SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
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SCSA3016 DATA SCIENCE

UNIT-I

LINEAR ALGEBRA

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DEPARTMENT OF CSE

SYLLABUS

UNIT 1 LINEAR ALGEBRA

Algebraic view – vectors 2D, 3D and nD, matrices, product of matrix & vector, rank, null space, solution of over determined set of equations and pseudo-inverse. Geometric view - vectors, distance, projections, eigenvalue decomposition, Equations of line, plane, hyperplane, circle, sphere, Hypersphere.

UNIT 2 PROBABILITY AND STATISTICS

Introduction to probability and statistics, Population and sample, Normal and Gaussian distributions, Probability Density Function, Descriptive statistics, notion of probability, distributions, mean, variance, covariance, covariance matrix, understanding univariate and multivariate normal distributions, introduction to hypothesis testing, confidence interval for estimates.

SYLLABUS

UNIT 3 EXPLORATORY DATA ANALYSIS AND THE DATA SCIENCE PROCESS

Exploratory Data Analysis and the Data Science Process - Basic tools (plots, graphs and summary statistics) of EDA -Philosophy of EDA - The Data Science Process – Data Visualization - Basic principles, ideas and tools for data visualization - Examples of exciting projects- Data Visualization using Tableau.

UNIT 4 MACHINE LEARNING TOOLS, TECHNIQUES AND APPLICATIONS

Supervised Learning, Unsupervised Learning, Reinforcement Learning, Dimensionality Reduction, Principal Component Analysis, Classification and Regression models, Tree and Bayesian network models, Neural Networks, Testing, Evaluation and Validation of Models.

UNIT 5 INTRODUCTION TO PYTHON

Data structures-Functions-Numpy-Matplotlib-Pandas- problems based on computational complexity-Simple case studies based on python (Binary search, common elements in list), Hash tables, Dictionary.

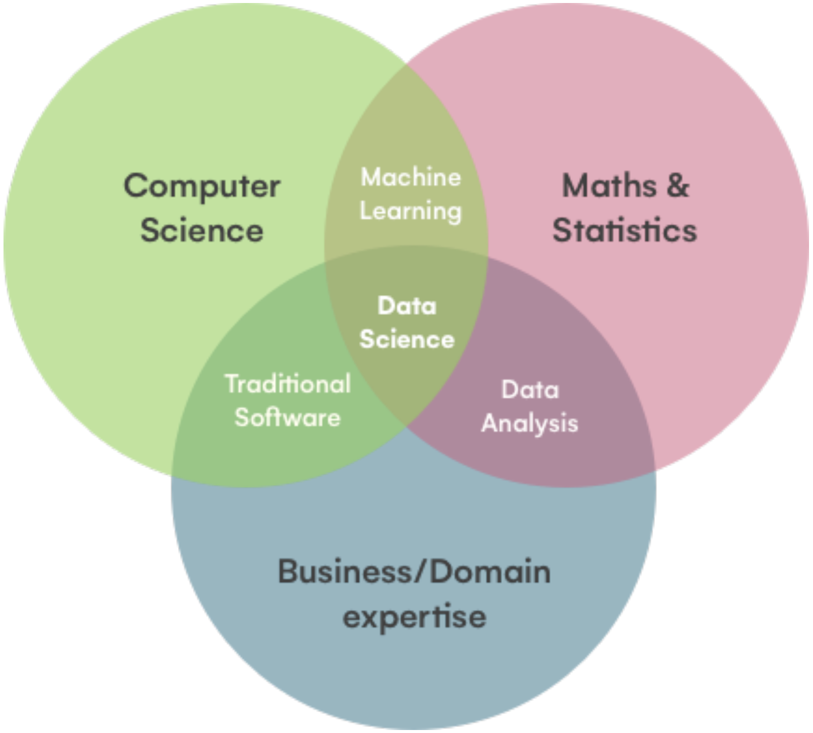
TEXT / REFERENCE BOOKS

1. Cathy O'Neil and Rachel Schutt. Doing Data Science, Straight Talk From The Frontline. O'Reilly. 2014.
2. Introduction to Linear Algebra - By Gilbert Strang, Wellesley-Cambridge Press, 5th Edition.2016.
3. Applied Statistics and Probability For Engineers – By Douglas Montgomery.2016.
4. Jure Leskovek, Anand Rajaraman and Jeffrey Ullman. Mining of Massive Datasets. v2.1, Cambridge University Press. 2014. (free online)
5. Avrim Blum, John Hopcroft and Ravindran Kannan. Foundations of Data Science.
6. Jiawei Han, Micheline Kamber and Jian Pei. Data Mining: Concepts and Techniques, 3rd Edition. ISBN 0123814790, 2011.
7. Trevor Hastie, Robert Tibshirani and Jerome Friedman. Elements of Statistical Learning, 2nd Edition. ISBN 0387952845. 2009. (free online)

What is Data Science?

- **Data Science** is the science of analyzing **raw data** using statistics and machine learning techniques with the purpose of drawing insights from the data
- **Data Science** is used in many industries to allow them to make better business decisions, and in the sciences to test models or theories
- This requires a process of inspecting, cleaning, transforming, modeling, analyzing, and interpreting raw data

DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING



Data perspective

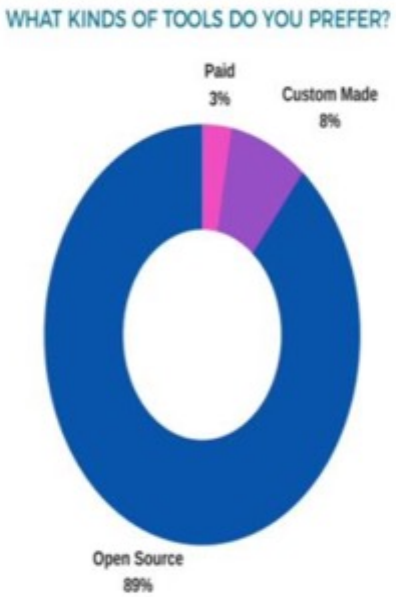
- Read data
- Data processing and cleaning
- Summarizing data
- Visualization
- Deriving insights from data

Data science using Python

- Python libraries provide key feature sets which are essential for data science
- Data manipulation and pre-processing
  - Python's 'pandas' library offers a variety of functions for data wrangling and manipulation
- Data summary
- Visualization
  - Plotting libraries like 'matplotlib' and 'seaborn' aid in condensing statistical information and help in identifying trends and relationships

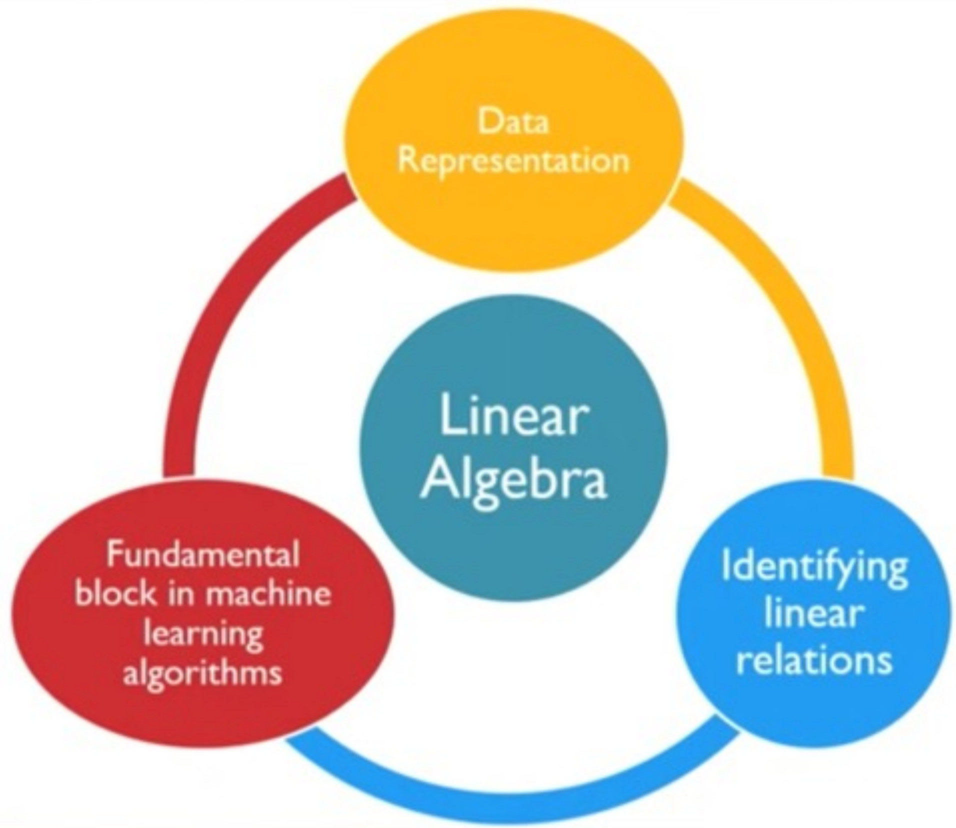
Trends in tools used for data science

As of 2018, most Indian data scientists prefer to use open source tools over paid or custom made tools



https://www.analyticsindiamag.com/wp-content/uploads/2018/12/Data-Science-Skills-Study-2018-By-AIM-Great-Learning.pdf

Overview


Data Representation
Linear Algebra
Fundamental block in machine learning algorithms
Identifying linear relations

Matrix theory and linear algebra

Matrix Theory and Linear Algebra

- Matrices can be used to represent samples with multiple attributes in a compact form
- Matrices can also be used to represent linear equations in a compact and simple fashion
- Linear algebra provides tools to understand and manipulate matrices to derive useful knowledge from data

Matrices for data science: Data representation

- Usually matrices are used to store and represent the data on machines
- Matrix is a very natural approach for organizing data
- In general, data is organized in the following fashion
  - Rows represent samples
  - Columns represent the values of the variables (or attributes)
  - It is also possible to use rows for variables and columns for samples
  - However, we will stick to rows as samples and columns as variables in all of the material that will be presented

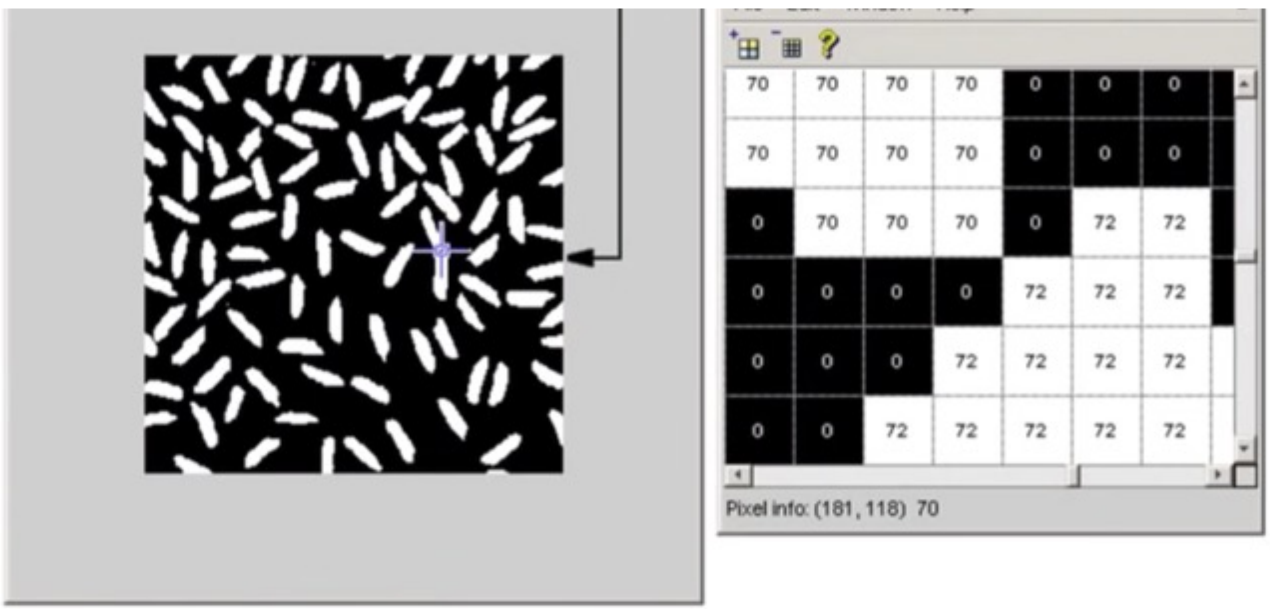
Data representation: Examples

- A real life example
  - Consider a reactor which needs to be controlled using multiple attributes from various sensors like Pressure (Pa), Temperature (K), Density (gm/m^3) etc.
  - Independently, the sensors have generated 1,000 data points
  - This complete set of information is contained in

$$
\begin{array}{c c}
 & \begin{array}{ccc} P & T & \rho \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ 1000 \end{array} & \begin{bmatrix} 300 & 300 & 1000 \\ \vdots & \vdots & \vdots \\ 500 & 1000 & 5000 \end{bmatrix}
\end{array}
$$

Data representation: Examples

- The simplicity in representation will become apparent when the image below is considered



Data as matrix: Summary

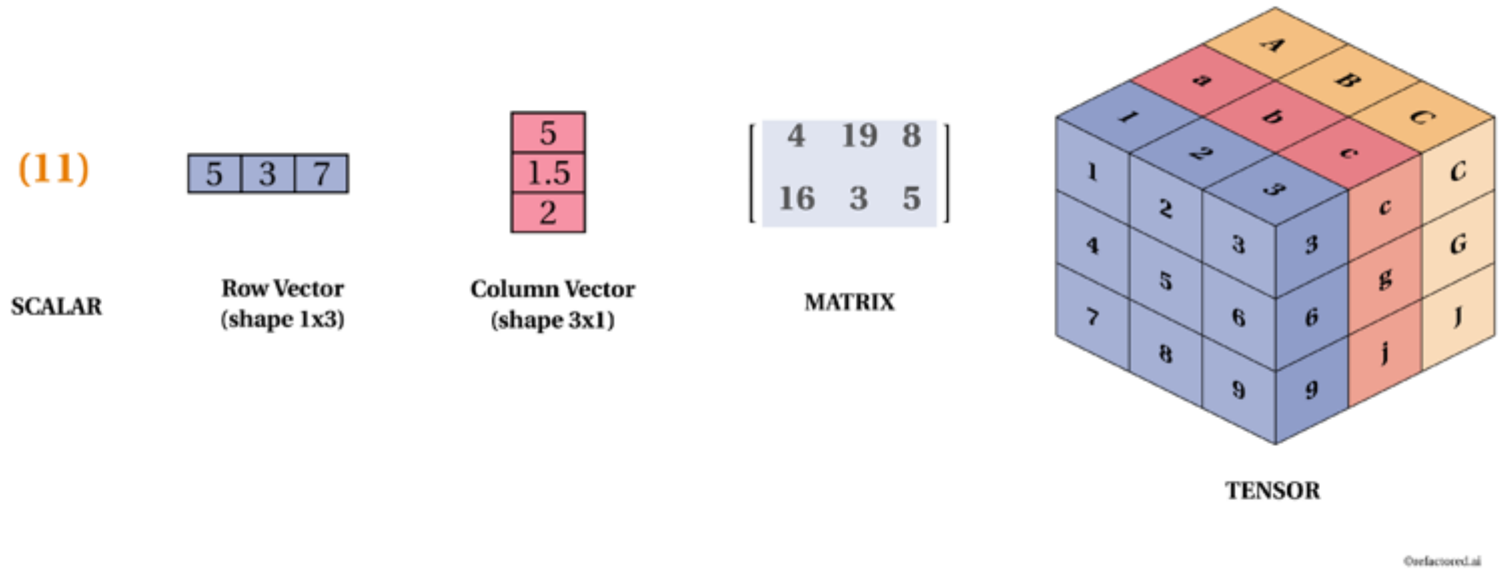
- **Scalar:** Any single numerical value is a scalar as shown in the image above. It is simply denoted by lowercase and italics. For example: n
- **Vector:** An array of numbers(data) is a vector. You can assume a column in a dataset to be a feature vector.

