QF605 Fixed-Income Securities
Assignment 3, Due Date: 12-Mar-2025

1. (a) Write down the LIBOR Market Model (LMM), and identify under what numeraire is the LIBOR process a martingale.

   (b) A contract pays
   $$\Delta_i \times \sqrt{L_i(T)}$$
   at $T = T_{i+1}$. Derive a valuation formula for this contract using LIBOR market model.

   (c) Consider a contract with the following payoff at time $T = T_{i+1}$:
   $$\begin{cases} \$1 & \text{if } K_1 \leq L_i(T) \leq K_2 \\ 0 & \text{otherwise} \end{cases}$$
   Derive a valuation formula for this contract using LIBOR market model.

2. Under the Swap Market Model (SMM), the forward swap rate follows the stochastic differential equation
   $$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}.$$

   (a) What is the numeraire security associated with the risk-neutral measure $\mathbb{Q}^{n+1,N}$, under which $W^{n+1,N}$ is a standard Brownian motion?

   (b) A floating-leg-or-nothing digital option pays
   $$P_{n+1,N}(T) S_{n,N}(T) \mathbb{1}_{S_{n,N}(T)>K}$$
   on maturity T, where $P_{n+1,N}$ is the *present value of a basis point*. Derive a valuation formula for this contract.

   (c) A contract pays
   $$S_{n,N}(T)$$
   on maturity T. Briefly explain why we cannot value this simple contract directly using the Swap Market Model without applying convexity correction.

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Derive a valuation formula for this contract using LIBOR market model.

ⓐ Denote $L_i(t) = L_t(T_i, T_{i+1})$ and $D_i(t) = D(t, T_i)$

Libor rate:

$$\Delta_i L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}$$

If we take the discount bond $D_{i+1}(t)$ as numeraire, then under the martingale measure Q^{i+1} associated with the numeraire $D_{i+1}(t)$, the process $\Delta_i L_i(t)$ must be a martingale.

Since Δ_i is a constant, the process $L_i(t)$ must be a martingale under Q^{i+1}.

ⓑ $\because$ $D(t, T_{i+1})$ is numeraire, and under T_{i+1} forward measure $Q^{T_{i+1}}$.

$$L_i(T) = L_i(t) e^{(-\frac{1}{2}\sigma_i^2 (T-t) + \sigma_i W_T^{(T_{i+1})})}$$

All assets for $D(t, T_{i+1})$ turned into a martingale.

$$\therefore V(t) = D(t, T_{i+1}) \cdot E^{T_{i+1}} [X(T_{i+1}) | F_t]$$

$$= D(t, T_{i+1}) E^{T_{i+1}} [\Delta_i \cdot \sqrt{L_i(T)} | F_t]$$

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t)$$

$$L_i(T) = L_i(t) e^{-\frac{1}{2}\sigma_i^2 (T-t) + \sigma_i W^{i+1}(T)} \sim \text{lognormal}(\ln L_i(t) - \tfrac{1}{2}\sigma_i^2 (T-t), \sigma_i^2 (T-t))$$

In LMM, we knew $E^{T_{i+1}}(\sqrt{L_i(T)}) = e^{\frac{1}{8}\sigma_i^2 (T-t)} \cdot \sqrt{L_i(t)}$

plug it into $V(t) = D(t, T_{i+1}) \Delta_i \cdot e^{\frac{1}{8}\sigma_i^2 (T-t)} \cdot \sqrt{L_i(t)}$

