
  - To find zones of stability, we could then solve for where $\frac{d^2V_{\text{eff}}}{dr^2} > 0$, but this gives the same boundary point as where k has a minimum for circular orbits.
  - In the Schwarzschild case, we find

$$V_{\text{eff}}^{\text{Schw}}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2} - \frac{GMh^2}{c^2r^3}$$

    - Differentiating this, we find the extrema at

$$r = \frac{h^2}{2GM}\left\{1 \pm \sqrt{1 - 12\left(\frac{GM}{hc}\right)^2}\right\}$$

- Hence, we find there are two stationary points for $h > \sqrt{12}\mu c$ but none for smaller h. The outermost of these is a minimum giving the innermost stable circular orbit (ISCO) for a given h. The innermost of these is a maximum corresponding to unstable circular orbits.



- With $h = \sqrt{12}\mu c$, we find the smallest value of the ISCO as $r = 6\frac{GM}{c^2} = 3R_S$.

- Consider the fractional binding energy $\frac{E}{mc^2} - 1 = k - 1$ varying with circular orbit radius r around a Schwarzschild black hole.
  - Where $\frac{dE_{\text{circ}}}{dr} > 0$, an orbit will have a restoring force, so will be stable. In the Schwarzschild case, $\frac{dE_{\text{circ}}}{dr} = 0$ at $r = \frac{6GM}{c^2} = 3R_S$, so this is likely to be the inner edge of an accretion disk.
  - The energy liberated by material drifting to this orbit from infinity is about 6% of its rest mass - one of the most efficient processes in the universe, so a huge energy source.
- In the Schwarzschild-de Sitter case, we find

$$k = \frac{\left(1 - \frac{2GM}{c^2 r} - \frac{1}{3}\Lambda c^2 r^2\right)}{\sqrt{1 - \frac{3GM}{c^2 r}}}$$

  - It can be shown that in this case, and for reasonable values of Λ, stability breaks down below

$$r_{\text{stab}} \approx \left(\frac{3GM}{4\Lambda c^2}\right)^{1/3}$$

Photon Orbits (in a Schwarzschild metric?)

- To analyse photon orbits, we use $\mathcal{L} = 0$ instead of $\mathcal{L} = c^2$ as above. Now instead of (4), we find

$$\dot{r}^2 - k^2c^2 = -\frac{h^2}{r^2}\left(1 - \frac{2GM}{rc^2}\right)$$

  - Propagating this, we find the orbit shape equation loses the Newtonian term;

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2$$

    - This implies there is a photon orbit with $r = \frac{3GM}{c^2}$.
  - In this case, we may rewrite the energy equation as

$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r) = \frac{1}{b^2} \quad \text{with} \quad b = \frac{h}{ck} \quad \text{and}$$
$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right)$$

    - This $V_{\text{eff}}(r)$ only has a single maximum at $r = 3\mu$, so the circular orbit at $r = 3\mu$ is unstable and there are no stable circular photon orbits in the Schwarzschild geometry.

L6 & 7 - Rotating Black Holes

L5 - Motion and Energy in Schwarzschild Metrics

The Kerr Metric

- By assuming that particles around a black hole should have conserved energy and angular momentum, we require that the metric is independent of t and ϕ.
  - Also assuming we have a steadily rotating black hole implies the solution must be non-static (dt and $d\phi \neq 0$), but also that the spinning to not change with time. Therefore, we expect there not to be time reversal symmetry (under $t \to -t$ or $\phi \to -\phi$). Such a solution is called stationary.
  - To generate a symmetric, stationary metric, we therefore incorporate cross terms of $dtd\phi$ which are invariant under reversal of time **and** spin.
- Using this and doing a lot of maths, the Kerr Metric can be found as;

$$ds^2 = \left(1 - \frac{2\mu r}{\rho^2}\right)c^2dt^2 + \frac{4\mu r a \sin^2\theta}{\rho^2}cdtd\phi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2\mu r a^2 \sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 (1)$$
$$\text{where} \quad \begin{aligned} a &= \frac{J}{Mc} && \text{is the Kerr parameter, controlling spin and is proportional to angular momentum per unit mass} \\ \Delta &= r^2 - 2\mu r + a^2 && \text{is the discriminant (from quadratic forms)} \\ \rho^2 &= r^2 + a^2\cos^2\theta && \text{with } \rho \text{ defined for convenience} \end{aligned}$$

  - With some work, this can be rewritten as;

$$ds^2 = \frac{\Delta}{\rho^2}\left(cdt - a\sin^2\theta d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left((r^2 + a^2)d\phi - acdt\right)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$

    - Where $a \to 0$, both of the above reduce to the Schwarzschild metric.
- Making the substitution $\Delta \to \Delta + \frac{Q^2G}{4\pi\epsilon_0 c^4}$, this in fact recovers the Kerr-Newman metric for charged, spinning objects.

	Non-rotating $J = 0$	Rotating $J \neq 0$
Uncharged $Q = 0$	Schwarzschild	Kerr
Charged $Q \neq 0$	Reissner-Nordström	Kerr–Newman

Boyer-Lindquist Coordinates

- Setting $\mu = 0$, we see that the spatial part of (1) reduces to

$$\frac{\rho^2}{r^2 + a^2}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2.$$

  - From here, we can convert back to Cartesian coordinates using

$$\begin{aligned} z &= r\cos\theta \\ x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \end{aligned} \tag{2}$$

  - With $r = \text{const}$, this therefore gives oblate ellipsoids, and with $r = 0$ we find a disk of radius a. This is further complicated by matter, and the coordinates are also made to rotate with time.
    - We therefore find that r, θ, ϕ are only plane-polar coordinates in the large $r \gg a, \mu$ limit.

Event Horizon(s)

- An event horizon is defined as a null surface with a light cone lying in the surface.
  - Boyer-Lindquist coordinates are defined such that this surface is a function of r only, ie $F(r) = 0$.
  - As a null vector is perpendicular to itself, we therefore require the gradient of F to be null, so

$$g^{\mu\nu}(\partial_\mu F)(\partial_\nu F) = 0 \implies g^{rr}(\partial_r F)^2 = 0$$

- Therefore, an event horizon is where

$$g^{rr} = -\frac{\Delta}{\rho^2} = 0 \implies \Delta = r^2 - 2\mu r + a^2 = 0$$

  - This has solutions at $r = \mu \pm \sqrt{\mu^2 - a^2}$ - ie we have two event horizons!

Stationary Limit Surfaces

- Considering a photon sent off in one of the $\pm\phi$ directions (with $dr = d\theta = 0$ initially), the line element simplifies to (remember the factor of 2 in $g_{t\phi}$);

$$ds^2 = 0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2$$

  - Dividing by dt^2 and solving the quadratic then gives

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

  - Therefore, where $g_{tt} = 0$ there is a solution with $\frac{d\phi}{dt} = 0$, meaning this photon fails to go against the black hole rotation.
    - In cases where $g_{tt} < 0$, $\frac{d\phi}{dt} > 0$ in both cases, so the photon is dragged along in the same direction as the black hole spin. This is called frame dragging.
    - Therefore, within the stationary limit surface, nothing can resist being swept in the same direction as the hole spin.
  - Mathematically, this occurs where

$$g_{tt} = 0 \implies \rho^2 = r^2 + a^2\cos^2\theta = 2\mu r \implies r = \mu \pm \sqrt{\mu^2 - a^2\cos^2\theta}$$

Singularities

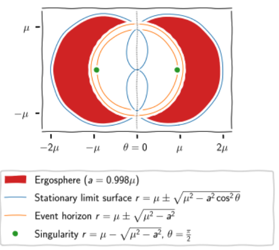
- Physical singularities occur where the curvature $K = R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ diverges.
  - In the Schwarzschild metric, $r = 0$ is a true singularity, whereas $r = R_S$ is a coordinate singularity.
  - In the Kerr metric, $R \to \infty$ where $\rho = 0$.
    - Hence, as $\rho^2 = r^2 + a^2\cos^2\theta$, this singularity is where

$$r = 0 \quad \text{and} \quad \theta = \frac{\pi}{2}$$

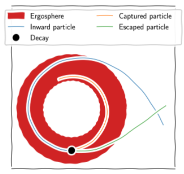
    - Then noting that $r = \text{const}$ gives oblate spheroids. Using equations (2), we see that this is in fact a ring singularity of **radius** a in the $\theta = \frac{\pi}{2}$ plane.

Structure of a Kerr Black Hole

- The ergosphere is the region between the outer stationary limit surface and the outer event horizon is.
  - Using the Penrose process, it is possible to extract angular momentum and energy from the black hole by injecting particles to this region.
  - At the point where $a = \mu$, the inner and outer event horizons are the same. Beyond this, we think there is no horizon and just a naked singularity.



The Penrose Process



- Suppose a particle A is emitted at infinity ($\mathcal{E}$) with energy, measured by an observer at infinity with $u_{\text{obs}} = (1,0,0,0)$;

$$E^{(A)} = p^{(A)} \cdot u_{\text{obs}} = p_t^{(A)}(\mathcal{E})$$

  - Suppose this particle eventually enters into the ergosphere of a black hole, and then decays (event $\mathcal{D}$) to produce particles B and C. At this point, conservation of 4-momentum gives

$$p^{(A)}(\mathcal{D}) = p^{(B)}(\mathcal{D}) + p^{(C)}(\mathcal{D})$$

  - If this decay occurs in such a way that particle C can escape, then it will be received at infinity ($\mathcal{R}$) with energy;

$$E^{(C)} = p_t^{(C)}(\mathcal{R}) = p_t^{(C)}(\mathcal{D})$$

    - Note, the second equality is due to the conservation of the covariant time component of a particle 4-momentum along geodesics in Kerr geometry due to the metric being stationary ($\partial_t g_{\mu\nu} = 0$).
  - Combining the three above equations then gives

$$E^{(C)} = E^{(A)} - p_t^{(B)}(\mathcal{D})$$

    - Note then that $p_t^{(B)} = e_t \cdot p^{(B)}$ where e_t is the t-coordinate basis vector with $e_t \cdot e_t = g_{tt}$.
    - As $p_t^{(B)}$ is conserved along the geodesic followed by particle B, if particle B never escapes the ergoregion, but falls into the black hole, then it will remain in a region where $g_{tt} < 0$ (definition of the ergosphere), making e_t spacelike and therefore $p_t^{(B)}$ a component of spatial momentum (positive or negative).
      - If $p_t^{(B)}$ is negative, then this gives $E^{(C)} > E^{(A)}$, so we extract energy from the black hole. This is the Penrose process.
- After such a process, the black hole is altered such that;

$$M \to M + \frac{p_t^{(B)}}{c^2}$$
$$J = Mac \to J - p_\phi^{(B)}$$

- Note that p_ϕ is minus the component of angular momentum of the particle along the rotation axis of the black hole.

Particle Motion Around a Kerr Black Hole

- Working in the $\theta = \pi/2$ plane and noting that the metric is stationary and axisymmetric, we use the same Lagrangian method as normal with

$$\begin{aligned} \mathcal{L} &= \left(1 - \frac{2\mu}{r}\right) c^2 \dot{t}^2 + \frac{4\mu a}{r} c\dot{t}\dot{\phi} - \left(1 - \frac{2\mu}{r} + \frac{a^2}{r^2}\right)^{-1} \dot{r}^2 - \left(r^2 + a^2 + \frac{2\mu a^2}{r}\right)\dot{\phi}^2 \\ \implies \frac{\partial \mathcal{L}}{\partial \dot{t}} &= 2\left(1 - \frac{2\mu}{r}\right) c^2 \dot{t} + \frac{4\mu a}{r} c\dot{\phi} = 2c^2 k, \\ \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \frac{4\mu a}{r} c\dot{t} - 2\left(r^2 + a^2 + \frac{2\mu a^2}{r}\right)\dot{\phi} = -2h. \end{aligned}$$

  - Combining these with $\mathcal{L} = c^2$ (or 0 for photons) gives first order equations for each coordinate which can then be integrated to find particle paths

$$\begin{aligned} c\dot{t} &= \frac{kc\left((r+2\mu)a^2 + r^3\right) - 2ha\mu}{r\Delta}, \\ \dot{\phi} &= \frac{h(r-2\mu) + 2kca\mu}{r\Delta}, \\ \dot{r}^2 &= c^2(k^2-1) + \frac{2\mu c^2}{r} - \frac{1}{r^2}\left(h^2 - a^2c^2(k^2-1)\right) + \frac{2\mu}{r^3}(h - ack)^2. \end{aligned}$$

  - The latter of these may be rewritten as

$$\begin{aligned} \frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r,h,k) &= \frac{1}{2}c^2(k^2-1) \\ \text{where} \quad V_{\text{eff}}(r,h,k) &= -\frac{\mu c^2}{r} + \frac{h^2 - a^2c^2(k^2-1)}{2r^2} - \frac{\mu(h-ack)^2}{r^3} (3) \end{aligned}$$

    - This matches the r dependence for the Schwarzschild case (), so they have the same general shape.

Circular Orbits

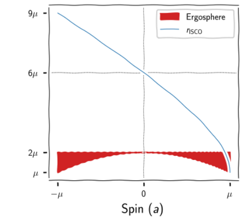
- Now demanding $\dot{r} = \ddot{r} = 0$ for prograde (circular) orbits, we find (where the upper signs correspond to counter-rotating orbits and the lower signs co-rotating orbits).

$$\begin{aligned} k &= \frac{1 - \frac{2\mu}{r} \mp a\sqrt{\frac{\mu}{r^3}}}{\sqrt{1 - \frac{3\mu}{r} \mp 2a\sqrt{\frac{\mu}{r^3}}}}, \\ h &= \mp\frac{c\left(\sqrt{\mu r} - \frac{2a\mu}{r} \pm a^2\sqrt{\frac{\mu}{r^3}}\right)}{\sqrt{1 - \frac{3\mu}{r} \mp 2a\sqrt{\frac{\mu}{r^3}}}}. \end{aligned}$$

  - Then finally solving for where the first derivative of (3) vanishes and its second derivative is positive gives the stability criterion;

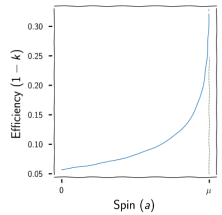
$$r^2 - 6\mu r \mp 8a\sqrt{\mu r} - 3a^2 > 0$$

    - In the $\theta = \pi/2$ plane, we can then find the ISCO radius for spinning black holes (below) note that $r = 6\mu$ corresponds to the Schwarzschild case.



- We can use the above to find k (below). Then taking efficiency as $1 - k$ we find the maximum attainable efficiency of with $a = 0.988\mu$ (the Throne limit) as around 32%.

$$\frac{a}{\mu} = \frac{2(2\sqrt{2}\sqrt{1-k^2} - k)}{3\sqrt{3}(1-k^2)}$$



  - The Thorne limit arises from the counteracting torque felt by the hole in absorbing radiation from the disk.

Photon Orbits

- For photons, instead of (3) we find

$$\begin{aligned} \frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r,b) &= \frac{1}{b^2} \quad \text{where} \quad b = \frac{h}{ck} \\ \text{and} \quad V_{\text{eff}}(r,b) &= \frac{1}{r^2}\left[1 - \left(\frac{a}{b}\right)^2 - \frac{2\mu}{r}\left(1 - \frac{a}{b}\right)^2\right] \end{aligned}$$

Black Hole Thermodynamics

- According to Heisenberg's uncertainty principle, $\Delta t \Delta E = \hbar$, allowing spontaneous pair production from the vacuum as long as the products annihilate within Δt.
  - Now assume that such a pair production event occurs just outside the event horizon of a black hole to generate particles with 4-momenta p and $\bar{p}$.
  - Assuming the Schwarzschild geometry, the spacetime is stationary, so $p_0 = e_0 \cdot p$ and $\bar{p}_0$ are conserved along particle worldlines. Hence, conservation requires

$$e_0 \cdot p + e_0 \cdot \bar{p} = 0 \quad \text{with} \quad e_0 \cdot e_0 = g_{00} = c^2\left(1 - \frac{2\mu}{r}\right)$$

  - This can therefore be satisfied if one of the e_0 is time-like and the other is space-like, which requires one of the particles to pass the even horizon to where $r < 2\mu$.
    - It then appears as though the black hole has emitted a particle of energy $e_0 \cdot p$, therefore also decreasing the black hole mass. This is the Hawking effect.
- Doing full QFT calculations finds that black holes radiate with a blackbody spectrum characterised by the Hawking temperature;

$$T = \frac{\hbar c^3}{8\pi k_B GM}$$

- Therefore, the black hole must lose mass (using standard blackbody results) at the rate;

$$\frac{dM}{dt} = -\frac{\sigma T^4 A}{c^2}$$

  - Here, A is the proper area of the event horizon - in the Schwarzschild case, $A = 16\pi\mu^2$.
  - We therefore also predict a burst of energy release at the end of a black hole's life.

