QF605 Fixed-Income Securities
Assignment 4, Due Date: 26-Mar-2025

1. Let S_t denote a forward swap rate at time t. Suppose a CMS product has the following payoff on maturity T:

$$g(S_T) = \begin{cases} 0, & S_T < K_1 \\ S_T - K_1 & K_1 \leq S_T \leq K_2 \\ K_2 - K_1 & S_T > K_2 \end{cases}$$

   Starting with

$$\int_0^\infty h(K) \frac{\partial^2 V^{pay}(K)}{\partial K^2} \, dK$$

   where $h(K) = \frac{g(K)}{\text{IRR}(K)}$, derive the static replication formula for this payoff.

2. The Ho-Lee interest rate model is given by

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

   where W_t^* is a standard Brownian motion under the measure $\mathbb{Q}^*$. Determine the mean and variance of the integral

$$\int_0^T r_u \, du.$$

3. Suppose we use a discrete ($\Delta t = 1y$) binomial-tree approximation of the Ho-Lee model, where at every step the rate can move up or down by 0.5%, and the risk-neutral probabilities of an up or down move are both 0.5. We observe the following discount factors:

Instrument	Value
$D(0, 1y)$	0.9656
$D(0, 2y)$	0.9224
$D(0, 3y)$	0.8903

   Draw the Ho-Lee binomial tree and determine the no-arbitrage values for θ_0 and θ_1.

1. Let S_t denote a forward swap rate at time t. Suppose a CMS product has the following payoff on maturity T:

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Step 1. split integral interval.

Payoff function $g(S_T)$ divided at K_1 and K_2, so we can split the orginal integal into two parts:

$$\int_0^{\infty} h(K) \frac{\partial^2 V^{pay}(K)}{\partial K^2} dK = \int_{K_1}^{K_2} \frac{K-K_1}{IRR(K)} \frac{\partial^2 V^{pay}(K)}{\partial K^2} dK + \int_{K_2}^{\infty} \frac{K_2-K_1}{IRR(K)} \frac{\partial^2 V^{pay}(K)}{\partial K^2} dK$$

Step 2. integrate by parts for each interval

interval $[K_1, K_2]$:

$$\int_{K_1}^{K_2} h_1(K) \frac{\partial^2 V^{pay}}{\partial K^2} dK = \left[h_1 \frac{\partial V^{pay}}{\partial K}\right]_{K_1}^{K_2} - \int_{K_1}^{K_2} h_1' \frac{\partial V^{pay}}{\partial K} dK$$

$$\int_{K_1}^{K_2} h_1' \frac{\partial^2 V^{pay}}{\partial K^2} dK = -\left[h_1' V^{pay}\right]_{K_1}^{K_2} + \int_{K_1}^{K_2} h_1'' V^{pay} dK$$

interval $[K_2, \infty)$:

$$\int_{K_2}^{\infty} h_2(K) \frac{\partial^2 V^{pay}}{\partial K^2} dK = \left[h_2 \frac{\partial V^{pay}}{\partial K}\right]_{K_2}^{\infty} - \int_{K_2}^{\infty} h_2' \frac{\partial V^{pay}}{\partial K} dK$$

$$-\int_{K_2}^{\infty} h_2' \frac{\partial V^{pay}}{\partial K} dK = -\left[h_2' V^{pay}\right]_{K_2}^{\infty} + \int_{K_2}^{\infty} h_2'' V^{pay} dK$$

Step 3. Merge the boundary term and intergal term.

① boundary term: K_1: $-h_1(K_1) \frac{\partial V^{pay}}{\partial K}\Big|_{K_1} + h_1'(K_1) V^{pay}(K_1)$

K_2: $h_1(K_2) \frac{\partial V^{pay}}{\partial K}\Big|_{K_2} - h_1'(K_2) V^{pay}(K_2) - h_2(K_2) \frac{\partial V^{pay}}{\partial K}\Big|_{K_2} + h_2'(K_2) V^{pay}(K_2)$

infinity: assume $h_2(\infty) \frac{\partial V^{pay}}{\partial K}\Big|_{\infty} = 0$

$h_2'(\infty) V^{pay}(\infty) = 0$

② intergal term: $\int_{K_1}^{K_2} h_1''(K) V^{pay}(K) dK + \int_{K_2}^{\infty} h_2''(K) V^{pay}(K) dK$

We have $h_1(K) = \frac{K-K_1}{IRR(K)}$

$$h_1'(K) = \frac{IRR(K) - (K-K_1) IRR'(K)}{IRR(K)^2}$$

$$h_1''(K) = \frac{-IRR''(K)(K-K_1) - 2IRR'(K)}{IRR(K)^2} + \frac{2 \cdot IRR'(K)^2 (K-K_1)}{IRR(K)^3}$$

$$h_2(K) = \frac{K_2-K_1}{IRR(K)}$$

$$h_2'(K) = -\frac{(K_2-K_1) IRR'(K)}{IRR(K)^2} + \frac{2 \cdot IRR'(K)^2 (K_2-K_1)}{IRR(K)^3}$$

