

SBAA5205

**APPLIED OPERATIONS
RESEARCH**

UNIT IV

REPLACEMENT MODEL AND GAME THEORY

UNIT - IV

Game Theory:

There are three Methods:

- (i) Min Max criteria
- (ii) Dominance property
- (iii) Graphical Method.

Min Max criteria:

Rules for determining saddle point

- (i) Select the minimum pay off in each row
And
(ii) Select the Maximum pay (Max value)
in each column
- (iii) Choose Maximum value ϱ_r (each row)
Row minimum.
- (iv) choose Minimum value ϱ_c column maximum

Result: If Maximum ϱ_r row minima = Minimum ϱ_c column maximum, then

game has a saddle point.

(i) If $\text{Max}(\min) = \text{Min}(\max)$, then the game has a saddle point. (ii) pure strategy

(ii) If $\text{max}(\min) \neq \text{min}(\max)$, then the game has no saddle point.
(mixed strategies).

①

Solve the same by using minimax criteria

		Group B				
		B ₁	B ₂	B ₃	B ₄	
Group A		A ₁	12	1	30	-10
		A ₂	20	3	10	5
		A ₃	-5	-2	25	0
		A ₄	15	-4	10	6

Name of players ↙

Solution:

		B ₁	B ₂	B ₃	B ₄	Row minimum
A ₁		12	1	30	-10	-10
A ₂		20	3	10	5	3 ✓
A ₃		-5	-2	25	0	-5
A ₄		15	-4	10	6	-4

Column max: 20 3 30 6

Here: Maximum { } Row minimum = 3

Minimum { } column maximum = 3

$$(ii) \max(\min) = \min(\max) = 3$$

⇒ The game has a saddle point
∴ Saddle point = (A₂, B₂)

Value of the game v = 3

Problem 2:

Solve the game $A_1 \begin{bmatrix} 4 & 2 \\ 0 & 8 \end{bmatrix}$ $B_1 \quad B_2$

Soln: Row minimum

$$\begin{bmatrix} 4 & 2 \\ 0 & 8 \end{bmatrix} \quad \begin{matrix} 2 \\ 0 \end{matrix}$$

Column max $\begin{matrix} 4 \\ 8 \end{matrix}$

\Rightarrow maximum & non-minima \neq minimum \Rightarrow column max
 \therefore the game has no saddle point

\Rightarrow ~~Let us~~ Let us reduce the game by
 using strategies

$$\text{Let } \begin{bmatrix} 4 & 2 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let the strategies for the players A & B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ with } p_1 + p_2 = 1$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix} \text{ with } q_1 + q_2 = 1$$

To find p_1, q_1

$$p_1 = \frac{d-c}{a-b-c+d} = \frac{8-0}{4-2-0+8} = \frac{8}{10} = \frac{2}{5}$$

$$p_2 = 1 - p_1 = 1 - \frac{2}{5} = \frac{3}{5}$$

$$q_1 = \frac{d-b}{a-b-c+d} = \frac{8-2}{4-2-0+8} = \frac{6}{10} = \frac{3}{5}$$

(ii)

$$V_2 = 1 - V_1 \\ = 1 - \frac{3}{5} = \frac{2}{5}$$

\therefore the strategies for players A & B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

Value of the game $V = \frac{ad - bc}{a+b+c+d}$

$$= \frac{4(8) - 0(0)}{4+2-0+8}$$

$$= \frac{32}{10}$$

$$\boxed{V = \frac{16}{5}}$$

Dominance property:

Result: (i) Row operation: Highest value (Row) dominates Least value (Row)

\Rightarrow omit least row

Note: (ii) Column operation: Least column (value) dominates the highest column (value)

\Rightarrow omit Highest column

(5)

Problem 3:

using the principle of Dominance

Solve the game
group B.

