

# PS1

## Q3: Exercise 5

### Profit maximization of the firm over time

From the assumption of the marginal costs which are  $\epsilon$  for the period 1, and  $c - \lambda q_1$  for the period 2, we can get the cost functions are

$$\begin{aligned}C(q_1) &= \epsilon q_1 + Const_1 \\C(q_1, q_2) &= q_2(c - \lambda q_1) + Const_2\end{aligned}$$

for  $q_1 > 0$  and  $q_2 > 0$  where  $Const_1$  and  $Const_2$  are constant variables. And let  $q_1 > 0$ ,  $q_2 > 0$ ,  $p_1 > 0$ , and  $p_2 > 0$ . This means  $0 < q_1 < 1$ ,  $0 < q_2 < 1$ ,  $0 < p_1 < 1$ , and  $0 < p_2 < 1$  from  $q_1 = 1 - p_1$ ,  $q_2 = 1 - p_2$ .

Next, the profit maximization of the firms over time is

$$\begin{aligned}\max_{p_1, p_2} (p_1 q_1 - C_1(q_1)) + (p_2 q_2 - C(q_1, q_2)) \\s.t. q_1 = 1 - p_1, \quad q_2 = 1 - p_2\end{aligned}$$

So, the FOC of this problem is

$$\begin{aligned}\mathcal{L} &= p_1(1 - p_1) - (\epsilon(1 - p_1) + Const_1) + \delta(p_2(1 - p_2) - ((1 - p_2)(c - \lambda(1 - p_1)) + Const_2)) \\ \frac{\partial \mathcal{L}}{\partial p_1} &= 1 - 2p_1 + \epsilon - \delta(1 - p_2)\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial p_2} &= 1 - 2p_2 + c - (1 - p_1)\lambda = 0\end{aligned}$$

### Compute the optimal pricing strategy

From the FOC, if  $\lambda \neq 2$  (from the assumption,  $\lambda$  is smaller than 2), then we can get the optimal pricing strategies of  $p_1$  and  $p_2$  are

$$\begin{aligned}p_1 &= \frac{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)}{\lambda^2 - 4} \\ p_2 &= \frac{\delta\lambda^2 + (1 - \epsilon)\lambda - 2(1 + c)}{\lambda^2 - 4}\end{aligned}$$

### Compute the Lerner Index by each periods

And then, the Lerner Index for the period 1 is

$$\begin{aligned}L_1 &= \frac{p_1 - \epsilon}{p_1} = 1 - \frac{\epsilon}{p_1} \\ &= 1 - \frac{\epsilon(\lambda^2 - 4)}{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)} \\ &= \frac{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1) - \epsilon(\lambda^2 - 4)}{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)} \\ &= \frac{(1 - \epsilon)\lambda^2 + (2\delta - 1 - c)\lambda - 2(1 - \epsilon)}{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)}\end{aligned}$$

We also have the FOC for the period 1 is

$$D(p_1) + p_1 D'(p_1) - C'_1 D'_1(p_1) - \delta C'_2 D'_1(p_1) = 0$$

And so, the elasticity of demand for the period 1 is

$$\begin{aligned} \Rightarrow -\frac{D(p_1)}{p_1 D'(p_1)} &= \frac{p_1 - C'_1 - \delta C'_2}{p_1} \\ &= L_1 - \frac{\delta C'_2}{p_1} \\ &= L_1 + \frac{\delta(1-p_2)\lambda}{p_1} > 0 \end{aligned}$$

where  $0 < p_1 < 1, 0 < p_2 < 1$ .

Therefore, in this situation, the relative profit margin (Lerner Index for the period 1) is lower than the elasticity of demand for the period 1.

Also, the Lerner Index for the period 2 is

$$\begin{aligned} L_2 &= \frac{p_2 - (c - \lambda(1 - p_1))}{p_2} \\ &= 1 - \frac{c - \lambda(1 - p_1)}{p_2} \\ &= 1 - \frac{2p_2 - 1}{p_2} \\ &= 1 - 2 + \frac{1}{p_2} \\ &= -1 + \frac{\lambda^2 - 4}{\delta\lambda^2 + (1 - \epsilon)\lambda - 2(1 + c)} \\ &= \frac{-\delta\lambda^2 - (1 - \epsilon)\lambda + 2(1 + c) + \lambda^2 - 4}{\lambda^2 + (1 - \epsilon)\lambda - 2(1 + c)} \\ &= \frac{(1 - \delta)\lambda^2 - (1 - \epsilon)\lambda - 2(1 - c)}{\lambda^2 + (1 - \epsilon)\lambda - 2(1 + c)} \end{aligned}$$

However, in the period 2, we can have the relative profit margin (Lerner Index for the period 2) is equal to the elasticity of demand for the period 2 because the FOC for the period 2 is

$$D(p_2) + p_2 D'(p_2) - C'_2 D'_2(p_2) = 0$$

So, the elasticity of demand for the period 2

$$\begin{aligned} \Rightarrow -\frac{D(p_2)}{p_2 D'(p_2)} &= \frac{p_2 - C'_2}{p_2} \\ &= L_2 \end{aligned}$$

