SBAA5205

APPLIED OPERATIONS RESEARCH

UNIT IV

REPLACEMENT MODEL
AND
GAME THEORY

UNIT – IV

Game Theory:

There are three Methods:

(i) Min Max criteria

(ii) Dominance property

(iii) Graphical Method.

Min Max criteria:

Rules for determining saddle point

(i) Select the minimum pay off in each row
And
(ii) Select the Maximum pay (Max value) in each column

(iii) Choose Maximum value of (each row) Row minimum.

(iv) Choose Minimum value of column maximum

Result: If Maximum of row minima = Minimum of column maximum, then game has a saddle point.

(i) If Max(Min) = Min(max), then the game has a saddle point. (ie) Pure strategies

(ii) If max(min) ≠ min(max), then the game has no saddle point. (ie) (mixed strategies).

① Solve the game by using minimax criteria

$$\begin{array}{cc|cccc}
 & & \multicolumn{4}{c}{\text{Group B}} \\
 & & B1 & B2 & B3 & B4 \\
\text{Group A} & A1 & 12 & 1 & 30 & -10 \\
 & A2 & 20 & 3 & 10 & 5 \\
 & A3 & -5 & -2 & 25 & 0 \\
 & A4 & 15 & -4 & 10 & 6
\end{array}$$

Name of players

Solution:

	B1	B2	B3	B4	Row minimum
A1	12	1	30	-10	-10
A2	20	3	10	5	3 ✓
A3	-5	-2	25	0	-5
A4	15	-4	10	6	-4
Column max:	20	3	30	6	

Here: Maximum of Row minimum = 3

Minimum of column maximum = 3

(ii) max(min) = min(max) = 3

$\Rightarrow$ The game has a saddle point

$\therefore$ Saddle point = (A_2, B_2)

Value of the game $V = 3$

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Problem 2:

Solve the game
$$\begin{array}{c|cc} & B_1 & B_2 \\ A_1 & 4 & 2 \\ A_2 & 0 & 8 \end{array}$$

Soln:

$$\begin{bmatrix} 4 & 2 \\ 0 & 8 \end{bmatrix} \quad \begin{array}{c} \text{Row minimum} \\ 2 \checkmark \\ 0 \end{array}$$

column max $\quad 4 \checkmark \quad 8$

$\Rightarrow$ maximum of row minima $\neq$ minimum of column max

$\therefore$ the game has no saddle point

$\Rightarrow$ ~~the~~ Let us reduce the game by using strategies

Let $\begin{bmatrix} 4 & 2 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Let the strategie for the players A & B are

$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ with $p_1 + p_2 = 1$

$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_1 + q_2 = 1$

To find p_1, q_1

$$p_1 = \frac{d-c}{a-b-c+d} = \frac{8-0}{4-2-0+8} = \frac{8}{10} = \frac{2}{5}$$

$$p_2 = 1 - p_1 = 1 - \frac{2}{5} = 3/5$$

$$q_1 = \frac{d-b}{a-b-c+d} = \frac{8-2}{4-2-0+8} = \frac{6}{10} = \frac{3}{5}$$

$q_2 = 1 - q_1$
$= 1 - \frac{3}{5} = \frac{2}{5}$

$\therefore$ The strategies for players A & B are

$$S_A = \begin{array}{cc} A_1 & A_2 \end{array}$$
$$S_A = \left[\begin{array}{cc} 2/5 & 3/5 \end{array} \right]$$

$$\begin{array}{cc} B_1 & B_2 \end{array}$$
$$S_B = \left[\begin{array}{cc} \frac{3}{5} & \frac{2}{5} \end{array} \right]$$

Value of the game $V = \dfrac{ad - bc}{a - b - c + d}$

$$= \frac{4(8) - 2(0)}{4 - 2 - 0 + 8}$$

$$= \frac{32}{10}$$

$$\boxed{V = \frac{16}{5}}$$

Dominance property:

Result: (i) Row operation: Highest value (Row) dominates Least value (Row)

$\Rightarrow$ omit least row

Note:

(ii) Column operation: Least column (value) dominates the highest column (value)

$\Rightarrow$ Omit Highest column

Problem 3:

Using the principle of Dominance Solve the game

$$\begin{array}{cc} & \text{group B} \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Group A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 8 & 10 & 9 & 14 \\ 10 & 11 & 8 & 12 \\ 13 & 12 & 14 & 13 \end{bmatrix} \end{array}$$

Step 1:

Row operation: (In group A)

Since the player A_3 dominates the player A_2

$\therefore$ omit A_2

the game reduces to

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_3 \end{array} & \begin{bmatrix} 8 & 10 & 9 & 14 \\ 13 & 12 & 14 & 13 \end{bmatrix} \end{array}$$

Step 2: column operation [In group B]

Since the player B_1 dominates both the player B_3 and B_4.

$\therefore$ omit B_3 and B_4.

The game reduces to

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_3 \end{array} & \begin{bmatrix} 8 & 10 \\ 13 & 12 \end{bmatrix} \end{array}$$

Step3: (Row operation)

Since the player A_3 dominates A_1

omit A_1

The game reduces to

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ A_3 & \begin{bmatrix} 13 & 12 \end{bmatrix} \end{array}$$

Step4: (column operation)

Since the player B_2 dominates the player B_1

∴ omit B_1

Hence the game reduces to

$$\begin{array}{cc} & B_2 \\ A_3 & \begin{bmatrix} 12 \end{bmatrix} \end{array}$$

∴ saddle point = $[A_3, B_2]$

Value of the game $V = 12$

Note:

$$\begin{array}{cccccc} & B_1 & B_2 & B_3 & B_4 & \text{Row minimum} \\ A_1 & 8 & 10 & 9 & 14 & 8 \\ A_2 & 10 & 11 & 8 & 12 & 8 \\ A_3 & 13 & 12 & 14 & 13 & 12 \\ \text{Column max} & 13 & 12 & 14 & 13 & \end{array}$$

saddle point = (A_3, B_2)

$V = 12$

Graphical Method:

consider the matrix $2 \times n$ or $n \times 2$.

order = Row x column.

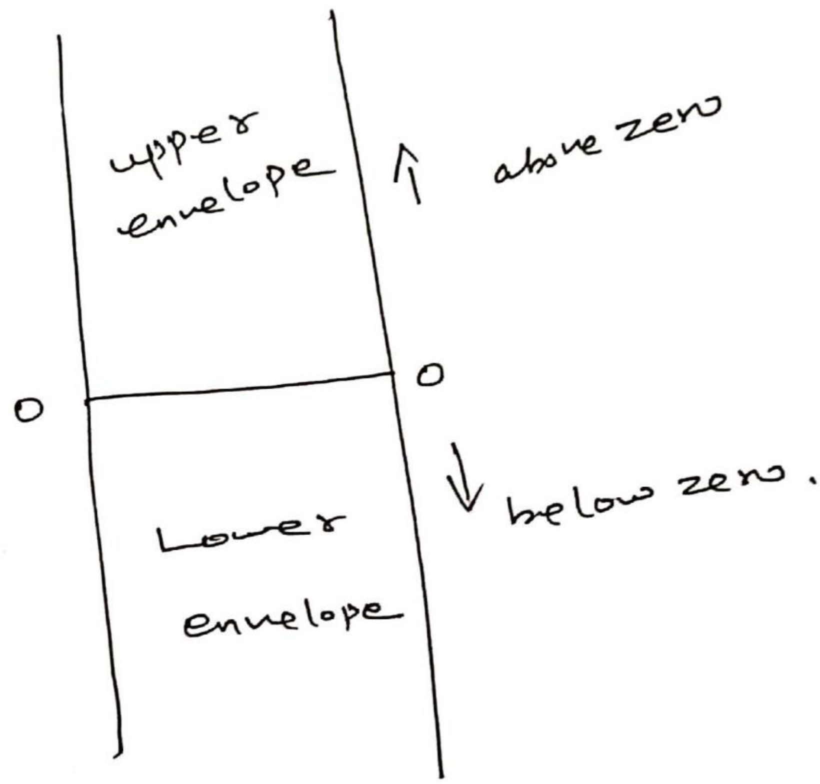
Case(i) : If the game is $2 \times n$ (2 Row x n column
then we select H.