- We can therefore use $dU = TdS$ to find the entropy of the black hole as

$$c^2 dM = \frac{\hbar c^3}{8\pi k_B GM} dS$$
$$\implies S = \frac{Gk_B}{\hbar c} 4\pi M^2$$

  - As this is proportional to M^2, it is thought that supermassive black holes (SMBH) in the centres of galaxies dominate the universe energy budget.

L8 - Compact Objects and the Life Cycles of Stars

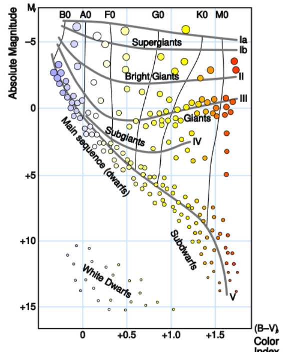
L6 & 7 - Rotating Black Holes L9 - The Electromagnetic Spectrum

Stars

- The stelar evolution of a star can be depicted on a Hertzsprung-Russel diagram.
  - The (B-V) colour index is the difference in radiated flux in the blue and visual (green) bands, so a bluer star will be towards the left hand side of the diagram.
    - For a black body, the peak emission wavelength is given by Wien's law (below), so hotter stars tend to be bluer. Hence, we could equivalently use a colour axis instead of B-V.

$$\lambda_{\text{peak}} = \frac{b}{T} \quad \text{with} \quad b \approx 3000\mu mK$$

  - The magnitude is an inverse brightness scale, so luminosity goes upwards in the diagram.



- Typically see that the number N of start with a particular mass M follow the Salpeter initial mass function;

$$\ln N \sim M^{-2.35}$$

  - The initial masses of stars dictates how they subsequently evolve.

- Normal stars are in hydrostatic equilibrium between gravity and the pressure from nuclear fusion such that they obey;

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{1}$$

  - Stars transition away from the "main sequence" when their composition has changed sufficiently to prevent nuclear binding energy release from overcoming the gravitational attraction.

The Eddington Limit

- This encodes the limit where a star becomes sufficiently bright to separate electrons and protons, so applies to any radiating body.

- The Luminosity L of a star is it's total power output, so at a radius R, an area σ receives power

$$W = \frac{L}{4\pi R^2}\sigma$$

  - Then, for photons, $E = pc$, so the rate of momentum transfer to this area is

$$\frac{dp}{dt} = \frac{W}{c} = \frac{L}{4\pi R^2 c}\sigma$$

  - Equating to the gravitational force on a proton and using the Thomson cross-section therefore gives the Eddington limit;

$$L_{\mathrm{edd}} = \frac{4\pi G M m_p c}{\sigma_T}$$

The Virial Theorem

- The general form of the Virial theorem is;

$$\langle 2T \rangle = -\sum_{i \in \mathrm{system}} \langle F_i \cdot r_i \rangle$$

- For a radial system with no external forces and potential $V \propto r^n$, the Virial Theorem (with much work) gives;

$$\langle 2T \rangle = n \langle V \rangle$$

  - Hence, for an inverse-square law force with $n = -1$, we find $2T \sim -V$.
  - This also gives the time-averaged total energy as $E = T + V \sim \frac{V}{2} = \sim -T < 0$, indicating such a system is bound.
- Note there are tricks in evaluating E_{grav} for a star like rewriting $GM(r)/r$ in terms of $\frac{dP}{dr}$ from (1) and then integrating by parts.

Solar Collapse Limit

- Stars typically self-regulate, with an increase in collapse resulting in an increase in nuclear burning such as to return the star to equilibrium.
- To determine where this self-regulation becomes impossible, we need to find where the total energy is zero;
  - Total internal energy, $U = C_V T$ and thermal energy is $E_T = \frac{3}{2}NkT$, so (where f is degrees of freedom);

$$\begin{gathered} C_P - C_V = Nk \\ \frac{C_P}{C_V} = \gamma = 1 + \frac{2}{f} \\ \implies \boxed{E_T = \frac{3}{2}(\gamma - 1)U}, \quad C_V = \frac{Nk}{\gamma - 1} \quad \text{and} \quad C_P = \gamma\frac{Nk}{\gamma - 1} \end{gathered}$$

  - The Virial theorem then gives $E_T = -\frac{E_{\mathrm{grav}}}{2}$, so the total energy of the a star is

$$E_{\mathrm{tot}} = U + E_{\mathrm{grav}} = -\frac{E_{\mathrm{grav}}}{3(\gamma - 1)} + E_{\mathrm{grav}}$$

$$\implies E_{\mathrm{tot}} = E_{\mathrm{grav}}\frac{3\gamma - 4}{3\gamma - 3}$$

  - If this is positive then a star is unbound, meaning it is unstable to collapse or infinite expansion. This occurs for

$$\gamma < \frac{4}{3} \quad \text{or} \quad f < 6$$

Degeneracy Pressure

Non-Relativistic

- Using the uncertainty principle $\Delta p \Delta x \sim \hbar$ gives a Fermi energy per particle of

$$E_{\mathrm{deg}} = \frac{p^2}{2m} \sim \frac{\hbar^2}{2m\Delta x^2}$$

  - Then taking $\Delta x \sim (V/N)^{1/3} \sim n^{-1/3}$ gives;

$$P_{\mathrm{deg}} \sim E_{\mathrm{deg}}\frac{N}{V} \sim \frac{\hbar^2}{2m}n^{5/3}$$

  - Since $m_p/m_e \sim 2000$, can take $m = m_e$. Therefore, this system behaves as an ideal gas with $\gamma = \frac{5}{3}$, so is stable.

Relativistic

- In the relativistic limit, we have $E = pc$, so

$$P_{\mathrm{deg}} \sim E_{\mathrm{deg}}\frac{N}{V} \sim pc\frac{N}{V} \sim \frac{\hbar c}{\Delta x}n \sim \hbar c n^{4/3}$$

  - Hence, in the relativistic case, the system behaves like an ideal gas with $\gamma = 4/3$, making it unstable to gravitational collapse.
  - Therefore, relativity breaks hydrostatic balance.

The Mass-Radius Relation for White Dwarfs

- For a constant density star of mass M and radius R, we find

$$E_{\mathrm{grav}} \sim -\frac{GM^2}{R}$$

  - The total energy of a star is $E = NE_{\mathrm{deg}} + E_{\mathrm{grav}}$. Using $N \sim M/m_p$, $V \sim R^3$ and $\Delta x^3 \sim V/N$, we find (assuming non-relativistic),

$$\begin{gathered} NE_{\mathrm{deg}} \sim N\frac{\hbar^2}{2m_e\Delta x^2} \sim N^{5/3}\frac{\hbar^2}{2m_e R^2} \sim \left(\frac{M}{m_p}\right)^{5/3}\frac{\hbar^2}{2m_e R^2} \\ \implies E = \left(\frac{M}{m_p}\right)^{5/3}\frac{\hbar^2}{2m_e R^2} - \frac{GM^2}{R} \end{gathered}$$

    - Then minimising wrt R gives (below), so radius decreases with increasing mass.

$$M \sim \frac{\hbar^6}{G^3 m_e^3 m_p^5}R^{-3}$$

    - If we instead do $m_e \to m_n$, then we can find that for a Neutron star

$$M \sim \frac{\hbar^6}{G^3 m_n^8}R^{-3}$$

        - Hence, $R_{NS} \approx R_{WD} \frac{m_e}{m_p}$.
    - equivalent relativistic calculation yields;

$$E_{\text{rel}} = \left(\frac{M}{m_p}\right)^{4/3} \frac{\hbar c}{R} - \frac{GM^2}{R}$$

        - As expected, this has no minimum. Finding where $E_{\text{rel}} = 0$ gives the maximum mass of a white dwarf star - the Chandrasekhar mass

$$M_{\text{Ch}} = \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2} = 1.4 M_\odot$$

Supernovae

Type II

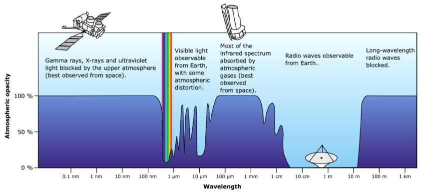
- Massive stars $M > M_\odot$ eventually form iron cores which cannot undergo fusion, so the star collapses.
    - A large fraction of the energy is released as neutrons from inverse β-decay ($e^- + p \to n + \nu$)
    - When core reaches nuclear density, the collapse bounces, ejecting matter at very high velocity and inducing an r-process.
        - In an r-process, neutrons pile into nuclei more rapidly than in weak decay, generating heavy elements.
    - Leaves a Neutron star or Black Hole.

Type Ia

- When a White Dwarf accretes enough mass to exceed the Chandrasekhar mass, so may be used as "standard candles".
- Distinguishable from type II by a lack of hydrogen features in the spectrum.
- Often around $5\times$ more luminous than Type II.

L9 - The Electromagnetic Spectrum

L8 - Compact Objects and the Life Cycles of Stars L10 - Accretion Discs



- Spatial resolution of a telescope is approximately (where D is the size of the telescope);

$$\alpha = 1.22 \frac{\lambda}{D}$$

- Defining four-acceleration as $a^\mu = \dot{u}^\mu$, we first note that four-acceleration and four-velocity are orthogonal, as $u^\mu u_\mu = \text{const}$, and secondly note that $f^\mu = m a^\mu$.
    - In electromagnetism, we identify the four-force as $f^\mu = \frac{e}{c} F^\mu_\nu u^\nu$ with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Radiative Processes

Suggested book: Radiative Processes in Astrophysics - Rybicki and Lightmann.

Bound-Bound Processes

- Occur when an electron in an atom or ion transitions from a higher bound energy state to a lower bound energy state.
    - This accounts for emission spectra and enable determination of the metallicity of a star.

Free-Free Processes

Bremsstrahlung

- Radiation produced by the deceleration of charge in matter, such as in cataclysmic processes like supernovae and shocks.
    - The Larmor formula in relativistic covariant form gives the power emitted by such an acceleration as;

$$P = -\frac{q^2 \dot{u}_\mu \dot{u}^\mu}{6\pi\epsilon_0 c^3} = \frac{q^2 \gamma^4}{6\pi\epsilon_0 c^3} \left(\dot{\beta}^2 + \frac{(\beta \cdot \dot{\beta})^2}{1 - \beta^2}\right) \tag{1}$$

    - This gives a continuous frequency spectrum with peak location proportional to $E_1 - E_2$, detected events are typically in the X-ray band.

Synchrotron Radiation

- The relativistic equations for charges moving in a magnetic field are;

$$\frac{d}{dt}(\gamma m v) = \frac{q}{c} v \times B, \quad \frac{d}{dt}(\gamma m c^2) = \frac{q}{c} v \cdot E$$

    - Hence, with $E = 0$ the second gives $\gamma = |v| = \text{const}$, so the first gives

$$m\gamma\frac{dv}{dt} = \frac{q}{c}v \times B$$

- Hence, there is only acceleration perpendicular to the velocity with cyclotron frequency

$$\omega_B = \frac{qB}{\gamma mc}$$

  - As $a_\perp = \omega_B v_\perp$, we therefore find using (1)

$$P = \frac{e^2\omega_B^2 v_\perp^2}{6\pi\epsilon_0 c^3}$$

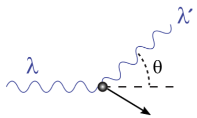
- Accounting for relativistic beaming gives the synchrotron frequency

$$\omega \sim \gamma^3\omega_B \sim 10\gamma^2 B \text{ GHz}$$

  - This generally produces long-range, polarised radio radiation.

Free-Bound

Compton Scattering



- For scattering of a photon from an electron with $p_\gamma = \frac{E}{c}(1, n)$, $p'_\gamma = \frac{E'}{c}(1, n')$, $p_e = (mc, 0)$ and $p'_e = (\epsilon/c, p)$, conserving energy and momentum, rearranging and squaring, eliminating the final electron momentum and using $\cos\theta = n \cdot n'$ gives

$$E' = \frac{E}{1 + \frac{E}{mc^2}(1-\cos\theta)} \implies \lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

  - Here, $\lambda_c = \frac{h}{mc}$ is the Compton wavelength of the electron, so for $\lambda \gg \lambda_c$ (gamma rays) we expect Compton scattering.

Inverse Compton Scattering

- If the electron above is moving, then the photon can also gain energy from the electron giving

$$\frac{\Delta E'}{E} \sim \frac{v_e^2}{c^2} = \frac{4kT}{m_e c^2}$$

  - This is generally at lower frequencies than gamma-rays, so is generally as or more important than Compton scattering.

Pair Production

- In a relativistic thermal gas, photons can collide to induce electron-positron pair production, but the particles normally annihilate quickly.
  - The constraint for this process is $E_\gamma^{(1)}E_\gamma^{(2)} > 2m_e^2c^4$.
  - For a source of radius R and luminosity L, the probability of two photons colliding is given by the optical depth $\tau_{\gamma\gamma} \sim n_\gamma \sigma_T R$.
    - Using $n_\gamma = \frac{L}{4\pi R^2 c \cdot 2m_e c^2}$ then gives $\tau_{\gamma\gamma} \sim \frac{L}{R}\frac{\sigma_T}{m_e c^3} = I$ - the compactness factor.
  - Significant pair production occurs for $I \gg 1$ - producing many rapidly varying γ-rays. Objects like this are called **compact**.

L10 - Accretion Discs

L9 - The Electromagnetic Spectrum L11 - Active Galactic Nuclei

Accretion

Bondi Accretion

- Using a fluid of constant density ρ and sound speed c_s around a mass M to solve the Euler equation;

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{d\rho} - \frac{GM}{r^2}$$

  - We find the mass accretion rate as

$$\dot{M} \propto \frac{\rho}{c_s^3}(GM)^2$$

- Considering only conservation of mass gives $\dot{M} \sim 4\pi R^2 \rho v$.
  - If we then equate this with the escape velocity, we find the Bondi radius

$$R_{\text{Bondi}} = \frac{2GM}{c_s^2}$$
$$\implies \dot{M} \sim \frac{4\pi\rho G^2 M^2}{c_s^2}$$

    - Notably, this mass loss rate is proportional to M^2.

Classical Accretion Disk Analysis

- We can consider solving the Navier-Stokes equation;

$$\frac{d\rho}{Dt} + \rho(\nabla \cdot \vec{u}) = \frac{\partial\rho}{\partial t} + (\nabla\rho)\cdot\vec{u} + \rho(\nabla\cdot\vec{u})$$

- Or considering a rotating disk with circular motion and $v_r \ll v_\phi = \Omega(r)r$, the viscous stress

$$s_\phi = \eta\frac{dv_\phi}{dr} = \eta\left(\Omega + r\frac{d\Omega}{dr}\right)$$

  - Considering only the second term, as the first represents uniform rotation and assuming Keplerian orbits with $\Omega = \frac{\sqrt{GM}}{r^{3/2}}$ gives;

$$s_\phi = \eta r\frac{d\Omega}{dr} = -\frac{3}{2}\eta\Omega \qquad (1)$$

  - Equating viscous torque to the rate of change of angular momentum (where r_* is the inner edge of an accretion disk and $\beta < 1 \implies$ angular momentum removal);

$$\overbrace{\underbrace{s_\phi}_{\text{stress}}\underbrace{2\pi r \cdot 2hr}_{\text{area}}}^{\text{torque}} = j^+ - j^- = -\dot{M}(\sqrt{GMr} - \underbrace{\beta\sqrt{GMr_*}}_{\text{ang mom absorbed by star}})$$

    - Eliminating s_ϕ using (1) and finding the rate of work per unit volume as force per unit area (s_ϕ) $\times$ velocity gradient ($\frac{3}{2}\Omega$) gives

$$\dot{Q} = \frac{3}{2}\eta\Omega\frac{\dot{M}(\sqrt{GMr} - \beta\sqrt{GMr_*})}{4\pi r^2 h\eta}$$

- Assuming locally radiated energy, we find the surface flux as

$$F(r) = \frac{1}{2} 2h\dot{Q} = \frac{3\dot{M}}{8\pi r^2}\frac{GM}{r}\left(1 - \beta\left(\frac{r_*}{r}\right)^{1/2}\right)$$

  - Factor of $\frac{1}{2}$ due to convention for taking flux in only one plane and note h is only half the total disk depth.

