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RESOURCE MANAGEMENT TECHNIQUES

For B.E. VII Semester CSC Branch

As per the Latest Syllabus of Anna University, Chennai

(Regulation 2013)

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PREFACE

We are very happy to present the first revised edition of this book. We are thankful to all the readers of the earlier editions of this book for their kind and continued support. Special thanks to the Members of the Department of Mathematics, SRM Engineering College for their constructive suggestions towards the improvement of this book.

This edition has been thoroughly revised taking into account the suggestions given by the students and teachers of various institutions. Also latest university questions have been added.

Hope this edition will also receive the same encouraging response. Suggestions towards for improvement of the book will be gratefully acknowledged.

June 2016 Authors

SYLLABUS

RESOURCE MANAGEMENT TECHNIQUES

UNIT I LINEAR PROGRAMMING **9**

Principal components of decision problem – Modeling phases – LP Formulation and graphic solution –Resource allocation problems – Simplex method – Sensitivity analysis.

UNIT II DUALITY AND NETWORKS **9**

Definition of dual problem – Primal – Dual relation ships – Dual simplex methods – Post optimality analysis – Transportation and assignment model - Shortest route problem.

UNIT III INTEGER PROGRAMMING **9**

Cutting plan algorithm – Branch and bound methods, Multistage (Dynamic) programming.

UNIT IV CLASSICAL OPTIMISATION THEORY: **9**

Unconstrained external problems, Newton – Ralphson method – Equality constraints – Jacobean methods – Lagrangian method – Kuhn – Tucker conditions – Simple problems.

UNIT V OBJECT SCHEDULING: **9**

Network diagram representation – Critical path method – Time charts and resource leveling – PERT.

TOTAL: 45 PERIODS

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UNIT - I : LINEAR PROGRAMMING

UNIT 1

Linear Programming

1.1 INTRODUCTION

Operations Research is the study of optimisation techniques. It is applied decision theory. The existence of optimisation techniques can be traced at least to the days of Newton and Lagrange. Rapid development and invention of new techniques occurred since the World War II essentially, because of the necessity to win the war with the limited resources available. Different teams had to do ***research*** on military ***operations*** in order to invent techniques to ***manage*** with available ***resources*** so as to obtain the desired objective. Hence the nomenclature ***Operations Research or Resource Management Techniques.***

Scope or Uses or Applications of O.R (Some O.R. Models)

O.R. is useful for solving

(1) Resource allocation problems
(2) Inventory control problems
(3) Maintenance and Replacement Problems
(4) Sequencing and Scheduling Problems
(5) Assignment of jobs to applicants to maximise total profit or minimize total cost.
(6) Transportation Problems
(7) Shortest route problems like travelling sales person problems
(8) Marketing Management problems
(9) Finance Management problems
(10) Production, Planning and control problems
(11) Design Problems
(12) Queuing problems, *etc.* to mention a few.

1.1.2 Role of operations research in Business and Management

1. ***Marketing Management Operations research techniques have definitely a role to play in***
   - *(a)* Product selection
   - *(b)* Competitive strategies
   - *(c)* Advertising strategy *etc.*

2. ***Production Management***

The O.R techniques are very useful in the following areas of production management.

- *(a)* Production scheduling
- *(b)* Project scheduling
- *(c)* Allocation of resources
- *(d)* Location of factories and their sizes
- *(e)* Equipment replacement and Maintenance
- *(f)* Inventory policy etc.

3. ***Finance Management***

The techniques O.R are applied to Budgeting and Investment areas and especially to

- *(a)* Cash flow analysis
- *(b)* Capital requirement
- *(c)* Credit policies
- *(d)* Credit risks *etc.*

4. ***Personal Management***
   - *(a)* Recruitment policies and
   - *(b)* Assignment of jobs are some of the areas of personnel management where O.R techniques are useful.

5. ***Purchasing and Procurement***
   - *(a)* Rules for purchasing
   - *(b)* Determining the quantity
   - *(c)* Determining the time of purchaser are some of the areas where O.R. techniques can be applied

6. ***Distribution***

In determining

- *(a)* location of warehouses
- *(b)* size of the warehouses
- *(c)* rental outlets
- *(d)* transportation strategies O.R. techniques are useful.

1.1.3 Role of O.R in Engineering

1. Optimal design of water resources systems
2. Optimal design of structures
3. Production, Planning, Scheduling and control
4. Optimal design of electrical networks
5. Inventory control
6. Planning of maintenance and replacement of equipment
7. Allocation of resources of services to maximise the benefit
8. Design of material handling
9. Optimal design of machines
10. Optimum design of control systems
11. Optimal selection of sites for an industry to mention a few.

1.1.4 Classification of Models

The first thing one has to do to use O.R. techniques after formulating a practical problem is to construct a suitable model to represent the practical problem. A model is a reasonably simplified representation of a real–world situation. It is an abstraction of reality. The models can broadly be classified as

Iconic (Physical) Models
Analogue Models
Mathematical Models

Also as

Static Models
Dynamic Models and, in addition, as
Deterministic Models
Stochastic Models

Models can further be subdivided as

Descriptive Models
Prescriptive Models
Predictive Models
Analytic Models
Simulation Models

Iconic Model :

This is a physical, or pictorial representation of various aspects of a system.

Example : Toy, Miniature model of a building, scaled up model of a cell in biology *etc.*

Analogue or Schematic model :

This uses one set of properties to represent another set of properties which a system under study has

Example : A network of water pipes to represent the flow of current in an electrical network or graphs, organisational charts etc.

Mathematical model or Symbolic model :

This uses a set of mathematical symbols (letters, numbers etc) to represent the decision variables of a system under consideration. These variables related by mathematical equations or inequalities which describes the properties of the system.

Example : A linear programming model, A system of equations representing an electrical network or differential equations representing dynamic systems etc.

Static Model :

This is a model which does not take time into account. It assumes that the values of the variables do not change with time during a certain period of time horizon.

Example : A linear programming problem, an assignment problem, transportation problem *etc.*

Dynamic model is a model which considers time as one of the important variables.

Example: A dynamic programming problem, A replacement problem

Deterministic model is a model which does not take uncertainty into account.

Example : A linear programming problem, an assignment problem *etc.*

Stochastic model is a model which considers uncertainty as an important aspect of the problem.

Example : Any stochastic programming problem, stochastic inventory models *etc.*

Descriptive model is one which just describes a situation or system.

Example : An opinion poll, any survey.

Predictive model is one which predicts something based on some data. Predicting election results before actually the counting is completed.

Prescriptive model is one which prescribes or suggests a course of action for a problem.

Example : Any programming (linear, nonlinear, dynamic, geometric *etc.*) problem.

Analytic model is a model in which exact solution is obtained by mathematical methods in closed form.

Simulation model is a representation of reality through the use of a model or device which will react in the same manner as reality under a given set of conditions. Once a simulation model is designed, it takes only a little time, in general, to run a simulation on a computer.

It is usually less mathematical and less time consuming and generally least expensive as well, in many situations.

Example : queuing problems, inventory problems.

1.1.5 Some characteristics of a good Model

(1) It should be reasonably simple.

(2) A good model should be capable of taking into account new changes in the situation affecting its frame significantly with ease *i.e.,* updating the models should be as simple and easy as possible,

(3) Assumptions made to simplify the model should be as small as possible.

(4) Number of variables used should be as small in number as possible.

(5) The model should be open to parametric treatment.

1.1.6 Principles of Modelling

(1) Do not build up a complicated model while a simple one will suffice.

(2) Beware of moulding the problems to fit a (favourite !) technique

(3) Deductions must be made carefully.

(4) Models should be validated prior to implementation.

(5) A model should neither be pressed to do nor criticised for failing to do that for which it was never intended.

(6) Beware of overselling the model in cases where assumption made for the construction of the model can be challenged.

(7) The solution of a model cannot be more accurate than the accuracy of the information that goes into the construction of the model.

(8) Models are only aids in decision making.

(9) Model should be as accurate as possible.

1.1.7 General Methods for solving O.R models

(1) ***Analytic Procedure :*** Solving models by classical Mathematical techniques like differential calculus, finite differences etc. to obtain analytic solutions.

(2) ***Iterative Procedure :*** Starts with a trial solution and a set of rules for improving it by repeating the procedure until further improvement is not possible.

(3) ***Monte–carlo technique :*** Taking sample observations, computing probability distributions for the variable using random numbers and constructing some functions to determine values of the decision variables.

1.1.8 Main Phases of O.R.

(1) ***Formulation of the Problems :*** Identifying the objective, the decision variables involved and the constraints that arise involving the decision variables

(2) ***Construction of a Mathematical Model:*** Expressing the measure of effectiveness which may be total profit, total cost, utility etc., to be optimised by a Mathematical function called ***objective function.*** Representing the constraints like budget constraints, raw materials constraints, ***resource constraints,*** quality constraints etc., by means of mathematical equations or inequalities.

(3) ***Solving the Model constructed:*** Determining the solution by analytic or iterative or Monto–carlo method depending upon the structure of the mathematical model.

(4) ***Controlling and updating :*** A solution which is optimum today may not be so tomorrow. The values of the variables may change, new variables may emerge. The structural relationship between the variables may also undergo a change. All these are determined in updating.

A solution from a model remains a solution only so long as the uncontrollable variables retain their values and the relationship between the variables does not change. The solution itself goes "out of control" if the values of one or more controlled variables vary or relationship between the variables undergoes a change. Therefore controls must be established to indicate the limits within which the model and its solution can be considered as reliable. This is called controlling.

(5) ***Testing the model*** and ***its solution*** *i.e.,* ***validating the model*** : checking as far as possible either from the past available data or by expertise and experience whether the model gives a solution which can be used in practice.

(6) ***Implementation :*** Implement using the solution to achieve the desired goal.

1.1.9 Limitation

Mathematical models which are the essence of OR do not take into account qualitative or emotional or some human factors which are quite real and influence the decision making. All such influencing factors find no place in O.R. This is the main limitation of O.R. Hence O.R. is only an aid in decision making.

EXERCISE

1. **What is O.R ?** *[MU. BE. Oct 96]*
2. **What is the scope of O.R ?**
3. **What are the applications of O.R. ?**
4. **List the uses of O.R.**
5. **Write a short note on the importance of operations research in production management** *[MU. MBA April, 97]*
6. **Write a short note on the role of operations research in marketing management.**

   [MU. MBA Nov.96, Nov. 97, April 98]
7. **Write a short note on the role of operations research in production planning.** *[MU. MBA April, 96]*

8. **What are the different phases of O.R.?** *[MKU. BE. Nov 97]*
9. **What are the characteristics of an O.R. problem ?** *[MKU. BE. Nov 97]*
10. **What is a model ?**
11. **What is a Mathematical Model ?**
12. **What are the different types of Models ?**
13. **What are the characteristics of a good model ?**
14. **What are the limitations of a mathematical model ?**
15. **What are the limitations of an O.R. Model ?** *[MKU. BE. Nov 97]*
16. **Explain the general methods of solving O.R. Models.**
17. **Explain the principles of modelling.**
18. **State the different types of Models used in O.R.** *[MU. BE. Oct 97]*
19. **What are the various phases in the study of operations research ?** *[BRU. BE. Apr 97]*
20. **Answer the following questions with examples wherever necessary.**
    (a) **Necessities of OR in industry,**
    (b) **Fields of application of OR in industry**
    (c) **Deterministic models,**
    (d) **Mention atleast eight mathematical models.** *[MKU. BE. Nov 97]*
21. **What is an iconic model in the study of operations research ?** *[MU. BE. Oct 96]*

1.2 LINEAR PROGRAMMING FORMULATION AND GRAPHICSOLUTION

1.2.1 Introduction

Linear programming problems deal with determining optimal allocations of limited resources to meet given objectives. The resources may be in the form of men, raw materials, market demand, money and machines etc. The objective is usually maximizing profit, minimizing total cost, maximizing utility etc. There are certain restrictions on the total amount of each resource available and on the quantity or quality of each product made.

Linear programming problem deals with the optimization (Maximization or Minimization) of a function of decision variables (The variables whose values determine the solution of a problem are called ***decision variables*** of the problem) known as ***objective function,*** subject to a set of simultaneous linear equations (or inequalities) known as ***constraints.*** The term ***linear*** means that all the variables occurring in the objective function and the constraints are of the first degree in the problems under consideration and the term ***programming*** means the process of determining a particular course of action.

Linear programming techniques are used in many industrial and economic problems. They are applied in product mix, blending, diet, transportation and assignment problems. Oil refineries, airlines, railways, textile industries, chemical industries, steel industries, food processing industries and defence establishments are also the users of this technique.

1.2.2 Requirements for employing LPP Technique :

[BRU. BE. Nov 96]

1. There must be a well defined objective function.
2. There must be alternative courses of action to choose.
3. At least some of the resources must be in limited supply, which give rise to constraints.
4. Both the objective function and constraints must be linear equations or inequalities.

1.2.3 Mathematical Formulation of L.P.P

If x_j ($j = 1, 2, \ldots, n$) are the n decision variables of the problem and if the system is subject to m constraints, the general Mathematical model can be written in the form :

$$\text{Optimize } Z = f(x_1, x_2, \ldots x_n)$$

$$\text{subject to } g_i(x_1, x_2, \ldots x_n) \le, =, \ge b_i, (i = 1,2, \ldots, m)$$

(called structural constraints)

$$\text{and } x_1, x_2, \ldots x_n \ge 0,$$

(called the non–negativity restrictions or constraints)

Procedure for forming a LPP Model :

Step 1 : Identify the unknown decision variables to be determined and assign symbols to them.

Step 2 : Identify all the restrictions or constraints (or influencing factors) in the problem and express them as linear equations or inequalities of decision variables.

Step 3 : Identify the objective or aim and represent it also as a linear function of decision variables.

Step 4 : Express the complete formulation of LPP as a general mathematical model.

We consider only those situations where this will help the reader to put proper inequalities in the formulation.

1. Usage of manpower, time, raw materials etc are always less than or equal to the availability of manpower, time, raw materials etc.

2. Production is always greater than or equal to the requirement so as to meet the demand.

Example 1 **A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines M_1 and M_2. Type A requires 1 minute of processing time on M_1 and 2 minutes on M_2. Type B requires 1 minute on M_1 and 1 minute on M_2. Machine M_1 is available for not more than 6 hours 40 minutes while machine M_2 is available for 10 hours during any working day. Formulate the problem as a LPP so as to maximize the profit.**

Solution : Let the firm decide to produce x_1 units of product A and x_2 units of product B to maximize its profit.

To produce these units of type A and type B products, it requires

$x_1 + x_2$ processing minutes on M_1

$2x_1 + x_2$ processing minutes on M_2

Since machine M_1 is available for not more than 6 hours and 40 minutes and machine B is available for 10 hours doing any working day, the constraints are

$$\begin{aligned} x_1 + x_2 &\le 400 \\ 2x_1 + x_2 &\le 600 \\ \text{and } x_1, x_2 &\ge 0. \end{aligned}$$

Since the profit from type A is Rs. 2 and profit from type B is Rs. 3, the total profit is $2x_1 + 3x_2$. As the objective is to maximize the profit, the objective function is maximize $Z = 2x_1 + 3x_2$.

∴ The complete formulation of the LPP is

$$\text{Maximize } Z = 2x_1 + 3x_2$$

subject to the constraints

$$\begin{aligned} x_1 + x_2 &\le 400 \\ 2x_1 + x_2 &\le 600 \\ \text{and } x_1, x_2 &\ge 0. \end{aligned}$$

Example 2 **(Production Allocation Problem)**

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below :

Machine	Time per unit (minutes)			Machine capacity (Minutes/day)
	Product 1	Product 2	Product3	
M_1	2	3	2	440
M_2	4	–	3	470
M_3	2	5	–	430

It is required to determine the number of units to be manufactured for each product daily. The profit per unit for product 1, 2 and 3 is Rs.4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical model for the problem. *[MU. B. Tech. Leather. Oct 96]*

Solution : Let x_1, x_2 and x_3 be the number units of products 1, 2 and 3 produced respectively.

To produce these amount of products 1, 2 and 3, it requires :

$$2x_1 + 3x_2 + 2x_3 \text{ minutes on } M_1$$
$$4x_1 + 3x_3 \text{ minutes on } M_2$$
$$2x_1 + 5x_2 \text{ minutes on } M_3.$$

But the capacity of the machines M_1, M_2 and M_3 are 440,470 and 430 (minutes/day).

∴ The constraints are

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &\leq 440 \\ 4x_1 + 3x_3 &\leq 470 \\ 2x_1 + 5x_2 &\leq 430 \\ \text{and } x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Since the profit per unit for product 1, 2, and 3 is Rs.4, Rs. 3 and Rs.6 respectively, the total profit is $4x_1 + 3x_2 + 6x_3$. As the objective is to maximize the profit, the objective function is maximize $Z = 4x_1 + 3x_2 + 6x_3$.

∴ The complete formulation of the LPP is

$$\text{Maximize } Z = 4x_1 + 3x_2 + 6x_3$$

subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &\leq 440 \\ 4x_1 + 3x_3 &\leq 470 \\ 2x_1 + 5x_2 &\leq 430 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Example 3 **(Blending Problem)**

A firm produces an alloy having the following specifications :

(i) **Specific gravity ≤ 0.98**

(ii) **Chromium ≥ 8%**

(iii) **Melting point ≥ 450°C**

Raw materials A, B and C having the properties shown in the table can be used to make the alloy.

Property	Raw material		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%
Melting point	440°C	490°C	480°C

Cost of the various raw materials per unit ton are : Rs. 90 for A, Rs. 280 for B and Rs. 40 for C. Find the proportions in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Solution : Let x_1, x_2 and x_3 be the tons of raw materials A,B and C to be used for making the alloy.

From these raw materials, the firm requires :

$$0.92\,x_1 + 0.97x_2 + 1.04\,x_3 \quad \text{specific gravity}$$
$$7x_1 + 13x_2 + 16x_3 \quad \text{chromium}$$
$$440\,x_1 + 490\,x_2 + 480\,x_3 \quad \text{melting point.}$$

∴ By the given specifications, the constraints are

$$\begin{aligned} 0.92\,x_1 + 0.97\,x_2 + 1.04\,x_3 &\leq 0.98 \\ 7\,x_1 + 13\,x_2 + 16\,x_3 &\geq 8 \\ 440\,x_1 + 490\,x_2 + 480\,x_3 &\geq 450 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Since the cost of the various raw materials per unit ton are : Rs. 90 for A, Rs. 280 for B and Rs. 40 for C, the total cost is $90x_1 + 280x_2 + 40x_3$. As the objective is to minimize the total cost, the objective function is

$$\text{Minimize } Z = 90\,x_1 + 280\,x_2 + 40\,x_3$$

∴ The complete formulation of the LPP is

$$\text{Minimize } Z = 90\,x_1 + 280\,x_2 + 40\,x_3$$

subject to

$$\begin{aligned} 0.92\,x_1 + 0.97\,x_2 + 1.04\,x_3 &\leq 0.98 \\ 7\,x_1 + 13\,x_2 + 16\,x_3 &\geq 8 \\ 440\,x_1 + 490\,x_2 + 480\,x_3 &\geq 450 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

To produce these amount of products 1, 2 and 3, it requires :

$$2x_1 + 3x_2 + 2x_3 \text{ minutes on } M_1$$
$$4x_1 + 3x_3 \text{ minutes on } M_2$$
$$2x_1 + 5x_2 \text{ minutes on } M_3.$$

But the capacity of the machines M_1, M_2 and M_3 are 440,470 and 430 (minutes/day).

$\therefore$ The constraints are

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &\leq 440 \\ 4x_1 + 3x_3 &\leq 470 \\ 2x_1 + 5x_2 &\leq 430 \\ \text{and } x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Since the profit per unit for product 1, 2, and 3 is Rs.4, Rs. 3 and Rs.6 respectively, the total profit is $4x_1 + 3x_2 + 6x_3$. As the objective is to maximize the profit, the objective function is maximize $Z = 4x_1 + 3x_2 + 6x_3$.

$\therefore$ The complete formulation of the LPP is

$$\text{Maximize } Z = 4x_1 + 3x_2 + 6x_3$$

subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &\leq 440 \\ 4x_1 + 3x_3 &\leq 470 \\ 2x_1 + 5x_2 &\leq 430 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Example 3 **(Blending Problem)**

A firm produces an alloy having the following specifications :

(i) **Specific gravity ≤ 0.98**

(ii) **Chromium $\geq 8\%$**

(iii) **Melting point $\geq 450°C$**

Raw materials A, B and C having the properties shown in the table can be used to make the alloy.

Property	Raw material		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%
Melting point	440°C	490°C	480°C

Cost of the various raw materials per unit ton are : Rs. 90 for A, Rs. 280 for B and Rs. 40 for C. Find the proportions in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Solution : Let x_1, x_2 and x_3 be the tons of raw materials A,B and C to be used for making the alloy.

From these raw materials, the firm requires :

$$\begin{aligned} &0.92\,x_1 + 0.97x_2 + 1.04\,x_3 && \text{specific gravity} \\ &7x_1 + 13x_2 + 16x_3 && \text{chromium} \\ &440\,x_1 + 490\,x_2 + 480\,x_3 && \text{melting point.} \end{aligned}$$

$\therefore$ By the given specifications, the constraints are

$$\begin{aligned} 0.92\,x_1 + 0.97\,x_2 + 1.04\,x_3 &\leq 0.98 \\ 7\,x_1 + 13\,x_2 + 16\,x_3 &\geq 8 \\ 440\,x_1 + 490\,x_2 + 480\,x_3 &\geq 450 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Since the cost of the various raw materials per unit ton are : Rs. 90 for A, Rs. 280 for B and Rs. 40 for C, the total cost is $90x_1 + 280x_2 + 40x_3$. As the objective is to minimize the total cost, the objective function is

$$\text{Minimize } Z = 90\,x_1 + 280\,x_2 + 40\,x_3$$

$\therefore$ The complete formulation of the LPP is

$$\text{Minimize } Z = 90\,x_1 + 280\,x_2 + 40\,x_3$$

subject to

$$\begin{aligned} 0.92\,x_1 + 0.97\,x_2 + 1.04\,x_3 &\leq 0.98 \\ 7\,x_1 + 13\,x_2 + 16\,x_3 &\geq 8 \\ 440\,x_1 + 490\,x_2 + 480\,x_3 &\geq 450 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Example 4 **(Diet Problem)**

A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in the following table:

Food type	Yield/unit			cost/unit (Rs)
	Proteins	Fats	carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate the L.P model for the problem.

Solution : Let x_1, x_2, x_3 and x_4 be the units of food of type 1, 2, 3 and 4 used respectively.

From these units of food of type 1, 2, 3 and 4 he requires

$3x_1 + 4x_2 + 8x_3 + 6x_4$ Proteins/day

$2x_1 + 2x_2 + 7x_3 + 5x_4$ Fats / day

$6x_1 + 4x_2 + 7x_3 + 4x_4$ Carbohydrates/day

Since the minimum requirement of these proteins, fats and carbohydrates are 800, 200 and 700 respectively, the constraints are

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$
$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$
$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0.$$

Since, the costs of these food of type 1, 2, 3 and 4 are Rs. 45, Rs.40, Rs.85 and Rs. 65 per unit, the total cost is Rs.$45x_1 + 40x_2 + 85x_3 + 65x_4$. As the objective is to minimize the total cost, the objective function is

$$\text{Minimize } Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$$

$\therefore$ The complete formulation of the L.P.P is

$$\text{Minimize } Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$$

subject to

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$
$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$
$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0.$$

Example 5 **A farmer has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can raise. The price he can obtain is Rs.1.00 per kg for tomatoes, Rs. 0.75 a head for lettuce and Rs. 2.00 per kg for radishes. The average yield per - acre is 2,000 kgs of tomatoes, 3000 heads of lettuce, and 1000 kgs of radishes. Fertilizer is available at Rs.0.50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs. 20.00 per man-day**

Formulate this problem as a L.P. model to maximise the farmer's total profit.

Solution : Let the farmer decide to allot x_1, x_2 and x_3 acre of his farm to grow tomatoes, lettuce, and radishes respectively to maximize his total profit.

Since the total area of the form is restricted to 100 acre and the total man-days labour is restricted to 400 man-days, the constraints are

$$x_1 + x_2 + x_3 \leq 100$$
$$5x_1 + 6x_2 + 5x_3 \leq 400$$
$$\text{and } x_1, x_2 \geq 0$$

The Farmer will produce 2000 x_1 kgs of tomatoes, 3000 x_2 heads of lettuce, and 1000 x_3 kgs of radishes.

$\therefore$ The total sale of farmer will be

$$= \text{Rs } [1 \times 2000\, x_1 + 0.75 \times 3000\, x_2 + 2 \times 1000\, x_3]$$

Fertilizer expenditure will be

$$= \text{Rs } 0.50\, [100\, x_1 + 100\, x_2 + 50\, x_3]$$

Labour expenditure will be $=$ Rs 20 $[5x_1 + 6x_2 + 5x_3]$

$\therefore$ Farmer's net profit will be

$$= \text{Rs [Sale} - \text{total expenditure]}$$
$$= \text{Rs } [2000\,x_1 + 2250\,x_2 + 2000\,x_3 - 50\,x_1 - 50\,x_2 - 25x_3 - 100\,x_1 - 120\,x_2 - 100\,x_3]$$
$$= \text{Rs } [1850\,x_1 + 2080\,x_2 + 1875\,x_3]$$

$\therefore$ The objective function is

$$\text{Maximize Z} = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$$

$\therefore$ The complete formulation of the L.P.P is

$$\text{Maximize Z} = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$$

subject to the constraints

$$x_1 + x_2 + x_3 \leq 100$$
$$5x_1 + 6x_2 + 5x_3 \leq 400$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

Example 6 **Old hens can be bought at Rs. 2 each and young ones at Rs.5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen costs Rs. 1 per week to feed. A person has only Rs.80 to spend for hens. How many of each kind should he buy to give a profit of more than Rs.6 per week, assuming that he cannot house more than 20 hens. Formulate this as a L.P.P.** *[MU. BE. 89]*

Solution : The person decides to buy x_1 old hens and x_2 young hens to maximize his profit.