(c) We knew that the risk-netural formula under LMM is:

$$V(t) = D(t,T_{i+1}) E^{T_{i+1}}[X(T_{i+1}) | F_t]$$

In this question, we have $K_1 \le L_i(T) \le K_2$

$$\therefore V(t) = D(t,T_{i+1}) P^{T_{i+1}}(K_1 \le L_i(T) \le K_2 | F_t)$$

$$\because L_i(T) \sim \log N(\ln L_i(t) - \tfrac{1}{2}\sigma_i^2(T-t), \sigma_i^2(T-t))$$

$$\ln L_i(T) \sim N(\ln L_i(t) - \tfrac{1}{2}\sigma_i^2(T-t), \sigma_i^2(T-t))$$

also, $P^{T_{i+1}}(K_1 \le L_i(T) \le K_2 | F_t)$

$$P^{T_{i+1}}(\ln K_1 \le \ln L_i(T) \le \ln K_2)$$

$$P^{T_{i+1}}\left(\frac{\ln K_1 - (\ln L_i(t) - \frac{1}{2}\sigma_i^2(T-t))}{\sigma_i\sqrt{T-t}} \le \frac{\ln L_i(T) - (\ln L_i(t) - \frac{1}{2}\sigma_i^2(T-t))}{\sigma_i\sqrt{T-t}} \le \frac{\ln K_2 - (\ln L_i(t) - \frac{1}{2}\sigma_i^2(T-t))}{\sigma_i\sqrt{T-t}}\right)$$

$$P^{T_{i+1}}\left(\frac{\ln K_1 - (\ln L_i(t) - \frac{1}{2}\sigma_i^2(T-t))}{\sigma_i\sqrt{T-t}} \le Z \le \frac{\ln K_2 - (\ln L_i(t) - \frac{1}{2}\sigma_i^2(T-t))}{\sigma_i\sqrt{T-t}}\right) = \Phi(d_2) - \Phi(d_1)$$

$$d_1 = \frac{\ln K_1 - (\ln L_i(t) - \frac{1}{2}\sigma_i^2(T-t))}{\sigma_i\sqrt{T-t}}$$

$$d_2 = \frac{\ln K_2 - (\ln L_i(t) - \frac{1}{2}\sigma_i^2(T-t))}{\sigma_i\sqrt{T-t}}$$

$$\therefore V(t) = D(t,T_{i+1})[\Phi(d_2) - \Phi(d_1)]$$

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(a) What is the numeraire security associated with the risk-neutral measure $\mathbb{Q}^{n+1,N}$, under which $W^{n+1,N}$ is a standard Brownian motion?

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on maturity T, where $P_{n+1,N}$ is the ***present value of a basis point.*** Derive a valuation formula for this contract.

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$$S_{n,N}(T)$$

on maturity T. Briefly explain why we cannot value this simple contract directly using the Swap Market Model without applying convexity correction.

ⓐ The numeraire is $P_{n+1,N}(t)$ under the risk-netural $Q^{n+1,N}$ and $W^{n+1,N}$.

ⓑ $S_{n,N} \sim GBM$

$\therefore dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t)\, dW^{n+1,N}$

$S_{n,N}(T) = S_{n,N}(0) e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N} W^{n+1,N}}$

$\ln S_{n,N}(T) \sim N(\ln S_{n,N}(0) - \frac{1}{2}\sigma_{n,N}^2 T,\ \sigma_{n,N}^2 T)$

we have a digital option pays: $P_{n+1,N}(T) S_{n,N}(T) 1_{S_{n,N}(T) > K}$

$\therefore V(t) = P_{n+1,N}(t) E^{n+1,N}[\,S_{n,N}(T) 1_{S_{n,N}(T) > K} | F_t]$

$= P_{n+1,N}(t) \cdot S_{n,N}(0)\, \Phi(d_1)$

$$d_1 = \frac{\ln \frac{S_{n,N}(0)}{K} + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}$$

$d_2 = d_1 - \sigma_{n,N}\sqrt{T}$

ⓒ

$\because S_{n,N}(T) = S_{n,N}(0) e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N} W^{n+1,N}(T)}$

$V(t) = P_{n+1,N}(t) E^{n+1,N}[S_{n,N}(T) | F_t]$

$E^{n+1,N}[S_{n,N}(T)] = S_{n,N}(0) e^{\frac{1}{2}\sigma_{n,N}^2 T}$

$e^{\frac{1}{2}\sigma_{n,N}^2 T}$ is correction term, if we only use $S_{n,N}(T)$

we will lead to pricing bias.