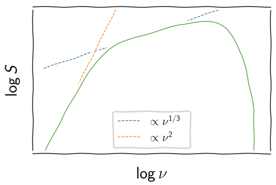
- This also gives total luminosity as

$$L = \int 2F \times 2\pi r dr = \left(\frac{3}{2} - \beta\right)\frac{GM\dot{M}}{r_*}$$

- Blackbody emission has a temperature $T = (F/\sigma_{SB})^{1/4}$.
  - Hence, where $r \gg r_*$ we have $T(r) \propto r^{-3/4}$.
  - Integrating the spectral radiance of a black body over the disk we then find the radiation spectrum of such a disk as having the distinctive X-ray band spectrum;

$$S(\nu) \propto \int_{r_*}^{\infty} \underbrace{\frac{\nu^3}{e^{h\nu/kT(r)} - 1}}_{B_\nu(T)} 2\pi r dr$$

$$\implies S(\nu) \propto \begin{cases} \nu^2 T & h\nu \ll kT \\ \nu^{1/3} & h\nu \sim kT \\ \nu^3 e^{-h\nu/kT} & h\nu \gg kT \end{cases}$$



- Within an accretion disk, magnetic fields link different radii and act as a shear viscosity acting to disrupt equilibrium.
  - This can therefore act to increase the ISCO.
  - Turbulent viscosity can be modelled in α-disks by assuming $\eta = \alpha c_s h$.

Radiative Efficiency

- Writing luminosity in terms of the radiative efficiency ϵ gives $L = \epsilon\dot{M}c^2$.
  - Then assuming that a mass dm falls from infinity to the surface of a star releasing $dE = GMdm/R^*$ and using $dm = \dot{M}dt$ gives

$$L = \frac{GM\dot{M}}{R_*} \implies \epsilon = \frac{GM}{Rc^2}$$

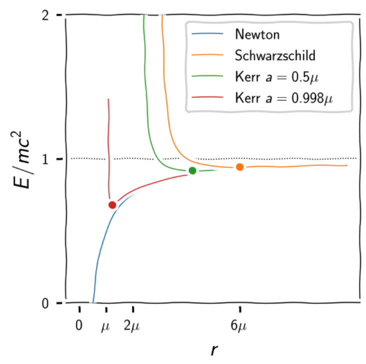
Circular Orbits

- In earlier lectures, we found that in a Schwarzschild metric,
  - With $h = \sqrt{12}\mu c$, we find the smallest value of the ISCO as $r = 6\frac{GM}{c^2} = 3R_S$.
  - Gas in an accretion disk slowly loses momentum due to turbulent viscosity (turbulence generated by MRI), slowly falling towards the black hole, gaining GPE and heating up, eventually falling into the black hole.
- The potentials found in previous lectures are;

$$\begin{aligned} V_{\text{Newton}}(r) &= -\frac{GM}{r} + \frac{h^2}{2r^2}, \\ V_{\text{Schwarzschild}}(r) &= -\frac{GM}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right), \\ V_{\text{Kerr}}(r) &= -\frac{GM}{r} + \frac{(h - ack)^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{hack}{r^2} - \frac{a^2c^2}{2r^2}. \end{aligned}$$

- Demanding circular orbits gives k and h in terms of r, giving energies;

$$\begin{aligned} E_{\text{Newton}} &= \left(1 - \frac{GM}{2c^2 r}\right)mc^2, \\ E_{\text{Schwarzschild}} &= \frac{1 - \frac{2GM}{c^2 r}}{\sqrt{1 - \frac{3GM}{c^2 r}}}mc^2, \\ E_{\text{Kerr}} &= \frac{1 - \frac{2GM}{c^2 r} + a\sqrt{\frac{GM}{c^2 r^3}}}{\sqrt{1 - \frac{3GM}{c^2 r} + 2a\sqrt{\frac{GM}{c^2 r^3}}}}mc^2. \end{aligned}$$

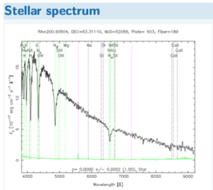


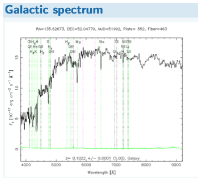
L11 - Active Galactic Nuclei

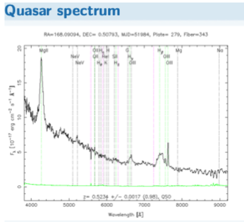
L10 - Accretion Discs L12 - Neutron Stars and Pulsars

AGNs

Emission Spectra







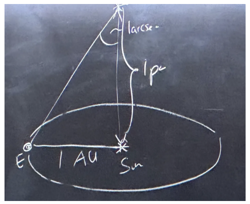
- Objects have distinctive emission spectra;
  - The Stellar spectrum is a blackbody spectrum with thermally broadened absorption lines.
  - The Galactic spectrum is very diverse due to a mixture of star-forming and quiescent galaxies. It also has sharper absorption lines from the intergalactic medium.
  - The Quasar spectrum shows strong broadened emission lines from material at high velocity near the AGN (more on this below).

AGN Size

- There are three main ways to estimate the size of the AGN
  - Turning the Eddington limit around gives (below), which predicts SMBHs with $M > 10^6 - 10^9 M_\odot$.

$$M > \frac{L\sigma_T}{4\pi G m_p c}$$

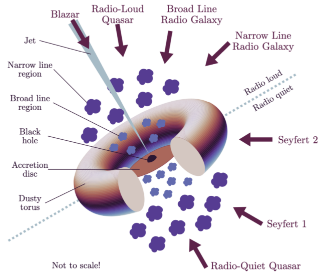
  - Using $d \sim c\Delta t$ where Δt is the timescale for the system gives an estimate of the size of the body. For $\Delta t = 1$ day we find $d \approx 0.001\text{pc}$.
    - 1 parsec is defined to be the distance to an object with parallax of one arcsecond, approximately $3 \times 10^{16}\text{m}$.



  - As the surface area of a disk is $A \sim M^2$ and in the Eddington limit, $L = \epsilon \dot{M} c^2 = \frac{4\pi G M m_p c}{\sigma_T}$, we find $\frac{\dot{M}}{M^2} \sim M^{-1}$.

    - As blackbody power radiated per area $\sim T^4$, the characteristic black-body temperature for an AGN, $T \sim M^{-1/4}$.

AGN Unification



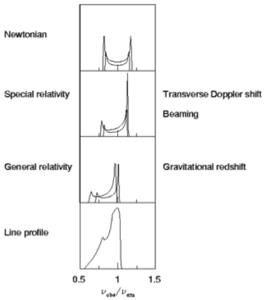
- An AGN has a dusty torus which re-emits high energy photons from the SMBH in the IR.
  - Broad line region is a "cold (1000K)" gas giving broad emission lines due to their high speed.
  - The narrow line region is further out so move slower, giving narrower emission lines, often highly ionised.
  - The jet can be up to $500kpc$ long and extremely powerful.
  - Also have hot corona of size $0.01\ \text{pc}$.
- Depending on the angle of view, we give the AGN different names - see above diagram.

Emission Profiles

- High luminosity AGNs which outshine their host galaxy are quasars.
  - When finding emission spectrum of an AGN, must account for both the relativistic doppler shift of the emitting gas as well as the gravitational redshift. This is expected to be two-peaked for a rotating disk;

$$(1+z)_{\text{orb}} = \gamma\left(1 + \frac{v}{c}\sin i\right); \quad \frac{dt}{d\tau} = (1+z)_{\text{grav}} = \left(1 - \frac{2GM}{rc^2}\right)^2$$
$$\implies (1+z) = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}\left(1 + \frac{v}{c}\sin i\right)$$

- Above the accretion disk, a hard X-ray continuum is produced by Comptonization.
  - This results in relativistic broadening of the iron emission lines in the spectra of nearby active galaxies, producing a reflection spectrum.
  - Predicting these reflection spectra and then fitting observations numerically it is then possible to find both the spin and inclination of an AGN;



Formation of Black Holes

- Assuming a black hole radiates at the Eddington limit with efficiency ϵ and accretes at rate $\dot{M}_{\text{bh}} = (1-\epsilon)\dot{M}$, then we can find the characteristic timescale of black hole growth as;

$$\tau \sim \frac{M_{\text{bh}}}{\dot{M}_{\text{bh}}} = \frac{M_{\text{bh}}}{L_{\text{Edd}}(1-\epsilon)/\epsilon c^2} = \frac{c\sigma_T \epsilon}{4\pi G m_p (1-\epsilon)} = 4\times 10^8 \frac{\epsilon}{1-\epsilon}\text{yr}$$

  - This is around 1% of the age of the universe - nicely matches that quasars are observed in $\sim 1\%$ of galaxies.
    - Hence, it is suspected that every galaxy experiences a quasar phase after it forms.
  - This also can't explain very old black holes which are too massive early on in cosmic time.

AGN Feedback

Gravitational Feedback

- By measuring the orbits and speeds (dispersion) of stars near to a black hole, it is possible to determine the SMBH mass, e.g 2020 Nobel prize.
  - Define dispersion here as $\sigma = \sqrt{\langle v^2 \rangle} \sim 300\text{km/s}$.
- We see that there is also a correlation between black hole mass and the mass of the spheroidal part of the host galaxy with $M_{\text{bh}} \approx 10^{-3} M_{\text{spheroid}}$.
  - This indicates coupling between the black hole and the surrounding galaxy - potentially due to feedback between the energy produced by the black hole when it's growing and the surrounding gases.

Radiative Feedback

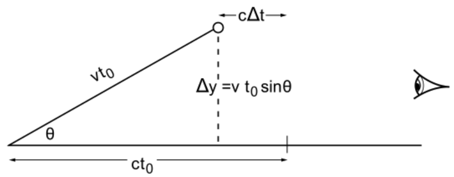
- Note that the binding energy of a galaxy is of the order $M_{\text{gal}}\sigma^2$ and the energy released by accretion from a black hole is around $0.1 M_{\text{BH}} c^2 \approx 10^{-4} M_{\text{gal}} c^2$. Comparing these then gives;

$$\frac{\text{BH Energy Release}}{\text{Galaxy Binding Energy}} = \frac{10^{-4} M_{\text{gal}} c^2}{M_{\text{gal}}\sigma^2} \sim 10^{-4}\frac{c^2}{\sigma^2} \sim 100$$

  - Hence, only around 1% of the accretion energy is needed to unbind the galaxy?!

Jets

- Considered to be magnetohydrodynamical effects.
- Jets normally appear to move superluminally (over the speed of light) due to geometric effects;



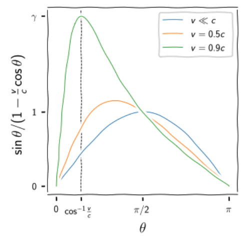
  - For a jet where particles travel with $v \sim c$, directed almost directly towards the observer who can only observe the transverse velocity;

$$\Delta y = vt_0 \sin\theta; \quad \Delta t = t_0\left(1 - \frac{v}{c}\cos\theta\right)$$

    - Here, Δt is time as measured by the observer. Hence, in the observer frame;

$$\frac{\Delta y}{\Delta t} = \frac{v\sin\theta}{1-\frac{v}{c}\cos\theta}$$

    - Therefore, for a blob moving at $v \sim c$ at $\cos\theta = \frac{v}{c}$ $\left(\implies \sin\theta = \gamma^{-1}\right)$, the apparent transverse speed is $\boxed{\gamma v}$



- For a blob emitting radiation with spectrum $P(\nu) \propto \nu^{-\alpha}$, the observed intensity scales as $D^{3+\alpha}$ with

$$D = \frac{1}{\gamma(1-\frac{v}{c}\cos\theta)}$$

  - Two powers of D (Doppler shift) are due to aberration changing the solid angle, one is due to compression of time and the final factor of α is due to the energy shift.
    - Note that the number n of photons in phase space is countable, so $I_\nu/\nu^3 \propto n$ must be Lorentz invariant.
  - This makes superluminal jets appear to an observer on Earth as much more intense than those pointed less directly towards us, making them unexpectedly common in any survey with an intensity cutoff.
    - In other words, most observable jets are directed towards us due to Doppler boosting.

Gamma-Ray Bursts

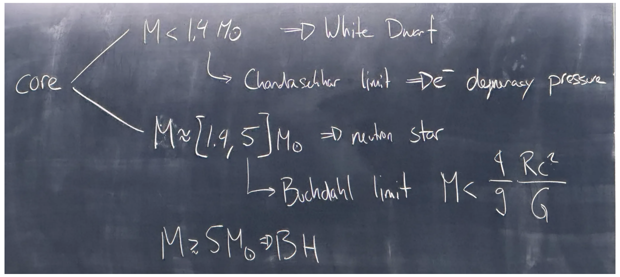
- Gamma ray bursts are found to originate in distant galaxies and still saturate gamma ray detectors around earth - around 10^{44} J.
  - They only last a few milliseconds, constraining them to only a few millilightseconds across.
  - This suggests that black-hole formation is causing these events, such as in NS-NS mergers (short bursts) or supernovae core collapse (long bursts).
    - In supernovae core collapse, jets with $\gamma > 100$ can form.
  - GRBs are hypothesised to explain mass-extinction events on Earth.

L12 - Neutron Stars and Pulsars

L11 - Active Galactic Nuclei

L13 - Precession, Binary Systems and Gravitational Waves

Neutron Stars



Neutron star formation and factors that affect the outcome; could also become a white dwarf or a black hole.

- Inside a neutron star, we expect inverse β decay to occur;

$$p^+ + e^- \to n + \nu$$

  - We would also expect n decay to occur regularly, as the lifetime of a free neutron is around 15 mins;

$$n \not\to p^+ + e^- + \nu$$

    - However, this does not occur due to the Pauli exclusion principle coupled with the fact that the lower energy proton and electron states are already filled.
- Using the mass-radius relationship derived for neutron stars in lecture 8 (below), we see that neutrons stars are of $R \sim 10\ \mathrm{km}$.

Spin

- Considering the angular momentum of a star being conserved, we find (using $L \sim MR^2\Omega$ and $P = \frac{2\pi}{\Omega}$) the rotational period for a neutron star as;

$$P_{NS} = P_\odot \left(\frac{R_{NS}}{R_\odot}\right)^2 \sim \text{milliseconds}$$

  - Equating the constraint this places on density, we find

$$v_{\text{escape}}^2 \sim R^2\Omega^2 \le \frac{GM}{R} \implies \boxed{\Omega^2 \le G\rho}$$

  - Hence, for $P < 1$ s, $\rho \sim \rho_{\text{nuclear}}$, so must be neutron stars.

Pulsars

- Rapidly spinning neutron stars emanate strong magnetic fields from their poles which produce radio waves.
  - Considering the star as a magnetic dipole, then it can be seen that the rate of energy loss from such a system is

$$\dot{E}_{\text{rot}} = -\frac{8\pi}{\mu_0}\frac{B_p^2 R^6 \Omega^4}{3c^3}\sin^2\alpha$$

    - Hence, we expect $\dot{\Omega} < 0$ such that we can find a characteristic spin-down time as

$$-\frac{\Omega}{\dot{\Omega}} = \frac{6Ic^3\mu_0}{B_p^2 R^6 \sin^2\alpha\Omega_0^2 4\pi} \implies \dot{\Omega} \propto -\Omega^{(n=3)}$$

      - In fact we measure $n \sim 2.5$, potentially due to particle outflow.
    - This energy loss can be extreme - enough to light up the crab nebula (10^{31} W).
  - Pulsars emit in the radio frequency due to the spinning of the magnetic field, which generates an electric field $E \sim v \times B \sim \Omega RB$ which in turn generates a voltage $V \sim \Omega R^2 B$.
    - This voltage rips electric charges from particle creation events, filling the magnetosphere with charged matter.
    - The motion of these charged particles in the strong magnetic field then generates synchrotron radiation.
    - The pulsar dies when the voltage is below the $e^\pm$ particle creation threshold.
- The period of neutron star rotations sometimes change rapidly in 'glitches' - potentially due to crustquakes rearranging the crust and therefore altering the moment of inertia.