Since he has only Rs. 80 to spend for hens and old hen costs Rs. 2 and young hen costs Rs. 5 each,

$$2x_1 + 5x_2 \leq 80$$

Also, since he can not house more than 20 hens,

$$x_1 + x_2 \leq 20$$

The total sale of eggs will be

$$= \text{Rs. } 0.3\,(3x_1 + 5x_2)$$

Expenditure on feeding will be

$$= \text{Rs.}1\,(x_1 + x_2)$$

$\therefore$ The net profit is $= \text{Rs. } [0.3(3x_1 + 5x_2) - 1(x_1 + x_2)]$

$$= \text{Rs. } (0.5x_2 - 0.1x_1)$$

$$\therefore \quad 0.5x_2 - 0.1\,x_1 \geq 6.$$

$\therefore$ The constraints are

$$2x_1 + 5x_2 \leq 80$$
$$x_1 + x_2 \leq 20$$
$$0.5x_2 - 0.1\,x_1 \geq 6.$$
$$\text{and } x_1, x_2 \geq 0.$$

$\therefore$ The complete formulation of the L.P.P is

$$\text{Maximize Z} = 0.5x_2 - 0.1x_1$$

subject to the constraints

$$2x_1 + 5x_2 \leq 80$$
$$x_1 + x_2 \leq 20$$
$$0.5x_2 - 0.1\,x_1 \geq 6.$$
$$\text{and } x_1, x_2 \geq 0.$$

Example 7 **A production planner in a soft drink plant has two bottling machines A and B. A is designed for 8–ounce bottles and B for 16-ounce bottles. However, each can be used on both types with some loss of efficiency. The following data is available :**

Machine	8–ounce bottles	16–ounce bottles
A	100/minute	40/minute
B	60/minute	75/minute

The machine can be run 8–hours per day, 5 days per week. Profit on 8–ounce bottle is 15 paise and on 16–ounce bottle is 25 paise. Weekly production of the drink cannot exceed 3,00,000 ounces and the market can absorb 25,000 eight–ounce bottles and 7000 sixteen–ounce bottles per week. The planner wishes to maximize his profit. Formulate this as a L.P.P. *[BRU. BE. Nov 96]*

Solution : Let the production planner decide to produce x_1 number of units of 8–ounce bottles and x_2 number of units of 16–ounce bottles to maximize his profit.

To produce these x_1 and x_2 units of 8–ounce and 16–ounce bottles, he requires

$$\frac{x_1}{100} + \frac{x_2}{40} \quad \text{minutes on machine A}$$

$$\frac{x_1}{60} + \frac{x_2}{75} \quad \text{minutes on machine B}$$

Since the machines A and B are available for 8–hours per day, 5–days per week, we have

$$\frac{x_1}{100} + \frac{x_2}{40} \leq 2400 \Rightarrow 2x_1 + 5x_2 \leq 4,80,000$$

$$\frac{x_1}{60} + \frac{x_2}{75} \leq 2400 \Rightarrow 5x_1 + 4x_2 \leq 7,20,000$$

Since the weekly production of the drink should not exceed 3,00,000 ounces and the market can absorb only upto 25,000 eight-ounce bottles and 7,000 sixteen–ounce bottles per week, we have

$$8x_1 + 16x_2 \leq 3,00,000$$

$$x_1 \leq 25,000 \text{ and } x_2 \leq 7000$$

$\therefore$ The constraints are :

$$\begin{aligned} 2x_1 + 5x_2 &\leq 4,80,000 \\ 5x_1 + 4x_2 &\leq 7,20,000 \\ 8x_1 + 16x_2 &\leq 3,00,000 \\ x_1 &\leq 25000 \\ x_2 &\leq 7000 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

Since the profit on 8–ounce bottle is Rs. 0.15 and on 16-ounce bottle is Rs.0.25, the total profit is Rs.$0.15x_1 + 0.25x_2$. As the objective is to maximize the profit, the objective function is

$$\text{Maximize } Z = 0.15x_1 + 0.25x_2$$

$\therefore$ The complete formulation of the L.P.P is

$$\text{Maximize } Z = 0.15x_1 + 0.25\,x_2$$

subject to the constraints

$$\begin{aligned} 2x_1 + 5x_2 &\leq 4,80,000 \\ 5x_1 + 4x_2 &\leq 7,20,000 \\ 8x_1 + 16x_2 &\leq 3,00,000 \\ x_1 &\leq 25000 \\ x_2 &\leq 7000 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

EXERCISE

1. A company makes two types of leather products A and B. Product A is of high quality and product B is of lower quality. The respective profits are Rs.4 and Rs.3 per product. Each product A requires twice as much time as product B and if all products were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 products per day (Both A and B combined). Product A requires a special spare part and only 400 per day are available. There are only 700 special spare parts a day available for product B. Formulate this as a L.P.P.

2. A firm engaged in producing two models A and B performs three operations–painting, Assembly and testing. The relevant data are as follows:

Model	Unit sale Price	Hours required for each unit		
		Assembly	Painting	Testing
A	Rs.50	1.0	0.2	0.0
B	Rs.80	1.5	0.2	0.1

Total number of hours available are : Assembly 600, painting 100, testing 30. Determine weekly production schedule to maximize the profit.

3. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs.100 for preparation, requires 7 man-days of work and yields a profit Rs. 30. An acre of wheat costs Rs.120 to prepare, requires 10 man-days of work and yields a profit Rs.40. An acre of soyabeans costs Rs.70 to prepare, requires 8 man-days of work and yields a profit Rs.20. The farmer has Rs.1,00,000 for preparation and 8000 man-days of work. Formulate this as a L.P.P

4. A television company operates two assembly sections, section A and section B. Each section is used to assemble the components of three types of televisions : Colour, Standard and Economy. The expected daily production on each section is as follows :

T.V. Model	Section A	Section B
Colour	3	1
Standard	1	1
Economy	2	6

The daily running costs for two sections average Rs. 6000 for section A and Rs.4000 for section B. It is given that the company must produce atleast 24 colours, 16 standard and 40 Economy TV sets for which an order is pending. Formulate this as a L.P.P so as to minimize the total cost.

5. A transistor Radio company manufactures four models A, B, C and D which have profit contributions of Rs.8, Rs.15 and Rs.25 on models A, B and C respectively and a loss of Rs.1 on model D. Each type of radio requires a certain amount of time for the manufacturing of components for assembling and for packing. Specially a dozen units of model A require 1 hour of manufacturing, 2 hours for assembling and 1 hour for packing. The corresponding figures for a dozen units of model B are 2, 1 and 2 and for a dozen units of model C are 3, 5 and 1, while a dozen units of model D require 1 hour of packing only. During the forthcoming week, the company will be able to make available 15 hours of manufacturing, 20 hours of assembling and 10 hours of packing time. Formulate this as a L.P.P.

6. A soft drinks firm has two bottling plants, one located at P and the other at Q. Each plant produces three different soft drinks A, B and C. The capacities of the two plants in number of bottles per day, are as follows :

Products	Plants	
	P	Q
A	3000	1000
B	1000	1000
C	2000	6000

A market survey indicates that during the month of April, there will be a demand for 24000 bottles of A, 16000 bottles of B and 48000 bottles of C. The operating costs per day of running plants P and Q respectively are Rs.6000 and Rs. 4000. How many days should the firm run each plant in April so that the production cost is minimized?

7. Consider two different types of food stuffs A and B. Assume that these food stuffs contain vitamins V_1, V_2 and V_3 respectively. Minimum daily requirements of these vitamins are 1 mg. of V_1, 50 mg. of V_2 and 10 mg. of V_3. Suppose that the food stuff A contains 1 mg. of V_1, 100 mg. of V_2 and 10 mg. of V_3, whereas foodstuff B contains 1 mg. of V_1, 10 mg. of V_2 and 100 mg. of V_3. Cost of one unit of food stuff A is Rs.1 and that of B is Rs.1.5. Find the minimum cost diet that would supply the body atleast the minimum requirements of each vitamin.

8. A company produces refrigerators in Unit I and heaters in Unit II. The two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 in Unit I and 36 in Unit II, due to constraints 60 workers are employed. A refrigerator requires 2 man – week of labour, while a heater requires 1 man–week of labour. The profit available is Rs. 600 per refrigerator and Rs.400 per heater. Formulate the LPP problem. *[BNU. BE. Nov 96]*

ANSWERS

1. Maximize $Z = 4x_1 + 3x_2$

subject to
$$\begin{aligned} 2x_1 + x_2 &\leq 1000 \\ x_1 + x_2 &\leq 800 \\ x_1 &\leq 400 \\ x_2 &\leq 700 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

2. Maximize $Z = 50x_1 + 80x_2$

subject to
$$\begin{aligned} x_1 + 1.5x_2 &\leq 600 \\ 0.2x_1 + 0.2x_2 &\leq 100 \\ 0.1x_2 &\leq 30 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

3. Maximize $Z = 30x_1 + 40x_2 + 20x_3$

subject to
$$\begin{aligned} 10x_1 + 12x_2 + 7x_3 &\leq 10000 \\ 7x_1 + 10x_2 + 8x_3 &\leq 8000 \\ x_1 + x_2 + x_3 &\leq 1000 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

4. Minimize Z $= 6000x_1 + 4000x_2$

subject to
$$3x_1 + x_2 \geq 24$$
$$x_1 + x_2 \geq 16$$
$$2x_1 + 6x_2 \geq 40$$
and $x_1, x_2 \geq 0.$

5. Maximize Z $= 8x_1 + 15x_2 + 25x_3 - x_4$

subject to
$$x_1 + 2x_2 + 3x_3 \leq 15$$
$$2x_1 + x_2 + 5x_3 \leq 20$$
$$x_1 + 2x_2 + x_3 + x_4 \leq 10$$
and $x_1, x_2, x_3, x_4 \geq 0.$

6. Minimize Z $= 6000x_1 + 4000x_2$

subject to
$$3x_1 + x_2 \geq 24$$
$$x_1 + x_2 \geq 16$$
$$x_1 + 3x_2 \geq 24$$
and $x_1, x_2 \geq 0.$

7. Minimize Z $= x_1 + 1.5x_2$

subject to
$$x_1 + x_2 \geq 1$$
$$100x_1 + 10x_2 \geq 50$$
$$10x_1 + 100x_2 \geq 10$$
and $x_1, x_2 \geq 0.$

8. Maximize Z $= 600x_1 + 400x_2$

subject to
$$2x_1 + x_2 \leq 60$$
$$x_1 \leq 25$$
$$x_2 \leq 36$$
and $x_1, x_2 \geq 0$

1.2.4 Basic Assumptions

[MU. MCA. Nov 98]

The linear programming problems are formulated on the basis of the following assumptions.

*1. **Proportionality :*** The contribution of each variable in the objective function or its usage of the resources is directly proportional to the value of the variable. *i.e.*, if resource availability increases by some percentage, then the output shall also increase by the same percentage.

*2. **Additivity :*** Sum of the resources used by different activities must be equal to the total quantity of resources used by each activity for all the resources individually or collectively.

*3. **Divisibility :*** The variables are not restricted to integer values.

*4. **Certainty or Deterministic :*** Coefficients in the objective function and constraints are completely known and do not change during the period under study in all the problems considered.

*5. **Finiteness :*** Variables and constraints are finite in number.

*6. **Optimality :*** In a linear programming problem we determine the decision variables so as to extremise (optimize) the objective function of the LPP.

7. The problem involves only one objective namely profit maximization or cost minimization.

1.2.5 Graphical method of solving a L.P.P (Graphic solution)

Linear Programming problems involving only two variables can be effectively solved by a graphical method which provides a pictorial representation of the problems and its solutions and which gives the basic concepts used in solving general L.P.P. which may involve any finite number of variables. This method is simple to understand and easy to use. The redundant constraints are automatically eliminated from the system. Multiple solutions, unbounded solutions and infeasible solutions get highlighted very clearly in graphical analysis. Sensitivity analysis can be illustrated easily by drawing the graph of the changes.

Graphical method is not a powerful tool of linear programming as most of the practical situations do involve more than two variables. But the method is really useful to explain the basic concepts of L.P.P to the persons who are not familiar with this. Though graphical method can deal with any number of constraints but since each constraint is shown as a line on a graph, a large number of lines make the graph difficult to read.

Working procedure for Graphical method :

Given a L.P.P, optimize Z = $f(x_i)$ subject to the constraints $g_j(x_i) \leq, =, \geq b_j$ and the non-negativity restrictions $x_i \geq 0$, $i = 1, 2$; $j = 1,2,3......m$

Step 1 : Consider the inequality constraints as equalities. Draw the straight lines in the XOY plane corresponding to each equality and non–negativity restrictions.

Step 2 : Find the permissible region (feasible region or solution space or convex region) for the values of the variables which is the region bounded by the lines drawn in step 1.

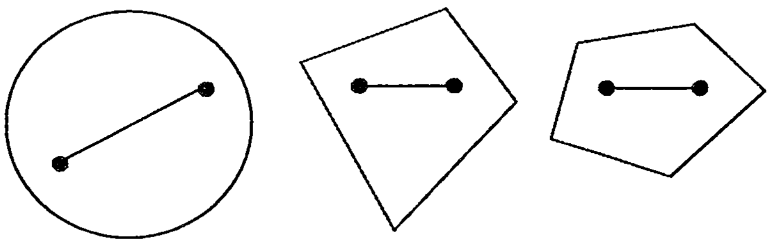
Step 3 : Find the points of intersection of the bounded lines (vertices of the permissible region) by solving the equations of the corresponding lines.

Step 4 : Find the values of Z at all vertices of the permissible region.

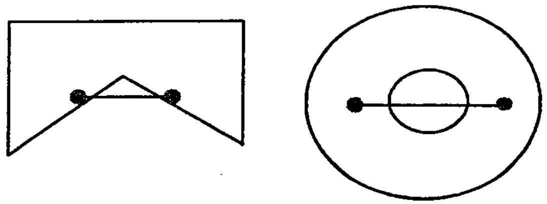
Step 5 : *(i)* For maximization problem, choose the vertex for which Z is maximum. *(ii)* For minimization problem, choose vertex for which Z is minimum.

Note : A ***region*** or a ***set*** of points is said to be ***convex*** if the line joining any two of its points lies completely within the region (or the set).

Example :



Convex Convex Convex



Not convex Not convex

Example 1 **Solve the following L.P.P by the graphical method**

$$\text{Max Z} = 3x_1 + 2x_2$$

subject to

$$-2x_1 + x_2 \leq 1$$
$$x_1 \leq 2$$
$$x_1 + x_2 \leq 3$$
$$\text{and } x_1, x_2 \geq 0$$

[MU. BE. Nov 94]

Solution : First consider the inequality constraints as equalities.

$$-2x_1 + x_2 = 1 \quad (1)$$
$$x_1 = 2 \quad (2)$$
$$x_1 + x_2 = 3 \quad (3)$$
$$\text{and } x_1 = 0 \quad (4)$$
$$x_2 = 0 \quad (5)$$

For the line, $-2x_1 + x_2 = 1$,

put $x_1 = 0 \Rightarrow x_2 = 1 \Rightarrow (0,1)$

put $x_2 = 0 \Rightarrow -2x_1 = 1 \Rightarrow x_1 = -0.5 \Rightarrow (-0.5, 0)$

So, the line (1) passes through the points (0,1) and (–0.5,0). The points on this line satisfy the equation $-2x_1 + x_2 = 1$. Now origin (0,0), on substitution, gives $-0 + 0 = 0 < 1$; hence it also satisfies the inequality $-2x_1 + x_2 \leq 1$. Thus all points on the origin side and on this line satisfy the inequality $-2x_1 + x_2 \leq 1$. Similarly interpreting the other constraints we get the feasible region OABCD. The feasible region is also known as solution space of the L.P.P. Every point within this area satisfies all the constraints.

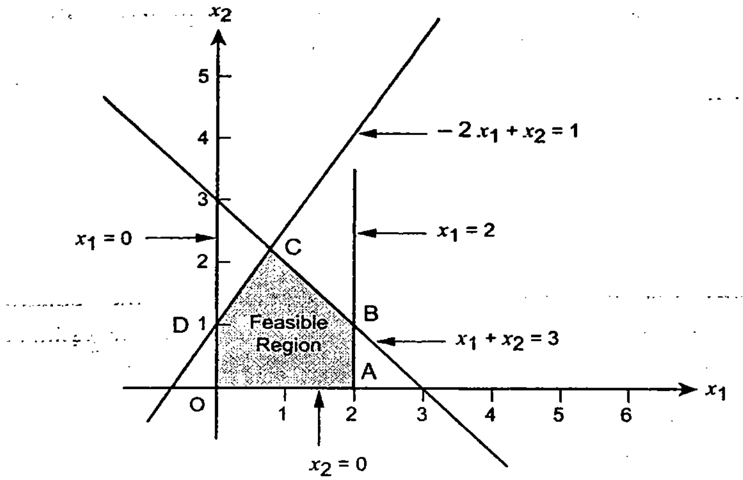


Fig. 2.1

Now our aim is to find the vertices of the solution space. B is the point of intersection of $x_1 = 2$ and $x_2 + x_2 = 3$. Solving these two equations, we have $x_1 = 2$, $x_2 = 1$. $\therefore$ We have the vertex B(2,1). Similarly, C is the intersection of $-2x_1 + x_2 = 1$ and $x_1 + x_2 = 3$. Solving these we have C $\left(\frac{2}{3}, \frac{7}{3}\right)$.

$\therefore$ The vertices of the solution space are O (0,0), A (2,0), B (2,1), C $\left(\frac{2}{3}, \frac{7}{3}\right)$ and D (0,1)

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A(2,0)	6
B(2,1)	8
C $\left(\frac{2}{3}, \frac{7}{3}\right)$	$\frac{20}{3}$
D(0,1)	2

($\because$ $Z = 3x_1 + 2x_2$)

Since the problem is of maximization type, the optimum solution to the L.P.P is

$$\text{Maximum } Z = 8, x_1 = 2, x_2 = 1.$$

Example 2 **Solve the following L.P.P by the graphical method**

$$\text{Minimize } Z = 3x_1 + 5x_2$$

subject to

$$-3x_1 + 4x_2 \quad 12$$
$$x_1 \leq 4$$
$$2x_1 - x_2 \geq -2$$
$$x_2 \geq 2$$
$$2x_1 + 3x_2 \geq 12 \text{ and } x_1, x_2 \geq 0.$$

Solution: Let us consider $-3x_1 + 4x_2 \leq 12$ as the equation $-3x_1 + 4x_2 = 12$, then it gives a line in the $x_1\,0\,x_2$ plane. All the points on the origin side and on this line satisfy the inequality $-3x_1 + 4x_2 \leq 12$. If we consider the constraint $2x_1 + 3x_2 \geq 12$ as the equation $2x_1 + 3x_2 = 12$, then it gives a line in the $x_1\,0\,x_2$ plane. All the points on the other side of the origin and on this line satisfy the inequality $2x_1 + 3x_2 \geq 12$. Similarly interpreting the other constraints, we get the feasible region ABCDE. Every point with in this area satisfies all the constraints. The vertices of the solution space are A (3, 2), B (4, 2), C (4, 6) D $\left(\frac{4}{5}, \frac{18}{5}\right)$, and E $\left(\frac{3}{4}, \frac{7}{2}\right)$.

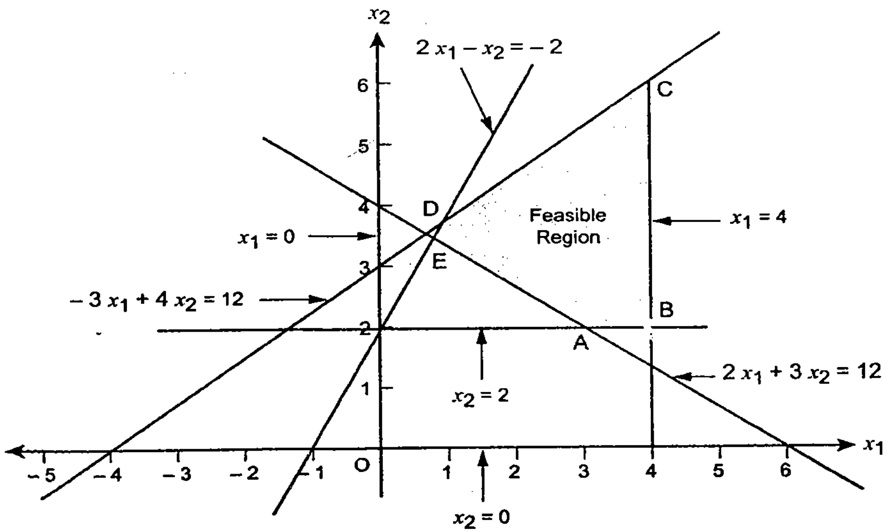


Fig 2.2

The values of Z at these vertices are given by

Vertex	Value of Z
A (3, 2)	19
B (4, 2)	22
C (4, 6)	42
D $\left(\frac{4}{5}, \frac{18}{5}\right)$	$\frac{102}{5}$
E $\left(\frac{3}{4}, \frac{7}{2}\right)$	$\frac{79}{4}$

$(\because Z = 3x_1 + 5x_2)$

Since the problem is of minimization type, the optimum solution is

$$\text{Minimum } Z = 19, x_1 = 3, x_2 = 2.$$

Example 3 **A pineapple firm produces two products canned pineapple and canned juice. The specific amounts of material, labour and equipment required to produce each product and the availability of each of these resources are shown in the table given below :**

	Canned Juice	Canned Pineapple	Available resources
Labour (Man hours)	3	2.0	12.0
Equipment (M/c hours)	1	2.3	6.9
Material (Unit)	1	1.4	4.9

Assuming one unit of canned juice and canned Pineapple has profit margins Rs.2 and Rs.1 respectively. Formulate this as a L.P.P. and solve it graphically also. *[MU.BE. Nov 91]*

Solution : Let x_1 be the number of units of canned juice and x_2 be the number of units of canned pineapple to be produced.

The constraints or restrictions in this problem are the labour, equipment and material.

$$\text{For labour, } 3x_1 + 2x_2 \le 12$$
$$\text{For Equipment, } x_1 + 2.3x_2 \le 6.9$$
$$\text{For material, } x_1 + 1.4\,x_2 \le 4.9$$
$$\text{and } x_1, x_2 \ge 0.$$

Here our objective is to maximize the profit.

$\therefore$ The objective function is Maximize $Z = 2x_1 + x_2$

$\therefore$ The complete formulation of the L.P.P. is Maximize $Z = 2x_1 + x_2$ subject to the constraints

$$3x_1 + 2x_2 \le 12$$
$$x_1 + 2.3x_2 \le 6.9$$
$$x_1 + 1.4\,x_2 \le 4.9 \text{ and } x_1, x_2 \ge 0.$$

The solution space is given below with the shaded area with vertices O (0, 0), A (0, 3), B (1.8, 2.2), C (3.2, 1.2) and D (4, 0)

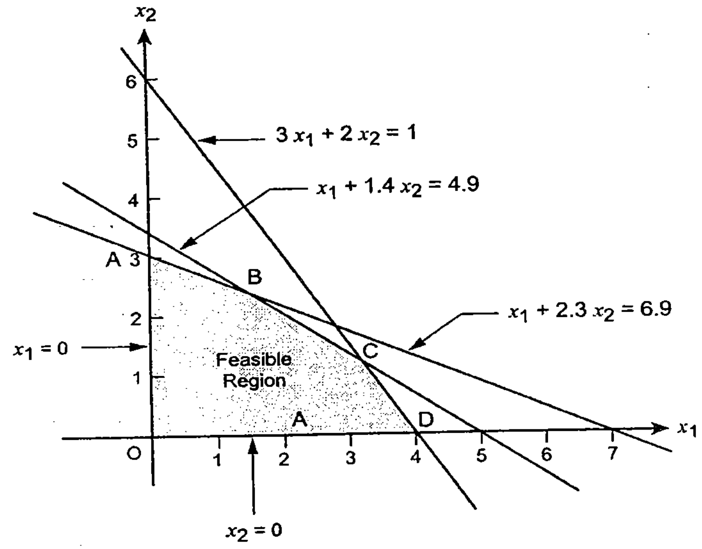


Fig. 2.3

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (0, 3)	3
B (1.8, 2.2)	5.8
C (3.2, 1.2)	7.6
D (4,0)	8

$(\because Z = 2x_1 + x_2)$

Since the problem is of maximization type, the optimum solution is

$$\text{Maximum } Z = 8, x_1 = 4, x_2 = 0.$$

Example 4 **A Company manufactures 2 types of printed circuits. The requirements of transistors, resistors and capacitors for each type of printed circuits along with other data are given below :**

	Circuit		Stock available
	A	**B**	
Transistor	**15**	**10**	**180**
Resistor	**10**	**20**	**200**
Capacitor	**15**	**20**	**210**
Profit	**Rs.5**	**Rs.8**	

How many circuits of each type should the company produce from the stock to earn maximum profit. *[MU. BE. Oct 95]*

Solution : Let x_1 be the number of type A circuits and x_2 be the number of type B circuits to be produced.