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

- **Matrix**: A matrix is a 2-D array of shape (m×n) with m rows and n columns.

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{bmatrix}$$

- **Tensor:** Generally, an n-dimensional array where n>2 is called a Tensor. But a matrix or a vector is also a valid tensor.


(11)
SCALAR
5 3 7
Row Vector
(shape 1x3)
5
1.5
2
Column Vector
(shape 3x1)
4 19 8
16 3 5
MATRIX
TENSOR
©refactored.ai

What is a vector?

- A **vector** is an object that has both a magnitude and a direction. Geometrically, we can picture a **vector** as a directed line segment, whose length is the magnitude of the **vector** and with an arrow indicating the direction. The direction of the **vector** is from its tail to its head.
- In **machine learning**, feature **vectors** are used to represent numeric or symbolic characteristics, called features, of an object in a mathematical, easily analyzable way.

Types of Vectors

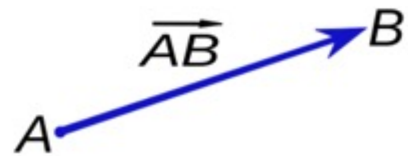
We can represent a vector as a line segment oriented from an initial point, called the tail, to a final point, called the head.

Geometric Vectors

Geometric vectors are not related to any coordinate system.

Algebraic vectors

Algebraic vectors are related to a coordinate system.



A geometric vector is not related to any coordinate system. A is the tail, B is the head.

$$\vec{v} = (1, 2)$$

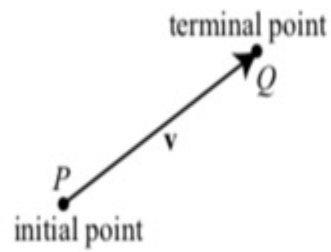
(1,2) is the point located in the coordinate system, where 1 corresponds to x and 2 to y.

Within this algebraic vectors, we can have a **position vector** that connects the origin of the coordinate system (O) with any point (P), and we write it like this:

$$\vec{OP}$$

Geometric Representation of Vectors

Vectors can be represented geometrically by arrows (directed line segments). The **arrowhead** indicates the direction of the vector, and the **length** of the arrow describes the magnitude of the vector.



A vector with initial point P (the tail of the arrow) and terminal point Q (the tip of the arrowhead) can be represented by

$$\overrightarrow{PQ}, \mathbf{v}, or\ \vec{v}. \tag{3.5.1}$$

We often write $v = \overrightarrow{PQ}$. In this text, we will use boldface font to designate a vector. When writing with pencil and paper, we always use an arrow above the letter (such as $\vec{v}$) to designate a vector. The **magnitude** (or **norm** or **length**) of the vector $\mathbf{v}$ is designated by $|\mathbf{v}|$. It is important to remember that $|\mathbf{v}|$ is a number that represents the magnitude or length of the vector $\mathbf{v}$.

According to our definition, a vector possesses the attributes of length (magnitude) and direction, but position is not mentioned. So we will consider two vectors to be equal if they have the same magnitude and direction. For example, if two different cars are both traveling at 45 miles per hour northwest (but in different locations), they have equal velocity vectors. We make a more formal definition.

Perpendicular vectors and Orthogonal vectors

- Two vectors are **perpendicular** if the angle between them is 90 degrees. That means if two vectors are nonzero and their dot product is equal to 0, then they are perpendicular.

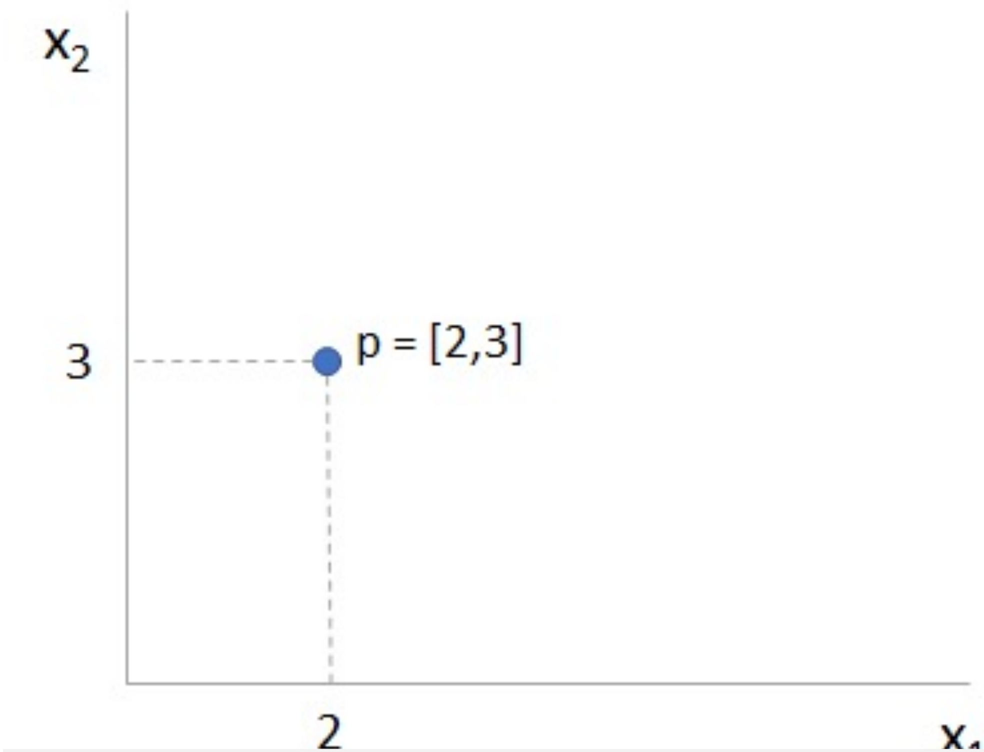
$$\vec{x} \text{ and } \vec{y} \text{ are perpendicular } \rightarrow \vec{x}.\vec{y} = 0$$

$$\vec{x}.\vec{y} = 0\,,\ \vec{x},\ \vec{y} \neq \vec{0} \rightarrow \vec{x} \text{ and } \vec{y} \text{ are perpendicular}$$

- All perpendicular vectors are **orthogonal**.
- The 0 vector is orthogonal to everything else (even to itself).

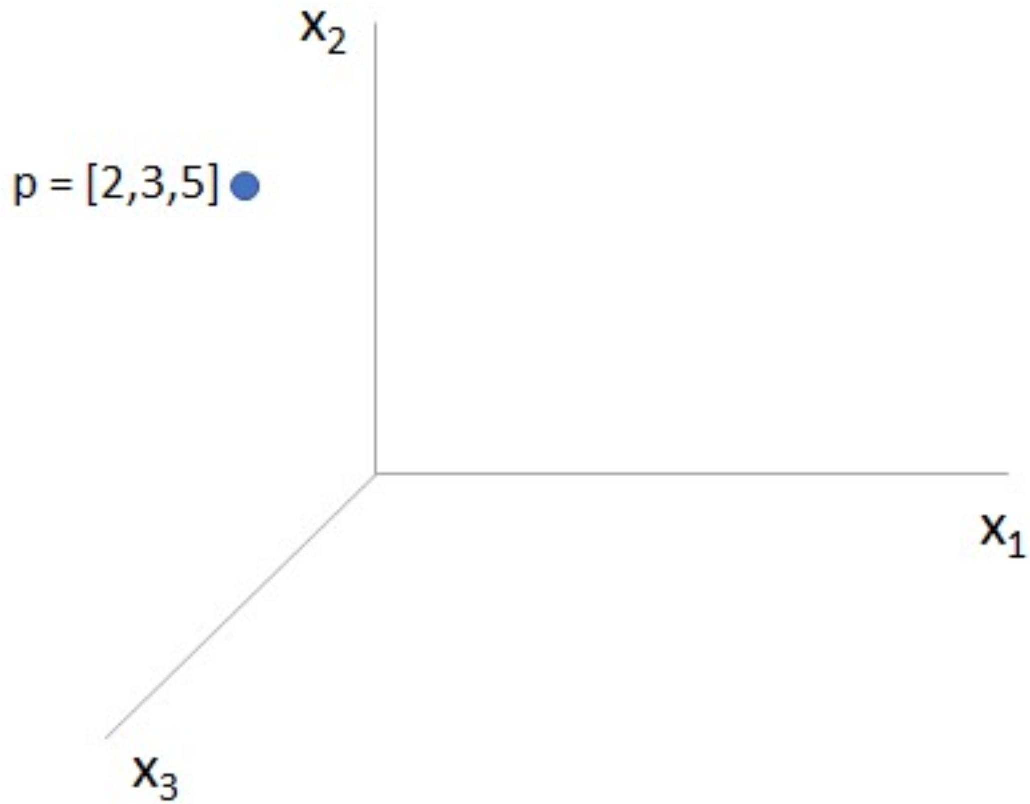
$$\vec{0}.\vec{x} = 0$$

Defining a 2D point/Vector:



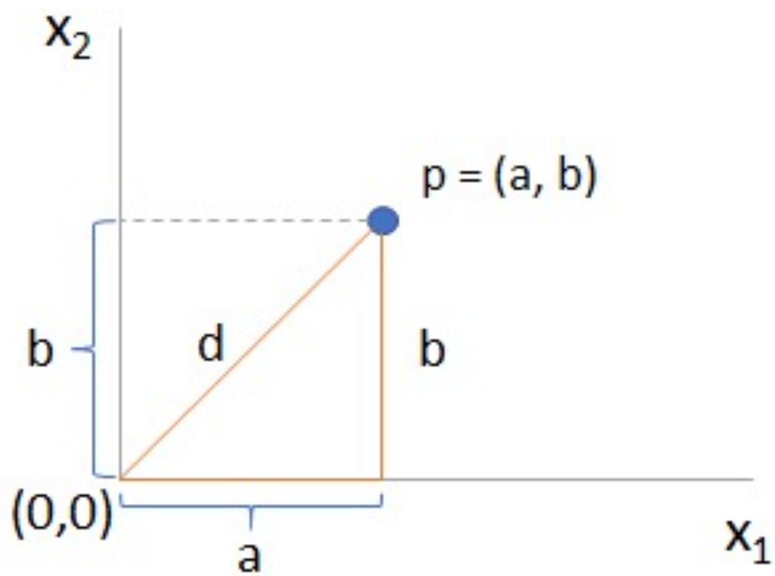
In 2D space, a point is defined as the (x,y) coordinates as shown above.
Here, the x_1 coordinate (x coordinate) is 2, and the x_2 coordinate (y coordinate) is 3.

Defining a point in 3D space:



Extending the 2D concept in 3D space point ‘p’ is defined by (x,y,z) coordinates, where 2 is x_1 coordinate(x coordinate), 3 is x_2 coordinate (y coordinate), and 5 is x_3 coordinate (z coordinate).

Distance of a point from Origin:
(a) In 2D:

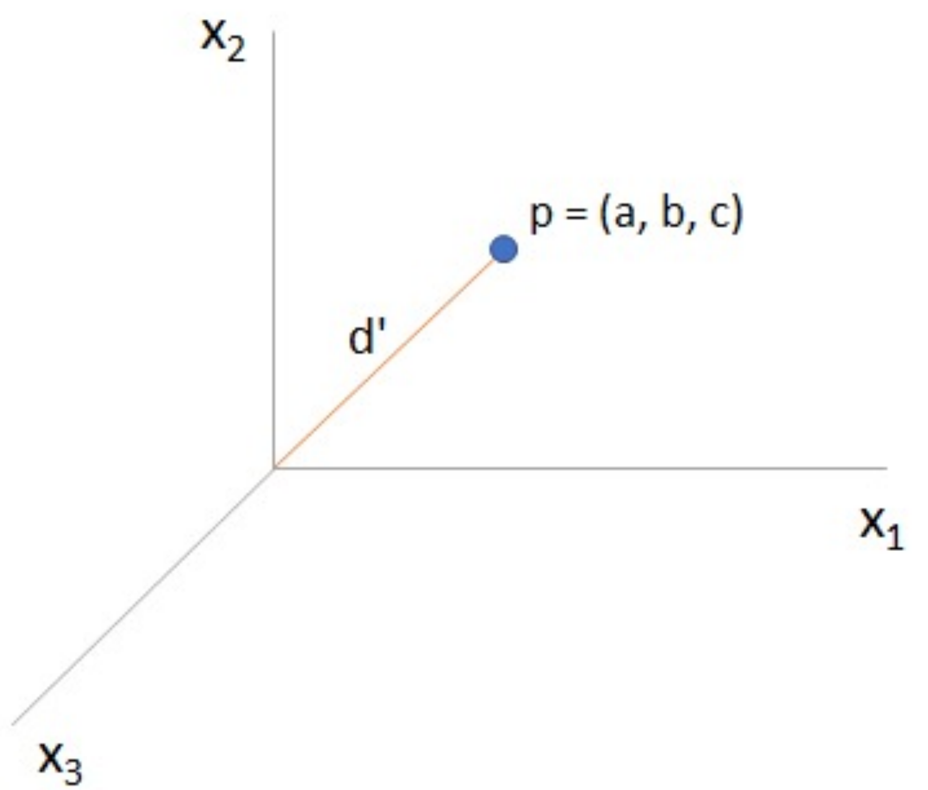


In 2D space, the distance d is given by,

$$d = \sqrt{(a^2 + b^2)} \left\{ \quad \text{where, } p = (a, b) \right.$$

This can be extended to 3D space and beyond for nD space.