## Does output increase or decrease over time?

From the above result, the output for the period 1 ( $q_1$ ) is

$$\begin{aligned} q_1 &= 1 - p_1 \\ &= 1 - \frac{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)}{\lambda^2 - 4} \\ &= \frac{\lambda^2 - 4 - \lambda^2 - (2\delta - 1 - c)\lambda + 2(\epsilon + 1)}{\lambda^2 - 4} \\ &= \frac{(c + 1 - 2\delta)\lambda + 2(\epsilon - 1)}{\lambda^2 - 4} \end{aligned}$$

From the above result, the output for the period 2 ( $q_2$ ) is

$$\begin{aligned} q_2 &= 1 - p_2 \\ &= 1 - \frac{\delta\lambda^2 + (1 - \epsilon)\lambda - 2(1 + c)}{\lambda^2 - 4} \\ &= \frac{\lambda^2 - 4 - \delta\lambda^2 - (1 - \epsilon)\lambda + 2(1 + c)}{\lambda^2 - 4} \\ &= \frac{(1 - \delta)\lambda^2 - (1 - \epsilon)\lambda + 2(c - 1)}{\lambda^2 - 4} \end{aligned}$$

And then, the difference between the output for the period 1 and that for the period 2 is

$$\begin{aligned} q_2 - q_1 &= \frac{(1 - \delta)\lambda^2 - (1 - \epsilon)\lambda + 2(c - 1)}{\lambda^2 - 4} - \frac{(c + 1 - 2\delta)\lambda + 2(\epsilon - 1)}{\lambda^2 - 4} \\ &= \frac{(1 - \delta)\lambda^2 - (1 - \epsilon) - (c + 1 - 2\delta) + 2c - 2\epsilon}{\lambda^2 - 4} \\ &= \frac{(1 - \delta)\lambda^2 + (-1 + \epsilon - c - 1 + 2\delta)\lambda + 2c - 2\epsilon}{\lambda^2 - 4} \\ &= \frac{(1 - \delta)\lambda^2 - (2\delta - 2 + c - \epsilon)\lambda + 2c - 2\epsilon}{\lambda^2 - 4} \\ &= \frac{\{(1 - \delta)\lambda + (c - \epsilon)\}(\lambda - 2)}{\lambda^2 - 4} \\ &= \frac{(1 - \delta)\lambda + (c - \epsilon)}{\lambda + 2} \end{aligned}$$

Therefore, if  $(1 - \delta)\lambda + (c - \epsilon) > 0$ , then the output increases over time. Also, if  $(1 - \delta)\lambda + (c - \epsilon) < 0$ , then the output decreases over time. if  $(1 - \delta)\lambda + (c - \epsilon) = 0$ , then the output is equal overtime.

## Supplement: Derivative of the prices

The price for the period 1 is

$$\begin{aligned}
1 - 2p_1 + \epsilon &= \delta(1 - p_2)\lambda \\
\Leftrightarrow p_2 &= 1 - \frac{1}{\delta\lambda}(1 - 2p_1 + \epsilon) = \frac{\delta\lambda - 1 + 2p_1 - \epsilon}{\lambda} \\
1 - 2p_2 + c &= (1 - p_1)\lambda \\
\Leftrightarrow p_2 &= \frac{1}{2}(1 + c - (1 - p_1)\lambda) \\
2(\delta\lambda - 1 + 2p_1 - \epsilon) &= \lambda(1 + c - (1 - p_1)\lambda) \\
2\delta\lambda - 2 + 4p_1 - 2\epsilon &= \lambda + \lambda c - \lambda^2 + p_1\lambda^2 \\
(\lambda^2 - 4)p_1 &= \lambda^2 + (2\delta - 1 - c)\lambda - 2\epsilon - 2 \\
p_1 &= \frac{\lambda^2 + (2\delta - 1 - c)\lambda - 2(\epsilon + 1)}{\lambda^2 - 4}
\end{aligned}$$

The price for the period 2 is

$$\begin{aligned}
p_2 &= \frac{1}{2}(1 + c - \left(1 - \frac{\lambda^2 + (2\delta - 1 - c)\lambda - 2\epsilon - 2}{\lambda^2 - 4}\right)\lambda) \\
&= \frac{1}{2}(1 + c - \left(\frac{-(2\delta - 1 - c)\lambda^2 + 2\epsilon\lambda - 2\lambda}{\lambda^2 - 4}\right)) \\
&= \frac{1}{2}\left(\frac{\lambda^2 - 4 + c\lambda^2 - 4c + (2\delta - 1 - c)\lambda^2 - 2\epsilon\lambda + 2\lambda}{\lambda^2 - 4}\right) \\
&= \frac{1}{2}\left(\frac{2\delta\lambda^2 - 4 - 4c + 2\lambda - 2\lambda\epsilon}{\lambda^2 - 4}\right) \\
&= \frac{\delta\lambda^2 + (1 - \epsilon)\lambda - 2(1 + c)}{\lambda^2 - 4}
\end{aligned}$$