To produce these units of type A and type B circuits, the company requires

$$\begin{aligned} \text{Transistors} &= 15x_1 + 10x_2 \\ \text{Resistor} &= 10x_1 + 20x_2 \\ \text{Capacitors} &= 15x_1 + 20x_2 \end{aligned}$$

Since the availability of these transistors, resistors and capacitors are 180, 200 and 210 respectively, the constraints are

$$\begin{aligned} 15x_1 + 10x_2 &\le 180 \\ 10x_1 + 20x_2 &\le 200 \\ 15x_1 + 20x_2 &\le 210 \\ \text{and } x_1 \ge 0, \; x_2 &\ge 0. \end{aligned}$$

Since the profit from type A is Rs.5 and from type B is Rs.8 per units, the total profit is $5x_1 + 8x_2$

$\therefore$ The complete formulation of the L.P.P. is

$$\begin{aligned} \text{Maximize } Z &= 5x_1 + 8x_2 \\ \text{subject to} \quad 15x_1 + 10x_2 &\le 180 \\ 10x_1 + 20x_2 &\le 200 \\ 15x_1 + 20x_2 &\le 210 \\ \text{and } x_1, x_2 &\ge 0. \end{aligned}$$

By using graphical method, the solution space is given below with shaded area OABCD with vertices O (0, 0), A (12, 0), B (10, 3), C (2, 9) and D (0, 10).

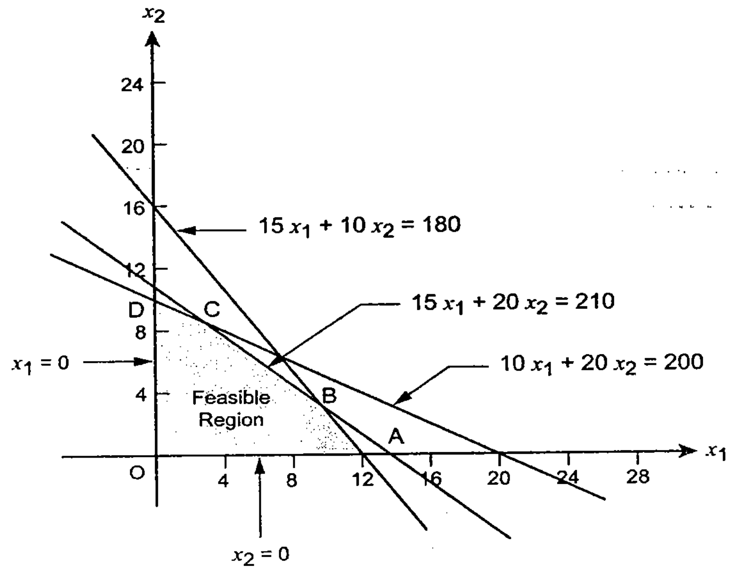


Fig. 2.4

The values of Z of these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (12, 0)	60
B (10, 3)	74
C (2, 9)	82
D (0,10)	80

$(\because \; Z = 5x_1 + 8x_2)$

Since the problem is of maximization type, the optimum solution is

$$\text{Maximum } Z = 82, \; x_1 = 2, \; x_2 = 9.$$

Example 5 **Apply graphical method to solve the LPP: Maximise $Z = x_1 - 2x_2$ subject to $-x_1 + x_2 \le 1$, $6x_1 + 4x_2 \ge 24$, $0 \le x_1 \le 5$ and $2 \le x_2 \le 4$.** ***[MU.MBA.Nov.96]***

Solution : By using graphical method, the solution space is given below with shaded area ABCDE with vertices A $\left(\frac{8}{3}, 2\right)$, B (5, 2), C (5, 4), D (3, 4) and E (2, 3).

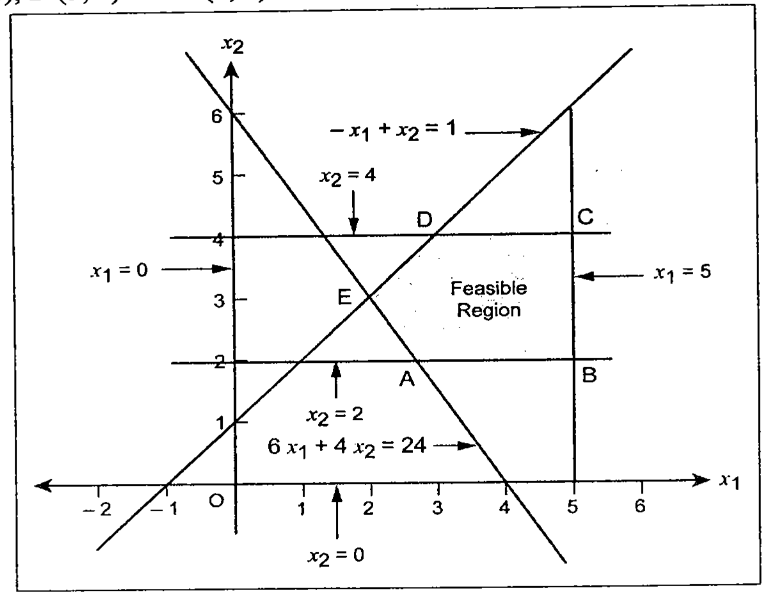


Fig. 2.5

The values of Z at these vertices are given by

Vertex	Value of Z
A $\left(\frac{8}{3}, 2\right)$	$-\frac{4}{3}$
B (5, 2)	1
C (5, 4)	-3
D (3, 4)	-5
E (2, 3)	-4

$(\because Z = x_1 - 2x_2)$

Since the problem is of maximization type, the optimal solution is

Maximum Z $= 1$, $x_1 = 5$, $x_2 = 2$.

Example 6 **A Company manufactures two types of cloth, using three different colours of wool. One yard length of type A cloth require 4 OZ of red wool, 5 OZ of green wool and 3 OZ of yellow wool. One yard length of type B cloth requires 5 OZ of red wool, 2 OZ of green wool and 8 OZ of yellow wool. The wool available for manufacturer is 1000 OZ of red wool, 1000 OZ of green wool and 1200 OZ of yellow wool. The manufacturer can make a profit of Rs. 5 on one yard of type A cloth and Rs.3 on one yard of type B cloth. Find the best combination of the quantities of type A and type B cloth which gives him maximum profit by solving the L.P.P. graphically.** ***[MU. BE. Apr 92]***

Solution : Let the manufacturer decide to produce x_1 yards of type A cloth and x_2 yards of type B cloth.

To produce these yards of type A and type B cloth, he requires

$$\begin{aligned} \text{red wool} &= 4x_1 + 5x_2 \text{ OZ} \\ \text{green wool} &= 5x_1 + 2x_2 \text{ OZ} \\ \text{yellow wool} &= 3x_1 + 8x_2 \text{ OZ} \end{aligned}$$

Since the availability of these red wool, green wool and yellow wool are 1000 OZ, 1000 OZ and 1200 OZ respectively, the constraints are

$$\begin{aligned} 4x_1 + 5x_2 &\leq 1000 \\ 5x_1 + 2x_2 &\leq 1000 \\ 3x_1 + 8x_2 &\leq 1200 \\ \text{and } x_1 \geq 0, \ x_2 &\geq 0. \end{aligned}$$

Since the profit from one yard of type A cloth is Rs.5 and the profit from one yard of type B cloth is Rs. 3, the total profit is $5x_1 + 3x_2$

$\therefore$ The complete formulation of the L.P.P. is

$$\begin{aligned} \text{Maximize } Z &= 5x_1 + 3x_2 \\ \text{subject to} \quad 4x_1 + 5x_2 &\leq 1000 \\ 5x_1 + 2x_2 &\leq 1000 \\ 3x_1 + 8x_2 &\leq 1200 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

By using graphical method, the feasible region is given below with shaded area OABCD with vertices O (0, 0), A (200, 0), B $\left(\frac{3000}{17}, \frac{1000}{17}\right)$, C $\left(\frac{2000}{17}, \frac{1800}{17}\right)$ and D (0, 150)

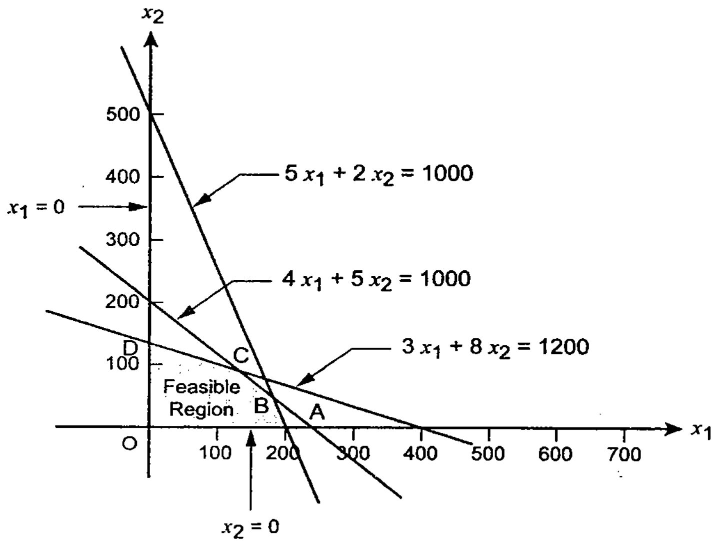


Fig 2.6

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (200, 0)	1000
B $\left(\frac{3000}{17}, \frac{1000}{17}\right)$	1058.8
C $\left(\frac{2000}{17}, \frac{1800}{17}\right)$	905.8
D (0, 150)	450

$(\because\ Z = 5x_1 + 3x_2)$

Since the problem is of maximization type, the optimum solution is

$$\text{Maximum } Z = 1058.8,\ x_1 = \frac{3000}{17},\ x_2 = \frac{1000}{17}.$$

Note : **In a given L.P.P, if any constraint does not affect the feasible region (or solution space), then the constraint is said to be a *redundant constraint.***

Example 7 **A Company making cold drinks has two bottling plants located at towns T_1 and T_2. Each plant produces three drinks A, B and C and their production capacity per day is given below :**

Cold drinks	Plant at	
	T_1	T_2
A	6000	2000
B	1000	2500
C	3000	3000

The marketing department of the company forecasts a demand of 80,000 bottles of A, 22,000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants at T_1 and T_2 are Rs. 6000 and Rs.4000 respectively. Find graphically, the number of days for which each plant must be run in June so as to minimize the operating costs while meeting the market demand.

Solution : Let the plant at T_1 and T_2 be run for x_1 and x_2 days respectively.

Since the plants at T_1 and T_2 run for x_1 and x_2 days, they will produce,

$6000x_1 + 2000x_2$ bottles of A

$1000x_1 + 2500x_2$ bottles of B

$3000x_1 + 3000x_2$ bottles of C

Since the demand for the cold drinks A, B and C are 80,000, 22,000 and 40,000 respectively and the production is always greater than or equal to the demand, the constraints are

$$6000x_1 + 2000x_2 \geq 80000 \Rightarrow 6x_1 + 2x_2 \geq 80$$
$$\Rightarrow 3x_1 + x_2 \geq 40$$
$$1000x_1 + 2500x_2 \geq 22000 \Rightarrow x_1 + 2.5x_2 \geq 22$$
$$3000x_1 + 3000x_2 \geq 40000 \Rightarrow 3x_1 + 3x_2 \geq 40$$

and $x_1, x_2 \geq 0$.

Since the operating costs per day at T_1 is Rs. 6000 and at T_2 is Rs. 4000 and T_1, T_2 run for x_1 and x_2 days, the total operating cost is Rs. $6000x_1 + 4000x_2$.

Here our objective is to minimize the total operating cost. Therefore the objective function is minimize $Z = 6000\,x_1 + 4000\,x_2$.

$\therefore$ The complete formulation of the L.P.P. is

$$\text{Minimize } Z = 6000x_1 + 4000x_2$$

subject to

$$3x_1 + x_2 \geq 40$$
$$x_1 + 2.5x_2 \geq 22$$
$$3x_1 + 3x_2 \geq 40$$
$$\text{and } x_1,\ x_2 \geq 0.$$

By using graphical method, the feasible region is given below with shaded area with vertices A (22, 0), B (12, 4), C (0, 40)

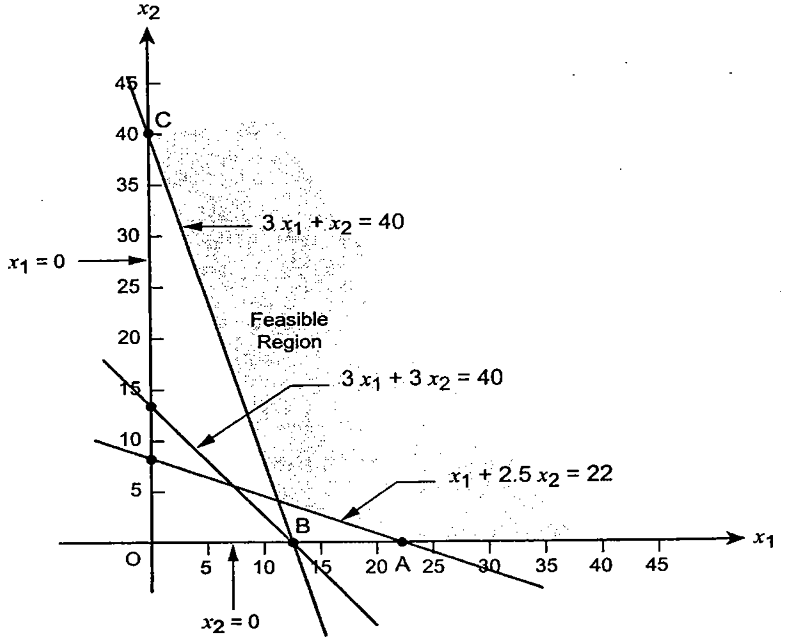


Fig 2.7

From the figure, we see that the constraint $3x_1 + 3x_2 \geq 40$ does not affect the solution space. So $3x_1 + 3x_2 \geq 40$ is a ***redundant constraint.*** Also from the direction of the arrows, we see that the solution space is unbounded above.

The values of Z at these vertices A (22,0), B(12,4) and C(0,40) are given by

Vertex	Value of Z
A (22, 0)	1,32,000
B (12, 4)	88,000
C (0, 40)	1,60,000

($\because$ $Z = 6000\,x_1 + 4000\,x_2$)

Since the problem is of minimization type, the optimum solution is

$$\text{Minimum } Z = \text{Rs. } 88{,}000,\ x_1 = 12 \text{ days},\ x_2 = 4 \text{ days}.$$

Note: From the above examples, for problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region. That is, ***"if there exists an optimal solution of an L.P.P, it will be at one of the vertices of the feasible region".***

1.2.6. Some more cases

The constraints generally, give region of feasible solution which may be bounded or unbounded. We discussed seven linear programming problems and the optimal solution for either of them was unique. However, it may not be true for every problem. In general, a linear programming problem may have :

(i) a unique optimal solution

(ii) an infinite number of optimal solutions

(iii) an unbounded solution

(iv) no solution

We now give a few examples to illustrate the cases.

Example 8 **A firm manufactures two products A and B on which the profits earned per unit are Rs.3 and Rs.4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2 while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours 30 minutes while machine M_2 is available for 10 hours during any working day. Find the number of units of products A and B to be manufactured to get maximum profit. Formulate the above as a L.P.P. and solve by graphical method.** ***[MU. BE. Apr 92, MSU. BE. Nov 96***

Solution: Let the firm decide to manufacture x_1 units of product A and x_2 units of product B.

To produce these units of products A and B, it requires

$x_1 + x_2$ hours of processing times on M_1

$2x_1 + x_2$ hours of processing times on M_2

But the availability of these two machines M_1 and M_2 are 450 minutes and 600 minutes respectively, the constraints are

$$x_1 + x_2 \le 450$$
$$2x_1 + x_2 \le 600$$
$$\text{and } x_1, x_2 \ge 0$$

Since the profit from product A is Rs.3 per unit and from product B is Rs. 4 per unit, the total profit is Rs. $3x_1 + 4x_2$ and our objective is to maximize the profit

∴ The complete formulation of the L.P.P is

$$\text{Maximize } Z = 3x_1 + 4x_2$$

subject to

$$x_1 + x_2 \le 450 \quad \ldots (i)$$
$$2x_1 + x_2 \le 600 \quad \ldots (ii)$$
$$\text{and } x_1, x_2 \ge 0 \quad \ldots (iii)$$

By graphical method, the solution space satisfying the constraints *(i), (ii)* and meeting the non-negativity restriction *(iii)* is shown shaded in the following figure.

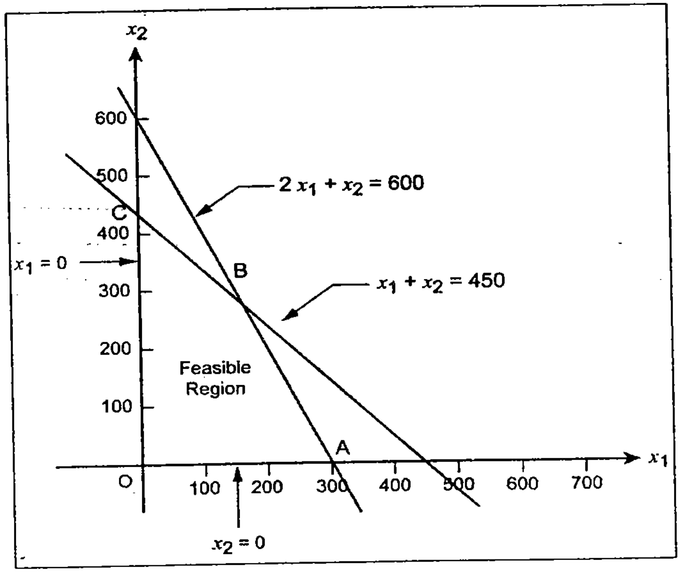


Fig. 2.8

The solution space is the region OABC. The vertices of this solution space are O (0, 0), A (300, 0), B (150, 300) and C (0, 450)

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (300, 0)	900
B (150, 300)	1650
C (0, 450)	1800

($\because$ $Z = 3x_1 + 4x_2$)

Since the problem is of maximization type and the maximum value of Z is attained at a single vertex, this problem has a ***unique optimal solution.***

∴ The optimal solution is

$$\text{Maximum } Z = 1800, x_1 = 0, x_2 = 450.$$

Example 9 **Solve the following L.P.P. graphically.**

$$\text{Maximize } Z = 100x_1 + 40x_2$$

subject to
$$5x_1 + 2x_2 \leq 1000$$
$$3x_1 + 2x_2 \leq 900$$
$$x_1 + 2x_2 \leq 500$$
$$\text{and } x_1, x_2 \geq 0.$$

Solution : By using graphical method, the solution space OABC shown shaded in the following figure.

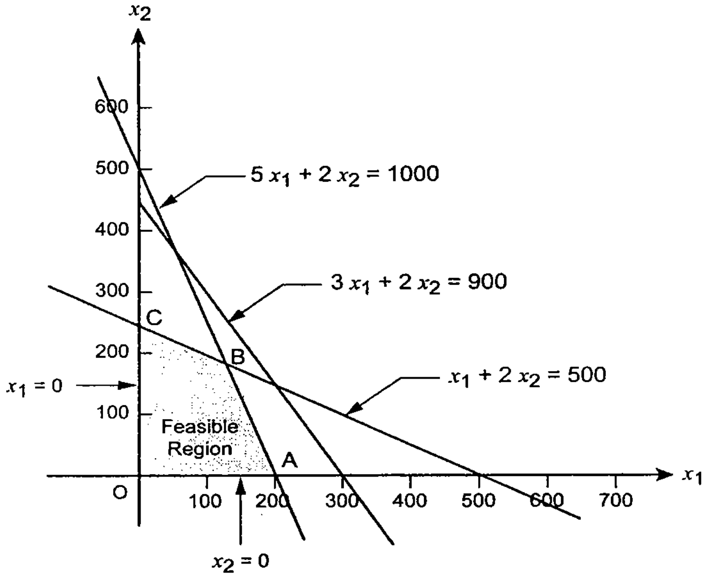


Fig. 2.9

The vertices of this convex region are O (0, 0), A (200, 0), B (125, 187.5) and C (0, 250)

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (200, 0)	20,000
B (125, 187.5)	20,000
C (0, 250)	10,000

($\because Z = 100x_1 + 40x_2$)

Here the maximum value of Z occurs at two vertices A and B.

Any point on the line joining A and B will also give the same maximum value of Z.

Since, there are infinite number of points between any points, there are infinite number of points on the line joining A and B gives the same maximum value of Z.

Thus, there are ***infinite number of optimal solutions*** for this L.P.P.

Note : An L.P.P having more than one optimal solution is said to have ***alternative*** or ***multiple optimal solutions.*** That is, the resources can be combined in more than one way to maximize the profit.

Example 10 **Using graphical method, solve the following L.P.P.**

$$\textbf{Maximize } Z = 2x_1 + 3x_2$$

subject to
$$x_1 - x_2 \leq 2$$
$$x_1 + x_2 \geq 4 \text{ and } x_1, x_2 \geq 0.$$

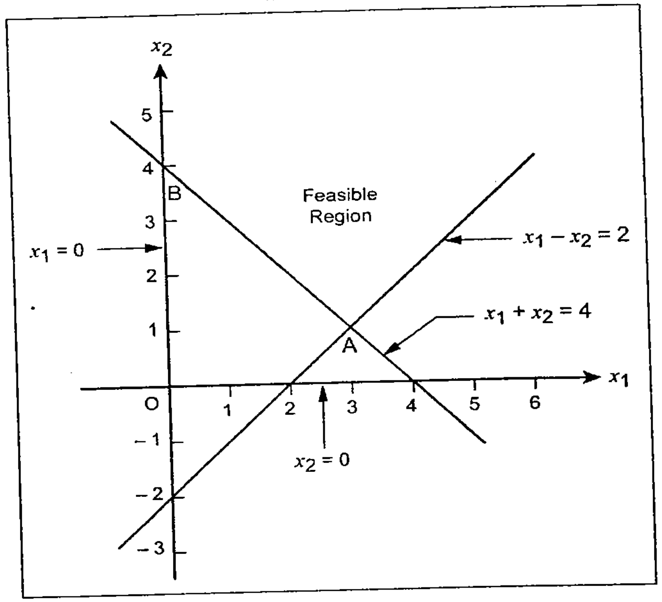


Fig. 2.10

Solution : By using graphical method, the solution space is shaded in the following figure.

Here the solution space is unbounded. The vertices of the feasible region (in the finite plane) are A (3,1) and B (0,4)

Value of the objective function $Z = 2x_1 + 3x_2$ at these vertices are $Z(A) = 9$ and $Z(B) = 12$.

But there are points in this convex region for which Z will have much higher values. In fact, the maximum value of Z occurs at infinity. Hence this problem has an ***unbounded solution.***

Example 11 **Solve graphically the following L.P.P.:**

$$\text{Maximize } Z = 4x_1 + 3x_2$$

subject to

$$x_1 - x_2 \leq -1 \quad \ldots (i)$$

$$-x_1 + x_2 \leq 0 \quad \ldots (ii)$$

$$\text{and } x_1, x_2 \geq 0 \quad \ldots (iii)$$

Solution : Any point satisfying the non–negativity restrictions *(iii)* lies in the first quadrant only. The two solution spaces, one satisfying *(i)*, and other satisfying *(ii)* are shown shaded in the following figure.

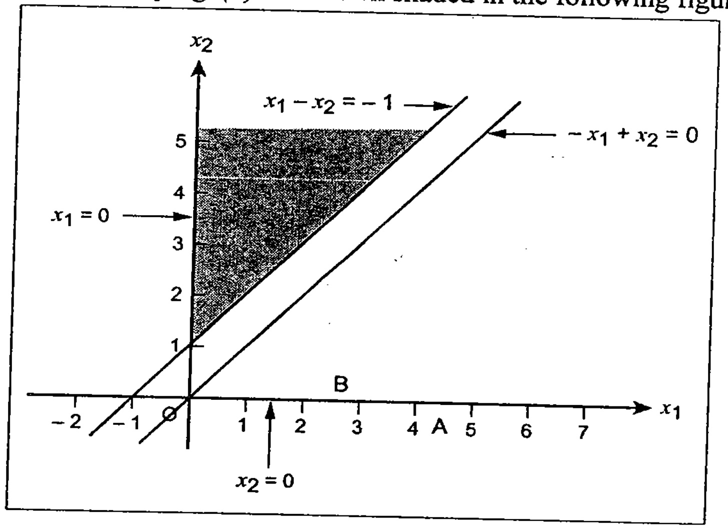


Fig. 2.11

There being no point (x_1, x_2) common to both the shaded regions. That is, we can not find a convex region for this problem. So the problem cannot be solved. Hence the problem have ***no feasible solution.***

1.2.7. Advantage of Linear Programming :

1. It provides an insight and perspective in to the problem environment. This generally results in clear picture of the true problem.
2. It makes a scientific and mathematical analysis of the problem situations.
3. It gives an opportunity to the decision maker to formulate his strategies consistent with the constraints and the objectives.
4. It deals with changing situations. Once a plan is arrived through the linear programming it can also be revaluated for changing conditions.
5. By using linear programming the decision maker makes sure that he is considering the best solution.