	B ₁	B ₂	B ₃	B ₄
A ₁	8	10	9	14
A ₂	10	11	8	12
A ₃	13	12	14	13

Group A

Step 1:

Row operation: (In group A)

since the player A₃ dominates the player A₂∴ omit A₂

	B ₁	B ₂	B ₃	B ₄
A ₁	8	10	9	14
A ₃	13	12	14	13

Step 2: column operation [In group B]Since the player B₁ dominatesboth the players B₃ and B₄.∴ omit B₃ and B₄.

The game reduces to

	B ₁	B ₂
A ₁	8	10
A ₃	13	12

(b)

Step 3: (Row operation)

since the player A_3 dominates A_1
 omit A_1

the game reduces to

$$B_1 \quad B_2 \\ A_3 [13 \quad 12]$$

Step 4: (column operation)

since the player B_2 dominates the
 player B_1

\therefore omit B_1

Hence the game reduces to

$$A_3 [\begin{smallmatrix} B_2 \\ 12 \end{smallmatrix}]$$

\therefore saddle point = $[A_3, B_2]$

value of the game $v = 12$

		B_1	B_2	B_3	B_4	Row minimum
		8	10	9	14	8
A_1	A_2	10	11	8	12	8
	A_3	13	12	14	13	12

Note:
 A_1 $B_1 \quad B_2 \quad B_3 \quad B_4$
 A_2 8 10 9 14 Row minimum
 A_3 10 11 8 12 8
 Column max 13 12 14 13
 saddle point = (A_3, B_2)
 $v = 12$

Graphical Method:

consider the matrix $2 \times n$ or $n \times 2$

order = Row \times column.

case(i) : If the game is $2 \times n$ (2 row \times column)
then we select H.

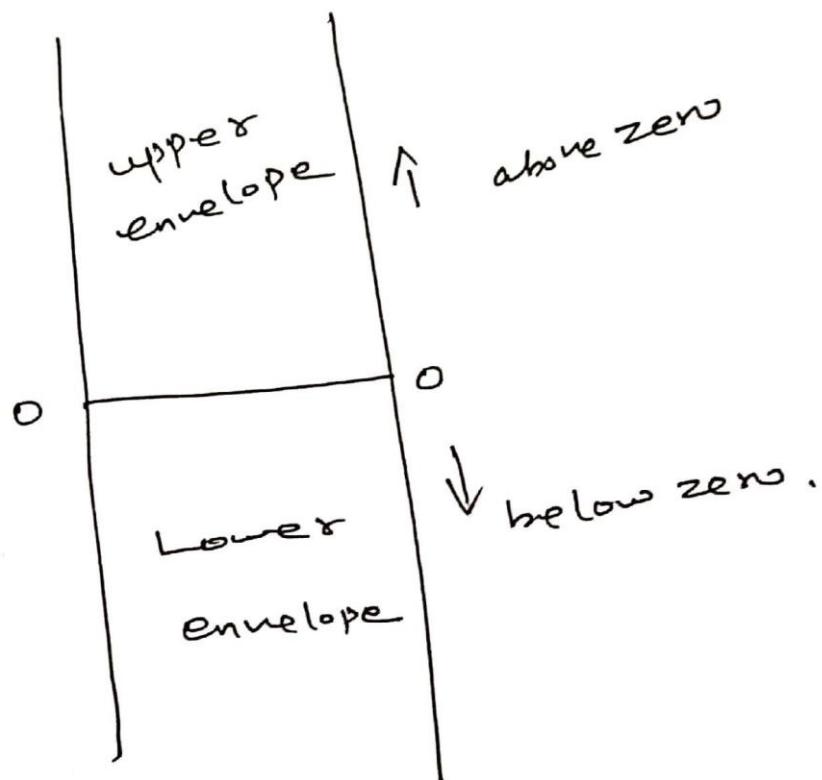
case(ii) : If the game is $n \times 2$, then
select L.