L13 - Precession, Binary Systems and Gravitational Waves

L12 - Neutron Stars and Pulsars L14 - Gravitational Waves

Precession

- Optimising the Lagrangian around a central body and requiring ϕ axis symmetry gives the relativistic orbit equation (with $u = \frac{1}{r}$)

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2 \tag{1}$$

  - Renormalising this with $U = \frac{h^2}{GM}u$ gives

$$\frac{d^2U}{d\phi^2} + U = 1 + \underbrace{\frac{3(GM)^2}{c^2h^2}}_{=\alpha}U \tag{2}$$

    - Assuming the relativistic correction is small ($GM \ll rc^2$) allows us to solve by expanding $U(\phi)$ around the Newtonian solution $U_0 = 1 + e\cos\phi$ with

$$U = U_0 + \alpha U_1 + \alpha^2 U_2 + \dots$$

    - Substituting into (2) and keeping terms to first order in α gives;

$$\begin{aligned}\frac{d^2U_1}{d\phi^2} + U_1 &= (1 + e\cos\phi)^2 \\ &= 1 + \frac{e^2}{2} + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\end{aligned}$$

  - Solving the above, we find

$$U_1(\phi) = (1 + \frac{1}{2}e^2) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi$$

    - Noticing that the $e\phi\sin\phi$ term is the only significant contribution as ϕ is unbounded, we can then write the approximate solution

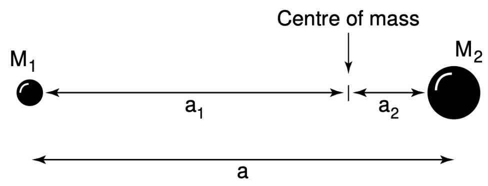
$$\begin{aligned}u(\phi) &\approx \frac{GM}{h^2}(1 + e\cos\phi + e\alpha\phi\sin\phi) \\ &\approx \frac{GM}{h^2}[1 + e\cos(\phi - \alpha\phi)]\end{aligned}$$

  - Hence, the r values repeat on a cycle slightly larger than 2π, giving precession as;

$$\Delta\phi = \frac{2\pi}{1-\alpha} - 2\pi \approx 2\pi\alpha = \frac{6\pi GM}{c^2a(1-e^2)} \tag{3}$$

Binaries

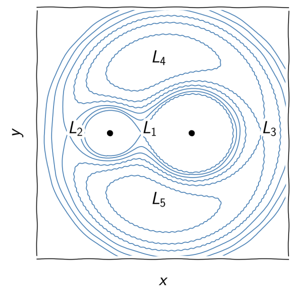
- Using redshift from the orbiting stars as well as their transits of one another, we can determine the period, velocity, mass and distance of binary systems.



- Considering a system (above) with M_1 and M_2 in circular orbits, we have the following standard properties following from $M_1a_1 = M_2a_2$;

$$\begin{aligned}
&\text{Kepler's Law:} && \Omega^2 = \frac{GM}{a^3} && (4)\\
&\text{Moment of Interia:} && I = \frac{M_1M_2}{M}a^2 && (5)\\
&\text{Total energy:} && E = -\frac{GM_1M_2}{2a} && (6)\\
&\text{Angular Momentum:} && J = \frac{M_1M_2}{M}a^2\Omega && (7)\\
&\text{Reduced Mass:} && \mu = \frac{M_1M_2}{M} && (8)
\end{aligned}$$

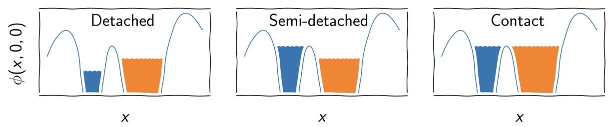
- Consider a restricted 3-body problem, we may solve for where gravitational and centrifugal forces balance.



  - Writing down the potential for one star M_1 at $(-a_1, 0, 0)$ and another with M_2 at $(a_2, 0, 0)$ (taking $(0,0,0)$ as the COM position whilst also including the centrifugal contribution gives;

$$\phi = -\frac{\Omega^2(x^2+y^2)}{2} - \frac{GM_1}{\sqrt{(x+a_1)^2+y^2+z^2}} - \frac{GM_2}{\sqrt{(x-a_2)^2+y^2+z^2}}$$

  - Then minimising this, we find there are five positions where gravity and centrifugal force cancel - **the Lagrange points**.
    - $L_{1,2,3}$ are pseudo-stable and $L_{4,5}$ are stable if $\frac{M_1}{M_2} > 24.96$.
- If the matter of a star in a binary system passes L_1 (the first equipotential common to both stars - the Roche Lobe), they may begin to exchange matter through a Roche lobe overflow process.
  - These systems can therefore be categorised as (below);



- In these mass transfer processes, angular momentum (7) must be conserved, causing a to change.
- Requiring $\dot{J} = \dot{M} = 0$ for such processes implies

$$0 = \frac{d}{dt}(JM) = \dot{M}_1 M_2 a^2 \Omega + M_1 \dot{M}_2 a^2 \Omega + 2M_1 M_2 a\dot{a}\Omega + M_1 M_2 a^2 \dot{\Omega}$$
$$\text{and} \quad \dot{M}_1 + \dot{M}_2 = 0$$

- Similarly, differentiating Kepler's law (4) gives $\frac{\dot{a}}{a} = -\frac{2}{3}\frac{\dot{\Omega}}{\Omega}$. Combining these we find the rate of spin-up and radius decrease:

$$\frac{3\dot{M}_1(M_1 - M_2)}{M_1 M_2} = -\frac{\dot{\Omega}}{\Omega} = \frac{\dot{P}}{P} = \frac{3}{2}\frac{\dot{a}}{a}$$

- The precession of binary pulsars can also be observed, and their precession is found to match that predicted by (3).
  - This provides another good test of GR.

Gravitational Waves

Linearising the Gravitational Field Equations

- Working in the weak field limit, the metric becomes;

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1$$

  - Then also note that;

$$(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\sigma} - h^{\nu\sigma}) = \delta^\sigma_\mu + O(h^2)$$
$$\implies g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

- After some calculation, and defining $\Box^2 = \partial_\sigma \partial^\sigma$ and $\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h}$, we find the Einstein equations in this limit linearise to;

$$\Box^2 \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho \partial_\sigma \bar{h}^{\rho\sigma} - \partial_\nu \partial_\rho \bar{h}^\rho_\nu = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

  - Applying gauge freedom in GR with coordinate perturbations $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ results in the metric changing under transformation as

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}$$

    - Then demanding that $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu$ requires;

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

  - Furthermore, applying derivatives to $\bar{h}$, we find that it transforms as

$$\partial_\rho \bar{h}^{\mu\rho} \to \partial_\rho \bar{h}'^{\mu\rho} = \partial_\rho \bar{h}^{\mu\rho} - \Box^2 \xi^\mu$$

    - We may then choose $\xi^\mu(x)$ such that $\partial_\rho \bar{h}'^{\mu\rho} = 0$ (the **Lorenz gauge**) independent of the primes. Hence, the field equations simplify to

$$\Box^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad \partial_\mu \bar{h}^{\mu\nu} = 0 \tag{9}$$

Vacuum Solution

- In the vacuum, we expect the RHS of (9) to be zero;

$$\Box^2 \bar{h}^{\mu\nu} = 0 \tag{10}$$

  - Hence, we expect plane wave solutions;

$$\bar{h}^{\mu\nu} = A^{\mu\nu} e^{ik_\rho x^\rho}$$
$$\text{Hence, (9)} \implies \begin{cases} k_\rho k^\rho = 0 & \text{(k is null)} \\ A^{\mu\nu} k_\nu = 0 \end{cases}$$

    - This also implies the dispersion relation $\omega^2 = c^2 |\vec{k}|^2$ for photons.
    - As the field equations are linear, we may therefore write a general solution by superposition as;

$$\bar{h}^{\mu\nu}(x) = \int A^{\mu\nu}(\vec{k}) \exp(ik_\rho x^\rho) d^3\vec{k}$$

- $A^{\mu\nu}$ initially has 16 independent components. However, this is reduced to 10 by symmetry $A^{\mu\nu} = A^{\nu\mu}$ and further to 6 by the Lorentz condition $A^{\mu\nu} k_\nu = 0$.
  - Then solving (10) for perturbations of the form $\xi^\mu = \epsilon^\mu e^{ik\cdot x}$ with $k^2 = 0$, we find the amplitudes must transform as;

$$A^{\mu\nu} \to A'^{\mu\nu} = A^{\mu\nu} - i\epsilon^\mu k^\nu - i\epsilon^\nu k^\mu + i\eta^{\mu\nu}\epsilon^\rho k_\rho$$

  - This can be used to further reduce the degrees of freedom; typically we chose ϵ^μ such that $A^{\mu\nu}$ has only two non-zero components, allowing them to take the transverse and traceless form (gauge)

$$A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a^+ & a^\times & 0 \\ 0 & a^\times & -a^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}$$

    - A z-travelling gravitational wave is then parameterised by $k, a^+, a^\times$ (note $A^{\mu\nu} e^{ik_\rho x^\rho}$ is also the solution for h as it is traceless).
    - The constants a^+ and $a^\times$ control polarisations of the wave;
      - a^+ controls the expansion of **proper** (not coordinate) distances in the x direction as they decrease in the y direction.
      - $a^\times$ controls the same thing, but rotated $45°$.

L14 - Gravitational Waves

L13 - Precession, Binary Systems and Gravitational Waves   L15 - Gravitational Lensing

General Linearised Solutions to the Field Equations

- We now wish to solve;

$$\Box^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \quad \partial_\mu \bar{h}^{\mu\nu} = 0$$

  - Defining a Greens function (where θ is the Heaviside function);

$$G = \frac{\delta(ct - |\vec{x}|)\theta(ct)}{4\pi|\vec{x}|}$$
$$\implies \Box^2 G = \delta^{(4)}(x)$$

  - We see that the general solution is;

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y}$$

    - Note that here, we are integrating over a set of source locations $\vec{y}$ and retarded times $ct - |\vec{x} - \vec{y}|$.
    - Assuming we are dealing with a compact source (source of gravitational wave $\ll$ wavelength of gravitational wave), this simplifies to

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4 r} \int T^{\mu\nu}(ct - r, \vec{y}) d^3\vec{y}$$

- Considering the above component by component, we recall that

$$\int T^{00} d^3\vec{y} = Mc^2 \qquad \text{total energy} \implies \boxed{\bar{h}^{00} = -\frac{4GM}{c^4 r}} \tag{1}$$
$$\int T^{0i} d^3\vec{y} = P^i c \qquad \text{total momentum in direction i} \implies \boxed{\bar{h}^{0i} = \bar{h}^{i0} = -\frac{4GP^i}{c^3 r}} \tag{2}$$
$$\int T^{ij} d^3\vec{y} = \frac{1}{2c^2}\frac{d^2}{dt^2}\int T^{00} y^i y^j d^3\vec{y} = \frac{1}{2c^2}\ddot{I}^{ij} \implies \boxed{\bar{h}^{ij} = -\frac{2G}{c^6 r}\ddot{I}^{ij}} \tag{3}$$

  - Note, the final line represents the internal stresses in the source and the algebra follows from requiring $\partial_\mu T^{\mu\nu} = 0$ and using the quadrupole moment, I;

$$I^{ij} = c^2 \int \rho y^i y^j d^3\vec{y} \tag{4}$$

    - The maths here is non-examinable.
  - Examining the $h^{\mu\nu}$ above, we see that the far-field of a compact source is a combination of a steady field from the mass M and a potentially time-varying field arising from integrated internal stresses (responsible for gravitational waves).
  - Note also that strain $h^{\mu\nu}$ decreases with $1/r$, so there is a higher abundance of gravitational waves to measure than electromagnetic waves decaying at $1/r^2$.
    - On the other hand, as quadrupoles are responsible for emission and gravity couples weakly to matter, gravitational radiation is very weak.

Energy and Momentum in Gravitational Fields

- It is difficult to localise the energy of a gravitational field, as assigning an energy results in mass-energy equivalence generating more gravity. In a general spacetime there is no symmetry, so we shouldn't expect conserved quantities.
- In a linearised theory, however, we can assign the second order terms, quantifying how much the wave fails to be linear, to a stress energy tensor $t_{\mu\nu}$ such that

$$G_{\mu\nu} = G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} + t_{\mu\nu}) \implies t_{\mu\nu} = \frac{c^4}{8\pi G}\langle G^{(2)}_{\mu\nu}\rangle$$

  - In the transverse traceless gauge this reduces to

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h^{TT}_{ij} \partial_\nu h^{ij}_{TT}\rangle$$

  - Hence, we find the rate of energy loss (using $F = -ct^{0k} n_k$)

$$\frac{dE}{dt} = -L_{GW} = -r^2 \int F d\Omega = -\frac{G}{5c^9}\langle \dddot{I}_{ij} \dddot{I}^{ij}\rangle \tag{5}$$

    - More generally this can be written with the traceless equivalent of I^{ij} with

$$Q^{ij} = \int T^{00}(y^i y^j - \frac{1}{3}\delta^{ij}) d^3\vec{y}$$

Example: A Binary System

2-Body Problem

- Assuming we have two stars orbiting each other with angular speed $\Omega = \frac{GM}{16a^3}$, masses $M_1 = M_2 = M/2$, separation $2a$ and coordinates $\vec{x}_a = (a\cos\Omega t, a\sin\Omega t) = -\vec{x}_B$, substituting into (3) and (4) we find;

$$I^{ij} = \frac{Mc^2a^2}{2}\begin{pmatrix} 1 + \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & 1 - \cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\implies \bar{h}^{\mu\nu}_{\text{rad}} = \frac{4GMa^2\Omega^2}{c^4 r}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos 2\Omega t_r & \sin 2\Omega t_r & 0 \\ 0 & \sin 2\Omega t_r & -\cos 2\Omega t_r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  - Note here $t_r = t - \frac{r}{c}$ is the retarded time and we only keep radiative parts of $\bar{h}^{\mu\nu}$
  - Then computing the third time derivative of I^{ij} and using (5) we find

$$\frac{dE}{dt} = -L_{GW} \propto \frac{a^4 M^2 \Omega^6}{c^5}$$

    - We then use Kepler's law to replace a in the total energy by Ω, and we find the period $P \propto t^{3/8}$ and $\dot{P} \propto P^{-5/3}$.
- For a general binary system, we again find $\dot{\Omega} \propto \mathcal{M}^{5/3}$ where $\mathcal{M}$ is the Chirp Mass;

$$\mathcal{M} = \frac{(M_1 M_2)^{3/5}}{M^{1/5}}$$

Detecting Gravitational Waves

- The separation of two particles is (where n^i are unit vectors in the direction of gravitational wave propagation)

$$L^2 = -g_{ij}\Delta x^i \Delta x^j = (\delta_{ij} - h_{ij})\Delta x^i \Delta x^j = L_0^2(1 - h_{ij} n^i n^j)$$

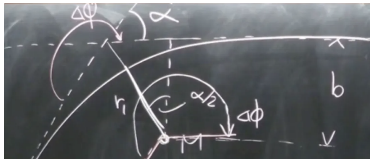
  - Hence, we expect to be able to measure a strain, e.g. at LIGO (to first order) of;

$$\frac{\delta L}{L_0} = -\frac{1}{2} h_{ij} n^i n^j \sim 10^{-21}$$

L15 - Gravitational Lensing

L14 - Gravitational Waves L16 - The Geometry of the Universe

Newtonian Lensing



- Consider the above configuration. In the Newtonian case, we must solve the orbit equation subject to $u = 1/r$ and $\sin\phi \to b/r$ as as $\phi \to 0$.

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} \implies u = \frac{1}{r} = \frac{GM}{h^2}(1-\cos\phi) + \frac{1}{b}\sin\phi$$

  - Then finding the point of closest approach, and using $h = \vec{r}\times\vec{v} = bc$, gives

$$\frac{du}{d\phi} \implies \phi_* - \frac{\pi}{2} = \arctan\left(\frac{GM}{bc^2}\right) = \frac{\alpha}{2}$$

  - Hence, for small α, we have

$$\alpha_{\text{Newt}} = \frac{2GM}{bc^2} = \frac{R_s}{b}$$

Lensing in GR

The Weak Field Metric for Stationary Sources

- We now wish to solve the linearised field equations under the approximation that the source is stationary; $\partial_t T^{\mu\nu} = 0$. This implies $T^{\mu\nu} = T^{\mu\nu}(\vec{y})$, so

$$\bar{h}^{\mu\nu}(\vec{x}) = -\frac{4G}{c^4}\int \frac{T^{\mu\nu}(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}$$

  - Then supposing we are dealing with a dust (a pressureless, non-relativistic fluid), the stress-energy tensor is diagonal can be written as $T^{\mu\nu} = \rho u^\mu u^\nu$.
    - Note that this also means $\left|\frac{T^{ij}}{T^{00}}\right| \sim \frac{u^2}{c^2} \ll 1$, so $u^\mu \approx (c, \vec{u})$. Therefore, the above metric components simplify to;

$$\bar{h}^{00} = \frac{4\Phi}{c^2} \quad \text{where} \quad \Phi(\vec{x}) = -G\int\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}$$
$$\bar{h}^{0i} = \frac{A^i}{c} \quad \text{where} \quad \vec{A}(\vec{x}) = -\frac{4G}{c^2}\int\frac{\rho(\vec{y})\vec{u}(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}$$
$$\bar{h}^{ij} = 0$$

  - Then, since the trace is only $\bar{h} = \bar{h}^{00}$, using $h^{\mu\nu} = \bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\bar{h}$, and lowering indices with $\eta_{\mu\nu}$, we find

$$h_{00} = h_{11} = h_{22} = h_{33} = \frac{2\Phi}{c^2}$$
$$h_{0i} = \frac{A_i}{c}$$

    - Using this, we find the stationary weak field metric valid for relativistic motion around non-relativistic sources;

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 + 2\vec{A}\cdot d\vec{x}cdt - \left(1 - \frac{2\Phi}{c^2}\right)d\vec{x}^2 \tag{1}$$

      - Note that this can be made static (invariant under $t \to -t$) with $\vec{A} = 0$.