1.2.8. Limitations of Linear Programming :

1. The major limitation of linear programming is that it treats all relationships as linear. But it is not true in many real life situations.
2. The decision variables in some LPP would be meaningful only if they have integer values. But some times we get fractional values to the optimal solution, where only integer values are meaningful.
3. All the parameters in the linear programming model are assumed to be known constants. But in real life they may not be known completely or they may be probabilistic and they may be liable for changes from time to time.
4. The problems are complex if the number of variables and constraints are quite large.
5. Linear Programming deals with only a single objective problems, whereas in real life situations, there may be more than one objective.

EXERCISE

1. Explain the essential characteristics and limitations of Linear Programming Problem.
2. What is feasibility region in an LP problem ? Is it necessary that it should always be a convex set ?
3. What is a redundant constraint ? What does it imply ? Does it affect the optimal solution to an LPP ?
4. Explain the advantages of linear programming problems.
5. State the limitations of the graphical method of solving a LPP.
6. Solve the following by graphical method :

Maximize $x_1 - 3x_2$

subject to
$$\begin{aligned} x_1 + x_2 &\le 300 \\ x_1 - 2x_2 &\le 200 \\ 2x_1 + x_2 &\ge 100 \\ x_2 &\le 200 \\ \text{and } x_1, x_2 &\ge 0. \end{aligned}$$

7. By graphical method solve the following problem :

Maximize $Z = 3x_1 + 4x_2$

subject to
$$\begin{aligned} 5x_1 + 4x_2 &\le 200 \\ 3x_1 + 5x_2 &\le 150 \\ 5x_1 + 4x_2 &\ge 100 \\ 8x_1 + 4x_2 &\ge 80 \\ \text{and } x_1, x_2 &\ge 0. \end{aligned}$$

8. Solve the following problem by graphical method :

Maximize $5x + 8y$

subject to the constraints
$$\begin{aligned} 3x + 2y &\le 36 \\ x + 2y &\le 20 \\ 3x + 4y &\le 42 \\ \text{and } x &\ge 0 \\ y &\ge 0 \end{aligned}$$

9. Maximize $2x_1 + x_2$

subject to
$$\begin{aligned} 3x_1 + 2x_2 &\le 12.0 \\ x_1 + 2.3x_2 &\le 6.9 \\ x_1 + 1.4x_2 &\le 4.9 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

Using graphical method.

10. Using graphical method to solve L.P.P :

Minimize $Z = 3x_1 + 2x_2$

subject to
$$\begin{aligned} 5x_1 + x_2 &\ge 10 \\ x_1 + x_2 &\ge 6 \\ x_1 + 4x_2 &\ge 12 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

11. Solve the following problem graphically.

Maximize $Z = 2x_1 + x_2$

subject to
$$\begin{aligned} 3x_1 + 2x_2 &\le 12.0 \\ x_1 + 2x_2 &\le 7 \\ x_1 + x_2 &\le 5 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

12. Use graphical method to Maximize $Z = 6x_1 + 4x_2$

subject to
$$\begin{aligned} -2x_1 + x_2 &\le 2 \\ x_1 - x_2 &\le 2 \\ 3x_1 + 2x_2 &\le 9 \\ x_1 \ge 0,\ x_2 &\ge 0 \end{aligned}$$

13. Minimize $Z = x - 3y$

subject to the constraints
$$\begin{aligned} x + y &\le 300 \\ x - 2y &\le 200 \\ 2x + y &\ge 100 \\ y &\ge 200 \\ \text{and } x, y &\ge 0 \end{aligned}$$

by graphical method.

14. Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram, and costs 20 paise per gram. The daily minimum requirement of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix.

15. Solve graphically the L.P.P

$$\text{Maximize } Z = 3x_1 + 4x_2$$

subject to the constraints

$$\begin{aligned} 2x_1 + 5x_2 &\le 120 \\ 4x_1 + 2x_2 &\le 80 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

16. A company produces 2 types of hats. Each hat A require twice as much labour time as the second hat B. If all are of hat B only, the company can produce a total of 500 hats a day. The market limits daily sales of the hat A and hat B to 150 and 250 hats. The profits on hat A and B are Rs. 8 and Rs.5 respectively. Solve graphically to get the optimal solution.

17. Solve graphically the following L.P.P :

$$\text{Maximize } Z = 5x_1 + 3x_2$$

subject to constraints

$$\begin{aligned} 3x_1 + 5x_2 &\le 15 \\ 5x_1 + 2x_2 &\le 10 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

18. Solve graphically the following L.P.P.

$$\text{Minimize } Z = 20x_1 + 10x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\le 40 \\ 3x_1 + x_2 &\ge 30 \\ 4x_1 + 3x_2 &\ge 60 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

19. Solve graphically the following L.P.P.

$$\text{Max } Z = 3x + 2y$$

subject to

$$\begin{aligned} -2x + 3y &\le 9 \\ x - 5y &\ge -20 \\ \text{and } x, y &\ge 0 \end{aligned}$$

20. Solve graphically the following L.P.P :

$$\text{Minimize } Z = -6x_1 - 4x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\ge 30 \\ 3x_1 + 2x_2 &\le 24 \\ x_1 + x_2 &\ge 3 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

21. Solve graphically the following L.P.P.

$$\text{Maximize } Z = 3x_1 - 2x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\le 1 \\ 2x_1 + 2x_2 &\ge 4 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

22. Solve graphically the following L.P.P.

$$\text{Maximize } Z = x_1 + x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\ge 0 \\ -3x_1 + x_2 &\ge 3 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

23. A manufacturer of furniture makes two products, chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machines B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs. 2 and Rs. 10 respectively. What should be the daily production of each of the products.

24. A person requires 10, 12 and 12 units of chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C respectively per Jar. A dry product contains 1,2 and 4 units of A, B and C per carton. How many of each should he purchase in order to minimize the cost and meet the requirement.

25. Solve the following problem graphically,

$$\text{Maximize } Z = 40x_1 + 100x_2$$

subject to

$$12x_1 + 6x_2 \le 3000$$
$$4x_1 + 10x_2 \le 2000$$
$$2x_1 + 3x_2 \le 900$$
$$\text{and } x_1, x_2 \ge 0$$

26. A garment manufacturing company can make two products, Prima and Seconda. Each of the products requires time on a cutting machine and a finishing Machine. Relevant data are

	Product	
	Prima	seconda
Cutting hours (per unit)	2	1
Finishing hours (per unit)	3	3
Unit cost Rs.	128	120
Selling price Rs.	134	129
Maximum sales (units per week)	200	200

The number of cutting hours available per week is 390 and the number of finishing hours available per week is 810. How much should each product be produced in order to maximize the profit ?

27. A firm manufactures refrigerators and air coolers. Production takes place in two separate departments. Refrigerators are produced in department I and air coolers are produced in department II. The firm's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 refrigerators in department I and 35 air coolers in department II, because of the limited available facilities in these two departments. The firm regularly employs a total of 60 workers in two departments. A refrigerator requires 2 man–weeks of labour, while an air coolers requires 1 man–week of labour. The firm receives a profit margin of Rs.300 and Rs.200 per refrigerator and cooler respectively. Determine the product mix in order to maximize profit.

28. The production and planning department of a soft drink plant faces the following problem. The bottling plant has two bottling machines A and B. A is designed for 80cc bottle and B for 160 cc bottles. However each can be used on both types with loss of efficiency. The following data is available.

Machine	80 cc bottle	160 cc bottle
A	100 per min	40 per min
B	60 per min	75 per min

The machine can run 8 hour per day 5 days per week. Profit on 80 cc bottle is 15 paise and 160 cc bottle is 25 paise. Weekly production on the drink cannot exceed 3 million cc and the market can absorb 250000 of 80 cc bottles and 70000 of 160 cc bottles per week. The department wishes to maximize the profit subject to production and marketing restrictions. Solve the problem by graphical method or by simplex method.

29. A company manufactures two products A and B. Each unit of B takes twice as long to produce as one unit of A and if the company were to produce only A it would have time to produce 2000 units per day. The availability of the raw material is sufficient to produce 1500 units per day of both A and B combined. Product B requiring a special ingredient only 600 units can be made per day. If A fetches a profit of Rs. 2 per unit and B a profit of Rs. 4 per unit find the optimum product mix by graphical method.

30. A company produces two different products A and B. The company makes a profit of Rs. 40 and Rs. 30 per unit on A and B respectively. The production process has a capacity of 30,000 man hours. It takes 3 hours to produce one unit of A and one hour to produce one unit of B. The market survey indicates that the maximum number of units A that can be sold is 8,000 and those of B is 12,000 units. Formulate the problem and solve it by graphical method to get maximum profit.

31. Apply graphical method to find non–negative values of x_1 and x_2 which minimise $z = 10x_1 + 25x_2$ subject to $x_1 + x_1 \ge 50, x_1 \ge 20$, and $x_2 \le 40$.

ANSWERS

6. Max Z $= 200$, $x_1 = 200$, $x_2 = 0$.

7. Max Z $= \frac{1800}{13}$, $x_1 = \frac{400}{13}$, $x_2 = \frac{150}{13}$

8. Max Z $= 82$, $x = 2$, $y = 9$.

9. Max Z $= 8$, $x_1 = 4$, $x_2 = 0$.

10. Min Z $= 13$, $x_1 = 1$, $x_2 = 5$. 11. Max Z $= 8$, $x_1 = 4$, $x_2 = 0$.

12. An infinite number of solutions with Max Z = 18,

    (i) $x_1 = \frac{13}{5}$, $x_2 = \frac{3}{5}$ *(ii)* $x_1 = \frac{5}{7}$, $x_2 = \frac{24}{7}$ etc.

13. Min Z $= -600$, $x = 0$, $y = 200$.

14. Min Z $= 12x_1 + 20x_2$

    subject to
    $$6x_1 + 8x_2 \geq 100$$
    $$7x_1 + 12x_2 \geq 120$$
    $$\text{and } x_1, x_2 \geq 0.$$
    Also Min Z $= 205$, $x_1 = 15$, $x_2 = 1.25$

15. Max Z $= 110$, $x_1 = 10$, $x_2 = 20$

16. Max Z $= 8x_1 + 5x_2$

    subject to
    $$2x_1 + x_2 \leq 500$$
    $$x_1 \leq 150$$
    $$x_2 \leq 250$$
    $$\text{and } x_1, x_2 \geq 0.$$
    Also Max Z $= 2250$, $x_1 = 125$, $x_2 = 250$

17. Max Z $= \frac{235}{19}$, $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$

18. Min Z $= 240$, $x_1 = 6$, $x_2 = 12$

19. Unbounded solution

20. An infinite number of optimal solutions with Min Z $= -48$

    (i) $x_1 = 8$, $x_2 = 0$

    (ii) $x_1 = \frac{12}{5}$, $x_2 = \frac{42}{5}$, *etc* ......

21. No feasible solution

22. No feasible solution

23. Max Z $= 2x_1 + 10x_2$

    subject to
    $$2x_1 + 5x_2 \leq 16$$
    $$6x_1 \leq 30$$
    $$\text{and } x_1, x_2 \geq 0.$$
    Also Max Z $= 32$, $x_1 = 0$, $x_2 = 3.2$.

24. Min Z $= 3x_1 + 2x_2$

    subject to
    $$5x_1 + x_2 \geq 10$$
    $$2x_1 + 2x_2 \geq 12$$
    $$x_1 + 4x_2 \geq 12$$
    $$\text{and } x_1, x_2 \geq 0.$$
    Also Min Z $= 13$, $x_1 = 1$, $x_2 = 5$.

25. An infinite number of optimal solutions with Max Z = 20,000

    (i) $x_1 = 0$, $x_2 = 200$

    (ii) $x_1 = \frac{375}{2}$, $x_2 = 125$, *etc.*

26. Max Z $= 6x_1 + 9x_2$

    subject to
    $$2x_1 + x_2 \leq 390$$
    $$3x_1 + 3x_2 \leq 810$$
    $$x_1 \leq 200$$
    $$x_2 \leq 200 \text{ and } x_1, x_2 \geq 0.$$
    Also Max Z $= 2220$, $x_1 = 70$, $x_2 = 200$.

27. Max Z $= 300x_1 + 200x_2$

    subject to
    $$2x_1 + x_2 \leq 60$$
    $$x_1 \leq 25$$
    $$x_2 \leq 35 \quad \text{and } x_1, x_2 \geq 0.$$
    Also Max Z = Rs.10750, $x_1 = \frac{25}{2}$, $x_2 = 35$.

28. Max $Z = 0.15x_1 + 0.25x_2$

subject to
$$\begin{aligned} 2x_1 + 5x_2 &\leq 4{,}80{,}000 \\ 5x_1 + 4x_2 &\leq 7{,}20{,}000 \\ 80x_1 + 160x_2 &\leq 3{,}00{,}000 \\ x_1 &\leq 2{,}50{,}000 \\ x_2 &\leq 70{,}000 \quad \text{and } x_1, x_2 \geq 0. \end{aligned}$$

Also Max $Z = 562.50, x_1 = 3750, x_2 = 0.$

29. Max $Z = 2x_1 + 4x_2$

subject to
$$\begin{aligned} x_1 + 2x_2 &\leq 2000 \\ x_1 + x_2 &\leq 1500 \\ x_2 &\leq 600 \quad \text{and } x_1, x_2 \geq 0. \end{aligned}$$

Also Max $Z = 4000$ with

(i) $x_1 = 800,\ x_2 = 600$, *(ii)* $x_1 = 1000, x_2 = 500$ etc.

30. Max $Z = 40x_1 + 30x_2$

subject to
$$\begin{aligned} 3x_1 + x_2 &\leq 30{,}000 \\ x_1 &\leq 8000 \\ x_2 &\leq 12000 \quad \text{and } x_1, x_2 \geq 0. \end{aligned}$$

Also Max $Z = 6{,}00{,}000, x_1 = 6000, x_2 = 12000$

31. Min $Z = 500,\ x_1 = 50,\ x_2 = 0.$

1.3 GENERAL LINEAR PROGRAMMING PROBLEM

1.3.1 General Linear Programming Problem

The linear programming involving more than two variables may be expressed as follows :

Maximize (or) Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots c_nx_n$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{1n}x_n &\leq \text{ or } = \text{ or } \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{2n}x_n &\leq \text{ or } = \text{ or } \leq b_2 \\ a_{31}x_1 + a_{32}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{3n}x_n &\leq \text{ or } = \text{ or } \leq b_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{mn}x_n &\leq \text{ or } = \text{ or } \geq b_m \end{aligned}$$

and the non-negativity restrictions

$$x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots x_n \geq 0.$$

Note : Some of the constraints may be equalities, some others may be inequalities of ($\leq$) type and remaining ones inequalities of ($\geq$) type or all of them are of same type.

Definition (1) : A set of values $x_1, x_2 \ldots\ldots x_n$ which satisfies the constraints of the LPP is called its ***solution.***

Definition (2) : Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its ***feasible solution.***

Definition (3) : Any feasible solution which optimizes (maximizes or minimizes) the objective function of the LPP is called its ***optimum solution*** or ***optimal solution.***

Definition (4) : If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad (i = 1, 2, 3, \ldots k) \qquad \ldots(1)$$

then the non-negative variables s_i which are introduced to convert the inequalities (1) to the equalities

$$\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i \quad (i = 1, 2, 3, \ldots k)$$

are called ***slack variables.*** The value of these variables can be interpreted as the amount of unused resource.

Definition (5) : If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad (i = k, k+1, \ldots) \qquad \ldots(1)$$

then the non-negative variables s_i which are introduced to convert the inequalities (1) to the equalities

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i \qquad (i = k, k+1, \ldots)$$

are called ***surplus variables.*** The value of these variables can be interpreted as the amount over and above the required level.

1.3.2 Canonical and Standard forms of LPP :

After the formulation of LPP, the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable from. Two forms are dealt with here, the canonical form and the standard form.

The canonical form : The general linear programming problem can always be expressed in the following form :

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots c_nx_n$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \leq b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \leq b_m$$

and the non-negativity restrictions

$$x_1, x_2, \ldots\ldots\ldots\ldots, x_n \geq 0.$$

This form of LPP is called the ***canonical form*** of the LPP.

In matrix notation the canonical form of LPP can be expressed as :

$$\begin{aligned} \text{Maximize } Z &= CX \text{ (objective function)} \\ \text{subject to } AX &\leq b \text{ (constraints)} \\ \text{and } X &\geq 0 \text{ (non-negativity restrictions)} \end{aligned}$$

where $C = (c_1\ c_2 \ldots\ldots c_n)$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} \\ a_{12} & a_{22} & \cdots\cdots\cdots & a_{2n} \\ \cdots & \cdots\cdots\cdots & & \cdots \\ \cdots & \cdots\cdots\cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots\cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}$$

Characteristics of the Canonical form :

(i) The objective function is of maximization type.

(ii) All constraints are of ($\leq$) type.

(iii) All variables x_i are non-negative.

The Standard Form :

The general linear programming problem in the form

Maximize $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n = b_m$$

and $x_1, x_2, \ldots\ldots\ldots\ldots x_n \geq 0$ is known as ***standard form.***

In matrix notation the standard form of LPP can be expressed as :

$$\text{Maximize } Z = CX \text{ (objective function)}$$

subject to constraints

$$AX = b \text{ and } X \geq 0$$

Characteristics of the standard form :

(i) The objective function is of maximization type.

(ii) All constraints are expressed as equations.

(iii) Right hand side of each constraint is non-negative.

(iv) All variables are non-negative.

Note :

(1) : The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function.

i.e., Min $f(x)$ = – Max $\{-f(x)\}$

i.e., Min Z = – Max (– Z)

e.g : Min $Z = c_1x_1 + c_2x_2$ is equivalent to

$$\text{Max } (-Z) = -c_1x_1 - c_2x_2$$

(2) : An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by (–1).

e.g : $$ax_1 + bx_2 \geq c$$
$$\Rightarrow -ax_1 - bx_2 \leq -c$$

(3) : An equality constraint can be expressed as two inequalities.

e.g : $$ax_1 + bx_2 = c \Rightarrow \left.\begin{matrix} ax_1 + bx_2 \leq c \\ ax_1 + bx_2 \geq c \end{matrix}\right\} \Rightarrow \begin{matrix} ax_1 + bx_2 \leq c \\ -ax_1 - bx_2 \leq -c \end{matrix}$$

(4) : An inequality constraint with its left hand side in the absolute form can be expressed as two inequalities.

e.g. : $$|ax_1 + bx_2| \leq c \Rightarrow \begin{matrix} ax_1 + bx_2 \leq c \\ ax_1 + bx_2 \geq -c \end{matrix}$$

(5) : If a variable is unconstrained or unrestricted (without specifying its sign), it can always be expressed as the difference of two non-negative variables.

e.g. : If x_2 is unrestricted, then

$$x_2 = x_2' - x_2'' \text{ where } x_2', x_2'' \geq 0.$$

(6) : Whenever slack / surplus variables are introduced in the constraints, they should also appear in the objective function with zero coefficients.

Example 1 : Express the following LPP in the canonical form.

$$\text{Maximize } Z = 2x_1 + 3x_2 + x_3$$

subject to the constraints
$$\begin{aligned} 4x_1 - 3x_2 + x_3 &\leq 6 \\ x_1 + 5x_2 - 7x_3 &\geq -4 \\ \text{and } x_1, x_3 &\geq 0, x_2 \text{ is unrestricted} \end{aligned}$$

Solution : As x_2 is unrestricted, $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$. ∴ The given LPP becomes

$$\text{Maximize } Z = 2x_1 + 3(x_2' - x_2'') + x_3$$

subject to
$$\begin{aligned} 4x_1 - 3x_2' + 3x_2'' + x_3 &\leq 6 \\ x_1 + 5x_2' - 5x_2'' - 7x_3 &\geq -4 \\ x_1, x_2', x_2'', x_3 &\geq 0. \end{aligned}$$

Convert the second constraint ≤ type by multiplying both sides by –1. Now the LPP becomes

$$\text{Maximize } Z = 2x_1 + 3x_2' - 3x_2'' + x_3$$

subject to
$$\begin{aligned} 4x_1 - 3x_2' + 3x_2'' + x_3 &\leq 6 \\ -x_1 - 5x_2' + 5x_2'' + 7x_3 &\leq 4 \\ \text{and } x_1, x_2', x_2'', x_3 &\geq 0. \end{aligned}$$

which is in the canonical form.

Example 2 : Express the following LPP in standard form

$$\text{Minmize } Z = 5x_1 + 7x_2$$

subject to the constraints
$$\begin{aligned} x_1 + x_2 &\leq 8 \\ 3x_1 + 4x_2 &\geq 3 \\ 6x_1 + 7x_2 &\geq 5 \text{ and } x_1, x_2 \geq 0. \end{aligned}$$

Solution : Since Min Z = – Max (–Z) = – Max Z*,

The given LPP becomes $\text{Maximize } Z^* = -5x_1 - 7x_2$

subject to
$$\begin{aligned} x_1 + x_2 &\leq 8 \\ 3x_1 + 4x_2 &\geq 3 \\ 6x_1 + 7x_2 &\geq 5 \text{ and } x_1, x_2 \geq 0. \end{aligned}$$

By introducing slack variable s_1 and surplus variables s_2, s_3 the standard form of the LPP is given by

$$\text{Maximize } Z^* = -5x_1 - 7x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to
$$x_1 + x_2 + s_1 = 8$$
$$3x_1 + 4x_2 - s_2 = 3$$
$$6x_1 + 7x_2 - s_3 = 5$$
$$\text{and } x_1, x_2, s_1, s_2, s_3 \geq 0.$$

Example 3 : Express the following LPP in standard (Matrix) form

$$\text{Maximize } Z = 4x_1 + 2x_2 + 6x_3$$

subject to
$$2x_1 + 3x_2 + 2x_3 \geq 6$$
$$3x_1 + 4x_2 = 8$$
$$6x_1 - 4x_2 + x_3 \leq 10 \text{ and } x_1, x_2, x_3 \geq 0.$$

Solution : By introducing the surplus variable s_1 and slack variable s_2, the standard form of the LPP becomes

$$\text{Maximize } Z = 4x_1 + 2x_2 + 6x_3 + 0s_1 + 0s_2$$

subject to
$$2x_1 + 3x_2 + 2x_3 - s_1 + 0s_2 = 6$$
$$3x_1 + 4x_2 + 0x_3 + 0s_1 + 0s_2 = 8$$
$$6x_1 - 4x_2 + x_3 + 0s_1 + s_2 = 10$$
$$\text{and } x_1, x_2, x_3, s_1, s_2 \geq 0.$$

Thus the given problem in matrix form is

$$\text{Maximize } Z = CX$$
$$\text{subject to } AX = b$$
$$X \geq 0$$
$$\text{where } C = (4, 2, 6, 0, 0)$$

$$A = \begin{pmatrix} 2 & 3 & 2 & -1 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 6 & -4 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 8 \\ 10 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix}.$$

1.4 SIMPLEX METHOD

1.4.1 The Simplex Method

While solving a LPP graphically, the region of feasible solutions was found to be convex. The optimal solution if it exists, occured at some vertex. If the optimal solution was not unique, the optimal points were on an edge. These observations also hold for the general LPP. Essentially the problem is that of finding the particular vertex of the convex region which corresponds to the optimal solution. The most commonly used method for locating the optimal vertex is the ***simplex method or simplex technique or simplex algorithm*** which was developed by G. Dantzig in 1947.

The simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another vertex in such a way that the value of the objective function at the succeeding vertex is more (or less, as the case may be) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of an unbounded solution.

Definition (1): Given a system of m linear equations with n variables ($m < n$). The solution obtained by setting ($n - m$) variables equal to zero and solving for the remaining m variables is called a ***basic solution.***

The m variables are called ***basic variables*** and they form the basic solution. The ($n - m$) variables which are put to zero are called as ***non-basic variables.***

Definition (2) : A basic solution is said to be a ***non–degenerate basic solution*** if none of the basic variables is zero.

Definition (3) : A basic solution is said to be a ***degenerate basic solution*** if one or more of the basic variables are zero.

Definition (4) : A feasible solution which is also basic is called a ***basic feasible solution.***

Example 1 : **Find all the basic solutions to the following problem :**

$$\text{Maximize } Z = x_1 + 3x_2 + 3x_3$$
$$\text{subject to } x_1 + 2x_2 + 3x_3 = 4$$
$$2x_1 + 3x_2 + 5x_3 = 7$$

Also find which of the basic solutions are

(i) basic feasible

(ii) non–degenerate basic feasible

(iii) optimal basic feasible.

Solution : Since there are $m = 2$ equations with $n = 3$ variables, the basic solutions are obtained by setting $(n - m) = (3 - 2) = 1$ variable equal to zero and solving for the remaining two variables. Since there are 3 variables with 2 equations. We shall have $3C_2 = 3$ different basic solutions, which are given in the following table.

S. No	Basic variables	Non-basic variables	Values of the basic variables given by the constraint equations	Value of the objective function	Is the solution feasible ?	Is the solution non–degenerate ?	Is the solution feasible and optimal ?
1	x_1, x_2	$x_3 = 0$	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $\Rightarrow x_1 = 2, x_2 = 1$	5	yes	yes	yes
2	x_1, x_3	$x_2 = 0$	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $\Rightarrow x_1 = 1, x_3 = 1$	4	yes	yes	No
3	x_2, x_3	$x_1 = 0$	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $\Rightarrow x_2 = -1, x_3 = 2$	3	No	yes	No

From the table, we see that, the first two solutions are non–degenerate basic feasible solutions and the third is non–degenerate and infeasible. The first solution is the optimal one.

$\therefore$ The optimal solution is

Maximize Z $=$ 5, $x_1 = 2, x_2 = 1, x_3 = 0$.

Example 2 : Obtain all the basic solutions to the following system of linear equations :

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

Which of them are basic feasible solutions and which are non-degenerate basic solutions ? Is the non–degenerate solution feasible ?

