Capital Asset Pricing Model

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Efficient Frontier with Riskless Asset – Part 1

- Financial market consists of $n \geq 2$ risky assets (with normal returns) and riskless asset with risk-free rate of R_f

  n个风险资产的expected return
- Let $\mathbf{R}$ be $n \times 1$ vector of expected returns for risky assets

  风险之间的收益率如何相关联，反映asset之间的关系
- Let $\mathbf{V}$ be $n \times n$ is covariance matrix of returns for risky assets
- Let $\mathbf{w}$ be $n \times 1$ vector of portfolio weights for risky assets
- Frontier portfolio weights for expected return of R_p:

用来调整portfolio的整体风险水平

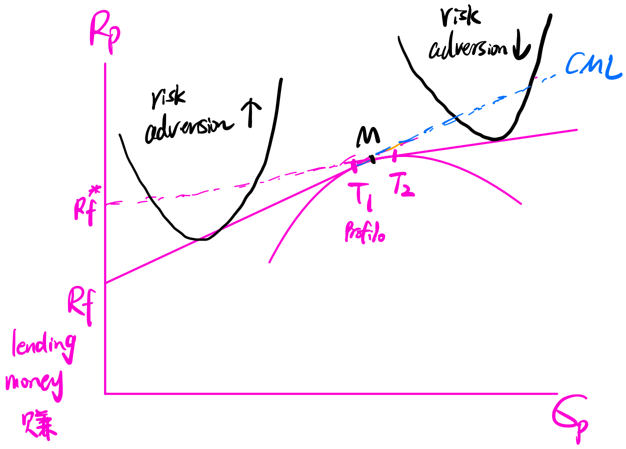
$$\mathbf{w} = \lambda \mathbf{V}^{-1} \left(\mathbf{R} - R_f \mathbf{e} \right); \qquad \lambda = \frac{R_p - R_f}{\zeta - 2\alpha R_f + \delta R_f^2};$$

$$\alpha = \mathbf{R}'\mathbf{V}^{-1}\mathbf{e}; \qquad \zeta = \mathbf{R}'\mathbf{V}^{-1}\mathbf{R}; \qquad \delta = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

与risk asset的expected return和无风险收益有关.

risk asset之间的cov与expected return的关系

all asset的cov结构的整体权衡.



如何画CML: 确定无风险收·

Efficient Frontier with Riskless Asset – Part 2

- Efficient frontier is tangent to top half of risky-asset-only frontier when $R_f < R_{mv} = \frac{\alpha}{\delta}$, so efficient frontier is given by capital allocation line for tangency portfolio:

$$R_p = R_f + \left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}} \sigma_p$$

- Tangency portfolio has no investment in riskless asset, so let $\mathbf{e}'\mathbf{w}_m = 1$ to get portfolio weights for tangency portfolio:

$$\mathbf{w}_m = \lambda_m \mathbf{V}^{-1} \left(\mathbf{R} - R_f \mathbf{e}\right); \qquad \lambda_m = \frac{1}{\alpha - \delta R_f}$$

- Efficient frontier is also known as capital market line (CML)

Capital Asset Pricing Model

- Previously, assumed that all investors are "price takers", in sense that allocation choices have no effect on asset prices
- Now assume that asset prices adjust to produce market equilibrium, where supply of risky assets equals demand
- Also assume that all investors hold mean-variance efficient portfolios, agree on **R** and **V**, and can borrow and lend (without limit) at risk-free rate
- All investors will choose to hold tangency portfolio (in combination with riskless asset) $\implies$ tangency portfolio becomes market portfolio: aggregate portfolio of risky assets
- Hence market portfolio must have highest possible Sharpe ratio in this capital asset pricing model (CAPM)

CAPM Pricing Formula

- Let R_m be expected market return, and let $\vec{\sigma}_m$ be $n \times 1$ vector of covariances between asset returns and market return:

$\vec{\sigma}_m = \begin{bmatrix} \sigma_{1m} \\ \vdots \\ \sigma_{nm} \end{bmatrix}$

$$\vec{\sigma}_m = \mathbf{V}\mathbf{w}_m = \lambda_m (\mathbf{R} - R_f \mathbf{e}) ; \quad 1\times1 \text{ vetor}$$

$$\sigma_m^2 = \mathbf{w}'_m \mathbf{V} \mathbf{w}_m = \lambda_m (\underbrace{R_m}_{W'_m R} - R_f) \quad \text{scale (Positive)}$$

