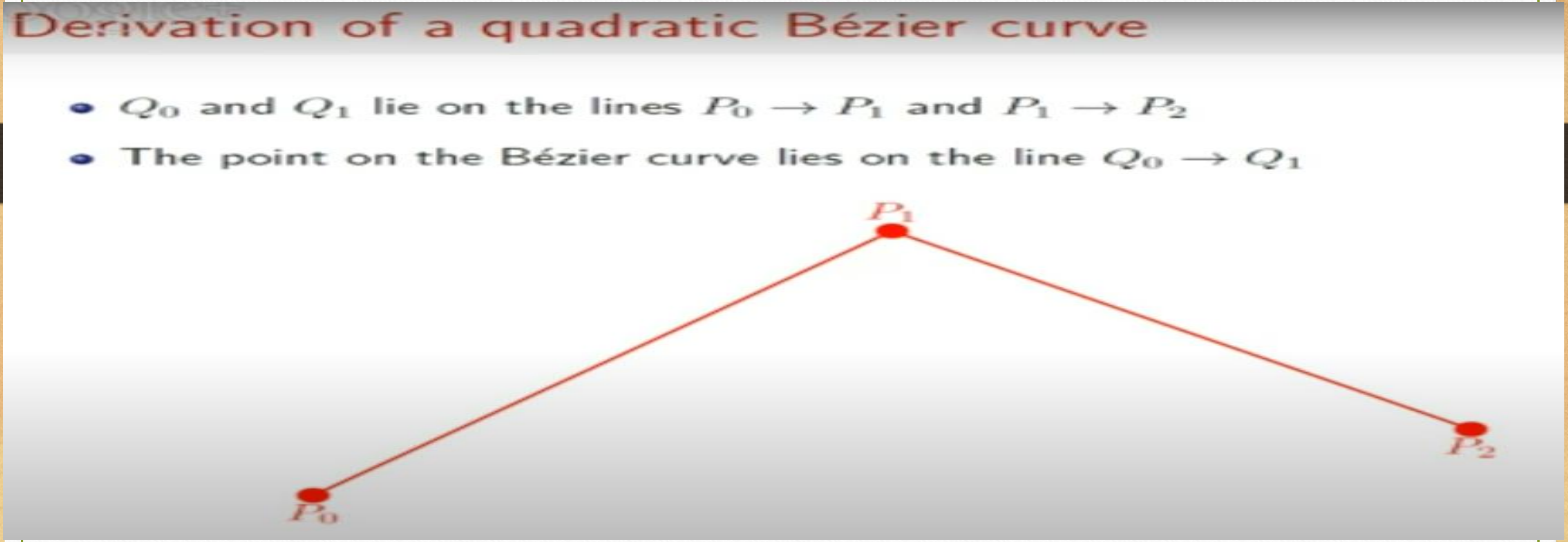
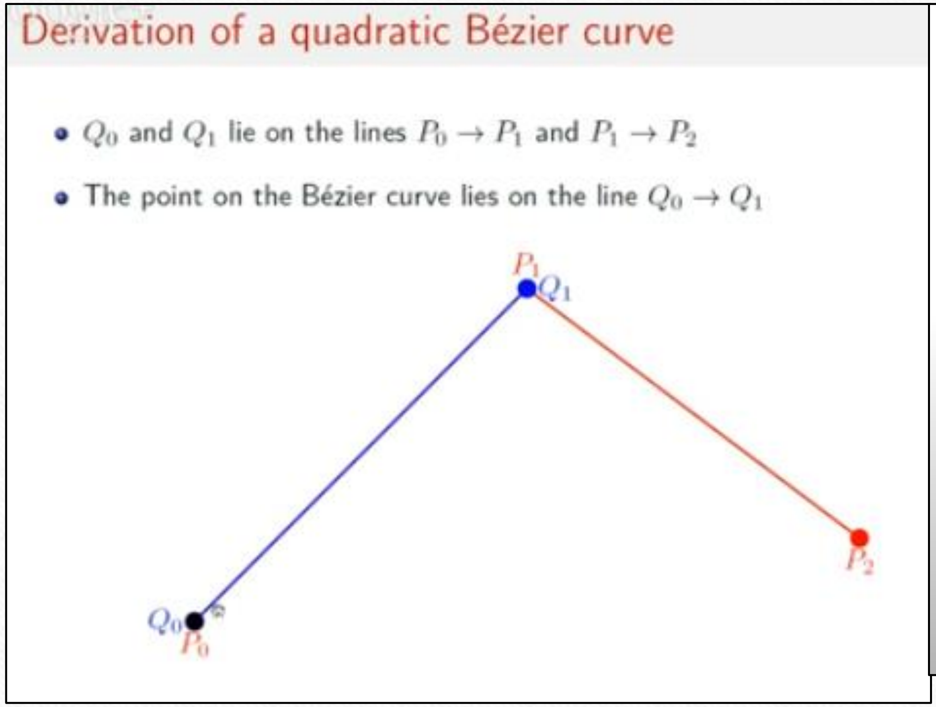
