A Simplex based approach to Vertex Enumeration of Arrangements and Polyhedra

Tilman Hinnerichs

Seminar „Selected Topics in Logic and Verification" – TU Dresden

July 27, 2020

The paper

„A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra“

Authors: David Avis and Komei Fukuda
Publication year: 1992
Journal: Discrete & Computational Geometry

The paper

„A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra"

Authors: David Avis and Komei Fukuda
Publication year: 1992
Journal: Discrete & Computational Geometry
Cited 751 times (according to Google Scholar)

Outline

1. The Vertex Enumeration Problem

2. Approaches to the VEP

3. The SIMPLEX Algorithm

The Vertex Enumeration Problem

Definition (Convex Polyhedron)

Given a matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$.
A *convex polyhedron* P is defined as

$$P = \{x \in \mathbb{R}^n : Ax + b \geq 0\} \tag{1}$$

The Vertex Enumeration Problem

Definition (Convex Polyhedron)

Given a matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$.
A *convex polyhedron* P is defined as

$$P = \{x \in \mathbb{R}^n : Ax + b \geq 0\} \tag{1}$$

Definition (Polyhedron Vertex)

A point $x \in P$ is a *vertex* of P **iff**
it cannot be written as a convex combination of two other solutions **iff**
it is the unique solution to a subset of m inequalities solved as equations.

The Vertex Enumeration Problem (VEP)

Definition (Vertex Enumeration Problem)

For a given polytope/polyhedron, hyperplane arrangement P return all vertices of P.

[1] Khachiyan et al. [2008]

The Vertex Enumeration Problem (VEP)

Definition (Vertex Enumeration Problem)

For a given polytope/polyhedron, hyperplane arrangement P return all vertices of P.

Theorem[1]

The VEP is NP-complete.

[1] Khachiyan et al. [2008]

The Vertex Enumeration Problem (VEP)

Definition (Vertex Enumeration Problem)

For a given polytope/polyhedron, hyperplane arrangement P return all vertices of P.

Theorem[1]

The VEP is NP-complete.

Goals:

- should be deterministic
- should be fast
- shouldn't use too much space

[1] Khachiyan et al. [2008]

An Example Problem

Example: Meals for 1 Euro

Var	Food type	Energy (kcal/100g)	Protein (g)	Calcium (mg)	Price (Euro cents)	Maximum
x_1	Oatmeal[2]	250	15	40	10	500g
x_2	Milk[3]	80	4	130	4	1000g
x_3	DD Stollen[4]	340	7	30	20	200g
	Min. daily	**2000**	**55**	**800**		

[2] Regular LIDL oatmeal
[3] Regular Bio LIDL milk
[4] https://fddb.info

Putting data into inequalities

Example: Meals for 1 Euro

$$A = \begin{bmatrix} 350 & 80 & 340 \\ 15 & 4 & 7 \\ 40 & 130 & 30 \\ -10 & -4 & -20 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, -b = \begin{bmatrix} 2000 \\ 55 \\ 800 \\ -100 \\ -5 \\ -10 \\ -2 \end{bmatrix}, \begin{pmatrix} Energy \\ Protein \\ Calcium \\ Price \\ x_1 Max \\ x_2 Max \\ x_3 Max \end{pmatrix}$$

Polytope visualized

Example: Meals for 1 Euro

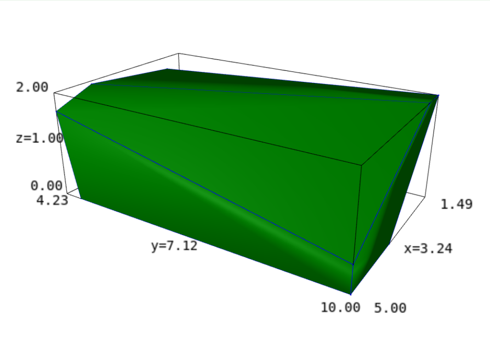


Figure: The exact shape is unknown/expensive

An Example Problem

- Use convex combination to yield infinitely many food combinations

An Example Problem

- Use convex combination to yield infinitely many food combinations

Definition (Convex combination)

A convex combination is a linear combination of points.
Let $x_1, x_2, \ldots, x_n$ be a finite set of points. A convex combination is a point of the form

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

while $a_1, \ldots, a_n \in \mathbb{R}^+$, and

$$a_1 + \cdots + a_n = 1$$

Approaches to the VEP

1. Double-description/Motzkin-based method
2. Pivot-based methods
3. Newer methods

Linear optimization problems

Definition

Let $G \subseteq \mathbb{R}^n$ be a polyhedron and $c \in \mathbb{R}^n$. Then a problem of the form

$$z = f(x) = c^T \cdot x \to min \qquad with\ x \in G$$

is called a *linear* optimization problem.

Linear optimization problems

Definition

Let $G \subseteq \mathbb{R}^n$ be a polyhedron and $c \in \mathbb{R}^n$. Then a problem of the form

$$z = f(x) = c^T \cdot x \to min \qquad with\ x \in G$$

is called a *linear* optimization problem.

