QF605 Fixed Income and Securities Project

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Content Table:

Part I (Bootstrapping Swap Curves)

Overview

The objective of this part is to bootstrap OIS and Libor discount factors using the interest rate data provided. The discount curves are then plotted for maturities T ∈ [0, 30]. It is assumed that the swap market is collateralized in cash and overnight interest is paid on collateral posted. Subsequently, the forward swap rates are calculated for the specified expiry-tenor combinations.

Methodology

Bootstrapping the OIS Discount Curve

The objective is to construct the OIS discount curve ***D_0(0,T)*** over the time range of 0 to 30 years. For short-end bootstrapping (Deposits), which is for short maturities, the discount factor is derived using the following equation:

$$D_0(T) = \frac{1}{1 + R \cdot \Delta}$$

Where ***D_0(T)*** is the discount factor at time T, Δ is the year fraction between 0 and T, and R is the interest rate quoted.

For longer-end bootstrapping (Par Swaps), which for tenors more than 1 year, the OIS is calculated as a Present Value (PV) of payer swap.

$$\mathrm{PV}_{\text{payer}} = \mathrm{PV}_{\text{float}} - \mathrm{PV}_{\text{fixed}}$$

The PV of floating leg and fixed leg are calculated using the following equations:

$$\mathrm{PV}_{\text{float}} = 1 - D_0(t_n) \quad \text{and} \quad \mathrm{PV}_{\text{fixed}} = K \cdot \sum_{i=1}^{n} \alpha_i D_0(t_i)$$

Where ***D_0(t_n)***is the discount factor at payment final maturity **t_n**, **K** is the fixed rate, **α_i** is the accrual factor (0.5 for semiannual payments). ***D_0(t_n)*** solved for such that **K** · $\sum_{i=1}^{n} \alpha_i D(t_i) = 1 - D_0(t_n)$. Rewriting the equation:

$$K \cdot \left(\sum_{i=1}^{n-1} \alpha_i D_0(t_i) + \alpha_n D_0(t_n) \right) = 1 - D_0(t_n)$$

$D_0(t_n)$ is then solved using the root-finding numerical method, Brent's method.

A full tenor grid from 0.5 year to 30 years, with 0.5 year time steps is constructed based on the results calculated. Missing discount factors along this curve are estimated using linear interpolation to ensure a smooth and continuous discount curve.

Bootstrapping the LIBOR Discount Curve

The objective of this section is to construct the LIBOR discount curve over the time range of 0 to 30 years. The key assumptions made are:

a) The swap market is collateralized in cash, and collateral earns the OIS rate.
b) Hence, all cash flows are discounted using the OIS curve constructed earlier.

The LIBOR discount factors, D(0,T), are calculated using the PV of payer swap as with the OIS, but instead the IRS rates are used to compute.

$$\mathrm{PV}_{\mathrm{payer}} = (1 - D(0,t_n)) - K \cdot \sum_{i=1}^{n} \alpha_i D(0,t_i)$$

The unknown discount factor ***D(t□)*** is again numerically solved using Brent's method.
Like in the case of OIS, a full tenor grid from 0.5 year to 30 years, with 0.5 year time steps is constructed and missing discount factors along this curve are estimated using linear interpolation.

Calculating Forward Swap Rates

The objective of this section is to compute the forward swap rate starting at *t* with maturity at *T*, denoted as ***t X T*** forward swap, using the formula below:

$$\text{Forward Swap Rate}(t,T) = \frac{\sum_{i=1}^{N} \alpha_i D_0(t_i) \cdot F(t_i)}{\sum_{i=1}^{N} \alpha_i D_0(t_i)}$$