Case(ii) : If the game is $n \times 2$, then
select L.

Here H : Highest vertex from the
Lower envelope region

L : Least vertex from the
upper envelope region

Note:



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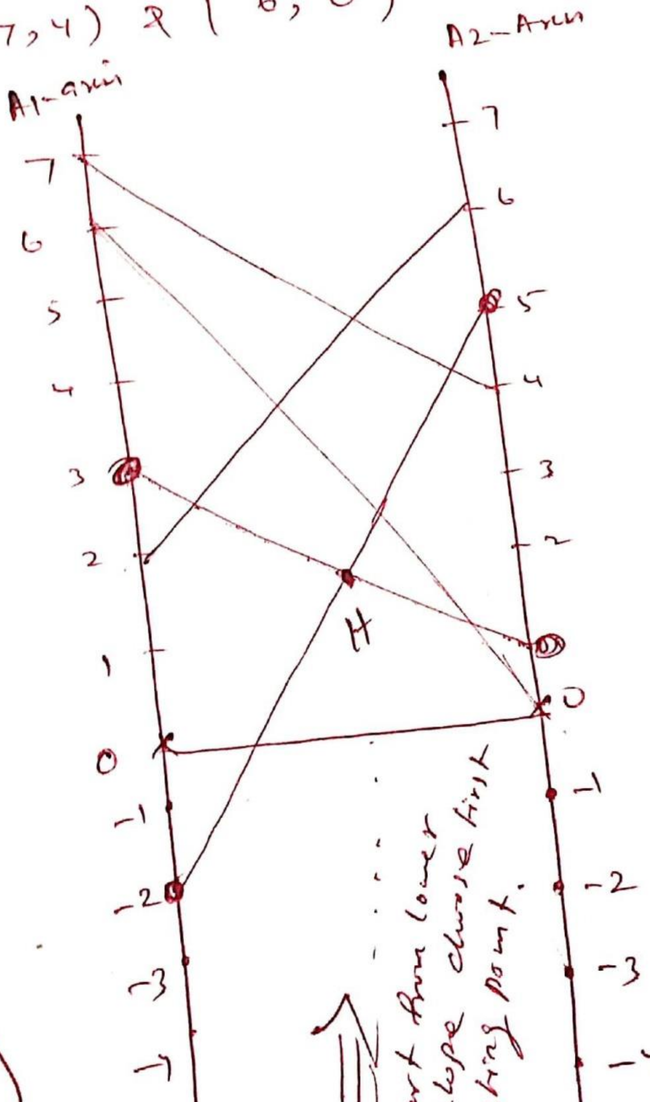
1) Solve the game by using graphical method:

$$
\begin{array}{c} \\ A_1 \\ A_2 \end{array}
\begin{array}{c}
\begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\
\begin{bmatrix} 2 & -2 & 3 & 7 & 6 \\ 6 & 5 & 1 & 4 & 0 \end{bmatrix}
\end{array}
$$

Soln:

This $2 \times n$ game (2×5). we choose Highest vertex H.

plot pair of points $(2,6)$ $(-2,5)$ $(3,1)$ $(7,4)$ & $(6,0)$



Start from lower envelope choose first meeting point.