Protein rich meals for 1 Euro

We are trying to maximize our protein intake:

$$c = \begin{bmatrix} 15 \\ 4 \\ 7 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$z = c^T x \to max$$

The SIMPLEX Algorithm

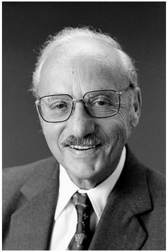


Figure: George Dantzig (1914 - 2005)

Algorithm 1 The SIMPLEX Algorithm

1: Find a first feasible vertex
2: Calculate the optimal vertex

From $\geq$ to $=$

The inequalities

$$Ax + b \geq 0$$

describing P are transformed to the equivalent equality system

$$Mx + b = x_p$$

with $x \in \mathbb{R}^n_+$, $x_p \in \mathbb{R}^m_+$, and $M \in \mathbb{R}^{m \times n}$ (if $x \geq 0$)

From $\geq$ to $=$

The inequalities

$$Ax + b \geq 0$$

describing P are transformed to the equivalent equality system

$$Mx + b = x_p$$

with $x \in \mathbb{R}_+^n$, $x_p \in \mathbb{R}_+^m$, and $M \in \mathbb{R}^{m \times n}$ (if $x \geq 0$)

Definition (Tableau form)

A linear program of the form

$$z = f'(x') = c'^T \cdot x' \to max \qquad \text{with } Mx_N + b = x_B \text{ with } \begin{array}{l} x_N := x, \\ x_B := x_p, \\ x' \leftrightarrow \begin{pmatrix} x_N \\ x_B \end{pmatrix} \\ b \geq 0 \end{array}$$

is said to be in **tableau form**.

Another example

$$-z = -30x_1 - 45x_2 \rightarrow min$$

$$\text{with} \quad \begin{array}{rcrcrcr} x_3 & = & -4x_1 & - & 3x_2 & + & 100 \\ x_4 & = & -x_1 & - & 2x_2 & + & 50 \\ x_3, x_4 & , & x_1 & , & x_2 & \geq & 0 \end{array}$$

non-base variables, with indices $N = \{1, 2\}, |N| = n$
base variables, with indices $B = \{3, 4\}, |B| = m$

SIMPLEX-Tableau:

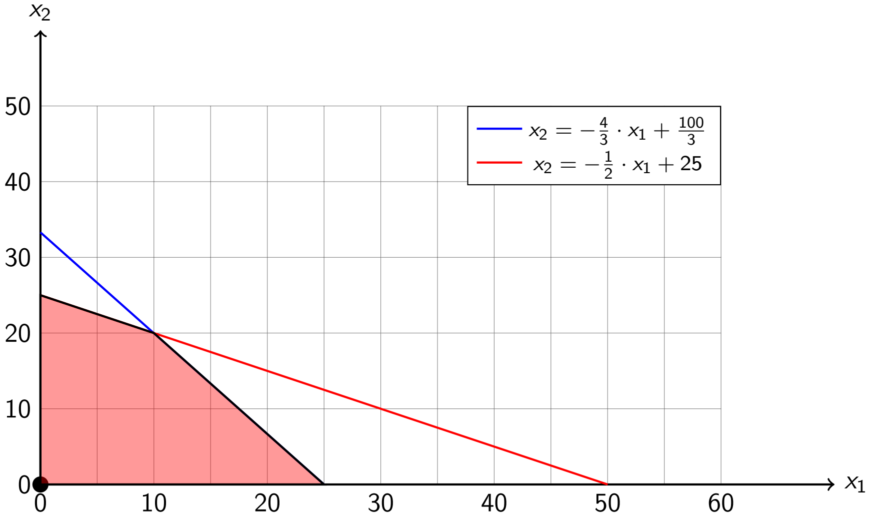
ST_0	x_1	x_2	1
$x_3 =$	-4	-3	100
$x_4 =$	-1	-2	50
$-z =$	-30	-45	0

ST_0	x_N	1
$x_B =$	M	p
$z =$	q^T	q_0

ST_0	x_N	1
$x_B =$	M	p
$z =$	q^T	q_0

x_N results from x and N
x_B results from x and B
M results from A
$p := b$
$q := c'$
$q_0 := 0$

Visualization of new polytope P



Phase 1: Determine first feasible basic solution

Algorithm 2 Phase 1: Determine first feasible basic solution (vertex)

1: **if** trivial solution visible **then**
2: choose it
3: **end if**
4: ...

Definition

The point

$$x \leftrightarrow \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$$

is called **basic solution**.
If $p \geq 0$, x is called **basic feasible solution(BFS)**.