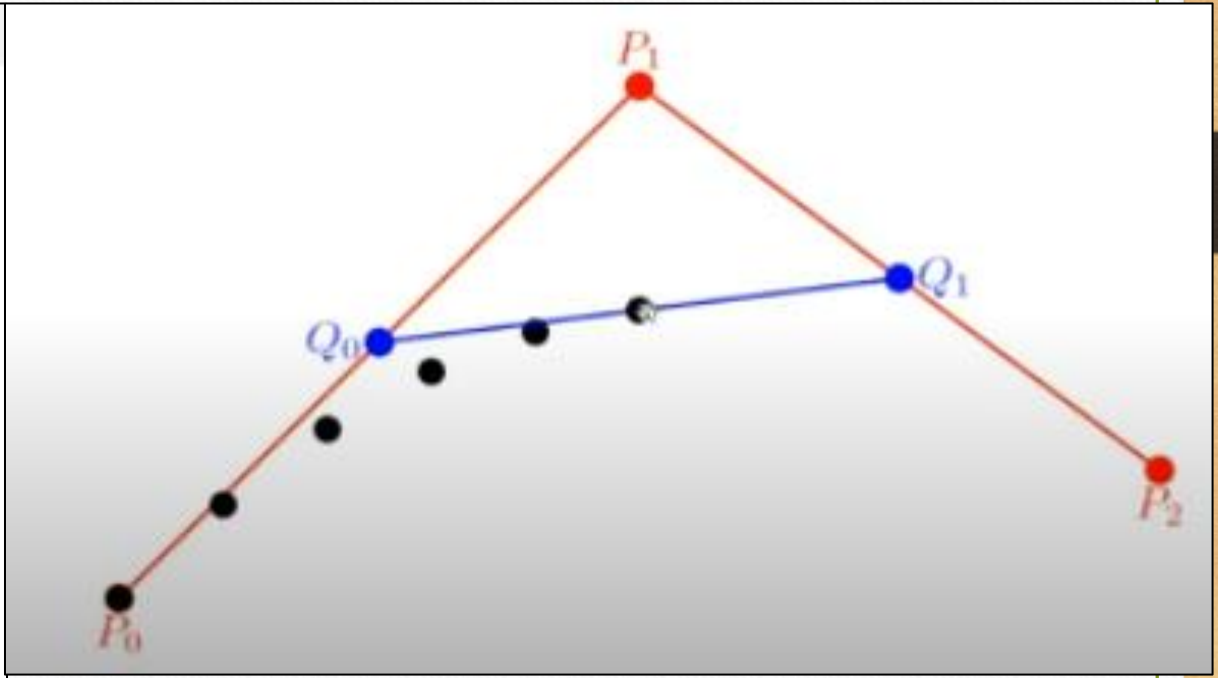
Bezier Curves derivation