Solution : Since there are 4 variables with 2 equations, we shall have $4C_2 = 6$ different basic solutions, which are given in the following table.

S.No	Basic variables	Non-basic variables	Values of the basic variables given by the constraint equations	Is the solution feasible ?	Is the solution non-degenerate ?	Is the solution feasible and non-degenerate?
1	x_1, x_2	$x_3 = x_4 = 0$	$\left.\begin{array}{l}2x_1 + 6x_2 = 3\\6x_1 + 4x_2 = 2\end{array}\right\}$ $\Rightarrow x_1 = 0,\ x_2 = 1/2$	yes	No	No
2	x_1, x_3	$x_2 = x_4 = 0$	$\left.\begin{array}{l}2x_1 + 2x_3 = 3\\6x_1 + 4x_3 = 2\end{array}\right\}$ $\Rightarrow x_1 = -2, x_3 = 7/2$	No	yes	No
3	x_1, x_4	$x_2 = x_3 = 0$	$\left.\begin{array}{l}2x_1 + x_4 = 3\\6x_1 + 6x_4 = 2\end{array}\right\}$ $\Rightarrow x_1 = \frac{8}{3}, x_4 = \frac{-7}{3}$	No	yes	No
4	x_2, x_3	$x_1 = x_4 = 0$	$\left.\begin{array}{l}6x_2 + 2x_3 = 3\\4x_2 + 4x_3 = 2\end{array}\right\}$ $\Rightarrow x_2 = \frac{1}{2}, x_3 = 0$	yes	No	No
5	x_2, x_4	$x_1 = x_3 = 0$	$\left.\begin{array}{l}6x_2 + x_4 = 3\\4x_2 + 6x_4 = 2\end{array}\right\}$ $\Rightarrow x_2 = \frac{1}{2}, x_4 = 0$	yes	No	No
6	x_3, x_4	$x_1 = x_2 = 0$	$\left.\begin{array}{l}2x_3 + x_4 = 3\\4x_3 + 6x_4 = 2\end{array}\right\}$ $\Rightarrow x_3 = 2, x_4 = -1$	No	yes	No

Definition (5) : Let X_B be a basic feasible solution to the LPP :

Maximize Z $=$ CX

subject to AX $=$ b

and X $\geq$ 0.

Then the vector $C_B = (C_{B_1}, C_{B_2}, \ldots C_{B_m})$, where C_{B_i} are components of C associated with the basic variables, is called the ***cost vector associated*** with the basic feasible solution X_B.

Remarks:

1 : If a LPP has a feasible solution, then it also has a basic feasible solution.

2 : There exists only finite number of basic feasible solutions to a LPP.

3 : Let a LPP have a feasible solution. If we drop one of the basic variables and introduce another variable in the basis set, then the new solution obtained is also a basic feasible solution.

Definition (6): Let $X_B = B^{-1} b$ be a basic feasible solution to the LPP: Maximize $Z = CX$, where $AX = b$ and $X \geq 0$,

Let C_B be the cost vector corresponding to X_B. For each column vector a_j in A, which is not a column vector of B, let

$$a_j = \sum_{i=1}^{m} a_{ij} b_i$$

Then the number $$Z_j = \sum_{i=1}^{m} C_{B_i} a_{ij}$$

is called the ***evaluation*** corresponding to a_j and the number $(Z_j - C_j)$ is called the ***net evaluation*** corresponding to a_j.

Remark 1 : If $(Z_j - C_j) = 0$ for atleast one j for which $a_{ij} > 0$, $i = 1,2, \ldots, m$; then another basic feasible solution is obtained which gives an unchanged value of the objective function.

Remark 2 : (Unbounded solution). Let there exist a basic feasible solution to a given LPP. If for atleast one j, for which $a_{ij} \leq 0$ $(i = 1,2,\ldots, m)$ and $(Z_j - C_j)$ is negative, then there does not exist any optimum solution to this LPP.

Remark 3 : A necessary and sufficient condition for a basic feasible solution to a LPP to be an optimum (maximum) is that $(Z_j - C_j) \geq 0$ for all j, for which $a_j \notin B$.

Remark 4 : The two fundamental conditions on which the simplex method is based are : ***(i) Feasibility condition :*** It ensures that if the initial (Starting) solution is basic feasible then during computation only basic feasible solutions will be obtained. ***(ii) Optimality condition:*** It guarantees that only improved solutions will be obtained.

1.4.2 The Simplex Algorithm

Assuming the existence of an initial basic feasible solution, an optimal solution to any LPP by simplex method is found as follows :

Step 1 : Check whether the objective function is to be maximized or minimized. If it is to be minimized, then convert it into a problem of maximization, by

$$\text{Minimize } Z = -\text{Maximize } (-Z)$$

Step 2 : Check whether all b_i's are positive. If any of the b_i's is negative, multiply both sides of that constraint by -1 so as to make its right hand side positive.

Step 3 : By introducing slack / surplus variables, convert the inequality constraints into equations and express the given LPP into its standard form.

Step 4 : Find an initial basic feasible solution and express the above information conveniently in the following simplex table.

		C_j	$(C_1$	C_2	C_3	0	0	0	......)
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	....
C_{B_1}	s_1	b_1	a_{11}	a_{12}	a_{13}	1	0	0	....
C_{B_2}	s_2	b_2	a_{21}	a_{22}	a_{23}	0	1	0	....
C_{B_3}	s_3	b_3	a_{31}	a_{32}	a_{33}	0	0	1	....
:	:	:	:	:	:				....
:	:	:	:	:	:				....
:	:	:	:	:	:				....
:	:	:	:	:	:				....
:	:	:	body matrix				unit matrix		
:	:	:							
$(Z_j - C_j)$		Z_0	$Z_1 - C_1$	$Z_2 - C_2$	... ...				....

(Where C_j - row denotes the coefficients of the variables in the objective function. C_B – column denotes the coefficients of the basic variables in the objective function. Y_B – column denotes the basic variables. X_B – column denotes the values of the basic variables. The coefficients of the non-basic variables in the constraint equations constitute the body matrix while the coefficients of the basic variables constitute the unit matrix. The row $(Z_j - C_j)$ denotes the net evaluations (or) index for each column).

Step 5 : Compute the net evaluations $(Z_j - C_j)$ $(j = 1, 2, \ldots n)$ by using the relation $Z_j - C_j = C_B\, a_j - C_j$.

Examine the sign of $Z_j - C_j$

(a) If all $(Z_j - C_j) \geq 0$ then the current basic feasible solution X_B is optimal.

(b) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal, go to the next step.

Step 6 : (To find the entering variable)

The entering variable is the non-basic variable corresponding to the most negative value of $(Z_j - C_j)$. Let it be x_r for some $j = r$. The entering variable column is known as the ***key column*** (or) ***pivot column*** which is shown marked with an arrow at the bottom. If more than one variable has the same most negative $(Z_j - C_j)$, any of these variables may be selected arbitrarily as the entering variable.

Step 7 : (To find the leaving variable)

Compute the ratio $\theta = \text{Min}\left\{\dfrac{X_{B_i}}{a_{ir}},\ a_{ir} > 0\right\}$ (*i.e.*, the ratio between the solution column and the entering variable column by considering only the positive denominators)

(a) If all $a_{ir} \leq 0$, then there is an unbounded solution to the given LPP.

(b) If atleast one $a_{ir} > 0$, then the leaving variable is the basic variable corresponding to the minimum ratio θ. If $\theta = \dfrac{X_{B_k}}{a_{kr}}$, then the basic variable x_k leaves the basis. The leaving variable row is called the ***key row*** or ***pivot row*** (or) ***pivot equation***, and the element at the intersection of the pivot column and pivot row is called the ***pivot element or key element (or) leading element.***

Step 8 : Drop the leaving variable and introduce the entering variable along with its associated value under C_B column. Convert the pivot element to unity by dividing the pivot equation by the pivot element and all other elements in its column to zero by making use of

(i) New pivot equation = old pivot equation ÷ pivot element

(ii) New equation (all other rows including $(Z_j - C_j)$ row)

$$= \text{Old equation} - \begin{pmatrix}\text{Corresponding}\\ \text{column}\\ \text{coefficient}\end{pmatrix} \times \begin{pmatrix}\text{New pivot}\\ \text{equation}\end{pmatrix}$$

Step 9 : Go to step (5) and repeat the procedure unitl either an optimum solution is obtained or there is an indication of an unbounded solution.

Note (1) : For maximization problems:

(i) If all $(Z_j - C_j) \geq 0$, then the current basic feasible solution is optimal.

(ii) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non-basic variable corresponding to the most negative value of $(Z_j - C_j)$.

Note (2) : For minimization problems :

(i) If all $(Z_j - C_j) \leq 0$, then the current basic feasible solution is optimal.

(ii) If atleast one $(Z_j - C_j) > 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non-basic variable corresponding to the most positive value of $(Z_j - C_j)$.

Note (3) : For both maximization and minimization problems, the leaving variable is the basic variable corresponding to the minimum ratio θ.

Example 1 : Use simplex method to solve the LPP

$$\begin{aligned} \textbf{Maximize } Z &= 4x_1 + 10x_2 \\ \textbf{subject to} \quad 2x_1 + x_2 &\leq 50 \\ 2x_1 + 5x_2 &\leq 100 \\ 2x_1 + 3x_2 &\leq 90 \text{ and } x_1, x_2 \geq 0. \end{aligned}$$

Solution : By introducing the slack variables s_1, s_2 and s_3, the problem in standard form becomes

$$\begin{aligned} \text{Maximize } Z &= 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3 \\ \text{subject to} \quad 2x_1 + x_2 + s_1 + 0s_2 + 0s_3 &= 50 \\ 2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 &= 100 \\ 2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 &= 90 \\ \text{and } x_1, x_2, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

Since there are 3 equations with 5 variables, the initial basic feasible solution is obtained by equating $(5-3)=2$ variables to zero.

$\therefore$ The initial basic feasible solution is $s_1 = 50,\ s_2 = 100,\ s_3 = 90$

$$(x_1 = 0,\ x_2 = 0,\ \text{non-basic})$$

The initial simplex table is given by

$C_j \quad (4 \quad 10 \quad 0 \quad 0 \quad 0)$

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	$\theta = \min \frac{X_{Bi}}{a_{ir}}$
0	s_1	50	2	1	1	0	0	50
0	s_2	100	2	(5)	0	1	0	20*
0	s_3	90	2	3	0	0	1	30
$Z_j - C_j$		0	–4	–10	0	0	0	

Here the net evaluations are calculated as $Z_j - C_j = C_B\, a_j - C_j$.

$$Z_1 - C_1 = C_B\, a_1 - C_1 = (0\ 0\ 0)\ [2\ 2\ 2]^T - 4$$

[where T denotes transpose]

$$= (0\ 0\ 0)\begin{pmatrix}2\\2\\2\end{pmatrix} - 4 = -4$$

$$Z_2 - C_2 = C_B\, a_2 - C_2 = (0\ 0\ 0)\ [1\ 5\ 3]^T - 10 = -10$$

$$Z_3 - C_3 = C_B\, a_3 - C_3 = (0\ 0\ 0)\ [1\ 0\ 0]^T - 10 = 0$$

$$Z_4 - C_4 = C_B\, a_4 - C_4 = (0\ 0\ 0)\ [0\ 1\ 0]^T - 0 = 0$$

$$Z_5 - C_5 = C_B\, a_5 - C_5 = (0\ 0\ 0)\ [0\ 0\ 1]^T - 0 = 0$$

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

To find the entering variable :

Since $(Z_2 - C_2) = -10$ is the most negative, the corresponding non-basic variable x_2 enters the basis. The column corresponding to this x_2 is called the key column or pivot column.

To find the leaving variable :

$$\text{Find the ratio } \theta = \min\left\{\frac{X_{Bi}}{a_{ir}},\ a_{ir} > 0\right\}$$

$$= \min\left\{\frac{X_{Bi}}{a_{i2}},\ a_{i2} > 0\right\}$$

$$\theta = \min\left\{\frac{50}{1},\ \frac{100}{5},\ \frac{90}{3}\right\}$$

$$= \min\{50, 20, 30\} = 20, \text{ which corresponds to } s_2$$

$\therefore$ The leaving variable is the basic variable s_2 which corresponds to the minimum ratio $\theta = 20$. The leaving variable row is called the pivot – row or key row or pivot equation and 5 is the pivot element. Now, New pivot equation = old pivot equation $\div$ pivot element.

$$= (100\ \ 2\ \ 5\ \ 0\ \ 1\ \ 0) \div 5$$

$$= 20\ \ \frac{2}{5}\ \ 1\ \ 0\ \ \frac{1}{5}\ \ 0$$

$$\text{New } s_1 \text{ equation} = \text{old } s_1 \text{ equation} - \begin{pmatrix}\text{corresponding}\\ \text{column}\\ \text{coefficient}\end{pmatrix} \times \begin{pmatrix}\text{New}\\ \text{pivot}\\ \text{equation}\end{pmatrix}$$

$$\begin{array}{rrrrrrr}
 & = 50 & 2 & 1 & 1 & 0 & 0 \\
(-) & 20 & \frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 \\
\hline
 & = 30 & \frac{8}{5} & 0 & 1 & -\frac{1}{5} & 0
\end{array}$$

$$\begin{array}{lrrrrrrr}
\text{New } s_3 \text{ equation} & = 90 & 2 & 3 & 0 & 0 & 1 \\
(-) & 60 & \frac{6}{5} & 3 & 0 & \frac{3}{5} & 0 \\
\hline
 & = 30 & \frac{4}{5} & 0 & 0 & \frac{-3}{5} & 1
\end{array}$$

$$\begin{array}{lrrrrrrr}
\text{New } (Z_j - C_j) \text{ eqn.} & = 0 & -4 & -10 & 0 & 0 & 0 \\
(-) & -200 & \frac{-20}{5} & -10 & 0 & \frac{-10}{5} & 0 \\
\hline
 & = 200 & 0 & 0 & 0 & 2 & 0
\end{array}$$

$\therefore$ The improved basic feasible solution is given in the following simplex table.

First Iteration :

C_B	Y_B	C_j → X_B	4 x_1	10 x_2	0 s_1	0 s_2	0 s_3
0	s_1	30	$^8/_5$	0	1	$^{-1}/_5$	0
10	x_2	20	$^2/_5$	1	0	$^1/_5$	0
0	s_3	30	$^4/_5$	0	0	$^{-3}/_5$	1
$Z_j - C_j$		200	0	0	0	2	0

Since all $Z_j - C_j \geq 0$ the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = 200, x_1 = 0, x_2 = 20$.

Example 2 : Find the non-negative values of x_1, x_2 and x_3 which Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to

$$x_1 + 4x_2 \leq 420$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1 + 2x_2 + x_3 \leq 430$$

Solution : Given the LPP

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$x_1 + 4x_2 \leq 420$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1 + 2x_2 + x_3 \leq 430$$
$$x_1, x_2, x_3 \geq 0.$$

By introducing non-negative slack variables s_1, s_2 and s_3, the standard form of the LPP becomes

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$x_1 + 4x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 = 420$$
$$3x_1 + 0x_2 + 2x_3 + 0s_1 + s_2 + 0s_3 = 460$$
$$x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + s_3 = 430$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Since there are 3 equations with 6 variables, the initial basic feasible solution is obtained by equating $(6 - 3) = 3$ variables to zero.

$\therefore$ The initial basic feasible solution is $s_1 = 420$, $s_2 = 460$, $s_3 = 430$ ($x_1 = x_2 = x_3 = 0$, non-basic)

The initial simplex table is given by

Initial iteration :

C_B	Y_B	C_j → X_B	3 x_1	2 x_2	5 x_3	0 s_1	0 s_2	0 s_3	θ
0	s_1	420	1	4	0	1	0	0	–
0	s_2	460	3	0	(2)	0	1	0	$\frac{460}{2} = 230^*$
0	s_3	430	1	2	1	0	0	1	$\frac{430}{1} = 430$
$Z_j - C_j$		0	–3	–2	–5	0	0	0	

Since there are some $(Z_j - C_j) < 0$, The current basic feasible solution is not optimal.

To find the entering variable : Since $(Z_3 - C_3) = -5$ is the most negative, the corresponding non-basic variable x_3 enters into the basis. The column corresponding to this x_3 is called the key column or pivot column.

To find the leaving variable :

$$\text{Find the ratio } \theta = \min\left\{\frac{X_{Bi}}{a_{ir}}, a_{ir} > 0\right\} = \min\left\{\frac{X_{Bi}}{a_{i3}}, a_{i3} > 0\right\}$$

$$\theta = \min\left\{\frac{460}{2}, \frac{430}{1}\right\} = \min\{230, 430\} = 230$$

$\therefore$ The leaving variable is the basic variable s_2 which corresponds to the minimum ratio $\theta = 230$. The leaving variable row is called the key row or pivot equation and 2 is the pivot element.

New pivot equation = old pivot equation ÷ pivot element

$$= (460 \quad 3 \quad 0 \quad 2 \quad 0 \quad 1 \quad 0) \div 2$$
$$= 230 \quad \frac{3}{2} \quad 0 \quad 1 \quad 0 \quad \frac{1}{2} \quad 0$$

$$\text{New } s_1 \text{ equation} = \text{old } s_1 \text{ equation} - \begin{pmatrix}\text{its entering}\\ \text{column}\\ \text{coefficient}\end{pmatrix} \times \begin{pmatrix}\text{New}\\ \text{pivot}\\ \text{equation}\end{pmatrix}$$

	=	420	1	4	0	1	0	0
(–)		0	0	0	0	0	0	0
	=	420	1	4	0	1	0	0

New s_3 equation $=$ old s_3 equation $- \begin{pmatrix}\text{its entering}\\ \text{column}\\ \text{coefficient}\end{pmatrix} \times \begin{pmatrix}\text{New}\\ \text{pivot}\\ \text{equation}\end{pmatrix}$

	=	430	1	2	1	0	0	1
(–)		230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0
	=	200	$-\frac{1}{2}$	2	0	0	$-\frac{1}{2}$	1

New $(Z_j - C_j)$ eqn. $=$ old $(Z_j - C_j)$ equation $- \begin{pmatrix}\text{its entering}\\ \text{column}\\ \text{coefficient}\end{pmatrix} \times \begin{pmatrix}\text{New}\\ \text{pivot}\\ \text{equation}\end{pmatrix}$

	=	0	–3	–2	–5	0	0	0
(–)		–1150	$\frac{-15}{2}$	0	–5	0	$\frac{-5}{2}$	0
	=	1150	$\frac{9}{2}$	–2	0	0	$\frac{5}{2}$	0

The improved basic feasible solution is given in the following simplex table.

First iteration :

		C_j	(3	2	5	0	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	420	1	4	0	1	0	0	$\frac{420}{4} = 105$
5	x_3	230	$3/2$	0	1	0	$1/2$	0	–
0	s_3	200	$-1/2$	(2)	0	0	$-1/2$	1	$\frac{200}{2} = 100$*
$Z_j - C_j$		1150	$9/2$	–2	0	0	$5/2$	0	

Since there is an $(Z_2 - C_2) = -2$, the current basic feasible solution is not optimal.

$\therefore$ Here the non-basic variable x_2 enters into the basis and the basic variable s_3 leaves the basis.

Second iteration :

		C_j	(3	2	5	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	20	2	0	0	1	1	–2
5	x_3	230	$3/2$	0	1	0	$1/2$	0
2	x_2	100	$-1/4$	1	0	0	$-1/4$	$1/2$
$Z_j - C_j$		1350	4	0	0	0	2	1

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = 1350, x_1 = 0, x_2 = 100, x_3 = 230$.

Example 3 : Solve the following :

Maximize : $15x_1 + 6x_2 + 9x_3 + 2x_4$

subject to

$$2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$$
$$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$$
$$7x_1 + x_4 \leq 70$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Solution : By introducing non-negative slack variables s_1, s_2 and s_3, the standard form of the LPP becomes.

Maximize $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0s_1 + 0s_2 + 0s_3$

subject to

$$2x_1 + x_2 + 5x_3 + 6x_4 + s_1 + 0s_2 + 0s_3 = 20$$
$$3x_1 + x_2 + 3x_3 + 25x_4 + 0s_1 + s_2 + 0s_3 = 24$$
$$7x_1 + 0x_2 + 0x_3 + x_4 + 0s_1 + 0s_2 + s_3 = 70$$
$$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0.$$

$\therefore$ The initial basic feasible solution is $s_1 = 20$, $s_2 = 24$, $s_3 = 70$ $(x_1 = x_2 = x_3 = x_4 = 0$, non-basic)

The initial simplex table is given by

Initial iteration :

		C_j	15	6	9	2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	θ
0	s_1	20	2	1	5	6	1	0	0	$\frac{20}{2} = 10$
0	s_2	24	(3)	1	3	25	0	1	0	$\frac{24}{3} = 8^*$
0	s_3	70	7	0	0	1	0	0	1	$\frac{70}{7} = 10$
$Z_j - C_j$		0	-15	-6	-9	-2	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters into the basis and the basic variable s_2 leaves the basis.

First iteration :

		C_j	15	6	9	2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	θ
0	s_1	4	0	$(^1/_3)$	3	$-\frac{32}{3}$	1	$^{-2}/_3$	0	12*
15	x_1	8	1	$^1/_3$	1	$^{25}/_3$	0	$^1/_3$	0	24
0	s_3	14	0	$^{-7}/_3$	-7	$\frac{-172}{3}$	0	$^{-7}/_3$	1	–
$Z_j - C_j$		120	0	-1	6	123	0	5	0	

Since $(Z_2 - C_2) = -1 < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable s_1 leaves the basis.

Second iteration :

		C_j	15	6	9	2	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
6	x_2	12	0	1	9	-32	3	-2	0
15	x_1	4	1	0	-2	$\frac{57}{3}$	-1	1	0
0	s_3	42	0	0	14	-132	7	-7	1
$Z_j - C_j$		132	0	0	15	91	3	3	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

Max $Z = 132$, $x_1 = 4$, $x_2 = 12$, $x_3 = 0$ and $x_4 = 0$.

Example 4 : Solve the following LPP by simplex method :

Minimize $Z = 8x_1 - 2x_2$.

$$\begin{aligned} \textbf{subject to} \quad -4x_1 + 2x_2 &\leq 1 \\ 5x_1 - 4x_2 &\leq 3 \\ \textbf{and } x_1, x_2 &\geq 0. \end{aligned}$$

Solution : Since the given objective function is of minimization type, we shall convert it in to a maximization type as follows :

$$\text{Maximize } (-Z) = \text{Maximize } Z^* = -8x_1 + 2x_2$$

$$\begin{aligned} \text{subject to} \quad -4x_1 + 2x_2 &\leq 1 \\ 5x_1 - 4x_2 &\leq 3 \\ x_1, x_2 &\geq 0. \end{aligned}$$

By introducing non-negative slack variables s_1, s_2, the standard form of the LPP becomes

$$\text{Maximize } Z^* = -8x_1 + 2x_2 + 0s_1 + 0s_2$$

subject to the constraints

$$\begin{aligned} -4x_1 + 2x_2 + s_1 + 0s_2 &= 1 \\ 5x_1 - 4x_2 + 0s_1 + s_2 &= 3 \\ \text{and } x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

$\therefore$ The initial basic feasible solution is given by $s_1 = 1$, $s_2 = 3$, ($x_1 = x_2 = 0$, non-basic).

Initial iteration :

	C_j		(−8	2	0	0)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
0	s_1	1	−4	(2)	1	0	$\frac{1}{2}$*
0	s_2	3	5	−4	0	1	–
$Z_j^* - C_j$		0	8	−2	0	0	

Since $(Z_2^* - C_2) = -2 < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters the basis and the basic variable s_1 leaves the basis.

First iteration :

	C_j		(−8	2	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
2	x_2	$\frac{1}{2}$	−2	1	$\frac{1}{2}$	0
0	s_2	5	−3	0	2	1
$Z_j^* - C_j$		1	4	0	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by maximize $Z^* = 1$, $x_1 = 0$, $x_2 = {}^1/_2$,

But Minimize $Z = -$ Maximize $(-Z) = -$ Maximize $Z^* = -1$

$\therefore$ Min $Z = -1$, $x_1 = 0$, $x_2 = \frac{1}{2}$.

Aliter : The above problem can be solved without converting the objective function in to maximization type.

$$\text{Given Minimize } Z = 8x_1 - 2x_2$$

$$\begin{aligned} \text{subject to the constraints} \quad -4x_1 + 2x_2 &\leq 1 \\ 5x_1 - 4x_2 &\leq 3 \\ x_1, x_2, &\geq 0. \end{aligned}$$

$\therefore$ By introducing the non-negative slack variable s_1, s_2 the LPP becomes

$$\text{Minimize } Z = 8x_1 - 2x_2 + 0s_1 + 0s_2$$

$$\begin{aligned} \text{subject to the constraints} \quad -4x_1 + 2x_2 + s_1 + 0s_2 &= 1 \\ 5x_1 - 4x_2 + 0s_1 + s_2 &= 3 \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

The initial basic feasible solution is given by

$s_1 = 1$, $s_2 = 3$ (basic) ($x_1 = x_2 = 0$, non-basic)

Initial iteration :

	C_j		(8	−2	0	0)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
0	s_1	1	−4	(2)	1	0	$\frac{1}{2}$*
0	s_2	3	5	−4	0	1	–
$Z_j - C_j$		0	−8	2	0	0	

Since $(Z_2 - C_2) = 2 > 0$, the current basic feasible solution is not optimal.