$\frac{\vec{\sigma}_m}{\sigma_m^2} = \frac{\vec{R} - R_f\vec{e}}{R_m - R_f}$

$\underbrace{\vec{R} - R_f\vec{e}}$ $= \frac{\vec{\sigma}_m}{\sigma_m^2} \cdot (R_m - R_f)$

- Divide and rearrange to get relationship between asset risk premiums and market risk premium:

$V\vec{W}_m = \vec{\sigma}_m = \begin{bmatrix} \sigma_{1m} \\ \vdots \\ \sigma_{nm} \end{bmatrix} = \lambda_m (\vec{R} - R_f \cdot \vec{e})$

$\vec{W}'_m V \vec{w}_w = \sigma_m^2 = \lambda_m (R_m - R_f)$

$$\begin{bmatrix} R_1 - R_f \\ \vdots \\ R_n - R_f \end{bmatrix} \leftarrow \mathbf{R} - \underset{n\times1}{R_f \mathbf{e}} = \frac{\vec{\sigma}_m}{\sigma_m^2} (R_m - R_f) = \vec{\beta} (R_m - R_f)$$

$\vec{\beta} = \frac{\vec{\sigma}_m}{\sigma_m^2}$

$R_i - R_f = \beta_i (R_m - R_f), \; i = 1, \ldots m$

$\beta_m = 1$

Asset Beta 系统性风险

β: 潜在 risk

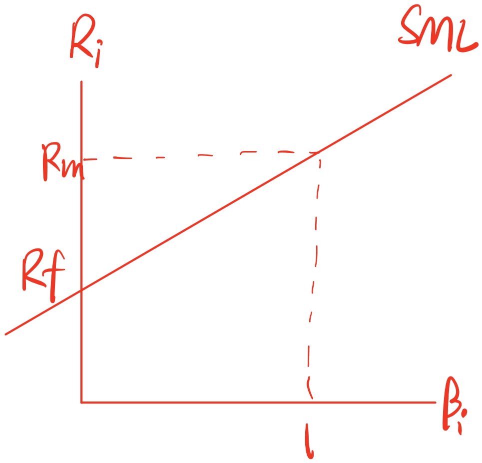
- Here $\vec{\beta}$ is $n \times 1$ vector with $\beta_i = \sigma_{im}/\sigma_m^2$, where σ_{im} is covariance of return between i'th asset and market portfolio
- By definition, market portfolio has unit beta: $\beta_m = 1$ $= \frac{\sigma_m^2}{\sigma_m^2}$
- Hence CAPM pricing formula for individual risky asset:

if $\beta_i < 0$: mean return $R_i < R_f$.

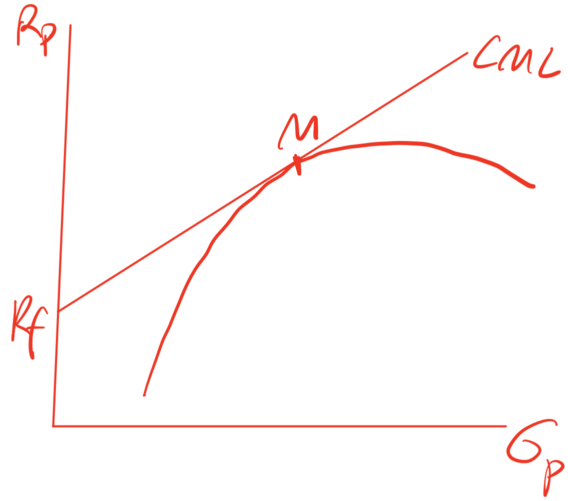
$$R_i - R_f = \underbrace{\beta_i}_{=\frac{\sigma_{im}}{\sigma_m^2}}(R_m - R_f)$$

$\beta_p = \sum_{i=1}^{n} w_i \beta_i$

- As explained later, β_i measures amount of exposure to systematic "market" risk, when investing in i'th asset
- Market risk premium represents appropriate economic compensation for taking on average amount of exposure to market risk (i.e., corresponding to unit beta)



Slope: $R_m - R_f$



Portfolio Beta

- CAPM pricing formula for portfolio of risky assets:

$$
\begin{aligned}
R_p - R_f &= \mathbf{w}' (\mathbf{R} - R_f \mathbf{e}) \\
&= \mathbf{w}' \vec{\beta} (R_m - R_f) = \beta_p (R_m - R_f)
\end{aligned}
$$