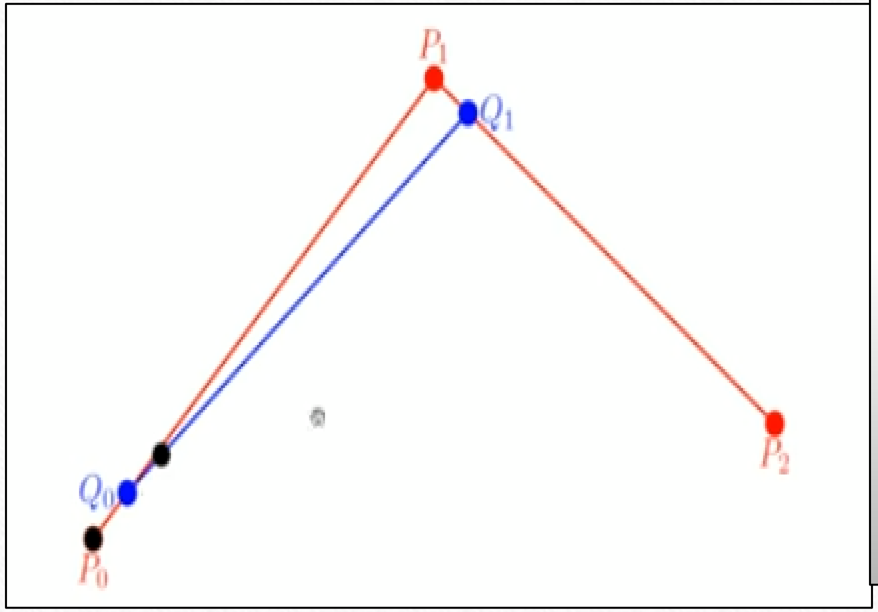
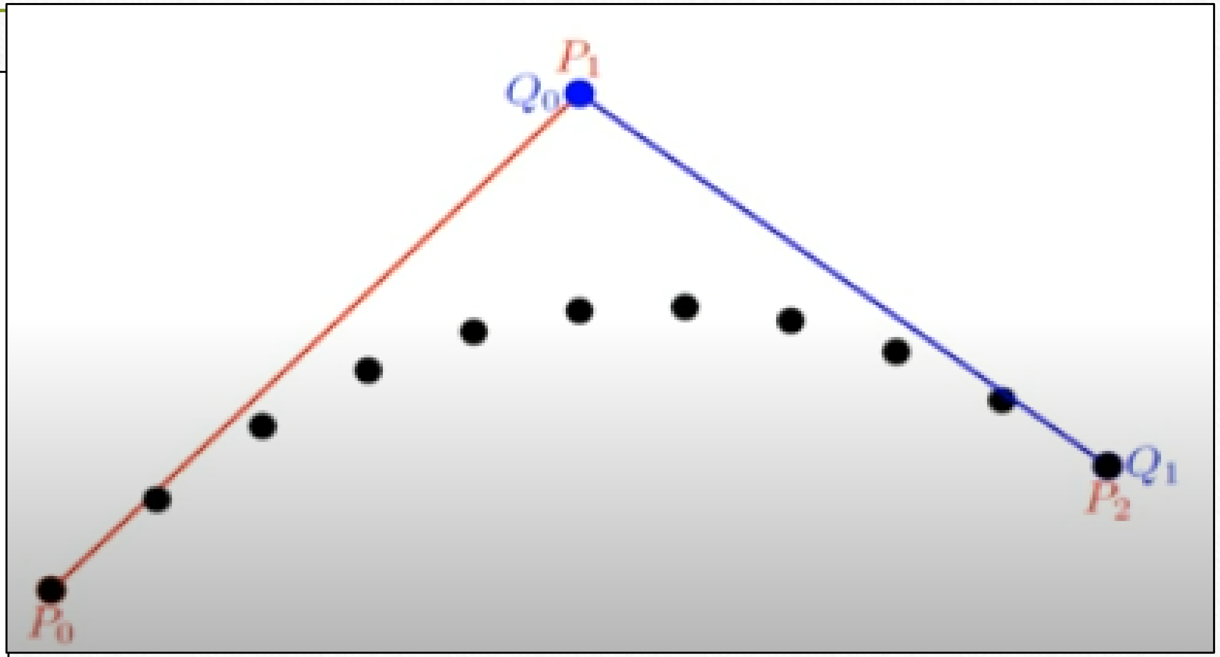
UNIT 3