(b) In 3D:



In 3D space, the distance d’ is given by,

$$d' = \sqrt{(a^2 + b^2 + c^2)} \Big\{ \quad \text{where } p = (a, b, c)$$

(c) In nD:

In n-Dimensional space applying Pythagoras theorem on point 'p' we get,

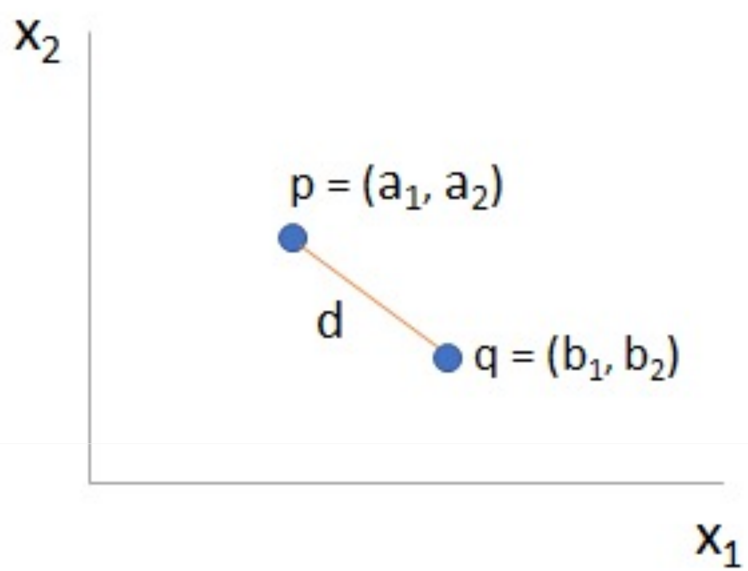
$$d = \sqrt{(a_1^2 + a_2^2 + \ldots + a_n^2)} \Big\{ \quad \text{where } p = (a_1, a_2, .., a_n)$$

Distance between two points

Consider we have two points say, p and q then the distance d for p=(a,b) is given by

$$d = \sqrt{((a_1 - b_1)^2 + (a_2 - b_2)^2)} \begin{cases} \text{where } p = (a_1, a_2) \\ \text{and } q = (b_1, b_2) \end{cases}$$

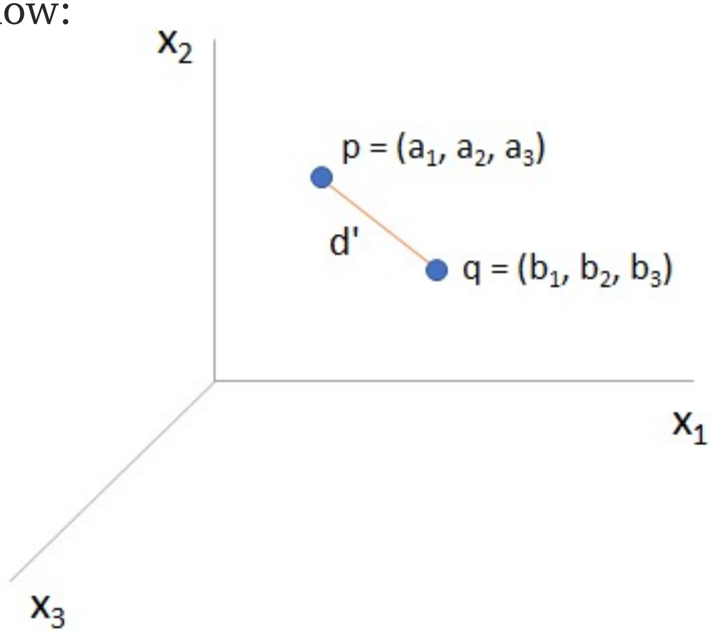
The image for 2D space is as shown below:



Extending the same concept in 3D space we get the distance d' for the points p and q as follows:

$$d = \sqrt{((a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2)} \begin{cases} \text{where } p = (a_1, a_2, a_3) \\ \text{and } q = (b_1, b_2, b_3) \end{cases}$$

The image for 3D space is as shown below:



Extending the above concept in nD space, we get the distance formulae as,

$$d_{pq} = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2} \begin{cases} \text{where } p = (a_1, a_2, \ldots, a_n) \\ \text{and } q = (b_1, b_2, \ldots, b_n) \end{cases}$$

Row-Vector Representation:

The row vector is a (1xn) matrix where the number of rows is 1 and the number of columns is ‘n’.

$$\begin{bmatrix} a_1 & a_2 & a_3 \ldots a_n \end{bmatrix}$$

Column-Vector Representation:

The column vector is an (nx1) matrix where the number of rows is ‘n’ and the number of columns is 1.

$$\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$$

Dot product of Vectors

Dot product

If we have two vectors **u** and **v**:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The dot product (or inner product) of these vectors is defined as the transpose of **u** multiplied by **v**:

$$\mathbf{u}.\mathbf{v} = \mathbf{u}^T\mathbf{v} = [u_1 u_2 \cdots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Based on this definition the dot product is commutative so:

$$\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$$

Dot product of Vectors

Lengh of vector **u**,**v**

Symbol for inner product

Angle between **u** and **v**

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta) \quad (1)$$

$$= x_1 \times x_2 + y_1 \times y_2 \quad (2)$$

$$= \mathbf{u}\mathbf{v}^T \quad (3)$$

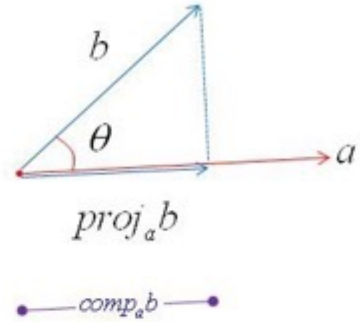
Transpose of vector **v**

(Why do we have to transpose ?)

Projection

Vector Projection

$$proj_a b = \frac{a \cdot b}{|a|^2} a$$

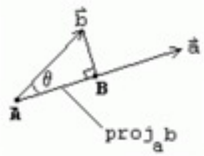


Scalar Projection

$$comp_a b = \frac{a \cdot b}{|a|}$$

Projections

One important use of dot products is in projections. The scalar projection of **b** onto **a** is the *length* of the segment AB shown in the figure below. The vector projection of **b** onto **a** is the *vector* with this length that begins at the point A points in the same direction (or opposite direction if the scalar projection is negative) as **a**.



Thus, mathematically, the scalar projection of **b** onto **a** is |**b**|cos(theta) (where theta is the angle between **a** and **b**) which from (*) is given by

$$comp_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

This quantity is also called the component of **b** in the **a** direction (hence the notation comp). And, the vector projection is merely the unit vector **a**/|**a**| times the scalar projection of **b** onto **a**:

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$

Thus, the scalar projection of **b** onto **a** is the magnitude of the vector projection of **b** onto **a**.

When are two vectors `x` and `y` orthogonal?

Answer

Two vectors are said to be orthogonal if the dot product of them is equal to zero,

$$\vec{x} \cdot \vec{y} = 0$$

This is because the definition of the dot product:

$$\vec{x} \cdot \vec{y} = |x||y| \cos(\theta)$$

where `θ` is the angle between the two vectors, therefore if *x* and *y* are orthogonal, the angle between them is `90` and `cos(90) = 0`.

Projection of a Vector on another vector

The projection of a vector onto another vector is given as $proj_{\vec{v}}(\vec{u}) = \frac{\vec{u}.\vec{v}}{||\vec{v}||^2}\vec{v}$

```
 # import numpy to perform operations on vector
import numpy as np
  u = np.array([1, 2, 3])   # vector u
v = np.array([5, 6, 2])   # vector v:

# Task: Project vector u on vector v

# finding norm of the vector v
v_norm = np.sqrt(sum(v**2))

# Apply the formula as mentioned above
# for projecting a vector onto another vector
# find dot product using np.dot()
proj_of_u_on_v = (np.dot(u, v)/v_norm**2)*v

print("Projection of Vector u on Vector v is: ", proj_of_u_on_v)
```

Output:

Projection of Vector u on Vector v is: [1.76923077 2.12307692 0.7076923

Projection of a Vector onto a Plane

The projection of a vector onto a plane is calculated by subtracting the component of which is orthogonal to the plane from $proj_{Plane}(\vec{u}) = \vec{u} - proj_{\vec{n}}(\vec{u}) = \vec{u} - \frac{\vec{u}.\vec{n}}{||\vec{n}||^2}\vec{n}$

where, $\vec{n}$ is the plane normal vector.

```
# import numpy to perform operations on vector
import numpy as np

u = np.array([1, 2, 3])   # vector u
v = np.array([5, 6, 2])   # vector v:

# Task: Project vector u on vector v

# finding norm of the vector v
v_norm = np.sqrt(sum(v**2))

# Apply the formula as mentioned above
# for projecting a vector onto another vector
# find dot product using np.dot()
proj_of_u_on_v = (np.dot(u, v)/v_norm**2)*v

print("Projection of Vector u on Vector v is: ", proj_of_u_on_v)
```

Output:

Projection of Vector u on Vector v is: [1.76923077 2.12307692 0.7076

Terms related to Matrix

- **Order of matrix** – If a matrix has 3 rows and 4 columns, order of the matrix is 3*4 i.e. row*column.
- **Square matrix** – The matrix in which the number of rows is equal to the number of columns.
- **Diagonal matrix** – A matrix with all the non-diagonal elements equal to 0 is called a diagonal matrix.
- **Upper triangular matrix** – Square matrix with all the elements below diagonal equal to 0.
- **Lower triangular matrix** – Square matrix with all the elements above the diagonal equal to 0.
- **Scalar matrix** – Square matrix with all the diagonal elements equal to some constant k.
- **Identity matrix** – Square matrix with all the diagonal elements equal to 1 and all the non-diagonal elements equal to 0.
- **Column matrix** – The matrix which consists of only 1 column. Sometimes, it is used to represent a vector.
- **Row matrix** – A matrix consisting only of row.
- **Trace** – It is the sum of all the diagonal elements of a square matrix

Matrix Addition, Subtraction, Multiplication

Given the following: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

Matrix **Addition**:

$$A + \mathrm{B} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0+1 & 1+0 \\ -2+3 & -3+(-2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix}$$

Matrix **Subtraction**:

$$A - \mathrm{B} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0-1 & 1-0 \\ -2-3 & -3-(-2) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & -1 \end{bmatrix}$$

Matrix **Multiplication**:

$$AB = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} = = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 3 & 0 \cdot 0 + 1 \cdot (-2) \\ -2 \cdot 1 - 3 \cdot 3 & -2 \cdot 0 - 3 \cdot (-2) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -11 & 6 \end{bmatrix}$$

Basic operations on matrix

- **Addition** – Addition of matrices is almost similar to basic arithmetic addition.

Suppose we have 2 matrices 'A' and 'B' and the resultant matrix after the addition is 'C'. Then

$C_{ij} = A_{ij} + B_{ij}$

For example, let's take two matrices and solve them.

$$A = \begin{matrix} 1 & 0 \\ 2 & 3 \end{matrix}$$

$$B = \begin{matrix} 4 & -1 \\ 0 & 5 \end{matrix} \qquad C = \begin{matrix} 5 & -1 \\ 2 & 8 \end{matrix}$$

Matrix Multiplication

- In matrix multiplication the matrices don't need to be quadratic, but the inner dimensions need to be the same.
- The size of the resulting matrix will be the outer dimensions.

$$[A] \times [B] = [C]$$

$$(n \times m) \qquad (m \times p) \qquad (n \times p)$$

Inner dimensions need to be the same

The resulting matrix will be the **outer** dimensions

A

A(0,0)	A(0,1)
A(1,0)	A(1,1)

B

B(0,0)	B(0,1)
B(1,0)	B(1,1)

A(0,0) * B(0,0)	A(0,1) * B(0,1)
A(1,0) * B(1,0)	A(1,1) * B(1,1)

numpy.multiply(A, B)

```
import numpy as np
# first 2-D array arr1
arr1 = np.array([[3, 0], [0, 4]])
print("first array is :")
print(arr1)
print("Shape of first array is: ", arr1.shape)

# second 2-D array arr1
arr2 = np.array([1, 2])
print("second array is :")
print(arr2)
print("Shape of second array is: ", arr2.shape)

# calculating matrix product
res = np.matmul(arr1, arr2)
print("Resultant array is :")
print(res)
print("Shape of resultant array is: ", res.shape)
```

Dot Product

$$\begin{bmatrix} a & b \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

Dot Product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \bullet \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (1\cdot 5)+(2\cdot 7) & (1\cdot 6)+(2\cdot 8) \\ (3\cdot 5)+(4\cdot 7) & (3\cdot 6)+(4\cdot 8) \end{bmatrix}$$

$$= \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

DoT Matrix

```
A = np.random.randint(5, size=(3,2))
B = np.random.randint(5, size=(2,3))
```

```
A
```

```
array([[2, 2],
       [0, 3],
       [0, 4]])
```

```
B
```

```
array([[2, 1, 2],
       [3, 2, 4]])
```

```
np.dot(A,B)
```

```
array([[10,  6, 12],
       [ 9,  6, 12],
       [12,  8, 16]])
```

Transpose of a Matrix

A general matrix is given by:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Where n is number of rows and m is number of columns

$(n \times m)$

The transpose of matrix A is then:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

$(m \times n)$

Transpose of a Matrix - Examples

$$A = \begin{bmatrix} 1 & 3 & 7 & 2 \\ 5 & 8 & -9 & 0 \\ 6 & -7 & 11 & 12 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 5 & 6 \\ 3 & 8 & -7 \\ 7 & -9 & 11 \\ 2 & 0 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 5 \\ 4 & 5 \\ 3 & 2 \\ 7 & 8 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 1 & 4 & 3 & 7 \\ 5 & 5 & 2 & 8 \end{bmatrix}$$

A

A(0,0)	A(0,1)	A(0,2)
A(1,0)	A(1,1)	A(1,2)

A.transpose()

A(0,0)	A(1,0)
A(0,1)	A(1,1)
A(0,2)	A(1,2)

Transpose

```
import numpy as np
arr1 = np.array([[1, 2, 3], [4, 5, 6]])
print(f'Original Array:\n{arr1}')
arr1_transpose = arr1.transpose()
print(f'Transposed Array:\n{arr1_transpose}’)
```

Output:

```
Original Array:
[[1 2 3]
 [4 5 6]]
Transposed Array:
[[1 4]
 [2 5]
 [3 6]]
```

Determinant of a Matrix

- The Determinant of a matrix is a special number that can be calculated from square matrices

What is the Determinant used for?

- The determinant helps us find the inverse matrix (which we will cover later)
- The Determinant will give us useful information when dealing with Systems of Linear Equations (which we will cover later)
- Used in advanced Control Engineering theory
- Etc.

Determinant of a Matrix

Given a matrix A the Determinant is given by:

$$det(A) = |A|$$

For a $2x2$ matrix we have:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \quad det(A) = |A| = a_{11}\, a_{22} - a_{21}\, a_{12}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \quad det(A) = |A| = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = \underline{\underline{-2}}$$

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc)

Example: find the determinant of

$$C = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

Answer:

$$\begin{aligned} |C| &= 4\times8 - 6\times3 \\ &= 32 - 18 \\ &= 14 \end{aligned}$$

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

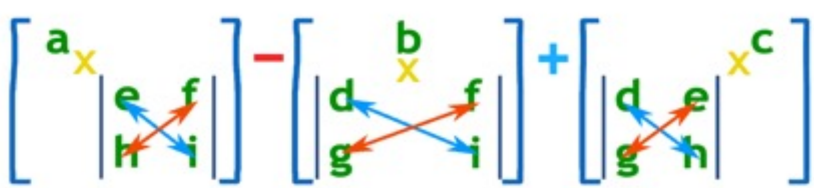
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

It may look complicated, but **there is a pattern**:



$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Example:

$$D = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$\begin{aligned} |D| &= 6\times(-2\times7 - 5\times8) - 1\times(4\times7 - 5\times2) + 1\times(4\times8 - (-2\times2)) \\ &= 6\times(-54) - 1\times(18) + 1\times(36) \\ &= \mathbf{-306} \end{aligned}$$

For 4×4 Matrices and Higher

$$\begin{bmatrix} a & & & \\ & f & g & h \\ & j & k & l \\ & n & o & p \end{bmatrix} - \begin{bmatrix} & b & & \\ e & & g & h \\ i & & k & l \\ m & & o & p \end{bmatrix} + \begin{bmatrix} & & c & \\ e & f & & h \\ i & j & & l \\ m & n & & p \end{bmatrix} - \begin{bmatrix} & & & d \\ e & f & g & \\ i & j & k & \\ m & n & o & \end{bmatrix}$$

As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the **+−+−** pattern (+ a... − b... + c... − d...). This is important to remember.

Determinant of a Matrix

Given the following Matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$det(A) = -2$$

$$B = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & -5 \\ 1 & 4 & -2 \end{bmatrix}$$

$$det(B) = -21$$

Python Solution:

```
-2.0000000000000004
-21.0000000000000001
```

```python
import numpy as np
import numpy.linalg as la

A = np.array([[1, 2],
              [3, 4]])

Adet = la.det(A)

print(Adet)

B = np.array([[-1, 3, 0],
              [2, 1, -5],
              [1, 4, -2]])

Bdet = la.det(B)

print(Bdet)
```

Inverse Matrices

The **inverse** of a quadratic matrix A is defined by: A^{-1} Note: $AA^{-1} = A^{-1}A = I$

For a $2x2$ matrix we have:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The inverse A^{-1} is then given by

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{det(A)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Where:

$$det(A) = |A| = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2$$

This gives: $$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

It will be more complicated for systems with larger order, so we need a programming language like python to handle this.

How do you find the inverse of a 2x2 matrix?