- Overall beta for portfolio is weighted average of individual component asset betas: $\beta_p = \mathbf{w}' \vec{\beta}$
- Hence if all component assets have same beta, then so will portfolio $\Longrightarrow$ exposure to market risk cannot be reduced by diversifying investment across risky assets (with same beta)
- In summary, appropriate risk premium for any investment is given by market risk premium scaled up (or down) by beta

Negative Beta

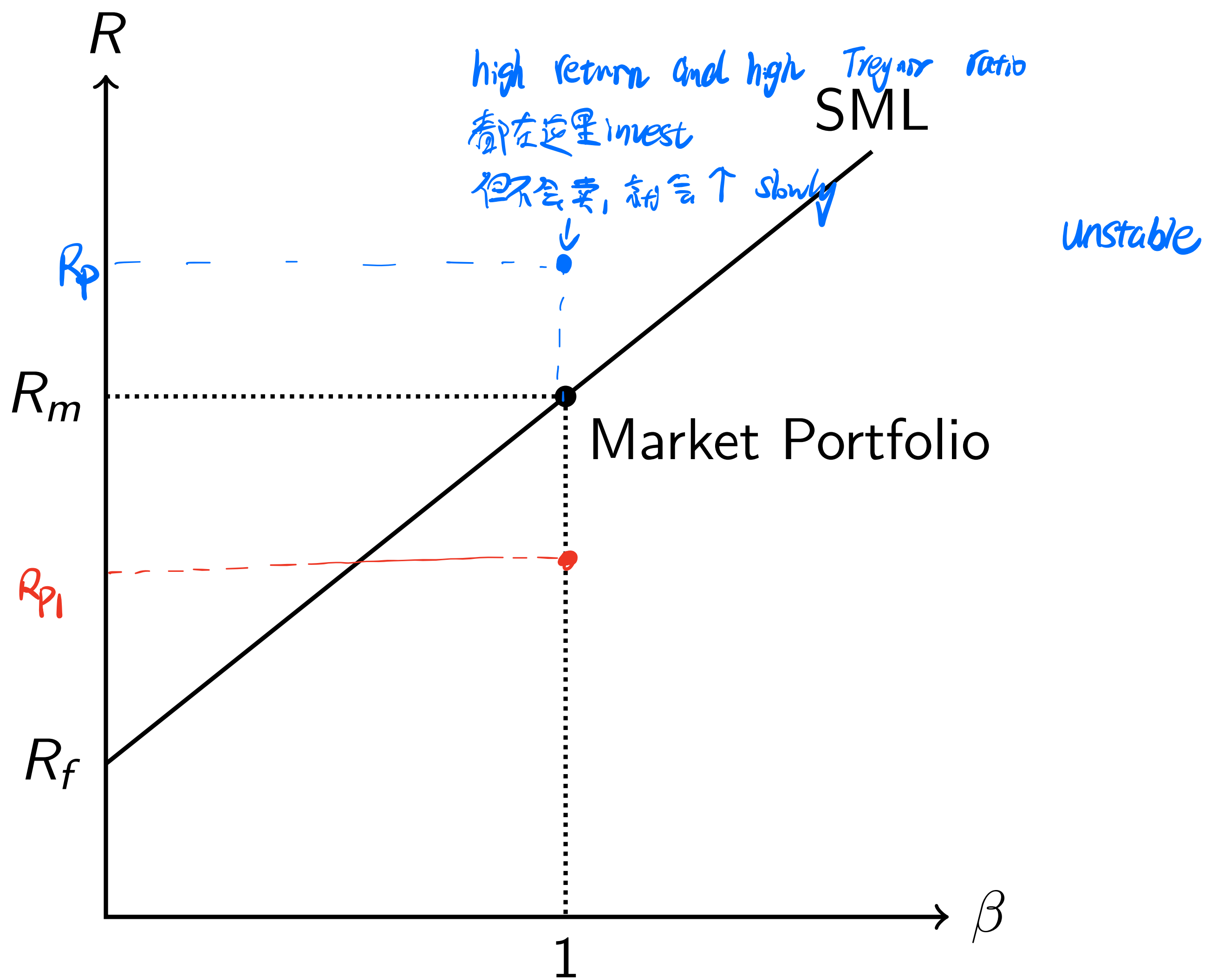
$\beta > 0 : \sigma_{im} > 0$

$\beta < 0 : \sigma_{im} < 0.$

- Asset will have negative beta when asset return has negative correlation with market return $\Longrightarrow$ "counter-cyclical" assets that usually perform well in bear market
- Create negative-beta asset by short-selling positive-beta asset
- Negative-beta assets can be used to reduce beta of portfolio and provide insurance against unexpected market downturn
- Hence investors are willing to pay premium and accept negative risk premium to invest in negative-beta assets
- Frontier portfolios with negative risk premium (i.e., mean return less than risk-free rate) must have negative beta

