PS1

Q3: Exercise 5

Profit maximization of the firm over time

From the assumption of the marginal costs which are ϵ for the period 1, and $c - \lambda q_1$ for the period 2, we can get the cost functions are

$$C(q_1) = \epsilon q_1 + Const_1$$
$$C(q_1, q_2) = q_2(c - \lambda q_1) + Const_2$$

for $q_1 > 0$ and $q_2 > 0$ where $Const_1$ and $Const_2$ are constant variables. And let $q_1, q_2 > 0$, $p_1, p_2 > 0$. This means $0 < q_1 < 1$, $0 < q_2 < 1$, $0 < p_1 < 1$, and $0 < p_2 < 1$ from $q_1 = 1 - p_1, \quad q_2 = 1 - p_2$.

Next, the profit maximization of the firms over time is

$$\max_{p_1, p_2} (p_1 q_1 - C_1(q_1)) + (p_2 q_2 - C(q_1, q_2))$$
$$s.t. q_1 = 1 - p_1, \quad q_2 = 1 - p_2$$

So, the FOC of this problem is

$$\mathcal{L} = p_1(1 - p_1) - (\epsilon(1 - p_1) + Const_1) + \delta(p_2(1 - p_2) - ((1 - p_2)(c - \lambda(1 - p_1)) + Const_2))$$
$$\frac{\partial \mathcal{L}}{\partial p_1} = 1 - 2p_1 + \epsilon - \delta(1 - p_2)\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial p_2} = 1 - 2p_2 + c - (1 - p_1)\lambda = 0$$

Compute the optimal pricing strategy

From the FOC, if $\lambda \neq 2$ (from the assumption, λ is smaller than 2), then we can get the optimal pricing strategies of p_1 and p_2 are

$$p_1 = \frac{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)}{\lambda^2 - 4}$$
$$p_2 = \frac{\delta\lambda^2 + (1 - \epsilon)\lambda - 2(1 + c)}{\lambda^2 - 4}$$

Compute the Lerner Index by each periods

And then, the Lerner Index for the period 1 is

$$\begin{aligned}
L_1 &= \frac{p_1 - \epsilon}{p_1} = 1 - \frac{\epsilon}{p_1} \\
&= 1 - \frac{\epsilon(\lambda^2 - 4)}{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)} \\
&= \frac{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1) - \epsilon(\lambda^2 - 4)}{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)} \\
&= \frac{(1 - \epsilon)\lambda^2 + (2\delta - 1 - c)\lambda - 2(1 - \epsilon)}{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)}
\end{aligned}$$

We also have the FOC for the period 1 is

$$D(p_1) + p_1 D'(p_1) - C_1' D_1'(p_1) - \delta C_2' D_1'(p_1) = 0$$

And so, the elasticity of demand for the period 1 is

$$\begin{aligned}
\Rightarrow -\frac{D(p_1)}{p^1 D'(p_1)} &= \frac{p_1 - C_1' - \delta C_2'}{p_1} \\
&= L_1 - \frac{\delta C_2'}{p_1} \\
&= L_1 + \frac{\delta(1-p_2)\lambda}{p_1} > 0
\end{aligned}$$

where $0 < p_1 < 1, 0 < p_2 < 1$.

Therefore, in this situation, the relative profit margin (Lerner Index for the period 1) is lower than the elasticity of demand for the period 1.

Also, the Lerner Index for the period 2 is

$$\begin{aligned}
L_2 &= \frac{p_2 - (c - \lambda(1-p_1))}{p_2} \\
&= 1 - \frac{c - \lambda(1-p_1)}{p_2} \\
&= 1 - \frac{2p_2 - 1}{p_2} \\
&= 1 - 2 + \frac{1}{p_2} \\
&= -1 + \frac{\lambda^2 - 4}{\delta\lambda^2 + (1-\epsilon)\lambda - 2(1+c)} \\
&= \frac{-\delta\lambda^2 - (1-\epsilon)\lambda + 2(1+c) + \lambda^2 - 4}{\lambda^2 + (1-\epsilon)\lambda - 2(1+c)} \\
&= \frac{(1-\delta)\lambda^2 - (1-\epsilon)\lambda - 2(1-c)}{\lambda^2 + (1-\epsilon)\lambda - 2(1+c)}
\end{aligned}$$

However, in the period 2, we can have the relative profit margin (Lerner Index for the period 2) is equal to the elasticity of demand for the period 2 because the FOC for the period 2 is

$$D(p_2) + p_2 D'(p_2) - C_2' D_2'(p_2) = 0$$

So, the elasticity of demand for the period 2

$$\begin{aligned}
\Rightarrow -\frac{D(p_1)}{p^1 D'(p_1)} &= \frac{p_1 - C_2'}{p_1} \\
&= L_2
\end{aligned}$$

Does output increase or decrease over time?