The game reduced to

$$\begin{bmatrix} -2 & 3 \\ 5 & 1 \end{bmatrix}$$

The game reduce to (2×2) game

$$\begin{array}{c|cc} & B_2 & B_3 \\ A_1 & -2 & 3 \\ A_2 & 5 & 1 \end{array} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ with } p_1 + p_2 = 1$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \\ 0 & q_1 & q_2 & 0 & 0 \end{bmatrix} \text{ with } q_1 + q_2 = 1.$$

$$p_1 = \frac{d-c}{a-b-c+d} = \frac{1-5}{-2-3-5+1} = \frac{-4}{-9} = \frac{4}{9}$$

$$p_2 = 1 - p_1 = 1 - \frac{4}{9} = 5/9.$$

$$q_1 = \frac{d-b}{a-b-c+d} = \frac{1-3}{-2-3-5+1} = \frac{-2}{-9} = \frac{2}{9}$$

$$q_2 = 1 - q_1 = 1 - \frac{2}{9} = \frac{7}{9}.$$

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{4}{9} & \frac{5}{9} \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \\ 0 & \frac{2}{9} & \frac{7}{9} & 0 & 0 \end{bmatrix}$$

Value of the game $v = \dfrac{ad-bc}{a-b-c+d}$

$$= \frac{-2(1) - 3(5)}{-2-3-5+1}$$

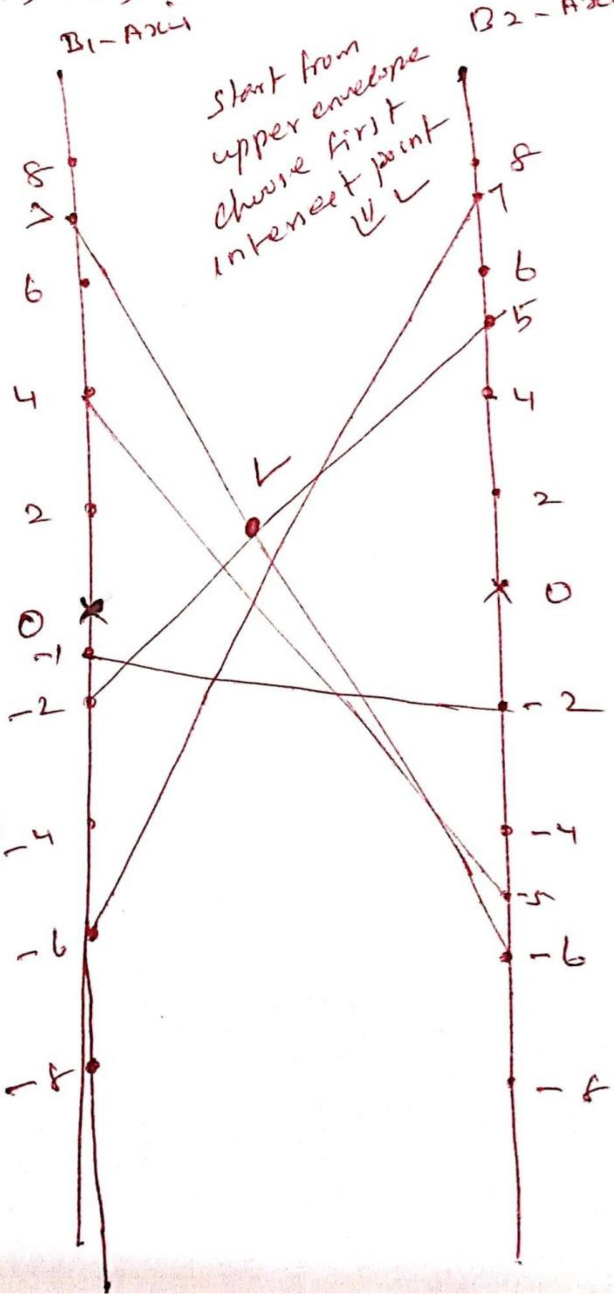
$$v = \frac{17}{9}$$

2) Solve the game using graphical Method :

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -6 & 7 \\ A_2 & 4 & -5 \\ A_3 & -1 & -2 \\ A_4 & -2 & 5 \\ A_5 & 7 & -6 \end{array}$$

This is $n \times 2$ game (5×2). we choose L

plot pair of points $(-6,7)$, $(4,-5)$, $(-1,-2)$, $(-2,5)$ and $(7,-6)$:



The game reduce to

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_4 & -2 & 5 \\ A_5 & 7 & -6 \end{array}$$

we formula to find the value of the game

Try to solve

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & 0 & \frac{13}{20} & \frac{17}{20} \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{11}{20} & \frac{9}{20} \end{bmatrix}$$