Answer

For an arbitrary A matrix, we can derive it's inverse by following the next steps:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

1. Check if the matrix if invertible by finding its determinant :

$$|A| = ad - bc.$$

   If $|A| \neq 0$ then the matrix is invertible.

2. Interchange the two elements on the diagonal.
3. Take the negatives of the other two elements out of the diagonal.
4. Divide each element of the matrix by $|A|$. The result of the inverse of the matrix A is then:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ c & a \end{pmatrix}$$

What is the *determinant* of a square matrix? How is it calculated?

Answer

The **determinant** is a scalar value that is a *function of the entries* of a square matrix. The determinant of a matrix **A** is denoted **det(A)**, **det A**, or **|A|**. Geometrically, it can be viewed as the *volume scaling factor* of the *linear transformation* described by the matrix.

In the case of a 2×2 matrix the determinant is calculated following the next diagram:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

That is, the determinant is equal to the *product* of the elements along the *plus-labeled arrow* minus the *product* of the elements along the *minus-labeled arrow*.

Similarly, for a 3×3 matrix **A**, its determinant is

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$

Each determinant of a 2×2 matrix in the equation above is called a *minor* of the matrix **A**.

For an $n \times n$ matrix, the previous procedure is extended and provides a *recursive definition* for the determinant, known as a *Laplace expansion*.

Find the *inverse* of the following matrix

Problem

Consider the matrix:

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 5 & 0 & 3 \\ -1 & 2 & 6 \end{bmatrix}$$

Answer

We will compute inverse using the following equation: $A^{-1} = \frac{1}{det(A)} adj(A)$ where *adj(A)* is the *adjugate* of matrix *A*. Now we follow the next steps:

1. **Calculate the determinant of *A*:**

$$det(A) = 4\begin{vmatrix} 0 & 3 \\ 2 & 6 \end{vmatrix} - (-2)\begin{vmatrix} 5 & 3 \\ -1 & 6 \end{vmatrix} + 1\begin{vmatrix} 5 & 0 \\ -1 & 2 \end{vmatrix}$$

$$= 4(0-6) + 2(30+3) + 1(10-0)$$

$$= 52 \neq 0 \therefore \text{the matrix is invertible}$$

2. **Calculate the cofactor** of each element:

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 3 \\ 2 & 6 \end{vmatrix} = -6,\ C_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 3 \\ -1 & 6 \end{vmatrix} = -33,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 5 & 0 \\ -1 & 2 \end{vmatrix} = 10,\ C_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 2 & 6 \end{vmatrix} = 14,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 4 & 1 \\ -1 & 6 \end{vmatrix} = 25,\ C_{23} = (-1)^{2+3}\begin{vmatrix} 4 & -2 \\ -1 & 2 \end{vmatrix} = -6,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 0 & 3 \end{vmatrix} = -6,\ C_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 1 \\ 5 & 3 \end{vmatrix} = -7,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 4 & -2 \\ 5 & 0 \end{vmatrix} = 10.$$

Thus, the cofactor matrix is: $C = \begin{bmatrix} -6 & -33 & 10 \\ 14 & 25 & -6 \\ -6 & -7 & 10 \end{bmatrix}$

3. **Obtain the adjugate matrix** by transposing cofactor matrix

$$adj(A) = \begin{bmatrix} -6 & 14 & -6 \\ -33 & 25 & -7 \\ 10 & -6 & 10 \end{bmatrix}$$

3. **Obtain the adjugate matrix** by transposing cofactor matrix

$$adj(A) = \begin{bmatrix} -6 & 14 & -6 \\ -33 & 25 & -7 \\ 10 & -6 & 10 \end{bmatrix}$$

4. Finally, the inverse matrix is the **adjugate matrix divided by the determinant**:

$$A^{-1} = \frac{1}{52} \cdot \begin{bmatrix} -6 & 14 & -6 \\ -33 & 25 & -7 \\ 10 & -6 & 10 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3/26 & 7/26 & -3/26 \\ -33/52 & 25/52 & -7/52 \\ 5/26 & -3/26 & 5/26 \end{bmatrix}$$

Linear Equations

Given the following linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = b_1$$
$$a_{21}x_1 + a_{21}x_2 + a_{23}x_3 + \cdots = b_2$$
$$\cdots$$

These equations can be set on the following general form:

$$Ax = b$$

Where A is a matrix, x is a vector with the unknowns and b is a vector of constants

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Solution:

$$x = A^{-1}b$$

(assuming A^{-1} is possible)

Solving Linear Equations in python

$$4x + 3y = 20$$

$$-5x + 9y = 26$$

- To solve the above system of linear equations, we need to find the values of the x and y variables. There are multiple ways to solve such a system, such as Elimination of Variables, Cramer's Rule, Row Reduction Technique, and the Matrix Solution.
- In the matrix solution, the system of linear equations to be solved is represented in the form of matrix AX = B.

A = [[4   3]
    [-5   9]]

X = [[x]
    [y]]

B = [[20]
    [26]]

```
import numpy as np
m_list = [[4, 3], [-5, 9]]
A = np.array(m_list)
#To find the inverse of a matrix, the matrix is passed to the linalg.inv() method of the Numpy module
inv_A = np.linalg.inv(A)
print(inv_A)
#find the dot product between the inverse of matrix A, and the matrix B.
B = np.array([20, 26])
X = np.linalg.inv(A).dot(B)
print(X)
```

Output:

[2. 4.]

Here, 2 and 4 are the respective values for the unknowns x and y in *Equation 1*.

Equation 2: $4x + 3y + 2z = 25$

$$-2x + 2y + 3z = -10$$

$$3x - 5y + 2z = -4$$

```
A = np.array([[4, 3, 2], [-2, 2, 3], [3, -5, 2]])
B = np.array([25, -10, -4])
X = np.linalg.inv(A).dot(B)
print(X)
```

Output:

```
[ 5.  3. -2.]
```

Using the solve() Method

```
A = np.array([[4, 3, 2], [-2, 2, 3], [3, -5, 2]])
B = np.array([25, -10, -4])
X2 = np.linalg.solve(A,B)
print(X2)
```

Output:

```
[ 5.  3. -2.]
```

A Real-World Example

- Let's see how a system of linear equation can be used to solve real-world problems.
- Suppose, a fruit-seller sold 20 mangoes and 10 oranges in one day for a total of $350. The next day he sold 17 mangoes and 22 oranges for $500. If the prices of the fruits remained unchanged on both the days, what was the price of one mango and one orange?
- This problem can be easily solved with a system of two linear equations.

- Let's say the price of one mango is x and the price of one orange is y. The above problem can be converted like this:

$$20x + 10y = 350$$

$$17x + 22y = 500$$

```
A = np.array([[20, 10], [17, 22]])
B = np.array([350, 500])
X = np.linalg.solve(A,B)
print(X)
```

Output:

```
[10. 15.]
```

Null space for data science

- The null space of a matrix **A** consists of all vectors $\boldsymbol{\beta}$ such that $\boldsymbol{A\beta} = \mathbf{0}$ and $\boldsymbol{\beta} \neq \mathbf{0}$
- Nullity of a matrix is the number of vectors in the null space of the given matrix
- The size of the null space of a matrix provides us with the number of linear relations among the attributes
- And the null space vectors $\boldsymbol{\beta}$ are useful to identify these linear relationships

Null space: The idea

- Notice that if $\boldsymbol{A\beta} = \mathbf{0}$, every row of A when multiplied by $\boldsymbol{\beta}$ goes to zero
- This implies that variable values in each sample (represented by a row) behave the same
- This helps in identifying the linear relationships in the attributes
- Every null space vector corresponds to one linear relationship
- This idea is demonstrated further using examples

Null space : general description

- Let us suppose
- $A = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}$ is a data matrix and there is one vector in the null space of A, i.e, $\beta = [\beta_1 \ \ldots \beta_m]^T$, then as per the definition, β satisfies all the equations given below

- $x_{11}\beta_1 + x_{12}\beta_2 + \cdots x_{1n}\beta_n = 0$

$$\vdots$$

- $x_{m1}\beta_1 + x_{m2}\beta_2 + \cdots x_{mn}\beta_n = 0$

Null space: An Example

- Consider the matrix A with attributes $\{x_1, x_2\}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Number of columns in A =2

Rank of A =2

Thus, nullity =0

- This implies that the null space of the matrix A does not contain any vectors
- Thus we can claim that all the attributes are linearly independent

Null space: Another example

- Now consider A with attributes $\{x_1, x_2, x_3\}$ such that

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

Number of columns in A =3

Rank of A =2

Thus, nullity =1

- Thus, we need to identify the vectors in the null space of A which is non-zero in this case

Null space: Further Example

$$A\boldsymbol{\beta} = 0$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Thus we obtain,

$$b_1 + 2b_2 = 0$$
$$b_3 = 0$$

- The null vector is $\boldsymbol{B} = [b_1 \;\; b_2 \;\; b_3 \;]^T = [-2b_2 \;\; b_2 \;\; 0]^T = k[-2\ 1\ 0]^T$
- We see that we obtain a direct linear relationship between the attributes of A using null space and rank-nullity theorem
- The same concept can be extended for bigger data set

Rank of Matrix

- **Rank of a matrix** – Rank of a matrix is equal to the maximum number of linearly independent row vectors in a matrix.
- A set of vectors is linearly dependent if we can express at least one of the vectors as a linear combination of remaining vectors in the set.
- **To Calculate Rank of Matrix There are Two Methods:**
    1. Minor method
    2. Echelon form

- The maximum number of linearly independent rows in a matrix A is called the **row rank** of A, and the maximum number of linearly independent columns in A is called the **column rank** of A.
- If A is an m by n matrix, that is, if A has m rows and n columns, then it is obvious that

$$\left.\begin{array}{r}\text{row rank of } A \le m \\ \text{column rank of } A \le n\end{array}\right\} \quad (*)$$

How to find the Rank of a Matrix?

- To find the rank of a matrix, we will transform that matrix into its echelon form.
- Then determine the rank by the number of non zero rows.
- Consider the following matrix.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$$

While observing the rows, we can see that the second row is two times the first row. Here we have two rows. But it does not count. The rank is considered as 1.

Consider the unit matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that the rows are independent. Hence the rank of this matrix is 3.

The rank of a unit matrix of order m is m.

If A matrix is of order $m \times n$, then $\rho(A) \leq \min\{m, n\}$ = minimum of m, n.

If A is of order $n \times n$ and $|A| \neq 0$, then the rank of A = n.

If A is of order $n \times n$ and $|A| = 0$, then the rank of A will be less than n

Rank of a Matrix by Row- Echelon Form

- We can transform a given non-zero matrix to a simplified form called a Row-echelon form, using the row elementary operations. In this form, we may have rows all of whose entries are zero. Such rows are called zero rows. A non-zero row is one in which at least one of the elements is not zero.

Row-Echelon Form

A matrix is said to be in row-echelon form if the following rules are satisfied.

- All the leading entries in each row of the matrix is 1
- If a column contains a leading entry then all the entries below the leading entry should be zero
- If any two consecutive non-zero rows, the leading entry in the upper row should occur to the left of the leading entry in the lower row.
- All rows which consist only of zeros should occur in the bottom of the matrix

We need to perform some operations on the rows to reduce the matrix. These operations are called **elementary row operations** and there are a certain rules to follow for these operations as given below,

1. Interchange one row of the matrix with another of the matrix.
2. Multiply one row of the matrix by a nonzero scalar constant.
3. Replace the one row with the one row plus a constant times another row of the matrix.

- A matrix A of order $m \times n$ is said to be in echelon form if

  (i) Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.

  (ii) The number of zeros before the first non-zero element in a row is less then the number of such zeros in the next row.

- For example, consider the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Here R_1 and R_2 are non zero rows.
- R_3 is a zero row.

- Note: A non-zero matrix is said to be in a row-echelon form, if all zero rows occur as bottom rows of the matrix and if the first non-zero element in any lower row occurs to the right of the first non-zero entry in the higher row.
- If a matrix is in row-echelon form, then all elements below the leading diagonal are zeros.
- Consider the following matrix. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$
- Check the rows from the last row of the matrix. The third row is a zero row. The first non-zero element in the second row occurs in the third column and it lies to the right of the first non-zero element in the first row which occurs in the second column. Hence the matrix A is in row echelon form.

 # Linear Algebra Learning Sequence # Rank of a Matrix

```
import numpy as np
a = np.array([[4,5,8], [7,1,4], [5,5,5], [2,3,6]])
rank = np.linalg.matrix_rank(a)
print('Matrix : ', a)
print('Rank of the given Matrix : ',rank)
```

Output

```
Matrix : [[4 5 8]
[7 1 4]
[5 5 5]
[2 3 6]]
Rank of the given Matrix : 3
```

Find the Rank and Nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$

Method 1: Normal method

Given $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$ and order $n = 3$

$|A| = 1(0-5) - 2(0-3) + 1(-10+9)$

$= -5 + 6 - 1 = 0$

i.e. $|A| = 0 \Rightarrow$ rank of matrix is less than order $= 3$ i.e. $\rho(A) < 3$

Consider $\begin{vmatrix} 1 & 2 \\ -2 & -3 \end{vmatrix} = (-3) + 4 = 1 \neq 0$

$\therefore$ The rank of a matrix, $r = 2$ ($\because$ high order of non zero minor order)

Nullity of a matrix $= n - r = 3 - 2 = 1$

Method 2: Echelon Form Method

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

$R_2 : R_2 + 2R_1$

$R_3 : R_3 - 3R_1$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$R_3 : R_3 + R_2$

$$\therefore A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of a matrix $= r = 2$ (Number of non $-$ zero rows)

Nullity of a matrix, $n - r = 3 - 2 = 1$

→ **TIP**:

1) If order of a matrix is ≤ 3. Then apply Normal Method to find the rank of a matrix.

2) If the order of matrix is > 3 , then apply Echelon method to find the rank of a matrix.

Find the rank of the matrix A= $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$

Solution:

The order of A is 3 × 4.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$	$R_3 \to R_3 + 5R_2$

The number of non zero rows is 3. $\therefore \rho(A) = 3$.

Find the rank of the matrix A= $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$

Solution:

The order of A is 3×4.

$\therefore\ \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The number of non zero rows is 3.

$\therefore\ \rho\ (A) = 3$.

Find the rank of matrix A by using the row echelon form.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

Solution:

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

Now we apply elementary transformations.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

We get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in row echelon form.

Number of non-zero rows = 2

Hence the rank of matrix A = 2

Find the rank of the matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Solution:

Given A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Now we transform the matrix A to echelon form by using elementary transformation.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 2

Hence the rank of matrix A = 2

Find the rank of the matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution:

Given

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

We get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here number of non zero rows = 1

Hence the rank of the matrix = 1

Given A = $\begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$

Find the rank of matrix A.

Solution:

Given

$$A = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$$

By observing the rows, we can see that elements of the second row are twice the elements of the first row.

$R_1 \rightarrow 2R_1 - R_2$

$$\begin{bmatrix} 0 & 0 \\ 8 & 14 \end{bmatrix}$$

Number of non zero rows = 1

Rank of matrix A = 1.

Given A = $\begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$

Find the rank of matrix A.

Solution:

Given

$$A = \begin{bmatrix} 4 & 7 \\ 8 & 14 \end{bmatrix}$$

By observing the rows, we can see that elements of the second row are twice the elements of the first row.

$R_1 \rightarrow 2R_1 - R_2$

$$\begin{bmatrix} 0 & 0 \\ 8 & 14 \end{bmatrix}$$

Number of non zero rows = 1

Rank of matrix A = 1.

The rank of the following matrix is

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$$

a) 1

b) 2

c) 3

4) 4

Solution:

Given $\begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$

We transform the matrix using elementary row operations.