Bezier Curves dervitation




Derivation of a quadratic Bézier curve
Q_0 and Q_1 lie on the lines $P_0 \to P_1$ and $P_1 \to P_2$
The point on the Bézier curve lies on the line $Q_0 \to Q_1$
P_1
Q_1
Q_0
P_0
P_2


P_1
Q_1
Q_0
P_0
P_2


P_1
Q_1
Q_0
P_0
P_2

P_1
Q_0
Q_1
P_2
P_0

Derivation of a quadratic Bézier curve

- Q_0 and Q_1 are points on the lines $P_0 \rightarrow P_1$ and $P_1 \rightarrow P_2$

$$Q_0 = (1-t)P_0 + tP_1,$$
$$Q_1 = (1-t)P_1 + tP_2.$$

- $C(t)$ is a point on the Bézier curve on the line $Q_0 \rightarrow Q_1$

$$C(t) = (1-t)Q_0 + tQ_1.$$

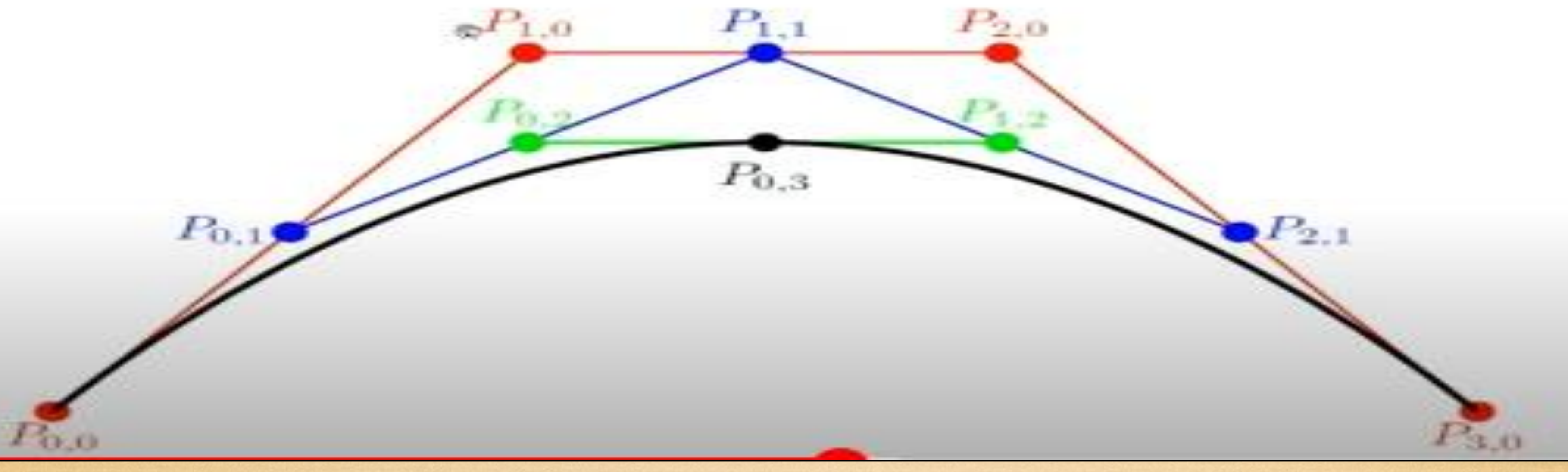
- Combining gives

$$C(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2.$$

Cubic Bézier curve

- A cubic Bézier curve is defined by 4 control points: $P_{0,0}$, $P_{1,0}$, $P_{2,0}$ and $P_{3,0}$

$$P_{0,3} = (1-t)^3 P_{0,0} + 3t(1-t)^2 P_{1,0} + 3t^2(1-t) P_{2,0} + t^3 P_{3,0}.$$



Degree n Bézier curves

- The general form of a degree n Bézier curve defined by the control points P_i (where $i = 0, 1, \ldots, n$) is

$$C(t) = \sum_{i=0}^{n} b_{i,n}(t) P_i,$$

where $b_{i,n}(t)$ are called **Bernstein polynomials** that are defined using

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

and $\binom{n}{i}$ is the Binomial coefficient.

The binomial coefficient

- The **Binomial coefficient** is written using $\binom{n}{i}$ and is read as "n choose i" since it gives the number of ways of choosing i items from a set of n items

$$\binom{n}{i} = \frac{n!}{i!(n-i)!},$$

where $n!$ denotes the factorial of n

Cubic Bernstein polynomials

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

- For example, the Bernstein polynomials for a cubic Bézier curve are

$$\begin{aligned}
b_{0,3}(t) &= \binom{3}{0} t^0 (1-t)^{3-0} = (1-t)^3, \\
b_{1,3}(t) &= \binom{3}{1} t^1 (1-t)^{3-1} = 3t(1-t)^2, \\
b_{2,3}(t) &= \binom{3}{2} t^2 (1-t)^{3-2} = 3t^2(1-t), \\
b_{3,3}(t) &= \binom{3}{3} t^3 (1-t)^{3-3} = t^3.
\end{aligned}$$

Matrix form of a quadratic Bézier curve

- Consider the quadratic Bézier curve with the brackets expanded out

$$C(t) = (t^2 - 2t + 1)P_0 + (-2t^2 + 2t)P_1 + t^2 P_2,$$

this can be expressed in matrix form as

$$C(t) = \begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}.$$

- Similarly a cubic Bézier curve can be expressed using

$$C(t) = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^3 \\ t^2 \\ t \\ 1 \end{pmatrix}.$$