To find the entering variable :

Since $(Z_2 - C_2) = 2$ is most positive, the corresponding non-basic variable x_2 enters into the basis.

To find the leaving variable :

The leaving variable is the basic variable s_1 corresponding to the minimum ratio $\theta = \frac{1}{2}$.

First iteration :

	C_j		(8	−2	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
−2	x_2	$\frac{1}{2}$	−2	1	$\frac{1}{2}$	0
0	s_2	5	−3	0	2	1
$Z_j - C_j$		−1	−4	0	−1	0

Since all $(Z_j - C_j) \leq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Min $Z = -1$, $x_1 = 0$, $x_2 = \frac{1}{2}$

Example 5 : Use simplex method to

$$\text{Min } Z = x_2 - 3x_3 + 2x_5$$

subject to

$$3x_2 - x_3 + 2x_5 \le 7$$
$$-2x_2 + 4x_3 \le 12$$
$$-4x_2 + 3x_3 + 8x_5 \le 10$$
$$\text{and } x_2, x_3, x_5 \ge 0$$

Solution : Since the given objective function is of minimization type, we shall convert it in to a maximization type as follows :

$$\text{Maximize } (-Z) = \text{Maximize } Z^* = -x_2 + 3x_3 - 2x_5$$

subject to

$$3x_2 - x_3 + 2x_5 \le 7$$
$$-2x_2 + 4x_3 \le 12$$
$$-4x_2 + 3x_3 + 8x_5 \le 10$$
$$x_2, x_3, x_5 \ge 0.$$

By introducing non-negative slack variables s_1, s_2 and s_3, the standard form of the LPP becomes

$$\text{Maximize } Z^* = -x_2 + 3x_3 - 2x_5 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints

$$3x_2 - x_3 + 2x_5 + s_1 + 0s_2 + 0s_3 = 7$$
$$-2x_2 + 4x_3 + 0x_5 + 0s_1 + s_2 + 0s_3 = 12$$
$$-4x_2 + 3x_3 + 8x_5 + 0s_1 + 0s_2 + s_3 = 10$$
$$\text{and } x_2, x_3, x_5, s_1, s_2, s_3 \ge 0.$$

$\therefore$ The initial basic feasible solution is given by $s_1 = 7$, $s_2 = 12$, $s_3 = 10$ ($x_2 = x_3 = x_5 = 0$, non-basic)

Initial iteration :

		C_j	-1	3	-2	0	0	0	
C_B	Y_B	X_B	x_2	x_3	x_5	s_1	s_2	s_3	θ
0	s_1	7	3	-1	2	1	0	0	–
0	s_2	12	-2	(4)	0	0	1	0	$\frac{12}{4} = 3^*$
0	s_3	10	-4	3	8	0	0	1	$\frac{10}{3} = 3.33$
$Z_j^* - C_j$		0	1	-3	2	0	0	0	

Since $(Z_2^* - C_2) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_3 enters into the basis and the basic variable s_2 leaves the basis.

First iteration :

		c_j	-1	3	-2	0	0	0	
C_B	Y_B	X_B	x_2	x_3	x_5	s_1	s_2	s_3	θ
0	s_1	10	$\left(\frac{5}{2}\right)$	0	2	1	$\frac{1}{4}$	0	$\frac{20}{5} = 4^*$
3	x_3	3	$\frac{-1}{2}$	1	0	0	$\frac{1}{4}$	0	–
0	s_3	1	$\frac{-5}{2}$	0	8	0	$\frac{-3}{4}$	1	–
$Z_j^* - C_j$		9	$\frac{-1}{2}$	0	2	0	$\frac{3}{4}$	0	

Since $(Z_1^* - C_1) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable s_1 leaves the basis.

Second iteration :

		C_j	-1	3	-2	0	0	0
C_B	Y_B	X_B	x_2	x_3	x_5	s_1	s_2	s_3
-1	x_2	4	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0
3	x_3	5	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0
0	s_3	11	0	0	10	1	$\frac{-1}{2}$	1
$(Z_j^* - C_j)$		11	0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0

Since all $(Z_j^* - C_j) \ge 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

$$\text{maximize } Z^* = 11, \quad x_2 = 4, \quad x_3 = 5, \quad x_5 = 0$$

$$\text{But Minimize } Z = -\text{Maximize } Z^* = -11$$

$$\therefore \text{Min } Z = -11, \; x_2 = 4, \; x_3 = 5, \; x_5 = 0.$$

Example 6 : **An automobile manufacturer makes auto–mobiles and trucks in a factory that is divided into two shops. Shop A, which performs the basic assembly operation must work 5 man–days on each truck but only 2 man–days on each automobile. Shop B, which performs finishing operation must work 3 man–days for each truck or automobile that it produces. Because of men and machine limitations shop A has 180 man–days per week available while shop B has 135 man–days per week. If the manufacturer makes a profit of Rs. 300 on each truck and Rs. 200 on each automobile, how many of each should he produce to maximize his profit?** *[MU. MBA 1982]*

Solution : Let x_1 units of trucks and x_2 units of automobiles be manufactured per week to maximize his profit.

Then the mathematical form of the given problem will be :

$$\text{Maximize } Z = 300x_1 + 200x_2$$

$$\text{subject to} \quad \begin{aligned} 5x_1 + 2x_2 &\leq 180 \\ 3x_1 + 3x_2 &\leq 135 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

By introducing non-negative slack variables s_1 and s_2, the standard form of the LPP becomes

$$\text{Maximize } Z = 300x_1 + 200x_2 + 0s_1 + 0s_2$$

$$\text{subject to} \quad \begin{aligned} 5x_1 + 2x_2 + s_1 + 0s_2 &= 180 \\ 3x_1 + 3x_2 + 0s_1 + s_2 &= 135 \\ \text{and } x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

$\therefore$ The initial basic feasible solution is given by $s_1 = 180$, $s_2 = 135$, ($x_1 = x_2 = 0$, non-basic)

Initial iteration :

		C_j	(300	200	0	0)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
0	s_1	180	(5)	2	1	0	$\frac{180}{5} = 36^*$
0	s_2	135	3	3	0	1	$\frac{135}{3} = 45$
$Z_j - C_j$		0	–300	–200	0	0	

First iteration : Introduce x_1 and drop s_1.

		c_j	(300	200	0	0)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
300	x_1	36	1	$^2/_5$	$^1/_5$	0	$36 \times \frac{5}{2} = 90$
0	s_2	27	0	$(^9/_5)$	$^{-3}/_5$	1	$27 \times \frac{5}{9} = 15^*$
$Z_j - C_j$		10,800	0	– 80	60	0	

Second iteration : Introduce x_2 and drop s_2.

		c_j	(300	200	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
300	x_1	30	1	0	$^1/_3$	$\frac{-2}{9}$
200	x_2	15	0	1	$^{-1}/_3$	$\frac{5}{9}$
$Z_j - C_j$		12,000	0	0	$\frac{100}{3}$	$\frac{400}{9}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = 12{,}000$, $x_1 = 30$, $x_2 = 15$.

The manufacturer should manufacture 30 trucks and 15 automobiles per week in order to get a maximum profit of Rs. 12000.

Example 7: **A gear manufacturing company received an order for three specific types of gears for regular supply. The management is considering to devote the available excess capacity to one or more of the three types, say A, B and C. The available capacity on the machines which might limit output and the number of machine hours required for each unit of the respective gear is also given below :**

Machine Type	Available machine hours/week	Productivity in machine hours/unit		
		Gear A	Gear B	Gear C
Gear Hobbing m/c	250	8	2	3
Gear Shaping m/c	150	4	3	0
Gear Grinding m/c	50	2	–	1

The unit profit would be Rs. 20, Rs.6 and Rs.8 respectively for the gears A, B and C. Find how much of each gear the company should produce in order to maximize profit ?

[BRU M.Sc. 86, MU. BE. Oct 96]

Solution : Let x_1, x_2 and x_3 be the number of units of gears A, B and C produced respectively to maximize the profit. The mathematical formulation of the LPP is given by

$$\text{Maximize } Z = 20x_1 + 6x_2 + 8x_3$$

subject to

$$\begin{aligned} 8x_1 + 2x_2 + 3x_3 &\leq 250 \\ 4x_1 + 3x_2 &\leq 150 \\ 2x_1 + x_3 &\leq 50 \\ \text{and } x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

By introducing non-negative slack variables s_1, s_2 and s_3, the standard form of the LPP becomes

$$\text{Maximize } Z^* = 20x_1 + 6x_2 + 8x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints

$$\begin{aligned} 8x_1 + 2x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 &= 250 \\ 4x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 &= 150 \\ 2x_1 + 0x_2 + x_3 + 0s_1 + 0s_2 + s_3 &= 50 \\ \text{and } x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

$\therefore$ The initial basic feasible solution is given by $s_1 = 250$, $s_2 = 150$, $s_3 = 50$ $(x_1 = x_2 = x_3 = 0$, non-basic).

Initial iteration :

C_j (20 6 8 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	250	8	2	3	1	0	0	$\frac{250}{8}$
0	s_2	150	4	3	0	0	1	0	$\frac{150}{4}$
0	s_3	50	(2)	0	1	0	0	1	$\frac{50}{2}$ *
$Z_j - C_j$		0	–20	–6	–8	0	0	0	

First iteration : Introduce x_1 and drop s_3.

C_j (20 6 8 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	50	0	2	–1	1	0	–4	$\frac{50}{2}$
0	s_2	50	0	(3)	–2	0	1	–2	$\frac{50}{3}$ *
20	x_1	25	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	–
$Z_j - C_j$		500	0	–6	2	0	0	10	

Second iteration : Introduce x_2 and drop s_2.

C_j (20 6 8 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	$\frac{50}{3}$	0	0	$\frac{1}{3}$	1	$\frac{-2}{3}$	$\frac{-8}{3}$	50
6	x_2	$\frac{50}{3}$	0	1	$\frac{-2}{3}$	0	$\frac{1}{3}$	$\frac{-2}{3}$	–
20	x_1	25	1	0	$\left(\frac{1}{2}\right)$	0	0	$\frac{1}{2}$	50*
$Z_j - C_j$		600	0	0	–2	0	2	6	

Third iteration : Introduce x_3 and drop x_1.

C_j (20 6 8 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	0	$\frac{-2}{3}$	0	0	1	$\frac{-2}{3}$	–3
6	x_2	50	$\frac{4}{3}$	1	0	0	$\frac{1}{3}$	0
8	x_3	50	2	0	1	0	0	1
$Z_j - C_j$		700	4	0	0	0	2	8

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max Z = 700, $x_1 = 0$, $x_2 = 50$, $x_3 = 50$

$\therefore$ The company should produce 50 units of gear B, 50 units of gear of C and none of gear A in order to have a maximum profit Rs. 700.

EXERCISE

1. Explain the mathematical formulation of a linear programming problem and its matrix formulation. *[MU. BE. Apr 90]*
2. Define *(i)* objective function *(ii)* optimal solution. *[MU. BE. Apr 90]*
3. Define *(i)* objective function *(ii)* optimal solution *(iii)* surplus variable. *[MU. BE. Apr 91]*
4. Define *(i)* Feasible solution *(ii)* Basic solution *(iii)* optimal solution of LPP. *[MU. BE. Apr 93, Apr 95]*
5. Write the standard form of LPP in matrix form. *[MU. BE. Nov 93]*
6. What is the canonial form of a LPP ? *[MU. B. Tech. Leather. Oct 96]*
7. Establish the difference between basic feasible solution and non–degenerate basic feasible solution. *[MKU. BE. 1979]*
8. What is the difference between slack variable and surplus variable ? *[BNU. BE. Nov 96]*
9. What are the slack and surplus variables ? *[MU. BE. Nov 93, MU.MBA.Nov 96, Apr. 97]*
10. Explain the problem of LPP in general. Define basic solution, basic feasible solution and optimum solution of LPP. *[MU.MBA. Apr 96, Nov 96, MU. MCA. May 95]*
11. State the general linear programming problem and define *(a)* feasible solution and *(b)* basic feasible solution *[MU.MBA. Apr 95]*
12. Describe simplex method of solving linear programming problem. *[MU.MBA. Apr 95]*
13. Define the general linear programming problem and explain the terms : slack variables, surplus variables, Feasible solution. *[MU. MCA. May 91]*
14. Express the following LPP in the canonical form,

$$\text{Maximize } Z = 3x_1 + x_2$$

subject to

$$x_1 + 2x_2 \geq -5$$
$$3x_1 + 5x_2 \leq 6$$
$$\text{and } x_1, x_2 \geq 0.$$

15. Express the following LPP in the canonical form,

$$\text{Minimize } Z = x_1 + 4x_2$$

subject to

$$3x_1 + x_2 \leq 5$$
$$-2x_1 + 4x_2 \geq -7$$
$$\text{and } x_1, x_2 \geq 0.$$

16. Express the following LPP in the standard form

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$2x_1 - 3x_2 \leq 3$$
$$x_1 + 2x_2 + 3x_3 \geq 5$$
$$3x_1 + 2x_3 \leq 2$$
$$x_1, x_2 \geq 0.$$

17. Express the following LPP in the standard form

$$\text{Minimize } Z = 3x_1 + 5x_2 + x_3$$

subject to

$$3x_1 + 4x_2 - 5x_3 \leq 8$$
$$2x_1 + 6x_2 + x_3 \geq 7$$
$$x_1 - 2x_2 + x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0.$$

18. Compute all the basic feasible solutions to the LPP

$$\text{Maximize } Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$$

subject to the constraints

$$2x_1 + 3x_2 - x_3 + 4x_4 = 8$$
$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

and choose that one which maximizes Z.

19. Using simplex method, find non-negative values of x_1, x_2 and x_3 which maximize $Z = x_1 + 4x_2 + 5x_3$ subject to the constraints $3x_1 + 6x_2 + 3x_3 \leq 22$, $x_1 + 2x_2 + 3x_3 \leq 14$ and $3x_1 + 2x_2 \leq 14$. *[MU. MBA. Apr 95, Apr. 97]*

20. **Apply the simplex method to solve the problem.**

$$\text{Maximize } Z = 100x_1 + 200x_2 + 50x_3$$
$$\text{subject to} \quad 5x_1 + 5x_2 + 10x_3 \le 1000$$
$$10x_1 + 8x_2 + 5x_3 \le 2000$$
$$10x_1 + 5x_2 \le 500$$
$$x_1, x_2, x_3 \ge 0.$$
[MU. BE. Apr 90]

21. **Using simplex method, find the non-negative values of x_1 and x_2 which maximize the objective function $Z = 2x_1 + x_2$ subject to the constraints**

$$x_1 - 2x_2 \le 1$$
$$x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 10$$
$$\text{and } x_1 - x_2 \le 2.$$
[MU. BE. Nov 91, BRU. BE. Apr 98]

22. **Solve the following LPP using simplex method :**

$$\text{Maximize } Z = x_1 + x_2 + 3x_3$$
$$\text{subject to} \quad 3x_1 + 2x_2 + x_3 \le 3$$
$$2x_1 + x_2 + 2x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0.$$
[MU. BE. Apr 92, Nov 92, MKU. BE. Nov 96, MSU. BE. Apr 97]

23. **Solve by simplex method :**

$$\text{Maximize } Z = 3x_1 + 5x_2 + 4x_3$$
$$\text{subject to} \quad 2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$x_1, x_2, x_3 \ge 0.$$
[MU. BE. Apr 92, Nov 93, Apr 97, B.Sc. 84, BRU. BE. Nov 96, Apr 97, Apr 98]

24. **Solve by simplex method :**

$$\text{Maximize } Z = 2x_1 - 4x_2 + 5x_3 - 6x_4$$
subject to the constraints
$$x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$$
$$x_1, x_2, x_3, x_4 \ge 0.$$
[MU. BE. Nov 92]

25. **Solve by simplex method :**

$$\text{Maximize } 3x_1 + 4x_2 + x_3 + 7x_4$$
subject to
$$8x_1 + 3x_2 + 4x_3 + x_4 \le 7$$
$$2x_1 + 6x_2 + x_3 + 6x_4 \le 3$$
$$x_1 + 4x_2 + 5x_3 + 2x_4 \le 8$$
$$x_1, x_2, x_3, x_4 \ge 0.$$
[MU. BE. Nov 90]

26. **Solve by simplex method :**

$$\text{Maximize } 10x_1 + x_2 + 2x_3$$
$$\text{subject to} \quad x_1 + x_2 - 2x_3 \le 10$$
$$4x_1 + x_2 + x_3 \le 20$$
$$\text{and } x_1, x_2, x_3 \ge 0.$$
[MKU. B.Sc.81, MU. BE. Apr 95]

27. **Solve the following LPP using simplex method :**

$$\text{Maximize } F = x_1 + 2x_2 + x_3$$
$$\text{subject to} \quad 2x_1 + x_2 - x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_i \ge 0, \; i = 1, 2, 3.$$
[MU. MCA. Nov 94]

28. **Solve the LPP:**

$$\text{Maximize } Z = 2x_1 + 4x_2 + 3x_3$$
$$\text{subject to} \quad 3x_1 + 4x_2 + 2x_3 \le 60$$
$$2x_1 + x_2 + 2x_3 \le 40$$
$$x_1 + 3x_2 + 2x_3 \le 80$$
$$\text{and } x_1, x_2, x_3 \ge 0.$$
[MU. BE. 81]

29. **A manufacturer of leather belts makes three types of belts A, B and C which are processed on three machines M_1, M_2 and M_3. Belt A requires 2 hours on machine M_1, and 3 hours on machine M_3 belt B requires 3 hours on Machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3. Belt C requires 5 on machine M_2 and 2 hours on machine M_3. There are hours on machine M_2 and 4 hours on machine M_3. There are 8 hours of time per day available on machine M_1, 10 hours per day available on M_2 and 15 hours per day available on M_3.**

The profit gained from belt A is Rs. 3 per unit, from belt B is Rs. 5 per unit, from belt C is Rs.4 per unit. What should be the daily production of each types of belts so that the profit is maximum?*[Annamalai BE 81, BRU. BE. Nov 96, MU. BSc. 84,]*

30. A farmer has 1000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn cost Rs.100 for preparation, requires 7 man–days of work and yields of profit of Rs.30. An acre of wheat costs Rs.120 to prepare and requires 10 man–days of work and yields a profit of Rs.40. An acre of soyabeans costs Rs.70 to prepare, requires 8 man–days of work and yields of profit of Rs.20. If the farmer has Rs.100000 for preparation and can count on 8000 man–days of work, how many acres should be allocated to each crop to maximize profit ? *[MU. B.Sc. 84]*

31. Using simplex method

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 43 \\ 3x_1 + 2x_3 &\le 46 \\ x_1 + 4x_2 &\le 42 \\ \text{and } x_1, x_2, x_3 &\ge 0. \end{aligned}$$

[MU. BE. Oct 96]

32. Use simplex method to solve the following LPP :

$$\text{Maximize } Z = 2x_1 + 4x_2 + x_3 + x_4$$

subject to

$$\begin{aligned} x_1 + 3x_2 + x_4 &\le 4 \\ 2x_1 + x_2 &\le 3 \\ x_2 + 4x_3 + x_4 &\le 3 \\ \text{and } x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}$$

[BRU. BE. Nov 96]

33. A firm has available 240, 370 and 180 kg of wood, plastic and steel respectively. The firm produces two products A and B. Each unit of A requires 1, 3 and 2 kg of wood, plastic and steel respectively. The corresponding requirement for each unit of B are 3, 4 and 1 respectively. If A sells for Rs. 4 and B for Rs.6, determine how many units of A and B should be produced in order to obtain the maximum gross income. Use the simplex method ? *[BRU. BE. Apr 95]*

34. Find non-negative values of x_1, x_2 and x_3 which maximize $Z = 2x_1 + 4x_2 + x_3$

subject to

$$\begin{aligned} x_1 + 3x_2 &\le 4 \\ 2x_1 + x_2 &\le 3 \\ x_2 + 4x_3 &\le 3. \end{aligned}$$

[MU. MBA. Nov 96]

35. A manufacturer is engaged in producing 2 products X and Y, the contribution margin being Rs. 15 and Rs. 45 respectively. A unit of product X requires 1 unit of facility A and 0.5 unit of facility B. A unit of product Y requires 1.6 units of facility A, 2.0 units of facility B and 1 unit of raw material C. The availability of total facility A and B and raw material C during a particular time period are 240, 162 and 50 units respectively. Find out the product mix which will maximize the contribution margin, by simplex method.

[BRU. BE. Nov 95]

36. Find non-negative values of x_1, x_2 and x_3 which

$$\text{Maximize } Z = 9x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 &\le 12 \\ 2x_1 - x_2 + 3x_3 &\le 3 \\ 3x_1 + x_2 - 2x_3 &\le 2. \end{aligned}$$

[MU. MBA. April 1996 & April 1998]

ANSWERS

14.

$$\text{Maximize } Z = 3x_1 + x_2$$

subject to

$$\begin{aligned} -x_1 - 2x_2 &\le 5 \\ 3x_1 + 5x_2 &\le 6 \\ \text{and } x_1, x_2 &\ge 0. \end{aligned}$$

15.

$$\text{Maximize } Z^* = -x_1 - 4x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 &\le 5 \\ 2x_1 - 4x_2 &\le 7 \\ \text{and } x_1, x_2 &\ge 0. \end{aligned}$$

16. Maximize Z $= 3x_1 + 2x_2 + 5x_3' - 5x_3''$

subject to
$$\begin{aligned} 2x_1 - 3x_2 + s_1 &= 3 \\ x_1 + 2x_2 + 3x_3' - 3x_3'' - s_2 &= 5 \\ 3x_1 + 2x_3' - 2x_3'' + s_3 &= 2 \\ x_1, x_2, x_3', x_3'', s_1, s_2, s_3 &\geq 0. \end{aligned}$$

17. Maximize $(-Z) = -3x_1 - 5x_2 - x_3$

subject to
$$\begin{aligned} 3x_1 + 4x_2 - 5x_3 + s_1 &= 8 \\ 2x_1 + 6x_2 + x_3 - s_2 &= 7 \\ x_1 - 2x_2 + x_3 + s_3 &= 5 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

18. (i) $x_1 = 1, \quad x_2 = 2, \quad x_3 = 0, \quad x_4 = 0$

(ii) $x_1 = \frac{22}{9}, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = \frac{7}{9}$

(iii) $x_1 = 0, \quad x_2 = \frac{45}{16}, \quad x_3 = \frac{7}{16}, \quad x_4 = 0$

(iv) $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{44}{17}$, $x_4 = \frac{45}{17}$ and Max $Z = \frac{491}{17}$

19. Max $Z = \frac{74}{3}$, $x_1 = 0$, $x_2 = 2$, $x_3 = \frac{10}{3}$

20. Max $Z = 22500$, $x_1 = 0$, $x_2 = 100$, $x_3 = 50$

21. Max $Z = 10$, $x_1 = 4$, $x_2 = 2$

22. Max $Z = 3$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$

23. Max $Z = \frac{765}{41}$, $x_1 = \frac{89}{41}$, $x_2 = \frac{50}{41}$, $x_3 = \frac{62}{41}$

24. Max $Z_0 = 31$, $x_1 = 8$, $x_2 = 0$, $x_3 = 3$, $x_4 = 0$

25. Max $Z = \frac{187}{46}$, $x_1 = \frac{39}{46}$, $x_2 = 0$, $x_3 = 0$, $x_4 = \frac{10}{46}$

26. Max $Z = 50$, $x_1 = 5$, $x_2 = 0$, $x_3 = 0$

27. Max F = 10, $x_1 = 0$, $x_2 = 4$, $x_3 = 2$.

28. Max $Z = \frac{250}{3}$, $x_1 = 0$, $x_2 = \frac{20}{3}$, $x_3 = \frac{50}{3}$

29. Max $Z = 3x_1 + 5x_2 + 4x_3$ subject to

$2x_1 + 3x_2 \leq 8$, $2x_2 + 5x_3 \leq 10$, $3x_1 + 2x_2 + 4x_3 \leq 15$, $x_1, x_2, x_3 \geq 0$

Also Max $Z = \frac{765}{41}$, $x_1 = \frac{89}{41}$, $x_2 = \frac{50}{41}$, $x_3 = \frac{62}{41}$

30. Max $Z = 30x_1 + 40x_2 + 20x_3$ subject to

$10x_1 + 12x_2 + 7x_3 \leq 10000$,

$7x_1 + 10x_2 + 8x_3 \leq 8000$, $x_1 + x_2 + x_3 \leq 1000$, $x_1, x_2, x_3 \geq 0$

Also Max $Z = 32500$, $x_1 = 250$, $x_2 = 625$, $x_3 = 0$.

31. Max $Z = 135$, $x_1 = 0$, $x_2 = 10$, $x_3 = 23$

32. Max $Z = \frac{13}{2}$, $x_1 = 1$, $x_2 = 1$, $x_3 = \frac{1}{2}$, $x_4 = 0$

33. Max $Z = 4x_1 + 6x_2$

subject to
$$\begin{aligned} x_1 + 3x_2 &\leq 240, \\ 3x_1 + 4x_2 &\leq 370 \\ 2x_1 + x_2 &\leq 180 \text{ and } x_1, x_2 \geq 0 \end{aligned}$$

Also Max $Z = 540$, $x_1 = 30$, $x_2 = 70$.

34. Max $Z = \frac{13}{2}$, $x_1 = 1, x_2 = 1, x_3 = \frac{1}{2}$

35. Max $Z = 15x_1 + 45x_2$

subject to
$$\begin{aligned} x_1 + 1.6x_2 &\leq 240, \\ 0.5x_1 + 2x_2 &\leq 162 \\ x_2 &\leq 50 \\ \text{and } x_1, x_2 &\geq 0 \end{aligned}$$

Also Max $Z =$ Rs. 1851, $x_1 = 18.4$, $x_2 = 35$.