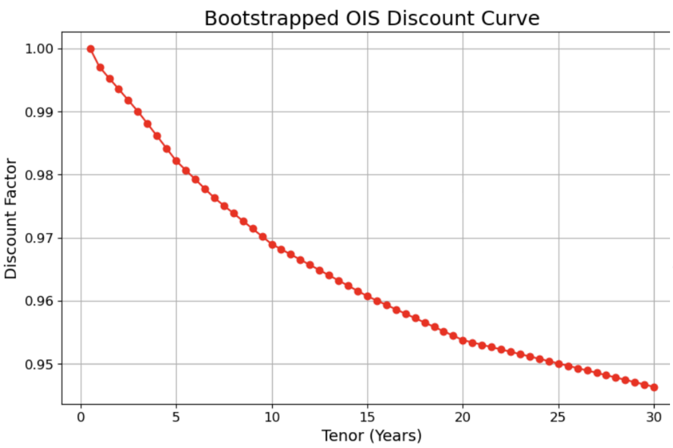
Where **t** is the expiry (start of the forward swap), **T** is the tenor (maturity), α_i is the accrual factor (0.5 for semiannual payments), $D_0(t_i)$ is the OIS discount factor at time t_i, $F(t_i)$ is the forward LIBOR rate at time t_i.
The LIBOR forward rate, **F(t)** is computed as:

$$F(t) = \frac{D(t-\delta) - D(t)}{D(t) \cdot \delta}$$

Where δ is 0.5.

Results

Bootstrapped OIS and LIBOR Discount Curves:



Forward Swap Rates

Expiry (Start of Swap)	Tenor	Forward Swap Rate
1Y	1Y	0.031811
	2Y	0.033082
	3Y	0.033852
	5Y	0.035043
	10Y	0.037802
5Y	1Y	0.038823
	2Y	0.039606
	3Y	0.039438
	5Y	0.040281
	10Y	0.042205
10Y	1Y	0.040428
	2Y	0.041279
	3Y	0.042179
	5Y	0.044149
	10Y	0.053165

Table 1: Forward Swap Rates for Different Starting and Maturity Tenors

Part II (Swaption Calibration)

Overview

This part focuses on modeling and analyzing swaption volatilities using two advanced financial models: the Displaced-Diffusion (DD) model and the Stochastic Alpha Beta Rho (SABR) model. The goal is to calibrate these models to market data, compare their performance, and use them to price swaptions under different strike scenarios. The methodology involves data preparation, model calibration, interpolation of parameters, and swaption pricing.

Methodology

Data preparation**:** The swaption volatility data is loaded from an Excel file and cleaned to ensure consistency. The data is structured with expiry and tenor as indices, and strike levels (ranging from -200bps to +200bps) as columns. The volatilities are converted from percentages to decimals for analysis.

Displaced-Diffusion Model:
The DD model adjusts the standard Black-Scholes model to account for non-linear volatility behavior. The objective is to obtain the **σ** and **β** parameters, which are **volatility** and **displacement coefficient** respectively. The following are the key calculations done under this model:

a) **Black-Scholes Formula for Swaptions(Black 76):**

$$\text{black76\_price}(F, K, \sigma, T, P) = P \cdot (F \cdot N(d_1) - K \cdot N(d_2))$$

Where F is the forward rate, K is the strike price, σ is the volatility, T is time to maturity, P is the discount factor, and N(.) is the cumulative standard normal distribution function.

b) Displaced-Diffusion Volatility Adjustment:

$$\text{displaced\_diffusion\_vol}(F, K, \sigma, \beta, T) = \sigma \cdot \frac{(F + \alpha) \cdot \phi(d_1)}{K + \alpha}$$

Where α=(1−β)*F and ϕ(d1) is the probability density function of d1.

The market volatilities and forward swap rates are fed into the model, and σ and β parameters are iteratively calibrated to minimize the sum of squared differences between market volatilities and model volatilities.