$\frac{23}{20}$

Let $\begin{array}{c} \\ A_4 \\ A_5 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -2 & 5 \\ 7 & -6 \end{bmatrix} \end{array} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & 0 & p_1 & p_2 \end{bmatrix}$ with $p_1 + p_2 = 1$

$$p_1 = \frac{d-c}{a-b-c+d} = \frac{-6-7}{-2-5-7-6} = \frac{-13}{-20} = \frac{13}{20}$$

$$p_2 = 1 - p_1 = 1 - \frac{13}{20} = \frac{7}{20}$$

$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_1 + q_2 = 1$

$$q_1 = \frac{d-b}{a-b-c+d} = \frac{-6-5}{-2-5-7-6} = \frac{11}{20}$$

$$q_2 = 1 - q_1 = 1 - \frac{11}{20} = \frac{9}{20}$$

$\therefore S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & 0 & \frac{13}{20} & \frac{7}{20} \end{bmatrix}$

$S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{11}{20} & \frac{9}{20} \end{bmatrix}$

value of the game $v = \dfrac{ad-bc}{a-b-c+d}$

$$= \frac{-2(-6)-5(7)}{-2-5-7-6}$$

$$v = \frac{23}{20}$$

(4) Solve the game by using dominance property: (or) Solve the game whose pay off

$$\text{Group } A = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 4 & -1 & 5 \\ 0 & 5 & 3 \\ 5 & 3 & 7 \end{bmatrix} \end{array}$$

step 1: use minimax criteria

$$\begin{array}{c} \begin{bmatrix} 4 & -1 & 5 \\ 0 & 5 & 3 \\ 5 & 3 & 7 \end{bmatrix} \\ \begin{array}{ccc} 5 & 5 & 7 \end{array} \end{array} \quad \begin{array}{c} \text{Row minima} \\ -1 \\ 0 \\ 3 \\ \\ \end{array}$$

Column max

maximum of Row minima = 3

minimum of colum max = 7

⇒ max (min) ≠ min (max)

⇒ the game has no saddle point.

note: (minmax criteria false)

Step 2

Let us reduce the game by using Dominance property:

Row operation : Hint: (Row operation false) go to column operation)

Step I:

Column operation: (In Group B)

Since the player B_1 dominates the player B_3.

⇒ omit the player B_3

Note:

$4 \to 5$
$0 \to 3$
$5 \to 2$

Smallest Highest

Note:

Hint: Column operation Least dominates Highest ⇒ omit Highest

∴ the game reduces to

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 4 & -1 \\ A_2 & 0 & 5 \\ A_3 & 5 & 3 \end{array}$$

(Go to ~~column~~ Row operation)

~~Column~~

Step II: (Row operation) (in) Group A

Since the player A_3 dominates the player A_1

⇒ omit A_1

∴ the game reduces to

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_2 & 0 & 5 \\ A_3 & 5 & 3 \end{array}$$

(Hint: Here after dominance property false)

Let us reduce the game by using mixed strategy:

(13)

Let the strategies for players A & B are

$$S_A = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} A_1 & A_2 & A_3 \\ \end{array}$$
$$S_A = \left[\begin{array}{ccc} 0 & p_1 & p_2 \end{array}\right] \text{ with } p_1+p_2=1$$

(Choose A_1 value 0 become A_1 out) B_3 out.

