

A Simplex based approach to Vertex Enumeration of Arrangements and Polyhedra

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The paper

„A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra“

Authors: David Avis and Komei Fukuda

Publication year: 1992

Journal: Discrete & Computational Geometry

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Outline

1. The Vertex Enumeration Problem
2. Approaches to the VEP
3. The SIMPLEX Algorithm

The Vertex Enumeration Problem

Definition (Convex Polyhedron)

Given a matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$.

A *convex polyhedron* P is defined as

$$P = \{x \in \mathbb{R}^n : Ax + b \geq 0\} \quad (1)$$

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Definition (Polyhedron Vertex)

A point $x \in P$ is a *vertex* of P **iff**

it cannot be written as a convex combination of two other solutions **iff**

it is the unique solution to a subset of m inequalities solved as equations.

The Vertex Enumeration Problem (VEP)

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For a given polytope/polyhedron, hyperplane arrangement P return all vertices of P .

¹Khachiyan et al. [2008]

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Theorem¹

The VEP is NP-complete.

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Goals:

- ▶ should be deterministic
- ▶ should be fast
- ▶ shouldn't use too much space

¹Khachiyan et al. [2008]

An Example Problem

Example: Meals for 1 Euro

Var	Food type	Energy (kcal/100g)	Protein (g)	Calcium (mg)	Price (Euro cents)	Maximum
x_1	Oatmeal ²	250	15	40	10	500g
x_2	Milk ³	80	4	130	4	1000g
x_3	DD Stollen ⁴	340	7	30	20	200g
	Min. daily	2000	55	800		

²Regular LIDL oatmeal

³Regular Bio LIDL milk

⁴<https://fdodb.info>

Putting data into inequalities

Example: Meals for 1 Euro

$$A = \begin{bmatrix} 350 & 80 & 340 \\ 15 & 4 & 7 \\ 40 & 130 & 30 \\ -10 & -4 & -20 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, -b = \begin{bmatrix} 2000 \\ 55 \\ 800 \\ -100 \\ -5 \\ -10 \\ -2 \end{bmatrix}, \begin{pmatrix} \text{Energy} \\ \text{Protein} \\ \text{Calcium} \\ \text{Price} \\ x_1 \text{Max} \\ x_2 \text{Max} \\ x_3 \text{Max} \end{pmatrix}$$

Polytope visualized

Example: Meals for 1 Euro

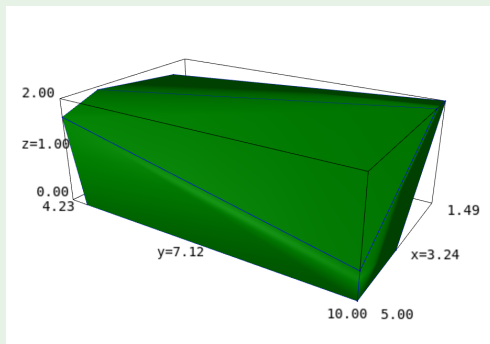


Figure: The exact shape is unknown/expensive

An Example Problem

- ▶ Use convex combination to yield infinitely many food combinations

An Example Problem

- Use convex combination to yield infinitely many food combinations

Definition (Convex combination)

A convex combination is a linear combination of points.

Let x_1, x_2, \dots, x_n be a finite set of points. A convex combination is a point of the form

$$y = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

while $a_1, \dots, a_n \in \mathbb{R}^+$, and

$$a_1 + \dots + a_n = 1$$

Approaches to the VEP

1. Double-description/Motzkin-based method
2. Pivot-based methods
3. Newer methods

Linear optimization problems

Definition

Let $G \subseteq \mathbb{R}^n$ be a polyhedron and $c \in \mathbb{R}^n$. Then a problem of the form

$$z = f(x) = c^T \cdot x \rightarrow \min \quad \text{with } x \in G$$

is called a *linear* optimization problem.

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Protein rich meals for 1 Euro

We are trying to maximize our protein intake:

$$c = \begin{bmatrix} 15 \\ 4 \\ 7 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$z = c^T x \rightarrow \max$$

The SIMPLEX Algorithm

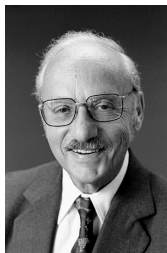


Figure: George Dantzig (1914 - 2005)

Algorithm 1 The SIMPLEX Algorithm

- 1: Find a first feasible vertex
 - 2: Calculate the optimal vertex
-

A visual example

Example: From Avis' lecture notes

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, -b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{array}{l} 1 - x_1 + x_3 \geq 0 \\ 1 - x_2 + x_3 \geq 0 \\ 1 + x_1 + x_3 \geq 0 \\ 1 + x_2 + x_3 \geq 0 \\ -x_3 \geq 0 \end{array}$$

A visual example

Example

Example: From Avis' lecture notes

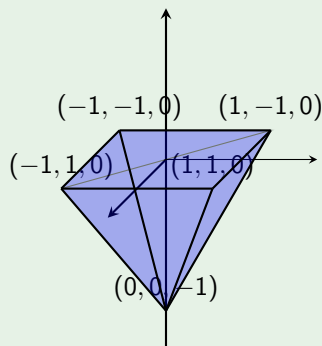


Figure: P is a regular pyramid

The SIMPLEX Algorithm

Example

A linear OP example content...

Sources



L. Khachiyan, E. Boros, K. Borys, K. Elbassioni, and V. Gurvich. Generating all vertices of a polyhedron is hard. *Discrete and Computational Geometry*, 39(1):174–190, 2008.