SABR Model

The SABR model captures the volatility smile by incorporating stochastic volatility dynamics.The objective is to obtain the **α (initial volatility), ρ (correlation), and v (vol-of-vol). β** was fixed at the value of 0.9 for all the calculations.   The following are the key calculations done under this model:

a) SABR Volatility Expansion:

$$\sigma_{\text{BS}}(F, K) = \begin{cases} \frac{\alpha}{F^{1-\beta}} \left[1 + \left(\frac{(1-\beta)^2}{24} \cdot \frac{\alpha^2}{F^{2(1-\beta)}} + \frac{\rho\beta\nu\alpha}{4F^{1-\beta}} + \frac{(2-3\rho^2)\nu^2}{24}\right) T\right], & \text{if } F = K \\ \frac{\alpha}{(FK)^{(1-\beta)/2}} \cdot \frac{z}{x(z)} \cdot \left[1 + \left(\frac{(1-\beta)^2}{24} \cdot \frac{\alpha^2}{(FK)^{1-\beta}} + \frac{\rho\beta\nu\alpha}{4(FK)^{(1-\beta)/2}} + \frac{(2-3\rho^2)\nu^2}{24}\right) T\right], & \text{if } F \neq K \end{cases}$$

The model is calibrated with fixed *β* (typically 0.9) and optimized for *α*, *ρ*, and *v*.

The model parameters are calibrated by minimizing the error between market and model volatilities using optimization techniques (scipy.optimize.minimize).Interpolation of parameters (for missing expiries/tenors) is performed using linear interpolation between available data points.

Swaption Pricing

Utilizing the optimized parameters from the previous sections, the following swaptions are priced using the calibrated DD and SABR model:

1. **Payer 2y × 10y** swaptions for strike rates: 1% to 8%
2. **Receiver 8y × 10y** swaptions for strike rates: 1% to 8%

Both payer and receiver swaptions are evaluated across different strike levels.

Results

The following tables summarize the values of the parameters obtained after calibrating the displaced-diffusion and SABR model parameters.

Expiry (Start of Swap)	Tenor	Sigma, σ	Beta, β
1Y	1Y	2.0000	0.8000
	2Y	2.0000	0.8000
	3Y	2.0000	0.8000
	5Y	0.9333	0.4663
	10Y	0.7759	0.2000
5Y	1Y	0.9201	0.8000
	2Y	0.9216	0.8000
	3Y	0.9104	0.8000
	5Y	0.9080	0.8000
	10Y	0.9056	0.8000
10Y	1Y	0.6425	0.8000
	2Y	0.6407	0.8000
	3Y	0.6399	0.8000
	5Y	0.6389	0.8000
	10Y	0.6376	0.8000

Table 2: DD Model Sigma and Beta

Expiry (Start of Swap)	Tenor	Alpha, α	Nu, ν	Rho, ρ
1Y	1Y	0.0068	0.1000	0.2928
	2Y	0.0102	0.1000	0.2914
	3Y	0.0112	0.1000	0.3013
	5Y	0.0096	0.1000	0.4338
	10Y	0.0104	0.1000	0.4929
5Y	1Y	0.0099	0.1000	0.2005
	2Y	0.0118	0.1000	0.2735
	3Y	0.0122	0.1000	0.2925
	5Y	0.0111	0.1000	0.3786
	10Y	0.0119	0.1208	0.8254
10Y	1Y	0.0122	0.1000	0.1822
	2Y	0.0134	0.1000	0.1720
	3Y	0.0164	0.1096	0.8055
	5Y	0.0148	0.1037	0.8473
	10Y	0.0126	0.1000	0.3007

Table 3: SABR Model Alpha, Nu and Rho

The tables below summarize the DD price and SABR price for payer and receiver swaptions respectively.

	Payer 2y X 10y		Receiver 8Y X 10Y	
Strike	DD Price	SABR Price	DD Price	SABR Price
0.01	0.234055	0.249250	0.006000	0.298185
0.02	0.148266	0.154291	0.049872	0.276137
0.03	0.079510	0.068007	0.158419	0.251004
0.04	0.034743	0.032701	0.349521	0.496876
0.05	0.011663	0.024296	0.625264	0.928791
0.06	0.002785	0.018623	0.969752	1.348741
0.07	0.000431	0.014343	1.357862	1.758275
0.08	0.000039	0.011068	1.767281	2.162482

Table 4: Payer and Receiver Swaption Prices under DD and SABR Model for Different Strikes

The volatility for 1Y x 5Y Swaptions and 5Y x 10Y Swaptions, as an example, are plotted for different strikes, for DD and SABR model to compare them with the market volatilities.

