Formalizing Fuzzy DL with DNN

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1 Definitions

Given this ontology (mainly interested in T-box) (for now only consider EL++), we want to do perform reasoning and eventually proof soundness, correctness, completeness.

Input: Some Ontology

$$\begin{aligned}
&\mathcal{O} = (\Sigma, \mathcal{T}, \mathcal{A})\\
&\Sigma = (\mathbf{C}, \mathbf{R}, \mathbf{I})\\
&\mathcal{T} = \mathcal{T}_{NF1} \cup \mathcal{T}_{NF2} \cup \mathcal{T}_{NF3} \cup \mathcal{T}_{NF4}\\
&\mathcal{A} = \{a : C\}
\end{aligned}$$

1. A Box elimination:

$$\begin{aligned}
\mathbf{C}' :=& \mathbf{C} \cup \mathbf{I}\\
\mathcal{T}' :=& \mathcal{T} \cup \{\{a\} \sqsubseteq \exists r.\{b\} | \forall r(a,b) \in \mathcal{A}\}\\
& \cup \{\{a\} \sqsubseteq C | \forall (a : C) \in \mathcal{A}\}\\
(\mathcal{A}' :=& \emptyset)
\end{aligned}$$

2. Add new individuals for each concept:

$$\mathbf{I} := \{c_C | C \in \mathbf{C}'\}$$

3. Introduce mappings/functions (learnable neural nets) (see [LTN]).
We basically embed all individuals into $\mathbb{R}^n$, and train neural nets to decide and describe fuzzy affiliation to concept. n is dimensionality of latent space and thus a hyperparameter.

   - $f_{emb} : \mathbf{C}' \cup \mathbf{I} \to \mathbb{R}^n$: Mapping all individuals into latent space
   - $f_{skol} : \mathbb{R}^n \to \mathbb{R}^n$: Skolemizing function, as we only consider EL++ only relevant for normal forms $C \sqsubseteq \exists R.D$ and $\exists R.C \sqsubseteq D$. We make $f_{skol}(c_C)$ dependent on both $f_{emb}(c_D)$ and the relation R, which should suffice through axiomatization theorem?

- $f_{class} : \mathbb{R}^n \to [0,1]^{|C'|}$: describes for all points in $\mathbb{R}^n$ its fuzzy association with each concept:

$$C^{\mathcal{I}}(c_C) = (f_{class} \circ f_{emb})(c_C)$$

4. Maximize satisfaction of all axioms over all individuals (we only care about **I**, e.g. given $C \sqsubseteq D \in \mathcal{T}$ we want to find above functions such that

$$\max(C^{\mathcal{I}} \to D^{\mathcal{I}})(d) = \to (C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)) \tag{1}$$

with fuzzy implication $\to: [0,1] \times [0,1] \to [0,1]$

We basically follow [LTN], but do not solve quantification over grounding, but over skolemization for $\exists$. Finiteness would be cool in order to compute satisfaction of $\forall$

Questions for Tobias:

- This should have an infinite Herbrand universe as model?
- is the skolemization correct? Does it yield feasible fresh elements?
- Are there finite models? One model could be subsets of $\mathbb{R}^n$ (crisp fuzzy sets $A_{\geq t}$ with global threshold t) and $\sqsubset = \subseteq$, but then skolemization will not yield fresh elements of domain
- correctness: This should be correct as for $loss = 1 - satisfaction = 0$ this is crisp, and hence this is a model of underlying EL++ instance

Knowledge base:

2 Questions to answer

- Proper definitions?
- infinite universe? If so what is the issue?
- can we just take finite subset of universe?
- decidability and completeness
- what are the individuals actually?
- formalize the three functions
- What are the mappings meaning?
  - embedding function
  - Skolemizer
  - fuzzy set descriptor
- What is the actual knowledge base?

References

[LTN] https://arxiv.org/abs/2012.13635