Total probability is always one.

$$S_B = \left[\begin{array}{ccc} q_1 & q_2 & 0 \end{array}\right] \text{ with } q_1+q_2=1$$

(columns B_1, B_2, B_3)

To find p_1 & q_1 Here $\begin{bmatrix} 0 & 5 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$p_1 = \frac{d-c}{a-b-c+d}$$

$$= \frac{3-5}{0-5-5+3} = \frac{-2}{-7} = \frac{2}{7}$$

$p_1 + p_2 = 1$

$p_2 = 1 - p_1 = 1 - \frac{2}{7} = 5/7.$

$$q_1 = \frac{d-b}{a-b-c+d} = \frac{3-5}{0-5-5+3} = \frac{-2}{-7} = \frac{2}{7}.$$

$q_1 + q_2 = 1$

$q_2 = 1 - q_1 = 1 - \frac{2}{7} = 5/7.$

$$\therefore S_A = \begin{array}{ccc} A_1 & A_2 & A_3 \end{array}$$
$$S_A = \left[\begin{array}{ccc} 0 & \frac{2}{7} & \frac{5}{7} \end{array}\right]$$

$$S_B = \begin{array}{ccc} B_1 & B_2 & B_3 \end{array}$$
$$S_B = \left[\begin{array}{ccc} \frac{2}{7} & \frac{5}{7} & 0 \end{array}\right]$$

Hence, Value of the game

$$V = \frac{ad-bc}{a-b-c+d}$$

$$= \frac{0(3)-5(5)}{0-5-5+3}$$

$$= \frac{25}{7}$$

Replacement

Model 1:

Replacement of an item whose maintenance cost increases with time and money value is not changed:

Note:

$$\text{Total cost} = \Sigma[f(t)] + C - S(t)$$

$$\text{Average cost} = \frac{\text{Total cost}}{\text{No of years}}.$$

where

C - purchase price

f(t) - Running cost

S(t) - Resale value.

Result:

» The best replacement period = Minimum Average cost

Problem:

Following table gives the running costs per year and resale value of a certain equipment whose purchase price Rs. 5000

year:	1	2	3	4	5	6	7	8
Run. cost:	1500	1600	1800	2100	2500	2900	3400	4000
Resale Value:	3500	2500	1700	1200	800	500	500	500

when should the machine be replaced?

Let $f(t)$ – Running cost : $c = 5000$, T = Total cost

$s(t)$ – Resale value

Year	$f(t)$	$\Sigma f(t)$ (A)	$s(t)$	$c - s(t)$ $5000 - s(t)$ (B)	Total cost = A+B	Avg cost
1	1500	1500	3500	5000−3500 = 1500	3000	$\frac{3000}{1} = 3000$
2	1600	3100	2500	2500	5600	$\frac{5600}{2} = 2800$
3	1800	4900	1700	3300	8200	2733
$\boxed{4}$	2100	7000	1200	3800	10800	$\boxed{2700}$
5	2500	9500	800	4200	13700	2740
6	2900	12400	500	4500	16900	2817
7	3400	15800	500	4500	20300	2900
8	4000	19800	500	4500	24300	3038

Hence the minimum Average cost = Rs. 2700

⇒ The best replacement period

= End of 4^{th} year.

Assignment:

Problem 2:

The cost of a machine is Rs. 6100 and its scrap value is Rs. 100. The maintenance costs found from experience are as follow:

Year:	1	2	3	4	5	6	7	8
Maintenance cost:	100	250	400	600	900	1200	1600	2000

At what year is the Replacement due?

Ans: end of 6^{th} year

Problem 3:

The data collected in running a machine the cost of which is Rs. ~~10~~,000 = 12000 are given below:

Year:	0	1	2	3	4	5	6
Cost of spares:	–	200	400	700	1000	1400	1600
Slary of Main: staff	–	1200	1200	1400	1600	2000	2600
Losses due to break down:	–	600	800	700	1000	1200	1600
Resale value	12000	6000	3000	1500	800	400	400

Determine the optimum period for replacement of the above machine.