Security Market Line

- Security market line (SML) is graphical representation of CAPM pricing formula
- Slope of SML represents ratio of risk premium to beta, or Treynor ratio $\implies$ all risky assets (and portfolios) must have same Treynor ratio, in equilibrium
- Asset that lies above SML is underpriced, so investors will buy asset, causing price to rise (and expected return to fall)
- Price will continue to rise until asset is back on SML
- Conversely, asset that lies below SML is overpriced, so all investors will sell asset, causing price to fall (and expected return to rise) until asset is back on SML



R
high return and high Treynor ratio
都在这里 invest
但不会卖，就会↑ slowly
SML
unstable
R_p
R_m
Market Portfolio
R_{p1}
R_f
β
1

Risk Factors – Part 1

- Let $\tilde{\nu}_i$ be unexpected component of (random) return for i'th asset, and apply result for CAPM:

$$\begin{aligned} \tilde{R}_i &= R_i + \tilde{\nu}_i \quad \sim N(0, ..) \\ &= R_f + \beta_i (R_m - R_f) + \tilde{\nu}_i \end{aligned}$$

Risk free

- Also let $\tilde{\nu}_m$ be unexpected component of market return:

$$\begin{aligned} \tilde{R}_m &= R_m + \tilde{\nu}_m \implies \\ \tilde{R}_i &= R_f + \beta_i \left(\tilde{R}_m - \tilde{\nu}_m - R_f \right) + \tilde{\nu}_i \end{aligned}$$

- By definition, $\tilde{\nu}_i$ and $\tilde{\nu}_m$ must have zero mean

- Let $\tilde{\epsilon}_i = \tilde{\nu}_i - \beta_i \tilde{\nu}_m$, which also has zero mean, and must be uncorrelated with $\tilde{R}_m$:

$$\mathrm{Cov}\left(\tilde{R}_m, \tilde{\epsilon}_i\right) = \mathrm{Cov}\left(\tilde{R}_m, \tilde{\nu}_i\right) - \beta_i \mathrm{Cov}\left(\tilde{R}_m, \tilde{\nu}_m\right)$$

tb3 → constant

$$= \mathrm{Cov}\left(\tilde{R}_m, \tilde{R}_i\right) - \beta_i \mathrm{Cov}\left(\tilde{R}_m, \tilde{R}_m\right) = 0$$

- Now substitute $\tilde{\epsilon}_i$ into previous result for asset return:

$R_i - R_f = \beta_i (R_m - R_f)$

asset return

$$\tilde{R}_i - R_f = \beta_i \left(\tilde{R}_m - R_f - \tilde{\nu}_m\right) + \tilde{\nu}_i$$

$$= \beta_i \left(\tilde{R}_m - R_f\right) + \tilde{\epsilon}_i$$

mkt return

risk mkt return

Risk Factors – Part 3

- CAPM defines relationship between asset risk premium and market risk premium
- Hence there must be corresponding relationship between (excess) asset return, and (excess) market return, where (random) excess return is measured relative to risk-free rate
- Interpret $\tilde{R}_m$ as risk factor for market risk, which captures effect of (random) market return on (random) asset return
- Market risk is systematic risk that affects all tradable assets, and β_i represents amount of exposure to market risk
- Then interpret $\tilde{\epsilon}_i$ as risk factor for idiosyncratic risk, which is specific to investing in i'th asset

Systematic Risk vs Idiosyncratic Risk

- Use variance of return to measure total amount of risk when investing in i'th asset:

total risk of investment

$$\sigma_i^2 = \text{Var}\left(\tilde{R}_i\right) = \text{Var}\left(\beta_i \tilde{R}_m + \tilde{\epsilon}_i\right)$$
$$= \text{Var}\left(\beta_i \tilde{R}_m\right) + \text{Var}(\tilde{\epsilon}_i) = \beta_i^2 \sigma_m^2 + \sigma_{\epsilon_i}^2$$

- $\beta_i^2 \sigma_m^2$ represents amount of systematic (market) risk, while $\sigma_{\epsilon_i}^2$ represents amount of idiosyncratic (asset-specific) risk
- In reality, will have many different types of systematic risk
- CAPM assumes that different types of systematic risk can be combined into aggregate pool of "market" risk