Here

H: Highest vertex from the
lower envelope region

L: Least vertex from the
upper envelope region

Note:



(#) (7)

1) Solve the game by using graphical

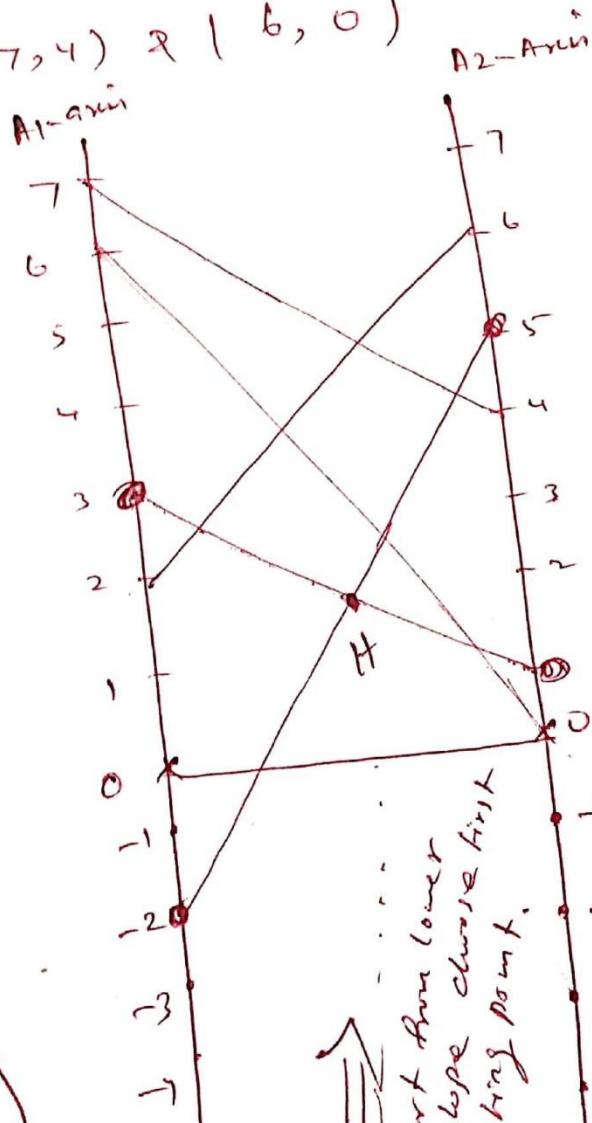
Method:

	B_1	B_2	B_3	B_4	B_5
A_1	2	-2	3	7	6
A_2	6	5	1	4	0

Soln: This $2 \times n$ game (2×5). we choose highest vertex H.

plot pair of point $(2, 6)$ $(-2, 5)$ $(3, 1)$

$(7, 4) \times (6, 0)$



Start from lower envelope choose first meeting point.

The game matrix is

$$\begin{bmatrix} -2 & 3 \\ 5 & 1 \end{bmatrix}$$

The same reduce to (2×2) game

$$\begin{matrix} & B_2 & B_3 \\ A_1 & \begin{bmatrix} -2 & 3 \\ 5 & 1 \end{bmatrix} & = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{matrix}$$

$$S_A = \begin{bmatrix} a_1 & a_2 \\ p_1 & p_2 \end{bmatrix} \text{ with } p_1 + p_2 = 1$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \\ 0 & q_1 & q_2 & 0 & 0 \end{bmatrix} \text{ with } q_1 + q_2 = 1.$$

$$p_1 = \frac{d-c}{a-b-c+d} = \frac{1-5}{-2-3-5+1} = \frac{-4}{-9} = \frac{4}{9}$$

$$p_2 = 1 - p_1 = 1 - \frac{4}{9} = \frac{5}{9}.$$

$$q_1 = \frac{d-b}{a-b-c+d} = \frac{1-3}{-2-3-5+1} = \frac{-2}{-9} = \frac{2}{9}$$

$$q_2 = 1 - q_1 = 1 - \frac{2}{9} = \frac{7}{9}.$$

$$S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{4}{9} & \frac{5}{9} \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \\ 0 & \frac{2}{9} & \frac{7}{9} & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{Value of the game } V &= \frac{ad - bc}{a-b-c+d} \\ &= \frac{-2(1) - 3(5)}{-2-3-5+1} \end{aligned}$$

$$V = \frac{17}{9}$$

2) Solve the game using graphical method:

(9)

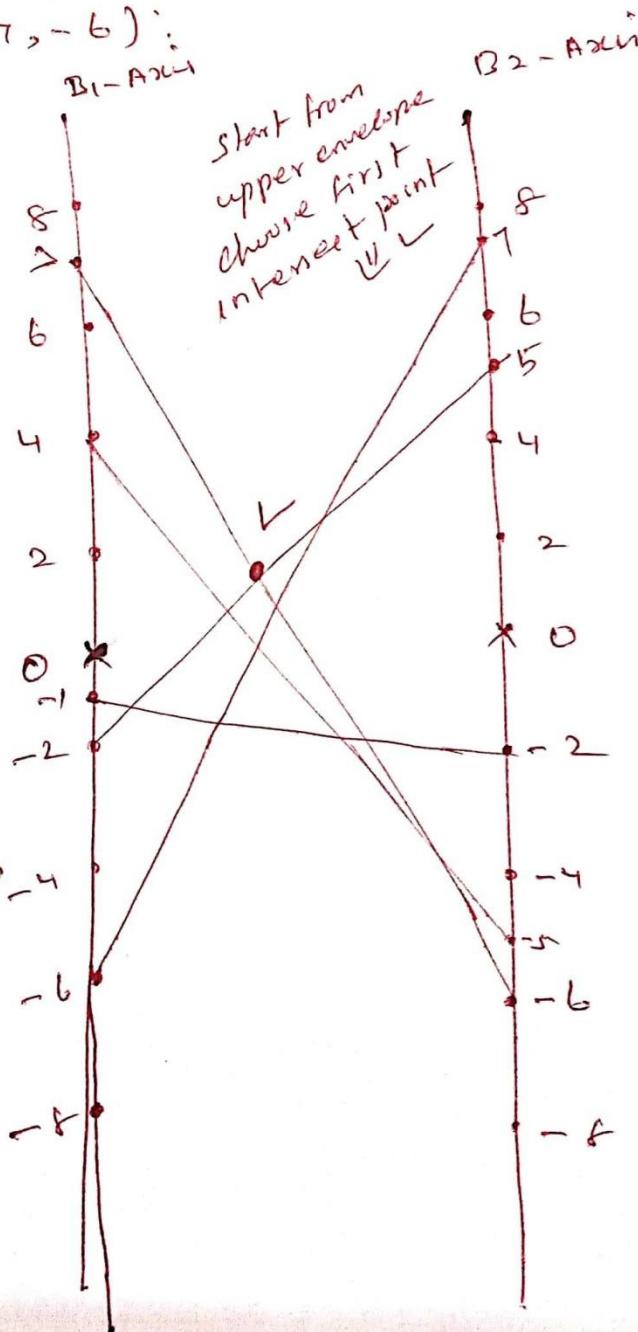
(12)

	B ₁	B ₂
A ₁	-6	7
A ₂	4	-5
A ₃	-1	-2
A ₄	-2	5
A ₅	-7	-6

This is $n \times 2$ game (5×2). We choose L.

Plot pairs of points $(-6, 7), (4, -5), (-1, -2), (-2, 5)$ and $(-7, -6)$.

"



The game reduces to
try to solve A₅.
 $S_A = \left\{ \begin{array}{l} 0 \\ 0 \\ 0 \\ 0 \\ \frac{13}{20} \end{array} \right\}$
 $S_B = \left\{ \begin{array}{l} \frac{1}{2} \\ \frac{1}{2} \\ \frac{23}{20} \end{array} \right\}$