Soln: Let Running cost $f(t) = (\text{cost of spares}) + (\text{Slary of Main. staff}) + (\text{Losses due to break down})$

$C = Rs. 12000$

Year	$f(t)$	(A) $\Sigma f(t)$	$s(t)$	(B) $C - s(t) = 12000 - s(t)$	Total cost $= A+B$	Avg cost
1	2000	2000	6000	6000	8000	$\frac{8000}{1} = 8000$
2	2400	4400	3000	9000	13400	$\frac{13400}{2} = 6700$
3	2800	7200	1500	10500	17700	5900
4	3600	10800	800	11200	22000	5500
$\boxed{5}$	4600	15400	400	11600	27000	$\boxed{5400}$
6	5800	21200	400	11600	32800	5467

$\therefore$ Minimum Avg cost = Rs. 5400

$\Rightarrow$ the best replacement year = The end of 5^{th} year

Model 2: Replacement of items due to sudden failure:

There are two types of policy

(i) Individual replacement policy

(ii) Group replacement policy

$$\text{Total cost} = N c_g + c_1 \sum_{i=1}^{n-1} N_i$$

$$\text{Average cost} = \frac{\text{Total cost}}{\text{No. of years}}.$$

where

N – Total no. of items in the system.

N_i – Number of items failed the i^{th} period

c_g = group replacement cost.

c_1 – indi. replace. cost.

Problem: Following mortality rates have been observed for a certain type of fuses

week:	1	2	3	4	5
% failing before end of week	5	15	35	75	100

There are 1000 fuses in use and it cost Rs. 5 to replace an individual fuse. If all fuses were replaced simultaneously it would cost Rs. 1.25 per fuse. It is proposed to replace all fuses at fixed intervals of time, whether or not they have burnt out, and to continue replacing burnt out fuses as they fail. At what intervals the group replacement should be made?

(5)

Let p_i be the probability that a

Soln: fuse fails during the ith week.

week	% failing before the end of week	% failing during the week
1	$5\% = \frac{5}{100} = 0.05$	5
2	15	10
3	35	20
4	75	40
5	100	25

$\therefore$

$p_1 = 0.05$

$p_2 = 0.10$

$p_3 = 0.20$

$p_4 = 0.40$

$p_5 = 0.25$

Let N_i be the Number of fuses failed during the ith week.

$N_0 = 1000$

$N_1 = N_0 p_1 = 1000(0.05) = 50$

$N_2 = N_0 p_2 + N_1 p_1 = 1000 \times 0.1 + 50 \times 0.05 \simeq 102$

$N_3 = N_0 p_3 + N_1 p_2 + N_2 p_1$

$= 1000 \times 0.2 + 50 \times 0.1 + 102 \times 0.05 \simeq 210$

$N_4 = N_0 p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1 \simeq 430$

$N_5 = N_0 p_5 + N_1 p_4 + N_2 p_3 + N_3 p_2 + N_4 p_1 \simeq 333$

To find Group Replacement period

$$\text{Total cost} = N C_g + C_1 \sum_{i=1}^{n-1} N_i$$

End of week	Total cost $= N C_g + C_1 \sum_{i=1}^{n-1} N_i$	Avg cost
1	$1000 \times 1.25 + 50 \times 5 = 1500$	$\frac{1500}{1} = 1500$
$\boxed{2}$	$1500 + 102 \times 5 = 2010$	$\frac{2010}{2} = \boxed{1005}$
3	$2010 + 210 \times 5 = 3060$	$\frac{3060}{3} = 1030$
4	$3060 + 430 \times 5 = 5210$	$\frac{5210}{4} = 1302.50$
5	$5210 + 333 \times 5 = 6875$	$\frac{6875}{5} = 1375$

$\therefore$ Minimum Avg cost = Rs. 1005

$\Rightarrow$ Group Replacement period is 2 week

To find individual Replacement $= \sum x\, p(x)$

week x	p_i $p(x)$	Expected life $E(x) = \sum x\, p(x)$
1	0.05	$1 \times 0.05 = 0.05$
2	0.10	$2 \times 0.1 = 0.20$
3	0.20	$3 \times 0.2 = 0.60$
4	0.40	$4 \times 0.4 = 1.60$
5	0.25	$5 \times 0.25 = 1.25$

Total = 3.70

$\therefore$ Average No of fuses failing per week $= \frac{1000}{3.7}$

$= 270.$

$\therefore$ Total Average cost $= 270 \times 5$

$= \text{Rs.}\ 1350$

Comparing this with average cost of group replacement we get,
the group replacement policy is superior to individual replacement policy.