Relativistic Lensing

- We previously found that for a photon orbit in a Schwarzschild metric,

$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r) = \frac{1}{b^2} \quad \text{with} \quad b = \frac{h}{ck} \quad \text{and}$$
$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right)$$

  - Substituting for h and μ and applying chain rule, we find

$$\left(\frac{dr}{d\phi}\right)^2 + r^2\left(1 - \frac{2\mu}{r}\right) = \frac{r^4}{b^2}$$

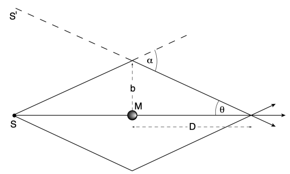
  - This can then be rearranged and integrated (noting that $\Delta\phi$ is twice the angle from the point of closest approach (r_1) - see this), giving

$$\Delta\phi = 2\int_{r_1}^{\infty}\frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right]^{-\frac{1}{2}}$$

  - This can then be integrated (in the limit of small M/b) to find;

$$\Delta\phi \approx \pi + \frac{4GM}{c^2b} \implies \alpha_{\text{Einstein}} = \Delta\phi - \pi = \frac{4GM}{c^2b} = \frac{2R_S}{b} = 2\alpha_{\text{Newt}}$$

    - Note, we could have found this result using the general weak field metric (1).



Lensing Effects

- Note that at cosmological distances, assuming the source and observer are equidistant from the lens, we have the geometry above.
  - Here, S is the source and M is the lensing mass, so we find;

$$\theta = \frac{\alpha}{2} = \frac{R_S}{b} = \frac{b}{D} \implies b = \sqrt{R_S D}$$

  - In the general case, where D_s and D_l are the distances of the source and lens from the lensing mass, we have;

$$b = \sqrt{\frac{2R_S(D_s - D_l)D_l}{D_s}}$$

  - If the source is slightly offset from the line of sight through the lensing mass, then multiple images separated by 2θ may be seen. This also allows for measurement of H_0.
- Microlensing occurs when a non-luminous object transits between earth and a distant luminous object. Lensing causes a spike in the magnitude.
  - Due to the very compact geometry of a microlens, we can find the amplification as (put more here after doing sheet 3);

$$A = \frac{u^2+2}{u\sqrt{u^2+4}}$$

- Weak lensing slightly distorts images of far away objects; by imaging thousands of galaxies, it is possible to project the mass distribution along the line of sight.
- To image a black hole, we must consider the potential here (and blue above). The stationary point of this potential is at $r = 3\mu$.
  - Then considering the situation where $\dot{r} = 0$, (using the same block), we find the smallest impact parameter where a black hole is observable as;

$$b_{\min} = 3\sqrt{3}\mu = \frac{3\sqrt{3}}{2}R_S$$

Aside on Telescopes and Radio Astronomy

- Telescopes are diffraction limited with angular resolution

$$\alpha = 1.22\frac{\lambda}{D}$$

  - Note that the smallest? practical wavelength for radio astronomy is $\lambda = 1.33 \text{ mm}$.

L16 - The Geometry of the Universe

L15 - Gravitational Lensing L17 - Dynamics of the Universe

Distance Ladder - make this into a table

Method	Outline of Operation	Scale
Radar	Send out pulses and measure doppler shifts / reflection times.	Solar System
Parallax	Measure angular differences at opposite sides of earth-sun orbit.	Solar Neighbourhood
Cepheid Variables	Exploit that star pulsations are highly correlated with luminosity.	Galaxy
IA Supernovae	Approximately constant brightness that can be further calibrated by the emission spectra.	Local Supercluster
Baryon Acoustic Oscillations	Oscillations in the primordial universe plasma which are present in the modern galaxy distribution.	Observable Universe
Gravitational Waves	Measure the intrinsic luminosity using the frequency of the gravitational wave.	Observable Universe
CMB		Observable Universe

Body	Earth	Sun	Local Galactic Group	Local Superclusters
Velocity wrt Next	30 km/s	100 km/s	1200 km/s	...

The Universe

- Age: 13.8 Gyr
- Observable universe size: 46.5 Gly.

Cosmology

- We believe the universe is 13.8 Gyr old and the size of the observable universe to be 46.5 Gly.
  - This apparent contradiction is explained by Hubble expansion.
- The Cosmological Principle states that the universe is homogeneous and isotropic. At constant time, these require (respectively) - see here for more details;

$$ds^2_{r=\text{const}} = r^2 d\Omega = r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$ds^2 = f(r)dr^2 + r^2 d\Omega \tag{1}$$

  - Furthermore, for spatial homogeneity, $f(r)$ must give constant curvature $K(r) = K$, so as before, applying Gauss' formula for curvature, we find

$$f(r) = \frac{1}{1 - Kr^2}$$

    - As K is the Gaussian curvature of the 2-dimensional surface $\theta = \pi/2$ and isotropy implies this is independent of the coordinate system, combining the above gives a metric which is genuinely homogenous over 3D space.

Kinds of Universe

- If $K = 0$, then we just have ordinary Euclidean space with $ds^2 = dr^2 + r^2 d\Omega$.
- If $K > 0$, we have proper intervals from the origin;

$$a(r) = \int_0^{a(r)} ds = \int_0^r \frac{dr}{\sqrt{q - Kr^2}} = \frac{1}{\sqrt{K}}\sin^{-1}\left(r\sqrt{K}\right)$$
$$\implies r = \frac{1}{\sqrt{K}}\sin\left(a\sqrt{K}\right)$$

  - We therefore conclude that the surface area of a sphere varies with a.

- Such universes are spheres with three-dimensional surfaces, so are termed **closed**.
- If $K < 0$, repeating the above gives

$$r = \frac{1}{\sqrt{-K}} \sinh\left(a\sqrt{-K}\right)$$

  - Hence, the area increases faster than in a flat space and tends to ∞ as $a \to \infty$.
  - This is called an **open universe**.

The Time Component of the FLRW Metric

- The universe is observed to be expanding, so the metric must change with time. This expansion is described by the Hubble-Lemaître Law describing the "velocity" v with which two object a "distance" d apart separate (note the parameters are discussed further below);

$$v = H_0 d$$

  - As non-relativistic Doppler shift gives $\lambda_{\text{em}} = \lambda_{\text{obs}}(1 - v/c)$, we find, where $H_0 = 50 - 80 \text{ kms}^{-1}\text{Mpc}^{-1}$, is the Hubble constant;

$$v \approx zc = H_0 d$$

  - Projecting this backwards in time, we find the age of the universe as

$$t = \frac{d}{v} = \frac{d}{H_0 d} = H_0^{-1} \approx 15 \text{ Gyr}$$

    - Including dynamics results in an even lower age. Note this could be problematic, as some stars are around 11 Gyr old.
    - Hubble tension refers to the discrepancy between CMB and standard-candle measurements of H_0.
- To account for this time dependence, we redefine $K = K(t)$ in a form retaining the form $K \sim 1/(\text{radius of space})^2$;

$$K(t) = \frac{k}{R^2(t)} \begin{cases} k = +1 & \text{if curvature positive} \\ k = 0 & \text{if space is flat} \\ k = -1 & \text{if curvature is negative} \end{cases}$$

  - We then define a dimensionless coordinate $\boxed{\sigma = r/R(t)}$ such that (1) becomes

$$R^2(t)\left[\frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

  - Object with constant σ are **fundamental observers**. They have zero velocity with respect to the frame defined by the CMB.
    - It is possible to find out motion relative to the CMB via the dipole anisotropy.
  - Then defining cosmic time as the proper time as measured by a fundamental observer allows us to write the Friedmann-Robertson-Walker metric as;

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

    - It can be shown that co-moving observers follow geodesics of this metric, so they are freely falling.
- Finally, we introduce a new co-moving radial coordinate χ to make this take a convenient form for radially propagating photons;

$$\boxed{d\chi = \frac{d\sigma}{\sqrt{1 - k\sigma^2}}} \implies \chi = \begin{cases} \sin^{-1}\sigma & k = +1 \\ \sigma & k = 0 \\ \sinh^{-1}\sigma & k = -1 \end{cases}$$

  - This results in the FLRW metric;

$$ds^2 = c^2dt^2 - R^2(t)\left[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right]$$
$$\text{with} \quad S^2(\chi) = \begin{cases} \sin^2\chi & k = +1 \\ \chi^2 & k = 0 \\ \sinh^2\chi & k = -1 \end{cases}$$

- Therefore, WLOG, working radially, we find a proper distance in the FLRW metric as

$$\int \sqrt{g_{\chi\chi}} d\chi = R(t)\chi$$

Redshift in the FLRW Metric

- A photon travelling towards us from an external galaxy has approximately radial motion, so can ignore θ and ϕ coordinates. Then requring the interval vanishes for a photon,

$$ds^2 = c^2dt^2 - R^2(t)d\chi^2 = 0 \implies \frac{d\chi}{dt} = -\frac{c}{R(t)}$$

  - Assuming this photon was emitted at comic time t_1 and is detected here at time t_0, we know the emitter has χ coordinate;

$$\chi_1 = \int_{t_1}^{t_0} \frac{cdt}{R(t)}$$

  - Now, with the interval between the emission of successive wavecrests as δt_1 and that between reception as δt_0, and noting that the co-moving distance χ between the emitter and receiver doesn't change with time, we have;

$$\chi_1 = \int_{t_1}^{t_0} \frac{cdt}{R(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{cdt}{R(t)} = \left(\int_{t_1+\delta t_1}^{t_1} + \int_{t_1}^{t_0} + \int_{t_0}^{t_0+\delta t_0}\right) \frac{cdt}{R(t)}$$
$$\implies \int_{t_1}^{t_1+\delta t_1} \frac{cdt}{R(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{cdt}{R(t)}$$

    - Then assuming $R(t)/\dot{R}(t) \gg \delta t$ (age of the universe $\gg$ period of waves), we find

$$\frac{\delta t_1}{R(t_1)} = \frac{\delta t_0}{R(t_0)} \implies \frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)}$$

    - We therefore find that redshift coincides with the conception of "stretching" the universe with;

$$1 + z = \frac{R(t_0)}{R(t_1)} = \frac{\text{scale factor of univser on reception}}{\text{scale factor of universe when emitted}}$$

      - NOTE, there is another derivation for this on slide 27 - worth looking at again.

L17 - Dynamics of the Universe

L16 - The Geometry of the Universe L18 - The Evolution of the Universe

- notably, FLRW metric is time dependent

Curvature of the FLRW Metric

- In L16, we defined the 3D spatial curvature of the FLRW metric as;

$$K_{\text{space}} = \frac{k}{R(t)^2}$$

- We can also determine the 4D spacetime curvature of the FLRW metric by applying Gauss' Curvature Theorem in the 2D subspace where $ds^2 = g_{tt}dt^2 + g_{\chi\chi}d\chi^2$, which gives;

$$c^2 K(t) = -\frac{\ddot{R}}{R}$$

The Cosmological Equations

Derivation of the Cosmological Field Equations

- Adding a constant number of $g_{\mu\nu}$ to the LHS of the gravitational field equation gives

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}$$
$$\implies R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) + \Lambda g_{\mu\nu}$$

  - Then substituting with a perfect fluid, noting that homogeneity and isotropy imply the density and pressure must be functions of only time, we find that in the frame of a co-moving observer with $u^\mu = (1,0,0,0) \implies u_\mu = c^2\delta^0_\mu$,

$$T_{\mu\nu} = (\rho c^2 + P)c^2\delta^0_\mu\delta^0_\nu - Pg_{\mu\nu}$$
$$\implies T = T^\mu_\mu = \rho c^2 - 3P$$

  - Note also that in these co-moving coordinates, the FLRW metric takes the form

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

  - Calculating the connection coefficients and substituting them for $R_{\mu\nu}$, we then eventually find the cosmological field equations;

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{1}{3}\Lambda c^2 R, \tag{1}$$
$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 + \frac{1}{3}\Lambda c^2 R^2 - c^2 k. \tag{2}$$

The Continuity Equation

- It is most common to rewrite (1-2) as

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}, \quad \text{(acceleration)} \tag{3}$$
$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2}. \quad \text{(velocity/energy)} \tag{4}$$

  - These may be combined to form the continuity equation;

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right), \quad \text{(continuity)} \tag{5}$$

    - The continuity equation can also be derived by using stress-energy conservation $\nabla_\mu T^{\mu\nu}$ in co-moving coordinates.

- Only two of these equations are independent, and there are three variables, R, ρ and P. Therefore, we use an equation of state relating P and ρ to solve.

w-fluids

- Assuming that the energy density ρ has contributions from matter, radiation and and cosmological constant, we write;

$$\rho(t) = \rho_m(t) + \rho_r(t) + \rho_\Lambda(t)$$

  - In cosmology it is then often useful to assume an equation of state;

$$\boxed{P_i = w_i\rho_i c^2} \quad \text{with}$$
$$w_m \approx 0 \quad \text{(matter is a pressureless dust)}$$
$$w_r = \frac{1}{3} \quad \text{(radiation is a gas of photons)}$$
$$w_\Lambda = -1 \quad \text{(for convenience)}$$
$$w_k = -\frac{1}{3} \quad \text{(for conveience)}$$

    - Note, this can also be parameterised in terms for $\epsilon = 3w$ and $c_s^2 = wc^2$.

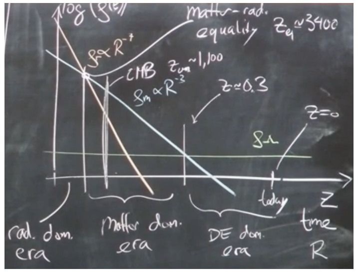
  - Using this equation of state, we see that (3-5) take the forms (with $i \in \{r, m, k, \Lambda\}$);

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\sum_i \rho_i(1+3w_i) \tag{6}$$
$$\dot{\rho}_i = -3\frac{\dot{R}}{R}\rho_i(1+w_i) \tag{7}$$
$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\sum_i \rho_i \tag{8}$$

    - Using (7) we then immediately see that

$$\boxed{\rho_i \propto R^{-3(1+w_i)}}$$

    - We therefore expect the contributions of the different energy density components of the universe to evolve with time as;



- Note, radiation has $\rho \propto R^{-4}$ - the combined contributions of number conservation and redshift evolution; and the vacuum (Λ) contribution is constant with $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$. We can therefore rewrite $\rho(t)$ as

$$\rho(t) = \rho_{r0}\left(\frac{R}{R_0}\right)^{-4} + \rho_{m0}\left(\frac{R}{R_0}\right)^{-3} + \rho_\Lambda$$

Dimensionless Form of the Dynamical Equations

- Now define the Hubble parameter H and the deceleration parameter q, along with the density parameters Ω_i;

$$H = \frac{\dot{R}}{R} \quad q = -\frac{\ddot{R}R}{\dot{R}^2} \quad \Omega = \frac{8\pi G\rho}{3H^2} \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H^2} \quad \Omega_k = -\frac{kc^2}{R^2H^2}$$

  - We can then rewrite the cosmological equations as

$$(6) \implies q = \frac{1}{2}(1+3w)\Omega - \Omega_\Lambda \qquad (9)$$
$$(8) \implies 1 = \Omega + \Omega_\Lambda + \Omega_k \qquad (10)$$

    - We may also write the total density parameter as the sum of each component contribution, $\Omega = \Omega_r + \Omega_m$.
    - Note that taking (4) where $\Lambda = k = 0$ gives the critical density required for the universe to be spatially flat as

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$$

      - The total density parameter Ω is therefore the ratio of the density to the critical density. Similarly, Ω_Λ is the ratio of energy density to the critical density.
        - This makes sense in the context of a dark energy fluid with $w = -1$ where $\rho_\Lambda = \Lambda c^2/8\pi G$.
- In the modern universe, we can neglect the radiation contribution, so rearranging (10) for Ω_k, we see that the universe is;

$$\text{universe}\begin{cases}\text{closed (+ve curvature)} & \text{if } \Omega_m + \Omega_\Lambda > 1 \\ \text{flat (zero curvature)} & \text{if } \Omega_m + \Omega_\Lambda = 1 \\ \text{open (-ve curvature)} & \text{if } \Omega_m + \Omega_\Lambda < 1\end{cases}$$

  - Furthermore, (9) implies that the universe is accelerating if

$$\frac{\Omega_m}{2}(1+3w) < \Omega_\Lambda$$

- The modern way of writing (10) is to take $a = R/R_0$ and Ω_i take their values at current time t_0 therefore allowing us to write;

$$\left(\frac{H}{H_0}\right)^2 = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda$$

L18 - The Evolution of the Universe

L17 - Dynamics of the Universe L19 - Measuring the Universe

Classifying Solutions of the Friedmann-Lemaître equation

- The cosmological field equations are;

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}, \qquad (1)$$
$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2}, \qquad (2)$$
$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right). \qquad (3)$$

- For a w-fluid, with $P = w\rho c^2$, where $i \in \{m, r, k, \Lambda\}$, these are written;

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\sum_i \rho_i(1+3w_i) \qquad (4)$$
$$\dot{\rho}_i = -3\frac{\dot{R}}{R}\rho_i(1+w_i) \qquad (5)$$
$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\sum_i \rho_i \qquad (6)$$

  - With $w_r = \frac{1}{3}$, $w_m \approx 0$, $w_k = -\frac{1}{3}$, $w_\Lambda = -1$, $\rho_k = 3kc^2/8\pi GR^2$ and $\rho_\Lambda = \Lambda c^2/8\pi G$.
- The Friedmann-Lemaître equation is;

$$\left(\frac{H}{H_0}\right)^2 = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda$$

  - This is easy to solve numerically, but non-linearity results in a large range of solutions.