36. Max $Z = \frac{637}{13}$, $x_1 = 0, x_2 = 12, x_3 = \frac{65}{13}$

1.4.3 Artificial Variables Techniques

To solve a LPP by simplex method, we have to start with the initial basic feasible solution and construct the initial simplex table. In the previous problems, we see that the slack variables readily provided the initial basic feasible solution. However, in some problems, the slack variables can not provide the initial basic feasible solution. In these problems atleast one of the contraints is of = or ≥ type. To solve such linear programming problems, there are two (closely related) methods available.

(i) The ***"Big M–method"*** or ***"M–technique"*** or the ***"Method of penalties"*** due to A. Charnes.

(ii) The ***"Two phase"*** method due to Dantzig, Orden and Wolfe.

1.4.4 The Big M – method :

Step (1) : Express the linear programming problem in the standard form by introducing slack and/or surplus variables, if necessary.

Step (2) : Introduce the non-negative artificial variables $R_1, R_2 \ldots$ to the left hand side of all the constraints of ≥ or = type. The purpose of introducing artificial variables is just to obtain an initial basic feasible solution. However, addition of these artificial variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the final solution. To achieve this we assign a very large penalty (–M for maximization problems and +M for minimization problems) as the coefficients of the artificial variables in the objective function.

Step (3) : Solve the modified linear programming problem by simplex method.

While making iterations, using simplex method, one of the following three cases may arise :

(i) If no artificial variable remains in the basis and the optimality condition is satisfied, then the current solution is an optimal basic feasible solution.

(ii) If atleast one artificial variable appears in the basis at zero level (with zero value of X_B column) and the optimality condition is satisfied, then the current solution is an optimal basic feasible (though degenerated) solution .

(iii) If atleast one artificial variable appears in the basis at non–zero level (with positive value in X_B column) and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function since it contains a very large penalty M and is called ***pseudo–optimal solution.***

Note : While applying simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Example 1 : **Solve the following LPP by simplex method :**

$$\text{Maximize } Z = 3x_1 + 2x_2$$

$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 &\le 2 \\ 3x_1 + 4x_2 &\ge 12 \\ \text{and } x_1, x_2 &\ge 0. \end{aligned}$$

[MKU. M.Sc 1985, MU. BE. Apr 97]

Solution : By introducing the non-negative slack variable s_1 and surplus variable s_2, the standard form of the LPP becomes

$$\text{Maximize } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 + s_1 + 0s_2 &= 2 \\ 3x_1 + 4x_2 + 0s_1 - s_2 &= 12 \\ x_1, x_2, s_1, s_2 &\ge 0. \end{aligned}$$

But this will not yield a basic feasible solution. To get the basic feasible solution, add the artificial variable R_1 to the left hand side of the constraint equation which does not possess the slack variable and assign – M to the artificial variable in the objective function. The LPP becomes

$$\text{Max } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MR_1$$

$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 + s_1 + 0s_2 &= 2 \\ 3x_1 + 4x_2 + 0s_1 - s_2 + R_1 &= 12 \\ x_1, x_2, s_1, s_2, R_1 &\ge 0 \end{aligned}$$

The initial basic feasible solution is given by

$s_1 = 2, R_1 = 12$ (basic) $(x_1 = x_2 = s_2 = 0$, non-basic)

Initial iteration :

	C_j	(3	2	0	0	–M)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1	θ
0 s_1	2	2	(1)	1	0	0	$\frac{2}{1}=2$
–M R_1	12	3	4	0	–1	1	$\frac{12}{4}=3$
$Z_j - C_j$	–12M	–3M–3	–4M–2	0	M	0	

Since there are some $(Z_j - C_j) < 0$, The current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable s_1 leaves the basis.

First Iteration :

	C_j	(3	2	0	0	–M)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1
2 x_2	2	2	1	1	0	0
–M R_1	4	–5	0	–4	–1	1
$Z_j - C_j$	–4M+4	5M +1	0	4M + 2	M	0

Since all $(Z_j - C_j) \geq 0$, and an artificial variable R_1 appears in the basis at non-zero level, the given LPP does not possess any feasible solution. But the LPP possess a ***pseudo optimal solution.***

Example 2 : Solve the following problem by simplex method :

$$\text{Maximize } Z = x_1 + 2x_2 + 3x_3 - x_4$$

subject to

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 \geq 20$$
$$x_1 + 2x_2 + x_3 + x_4 \geq 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

[MU. BE. Apr 93]

Solution: By introducing the non-negative surplus variables s_1 and s_2, the standard form of the LPP becomes

$$\text{Maximize } Z = x_1 + 2x_2 + 3x_3 - x_4 + 0s_1 + 0s_2$$

subject to

$$x_1 + 2x_2 + 3x_3 + 0s_1 + 0s_2 = 15$$
$$2x_1 + x_2 + 5x_3 - s_1 + 0s_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 + 0s_1 - s_2 = 10$$
$$x_1, x_2, x_3, x_4, s_1, s_2 \geq 0.$$

But this will not yield a basic feasible solution. To get the basic feasible solution, add the artificial variables R_1, R_2, R_3, to the left hand side of the constraint equations which does not possess the slack variables and assign –M to the artificial variables in the objective function. So the LPP becomes

$$\text{Max } Z = x_1 + 2x_2 + 3x_3 - x_4 + 0s_1 + 0s_2 - MR_1 - MR_2 - MR_3$$

subject to the constraints

$$x_1 + 2x_2 + 3x_3 + 0s_1 + 0s_2 + R_1 = 15$$
$$2x_1 + x_2 + 5x_3 - s_1 + 0s_2 + R_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 + 0s_1 - s_2 + R_3 = 10$$
$$x_1, x_2, x_3, x_4, s_1, s_2, R_1, R_2, R_3 \geq 0.$$

The initial basic feasible solution is given by

$R_1 = 15$, $R_2 = 20$, $R_3 = 10$ (basic) ($x_1 = x_2 = x_3 = x_4 = s_1 = s_2 = 0$, non-basic)

Initial iteration :

	C_j	(1	2	3	–1	0	0	–M	–M	–M)	
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	R_1	R_2	R_3	θ
–M R_1	15	1	2	3	0	0	0	1	0	0	$\frac{15}{3}=5$
–M R_2	20	2	1	(5)	0	–1	0	0	1	0	$\frac{20}{5}=4$
–M R_3	10	1	2	1	1	0	–1	0	0	1	$\frac{10}{1}=10$
$Z_j - C_j$	–45M	–4M –1	–5M –2	–9M –3	–M +1	M	M	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_3 enters into the basis and the basic variable R_2 leaves the basis. Also since the artificial variable R_2 leaves the basis, we drop that artificial variable R_2 and omit all the entries corresponding to its column from the simplex table.

First iteration :

C_B	Y_B	C_j → X_B	x_1 (1)	x_2 (2)	x_3 (3)	x_4 (−1)	s_1 (0)	s_2 (0)	R_1 (−M)	R_3 (−M)	θ
−M	R_1	3	$\frac{-1}{5}$	$\left(\frac{7}{5}\right)$	0	0	$\frac{3}{5}$	0	1	0	$\frac{15}{7}$
3	x_3	4	$\frac{2}{5}$	$\frac{1}{5}$	1	0	$\frac{-1}{5}$	0	0	0	20
−M	R_3	6	$\frac{3}{5}$	$\frac{9}{5}$	0	1	$\frac{1}{5}$	−1	0	1	$\frac{30}{9}$
$Z_j - C_j$		−9 M +12	$\frac{-2M+1}{5}$	$\frac{-16M-7}{5}$	0	−M +1	$\frac{-4M-3}{5}$	M	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable R_1 leaves the basis.

Second iteration :

C_B	Y_B	C_j → X_B	x_1 (1)	x_2 (2)	x_3 (3)	x_4 (−1)	s_1 (0)	s_2 (0)	R_3 (−M)	θ
2	x_2	$\frac{15}{7}$	$\frac{-1}{7}$	1	0	0	$\frac{3}{7}$	0	0	–
3	x_3	$\frac{25}{7}$	$\frac{3}{7}$	0	1	0	$\frac{-2}{7}$	0	0	–
−M	R_3	$\frac{15}{7}$	$\frac{6}{7}$	0	0	(1)	$\frac{-4}{7}$	−1	1	$\frac{15}{7}$
$Z_j - C_j$		$\frac{-15M+105}{7}$	$\frac{-6M}{7}$	0	0	−M+1	$\frac{4M}{7}$	M	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_4 enters into the basis and the basic variable R_3 leaves the basis.

Third iteration :

C_B	Y_B	C_j → X_B	x_1 (1)	x_2 (2)	x_3 (3)	x_4 (−1)	s_1 (0)	s_2 (0)	θ
2	x_2	$\frac{15}{7}$	$\frac{-1}{7}$	1	0	0	$\frac{3}{7}$	0	–
3	x_3	$\frac{25}{7}$	$\frac{3}{7}$	0	1	0	$\frac{-2}{7}$	0	$\frac{25}{3}$
−1	x_4	$\frac{15}{7}$	$\left(\frac{6}{7}\right)$	0	0	1	$\frac{-4}{7}$	−1	$\frac{15}{6}$
$Z_j - C_j$		$\frac{90}{7}$	$\frac{-6}{7}$	0	0	0	$\frac{4}{7}$	1	

Since there are some $(Z_j - C_j) = \frac{-6}{7} < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters the basis and the basic variable x_4 leaves the basis.

Fourth iteration :

C_B	Y_B	C_j → X_B	x_1 (1)	x_2 (2)	x_3 (3)	x_4 (−1)	s_1 (0)	s_2 (0)
2	x_2	$\frac{5}{2}$	0	1	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{-1}{6}$
3	x_3	$\frac{5}{2}$	0	0	1	$\frac{-1}{2}$	0	$\frac{1}{2}$
1	x_1	$\frac{5}{2}$	1	0	0	$\frac{7}{6}$	$\frac{-2}{3}$	$\frac{-7}{6}$
$Z_j - C_j$		15	0	0	0	1	0	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is Max Z = 15, $x_1 = \frac{5}{2}$, $x_2 = \frac{5}{2}$, $x_3 = \frac{5}{2}$, $x_4 = 0$.

Example 3 : Use Big – M method to solve

$$\text{Minimize } Z = 4x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$\text{and } x_1, x_2 \geq 0.$$

[BRU M.Sc 1988]

Solution : Given Min $Z = 4x_1 + 3x_2$

subject to
$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0.$$

That is Max $Z^* = -4x_1 - 3x_2$

subject to
$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0.$$

By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z^* = -4x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 - MR_1 - MR_2$$

subject to
$$2x_1 + x_2 - s_1 + 0s_2 + 0s_3 + R_1 = 10$$
$$-3x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 6$$
$$x_1 + x_2 + 0s_1 + 0s_2 - s_3 + R_2 = 6$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3, R_1, R_2 \geq 0.$$

(Here : s_1, s_3 – surplus, s_2 – slack, R_1, R_2 – artificials)

The initial basic feasible solution is given by

$R_1 = 10$, $s_2 = 6$, $R_2 = 6$ (basic) ($x_1 = x_2 = s_1 = s_3 = 0$, non-basic)

Initial iteration :

	C_j	(–4	–3	0	0	0	–M	–M)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
–M R_1	10	(2)	1	–1	0	0	1	0	$\frac{10}{2} = 5$
0 s_2	6	–3	2	0	1	0	0	0	–
–M R_2	6	1	1	0	0	–1	0	1	$\frac{6}{1} = 6$
$Z_j^* - C_j$	–16M	–3M+4	–2M+3	M	0	M	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters into the basis and the basic variable R_1 leaves the basis.

First iteration :

	C_j	(–4	–3	0	0	0	–M)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_2	θ
–4 x_1	5	1	$^1/_2$	$^{-1}/_2$	0	0	0	10
0 s_2	21	0	$^7/_2$	$^{-3}/_2$	1	0	0	$\frac{42}{7}$
–M R_2	1	0	$(^1/_2)$	$^1/_2$	0	-1	1	2
$Z_j^* - C_j$	–M–20	0	$\frac{-M+2}{2}$	$\frac{-M+4}{2}$	0	M	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable R_2 leaves the basis.

Second iteration :

	C_j	(–4	–3	0	0	0)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3
–4 x_1	4	1	0	–1	0	1
0 s_2	14	0	0	–5	1	7
–3 x_2	2	0	1	1	0	–2
$Z_j^* - C_j$	–22	0	0	1	0	2

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ Max $Z^* = -22$, $x_1 = 4$, $x_2 = 2$

But Min $Z = -$ Max $(-Z) = -$Max $Z^* = -(-22) = 22$.

$\therefore$ The optimal solution is Min $Z = 22$, $x_1 = 4$, $x_2 = 2$

Example 4: Use Penalty method to

$$\text{Maximize } Z = 2x_1 + x_2 + x_3$$

subject to

$$4x_1 + 6x_2 + 3x_3 \le 8$$
$$3x_1 - 6x_2 - 4x_3 \le 1$$
$$2x_1 + 3x_2 - 5x_3 \ge 4$$
$$\text{and } x_1, x_2, x_3 \ge 0.$$ *[MU. BE. Apr 90]*

Solution: By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 2x_1 + x_2 + x_3 + 0s_1 + 0s_2 + 0s_3 - MR_1$$

subject to

$$4x_1 + 6x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 8$$
$$3x_1 - 6x_2 - 4x_3 + 0s_1 + s_2 + 0s_3 = 1$$
$$2x_1 + 3x_2 - 5x_3 + 0s_1 + 0s_2 - s_3 + R_1 = 4$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3, R_1 \ge 0.$$

(Here : s_1, s_2 – slack, s_3 – surplus, R_1 – artificial)

The initial basic feasible solution is given by

$s_1 = 8,\ s_2 = 1,\ R_1 = 4$ (basic) ($x_1 = x_2 = x_3 = s_3 = 0$, non-basic)

Initial iteration :

	C_j	(2	1	1	0	0	0	–M)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1	θ
0 s_1	8	4	6	3	1	0	0	0	$\frac{8}{6} = 1.3$
0 s_2	1	3	–6	–4	0	1	0	0	–
–M R_1	4	2	(3)	–5	0	0	–1	1	$\frac{4}{3} = 1.33$
$Z_j - C_j$	–4M	–2M–2	–3M–1	5M–1	0	0	M	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable R_1 leaves the basis.

First iteration :

	C_j	(2	1	1	0	0	0)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0 s_1	0	0	0	(13)	1	0	2	0
0 s_2	9	7	0	–14	0	1	–2	–
1 x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{-5}{3}$	0	0	$\frac{-1}{3}$	–
$Z_j - C_j$	$\frac{4}{3}$	$\frac{-4}{3}$	0	$\frac{-8}{3}$	0	0	$\frac{-1}{3}$	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_3 enters into the basis and the basic variable s_1 leaves the basis.

Second iteration :

	C_j	(2	1	1	0	0	0)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
1 x_3	0	0	0	1	$\frac{1}{13}$	0	$\frac{2}{13}$	–
0 s_2	9	(7)	0	0	$\frac{14}{13}$	1	$\frac{2}{13}$	$\frac{9}{7}$
1 x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	0	$\frac{5}{39}$	0	$\frac{-1}{13}$	2
$Z_j - C_j$	$\frac{4}{3}$	$\frac{-4}{3}$	0	0	$\frac{8}{39}$	0	$\frac{1}{13}$	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters into the basis and the basic variable s_2 leaves the basis.

Third iteration :

	C_j	(2	1	1	0	0	0)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
1 x_3	0	0	0	1	$\frac{1}{13}$	0	$\frac{2}{13}$	
2 x_1	$\frac{9}{7}$	1	0	0	$\frac{2}{13}$	$\frac{1}{7}$	$\frac{2}{91}$	
1 x_2	$\frac{10}{21}$	0	1	0	$\frac{1}{39}$	$\frac{-2}{21}$	$\frac{-25}{273}$	
$Z_j - C_j$	$\frac{64}{21}$	0	0	0	$\frac{16}{39}$	$\frac{4}{21}$	$\frac{+29}{273}$	

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = \frac{64}{21}$, $x_1 = \frac{9}{7}$, $x_2 = \frac{10}{21}$, $x_3 = 0$.

1.4.5 The Two Phase Method *[BNU. BE. Nov 98]*

The two phase method is another method to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase I : In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 : Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function. Thus the new objective function is $Z^* = -R_1 - R_2 - R_3 - \ldots -R_n$

where R_i's are the artificial variables.

Step 2 : Construct the auxiliary LPP in which the new objective function Z^* is to be maximized subject to the given set of constraints.

Step 3 : Solve the auxiliary LPP by simplex method until either of the following three possibilities arise.

(i) Max $Z^* < 0$ and atleast one artificial variable appears in the optimum basis at a non–zero level. In this case the given LPP does not possess any feasible solution, stop the procedure.

(ii) Max $Z^* = 0$ and atleast one artificial variable appears in the optimum basis at zero level. In this case proceed to phase – II.

(iii) Max $Z^* = 0$ and no artificial variable appears in the optimum basis. In this case proceed to phase – II.

Phase II : Use the optimum basic feasible solution of Phase – I as a starting solution for the original LPP. Assign the actual costs to the variables in the objective function and a 0 cost to every artificial variable that appears in the basis at the zero level. Use simplex method to the modified simplex table obtained at the end of Phase – I, till an optimum basic feasible solution (if any) is obtained.

Note 1 : In Phase – I, the iterations are stopped as soon as the value of the new objective function becomes zero because this is its maximum value. There is no need to continue till the optimality is reached if this value becomes zero earlier than that.

Note 2 : The new objective function is always of maximization type regardless of whether the original problem is of maximization or minimization type.

Note 3 : Before starting phase – II, remove all artificial variables from the table which were non-basic at the end of phase – I.

Example 1 : **Use Two–phase simplex method to solve**

$$\textbf{Maximize } Z = 5x_1 + 8x_2$$

subject to the constraints

$$\begin{aligned} 3x_1 + 2x_2 &\geq 3 \\ x_1 + 4x_2 &\geq 4 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution: By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 - s_1 + 0s_2 + 0s_3 + R_1 &= 3 \\ x_1 + 4x_2 + 0s_1 - s_2 + 0s_3 + R_2 &= 4 \\ x_1 + x_2 + 0s_1 + 0s_2 + s_3 &= 5 \\ \text{and } x_1, x_2, s_1, s_2, s_3, R_1, R_2 &\geq 0. \end{aligned}$$

(***Here :*** s_1, s_2 – surplus, s_3 – slack, R_1, R_2 – artificials)

The initial basic feasible solution is given by

$R_1 = 3$, $R_2 = 4$, $s_3 = 5$ (basic) ($x_1 = x_2 = s_1 = s_2 = 0$, non-basic)

Phase–I : Assigning a cost -1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } Z^* = -R_1 - R_2$$

subject to the given constraints.

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	0	−1	−1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
−1 R_1	3	3	2	−1	0	0	1	0	$\frac{3}{2}$
−1 R_2	4	1	(4)	0	−1	0	0	1	$\frac{4}{4}$
0 s_3	5	1	1	0	0	1	0	0	$\frac{5}{1}$
$Z_j^* - C_j$	−7	−4	−6	1	1	0	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_2 and drop R_2.

	C_j	(0	0	0	0	0	−1	−1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
−1 R_1	1	$\left(\frac{5}{2}\right)$	0	−1	$\frac{1}{2}$	0	1	$\frac{-1}{2}$	$\frac{2}{5}$
0 x_2	1	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	0	0	$\frac{1}{4}$	4
0 s_3	4	$\frac{3}{4}$	0	0	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	$\frac{16}{3}$
$Z_j^* - C_j$	−1	$\frac{-5}{2}$	0	1	$\frac{-1}{2}$	0	0	$\frac{3}{2}$	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration : Introduce x_1 and drop R_1.

	C_j	(0	0	0	0	0	−1	−1)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2
0 x_1	$\frac{2}{5}$	1	0	$\frac{-2}{5}$	$\frac{1}{5}$	0	$\frac{2}{5}$	$\frac{-1}{5}$
0 x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$\frac{-3}{10}$	0	$\frac{-1}{10}$	$\frac{3}{10}$
0 s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	$\frac{-3}{10}$	$\frac{-1}{10}$
$Z_j^* - C_j$	0	0	0	0	0	0	1	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimum. Furthermore, no artificial variable appears in the optimum basis so we proceed to phase –II.

Phase – II :

Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are :

Initial iteration :

	C_j	(5	8	0	0	0	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	θ
5 x_1	$\frac{2}{5}$	1	0	$\frac{-2}{5}$	$\left(\frac{1}{5}\right)$	0	2
8 x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$\frac{-3}{10}$	0	–
0 s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	37
$(Z_j - C_j)$	$\frac{46}{5}$	0	0	$\frac{-6}{5}$	$\frac{-7}{5}$	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce s_2 and drop x_1

	C_j	(5	8	0	0	0)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	θ
0 s_2	2	5	0	−2	1	0	–
8 x_2	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{-1}{2}$	0	0	–
0 s_3	$\frac{7}{2}$	$\frac{-1}{2}$	0	$\left(\frac{1}{2}\right)$	0	1	7
$(Z_j - C_j)$	12	7	0	−4	0	0	

Since there are some $(Z_j - C_j) < 0$, current basic feasible solution is not optimal.

Second iteration : Introduce s_1 and drop s_3

	C_j	(5	8	0	0	0)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0 s_2	16	3	0	0	1	4
8 x_2	5	1	1	0	0	1
0 s_1	7	–1	0	1	0	2
$(Z_J - C_J)$	40	3	0	0	0	8

Since all $(Z_J - C_J) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $z = 40$, $x_1 = 0$, $x_2 = 5$.

Example 2 : Solve by two phase simplex method

$$\textbf{Maximize } X_0 = -4x_1 - 3x_2 - 9x_3$$

Subject to

$$2x_1 + 4x_2 + 6x_3 - s_1 + R_1 = 15$$
$$6x_1 + x_2 + 6x_3 - s_2 + R_2 = 12$$
$$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0.$$

[MU. BE. Nov 92]

Solution : The initial basic feasible solution is given by $R_1 = 15$, $R_2 = 12$, (basic) ($x_1 = x_2 = x_3 = s_1 = s_2 = 0$, non-basic)

Phase–I : Assigning a cost –1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } Z^* = -R_1 - R_2$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	0	–1	–1)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2	θ
–1 R_1	15	2	4	6	–1	0	1	0	$\frac{15}{6}$
–1 R_2	12	6	1	(6)	0	–1	0	1	$\frac{12}{6}$
$Z_J^* - C_J$	–27	–8	–5	–12	1	1	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_3 and drop R_2.

	C_j	(0	0	0	0	0	–1	–1)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2	θ
–1 R_1	3	–4	(3)	0	–1	1	1	–1	$\frac{3}{3}$
0 x_3	2	1	$\frac{1}{6}$	1	0	$\frac{-1}{6}$	0	$\frac{1}{6}$	12
$Z_j^* - C_j$	–3	4	–3	0	1	–1	0	2	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration : Introduce x_2 and drop R_1.

	C_j	(0	0	0	0	0	–1	–1
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2
0 x_2	1	$\frac{-4}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$
0 x_3	$\frac{11}{6}$	$\frac{22}{18}$	0	1	$\frac{1}{18}$	$\frac{-4}{18}$	$\frac{-1}{18}$	$\frac{4}{18}$
$Z_J^* - C_J$	0	0	0	0	0	0	1	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal. Further, no artificial variable appears in the basis, so we proceed to phase – II.

Phase – II :

Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } X_0 = -4x_1 - 3x_2 - 9x_3 + 0s_1 + 0s_2$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are :

Initial iteration :

C_B	Y_B	C_j → X_B	x_1 (−4)	x_2 (−3)	x_3 (−9)	s_1 (0)	s_2 (0)	θ
−3	x_2	1	$\frac{-4}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	–
−9	x_3	$\frac{11}{6}$	$\left(\frac{22}{18}\right)$	0	1	$\frac{1}{18}$	$\frac{-4}{18}$	$\frac{3}{2}$
$(X_0 - C_j)$		$\frac{-39}{2}$	−3	0	0	$\frac{1}{2}$	1	

Since there are some $(X_0 - C_J) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_1 and drop x_3

C_B	Y_B	C_j → X_B	x_1 (−4)	x_2 (−3)	x_3 (−9)	s_1 (0)	s_2 (0)
−3	x_2	3	0	1	$\frac{12}{11}$	$\frac{-3}{11}$	$\frac{1}{11}$
−4	x_1	$\frac{3}{2}$	1	0	$\frac{18}{22}$	$\frac{1}{22}$	$\frac{-4}{22}$
$(X_0 - C_j)$		−15	0	0	$\frac{27}{11}$	$\frac{7}{11}$	$\frac{5}{11}$

Since all $(X_0 - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $X_0 = -15, x_1 = \frac{3}{2}, x_2 = 3, x_3 = 0$

Example 3 : Use two–phase method to

$$\text{Maximize } Z = 2x_1 + x_2 + \frac{1}{4}x_3$$

subject to constraints

$$4x_1 + 6x_2 + 3x_3 \leq 8$$
$$3x_1 - 6x_2 - 4x_3 \leq 1$$
$$2x_1 + 3x_2 - 5x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0.$$

[MKU.B.Sc.1988]

Solution : By introducing slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 2x_1 + x_2 + \frac{1}{4}x_3$$

subject to

$$4x_1 + 6x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 8$$
$$3x_1 - 6x_2 - 4x_3 + 0s_1 + s_2 + 0s_3 = 1$$
$$2x_1 + 3x_2 - 5x_3 + 0s_1 + 0s_2 - s_3 + R_1 = 4$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0$$

(Here : s_1, s_2 – slack, s_3 – surplus, R_1 – artificial)

The initial basic feasible solution is given by

$s_1 = 8,\ s_2 = 1,\ R_1 = 4$ (basic) $(x_1 = x_2 = x_3 = s_3 = 0$, non-basic)

Phase – I : The objective function of the auxiliary LPP is

$$\text{Max } Z^* = -R_1$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

C_B	Y_B	C_j → X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	s_3 (0)	R_1 (−1)	θ
0	s_1	8	4	(6)	3	1	0	0	0	$\frac{8}{6}$
0	s_2	1	3	−6	−4	0	1	0	0	–
−1	R_1	4	2	3	−5	0	0	−1	1	$\frac{4}{3}$
$Z_j^* - C_j$		−4	−2	−3	5	0	0	1	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_2 and drop s_1.