Effect of Diversification – Part 1

- Consider return for two-asset portfolio with weight of $w \in (0, 1)$ on first asset and remainder on second asset:

$$\begin{aligned} \tilde{R}_p - R_f &= w\left(\tilde{R}_1 - R_f\right) + (1 - w)\left(\tilde{R}_2 - R_f\right) \\ &= \beta_p\left(\tilde{R}_m - R_f\right) + \tilde{\epsilon}_p \end{aligned}$$

- Portfolio beta is weighted average of individual asset betas, and portfolio idiosyncratic risk factor is also weighted average of individual asset-specific idiosyncratic risk factors:

$$\begin{aligned} \beta_p &= w\beta_1 + (1 - w)\,\beta_2 \\ \tilde{\epsilon}_p &= w\tilde{\epsilon}_1 + (1 - w)\,\tilde{\epsilon}_2 \end{aligned}$$

Effect of Diversification – Part 2

- Suppose that both assets have same amount of market risk: $\beta_1 = \beta_2 = \beta_p \implies$ market risk is undiversifiable since portfolio also has same amount of market risk
- Suppose that both assets have same amount of idiosyncratic risk: $\sigma_{\epsilon_1} = \sigma_{\epsilon_2} = \sigma_\epsilon$, and also that idiosyncratic risk factors are uncorrelated: $\text{Corr}(\tilde{\epsilon}_1, \tilde{\epsilon}_2) = 0$
- Idiosyncratic risk is diversifiable since portfolio has less idiosyncratic risk than either asset:

Portfolio:

+	+	+
−	−	−
+	−	0
−	+	0

$$\sigma^2_{\epsilon_p} = \text{Var}[w\tilde{\epsilon}_1 + (1-w)\,\tilde{\epsilon}_2] = \left(1 - 2w + 2w^2\right)\sigma^2_\epsilon < \sigma^2_\epsilon$$

- In fact, diversification applies as long as $\text{Corr}(\tilde{\epsilon}_1, \tilde{\epsilon}_2) < 1$

Effect of Diversification – Part 3

- Now consider equal-weighted portfolio of n different assets with same amount of idiosyncratic risk: $w_i = 1/n$
- Assuming idiosyncratic risk factors are uncorrelated:

$\tilde{\varepsilon}_p = \frac{1}{n}\sum_{i=1}^{n}\tilde{\varepsilon}_i$

$Var(\tilde{\varepsilon}_p) = \frac{1}{n^2}\cdot var\left(\sum_{i=1}^{n}\tilde{\varepsilon}_i\right)$

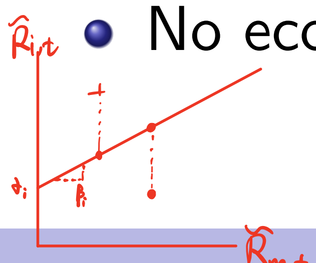
$= \frac{1}{n^2}\sum_{i=1}^{n} Var(\tilde{\varepsilon}_i)$ $= \sigma_\varepsilon^2$ $= n\sigma_\varepsilon^2$

$= \frac{\sigma_\varepsilon^2}{n}$

$$\sigma^2_{\epsilon_p} = \mathrm{Var}\left[\sum_{i=1}^{n} w_i \tilde{\epsilon}_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(\tilde{\epsilon}_i) = \frac{\sigma^2_\epsilon}{n}$$

$\sigma_{\varepsilon p} = \frac{\sigma_\varepsilon}{\sqrt{n}}$

- Hence portfolio will have 99% less idiosyncratic risk (by variance) than any individual asset for $n = 100$
- Optimal for risk-averse investor to hold "well-diversified" portfolio with $\sigma_{\epsilon_p} \approx 0$, such as market portfolio
- No economic compensation for bearing idiosyncratic risk



OLS

$Var(\hat{\varepsilon}_i)$

$\hat{\beta}_i = \frac{\sigma_{im}}{\sigma_m^2}$

Market Model

- Market model is one-factor linear regression model with (excess) asset return as dependent variable and (excess) market return as explanatory variable:

$$\tilde{R}_i - R_f = \alpha_i + \beta_i \left(\tilde{R}_m - R_f \right) + \tilde{\epsilon}_i$$

- Use well-diversified value-weighted stock index to represent market portfolio, such as S&P 500 index for U.S. market
- Slope coefficient provides estimate of asset beta
- If CAPM is correct, then intercept coefficient will be zero $\Longrightarrow$ intercept coefficient represents "pricing error" (relative to CAPM) for individual assets or "passive" portfolios