(10)

$$\text{Let } A_4 \begin{bmatrix} B_1 & B_2 \\ -2 & 5 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & 0 & p_1 & p_2 \end{bmatrix} \text{ with } p_1 + p_2 = 1$$

$$p_1 = \frac{d-c}{a-b-c+d} = \frac{-6-7}{-2-5-7-6} = \frac{-13}{-20} = \frac{13}{20}$$

$$p_2 = 1 - p_1 = 1 - \frac{13}{20} = \frac{7}{20}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix} \text{ with } q_1 + q_2 = 1$$

$$q_1 = \frac{d-b}{a-b-c+d} = \frac{-6-5}{-2-5-7-6} = \frac{11}{20}$$

$$q_2 = 1 - q_1 = 1 - \frac{11}{20} = \frac{9}{20}$$

$$\therefore S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & 0 & \frac{13}{20} & \frac{7}{20} \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{11}{20} & \frac{9}{20} \end{bmatrix}$$

value of the game $V = \frac{ad-bc}{a-b-c+d}$

$$= \frac{-2(-6)-5(7)}{-2-5-7-6}$$

$$V = \frac{23}{20}$$

④ ⑪

(u) Solve the game by using dominance
 property: ^(or) Solve the game whose pay off

	B ₁	B ₂	B ₃
A ₁	4	-1	5
A ₂	0	5	3
A ₃	5	3	7

Step 1: use minimax criteria
 Row minima

[4 -1 5]	-1
[0 5 3]	0
[5 3 7]	3

Column max 5 5 7

maximum Q_S Row minima = 3

minimum Q_B column max = 7

$\Rightarrow \text{max(min)} \neq \text{min(max)}$

\Rightarrow the game has no saddle point.

Note: (minimax criteria false)

Step 2 Let us reduce the game by using Dominance

Property:

Row operation : (Row operation false)
Hint:

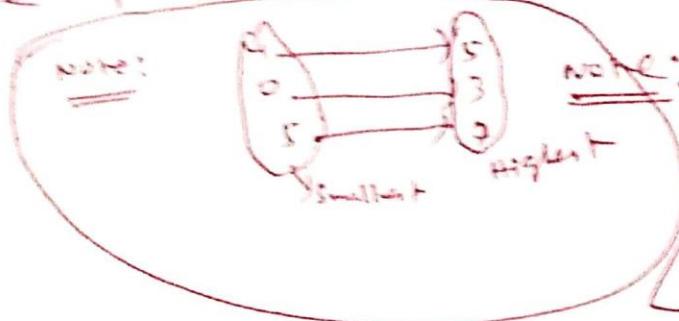
no \Rightarrow column operation

Step I: Column operation: (in group B)

(12)

since the player B_1 dominates the player B_2 .

\Rightarrow omit the player B_2 .



Hint:
Column operation
Least dominates
Highest
 \Rightarrow omit Highest

\therefore the game reduces to

	B_1	B_2
A_1	4	-1
A_2	0	5
A_3	5	3

(now to ~~row~~ operation)

Step II: (Row operation) (ii) group A

\Rightarrow since the player A_3 dominates the player A_1
 \Rightarrow omit A_1

\therefore the game reduces to

	B_1	B_2
A_2	0	5
A_3	5	3

(Note: Here after dominance property false)

Let us reduce the game by using mixed strategy:

(T3)

Let the strategies for players A & B are

$$S_A = \left\{ \begin{array}{c} A_1 \quad A_2 \quad A_3 \\ 0 \quad p_1 \quad p_2 \end{array} \right\} \xrightarrow{\text{minimax}} \left\{ \begin{array}{c} \text{choose } A_1 \text{ value 0} \\ \text{become } A_1 \text{ out} \\ B_3 \text{ out.} \end{array} \right.$$

$$S_B = \left\{ \begin{array}{c} B_1 \quad B_2 \quad B_3 \\ q_1 \quad q_2 \quad 0 \end{array} \right\} \xrightarrow{\text{with } q_1+q_2=1} \text{Total probability is always one.}$$