Einstein-de-Sitter Form - $\Lambda = k = 0$

- In this case, (4-5) give

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} \quad \text{and} \quad \rho \propto R^{-3(1+w)}$$
$$\implies \dot{R} \propto R^{-(1+3w)/2} \implies \boxed{R \propto t^{\frac{2}{3(1+w)}}}$$

  - Therefore, in a matter dominated EdS, $R \propto t^{2/3}$ and for a radiation dominated EdS, $R \propto t^{1/2}$.
    - Note, if not specified then EdS normally means matter dominated.

Aside: Conformal Time

- Define the **conformal time** η as

$$ds^2 = c^2dt^2 - R(t)^2dX^2 = R(t)^2(d\eta^2 - dX^2)$$

  - Therefore, we have

$$cdt = Rd\eta \implies \eta = \int \frac{cdt}{R}$$

- This can be through of as a dimensionless co-moving time. Is also time measured by a light clock expanding with the universe.

Friedmann Form - $\Lambda = 0$

- In a Friedmann universe, (4) therefore gives

$$\ddot{R} < 0 \quad \forall \quad t$$

  - Therefore, all $\Lambda = 0$ models correspond to a Big-Bang origin at a time before than the Hubble time.
  - As $R \to 0$, (2) gives

$$\frac{\dot{R}^2 + kc^2}{R^2} = \frac{8\pi G\rho}{3}$$

    - Hence, using $\rho \propto R^{-3(1+w)}$ from (5) gives

$$\dot{R}^2 + kc^2 \propto R^{-(1+3w)}$$

    - Therefore, as kc^2 is constant and $w \geq 0$, we find that $\dot{R} \to \infty$ as $R \to 0$, which further implies that for succifiently early epochs,

$$\dot{R} \propto R^{-(1+3w)/2}$$

      - As this is independent of k, space curvature can be ignored in the early universe. Hence, EdS is a good approximation for the early universe.

- Now noticing that we may rewrite

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{d\eta}\frac{d\eta}{dt} = \frac{c}{R}\frac{dR}{d\eta}$$

  - We can then rewrite (2), also using $a = R/R_0$;

$$\begin{aligned} &\frac{1}{R^2}\left(\frac{c}{R}\frac{dR}{d\eta}\right)^2 - \frac{8\pi G\rho_0 R_0^{3(1+w)}}{3R^{3(1+w)}} = -\frac{kc^2}{R^2} \\ \implies &\frac{1}{a^2R_0^2}\left(R_0\frac{da}{d\eta}\right)^2 - \frac{8\pi G\rho_0 R_0^2}{3c^2a^{1+3w}} = -k \\ \implies &\boxed{\left(\frac{1}{a}\frac{da}{d\eta}\right)^2 = \left(\frac{a_m}{a}\right)^{1+3w} - k} \quad \text{where} \quad \frac{8\pi G R_0^2\rho_0}{3c^2} = a_m^{1+3w} \end{aligned}$$

  - Performing separation of variables and integrating, we find

$$\eta = \int \frac{da}{a\sqrt{\left(\frac{a_m}{a}\right)^{1+3w} - k}}, \quad t = \frac{R_0}{c}\int a d\eta$$

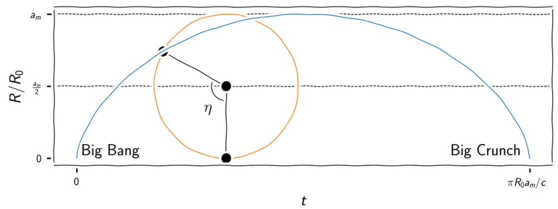
    - Integrating from $(0,0,0)$ to (η, a, t) and using the substitution $a = a_m \sin^2 x$ for the case $k=1$ and $w=0$ gives;

$$\begin{aligned} \eta &= \int_0^a \frac{da}{\sqrt{a(a_m - a)}} = 2\sin^{-1}\sqrt{a/a_m} \\ &\implies \boxed{R = R_0\frac{a_m}{2}(1 - \cos\eta)} \end{aligned}$$

- Note therefore that the maximum scale factor of the universe if $R = R_0 a_m$.
- Using this, we also find;

$$\boxed{t = R_0\frac{a_m}{2c}(\eta - \sin\eta)}$$

- Plotting these, we have;



  - For a radiation dominated $w = 1/3$ and closed $k=1$ universe, the solution is a semicircle;

$$R = R_0 a_m \sin\eta, \quad t = \frac{R_0}{c}a_m(1 - \cos\eta)$$

Einstein Static Universe - $\dot{R} = \ddot{R} = 0$

- Demanding $\dot{R} = \ddot{R} = 0$ forces solutions of (4) & (6) with;

$$R^2 = \frac{kc^2}{4\pi G\rho(1+w)}, \quad \Lambda = \frac{4\pi G}{c^2}\rho(1+3w)$$

  - Therefore, if the universe is closed with $k=1$, we can substitute for w and find the universe is static with a constant radius depending on the density. Using $\rho_0 = 1m_p\mathrm{m}^{-3}$, we find $R = 8.2\ \mathrm{Gpc}$ - large enough to encompass the known universe.
  - This is not obviously wrong, but disagrees with observed Hubble expansion and is unstable to small perturbations $R(t) = R_{\text{Einstein}} + \delta R(t)$.

Flat Λ universes - $k = 0$

- Taking $2\times(1)+(2)$ with $w=0$ and $k=0$, and using the Hubble parameter $H = \dot{R}/R$ such that;

$$\dot{H} = \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} \implies \dot{H} + H^2 = \frac{\ddot{R}}{R}$$

  - Allows us to write the reduced cosmological equation

$$2\dot{H} + 3H^2 = \Lambda c^2 \tag{7}$$

  - With $\Lambda > 0$, this has solutions

$$H(t) = \begin{cases} b\tanh(at) & (A) \\ b/\tanh(at) & (B) \\ b & (C) \end{cases}$$

    - Using case (B) and substituting into (7), we can see a solution is where;

$$\boxed{b=\sqrt{\frac{\Lambda}{3c}},\quad a=\frac{3}{2}\sqrt{\frac{\Lambda}{3c}}}$$

- These results turn out to hold in all cases.
- Furthermore, substituting these solutions (A-C) into (1) with $w=k=0$, we find

$$\frac{8\pi G}{c^2}\rho(t)=\begin{cases}-\Lambda\mathrm{sech}^2(at) & (A)\text{ negative density (unphysical)}\\ \Lambda\mathrm{cosech}^2(at) & (B)\text{ positive density}\\ 0 & (C)\text{ de Sitter, empty of matter}\end{cases}$$

- For the physical case (B), inverting (B) gives the age of the universe with $\Omega_\Lambda=\frac{\Lambda c^2}{3H^2}$ as

$$t=\frac{2}{3H}\frac{\tanh^{-1}\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_\Lambda}}$$

- Where $\Omega_\Lambda\to 0$ (EdS), we therefore have $t=\frac{2}{3H}$.
- Where $t\to\infty$, $H\to H_\infty=\sqrt{\frac{\Lambda}{3}}c$ and $\Omega_\Lambda\to 1$.

de Sitter Universes - $\Lambda\neq 0$

- de Sitter Universes are those where;

$$H=\frac{\dot{R}}{R}=\text{const}$$

- This implies

$$R\propto e^{Ht}$$

- Notably, this obeys the maximal Copernican principle - that nowhere and nowhen in spacetime is special.

L19 - Measuring the Universe

L18 - The Evolution of the Universe L20 - Constituents of the Universe

Measuring Distances

Proper Distance

- Within a homogeneous, isotropic universe, we have the FRW metric;

$$ds^2=c^2dt^2-R^2(t)dX^2 \tag{1}$$

- Taking a measurement at a fixed time t_0, we therefore find the proper distance to a galaxy with co-moving radial coordinate χ as;

$$d_{\text{prop}}(t_0,\chi)=R(t_0)=\int_0^\chi d\chi'=R_0\chi$$

- As co-moving χ is constant for a co-moving galaxy, we therefore have

$$\dot{d}_{\text{prop}}=\dot{R}\chi=H(t)R(t)\chi=H(t)d_{\text{prop}}$$

- Evaluating this at current time recovers Hubble's law, but in terms of unmeasurable quantities.

Linking χ, z, t and H

- Generally report the redshift of an object or event, as it is cosmologically invariant.
- Considering a galaxy at (z,χ,t) observed by us today at $(0,0,t_0)$ and using this;

$$dz=d(1+z)=d\left(\frac{R_0}{R}\right)=-\frac{R_0}{R^2}\dot{R}dt=-(1+z)H(z)dt \tag{2}$$

- Then noting that $ds^2=0$ for photons, we find using (1) and $R=\frac{R_0}{1+z}$ that

$$\frac{d\chi}{c}=\frac{dt}{R}=-\frac{dz}{R_0H(z)} \tag{3}$$

- Integrating, (2) and (3) we therefore find

$$\chi(t)=\int_t^{t_0}\frac{cdt'}{R(t')},\quad \chi(z)=\int_0^z\frac{cdz'}{R_0H(z')},$$
$$\text{and}\quad t_0-t=\int_t^{t_0}dt=\int_0^z\frac{dz}{(1+z)H(z)}$$

- These therefore relate χ, t, z and $H(z)$.
- In the EdS universe with $R\propto t^{2/3}$,

$$H=\frac{\dot{R}}{R}\propto t^{-1}\propto R^{-3/2}\propto(1+z)^{3/2}$$
$$\implies H(z)=H_0(1+z)^{3/2}$$

- Therefore, using the above and $H=\frac{2}{3t}\implies H_0=\frac{2}{3t_0}$ from here, we find;

$$\chi^{\text{EdS}}=\frac{3ct_0}{R_0}\left(1-\left(\frac{t}{t_0}\right)^{1/3}\right)=\frac{3ct_0}{R_0}\left(1-\frac{1}{\sqrt{1+z}}\right)$$

- Photons therefore appear to travel superluminally with proper distance $3ct_0$ with a maximum co-moving radius at infinite redshift.
- Also note that the Friedmann-Lemaître equation can be expressed in terms of redshift by using $a = (1+z)^{-1}$;

$$H = H_0 \left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0} \right)^{1/2}$$

  - In most cases, only a couple of these components are active or dominant and can be combined with the above for χ and R to fully understand a situation.

Luminosity Distance

- In Euclidean space, the flux from a source of luminosity L is (below) therefore allowing for the definition of an Euclidean luminosity distance d_L;

$$F = \frac{L}{4\pi d^2} \implies d_L = \left(\frac{L}{4\pi F} \right)^{1/2}$$

- Considering a general spacetime, we find
  1. the general surface area of a sphere in the FLRW metric is $4\pi(R_0 S(x))^2$.
  2. The energy of a red-shifted photon is $E_{\text{obs}} = \frac{E_{\text{em}}}{1+z}$.
  3. The rate at which photons arrive at the observer decreases $\propto (1+z)^{-1}$
  - Combining these, we find;

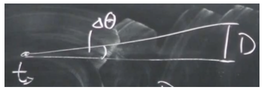
$$F = \frac{L}{4\pi (R_0 S(x))^2} (1+z)^{-2}$$

  - Therefore, the luminosity distance is

$$d_L = \left(\frac{L}{4\pi F} \right)^{1/2} = R_0 S(\chi)(1+z)$$

    - We can then use a standard candle with known L to calculate χ from observables - or can use this to fit for a cosmology.

Angular Diameter Distance



- In Euclidean space, considering the geometry above gives

$$\Delta\theta = \frac{D}{d} \implies d_\theta = \frac{D}{\Delta\theta}$$

- Then from the angular part of the FLRW metric, we see that the proper diameter D is given at the point of emission by;

$$D = R(t_1) S(\chi_1) \Delta\theta \implies d_\theta(t_0, \chi_1) = R(t_1) S(\chi_1)$$

  - Using the redshift relation $1 + z = R(t_0)/R(t_1)$, this can then be rewritten as

$$d_\theta = \frac{D}{\Delta\theta} = \frac{R(t_0) S(\chi)}{1+z}$$

    - We can find D using a standard ruler, allowing for measurement of geometric effects, or inversely, to determine χ from measurements.

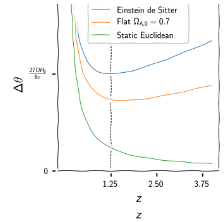
- We then find the relation between the luminosity and angular diameter distances;

$$d_L = d_\theta (1+z)^2$$

- Considering the angular diameter distances in an EdS universe where $S(\chi) = \chi$ gives

$$\chi^{\text{EdS}}(z) = \frac{2c}{R_0 H_0} \left\{ 1 - \frac{1}{\sqrt{1+z}} \right\}$$
$$\implies \Delta\theta = \frac{D(1+z)}{R_0 \chi} = \frac{DH_0(1+z)}{2c[1-(1+z)^{-1/2}]}$$

  - Interestingly, this increases with higher redshift with a minimum at $z = 1.25$.



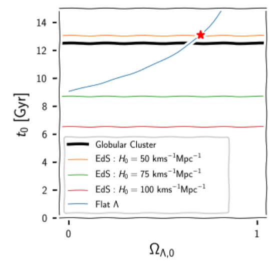
  - A similar result is found for flat Λ, shown in orange.

Ages in the Universe

- Using different cosmologies, we find different ages for the universe;

Type ($w = 0$)	Age t_0
Einstein de Sitter	$\frac{2}{3H_0}$
Flat Λ	$\frac{2\tanh^{-1}\sqrt{\Omega_{\Lambda,0}}}{3H_0\sqrt{\Omega_{\Lambda,0}}}$
de Sitter / Einstein	∞
Closed Friedmann	$\frac{q_0\cos^{-1}\left(\frac{1}{q_0}-1\right)-(2q_0-1)^{1/2}}{H_0(2q_0-1)^{3/2}}$
Open Friedmann	$\frac{(1-2q_0)^{1/2}-q_0\cosh^{-1}\left(\frac{1}{q_0}-1\right)}{H_0(1-2q_0)^{3/2}}$

  - We need these predictions to fall within other constraints.
    - One such constraint is the age of globular clusters, which can be determined by their 'turn off point' from main sequence stars in a Hertzprung-Russel diagram. These set a lower bound at around $12.5 \pm 1.5 \times 10^9$ years.
    - Similarly, we can use uranium radioactive dating to set a bound of $9 - 15$ Gyr.
    - The current favoured solution is with $\Omega_{\Lambda,0} = 0.7$ and $H_0 = 72 \text{ kms}^{-1}\text{Mpc}^{-1}$ giving $t_0 = 13.1 \text{ Gyr}$;

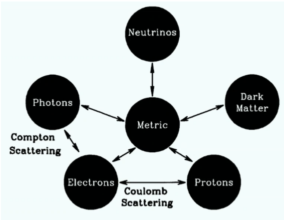


L20 - Constituents of the Universe

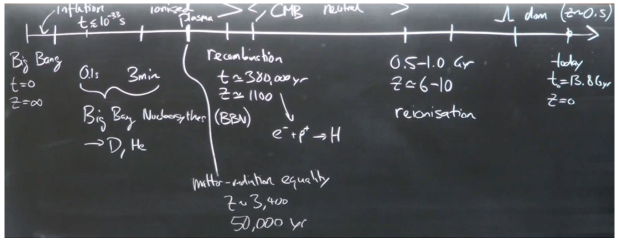
L19 - Measuring the Universe L21 - Modern Cosmological Data

Universe Evolution

Composition



- The universe is composed of the above constituents - note that Neutrinos are important in early universe analysis as they are relativistic particles. However, we will mostly neglect their contribution in this analysis.
- The composition of the universe changes with time according to the evolution of each the density constituent ρ_i. We can map this out as below.