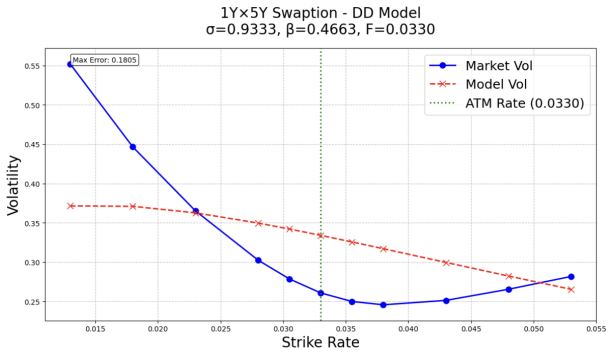


Figure 3: Comparison of DD Model Volatility and Market Volatility for 1Y x 5Y Swaption

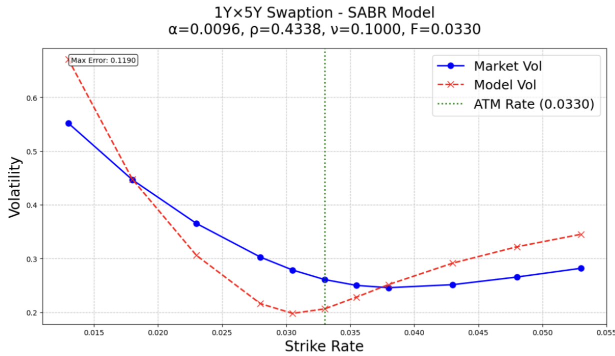


Figure 4: Comparison of SABR Model Volatility and Market Volatility for 1Y x 5Y Swaption

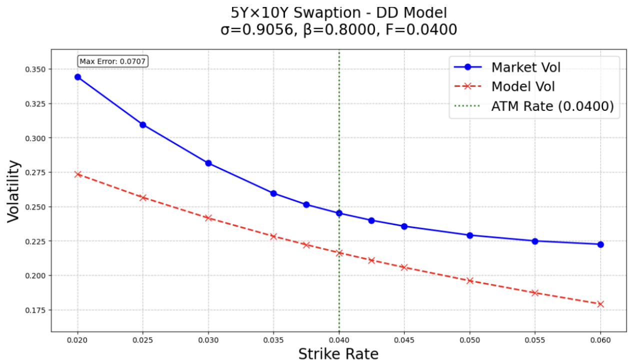


Figure 5: Comparison of DD Model Volatility and Market Volatility for 5Y x 10Y Swaption

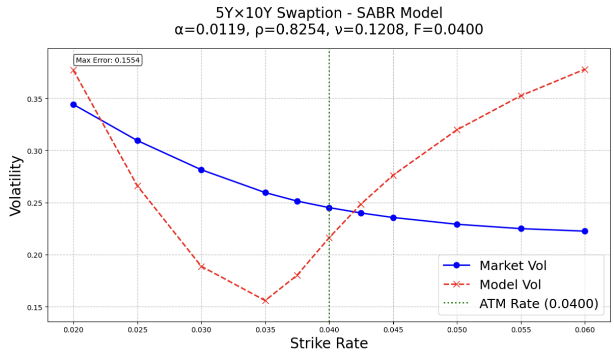


Figure 6: Comparison of SABR Model Volatility and Market Volatility for 5Y x 10Y Swaption

Based on the graphs, it is observed that DD model shows stable calibration with reasonable σ and β values, while the SABR model captures the volatility smile more accurately, especially for extreme strikes. It is also observed that at higher expiries, both models are less accurately able to match the market volatilities, as compared with shorter expiry of 1Y. For longer expiry, SABR was still able to capture the curvature of the smile.