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & -3 & 3 & 9 \end{bmatrix}$$

$R_2 \rightarrow R_2/-2$

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & -3 & 3 & 9 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 3R_2$

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the number of non zero rows is 2, rank = 2

Consider the following matrix as an example that we want to calculate its *Null-Space*:

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{bmatrix}$$

To find *N(A)* we should find all vectors *x such that:*

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, A\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By multiplying matrix *A* to vector *x* we have:

$$\begin{bmatrix} x_1 + 2x_2 + x_3 + x_4 \\ x_1 + 2x_2 + 2x_3 - x_4 \\ 2x_1 + 4x_2 + 6x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, we should solve this equation by considering it as a system of linear equations:

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 2x_3 - x_4 = 0 \\ 2x_1 + 4x_2 + 6x_4 = 0 \end{cases}$$

Therefore, the solution to the above equation is:

$$\begin{cases} x_1 + 2x_2 + 3x_4 = 0 \\ x_3 - 2x_4 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -2x_2 - 3x_4 \\ x_3 = 2x_4 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

The x2 and x4 are free variables in $\mathbb{R}$. So, all of the vectors in *Null-Space(A)*, which satisfy the original equation, *Ax=0*, can be represented as a linear combination of these two vectors.

Recall: The linear combinations of two vectors is the span of those two vectors.

Hence, it is true to say that the *Null-Space(A)* is:

$$Null - Space(A) = Span\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$$

Using Null-Space to show linearly independence of the column vectors of a matrix

If the column vectors of a matrix are ***linearly independent***, then the *Null-Space* of that matrix is only going to consist of the zero vector. Also, If the *Null-Space* of a matrix only contains the zero vector, that means that the columns of that matrix are linearly independent. To better understanding let us do a little math! Consider matrix *A(n*m)*. We can rewrite it as its column vectors (A1 to An) such as below:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [\overrightarrow{A_1} \ \overrightarrow{A_2} \ \ldots \ \overrightarrow{A_n}]$$

To calculate the *Null-Space*(A), we should find all vectors *x* that satisfy *Ax=0:*

$$A\vec{x} = \vec{0} \rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\overrightarrow{A_1} + x_2\overrightarrow{A_2} + \ldots + x_n\overrightarrow{A_n} = 0$$

Recall: The vectors v1,v2, …,vn are ***linearly independent*** if and only if the only solution to the equation of their linear combination to be zero is c1=c2=…=cn=0.

$$\{v_1, v_2, \ldots, v_n\} \text{ are linearly independent}$$

$$\Updownarrow$$

$$c_1 v_1 + c_2 v_2 + \ldots, c_n v_n = \vec{0}$$

$$\downarrow$$

$$c_i = 0 \text{ for } 1 \leq i \leq n \; (c_1 = c_2 = \ \ldots \ = c_n = 0)$$

Therefore, if the only solution for $Ax=0$ is $x_i=0$ (for $1 \leq i \leq n$), we can say that the column vectors of *A* are linearly independent. In other words:

$$Null-Space(A) = \{\vec{0}\}$$

$$\Updownarrow$$

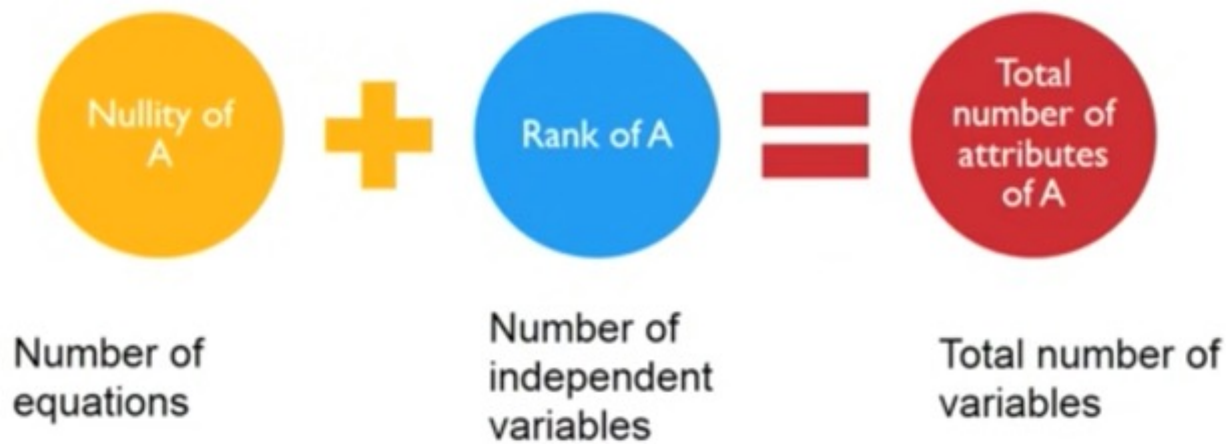
$$x_1 = x_2 = \ldots = x_n = 0$$

$$\Updownarrow$$

$$\overrightarrow{A_1}, \overrightarrow{A_2}, \ldots, \overrightarrow{A_n} \text{ are linearly independent}$$

Rank nullity theorem

- Consider the data matrix A with the null space and nullity as defined before
- The rank- nullity theorem helps us to relate the nullity of the data matrix to the rank and the number of attributes in the data
- According to the rank-nullity theorem



Summary till now

Data Matrix

- The available data is expressed in the form of a data matrix
- This data matrix is further used to do the necessary operations

Null Space

- Defined as a collection of vectors satisfying $A\beta = 0$
- Helps in identifying the linear relationships between the attributes directly

Nullity

- Nullity is the size of the null space of the data matrix
- Useful to identify the number of linear relationships in the attributes
- Rank- Nullity theorem

Hyperplanes

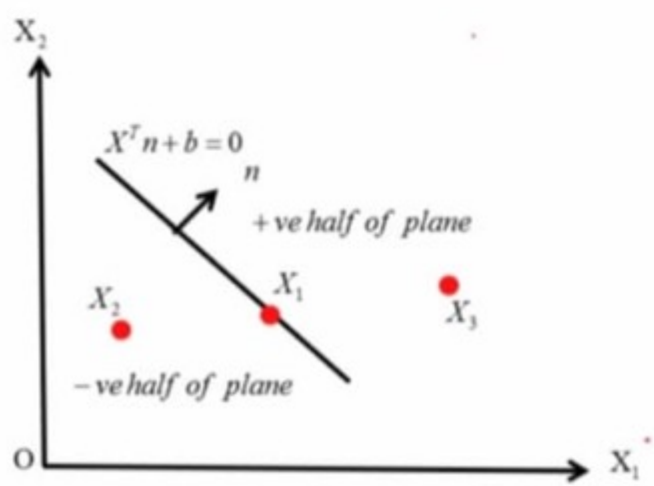
- Geometrically, hyperplane is a geometric entity whose dimension is one less than that of its ambient space.
- For instance, the hyperplanes for a 3D space are 2D planes and hyperplanes for a 2D space are 1D lines and so on.
- The hyperplane is usually described by an equation as follows

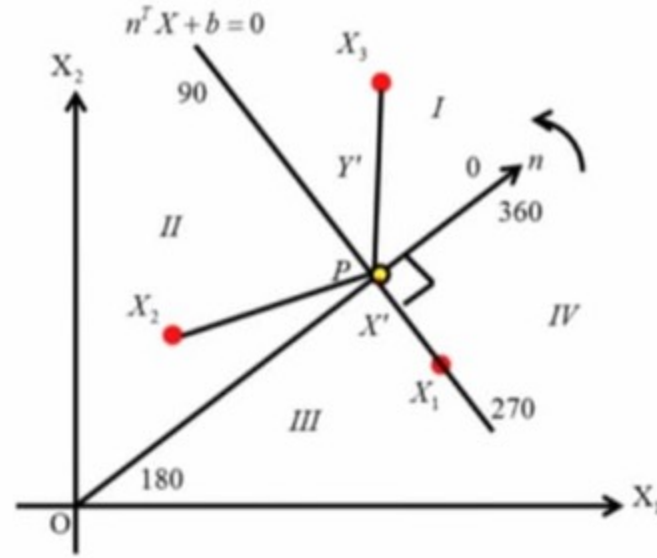
$$X^T n + b = 0$$

Halfspace

- We can observe that the equation can be evaluated for the two halfspaces
- It can be seen that

$$X^T n + b = 0 \; \forall \; X \in line$$
$$X^T n + b > 0 \; \forall \; X \in subspace\ in\ the\ n\ direction\ (X_3)$$
$$X^T n + b < 0 \; \forall \; X \in subspace\ in\ the\ -n\ direction\ (X_2)$$





Hyperplanes and halfspaces: Example

- Let us consider a 2D geometry with $n = \begin{bmatrix}1\\3\end{bmatrix}$ and $b = 4$

$$X = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$
$$X^T n + b = 0$$
$$[x_1\ x_2]\begin{bmatrix}1\\3\end{bmatrix} + 4 = 0$$
$$x_1 + 3x_2 + 4 = 0$$

- The hyperplane is the equation of a line
- The halfspaces corresponding to this hyperplane are

$$x_1 + 3x_2 + 4 > 0 : \text{Positive halfspace}$$
$$x_1 + 3x_2 + 4 < 0 : \text{Negative halfspace}$$

Eigenvalues and eigenvectors

- We have previously seen linear equations of the form $Ax = b$
- What is the geometrical interpretation of this equation?
- We can make an interpretation as follows
  - When vector x is operated on by A, we obtain a new vector b with a different orientation

Eigenvalues and eigenvectors

- Operator representation

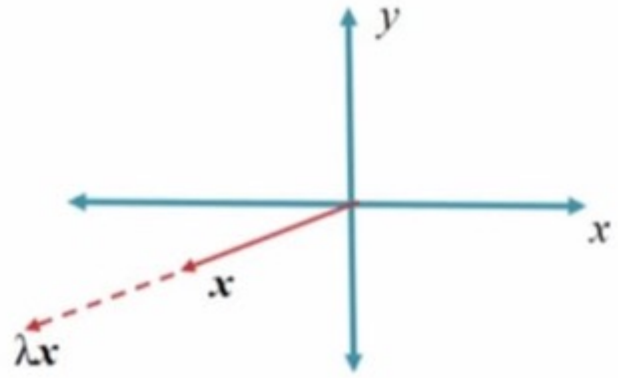
$$\xrightarrow{x} \boxed{\mathbf{A}} \longrightarrow \mathbf{A}x = \mathbf{b}$$

- The newly obtained *b* vector represents a new orientation. So we ask the following question
- Are there directions for a matrix *A* such that when the matrix operates on these directions they maintain their orientation save for multiplication by a scalar (positive or negative)?
- That is

$$\xrightarrow{x} \boxed{\mathbf{A}} \longrightarrow \mathbf{A}x = \mathbf{b} = \lambda x$$

Eigenvalues and eigenvectors

- The mathematical formulation of our question is
$$Ax = \lambda x$$
- The constant λ ($positive$) represents the amount of stretch or shrinkage the attributes x go through in the x direction
- The solutions (x) are known as eigenvectors and their corresponding λ are eigenvalues



Eigenvalues and eigenvectors

- We can find the eigenvalues as follows

$$Ax = \lambda x \quad A(n \times n); x(n \times 1)$$
$$Ax - \lambda I x = 0$$
$$(A - \lambda I)x = 0$$

- Thus the eigenvalues of the equation can be identified using

$$|A - \lambda I| = 0$$

- Substituting the eigenvalues in the original equation will help us find solutions for the eigenvector x

Eigenvalues and eigenvectors: Examples

- Consider the following example with the given A matrix

$$A = \begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$|A - \lambda I| = \left| \begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} 8-\lambda & 7 \\ 2 & 3-\lambda \end{bmatrix} \right| = 0$$

$$(8 - \lambda)(3 - \lambda) - 14 = 0$$

$$\lambda^2 - 11\lambda + 10 = 0$$

$$\lambda = (10,1)$$

- Thus we identify two eigenvalues and now we proceed to find the corresponding eigenvectors

Eigenvalues and eigenvectors: Examples

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\lambda = 1$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 8x_1 + 7x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

- Thus the eigenvector (unit) corresponding to $\lambda = 1$is

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Eigenvalues and eigenvectors: Examples

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\lambda = 1$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 8x_1 + 7x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$8x_1 + 7x_2 = x_1$

$7x_1 + 7x_2 = 0 \Rightarrow x_1 + x_2 = 0$ ✓

$2x_1 + 3x_2 = x_2$

$2x_1 + 2x_2 = 0 \rightarrow x_1 + x_2 = 0$

- Thus the eigenvector (unit) corresponding to $\lambda = 1$is

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad \begin{bmatrix} k \\ -k \end{bmatrix}$$

$\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2}$ ✓

$\sqrt{1/2 + 1/2} =$

Eigenvalues and eigenvectors: Examples

- $\lambda = 10$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$$

$$\begin{bmatrix} 8x_1 + 7x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$$

$$7x_2 = 2x_1$$

- Thus the eigenvector (unit) corresponding to $\lambda = 10$

$$X = \begin{bmatrix} \frac{7}{\sqrt{53}} \\ \frac{2}{\sqrt{53}} \end{bmatrix}$$

Eigen vectors are the vectors that does not change its orientation when multiplied by the transition matrix, but it just scales by a factor of corresponding eigenvalues.

Code for finding Eigen Vectors in python

```
import  numpy as np
#create an array
arr = np.arange(1,10).reshape(3,3)
#finding the Eigenvalue and Eigenvectors of arr
np.linalg.eig(arr)
```

```
from numpy import array
from numpy.linalg import eig
A = array([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
print(A)
values, vectors = eig(A)
print(values)
print(vectors)
```

Example: Solve for λ:

Start with $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Which is:

$$\begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

Calculating that determinant gets:

$$(-6-\lambda)(5-\lambda) - 3\times4 = 0$$

Which then gets us this Quadratic Equation:

$$\lambda^2 + \lambda - 42 = 0$$

And solving it gets:

$$\lambda = -7 \text{ or } 6$$

And yes, there are **two** possible eigenvalues.

Example : 2D

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

Step 1

$$\det(A - \lambda I) = 0$$

$$\downarrow$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) - 2\cdot 3$$

$$= -4 - \lambda + 4\lambda + \lambda^2 - 6$$

$$= \lambda^2 + 3\lambda - 10$$

$$= (\lambda - 2)(\lambda + 5) = 0$$

$$\therefore \lambda_1 = 2\ ,\ \lambda_2 = -5$$

Step 2

(i) $\lambda_1 = 2$

$$(A - \lambda I)v = 0$$

$$\downarrow$$

$$\begin{bmatrix} 1-\lambda_1 & 2 \\ 3 & -4-\lambda_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Nullspace!

$$\downarrow$$

$$\begin{cases} -v_1 + 2v_2 = 0 \\ 3v_1 - 6v_2 = 0 \end{cases}$$

$$\downarrow$$

$$\therefore\ v_1 = 2,\ v_2 = 1$$

(ii) $\lambda_2 = -5$

$$\begin{bmatrix} 1-\lambda_2 & 2 \\ 3 & -4-\lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{cases} 6v_1 + 2v_2 = 0 \\ 3v_1 + v_2 = 0 \end{cases}$$

$$\downarrow$$

$$\therefore v_1 = -1, \; v_2 = 3$$

Summary

eigenvalues $\begin{cases} \lambda_1 = 2 \\ \lambda_2 = -5 \end{cases}$

eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Diagonalization & Eigen decomposition

- A few applications of eigenvalues and eigenvectors that are very useful when handing the data in a matrix form because you could decompose them into matrices that are easy to manipulate.
- In order for the matrix “A” to be either diagonalized or eigen decomposed, it has to meet the following criteria:
  - Must be a Square matrix
  - Has to have linearly independent eigenvectors

Eigen/diagonal Decomposition

- Let $\mathbf{S} \in \mathbb{R}^{m \times m}$ be a square matrix with *m* linearly independent eigenvectors (a "non-defective" matrix)
- Theorem: Exists an eigen decomposition

$$\mathbf{S} = \mathbf{U}\underbrace{\mathbf{\Lambda}}_{\text{diagonal}}\mathbf{U}^{-1}$$

Unique for distinct eigen-values

  - (cf. matrix diagonalization theorem)
- Columns of *U* are eigenvectors of *S*
- Diagonal elements of $\mathbf{\Lambda}$ are eigenvalues of $\mathbf{S}$

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m), \quad \lambda_i \geq \lambda_{i+1}$$

Diagonal decomposition: why/how

Let ***U*** have the eigenvectors as columns: $U = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$

Then, ***SU*** can be written

$$SU = S\begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \dots & \lambda_n v_n \end{bmatrix} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{bmatrix}$$

Thus $SU=U\Lambda$, or $U^{-1}SU=\Lambda$

And $S=U\Lambda U^{-1}$.

