QF605 Fixed-Income Securities
Assignment 2, Due Date: 26-Feb-2025

session 4

1. Suppose the swap rate follows the stochastic differential equation

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(0) dW^{n+1,N},$$

derive a valuation formula for a payer swaption in the Black normal model

$$V^{pay}_{n,N}(0) = P_{n+1,N} \mathbb{E}^{n+1,N}[(S_{n,N}(T) - K)^+].$$

where $W^{n+1,N}$ is a Brownian Motion under measure $Q^{n+1,N}$.

The drift term is 0.

$$\int_0^T dS_{n,N}(t) = \int_0^T \sigma_{n,N} S_{n,N}(0)\, dW^{n+1,N}(t)$$

$$S_{n,N}(T) - S_{n,N}(0) = \sigma_{n,N} S_{n,N}(0) \int_0^T dW^{n+1,N}(t)$$

$$\int_0^T dW^{n+1,N}(t) = W^{n+1,N}(T) - W^{n+1,N}(0)$$

$$\text{let } W^{n+1,N}(0) = 0$$

So the solution is given by

$$S_{n,N}(T) = S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0)\, W^{n+1,W}(T)$$

$\because$ $W^{n+1,N}(T) \sim$ Black normal

$\therefore$ $S_{n,N}(T) \sim N(S_{n,N}(0), \sigma^2_{n,N} S^2_{n,N}(0) T)$

swap rate is a normal distribution R.V under the $Q^{n+1,N}$

Evaluating the expectation, we obtain:

$$V^{payer}_{n,N}(0) = P_{n+1,N}(0)\, E^{n+1,N}[(S_{n,N}(T) - K)^+]$$

$\because$ $S_{n,N}(T) \sim$ Normal distribution

$$\therefore\ Z = \frac{S_{n,N}(T) - S_{n,N}(0)}{\sigma_{n,N} S_{n,N}(0)\sqrt{T}} \sim N(0,1)$$

$$S_{n,N}(T) = S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}\, Z$$

where $d = \frac{S_{n,N}(0) - K}{\sigma_{n,N} S_{n,N}(0)\sqrt{T}}$

$$\therefore E\left[(S_{n,N}(T) - K)^+\right] = E\left[(S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0)\sqrt{T} Z - K)^+\right]$$

$$= \int_K^\infty (S_{n,N}(T) - K) f(S_{n,N}(T))\, dS_{n,N}(T)$$

$$= \int_K^\infty (S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0)\sqrt{T} Z - K)\, \phi(Z)\, dZ$$

$$= \int_K^\infty (S_{n,N}(0) - K)\, \phi Z\, dZ + \sigma_{n,N} S_{n,N}(0)\sqrt{T}\, Z \phi(Z)\, dZ$$

$$= (S_{n,N}(0) - K)\, \Phi(d) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}\, \phi(d)$$

$$\therefore V_{n,N}^{Payer}(0) = P_{n+1,N}(0)\, E^{n+1,N}\left[(S_{n,N}(T) - K)^+\right]$$

$$= P_{n+1,N}(0)\left[(S_{n,N}(0) - K)\, \Phi(d) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}\, \phi(d)\right]$$

2. Consider a displaced-diffusion LIBOR model, defined as

$$dL_i(t) = \sigma_i\big[\beta L_i(t) + (1-\beta)L_i(0)\big]\,dW_t^{i+1},$$

Evaluate the following expectations under the risk-neutral measure $\mathbb{Q}^{i+1}$, associated with the zero-coupon bond $D_{i+1}(t)$:

(a) $\mathbb{E}^{i+1}[L_i(T_i)]$

(b) $\mathbb{E}^{i+1}[(L_i(T_i)-K)^+]$

ⓐ $\because L_i(t)$ have not drift term, it means that $L_i(t)$ is a martingale under Q^{i+1}

$$L_i(t) = L_i(0) + \int_0^t \sigma_i[\beta L_i(u) + (1-\beta)L_i(0)]\,dW^{i+1}(u)$$

$$E^{i+1}[L_i(t)] = L_i(0) + \sigma_i E^{i+1}\left[\int_0^t [\beta L_i(u) + (1-\beta)L_i(0)]\,dW^{i+1}(u)\right]$$

$\because dW^{i+1}$ is a Brownian Motion.