Market Price of Risk – Part 1

$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$

$\beta_i(R_m - R_f) = \rho_{im}\sigma_i\left(\frac{R_m - R_f}{\sigma_m}\right)$

$= \frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2}$

$= \frac{\rho_{im}\sigma_i}{\sigma_m}$

- Let $\rho_{im} = \sigma_{im}/\sigma_i\sigma_m$ be correlation of return between asset i and market portfolio, so that $\beta_i = \rho_{im}\sigma_i/\sigma_m$:

$$R_i - R_f = \rho_{im}\sigma_i\left(\frac{R_m - R_f}{\sigma_m}\right) = \rho_{im}\sigma_i S_m$$

- Here $S_m = (R_m - R_f)/\sigma_m$ is Sharpe ratio of market portfolio, which represents market price of systematic risk
- Let w_{mi} be i'th element of $\mathbf{w}_m$ and let $\mathbf{v}_i$ be i'th row of $\mathbf{V}$:

$$\frac{\partial\sigma_m^2}{\partial w_{mi}} = \frac{\partial\mathbf{w}_m'\mathbf{V}\mathbf{w}_m}{\partial w_{mi}} = 2\mathbf{v}_i\mathbf{w}_m = 2\sum_{j=1}^{n} w_{mj}\sigma_{ij}$$

Market Price of Risk – Part 2

- Use $\tilde{R}_m = \sum_{j=1}^{n} w_{mj}\tilde{R}_j$ to determine covariance of returns:

$$\sigma_{im} = \text{Cov}\left(\tilde{R}_i, \tilde{R}_m\right) = \text{Cov}\left(\tilde{R}_i, \sum_{j=1}^{n} w_{mj}\tilde{R}_j\right) = \sum_{j=1}^{n} w_{mj}\sigma_{ij}$$

- Hence $\rho_{im}\sigma_i$ represents marginal increase in (systematic) market risk from marginal increase in weight on asset i:

$$\frac{\partial \sigma_m}{\partial w_{mi}} = \frac{1}{2\sigma_m}\frac{\partial \sigma_m^2}{\partial w_{mi}} = \frac{1}{\sigma_m}\sum_{j=1}^{n} w_{mj}\sigma_{ij} = \frac{\sigma_{im}}{\sigma_m} = \rho_{im}\sigma_i$$

Zero-Beta CAPM

- Original version of CAPM developed by William Sharpe and John Lintner (and others) assumes existence of riskless asset
- Later, Fischer Black developed “zero-beta CAPM” that doesn’t require existence of riskless asset
- Zero-beta portfolio should have same expected return as riskless asset, so can use in place of risk-free rate
- But how to find expected return for zero-beta portfolio?
- Zero-beta portfolio is orthogonal to market portfolio
- Hence expected return of zero-beta portfolio is given by y-intercept for line that is tangent to efficient frontier (for risky assets) at market portfolio

Pricing Anomalies

- Some types of assets consistently deliver higher risk premiums than predicted by CAPM $\implies$ consistently lies above SML
- Most likely explanation is that investing in these assets classes brings exposure to other types of systematic risk that are separate from market risk (and hence not reflected in beta)
- Investing in "small-cap" stocks (with low market capitalisation) consistently delivers higher Treynor ratio (but not Sharpe ratio) compared to investing in "big-cap" stocks
- Investing in "value" stocks (with low price-to-book ratio) consistently delivers higher Treynor ratio compared to investing in "growth" stocks (with high price-to-book ratio)

Size Effect – Part 1

- Comparison of reward-to-risk ratios for well-diversified portfolios of small-cap and big-cap stocks:

	Small-Cap	Big-Cap
Risk Premium	13.2%	8.6%
Std Dev of Ret	32%	20%
Beta	1.3	1.0
Sharpe Ratio	0.41	0.43
Treynor Ratio	10.2%	8.6%

- Based on arithmetic mean of annual returns in US financial markets from 1926 to 2016

Size Effect – Part 2

- Small-cap stocks are more risky than big-cap stocks, based on comparison of standard deviation of return and beta
- Small-cap stocks have about 60% more systematic risk than big-cap stocks, based on comparison of std dev of return
- Small-cap stocks have about 30% more exposure to market risk than big-cap stocks, based on comparison of beta
- Small-cap stocks are affected by systematic “size risk”, which is separate from market risk (and doesn’t affect big-caps)
- Hence CAPM doesn’t provide accurate prediction of risk premium for small-cap stocks