  - Recombination of protons and electrons occurred around 380,000 years after the big bang, reducing Compton scattering and therefore allowing light to propagate through the universe - this results in the emission of the CMB from the last scattering atoms.
- We can calculate the number density of photons in the CMB using blackbody theory as;

$$n_\gamma = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2 d\nu}{e^{h\nu/kT}-1} \approx 4.2\times 10^8 \mathrm{m}^{-3}$$

  - The number density of protons can be determined using the baryonic matter density parameter $\Omega_{b,0}$ and the critical density of the Universe;

$$n_p = \frac{\rho_{0,\mathrm{matter}}}{m_p} = \frac{3H_0^2\Omega_{b,0}}{8\pi G m_p} \tag{1}$$

  - We therefore have $n_\gamma/n_p = 1.67\times 10^{-9}$. This is thought to be related to the matter / anti-matter asymmetry, as in particle physics.
    - For a matter-antimatter asymmetry, we must have the Sakharov conditions; Baryon number violation, C- and CP-symmetry violation and non-equilibrium.

Temperature in the Early Universe

$$\rho = aT^4/c^2 \tag{2}$$

- The energy density from black body radiation is given by (2). Also, using $\rho_i \propto R^{-3(1+w_i)}$ with $w_i = \frac{1}{3}$ (from L17) for a radiation dominated early universe, we find $\rho \propto R^{-4}$.
  - Therefore, $RT = \mathrm{const} \implies \dot{R}T + R\dot{T} = 0 \implies \frac{\dot{R}}{R} = -\frac{\dot{T}}{T}$.
  - Ignoring spatial curvature (as is justified in the early universe), the velocity equation combined with the above gives;

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} \implies \left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi GaT^4}{3c^2} \implies T = \left(\frac{3c^2}{32\pi Ga}\right)^{1/4} t^{-1/2}$$

  - We can therefore parameterise the cosmic epoch by time with $t = 2.3\ \mathrm{s}\left(\frac{10^{10}\ \mathrm{K}}{T}\right)^2$, giving the relationships;

$$T \sim (1+z) \sim \frac{1}{a} \sim \frac{1}{R} \sim t^{-1/2} \sim \rho^{1/4} \sim E \tag{3}$$

  - Furthermore, using $E = k_BT \implies 1\ \mathrm{K} = 8.6\times 10^{-14}\ \mathrm{GeV}$ gives $E = (1.3\ \mathrm{MeV})(t/1\ \mathrm{s})^{-1/2}$.
    - This gives the time scale for grand unification around 10^{14} GeV at $t_{\mathrm{GUT}} = 2\times 10^{-34}$ s and around $t = 1$ s energy is approximately that needed for electron-positron pairs, allowing them to annihilate into radiation.

The Universe Timeline

Big-Bang Nucleosynthesis (Kinda non-examinable)

- From around 1 second to serval minutes after the big bang, the universe has cooled sufficiently to allow protons and neutrons to exist whilst also having enough energy to participate in nuclear reactions;

$$\begin{aligned} n &\leftrightarrow p + e^- + \bar{\nu}_e \\ p + \bar{\nu}_e &\leftrightarrow n + e^+ \\ p + e^- &\leftrightarrow n + \nu_e \end{aligned}$$

  - The thermal equilibrium of these reactions gives $N_n/N_p \sim e^{-\frac{1.5}{T/10^{10}\ \mathrm{K}}} = e^{-1.5/T_{10}}$. This would result in the decay of all neutrons, but the universe expands fast enough that the chance of particles colliding for such reactions to occur decreases, decreasing reaction efficiency - freezing them out. At this point, neutrons will decay at their free particle decay rate, which would still result in them decaying with a timescale of $\tau_n = 887$ s.
  - We obviously still have neutrons in the universe - this is because they reacted to form stable nuclei by the processes;

$$\begin{aligned} \text{D production} \quad & n + p^+ \leftrightarrow D + \gamma, \\ \text{He production} \quad & D + p^+ \leftrightarrow {}^3\mathrm{He} + \nu, \\ & D + {}^3\mathrm{He} \leftrightarrow {}^4\mathrm{He} + p^+. \end{aligned}$$

    - Note however that these processes cannot really start until $0.1\mathrm{MeV} \sim 330$ s due to the Deuterium bottleneck - there are too many photons, keeping the first reaction well away from forming Deuterium. After this time, all of the reactions above occur rapidly.
    - The helium mass fraction therefore depends on the initial baryon density and so can be used to investigate.

Epoch of Equivalence

- This is where the energy density of radiation and matter become equal; $\Omega_r = \Omega_m$. This occurs some time between nucleosynthesis and recombination.
  - Using $T = T_0(1+z)$ from (3) and (2) we find $\rho_{rad}c^2 = aT_0^4(1+z)^4$.
  - Noting that $\rho_m \propto R^{-3}$, using (3) and (1) we find $\rho_m = \rho_{m0}(1+z)^3 = \Omega_{m0}\frac{3H_0^2}{8\pi G}(1+z)^3$. Combining with the above gives;

$$(1+z_{eq}) = \frac{3\Omega_{m0}H_0^2c^2}{8\pi G a T_0^4}$$

    - Finally, noting that $H_0 = h \times 100 \text{ km s}^{-1}\text{Mpc}^{-1}$ gives $1 + z_{ez} = 4 \times 10^4 \Omega_{m0} h^2$.

Recombination

- As the universe cooled, there was a point where protons and electrons could combine to form neutral hydrogen. However, this is a complicated process - if an electron and proton recombine directly to the ground state then they emit a photon of 13.6 eV which can then ionise another hydrogen atom.
- A reasonable approximation can be found by requiring the number of photons with energy above 13.6 eV is low enough to prevent further ionisation. Take this to be where the number of high-energy photons per proton is less than one;

$$\exp\left(-\frac{13.6 \text{ eV}}{k_B T_{rec}}\right) < 10^{-9}$$
$$\implies T_{rec} \sim 7600 \text{ K}$$

CMB Emission

- After recombination, photons no longer couple to the (now neutral) Hydrogen, so begin to pass through space unobstructed. It was in thermal equilibrium with the universe when it was emitted - giving a black-body spectrum. Interestingly, it is still detected as a black-body spectrum despite now being in extreme disequilibrium with the matter in the universe;
  - Considering the number of photons in a comoving volume element $V(t)$ should be conserved if the photons travel unhindered from recombination (t_1) to the present time (t_0);

$$n_{\nu_1}(t_1)d\nu_1(t_1)V(t_1) = n_{\nu_0}(t_0)d\nu_0(t_0)V(t_0)$$

  - Then considering that $\nu \propto 1/R$ due to redshift, we have $d\nu \propto \frac{1}{R}$, so

$$n_{\nu_0}(t_0) = n_{\nu_1}(t_1)\left(\frac{R_0}{R_1}\right)\left(\frac{R_1}{R_0}\right)^3 = \left(\frac{R_1}{R_0}\right)^2 n_{\nu_1}(t_1)$$

  - Given that the number density of photons at recombination is given by the Planck distribution as;

$$n_\nu = \frac{8\pi\nu^2}{c^3\left(e^{\frac{h\nu}{kT}} - 1\right)}$$
$$\implies n_{\nu_0}(t_0) = \left(\frac{R_1}{R_2}\right)^2 \frac{8\pi\nu_1^2}{c^3}\left[\exp\left(\frac{h\nu_1}{kT_1}\right) - 1\right]^{-1}$$

    - Finally then noting that redshift gives $\nu_1 = \frac{R_0}{R_1}\nu_0$, we find;

$$n_{\nu_0}(t_0) = \frac{8\pi\nu_0^2}{c^3}\left[\exp\left(\frac{h\nu_0}{kT_1\frac{R_1}{R_0}}\right) - 1\right]^{-1}$$

    - This still has a Planckian form, so the present temperature is related to that at recombination by;

$$T_0 = T_1\frac{R_1}{R_0} = \frac{T_1}{1+z}$$

Reionisation

- It is believed that by around $z = 6$, the universe had reionised due to UV emission by early stars and quasars ionising the surrounding matter.
- The optical depth of the universe is a measure of the opaqueness of the universe to radiation from scattering by free electrons. It is calculated as using free-electron number density $n_e(\chi)$ and coordinate distances χ;

$$\tau(\chi) = \int_0^\chi n_e(\chi)\sigma_T R d\chi'$$

  - This corresponds to counting the number of scattering centres in a tube of constant proper area σ_T between the observer and coordinate radius χ. Note $Rd\chi$ is the element of proper distance at χ.
  - If the change from neutral to ionised matter was approximately instant, then (using this),

$$\tau(z_r) = \int_0^{z_r} n_e(z)\sigma_T R(z)\frac{d\chi}{dz}\,dz = \int_0^{z_r} n_e(0)(1+z)^3\sigma_T R(z)\frac{c}{R_0 H(z)}\,dz,$$
$$= \int_0^{z_r} \frac{n_e(0)(1+z)^2\sigma_T c}{H(z)}\,dz.$$

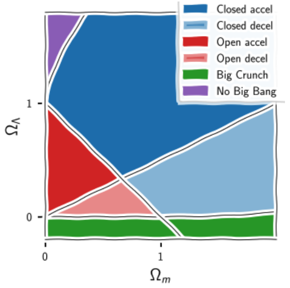
Dark Matter

- Estimates of the masses of stars in galaxy clusters give $M_\star \sim 10^{13} - 10^{14} m_\odot$, whereas predictions with virial theorem give $M \approx v^2R/G$, giving $M_{dyn} \sim 10^{14} - 10^{15}M_\odot$, so $M_{cluster} \gg \sum M_{galaxies}$.
- Rotational velocity around galaxies also don't obey Kepler's law.
  - Both suggest invisible matter - called dark matter.

L21 - Modern Cosmological Data

L20 - Constituents of the Universe L22 - The Primordial Universe

Parameterising The Universe



- When $\Omega_r \approx 0$, the universe is completely determined by its place in the $(\Omega_m, \Omega_\Lambda)$ plane.
  - Here, we found that

$$1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda$$

  - Hence, for a flat Λ universe with $\Omega_k = 0$, we require $\Omega_\Lambda + \Omega_k = 1$. With $\Omega_\Lambda + \Omega_k > 1$, we have a closed universe and for $\Omega_\Lambda + \Omega_k < 1$ we have an open universe.
  - For a Friedmann universe we have $\Lambda = \Omega_\Lambda = 0$.
  - Recalling the definition of q from here, we require that $q = 0 \implies 0 = \frac{1}{2}\Omega_m - \Omega_\Lambda$ and that for $q > 0$, the universe is accelerating and for $q < 0$, the universe is decelerating.
  - Also recalling the conditions for the de Sitter universe, with $\Omega_\Lambda = 1$ and $\Omega = m = 0$ gives a boundary to where no big bang occurred.

The CMB

- Has μK scale anisotropies, but these anisotropies are not white noise. We therefore wish to mathematically analyse these fluctuations $f(\vec{x}, t)$ varying with $\vec{x}$ at a given time t.
  - Define these fluctuations as

$$f(\vec{x}) = \frac{\rho(\vec{x}) - \langle\rho\rangle}{\langle\rho\rangle} = \delta(\vec{x})$$

  - If we then wish to describe this using a probability distribution where we have measurements at positions indexed $1, 2, \ldots N$ with $N \to \infty$, we require

$$P(f_1, f_2, \ldots, f_N)$$

    - The form of P can be determined if all of its moments are known;

$$\langle f_1^{l_1} f_2^{l_2} \ldots f_N^{l_N} \rangle = \int f_1^{l_N} \ldots f_N^{l_N} P(f_1, \ldots f_N) df_1 \ldots df_N, \quad l_1, \ldots l_N = 0, \ldots N$$

  - In cosmology, we have a Gaussian Random Field which is fully characterised by the two-point correlation function or the correlation function (power spectrum), which simplifies the above to;

$$P(f, f_N) = \frac{e^{-D}}{\left[(2\pi)^N \det(M)\right]^{1/2}}$$
$$\text{with} \quad D = \frac{1}{2}\sum_{ij} \delta_i (M^{-1}) \delta_j \quad \text{and} \quad M_{ij} = \langle \delta_i \delta_j \rangle$$

- We can then write the two point correlation function $\xi(\vec{x}) = \xi(r) = \langle f_1 f_2 \rangle$, where $r = |\vec{x}|$ follows from homogeneity. Finally, we can then relate this to the power spectrum $P(k)$ as;

$$\xi(r) = \int \frac{dk}{k} P(k) \frac{\sin(kr)}{r} \quad \text{with} \quad P(k) \propto |f(k)|^2$$

  - In the linear regime, these Fourier modes each evolve independently.

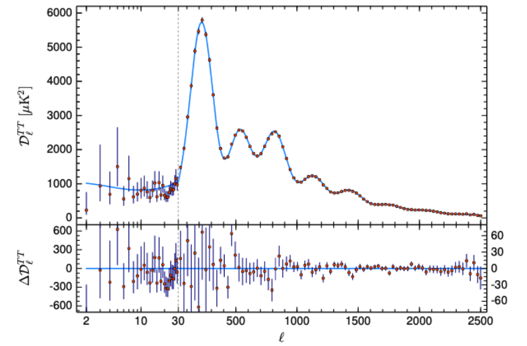
- Working on the sky, we use the appropriate transform for a sphere using spherical harmonics;

$$T(\theta, \phi) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi) \quad \leftrightarrow \quad a_{lm} = \int d\Omega Y^*_{lm}(\theta, \phi) T(\theta, \phi)$$

- Note that the reality of T requires $a^*_{lm} = a_{l-m}$, that for each l there are $2l+1$ different a_{lm}, and that the power spectrum for a spherical decomposition is $|a_{lm}|^2$. We can then calculate the variance for each a_{lm} by averaging over each multipole l;

$$\hat{C}_l = \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2 \sim C_l \pm \sqrt{\frac{C_l}{2l+1}}$$
$$\text{as} \quad \frac{\Delta C_k}{C_l} = \frac{\sqrt{\langle (C_l - \hat{C}_l)^2 \rangle}}{C_l} = \sqrt{\frac{2}{2l+1}}$$

  - At lower l, we see that the intrinsic scatter here gets larger with smaller l (larger spatial scale) - this is cosmic variance.
  - We measure CMB isotropies (here with $D_l = C_l \frac{l(l+1)}{2\pi}$) and find the spectrum below. The largest peaks are from the patchiness in the CMB.



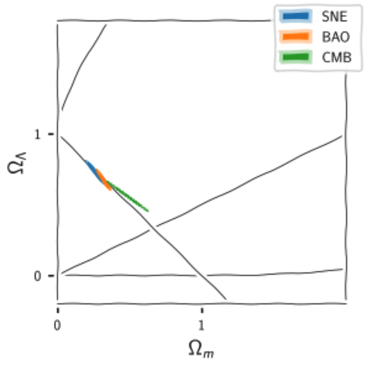
  - If the CMB were white noise then this would be expected to be flat $\sim l(l+1)$.
  - Using the Λ-CDM model, we can tune cosmological parameters to fit this spectrum.

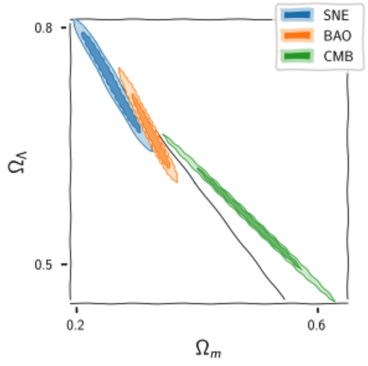
- It is also possible to measure the different polarisations of the CMB and compute the covariance between the different components;

$$C_l^{XY} = \frac{1}{2l+1}\sum_{m=-l}^{l} a_{lm}^{X*} a_{lm}^{Y}$$

Standard Rulers and Standard Candles

- Can use white dwarfs undergoing type Ia supernovae at their Chandrasekhar mass as standard candles.
  - Can standardise their light curves but potentially still have large systematic errors.
- Also possible to measure quasars and luminous red galaxies (LRGs) at high redshift to try to measure Baryon Acoustic Oscillations.
  - Prior to recombination, photons were tightly coupled to baryons in the plasma, which itself has a high and predictable sound speed around c.
  - This results in a length-scale associated with the wavelength - $L \sim ct_* \approx 450\ \mathrm{kly}$, which can act as a standard ruler.
    - This length-scale is "frozen out" after recombination and is statistically visible in distribution of galaxies - observable from the 3D spatial power spectrum of galaxies.
- Modern measurements from each of these methods disagree;





L22 - The Primordial Universe

L21 - Modern Cosmological Data L23 - The Perturbed Universe

Types of Horizon

Particle Horizon

- This is the furthest a particle can have travelled in the lifetime of the universe;

$$ds^2 = 0 \implies Rd\chi = \pm cdt$$
$$\implies \boxed{\chi_p = \int_0^{t_0} \frac{cdt}{R(t)}} \quad (= \eta,\ \text{conformal time})$$
$$\implies \chi_p^{EdS} = \begin{cases} 3ct_0 & \text{matter dom} \\ 2ct_0 & \text{rad dom} \end{cases} \tag{1}$$

  - Note that these EdS forms with $\Lambda = k = 0$ follow from here (rad dom) $\implies R_{rad} \propto t^{1/2}$ and here with $\rho \propto \frac{1}{R^3}$ (matter dom) $R_{mat} \propto t^{2/3}$.
  - By symmetry this also gives the furthest we can see.