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Diagonal decomposition - example

Recall $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \lambda_1 = 1, \lambda_2 = 3.$

The eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form $U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Inverting, we have $U^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Recall $UU^{-1} = 1$.

Then, $S = U \Lambda U^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Example continued

Let's divide ***U*** (and multiply ***U***$^{-1}$) by $\sqrt{2}$

Then, ***S***= $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$\boldsymbol{Q}$ $\qquad$ $\boldsymbol{\Lambda}$ $\qquad$ $(\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{T})$

Example

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Triangular Matrix

$$\det(A - \lambda I)$$

$$= \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2 - 1 \cdot 0$$

$$= (3-\lambda)^2$$

$$\therefore \lambda_1 = \lambda_2 = 3$$

Only has 1 eigenvalue!

(i) $\lambda_1 = \lambda_2 = 3$

$$(A - \lambda I)v = 0$$

$$\begin{bmatrix} 3-3 & 1 \\ 0 & 3-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_2 = 0$, v_1 could be anything.

↳ Let's say 1.

$$V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad U = [V_1 \;\; V_2] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\det U = 0 \rightarrow U$ is Singular.

Not invertible.

Solution to over determined set of equations

- If there are fewer equations than variables, then the system is called **underdetermined** and *cannot* have a unique solution. In this case, there are either infinitely many or no solutions.
- A system with more equations than variables is called **overdetermined**. If the number of equations equals the number of variables, we will say the system is **balanced** or **square**.
- A balanced system or an overdetermined system *may* have a unique solution (whereas an underdetermined system may not):

Systems of Linear Equations: Underdetermined and Overdetermined systems

Theorem

1. If the number of equations is greater than or equal to the number of variables then the system has no solution, one solution, or infinitely many solutions.
2. If the number of equations is less than the number of variables, then the system has no solution or infinitely many solutions.

Example. This system has infinitely many solutions.

$$x+2y=4$$

$$3x+6y=12$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 6 & 12 \end{array}\right]$$

$$\frac{1}{3}R_2 \rightarrow R_2 \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 2 & 4 \end{array}\right]$$

$$-R_1 + R_2 \rightarrow R_2 \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

- There is only one pivot column in this row-reduced matrix. The second column is not a pivot column, so we call y a **free variable**.
- A system with free variables is called **dependent**. Variables that *do* correspond to a pivot column are called **fixed variables**.
- In general, we can express the solution of the system as the fixed variables in terms of the free variables.

$$x=4-2y$$

- This is a dependent system, and there are infinitely many solutions, depending on the value of the free variable y. Once a value of y is selected, the value of x is automatically fixed.

Example. This system has no solution, so we call it **inconsistent**.

$$\begin{aligned} x + y + z &= 1 \\ 3x - y - z &= 4 \\ x + 5y + 5z &= -1 \end{aligned}$$

Now let's try to solve it!

$$\begin{array}{r} \\ \\ \end{array}\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & -1 \end{array}\right]$$

$$\begin{array}{r} -3R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array}\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & -2 \end{array}\right]$$

$$R_2 + R_3 \to R_3 \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We don't need to go any further than this. The last row reads $0x + 0y + 0z = -1$, that is, $0 = -1$. Such a false statement reveals that this system of equations has no solution. It is inconsistent!

Ex. A system with no solution:

$$\begin{aligned} x &+ y + 2z = 3 \\ 2x &- 3y - z = 2 \\ 2x &+ 2y + 4z = 5 \end{aligned}$$

Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & -3 & -1 & 2 \\ 2 & 2 & 4 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Notice the false statement $0 = 1$

The system is inconsistent and has **NO** solution.

Ex. A system with more equations than variables:

$$x + 3y = 5$$
$$-4x + 2y = 2$$
$$2x + 3y = 7$$

Matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ -4 & 2 & 2 \\ 2 & 3 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

No Solution.

Notice the false statement $0 = 1$.

Ex. A system with infinitely many solutions:

$$x + y + 2z = 3$$
$$3x - 2y + z = 4$$
$$2x - 3y - z = 1$$

Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 3 & -2 & 1 & 4 \\ 2 & -3 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice the row of zeros.

Singular Value Decomposition

- In linear algebra, the Singular Value Decomposition (SVD) of a matrix is a factorization of that matrix into three matrices.

$$A=U\Sigma V^T$$

- A is the input matrix
- U are the left singular vectors,
- sigma are the diagonal/eigenvalues
- V are the right singular vectors.

The shape of these three matrices will be

- A — m x n matrix
- U — m x k matrix
- Sigma — k x k matrix
- V — n x k matrix

Step 1- find eigenvalues of matrix A and as A can be a rectangular matrix, we need to convert it to a square matrix by multiplying A with its transpose. Here, for easier computation I have taken A as a 2 x 2 matrix.

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

Step 2: Now, that we have a square matrix, we can calculate the eigenvalues of A(transpose) A. We, can do so by calculating the determinant of A(transpose)A — (lambda)I where lambda are the two eigenvalues.

$$A^T A = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

$$A^T A - \lambda \underline{I} = \begin{bmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{bmatrix}$$

$$\det(A^T A - \lambda \underline{I}) = ((25-\lambda)(25-\lambda)) - (-15 \times -15)$$

$$= (625 - 25\lambda - 25\lambda + \lambda^2) - (225)$$

$$= \lambda^2 - 50\lambda + 400$$

Solving the equation, we get

$$\lambda^2 - 50\lambda + 400 = 0$$

$$\lambda^2 - 40\lambda - 10\lambda + 400 = 0$$

$$\lambda(\lambda - 40) - 10(\lambda - 40) = 0$$

$$(\lambda - 10)(\lambda - 40)$$

Two Eigen Values 10, 40

- **Step 3:** Once we have calculated the eigenvalues, calculate the two eigenvectors for each eigenvalue. So, let's start by calculating the eigenvector for 10.
- **Step 3.1:** Plug the value of lambda in the A(transpose)A — (lambda)I matrix

$$A^TA - 10I = \begin{bmatrix} 25-10 & -15 \\ -15 & 25-10 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

In order to find the eigenvector, we need to find the null space of a matrix where AB = 0. In other words,

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Next, reduce this matrix to the Row-Echelon Form to solve the equation easily.

row-echelon form

Row 1 + Row 2

$$\begin{bmatrix} 15 & -15 \\ 0 & 0 \end{bmatrix}$$

Row 1 x 1/15

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Now, we can solve for null space as below to find the eigenvector for eigenvalue 10

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve:

$$1v_1 - 1v_2 = 0$$

$$1v_1 = v_2$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } v2 = 1$$

Once we get this vector, we need to convert it to a **unit vector** . The way we do that is by taking the columnar values and dividing them by taking the square root of the sum of squares of the values. So, in this case we do the following,

$$\sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

So the final eigenvector for eigenvalue is

So the final eigenvector for eigenvalue is

Convert to Unit Vector

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \text{EigenVector of EigenValue } 10$$

We do the similar steps to get the eigenvector for eigenvalue 40

$$A^T A - 40I = \begin{bmatrix} 25-40 & -15 \\ -15 & 25-40 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

Row 1 + (-1)(Row 2)

$$\begin{bmatrix} -15 & -15 \\ 0 & 0 \end{bmatrix}$$

$-1/15$ Row 1

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$1v_1 + 1v_2 = 0$$

$$v_1 = -v_2$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \text{Eigenvector for Eigenvalue } 40$$

Now that we have got both the eigenvectors, let's put it together.

EigenVectors & EigenValues are

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = V$$

$$\begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \Sigma$$

Step 4

- Now that we have our V and Sigma matrices, now it's time to find U. We can just multiply the equation by sigma(inverse) and V on both sides to get the equation for U.
- In this case, as V is an orthogonal matrix, the transpose and inverse of V are the same, therefore, V(transpose) multiplied by V becomes an identity matrix.

$$A = U \Sigma V^T$$

$$A V \Sigma^T = U$$

$$AV = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -4/\sqrt{2} & 4/\sqrt{2} \\ -8/\sqrt{2} & -2/\sqrt{2} \end{bmatrix}$$

Next up, we need to convert this to unit vectors using the steps described above.

$$AV = \begin{bmatrix} -4/\sqrt{2} & 4/\sqrt{2} \\ -8/\sqrt{2} & -2/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -4/\sqrt{2} \times 1/2\sqrt{10} & 4/\sqrt{2} \times 1/\sqrt{10} \\ -8/\sqrt{2} \times 1/2\sqrt{10} & -2/\sqrt{2} \times 1/\sqrt{10} \end{bmatrix}$$

$$AV = \begin{bmatrix} -2/2\sqrt{5} & 4/2\sqrt{5} \\ -4/2\sqrt{5} & -2/2\sqrt{5} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

Next up we multiply this matrix with Sigma (transpose) which is sigma in itself because its a diagonal matrix.

$$AV\Sigma^T = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{5} \times \sqrt{40} & 2/\sqrt{5} \times \sqrt{10} \\ -2/\sqrt{5} \times \sqrt{40} & -1/\sqrt{5} \times \sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{\frac{40}{5}} & 2\sqrt{\frac{10}{5}} \\ -2\sqrt{\frac{40}{5}} & -\sqrt{\frac{10}{5}} \end{bmatrix}$$

$$= \begin{bmatrix} -2\sqrt{2} & 2\sqrt{2} \\ -4\sqrt{2} & -\sqrt{2} \end{bmatrix}$$

Now, we need to convert this to unit vectors to get the final U matrix.

$$U = \begin{bmatrix} -2\sqrt{2} & 2\sqrt{2} \\ -4\sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2\sqrt{2}}{\sqrt{40}} & \frac{2\sqrt{2}}{\sqrt{10}} \\ -\frac{4\sqrt{2}}{\sqrt{40}} & -\frac{\sqrt{2}}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} -2\sqrt{\frac{2}{40}} & 2\sqrt{\frac{2}{10}} \\ -4\sqrt{\frac{2}{40}} & -\sqrt{\frac{2}{10}} \end{bmatrix}$$

$$= \begin{bmatrix} -2/\sqrt{20} & 2/\sqrt{5} \\ -4/\sqrt{20} & -1/\sqrt{5} \end{bmatrix}$$

$$U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

Decomposed the matrix A into three matrices as given below.

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Applications

- **Calculation of Pseudo-inverse:** Pseudo inverse or Moore-Penrose inverse is the generalization of the matrix inverse that may not be invertible (such as low-rank matrices). If the matrix is invertible then its inverse will be equal to Pseudo inverse but pseudo inverse exists for the matrix that is not invertible. It is denoted by A^+

Pseudo Inverse Matrix

- If the columns of a matrix A are linearly independent, so $A^T \cdot A$ is invertible and we obtain with the following formula the pseudo inverse:

$$\mathbf{A^+ = (A^T \cdot A)^{-1} \cdot A^T}$$

- Here A^+ is a left inverse of A, what means: $A^+ \cdot A = E$.
- However, if the rows of the matrix are linearly independent, we obtain the pseudo inverse with the formula:

$$\boldsymbol{A^+ = A^T \cdot (A \cdot A^T)^{-1}}$$

- This is a right inverse of A, what means: $A \cdot A^+ = E$.
- If both the columns and the rows of the matrix are linearly independent, then the matrix is invertible and the pseudo inverse is equal to the inverse of the matrix.

The pseudoinverse is defined and unique for all matrices whose entries are real or complex numbers. Matrix inverse exists for square matrices only. Real world data is not always square. Furthermore, real world data is not always consistent and might contain repetitions. To deal with real world data generalized inverse for rectangular matrix is needed. It can be computed using the singular value decomposition also

Let A be a matrix of order $m \times n$ then the pseudoinverse of A is defined as

1. If the columns of a matrix A are linearly independent, then the pseudo inverse of A is: $A^+ = (A^T A)^{-1} \cdot A^T$

2. If the rows of the matrix are linearly independent, the pseudo inverse of A is:

$$A^+ = A^T \cdot (A A^T)^{-1}$$

3. If A has rank deficient, then the Pseudo inverse of A id defined as

$$\boldsymbol{A}^{+} = (\boldsymbol{U\Sigma V^{T}})^{-1} = (\boldsymbol{V^{T}})^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^{-1} = \boldsymbol{V\Sigma^{-1}U^{T}}$$

If $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$ then $\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sigma_2} & \mathbf{0} \end{bmatrix}$

1. $Find\ the\ pseudo\ inverse\ of\ A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

Sol:

$$Given\ \ A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$Here \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5 \neq 0$$

$$rank(A) = 2$$

Since for the given matrix the rank is equal to the number of rows therefore

the matrix has row rank which implies rows are linearly independent

$$Then\ the\ pseudo\ inverse\ of\ A\ is\ A^{+} = A^{T}(AA^{T})^{-1}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad A^{T} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$AA^{T} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 15 \\ 15 & 30 \end{bmatrix}$$

$$| AA^{T} | = 15(30-15) = 225$$

$$(AA^{T})^{-1} = \frac{1}{225}\begin{vmatrix} 30 & -15 \\ -15 & 15 \end{vmatrix} = \begin{bmatrix} 2/15 & -1/15 \\ -1/15 & 1/15 \end{bmatrix}$$

$$A^{+}=A^{T}(AA^{T})^{-1}=\begin{bmatrix}1&4\\2&3\\1&2\\3&1\end{bmatrix}\begin{bmatrix}2/15&-1/15\\-1/15&1/15\end{bmatrix}=\begin{bmatrix}-2/15&3/15\\1/15&1/15\\0&1/15\\5/15&-2/15\end{bmatrix}$$

$$=\frac{1}{15}\begin{bmatrix}-2&3\\1&1\\0&1\\5&-2\end{bmatrix}$$

$$3.\ \text{Find the pseudo inverse of } A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{bmatrix}$$

Sol:

$$\text{Given } A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{bmatrix}$$

$$\text{Here all minors of order two are zero}$$

$$\therefore\ rank(A)=1$$

Since for the given matrix the rank is not equal to the number of rows or columns

therefore it has to be solved by singular value decomposition.

$$\text{Then the pseudo inverse of } A \text{ using SVD is } A^{+}=V\Sigma^{+}U^{T}=V\Sigma^{-1}U^{T}$$

Compute $A^T A = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 45 \end{bmatrix}$

It has eigen values $\lambda_1 = 70, \lambda_2 = 0, \lambda_3 = 0$ and the corresponding

eigen vectors are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$

The singular values of A are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{70}, \sigma_2 = \sqrt{\lambda_1} = \sqrt{0}, \sigma_3 = \sqrt{\lambda_3} = \sqrt{0}$$

Normalized vectors are $v_1 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}$, $v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, v_2 = \begin{bmatrix} 3/\sqrt{70} \\ 6/\sqrt{70} \end{bmatrix}$

Normalized vectors are $v_1 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$, $v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3/\sqrt{70} \\ 6/\sqrt{70} \\ -5/\sqrt{70} \end{bmatrix}$

So Thus $V = \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & 3/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{5} & 6/\sqrt{70} \\ 3/\sqrt{14} & 0 & -5/\sqrt{70} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{70} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Also we can find $u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$$U = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$\therefore A^{+} = V\Sigma^{+}U^{T}$$

$$= \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & 3/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{5} & 6/\sqrt{70} \\ 3/\sqrt{14} & 0 & -5/\sqrt{70} \end{bmatrix} \begin{bmatrix} 1/\sqrt{70} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

Compute the Moore-Penrose pseudo-inverse of a matrix

```
import numpy as np
m = np.matrix([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
print np.linalg.pinv(m)
```

OUTPUT

[[-6.38888889e-01 -1.66666667e-01 3.05555556e-01] [-5.55555556e-02 -2.63677968e-16 5.55555556e-02] [5.27777778e-01 1.66666667e-01 -1.94444444e-01]]

Basic Equations of Lines

- The equation of a line means an equation in x and y whose solution set is a line in the (x,y) plane.
- The most popular form in algebra is the "slope-intercept" form

$$\mathbf{y = mx + b.}$$

This in effect uses x as a parameter and writes y as a function of x: $y = f(x) = mx+b$. When $x = 0$, $y = b$ and the point $(0,b)$ is the intersection of the line with the y-axis.

