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Exercise sheet 7

TASK 1: POLYNOMIAL KERNELS

Task 1: Polynomial kernels (10 points)

The general form of a polynomial kernel is given by

$$K(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^T \mathbf{z} + \gamma)^d.$$

a) Assume $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. For $d = 2$, $\gamma = 1$ and $F : \mathbb{R}^2 \to \mathbb{R}^6$ with

$$F(\mathbf{x}) = (x_1^2, \sqrt{2}x_1x_2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2, 1)^T,$$

show that $K(\mathbf{x}, \mathbf{z}) = F(\mathbf{x})^T \cdot F(\mathbf{z})$. (5 points)

b) Assume $\mathbf{x}, \mathbf{z} \in \mathbb{R}^2$. For $d = 3$ and $\gamma = 1$, find a mapping F, so that $K(\mathbf{x}, \mathbf{z}) = F(\mathbf{x})^T F(\mathbf{z})$. (5 points)

Solution:

a) Let us first find the result of the kernel function:

$$\begin{aligned} K(\boldsymbol{x}, \boldsymbol{z}) = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + 1 \right)^2 &= (x_1z_1 + x_2z_2 + 1)^2 = \\ &= x_1^2z_1^2 + x_1x_2z_1z_2 + x_1z_1 + x_1x_2z_1z_2 + x_2^2z_2^2 + x_2z_2 + x_1z_1 + x_2z_2 + 1 = \\ &= x_1^2z_1^2 + x_2^2z_2^2 + 2x_1x_2z_1z_2 + 2x_1z_1 + 2x_2z_2 + 1 \end{aligned}$$

Now we can find the dot product of $F(\boldsymbol{x}) \cdot F(\boldsymbol{z})$ and compare the results:

$$\begin{aligned} F(\boldsymbol{x}) \cdot F(\boldsymbol{z}) &= \left(x_1^2, \sqrt{2}x_1x_2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2, 1\right)^T \left(z_1^2, \sqrt{2}z_1z_2, z_2^2, \sqrt{2}z_1, \sqrt{2}z_2, 1\right) = \\ &= x_1^2z_1^2 + 2x_1x_2z_1z_2 + x_2^2z_2^2 + 2x_1z_1 + 2x_2z_2 + 1 = \\ &= x_1^2z_1^2 + x_2^2z_2^2 + 2x_1x_2z_1z_2 + 2x_1z_1 + 2x_2z_2 + 1 = K(\boldsymbol{x}, \boldsymbol{z}) \end{aligned}$$

b) Let us first find the result of the second kernel function:

$$K(\boldsymbol{x}, \boldsymbol{z}) = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + 1 \right)^3 = (x_1z_1 + x_2z_2 + 1)^3 =$$
$$= (x_1z_1 + x_2z_2 + 1)(x_1^2z_1^2 + x_2^2z_2^2 + 2x_1x_2z_1z_2 + 2x_1z_1 + 2x_2z_2 + 1) =$$
$$= x_1^3z_1^3 + x_1x_2^2z_1z_2^2 + 2x_1^2x_2z_1^2z_2 + 2x_1^2z_1^2 + 2x_1x_2z_1z_2 + x_1z_1 +$$
$$+x_2^3z_2^3 + x_1^2x_2z_1^2z_2 + 2x_1x_2^2z_1z_2^2 + 2x_2^2z_2^2 + 2x_1x_2z_1z_2 + x_2z_2 +$$
$$+x_1^2z_1^2 + x_2^2z_2^2 + 2x_1x_2z_1z_2 + 2x_1z_1 + 2x_2z_2 + 1 =$$
$$= x_1^3z_1^3 + x_2^3z_2^3 + 3x_1^2z_1^2 + 3x_2^2z_2^2 + 3x_1^2x_2z_1^2z_2 + 3x_1x_2^2z_1z_2^2 + 6x_1x_2z_1z_2 + 3x_1z_1 + 3x_2z_2 + 1$$

Assume the following mapping function F and check to see if it fits the criteria:

$$F(\boldsymbol{x}) = \left(x_1^3, x_2^3, \sqrt{3}x_1^2, \sqrt{3}x_2^2, \sqrt{3}x_1^2x_2, \sqrt{3}x_1x_2^2, \sqrt{6}x_1x_2, \sqrt{3}x_1, \sqrt{3}x_2, 1\right)$$

$$F(\boldsymbol{x}) \cdot F(\boldsymbol{z}) = \left(x_1^3, x_2^3, \sqrt{3}x_1^2, \sqrt{3}x_2^2, \sqrt{3}x_1^2x_2, \sqrt{3}x_1x_2^2, \sqrt{6}x_1x_2, \sqrt{3}x_1, \sqrt{3}x_2, 1\right) \cdot$$
$$\cdot \left(z_1^3, z_2^3, \sqrt{3}z_1^2, \sqrt{3}z_2^2, \sqrt{3}z_1^2z_2, \sqrt{3}z_1z_2^2, \sqrt{6}z_1z_2, \sqrt{3}z_1, \sqrt{3}z_2, 1\right) =$$
$$= x_1^3z_1^3 + x_2^3z_2^3 + 3x_1^2z_1^2 + 3x_2^2z_2^2 + 3x_1^2x_2z_1^2z_2 + 3x_1x_2^2z_1z_2^2 + 6x_1x_2z_1z_2 + 3x_1z_1 + 3x_2z_2 + 1 =$$

$$= K(\boldsymbol{x}, \boldsymbol{z})$$

Which means the mapping function is correct.