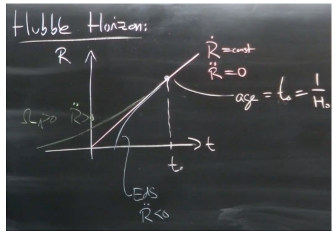
Event Horizon

- The even horizon is the furthest a photon will be able to travel by the end of the universe;

$$\boxed{\chi_e = \int_{t_0}^{\infty} \frac{cdt}{R(t)}}$$

  - By symmetry, we will never see objects further away than this.
  - In EdS, we have $\chi_e = \infty$. However, χ_e is finite for universes with $\Lambda \neq 0$ (ΛCDM gives $z_e \approx 1.8$, so we can never reach beyond this).

Hubble Horizon



Depending on the acceleration $\ddot{R}$, the real age is larger or smaller than that given by H^{-1}.

- The Hubble horizon isn't really a horizon, more a definition - equivalent to the Hubble radius;

$$\boxed{R_h = cH^{-1}}$$

  - Objects further than the Hubble radius have a recession velocity greater than the speed of light. We can also define the **co-moving Hubble radius;**

$$\chi_h = \frac{R_h}{R} = \frac{c}{RH} = \frac{c}{\dot{R}}$$

Properties of Horizons

- Considering the rate of change of the particle horizon, we find;

$$\frac{d\chi_p}{dt_0} = \frac{c}{R(t_0)} > 0$$

  - Therefore areas of the universe slowly come into view with time.
  - Using the result below from L19,

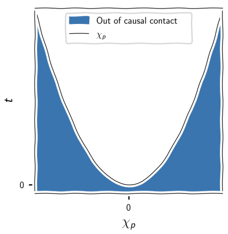
$$\chi^{\text{EdS}} = \frac{3ct_0}{R_0}\left(1 - \left(\frac{t}{t_0}\right)^{1/3}\right) = \frac{3ct_0}{R_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

    - Then taking $R_0 = 1$ by convention and equating with the $\chi_p^{EdS} = 3ct_0$ matter result from (1), we require $z \to \infty$.
    - Therefore, when object appear over the horizon they have infinite redshift, so don't suddenly appear.
  - Similarly, considering the velocity of the event horizon we see

$$\frac{d\chi_e}{dt} = -\frac{c}{R} < 0$$

    - Therefore, the event horizon is approaching us - this however is a future surface, not something we see.

Particle Horizon in w-Cosmologies



- Recalling that in a w-cosmology, we have (first from here);

$$\frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, \quad \implies H = \frac{\dot{R}(t)}{R(t)} = \frac{2}{3(1+w)t}$$
$$\implies \chi_p = \int_0^{t_0} \frac{c}{R(t)}dt = \frac{3ct_0}{R_0}\frac{1+w}{1+3w} = \frac{2cH_0^{-1}}{R_0(1+3w)} = \frac{2\chi_h}{1+3w}$$

  - This is finite for $w > -\frac{1}{3}$ (note $t^{\frac{2}{3(1+w)}}$ must be integrable at $t = 0$). Therefore, every Friedmann model has a particle horizon at the early stages where $w = 1/3$. Points beyond the χ_p horizon have never been in causal contact - potentially unexpected giben the universe comes from an initial singularity.
    - This causal disconnection is due to the very large spacetime curvature $\ddot{R}/R$ in the early universe.

Inflation

Problems

- Observations of the CMB show it to be near-perfectly uniform. However, assuming the universe is radiation dominated before recombination gives $R \propto t^{1/2}$, gives that at recombination, $\chi_p = 2ct_{rec}$.
  - Also assuming that the universe then expanded with $R \propto t^{2/3}$ after recombination gives the angle subtended by this distance today as $\Delta\theta = \frac{2}{3}(1+z_{rec})^{-1/2}$ - around 1°

  - This suggests photons from different 1° regions have never been in causal contact - problematic for how these regions synchronise their temperatures so accurately at recombination.
- Without a cosmological constant, and without exact parameter fitting, the universe undergoes a runaway behaviour with Ω_m moving increasingly rapidly away from flatness.
  - Using $\Omega_k = -\frac{kc^2}{(HR)^2}$ from here, we can find that

$$\frac{d\log|\Omega_k|}{d\log R} = (1-\Omega_k)(1+3w)$$

    - This implies the flatness of the universe has always increased exponentially, so must have been tiny initially - why would it be this way?

The Solution - Acceleration with $w < -\frac{1}{3}$

- If $w < -\frac{1}{3}$, which implies $P < -\frac{1}{3}\rho c^2$, the particle horizon diverges and the curvature parameter is driven towards flatness. The acceleration equation then gives

$$\frac{\ddot{R}}{R} = -\frac{8\pi G}{c^2}(P + \frac{1}{3}\rho c^2)$$

  - Substituting for P, we see $\ddot{R} > 0$, so the universe is accelerating.
  - Note, an accelerating universe causes the Hubble horizon to shrink with $\dot{\chi}_h = -\frac{\ddot{R}c}{\dot{R}^2}$, so the universe expands superluminally.
    - Hence, material is carried far outside its usual causal patch and spatial curvature is flattened.
- It is thought this was manifested as a period of hyperinflation in the early universe.
  - Note, Ekypyrotic models such as a bouncing universe can solve some of these issues, but require curvature, so don't solve the flatness problem.

Scalar Fields in Cosmology

- Setting $c = \hbar = 1$ and $\Lambda = 0$, and noting that $\nabla = \partial + \Gamma$, we may write the Lagrangian, energy-momentum tensor and equation of motion;

$$\mathcal{L} = \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi), \quad T^{\mu\nu} = \nabla^\mu\phi\nabla^\nu\phi - \left(\frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)\right)g^{\mu\nu}$$
$$\nabla^\mu\nabla_\mu\phi + \frac{d}{d\phi}V(\phi) = 0$$

  - Where $V(\phi) = \frac{1}{2}m^2\phi^2$, the latter equation is the Klein-Gordon equation for a free scalar field.
  - Assuming the scalar field is homogeneous and fills the universe, $\phi(x^\mu) = \phi(t)$ allows us to remove spatial derivatives, simplifying the stress-energy tensor;

$$T^{\mu\nu} = \dot{\phi}^2\delta_t^\mu\delta_t^\nu - \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)g^{\mu\nu}$$

    - Comparing with the form in L4, we see this is the same as a perfect fluid co-moving with fundamental observers with;

$$\boxed{u^\mu = \delta_t^\mu} \quad \boxed{\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)} \quad \boxed{P = \frac{1}{2}\dot{\phi}^2 - V(\phi)}$$

  - Considering the equation of motion and using $\nabla_t = \partial_t + 3H$ gives

$$\ddot{\phi} + 3H\dot{\phi} + \frac{d}{d\phi}V(\phi) = 0$$

- This is the equation of a particle in a potential with friction $\propto H$.
- This is also the continuity equation $\dot{\rho} = -3H(P+\rho)$ divided by $\dot{\phi}$.

- Using the velocity equation from L18, we see $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2}$, giving

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{k}{R^2}$$

- Using the acceleration equation from L18, we see $\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P)$, giving

$$\frac{\ddot{R}}{R} = \dot{H} + H^2 = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V(\phi)\right)$$

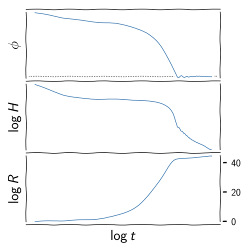
- If the field is trapped in a local minimum V_0 with $\dot{\phi} = 0$, then we must have

$$H = \sqrt{\frac{8\pi G V_0}{3}}$$

- This constant H implies exponential growth characteristic of de Sitter evolution;

$$R(t) \propto e^{Ht}$$

- Alternatively, we could consider ϕ perturbed away from a minimum such that it rolls downhill being slowed by friction until $\dot{\phi}^2 \ll V(\phi)$ - the slow-roll approximation.
  - This then gives $\frac{\dot{R}}{R} = H \approx \text{const}$, suggesting an approximate de Sitter state with $R \sim e^{Ht}$. This gives solutions like those below;



  - This approach drives the universe towards flatness, dilutes monopoles and topological defects and adds enough conformal time to resolve the horizon problem. However, it also introduced a new scalar field - the inflaton ϕ.
  - The primary backing for the inflation model is how it predicts exactly how non-universe the universe is afterwards.

L23 - The Perturbed Universe

L22 - The Primordial Universe

Perturbations

Jeans Analysis

$$\begin{aligned} \text{Continuity:} \quad & \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0, \\ \text{Euler:} \quad & \rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P - \rho\nabla\Phi, \\ \text{Poisson:} \quad & \nabla^2\Phi = 4\pi G\rho. \end{aligned}$$

- Considering the key fluid equations above, solving for a static an homogenous universe where ρ and P are constant and $\vec{v} = 0$, we require $\nabla\Phi = 0 \implies \nabla^2\Phi = \rho = 0$ - ie an empty universe. Hence, the universe must be either expanding or contracting.
- The Jeans swindle ignores this problem and proceeds anyway with perturbation analysis, setting

$$\begin{aligned} \rho &= \rho_0 + \delta\rho, \\ P &= P_0 + \delta P, \\ \vec{v} &= \delta\vec{v}, \\ \Phi &= \Phi_0 + \delta\Phi. \end{aligned}$$

  - Substituting into the fluid equations we find (respectively)

$$\begin{aligned} \frac{\partial \delta\rho}{\partial t} &= -\nabla\cdot(\rho_0\vec{v}), & (1) \\ \rho_0\frac{\partial \vec{v}}{\partial t} &= -\nabla(\delta P) - \rho_0\nabla(\delta\Phi), & (2) \\ \nabla^2(\delta\Phi) &= 4\pi G\delta\rho & (3) \end{aligned}$$

  - To solve we require an equation of state, so take $P = P(\rho)$, giving

$$\delta P = \frac{\partial P}{\partial \rho}\delta\rho = c_s^2\delta\rho$$

- Considering a combination of $\partial_t(1)$ and $\nabla(2)$, we find the equation describing density fluctuations in $\delta\rho$;

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 - 4\pi G\rho_0\right)\delta\rho = 0$$

- Supposing each perturbation evolves like $\delta\rho \to \delta\rho e^{i(kx-\omega t)}$ we can rewrite this as

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2k^2 - 4\pi G\rho_0\right)\delta\rho = 0$$

- Then defining the Jeans scale such as to differentiate between which term is dominant;

$$k_J = \frac{\sqrt{r\pi G\rho_0}}{c_s}$$

    - If $k \gg k_J$, pressure dominates, giving oscillatory solutions.
    - If $k \ll k_J$, gravity dominated, giving exponential growth with $\delta\rho \propto e^{\pm t/\tau}$ with $\tau = \sqrt{4\pi G\rho_0}$.
  - This can also be interpreted with $\omega^2 = c_s^2(k^2 - k_J^2)$ with $c_s^2 = P_0/\rho_0$.
  - Also note we can define the Jeans mass as $M_J = \frac{4\pi}{3}\lambda_J^2\rho_0$.
- Considering this again but instead in an expanding universe, and defining $\delta = \delta\rho/\rho_0$, we find;

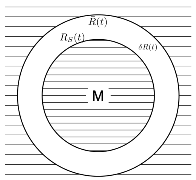
$$\ddot{\delta} + 2H\dot{\delta} - \left(\frac{c_s^2}{R^2}\nabla^2 + 4\pi G\rho_0\right)\delta = 0$$

  - We can then consider a case like $k/R \ll k_J$ which gives

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_0\delta = 0$$

    - Here, perturbations struggle to grow if they are too small. We find that all of those below the scale of a galaxy cluster do not grow effectively.

Collapse of Spherical Overdensities



- Considering a spherical, overdense region and noting that in the spherically symmetric case, material inside a sphere knows nothing about what is outside (Gauss and dynamical equations), we conclude that the equations governing the evolution of the radius of the sphere R_s must be the dynamical equations for the scale factor.
  - Using the acceleration equation for an EdS universe and defining $\delta R = R_s - \bar{R}$, we find (where the equality follows from the Friedmann equation);

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\rho}{3} = -\frac{GM}{R^3}$$
$$\implies \delta\ddot{R} = \ddot{\bar{R}} - \ddot{R}_s = -GM\left(\frac{1}{\bar{R}^2} - \frac{1}{R_s^2}\right) \approx \frac{2GM}{\bar{R}^3}\delta R = \frac{8\pi G\bar{\rho}}{3}\delta R = \bar{H}^2\delta R$$

    - Hence, for an EdS universe, we have

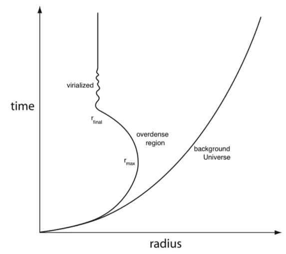
$$\frac{d}{dt^2}(\delta R) \approx \bar{H}^2\delta R$$

  - For matter dominated EdS we then have $H = \frac{2}{3t}$, so $\delta\ddot{R} = \frac{4}{9t^2}\delta R$. Then trying solutions of the form $\delta R = t^n$, we find $n = 4/3,\ -1/3$.
    - For the growing $n = 4/3$ mode, and noting that for matter dominated EdS, $\bar{R} \propto t^{2/3}$;

$$\frac{\delta R}{\bar{R}} = -\frac{1}{3}\frac{\delta\rho}{\bar{\rho}} \propto \frac{t^{4/3}}{t^{2/3}} \propto t^{2/3} \propto \bar{R}(t)$$

    - These perturbations therefore grow algebraically with time, so we require much larger density perturbations in the early universe than required by a static-universe-based Jeans analysis.

Overdensity Collapse



The stages of collapse for a perturbation. Divergence occurs around $\frac{\delta\rho}{\rho} \sim 1$.

- At r_{max}, most energy is in potential form with $V \sim -\frac{GM}{r_{max}}$, whereas at r_{final}, the system is virialised with $2T + V = 0 \implies E = T + V = \frac{V}{2}$.
  - Therefore, equating gives;

$$\boxed{r_{\text{final}} = \frac{1}{2}r_{\text{max}}}$$

  - This predicts the mean density after collapse to be $8\times$ that at its maximum size.
  - Simulations find the virialised object to have density around $200\times$ the background density, and allow us to estimate the redshift where galaxies formed;
    - Noting that the Virial theorem gives $v_c = \sqrt{\frac{GM}{r}}$ and we have $\rho_b = \frac{3H_0^2}{8\pi G}\Omega_b(1+z_f)^3$, we can link the size and velocity of clusters to their formation z_f;

$$\frac{\rho}{\rho_b} = \frac{2v_c^2}{\Omega_b(H_0 r)^2(1+z_f)^3} >\approx 200$$

The Need for Dark Matter

- In the early universe, when perturbations arise, radiation is dominant and is is thermal equilibrium with matter. We therefore assume adiabatic perturbations with

$$-\frac{1}{4}\frac{\delta\rho_r}{\rho_r} = \frac{\delta R}{R} = \frac{\delta T}{T} = -\frac{1}{3}\frac{\delta\rho_m}{\rho_m}$$

  - Note that these follow from $\rho_r \propto R^{-4}$ and $\rho_m \propto R^{-3}$.
  - We also know that $\delta T/T \sim 10^{-5}$ at recombination, giving $\delta\rho_m/\rho_m \sim 3\times 10^{-5}$ at $z \sim 1000$. Propogating to today $\propto (1+z)$ gives $\delta\rho_m/\rho_m \sim 0.03$.
    - This is $\ll 1$, so we should expect bound structures to not yet have formed. However, we observe bound structures with $\delta\rho/\bar{\rho} > 200$!
    - We use dark matter to solve this issue, forming potential wells far before recombination (as it decouples earlier), which baryons then fall into.
    - These dark matter fluctuations therefore give rise to the bound non-linear structures in the universe.