- Line as a geometrical object and not the graph of a function, it makes sense to treat x and y more evenhandedly. The general equation for a line (normal form) is

$$\mathbf{ax + by = c,}$$

- This can easily be converted to slope-intercept form by solving for y:

$$\mathbf{y = (-a/b)\,x + c/b,}$$

except for the special case b = 0, when the line is parallel to the y-axis.

- If the coefficients on the normal form are multiplied by a nonzero constant, the set of solutions is exactly the same, so, for example, all these equations have the same line as solution.

$$2x + 3y = 4$$
$$4x + 6y = 8$$
$$-x - (3/2)y = -2$$
$$(1/2)x + (3/4)y = 1$$

Finding the equation of a line through 2 points in the plane

- For any two points P and Q, there is exactly one line PQ through the points. If the coordinates of P and Q are known, then the coefficients a, b, c of an equation for the line can be found by solving a system of linear equations.

Example: For P = (1, 2), Q = (-2, 5), find the equation ax + by = c of line PQ.

- Since P is on the line, its coordinates satisfy the equation:

$$a(1) + b(2) = c, \text{ or } a + 2b = c$$

Since Q is on the line, its coordinates satisfy the equation: a(-2) + b5 = c,

or $-2a + 5b = c$.

- Multiply the first equation by 2 and add to eliminate a from the equation:

$4b + 5b = 9b = 2c + c = 3c$, so $b = (1/3)c$.

- Then substituting into the first equation, $a = c - 2b = c - (2/3)c = (1/3)c$.
- This gives the equation **[(1/3)c]x + [(1/3)c}y = c**.

Equation of a Plane

- A plane in 3D-space has the equation

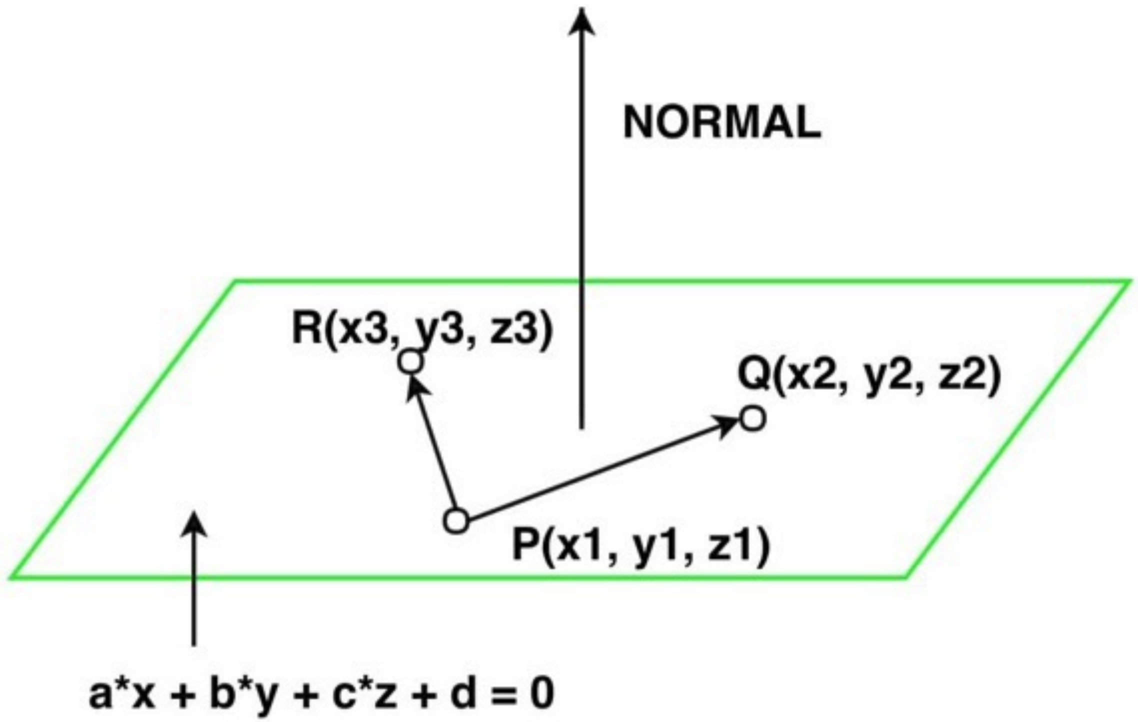
$$ax + by + cz = d,$$

- where at least one of the numbers a, b, c must be nonzero.
- If c is not zero, it is often useful to think of the plane as the graph of a function z of x and y. The equation can be rearranged like this:

$$z = -(a/c)x + (-b/c)\,y + d/c$$

- Another useful choice, when d is not zero, is to divide by d so that the constant term = 1.

$$(a/d)x + (b/d)y + (c/d)z = 1.$$


NORMAL
R(x3, y3, z3)
Q(x2, y2, z2)
P(x1, y1, z1)
a*x + b*y + c*z + d = 0

Finding the equation of a plane through 3 points in space

- Given points P, Q, R in space, find the equation of the plane through the 3 points.

Example: P = (1, 1, 1), Q = (1, 2, 0), R = (-1, 2, 1). We seek the coefficients of an equation $ax + by + cz = d$, where P, Q and R satisfy the equations, thus:

$$a + b + c = d$$
$$a + 2b + 0c = d$$
$$-a + 2b + c = d$$

- Subtracting the first equation from the second and then adding the second equation to the third, we eliminate a to get

$$b - c = 0$$
$$4b + c = 2d$$

- Adding the equations gives $5b = 2d$, or $b = (2/5)d$, then solving for $c = b = (2/5)d$ and then $a = d - b - c = (1/5)d$.
- So the equation (with a nonzero constant left in to choose) is $d(1/5)x + d(2/5)y + d(2/5)z = d$, so one choice of constant gives

$$x + 2y + 2z = 5$$

- or another choice would be $(1/5)x + (2/5)y + (2/5)z = 1$

P(x1, y1, z1), Q(x2, y2, z2), and R (x3, y3, z3) are three non-collinear points on a plane. Find equation of plane.

We know that: $ax + by + cz + d = 0$ ————————(i)

By plugging in the values of the points P, Q, and R into equation (i), we get the following:

$a(x_1) + b(y_1) + c(z_1) + d = 0$

$a(x_2) + b(y_2) + c(z_2) + d = 0$

$a(x_3) + b(y_3) + c(z_3) + d = 0$

Suppose, $P = (1,0,2)$, $Q = (2,1,1)$, and $R = (-1,2,1)$

Then, by substituting the values in the above equations, we get the following:

$a(1) + b(0) + c(2) + d = 0$

$a(2) + b(1) + c(1) + d = 0$

$a(-1) + b(2) + c(1) + d = 0$

Solving these equations gives us $b = 3a$, $c = 4a$, and $d = (-9)a$———————————(ii)

By plugging in the values from (ii) into (i), we end up with the following:

$ax + by + cz + d = 0$

$ax + 3ay + 4az - 9a$

$x + 3y + 4z - 9$

Therefore, the equation of the plane with the three non-collinear points P, Q, and R is $x + 3y + 4z - 9$.

A (3,1,2), B (6,1,2), and C (0,2,0) are three non-collinear points on a plane. Find the equation of the plane.

Solution:

We know that: $ax + by + cz + d = 0$ ————————(i)

By plugging in the values of the points A, B, and C into equation (i), we get the following:

$a(3) + b(1) + c(2) + d = 0$

$a(6) + b(1) + c(2) + d = 0$

$a(0) + b(2) + c(0) + d = 0$

Solving these equations gives us

$a = 0$, $c = 1/2b$, $d = -2b$ ——————————(ii)

By plugging in the values from (ii) into (i), we end up with the following:

$ax + by + cz + d = 0$

$(0)x + (-by) + \frac{1}{2}bz - 2b = 0$

$x - y + \frac{1}{2}z - 2 = 0$

$2x - 2y + z - 4 = 0$

Therefore, the equation of the plane with the three non-collinear points A, B and C is

$2x - 2y + z - 4 = 0$.

Equation of Hyperplane

- A hyperplane in an n−dimensional vector space satisfying the equation:

$$a_1x_1+\cdots+a_nx_n=b$$

Where $a_1,\ldots,a_n$ and b are real numbers with at least $a_1,\ldots,a_n$ non-zero.

n-D (hyperplane)

$$w_0 + w_1x_1 + w_2x_2 + \dots + w_nx_n = 0$$

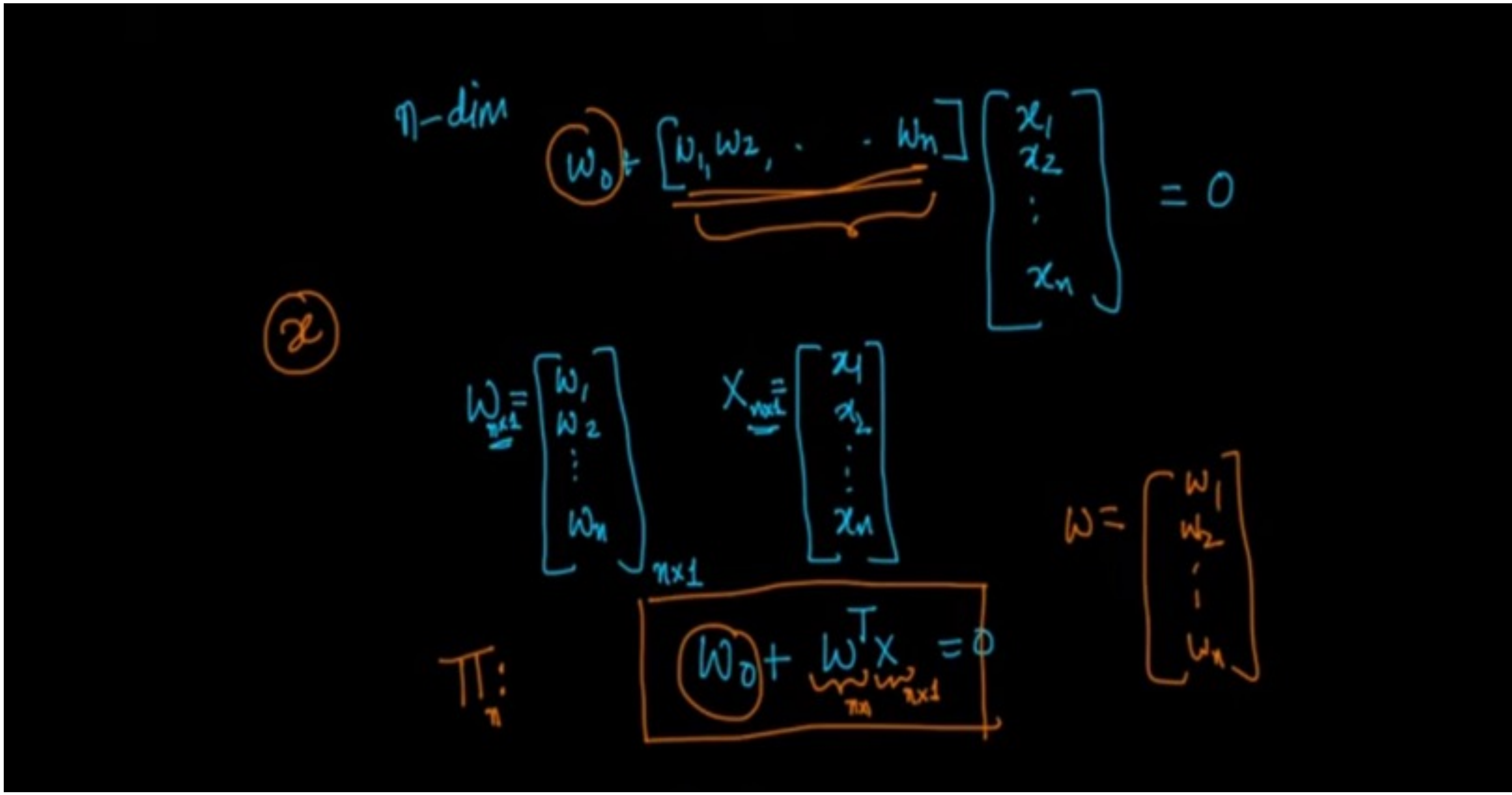
Summation ← ✓ $w_0 + \sum_{i=1}^{n} w_i x_i = 0$

Vector notation ← $w_0 + \underbrace{\begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}}_{W_{1\times n}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}^{x_{n\times 1}} = 0$

$$w_0 + w_1x_1 + w_2x_2 + \dots + w_nx_n = 0$$

✓

Equation of Hyperplane



ℓ passing through origin

line: 2D: $w_1 x_1 + w_2 x_2 = 0$

plane: 3D: $w_1 x_1 + w_2 x_2 + w_3 x_3 = 0$

hyperplane: nD: $w_1 x_1 + w_2 x_2 + \cdots + w_n x_n = 0$

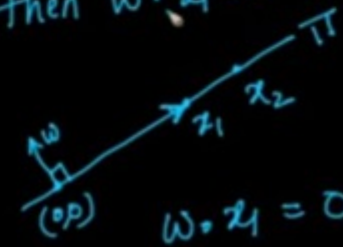
$$\boxed{w^T x = 0}$$ — eqn. of a plane passing through origin

$$\boxed{w^T x + w_0 = 0}$$

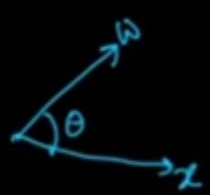
$$\pi_n : \boxed{w^T \underset{\sim}{x} = 0} \checkmark$$

If $w \perp \pi$
then $w \cdot x_i = 0 \quad \forall\, x_i \in \pi$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$



$$w \cdot x_1 = 0$$



$$w \cdot x = w^T x = \|w\| \, \|x\| \cos\theta_{w,x} = 0$$

$$w \perp x \Rightarrow \theta_{w,x} = 90^\circ$$

Equation of Circle

- The standard **equation of a circle** is given by:

$$\mathbf{(x-h)^2 + (y-k)^2 = r^2}$$

Where (h,k) is the coordinates of center of the circle and r is the radius.