From the above result, the output for the period 1 (q_1) is

$$\begin{aligned}
q_1 &= 1 - p_1 \\
&= 1 - \frac{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)}{\lambda^2 - 4} \\
&= \frac{\lambda^2 - 4 - \lambda^2 - (2\delta - 1 - c)\lambda + 2(\epsilon + 1)}{\lambda^2 - 4} \\
&= \frac{(c + 1 - 2\delta)\lambda + 2(\epsilon - 1)}{\lambda^2 - 4}
\end{aligned}$$

From the above result, the output for the period 2 (q_2) is

$$\begin{aligned}
q_2 &= 1 - p_2 \\
&= 1 - \frac{\delta\lambda^2 + (1 - \epsilon)\lambda - 2(1 + c)}{\lambda^2 - 4} \\
&= \frac{\lambda^2 - 4 - \delta\lambda^2 - (1 - \epsilon)\lambda + 2(1 + c)}{\lambda^2 - 4} \\
&= \frac{(1 - \delta)\lambda^2 - (1 - \epsilon)\lambda + 2(c - 1)}{\lambda^2 - 4}
\end{aligned}$$

And then, the difference between the output for the period 1 and that for the period 2 is

$$\begin{aligned}
q_2 - q_1 &= \frac{(1 - \delta)\lambda^2 - (1 - \epsilon)\lambda + 2(c - 1)}{\lambda^2 - 4} - \frac{(c + 1 - 2\delta)\lambda + 2(\epsilon - 1)}{\lambda^2 - 4} \\
&= \frac{(1 - \delta)\lambda^2 - (1 - \epsilon) - (c + 1 - 2\delta) + 2c - 2\epsilon}{\lambda^2 - 4} \\
&= \frac{(1 - \delta)\lambda^2 + (-1 + \epsilon - c - 1 + 2\delta)\lambda + 2c - 2\epsilon}{\lambda^2 - 4} \\
&= \frac{(1 - \delta)\lambda^2 - (2\delta - 2 + c - \epsilon)\lambda + 2c - 2\epsilon}{\lambda^2 - 4} \\
&= \frac{\{(1 - \delta)\lambda + (c - \epsilon)\}(\lambda - 2)}{\lambda^2 - 4} \\
&= \frac{(1 - \delta)\lambda + (c - \epsilon)}{\lambda + 2}
\end{aligned}$$

Therefore, if $(1 - \delta)\lambda + (c - \epsilon) > 0$, then the output increases over time. Also, if $(1 - \delta)\lambda + (c - \epsilon) < 0$, then the output decreases over time. if $(1 - \delta)\lambda + (c - \epsilon) = 0$, then the output is equal overtime.

Supplement: Derivative of the prices

The price for the period 1 is

$$
\begin{aligned}
1-2p_1+\epsilon &= \delta(1-p_2)\lambda \\
\Leftrightarrow p_2 &= 1-\frac{1}{\delta\lambda}(1-2p_1+\epsilon) = \frac{\delta\lambda-1+2p_1-\epsilon}{\lambda} \\
1-2p_2+c &= (1-p_1)\lambda \\
\Leftrightarrow p_2 &= \frac{1}{2}(1+c-(1-p_1)\lambda) \\
2(\delta\lambda-1+2p_1-\epsilon) &= \lambda(1+c-(1-p_1)\lambda) \\
2\delta\lambda-2+4p_1-2\epsilon &= \lambda+\lambda c-\lambda^2+p_1\lambda^2 \\
(\lambda^2-4)p_1 &= \lambda^2+(2\delta-1-c)\lambda-2\epsilon-2 \\
p_1 &= \frac{\lambda^2+(2\delta-1-c)\lambda-2(\epsilon+1)}{\lambda^2-4}
\end{aligned}
$$

The price for the period 2 is

$$
\begin{aligned}
p_2 &= \frac{1}{2}(1+c-\left(1-\frac{\lambda^2+(2\delta-1-c)\lambda-2\epsilon-2}{\lambda^2-4}\right)\lambda) \\
&= \frac{1}{2}(1+c-\left(\frac{-(2\delta-1-c)\lambda^2+2\epsilon\lambda-2\lambda}{\lambda^2-4}\right)) \\
&= \frac{1}{2}\left(\frac{\lambda^2-4+c\lambda^2-4c+(2\delta-1-c)\lambda^2-2\epsilon\lambda+2\lambda}{\lambda^2-4}\right) \\
&= \frac{1}{2}\left(\frac{2\delta\lambda^2-4-4c+2\lambda-2\lambda\epsilon}{\lambda^2-4}\right) \\
&= \frac{\delta\lambda^2+(1-\epsilon)\lambda-2(1+c)}{\lambda^2-4}
\end{aligned}
$$