$\therefore E^{i+1}[L_i(t)] = L_i(0)$

$\because t = T_i$

$\therefore E^{i+1}[L_i(T_i)] = L_i(0)$

ⓑ $$dL_i(t) = \sigma_i[\beta L_i(t) + (1-\beta)L_i(0)]\,dW_t^{i+1}$$

$$= \sigma_i\beta L_i(t)\,dW_t^{i+1} + \sigma_i(1-\beta)L_i(0)\,dW_t^{i+1}$$

there is a GBM: $dL_i(t) = \sigma_i\beta L_i(t)dW_t^{i+1}$, and $\int_0^T dW_t^{i+1} = 0$

$$d(\ln L_i(t)) = \frac{1}{L_i(t)}dL_i(t) - \frac{1}{2}\frac{1}{L_i^2(t)}(dL_i(t))^2$$

plug $dL_i(t)$ into:

$$d(\ln L_i(t)) = \sigma_i\beta\,dW_t^{i+1} - \frac{1}{2}\sigma_i^2\beta^2 dt$$

$$\ln[L_i(t)] = \ln L_i(0) + \sigma_i\beta W_T^{i+1} - \frac{1}{2}\sigma_i^2\beta^2 T \sim N\left(\ln L_i(0) - \frac{1}{2}\sigma_i^2\beta^2 T,\ \sigma_i^2\beta^2 T\right)$$

$$L_i(t) = L_i(0) + e^{\sigma_i\beta W_t^{i+1} - \frac{1}{2}\sigma_i^2\beta^2 T}$$

$$E^{i+1}[(L_i(T_i)-K)^+] = E^{i+1}[L_i(T_i)\mathbb{1}_{L_i(T_i)>K}] - KE^{i+1}[\mathbb{1}_{L_i(T_i)>K}]$$

$$= L_i(0)e^{-\frac{1}{2}\sigma_i^2\beta^2 T}\Phi(d_1) - K\Phi(d_2)$$

$$\therefore d_1 = \frac{\ln\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma_i^2\beta^2 T}{\sigma_i\beta\sqrt{T}} \qquad d_2 = d_1 - \sigma_i\beta\sqrt{T}$$

3. Write down the expectation of a receiver swaption payoff maturing at T and struck at K. Show that we cannot evaluate the expectation under $\mathbb{Q}^*$, the risk-neutral measure associated with the risk-free money market account numeraire $B_t = B_0 e^{\int_0^t r_u\,du}$, but by changing the measure to $\mathbb{Q}^{n+1,N}$, the risk-neutral measure associated with the present value of a basis point (PVBP) numeraire $P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)$, we can derive an analytical expression for the receiver swaption.

Payoff of the receiver swaption is:

$$V^{rec}_{n,N}(T) = P_{n+1,N}(T)\,(K - S_{n,N}(T))$$

$S_{n,N}(T) \sim$ lognormal under the risk-netural measure Q^*,

but the drift term is complicated, and it's not easy to slove.

Discount factor $D_i(T)$ is affected by the short term interest rate model,

and it's not easy to slove the expected value.

So if we need to change the measure to $Q^{n+1,N}$.

$S_{n,N}(t)$ became to the martingle.

$$\therefore \frac{dS_{n,N}(t)}{S_{n,N}(t)} = \sigma_{n,N}\, dW^{n+1,N}(t)$$

$$V^{rec}_{n,N}(0) = P_{n+1,N}(0)\, E^{Q^{n+1,N}}\left[(K - S_{n,N}(T))^+\right]$$

PVBP can be discount the swaption cashflow better.

$$S_{n,N}(T) = S_{n,N}(0)\, e^{-\frac{1}{2}\sigma^2_{n,N}T + \sigma_{n,N} W_T^{n+1,N}}$$

$$W_T^{n+1,N} \sim N(0,T)$$

$$E^{Q_{n+1,N}}\left[(K - S_{n,N}(T))^+\right] = K\Phi(-d_2) - S_{n,N}(0)\Phi(-d_1)$$

$$d_1 = \frac{\log\frac{S_{n,N}(0)}{K} + \frac{1}{2}\sigma^2_{n,N}T}{\sigma_{n,N}\sqrt{T}}$$

$$d_2 = d_1 - \sigma_{n,N}\sqrt{T}$$

$$\therefore V^{rec}_{n,N}(0) = P_{n+1,N}(0)\left[K\Phi(-d_2) - S_{n,N}(0)\Phi(-d_1)\right]$$