Part III (Convexity Correction)

Overview

In this section, the objective is to explore the impact of convexity in the pricing of Constant Maturity Swaps (CMS). Firstly, the present value (PV) of the two CMS legs using convexity-adjusted CMS rates is calculated. Subsequently, the forward swap rates calculated in Part 1 are compared with CMS rates calculated for different combinations of expiry-tenors. The results are later visualized.

Methodology

For static replication of CMS payoff, g(F), where F is the swap rate, the following formula is used.

$$V_0 = D(0,T)g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)] + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

Where

$$h(K) = \frac{g(K)}{\mathrm{IRR}(K)}$$
$$h'(K) = \frac{\mathrm{IRR}(K)g'(K) - g(K)\mathrm{IRR}'(K)}{\mathrm{IRR}(K)^2}$$
$$h''(K) = \frac{\mathrm{IRR}(K)g''(K) - \mathrm{IRR}''(K)g(K) - 2 \cdot \mathrm{IRR}'(K)g'(K)}{\mathrm{IRR}(K)^2} + \frac{2 \cdot \mathrm{IRR}'(K)^2 g(K)}{\mathrm{IRR}(K)^3}.$$

To calculate the CMS rate payoff, the payoff function is simplified to g(F) = F, and the static replication formula is simplified to:

$$D(0,T)F + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

The value of receiver and payer swaption are calculated using the Black-76 formula, with SABR-implied volatilities. The SABR parameters, alpha, rho and nu calculated earlier, are interpolated for each point.

Calculating PV of CMS10y and CMS2y leg

To calculate PV of leg receiving CMS10y semi-annually over 5 years and CMS2y quarterly over 10 years, the forward swap rates are calculated for 5y x 10y and 10y x 2y respectively. These forward swap rates are calculated based on the OIS and LIBOR discount factors. The discount factors are linearly interpolated where required.
The forward rates calculated are then substituted into the CMS rate payoff equation above to obtain the PV of the two legs.

Comparison of Forward Swap Rates with the CMS Rates

For the expiry-tenor combinations specified, the forward rate swaps calculated in Part 1 are used in this section. The CMS rates for the same expiry-tenor combinations are calculated using the payoff functions defined based on the equations above. The convexity adjustment is calculated using the equation below:

$$\text{Convexity Adjustment} = \mathrm{CMS}(t,T) - F(t,T)$$

Results

The PV of leg receiving CMS10y semi-annually over 5 years is calculated to be **0.2135** and for leg receiving CMS2y quarterly over 10 years is calculated to be **0.4269**.

The table below summarizes the convexity adjustment (convexity correction) values calculated for each expiry-tenor combinations.

Expiry (Start of Swap)	Tenor	Forward Swap rate	CMS Rate	Convexity Correction
1Y	1Y	0.031811	0.031820	0.000009
	2Y	0.033082	0.033094	0.000012
	3Y	0.033852	0.033864	0.000012
	5Y	0.035043	0.035054	0.000011
	10Y	0.037802	0.037817	0.000015
5Y	1Y	0.038823	0.039564	0.000741
	2Y	0.039606	0.040341	0.000735
	3Y	0.039438	0.040153	0.000715
	5Y	0.040281	0.040978	0.000697
	10Y	0.042205	0.043092	0.000887
10Y	1Y	0.040428	0.042874	0.002446
	2Y	0.041279	0.043444	0.002165
	3Y	0.042179	0.044485	0.002306
	5Y	0.044149	0.046414	0.002265
	10Y	0.053165	0.055746	0.002581