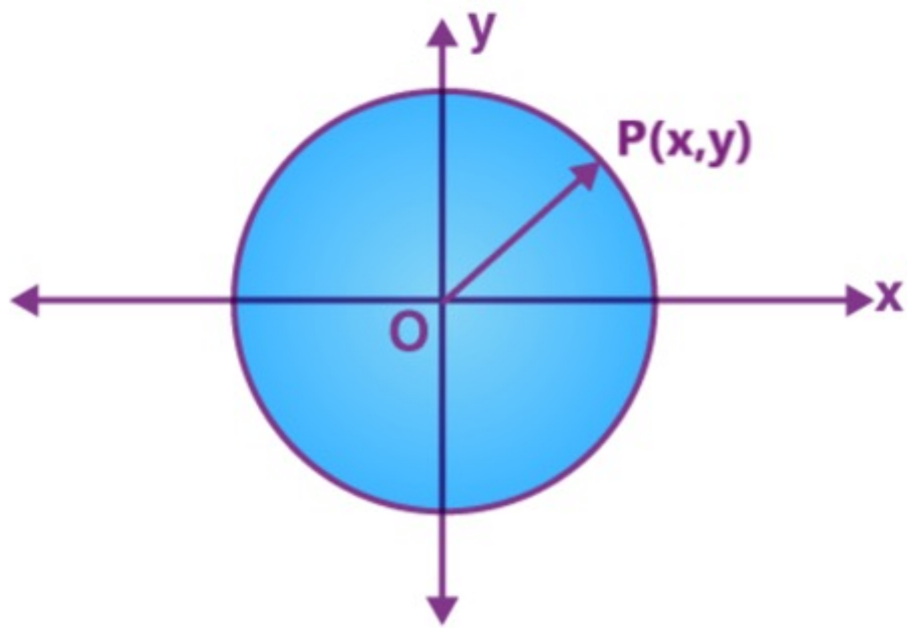
A circle is a set of all points which are equally spaced from a fixed point in a plane. The fixed point is called the center of the circle. The distance between the center and any point on the circumference is called the radius of the circle.

What is the Equation of a Circle?

- A circle is a closed curve that is drawn from the fixed point called the center, in which all the points on the curve are having the same distance from the center point of the center. The equation of a circle with (h, k) center and r radius is given by:

$$(x-h)^2 + (y-k)^2 = r^2$$

- This is the standard form of the equation. Thus, if we know the coordinates of the center of the circle and its radius as well, we can easily find its equation.



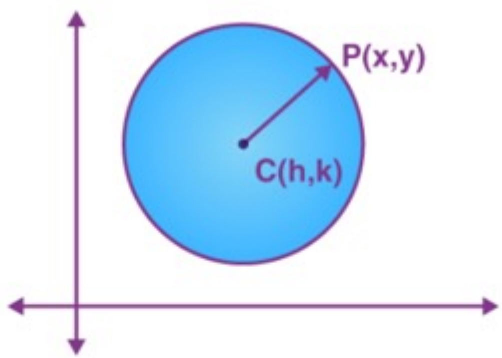
Consider an arbitrary point P(x, y) on the circle. Let 'a' be the radius of the circle which is equal to OP. The distance between the point (x, y) and origin (0,0)can be found using the distance formula which is equal to-

$$\sqrt{[x^2 + y^2]} = a$$

Therefore, the equation of a circle, with the center as the origin is,

$$\mathbf{x^2+y^2= a^2}$$

Where "a" is the radius of the circle.



Let C(h, k) be the centre of the circle and P(x, y) be any point on the circle.
Therefore, the radius of a circle is CP.
By using distance formula,

$$(x-h)^2 + (y-k)^2 = CP^2$$

Let radius be ‘a’.
Therefore, the equation of the circle with center (h, k)and the radius ‘a’ is,

$$\mathbf{(x-h)^2+(y-k)^2 = a^2}$$

which is called the **standard form for the equation of a circle.**

- **What is the equation of a circle when the center is at the origin?**

At origin, the value of coordinates is (0,0), therefore, the equation of circle becomes:

$(x-0)^2 + (y-0)^2 = r^2$

$x^2 + y^2 = r^2$

- **If $(x-4)^2+(y+7)^2=9$ is the equation of circle, then what is the center of circle?**

Given, $(x-4)^2+(y+7)^2=9$ is the equation of circle. If we compare this equation with the standard equation we get:

$(x-h)^2+(y-k)^2 = a^2$

$h=4$ and $y = -7$

Therefore, (4,-7) is the center of circle

- **Example 1:**

Consider a circle whose center is at the origin and radius is equal to 8 units.

Solution:

Given: Centre is (0, 0), radius is 8 units.

We know that the equation of a circle when the center is origin:

$\mathbf{x^2 + y^2 = a^2}$

For the given condition, the equation of a circle is given as

$x^2 + y^2 = 8^2$

$x^2 + y^2 = 64$, which is the equation of a circle

- **Example 2:**

Find the equation of the circle whose center is (3,5) and the radius is 4 units.

Solution:

Here, the center of the circle is not an origin.

Therefore, the general equation of the circle is,

$(x-3)^2 + (y-5)^2 = 4^2$

$x^2 – 6x + 9 + y^2 -10y +25 = 16$

$x^2 +y^2 -6x -10y + 18 =0$

- **Example 3:**

Equation of a circle is $x^2+y^2-12x-16y+19=0$. Find the center and radius of the circle.

Solution:

Given equation is of the form $x^2+ y^2 + 2gx + 2fy + c = 0$,

$2g = -12,\ 2f = -16, c = 19$

$g = -6, f = -8$

Centre of the circle is (6,8)

Radius of the circle $= \sqrt{[(-6)^2 + (-8)^2 - 19\ ]} = \sqrt{[100 - 19]} =$

$= \sqrt{81} = 9$ units.

Therefore, the radius of the circle is 9 units.

Sphere:

Sphere is the locus of points which are at constant distance from a fixed point known as Centre of the Sphere and the constant distance is known as Radius of the sphere.

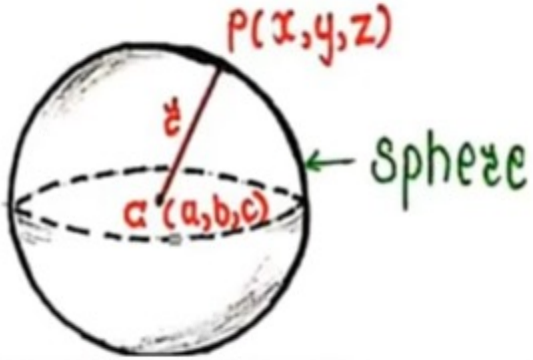
Equations of Sphere:

A) Centre-Radius form:

Let $P(x, y, z)$ be any point on the sphere, $C(a, b, c)$ be the centre of the sphere and r be the radius.

Then by distance formula,

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

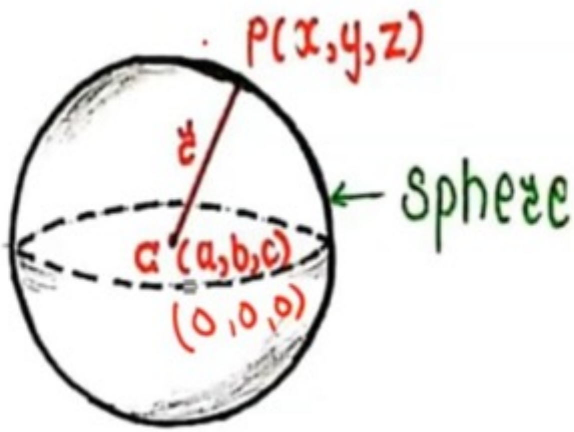


Equations of Sphere:

A) Centre-Radius form:

Note: If C$\equiv (a, b, c) \equiv (0,0,0)$ i.e. be the centre of the sphere is at origin and r be the radius.

Then equation of the sphere is,



$$x^2+y^2+z^2=r^2$$

Equations of Sphere:

B) General form:

Let P(x, y, z) be any point on the sphere, C(a, b, c) be the centre of the sphere and r be the radius.

Then equation of sphere by Centre-Radius form is:

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

$\Rightarrow x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2 - r^2) = 0$

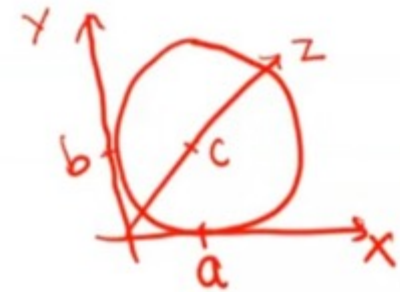
Put $a = -u, b = -v, c = -w$ & $d = a^2 + b^2 + c^2 - r^2$

$\therefore$ Equation of the sphere becomes,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Centre $C \equiv (a, b, c) \equiv (-u, -v, -w)$ & Radius: $r = \sqrt{u^2 + v^2 + w^2 - d}$

Equations of Sphere:



C) Intercept form:

Consider the equation of the sphere in general form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad .... (1)$$

Suppose this sphere cuts off the co-ordinate axes at a, b, c respectively and also it is passing through the Origin i.e. $O \equiv (0,0,0)$.

$\therefore O \equiv (0,0,0)$ satisfies eq.(1), which gives $d = 0$.

$\therefore$ eq.(1) becomes:

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \quad ....(2)$$

Equations of Sphere:

C) Intercept form:

$\therefore$ eq.(1) becomes:

$$\boldsymbol{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0} \quad ....(2)$$

Which is the equation of sphere passing through Origin.

Since, the sphere intercepts X-axis at $(a, 0,0)$, Y-axis at $(0, \text{b}, 0)$ and Z-axis at $(0,0, \text{c})$, we get,

For point $(a, 0,0)$, $\Rightarrow a^2 + 2ua = 0 \qquad \Rightarrow u = -\frac{a}{2}$

For point $(0, b, 0)$, $\Rightarrow b^2 + 2vb = 0 \qquad \Rightarrow v = -\frac{b}{2}$

For point $(0,0, c)$, $\Rightarrow c^2 + 2wc = 0 \qquad \Rightarrow u = -\frac{c}{2}$

$\therefore$ eq.(2) reduces to,

Equations of Sphere:

C) Intercept form:

$$\boldsymbol{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0} \quad ....(2)$$

$$\Rightarrow x^2 + y^2 + z^2 + 2\left(-\frac{a}{2}\right)x + 2\left(-\frac{b}{2}\right)y + 2\left(-\frac{c}{2}\right)z = 0$$

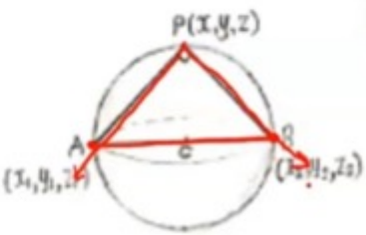
$$\Rightarrow \boldsymbol{x^2 + y^2 + z^2 - ax - by - cz = 0}$$

Hence, The equation of sphere in Intercept Form is:

$$\boxed{\boldsymbol{x^2 + y^2 + z^2 - ax - by - cz = 0}}$$

Equations of Sphere:

D) Diameter form:



Consider the sphere is defined on the line joining points A & B.
Let A (x_1, y_1, z_1) & B (x_2, y_2, z_2) be endpoints of the diameter. Also let P (x, y, z) be any point on the sphere.

$\therefore \angle APB = 90°$

$\Rightarrow$ d.r.s of AP $\equiv (x - x_1, y - y_1, z - z_1)$ &

d.r.s of BP $\equiv (x - x_2, y - y_2, z - z_2)$

As AP $\perp$ BP, By using condition for perpendicularity of two line,

Equations of Sphere:

E) Four Point form:

To find the equation of the sphere passing through four points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$.

Consider the equation of sphere in general form,

$$\# \quad x^2 + y^2 + z^2 + \underline{2ux} + \underline{2vy} + \underline{2wz} + \underline{d} = 0 \quad \text{....(1)}$$

As it passes through above four points, we get,

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \quad \text{....(2)}$$

$$x_2^2 + y_2^2 + z_2^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0 \quad \text{....(3)}$$

$$x_3^2 + y_3^2 + z_3^2 + 2ux_3 + 2vy_3 + 2w_3 + d = 0 \quad \text{....(4)}$$

$$x_4^2 + y_4^2 + z_4^2 + 2ux_4 + 2vy_4 + 2w_4 + d = 0 \quad \text{....(5)}$$

On solving eq.(2), (3), (4), (5) simultaneously to find u, v, w & d,

Example:

Find the equation of the sphere which passes through the points $(2, 1, 1)$ and $(0, 3, 2)$ and has its centre on the line

$$\mathbf{2x + y + 3z = 0 = x + 2y + 2z.}$$

Solution:

Let the equation of required sphere be:

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \ldots (1)$$

Since the sphere passes through points (2,1,1) and (0,3,2),

$\Rightarrow$ They satisfy eq.(1),

$\therefore$ for(2,1,1) $\Rightarrow 6 + 4u + 2v + 2w + d = 0$ ....(2)

$\therefore$ for(0,3,2) $\Rightarrow 13 + 0u + 6v + 4w + d = 0$ ....(3)

Example:

Also,

Centre of the sphere $(-u, -v, -w)$ lies on the line $2x + y + 3z = 0 = x + 2y + 2z$.

$\Rightarrow -2u - v - 3w = 0 \quad \& \quad -u - 2v - 2w = 0$

$\Rightarrow 2u + v + 3w = 0 \quad \text{....} (4)$

$\& \; u + 2v + 2w = 0 \quad \text{....} (5)$

On solving eq. (2), (3), (4), (5), we get,

$u = \frac{28}{18}, \quad v = \frac{7}{18}, \quad w = -\frac{21}{18}$

$\therefore$ from eq.(2) $\Rightarrow \; d = -\frac{96}{9}$

Example:

As $u = \frac{28}{18}$, $v = \frac{7}{18}$, $w = -\frac{21}{18}$, $d = -\frac{96}{9}$

$\therefore$ The required equation of the sphere is

$$x^2 + y^2 + z^2 + 2\left(\frac{28}{18}\right)x + 2\left(\frac{7}{18}\right)y + 2\left(-\frac{21}{18}\right)z + \left(-\frac{96}{9}\right) = 0$$

$$\Rightarrow \mathbf{9(x^2 + y^2 + z^2) + 28x + 7y - 21z - 96 = 0}$$

Equation of Hypersphere

- A hypersphere is a **four-dimensional** analog of a **sphere**; also known as a **4-sphere**.
- The intersection of a sphere with a **plane** is a circle, the intersection of a hypersphere with a hyperplane is a sphere. These analogies are reflected in the underlying mathematics.
- $x^2 + y^2 = r^2$ is the Cartesian equation of a circle of radius r; $x^2 + y^2 + z^2 = r^2$ is the corresponding equation of a sphere; $x^2 + y^2 + z^2 + w^2 = r^2$ is the equation of a hypersphere, where w is measured along a fourth dimension at right angles to the x-, y-, and z-axes.

The n-hypersphere (often simply called the n-sphere) is a generalization of the circle (called by geometers the 2-sphere) and usual sphere (called by geometers the 3-sphere) to dimensions n>=4. The n-sphere is therefore defined (again, to a geometer; see below) as the set of n-tuples of points (x_1, x_2, ..., x_n) such that

$$x_1^2+x_2^2+\ldots+x_n^2=R^2,$$

where R is the radius of the hypersphere.

- The hypersphere has a **hypervolume** (analogous to the volume of a sphere) of $\pi^2 r^4/2$, and a surface volume (analogous to the sphere's surface area) of $2\pi^2 r^3$.
- A solid angle of a hypersphere is measured in **hypersteradians**, of which the hypersphere contains a total of $2\pi^2$. The apparent pattern of 2π **radians** in a circle and 4π **steradians** in a sphere does not continue with 8π hypersteradians because the *n*-volume, *n*-area, and number of *n*-radians of an *n*-sphere are all related to gamma function and the way it can cancel out powers of π halfway between integers. In general, the term "hypersphere" may be used to refer to any *n*-sphere.

1. What is Ax=b? How to solve it?
2. How do we multiply matrices?
3. What is an Eigenvalue? And what is an Eigenvector? What is Eigenvalue Decomposition or The Spectral Theorem?
4. What is Singular Value Decomposition?

THANKYOU