To find p_1 & q_1 Here $\begin{bmatrix} 0 & 5 \\ s & 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$p_1 = \frac{d-c}{a-b-c+d}$$

$$= \frac{3-5}{0-5-5+3} = \frac{-2}{-7} = \frac{2}{7}$$

$$p_1 + p_2 = 1$$

$$p_2 = 1 - p_1 = 1 - \frac{2}{7} = \frac{5}{7}.$$

$$q_1 = \frac{d-b}{a-b-c+d} = \frac{3-5}{0-5-5+3} = \frac{-2}{-7} = \frac{2}{7}.$$

$$q_1 + q_2 = 1$$

$$q_2 = 1 - q_1 = 1 - \frac{2}{7} = \frac{5}{7}.$$

Hence, Value b the same

$$\therefore S_A = \left\{ \begin{array}{c} A_1 \quad A_2 \quad A_3 \\ 0 \quad \frac{2}{7} \quad \frac{5}{7} \end{array} \right\}$$

$$S_B = \left\{ \begin{array}{c} B_1 \quad B_2 \quad B_3 \\ \frac{2}{7} \quad \frac{5}{7} \quad 0 \end{array} \right\}$$

$$V = \frac{ad-bc}{a-b-c+d}$$

$$= \frac{0(5) - 5(5)}{0-5-5+3}$$

$$= \frac{25}{7} //$$

①

Replacement

Model I:

Replacement of an item whose maintenance cost increases with time and money value is not changed:

Note:

$$\text{Total cost} = \sum [f(t)] + c - sL(t) \quad \left| \begin{array}{l} \text{where} \\ c - \text{purchase price} \\ f(t) - \text{Running cost} \\ sL(t) - \text{Resale value.} \end{array} \right.$$

$$\text{Average cost} = \frac{\text{Total cost}}{\text{No of years}}$$

Result:

» The best replacement period = Minimum Average cost

Problem:

Following table gives the running costs per year and resale value of a certain equipment whose purchase price Rs. 5000

Year:	1	2	3	4	5	6	7	8
Run. cost:	1500	1600	1800	2100	2500	2900	3400	4000
Resale Value:	3500	2500	1700	1200	800	500	500	500

when should the machine be replaced?

(2)

Let $f(t)$ = Running cost : $c = 5000$, T = Total cost

$s(t)$ = Resale value

Year	$f(t)$	$\leq f(t)$ (A)	$s(t)$	$c - s(t)$ $5000 - s(t)$ (B)	Total cost $= A + B$	Avg cost
1	1500	1500	3500	$5000 - 3500$ = 1500	3000	$\frac{3000}{1} = 3000$
2	1600	3100	2500	2500	5600	$\frac{5600}{2} = 2800$
3	1800	4900	1700	3300	8200	2733
4	2100	7000	1200	3800	10800	2700
5	2500	9500	800	4200	13700	2740
6	2900	12400	500	4500	16900	2817
7	3400	15800	500	4500	20300	2900
8	4000	19800	500	4500	24300	3038

Hence the minimum Average cost = Rs. 2700

\Rightarrow The best replacement period
= End of 4th year.

Assignment:

Problem 2:

The cost of a machine is Rs. 6100
and its scrap value is Rs. 100. The
maintenance costs found from experience
are as follow:

Year:	1	2	3	4	5	6	7	8
Main maintenance cost	100	250	400	600	900	1200	1600	2000

At what year is the replacement due?
Ans: end of 6th year

(3)

Problem 3:

The data collected in running a machine the cost of which is Rs. $\frac{12000}{=12000}$ are given below:

Year:	0	1	2	3	4	5	6
Cost of spares:	-	200	400	700	1000	1400	1600
Salary of Main staff	-	1200	1200	1400	1600	2000	2600
Losses due to breakdown:	-	600	800	700	1000	1200	1600
Resale value	12000	6000	3000	1500	800	400	400

Determine the optimum period for replacement of the above machine.