Table 5: Convexity Correction Values for Different Expiry-Tenor Combinations

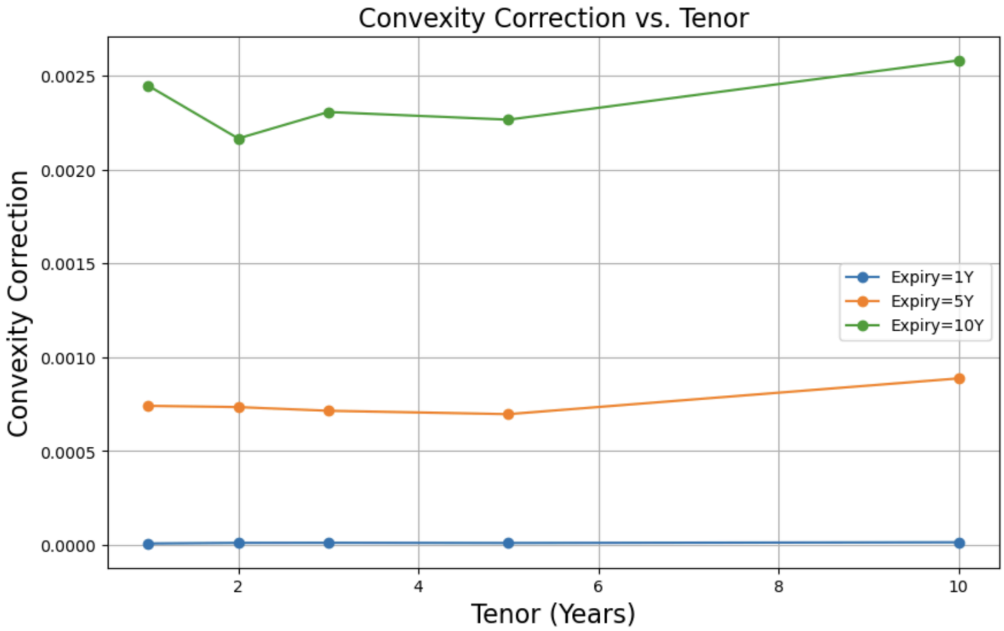


Figure 7: Plot of Convexity Correction vs Tenor for 1Y, 5Y & 10Y Expiries

It is observed that the convexity correction value increases with both longer tenors and expiries. With increasing tenor, there is a higher interest rate exposure as there are more cash flows, leading to more sensitivity to rate changes. With increasing expiries,the CMS leg has more future uncertainty, also leading to larger convexity adjustment required.

Part IV (Decompounded Options)

Overview

The objective of this part is to compute decompounded options' payoff using static replication.

Methodology

Calculating Payoff for CMS10y$^{1/p}$ - 0.04$^{1/q}$

Static replication is used to compute the CMS value, using the following equation:

$$V_0 = D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)\,dK + \int_F^\infty h''(K)V^{pay}(K)\,dK$$

The previously calculated 5Y x 10YForward Swap Rate is used as the input value and as well as the OIS discount factor for 5 years calculated earlier. The corresponding SABR parameters are used to calculate the SABR volatility. The volatility is then fed back into the Black 76 pricing formula to obtain the payer and receiver swaptions.

Putting it all together, the static replication approach, similar to Part 3, is used to compute the CMS pay off.

Calculating Payoff for (CMS 10y$^{1/p}$ - 0.04$^{1/q}$)$^+$

The payoff of this CMS is calculated like a Caplet, hence, only the positive region, which is the payer-style integral, is integrated, using the equation below:

$$CMS\ Caplet = h'(L)V^{pay}(L) + \int_L^\infty h''(K)V^{pay}(K)dK$$

Again, SABR volatility is used in the calculation.

Since this is decompounded CMS,

$\left(F^{1/4} - 0.2\right)^+ > 0$, hence F > 0.2^4 = 0.0016

And we have defined the lower limit L for replication to be L = 0.0016

Results

The Payoff for CMS10y$^{1/p}$ - 0.04$^{1/q}$, where p =4 and q = 2 is calculated to be **0.04220**

The payoff for the (CMS 10y$^{1/p}$ - 0.04$^{1/q}$)$^+$ caplet is calculated to be **0.128110.**