Therefore, the final static replication formula is:

$$\frac{1}{IRR(K_1)} V^{Pay}(K_1) - \frac{1}{IRR(K_2)} V^{Pay}(K_2) + \int_{K_1}^{K_2} h_1''(K) V^{Pay}(K) dK + \int_{K_2}^{\infty} h_2''(K) V^{Pay}(K) dK$$

2. The Ho-Lee interest rate model is given by

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$$\int_0^T r_u \, du.$$

integrating both side from 0 to t:

$$\int_0^t dr_s = \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$$

$$r_t - r_0 = \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$$

$$r_t = r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$$

plug r_u into:

$$\int_0^T r_u du = \int_0^T [r_0 + \int_0^u \theta(s)ds + \int_0^u \sigma dW_s^*] du$$

$$= r_0 T + \int_0^T (\int_0^u \theta(s)ds) du + \int_0^T (\int_0^u \sigma dW_s^*) du$$

$$\int_0^T (\int_0^u \theta(s)ds) du = \int_0^T \theta(s) (\int_s^T du) ds = \int_0^T \theta(s)(T-s)ds$$

$$\int_0^T (\int_0^u \sigma dW_s^*) du = \sigma \int_0^T (\int_s^T du) dW_s^* = \sigma \int_0^T (T-s) dW_s^*$$

$$\int_0^T r_u du = r_0 T + \int_0^T \theta(s)(T-s)ds + \sigma \int_0^T (T-s) dW_s^*$$

The expecation of a random integral is 0 (a property of incremental Brownian Motion,

So $E[\int_0^T r_u du] = r_0 T + \int_0^T \theta(s)(T-s)ds$

$V[\int_0^T r_u du] = Var[\sigma \int_0^T (T-s) dW_s^*]$

$= \sigma^2 \int_0^T (T-s)^2 ds$

$= \sigma^2 \int_0^T u^2 du$ let $u = T-s$

$= \sigma^2 \frac{T^3}{3}$

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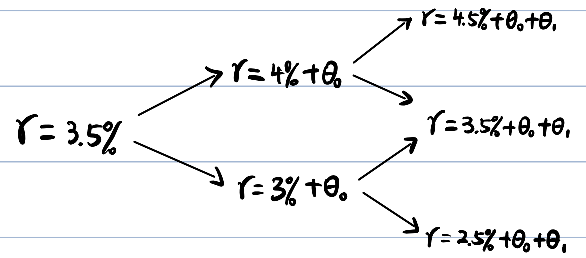

According to 1 year discount factor: $D(0,1y) = 0.9656$, then $r_0 = -\ln(0.9656) \approx 0.035$

$$D(0,2y) = D(0,1y) \times [0.5 \cdot e^{-(4\%+\theta_0)} + 0.5 \cdot e^{-(3\%+\theta_0)}]$$

$$e^{-\theta_0} = \frac{0.9224/0.9656}{0.5(e^{-0.04}+e^{-0.03})}$$

$$e^{-\theta_0} \approx 0.9892$$

$$\theta_0 = -\ln(0.9892) \approx 0.01078 = 1.078\%$$

Then, $D(0,3y) = 0.8903$.

$$D_u(1,3y) = e^{-(4\%+\theta_0)} \cdot [0.5 \cdot e^{-(4.5\%+\theta_0+\theta_1)} + 0.5 \cdot e^{-(3.5\%+\theta_0+\theta_1)}]$$

plug $\theta_0 = 0.01078$:

$$D_u(1,3y) \approx 0.90343 \cdot e^{-\theta_1}$$

$$D_d(1,3y) = e^{-(3\%+\theta_0)} \cdot [0.5 \cdot e^{-(3.5\%+\theta_0+\theta_1)} + 0.5 \cdot e^{-(2.5\%+\theta_0+\theta_1)}]$$

plug $\theta_0 = 0.01078$:

$$D_d(1,3y) = 0.92168 \cdot e^{-\theta_1}$$

$$\therefore D(0,3y) = 0.9656 \cdot [0.5 \times 0.90343 + 0.5 \times 0.92168] e^{-\theta_1}$$

$$e^{-\theta_1} = \frac{0.8903}{0.9656 \times 0.91256} \approx 1.0097$$

$$\theta_1 = -\ln(1.0097)$$

$$\approx -0.01032$$

Finally, $\theta_0 = 0.01078$ and $\theta_1 = -0.01032$