Phase 2: Bestimmung der optimalen Ecke

Algorithm 3 Phase 2: Bestimmung der optimalen Ecke

1: **while** $\tau \in N$ mit $q_\tau < 0$ existiert **do**
2:     Wähle das PIVOT-Element $P_{\sigma\tau}$
3:     Berechne mit Hilfe der Tauschungsregeln das neue Simplextableau
4: **end while**

Phase 2: Bestimmung der optimalen Ecke

Algorithm 4 Phase 2: Bestimmung der optimalen Ecke

1: **while** $\tau \in N$ mit $q_\tau < 0$ existiert **do**
2: Wähle das PIVOT-Element $P_{\sigma\tau}$
3: Berechne mit Hilfe der Tauschungsregeln das neue Simplextableau
4: **end while**

Satz

Das SIMPLEX-Tableau ist optimal gdw. $p \geq 0$ und $q \geq 0$

Die Wahl des PIVOT-Elements

ST_0	x_N	1
$x_B =$	P	p
$z =$	q^T	q_0

Wir wählen $\tau \in N$ mit $q_\tau < 0$ beliebig, und bilden daraufhin $-\dfrac{p_\sigma}{P_{\sigma\tau}} := \min\left\{-\frac{p_i}{P_{i\tau}} \mid -\frac{p_i}{P_{i\tau}} < 0, i \in B\right\}$:

ST_0	x_1	x_2	1
$x_3 =$	-4	-3	100
$x_4 =$	-1	-2	50
$-z =$	-30	-45	0

$\implies \tau := 2, \sigma := 4$

ST_0	x_1	x_2	1
$x_3 =$	-4	-3	100
$x_4 =$	-1	-2	50
$-z =$	-30	-45	0

ST_0	x_1	x_2	1
$x_3 =$	-4	-3	100
$x_4 =$	-1	-2	50
$-z =$	-30	-45	0

ST_1	x_1	x_4	1
$x_3 =$	$-\frac{5}{2}$	$\frac{3}{2}$	25
$x_2 =$	$-\frac{1}{2}$	$-\frac{1}{2}$	25
$-z =$	$-\frac{15}{2}$	$\frac{45}{2}$	-1125

ST_0	x_1	x_2	1
$x_3 =$	-4	-3	100
$x_4 =$	-1	-2	50
$-z =$	-30	-45	0

ST_1	x_1	x_4	1
$x_3 =$	$-\frac{5}{2}$	$\frac{3}{2}$	25
$x_2 =$	$-\frac{1}{2}$	$-\frac{1}{2}$	25
$-z =$	$-\frac{15}{2}$	$\frac{45}{2}$	-1125

ST_0	x_1	x_2	1
$x_3 =$	-4	-3	100
$x_4 =$	-1	-2	50
$-z =$	-30	-45	0

ST_1	x_1	x_4	1
$x_3 =$	$-\frac{5}{2}$	$\frac{3}{2}$	25
$x_2 =$	$-\frac{1}{2}$	$-\frac{1}{2}$	25
$-z =$	$-\frac{15}{2}$	$\frac{45}{2}$	-1125

ST_2	x_3	x_4	1
$x_1 =$	$-\frac{2}{5}$	$\frac{3}{5}$	10
$x_2 =$	$\frac{1}{5}$	$-\frac{8}{10}$	20
$-z =$	3	18	-1200

Properties of the SIMPLEX algorithm

- Does not always terminate
  - easily fixable by introducing an ordering among the nodes

Properties of the SIMPLEX algorithm

- Does not always terminate
  - easily fixable by introducing an ordering among the nodes

Theorem

For a given basic feasible solution (BFS) v the SIMPLEX algorithm deterministically determines a path from v to the optimal vertex $v*$.

A Simplex based approach to Vertex Enumeration of Arrangements and Polyhedra
21 / 26

Some thoughts

1. What if we choose a pair of indices different from the optimal one in a simplex step?
2.

A Simplex based approach to Vertex Enumeration of Arrangements and Polyhedra 22 / 26

A visual example

Example: From Avis' lecture notes

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, -b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{aligned} 1 - x_1 + x_3 &\geq 0 \\ 1 - x_2 + x_3 &\geq 0 \\ 1 + x_1 + x_3 &\geq 0 \\ 1 + x_2 + x_3 &\geq 0 \\ -x_3 &\geq 0 \end{aligned}$$

,

A visual example

Example

Example: From Avis' lecture notes

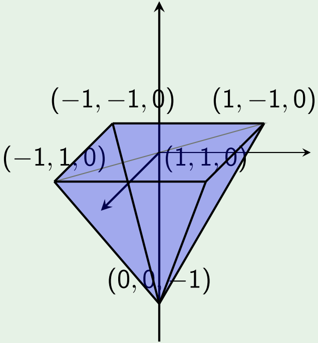


Figure: P is a regular pyramid

The SIMPLEX Algorithm

Example

A linear OP example content...

Sources

L. Khachiyan, E. Boros, K. Borys, K. Elbassioni, and V. Gurvich. Generating all verticesof a polyhedron is hard.Discrete and Computational Geometry, 39(1):174–190, 2008.