Soln: Let Running cost $f(t) = \text{Cost of spares} + \text{Salary of Main staff} + \text{Losses due to breakdown}$

Year	$f(t)$	(A) $\Sigma f(t)$	(B) $C - S(t)$ $= 12000 - S(t)$	Total cost $= A + B$	Avg cost
1	2000	2000	6000	8000	$\frac{8000}{1} = 8000$
2	2400	4400	3000	9000	$\frac{13400}{2} = 6700$
3	2800	7200	1500	10500	5900
4	3600	10800	800	11200	5500
5	4600	15400	400	11600	5400
6	5800	21200	400	11600	5467

\therefore Minimum Avg cost = Rs. 5400

\Rightarrow the best replacement year = the end of 5^{th} year

Model 2: Replacement of items due to sudden failure:

There are two types of policy

(i) Individual replacement policy

(ii) Group replacement policy

$$\text{Total cost} = N c_g + C_1 \sum_{i=1}^{n-1} N_i$$

where
 N - Total no. of items in the system.
 N_i - Number of items failed the i^{th} period
 c_g = group replacement cost.
 C_1 - Ind. replace. cost, rates have

$$\text{Average cost} = \frac{\text{Total cost}}{\text{no. of years}}.$$

Problem: Following mortality been observed for a certain type of fuses

week:	1	2	3	4	5
% failing before end of week	5	15	35	75	100

There are 1000 fuses in use and it cost Rs. 5 to replace an individual fuse. If all fuses were replaced simultaneously it would cost Rs. 1.25 per fuse. It is proposed to replace all fuses at fixed intervals of time, whether or not they have burnt out, and to continue replacing burnt out fuses as they fail. At what intervals the group replacement should be made?

(5)

Let p_i be the probability that a
solt: fuse fails during the i th week.

week	% failing before the end of week	% failing during the week	p_i
1	$5\% = \frac{5}{100} = 0.05$	5	$p_1 = 0.05$
2	15	10	$p_2 = 0.10$
3	35	20	$p_3 = 0.20$
4	75	40	$p_4 = 0.40$
5	100	25	$p_5 = 0.25$

Let N_i be the number of fuses failed during the i th week.

$$N_0 = 1000$$

$$N_1 = N_0 p_1 = 1000 (0.05) = 50$$

$$N_2 = N_0 p_2 + N_1 p_1 = 1000 \times 0.1 + 50 \times 0.05 \approx 102$$

$$\begin{aligned} N_3 &= N_0 p_3 + N_1 p_2 + N_2 p_1 \\ &= 1000 \times 0.2 + 50 \times 0.1 + 102 \times 0.05 \approx 210 \end{aligned}$$

$$N_4 = N_0 p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1 \approx 430$$

$$N_5 = N_0 p_5 + N_1 p_4 + N_2 p_3 + N_3 p_2 + N_4 p_1 \approx 333$$

(6)

To find Group Replacement period

$$\text{Total cost} = N C_g + C_1 \sum_{i=1}^{n-1} N_i$$

End of week	Total cost $= N C_g + C_1 \sum_{i=1}^{n-1} N_i$	Annual cost
1	$1000 \times 1.25 + 50 \times 5 = 1500$	$\frac{1500}{1} = 1500$
2	$1500 + 102 \times 5 = 2010$	$\frac{2010}{2} = 1005$
3	$2010 + 210 \times 5 = 3060$	$\frac{3060}{3} = 1030$
4	$3060 + 430 \times 5 = 5210$	$\frac{5210}{4} = 1302.50$
5	$5210 + 333 \times 5 = 6875$	$\frac{6875}{5} = 1375$

\therefore Minimum Annual cost = Rs. 1005

\Rightarrow Group Replacement period is 2 week

To find Individual Replacement $t = \frac{\sum x p(x)}{\sum p(x)}$

week	p_i $p(x)$	Expected life $E(x) = \sum x p(x)$
1	0.05	$1 \times 0.05 = 0.05$
2	0.10	$2 \times 0.1 = 0.20$
3	0.20	$3 \times 0.2 = 0.60$
4	0.40	$4 \times 0.4 = 1.60$
5	0.25	$5 \times 0.25 = 1.25$
		Total = 3.70

\therefore Average no of fuses failing per week = $\frac{1000}{3.7} = 270$

⑦.

$$\therefore \text{Total Average cost} = 270 \times 5 \\ = \underline{\underline{Rs. 1350}}$$

Comparing this with average cost of group replacement we get,

The group replacement policy is superior to individual replacement policy.