C_B	Y_B	C_j → X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	s_3 (0)	R_1 (−1)
0	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0
0	s_2	9	7	0	−1	1	1	0	0
−1	R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	−1	1
$Z_j^* - C_j$		0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	0	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal for the auxiliary LPP.

But at the same time the artificial variable R_1 appears in the optimum basis at the zero level. This optimal solution may or may not be optimal to the given (original) LPP. So we proceed to phase – II.

Phase–II : Here, we consider the actual costs associated with the original variables and assign a cost 0 to the artificial variable R_1, which appeared at zero level in phase – I, in the objective function. The new objective function then becomes

$$\text{Max } Z = 2x_1 + x_2 + \tfrac{1}{4}x_3 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are :

Initial iteration :

	C_j	(2	1	$\frac{1}{4}$	0	0	0	0)	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1	θ
1 x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0	$\frac{4}{2}$
0 s_2	9	(7)	0	–1	1	1	0	0	$\frac{9}{7}$
0 R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	–1	1	–
$(Z_j - C_j)$	$\frac{4}{3}$	$\frac{-4}{3}$	0	$\frac{1}{4}$	$\frac{1}{6}$	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_1 and drop s_2

	C_j	(2	1	$\frac{1}{4}$	0	0	0	0)
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1
1 x_2	$\frac{10}{21}$	0	1	$\frac{25}{42}$	$\frac{1}{14}$	$\frac{-2}{21}$	0	0
2 x_1	$\frac{9}{7}$	1	0	$\frac{-1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	0	0
0 R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	–1	1
$(Z_j - C_j)$	$\frac{64}{21}$	0	0	$\frac{5}{84}$	$\frac{5}{14}$	$\frac{4}{21}$	0	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = \frac{64}{21}, x_1 = \frac{9}{7}, x_2 = \frac{10}{21}, x_3 = 0.$

Example 4 : Use two phase simplex method to

$$\textbf{Maximize } Z = 5x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \leq 1$$
$$x_1 + 4x_2 \geq 6$$
$$x_1, x_2 \geq 0.$$

Solution : By introducing slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 5x_1 + 3x_2$$

subject to

$$2x_1 + x_2 + s_1 + 0s_2 = 1$$
$$x_1 + 4x_2 + 0s_1 - s_2 + R_1 = 6$$
$$\text{and } x_1, x_2, s_1, s_2, R_1 \geq 0.$$

(***Here :*** s_1 - slack, s_2 –surplus , R_1 – artificial)

The initial basic feasible solution is given by $s_1 = 1$, $R_1 = 6$ (basic) $(x_1 = x_2 = s_2 = 0$, non-basic)

Phase – I : The objective function of the auxiliary LPP is

$$\text{Max } Z^* = -R_1$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	–1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1	θ
0 s_1	1	2	(1)	1	0	0	1
–1 R_1	6	1	4	0	–1	1	$\frac{6}{4}$
$Z_j^* - C_j$	–6	–1	–4	0	1	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_2 and drop s_1.

	C_j	(0	0	0	0	–1)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1
0 x_2	1	2	1	1	0	0
–1 R_1	2	–7	0	–4	–1	1
$(Z_j^* - C_j)$	–2	7	0	4	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal to the auxiliary LPP

But since, Max $Z^* < 0$ and one artificial variable R_1 appears in the optimum basis at non–zero level, the given (original) LPP has **no feasible solution.**

Example 5 : Using simplex algorithm.

Minimize $-2x_1 - x_2$ subject to the constraints

$$x_1 + x_2 \geq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

[MU. BE. Oct 95]

Solution : Let $Z = -2x_1 - x_2$

$$\therefore \text{Min } Z = -2x_1 - x_2$$

subject to
$$x_1 + x_2 \geq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

That is Max $Z^* = 2x_1 + x_2$

subject to
$$x_1 + x_2 \geq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

By introducing slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z^* = 2x_1 + x_2$$

subject to
$$x_1 + x_2 - s_1 + 0s_2 + R_1 = 2$$
$$x_1 + x_2 + 0s_1 + s_2 = 4$$
$$x_1, x_2, s_1, s_2, R_1 \geq 0$$

(Here : s_1 - surplus, s_2 –slack , R_1 – artificial)

The initial basic feasible solution is given by $R_1 = 2$, $s_2 = 4$ (basic) ($x_1 = x_2 = s_1 = 0$, non-basic)

Phase – I : The objective function of the auxiliary LPP is

$$\text{Max } Z^* = -R_1$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	–1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1	θ
–1 R_1	2	(1)	1	–1	0	1	$\frac{2}{1}$
0 s_2	4	1	1	0	1	0	$\frac{4}{1}$
$(Z_j^* - C_j)$	–2	–1	–1	1	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_1 and drop R_1.

	C_j	(0	0	0	0	–1)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1
0 x_1	2	1	1	–1	0	1
0 s_2	2	0	0	1	1	–1
$(Z_j^* - C_j)$	0	0	0	0	0	1

Since all $(Z_j^* - C_j) \geq 0$, and no artificial variable appears in the optimum basis, the current basic feasible solution is optimal to the auxiliary LPP and we proceed to Phase – II.

Phase – II : Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } Z^* = 2x_1 + x_2 + 0s_1 + 0s_2$$

The initial basic feasible solution for this phase is the one obtained at the end of phase – I. The iterative simplex tables for this phase are :

Initial iteration :

	C_j	(2	1	0	0)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	θ
2 x_1	2	1	1	–1	0	–
0 s_2	2	0	0	(1)	1	$\frac{2}{1}$
$(Z_j^* - C_j)$	4	0	1	–2	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce s_1 and drop s_2.

		C_j	(2	1	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
2	x_1	4	1	1	0	1
0	s_1	2	0	0	1	1
$(Z_j^* - C_j)$		8	0	1	0	2

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is max $Z^* = 8, \quad x_1 = 4, \quad x_2 = 0$

But $\text{Min } Z = -\text{Max}(-Z) = -\text{Max } Z^* = -8$

$\therefore \text{Min } Z = -8, \quad x_1 = 4, \quad x_2 = 0.$

Disadvantage of Big–M method over Two – phase method :

Even though Big–M method can always be used to check the existence of a feasible solution, it may be computationally inconvenient especially when a digital computer is used because of the manipulation of the constant M. On the other hand, Two-phase method eliminates the constant M from calculations.

EXERCISE

1. **Explain briefly the term "Artificial" variables.**
[MU. BE. 79, MU. MBA. Nov 96, Apr 97]
2. **Explain the use of artificial variables in LPP.**
3. **Describe briefly the Big–M method of solving a LPP with artificial variables.** *[MU. MCA. Nov 98]*
4. **Describe briefly the Two–phase method of solving a LPP with artificial variables.**
5. **Explain the disadvantage of Big–M method over Two –phase method.**
6. **Using simplex method, solve :**

$$\text{Maximize } Z = 5x_1 - 2x_2 + 3x_3$$
$$\text{subject to} \quad \begin{aligned} 2x_1 + 2x_2 - x_3 &\geq 2 \\ 3x_1 - 4x_2 &\leq 3 \\ x_2 + 3x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

[MU. BE. Nov 89, Apr 94, Apr 95]

7. **Solve by simplex method,**

$$\text{Max } Z = x_1 + 2x_2 + 3x_3 - x_4$$
$$\text{subject to} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 15 \\ 2x_1 + x_2 + 5x_3 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

[MU. BE. Apr 91, Annamalai M.Sc.82]

8. **Solve the following LPP**

$$\text{Min } Z = 12x_1 + 20x_2$$
$$\text{subject to} \quad \begin{aligned} 6x_1 + 8x_2 &\geq 100 \\ 7x_1 + 12x_2 &\geq 120 \\ x_1, x_2 &\geq 0. \end{aligned}$$

[BRU. B.Sc 90]

9. **Solve the following LPP Minimize $Z = 4x_1 + 2x_2$**

$$\text{subject to} \quad \begin{aligned} 3x_1 + x_2 &\geq 27 \\ x_1 + x_2 &\geq 21 \\ x_1 + 2x_2 &\geq 30 \\ x_1, x_2 &\geq 0. \end{aligned}$$

10. **Solve the following LPP : Minimize $Z = 2x_1 + 3x_2$**

$$\text{subject to} \quad \begin{aligned} x_1 + x_2 &\geq 5 \\ x_1 + 2x_2 &\geq 6 \\ x_1, x_2 &\geq 0. \end{aligned}$$

11. **Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily minimum requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the minimum cost of product mix by simplex method.**

12. **Sovle the following LPP :**

$$\text{Maximize } Z = x_1 + 1.5x_2 + 2x_3 + 5x_4$$

subject to

$$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$$
$$2x_1 + x_2 + x_3 + 5x_4 \leq 4$$
$$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$$
$$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

13. **A company possesses two manufacturing plants, each of which can produce three products A,B,C from common raw material. However, the proportions in which the products are produced are different in each plant and so are the plant's operating cost per hour. Data on production per hour and costs are given below, together with current orders in hand for each product :**

	Product			Operating cost
	X	Y	Z	per hour (Rs)
Plant I	2	4	3	9
Plant II	4	3	2	10
Order on hand	50	24	60	

You are required to use simplex method to find the number of production hours needed to fulfil the orders on hand at a minimum cost.

ANSWERS

6. Max $Z = \frac{85}{3}$, $x_1 = \frac{23}{3}$, $x_2 = 5$, $x_3 = 0$

7. Max $Z = 15$, $x_1 = \frac{5}{2}$, $x_2 = \frac{5}{2}$, $x_3 = \frac{5}{2}$, $x_4 = 0$

8. Min $Z = 205$, $x_1 = 15$, $x_2 = \frac{5}{4}$

9. Min $Z = 48$, $x_1 = 3$, $x_2 = 18$,

10. Min $Z = 11$, $x_1 = 4$, $x_2 = 1$

11. Minimize $Z = 12x_1 + 20x_2$

subject to

$$6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$\text{and } x_1, x_2 \geq 0$$

Also, Min $Z = 205$, $x_1 = 15$, $x_2 = \frac{5}{4}$

12. Max $Z = 3.6$, $x_1 = 0$, $x_2 = 1.2$, $x_3 = 0.9$, $x_4 = 0$

13. Min $Z = 9x_1 + 10x_2$

subject to

$$2x_1 + 4x_2 \geq 50$$
$$4x_1 + 3x_2 \geq 24$$
$$3x_1 + 2x_2 \geq 60$$
$$x_1, x_2 \geq 0.$$

Also, Min $Z = 195$, $x_1 = \frac{35}{2}$, $x_2 = \frac{15}{4}$.

1.4.6 Variants of the Simplex Method

Here we present certain complications and variations encountered in the application of the simplex method and how they are resolved. These are called the variants of the simplex method. The following variants are being considered.

1. Degeneracy and cycling
2. Unbounded solution
3. Multiple solutions
4. Non-existing feasible solution
5. Unrestricted variables.

1. Degeneracy and Cycling : The concept of obtaining a degenerate basic feasible solution in a LPP is known as ***Degeneracy.*** Degeneracy in a LPP may arise

(i) at the initial stage when atleast one basic variable is zero in the initial basic feasible solution.

(ii) at any subsequent iteration when more than one basic variable is eligible to leave the basis and hence one or more variables becoming zero in the next iteration and the problem is said to degenerate. There is no assurance that the value of the objective function will improve, since the new solutions may remain degenerate. As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solutions. This concept is known as ***cycling*** or ***circling.***

Perturbation rule to avoid cycling :

(a) Divide each element in the tied rows by the positive co-efficients of the key column in that row.

(b) Compare the resulting ratios, column by column, first in the ***unit martix*** and then in the ***body matrix*** from left to right.

(c) The row which first contains the smallest algebraic ratio contains the leaving variable.

Example 1 : Solve the following LPP by simplex method :

$$\text{Maximize F} = x_1 + 2x_2 + x_3$$

subject to
$$2x_1 + x_2 - x_3 \leq 2$$
$$-2x_1 + x_2 - 5x_3 \geq -6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0.$$
[MU. MCA. Nov 94]

Solution : Given

$$\text{Maximize F} = x_1 + 2x_2 + x_3$$

subject to
$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0.$$

By introducing the non-negative slack variables s_1, s_2, s_3 the standard form of LPP becomes

$$\text{Maximize F} = x_1 + 2x_2 + x_3$$

subject to
$$2x_1 + x_2 - x_3 + s_1 + 0s_2 + 0s_3 = 2$$
$$2x_1 - x_2 + 5x_3 + 0s_1 + s_2 + 0s_3 = 6$$
$$4x_1 + x_2 + x_3 + 0s_1 + 0s_2 + s_3 = 6$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

The initial basic feasible solution is given by

$s_1 = 2, s_2 = 6, s_3 = 6$ (basic) $(x_1 = x_2 = x_3 = 0$, non-basic)

Initial iteration :

		C_j	1	2	1	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	2	2	(1)	–1	1	0	0	$\frac{2}{1} = 1^*$
0	s_2	6	2	–1	5	0	1	0	–
0	s_3	6	4	1	1	0	0	1	$\frac{6}{1} = 6$
$(F_j - C_j)$		0	–1	–2	–1	0	0	0	

Since there are some $(F_j - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_2 and drop s_1.

		C_j	1	2	1	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
2	x_2	2	2	1	–1	1	0	0	–
0	s_2	8	4	0	4	1	1	0	$\frac{8}{4} = 2$
0	s_3	4	2	0	(2)	–1	0	1	$\frac{4}{2} = 2^*$
$(F_j - C_j)$		4	3	0	–3	2	0	0	

Since there are some $(F_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_3 enters in to the basis. Since both the basic variables s_2 and s_3 having the same minimum ratio 2, there is a tie in selecting the leaving variable. To resolve this degeneracy, we divide each entry corresponding to basic variables s_2 and s_3 and then corresponding to non-basic variables x_1, x_2, x_3 and s_1. We get the following quotients.

	s_2	s_3	x_1	x_2	x_3	s_1
Row 2:	$\frac{1}{4}$	$\frac{0}{4}$	$\frac{4}{4}$	$\frac{0}{4}$	$\frac{4}{4}$	$\frac{1}{4}$
Row 3:	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{-1}{2}$

The columnwise comparison of quotients starting with basic variables s_2 and s_3, we find that column s_2 gives algebraically smaller ratio : 0, for Row 3 and as such Row 3 is selected as key row. So the basic variable s_3 leaves the basis.

Second iteration : Introduce x_3 and drop s_3.

		C_j	(1	2	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	4	3	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
0	s_2	0	0	0	0	3	1	–2
1	x_3	2	1	0	1	$\frac{-1}{2}$	0	$\frac{1}{2}$
$(F_j - C_j)$		10	6	0	0	$\frac{1}{2}$	0	$\frac{3}{2}$

Since all $(F_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max F = 10, $x_1 = 0, x_2 = 4, x_3 = 2$.

$\boxed{\textbf{Example 2}}$ **: Solve the LPP Maximize $Z = 5x_1 - 2x_2 + 3x_3$**

subject to

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &\geq 2 \\ 3x_1 - 4x_2 &\leq 3 \\ x_2 + 3x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

[MKU. BSc. 1983]

Solution: By introducing the non-negative surplus variable s_1, slack variables s_2, s_3 and an artificial variable R_1, the standard form of the LPP becomes

$$\text{Maximize } Z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - MR_1$$

$$\begin{aligned} \text{subject to} \quad 2x_1 + 2x_2 - x_3 - s_1 + 0s_2 + 0s_3 + R_1 &= 2 \\ 3x_1 - 4x_2 + 0x_3 + 0s_1 + s_2 + 0s_3 &= 3 \\ 0x_1 + x_2 + 3x_3 + 0s_1 + 0s_2 + s_3 &= 5 \\ \text{and } x_1, x_2, x_3, s_1, s_2, s_3, R_1 &\geq 0. \end{aligned}$$

The initial basic feasible solution is given by

$R_1 = 2, s_2 = 3, s_3 = 5$ (basic) $(x_1 = x_2 = x_3 = s_1 = 0$, non-basic)

Initial iteration :

		C_j	(5	–2	3	0	–M	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	R_1	s_2	s_3	θ
–M	R_1	2	(2)	2	–1	–1	1	0	0	$\frac{2}{2}$ *
0	s_2	3	3	–4	0	0	0	1	0	$\frac{3}{3}$
0	s_3	5	0	1	3	0	0	0	1	–
$(Z_j - C_j)$		–2 M	–2M –5	–2M +2	M –3	M	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal. The non-basic variable x_1 enters into the basis.

Since there is a tie in selecting the leaving variable among R_1 and s_2, it is an indication of the existence of degeneracy. But since R_1 is an artificial basic variable, we select R_1 as the leaving variable.

First iteration : Introduce x_1 and drop R_1.

		C_j	(5	–2	3	0	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
5	x_1	1	1	1	$\frac{-1}{2}$	$\frac{-1}{2}$	0	0	–
0	s_2	0	0	–7	$\left(\frac{3}{2}\right)$	$\frac{3}{2}$	1	0	0*
0	s_3	5	0	1	3	0	0	1	$\frac{5}{3}$
$(Z_j - C_j)$		5	0	7	$\frac{-11}{2}$	$\frac{-5}{2}$	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration : Introduce x_3 and drop s_2.

C_B	Y_B	X_B	x_1 (5)	x_2 (−2)	x_3 (3)	s_1 (0)	s_2 (0)	s_3 (0)	θ
5	x_1	1	1	$\frac{-4}{3}$	0	0	$\frac{1}{3}$	0	–
3	x_3	0	0	$\frac{-14}{3}$	1	1	$\frac{2}{3}$	0	–
0	s_3	5	0	(15)	0	−3	−2	1	$\frac{5}{15}$
$(Z_j - C_j)$		5	0	$\frac{-56}{3}$	0	3	$\frac{11}{3}$	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

Third iteration : Introduce x_2 and drop s_3.

C_B	Y_B	X_B	x_1 (5)	x_2 (−2)	x_3 (3)	s_1 (0)	s_2 (0)	s_3 (0)	θ
5	x_1	$\frac{13}{9}$	1	0	0	$\frac{-4}{15}$	$\frac{7}{45}$	$\frac{4}{45}$	–
3	x_3	$\frac{14}{9}$	0	0	1	$\left(\frac{1}{15}\right)$	$\frac{2}{45}$	$\frac{14}{45}$	$\frac{14}{9} \times \frac{15}{1}$ *
−2	x_2	$\frac{1}{3}$	0	1	0	$\frac{-1}{5}$	$\frac{-2}{15}$	$\frac{1}{15}$	–
$(Z_j - C_j)$		$\frac{101}{9}$	0	0	0	$\frac{-11}{15}$	$\frac{53}{45}$	$\frac{56}{45}$	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

Fourth iteration : Introduce s_1 and drop x_3.

C_B	Y_B	X_B	x_1 (5)	x_2 (−2)	x_3 (3)	s_1 (0)	s_2 (0)	s_3 (0)
5	x_1	$\frac{23}{3}$	1	0	4	0	$\frac{1}{3}$	$\frac{4}{3}$
0	s_1	$\frac{70}{3}$	0	0	15	1	$\frac{2}{3}$	$\frac{14}{3}$
−2	x_2	5	0	1	3	0	0	1
$(Z_j - C_j)$		$\frac{85}{3}$	0	0	11	0	$\frac{5}{3}$	$\frac{14}{3}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = \frac{85}{3}, x_1 = \frac{23}{3}, \; x_2 = 5, x_3 = 0.$

1.4.7 Unbounded solution

In some linear programming problems, the solution space becomes unbounded so that the value of the objective function also can be increased indefinitely without a limit. But it is worng to conclude that just because the solution space is unbounded the solution also is unbounded. The solution space may be unbounded but the solution may be finite.

Example 1 : (Unbounded solution space but bounded optimal solution)

$$\text{Max } Z = 3x_1 - x_2$$

subject to

$$x_1 - x_2 \leq 10$$
$$x_1 \leq 20$$
$$\text{and } x_1, x_2 \geq 0.$$

Solution : By introducing the slack variables s_1, s_2, the standard form of the LPP becomes

$$\text{Max } Z = 3x_1 - x_2 + 0s_1 + 0s_2$$

subject to

$$x_1 - x_2 + s_1 + 0s_2 = 10$$
$$x_1 + 0x_2 + 0s_1 + s_2 = 20$$
$$x_1, x_2, s_1, s_2 \geq 0.$$

The initial basic feasible solution is given by

$s_1 = 10, \; s_2 = 20$ (basic) $(x_1 = x_2 = 0$, non-basic)

Initial iteration :

C_B	Y_B	X_B	x_1 (3)	x_2 (−1)	s_1 (0)	s_2 (0)	θ
0	s_1	10	(1)	−1	1	0	$\frac{10}{1}$ *
0	s_2	20	1	0	0	1	$\frac{20}{1}$
$(Z_j - C_j)$		0	−3	1	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_1 and drop s_1.

	C_j	3	–1	0	0	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	θ
3 x_1	10	1	–1	1	0	–
0 s_2	10	0	(1)	–1	1	$\frac{10}{1}$*
$(Z_j–C_j)$	30	0	–2	3	0	

Since $(Z_2 - C_2) = -2 < 0$, the current basic feasible solution is not optimal.

Second iteration : Introduce x_2 and drop s_2.

	C_j	3	–1	0	0
C_B Y_B	X_B	x_1	x_2	s_1	s_2
3 x_1	20	1	0	0	1
–1 x_2	10	0	1	–1	1
$(Z_j - C_j)$	50	0	0	1	2

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is Max Z = 50, $x_1 = 20$, $x_2 = 10$.

The above problem is represented graphically in the following figure to indicate that there is a bounded optimal solution even though the solution space is unbounded.

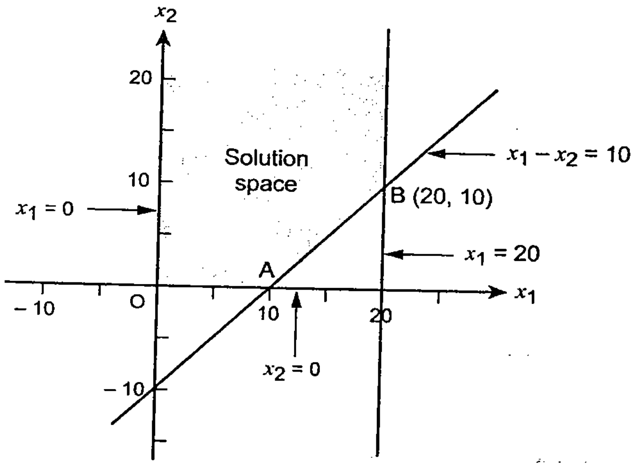


Fig. 3.1

Here the solution space is unbounded above, but the optimal solution occurs at the vertex (20,10).

Example 2 : Solve the following LPP by simplex method.

$$\text{Max } Z = 2x_1 + x_2$$

subject to the constraints

$$x_1 - x_2 \leq 10$$
$$2x_1 - x_2 \leq 40$$
$$\text{and } x_1, x_2 \geq 0$$

Solution : By introducing the non-negative slack variables s_1 and s_2, the standard form of the LPP becomes

$$\text{Max } Z = 2x_1 + x_2 + 0s_1 + 0s_2$$

subject to

$$x_1 - x_2 + s_1 + 0s_2 = 10$$
$$2x_1 - x_2 + 0s_1 + s_2 = 40$$
$$\text{and } x_1, x_2, s_1, s_2 \geq 0.$$

The initial basic feasible solution is given by

$s_1 = 10$, $s_2 = 40$ (basic) ($x_1 = x_2 = 0$, non-basic)

Initial iteration :

	C_j	2	1	0	0	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	θ
0 s_1	10	(1)	–1	1	0	$\frac{10}{1}$*
0 s_2	40	2	–1	0	1	$\frac{40}{2}$
$(Z_j - C_j)$	0	–2	–1	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters in to the basis and the basic variable s_1 leaves the basis.

First iteration :

	C_j	2	1	0	0	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	θ
2 x_1	10	1	–1	1	0	–
0 s_2	20	0	(1)	–2	1	$\frac{20}{1}$*
$(Z_j - C_j)$	20	0	–3	2	0	

Since $(Z_2 - C_2) = -3 < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters in to the basis and the basic variable s_2 leaves the basis.

Second iteration :

	C_j	(2	1	0	0)
C_B Y_B	X_B	x_1	x_2	s_1	s_2
2 x_1	30	1	0	−1	1
1 x_2	20	0	1	−2	1
$(Z_j - C_j)$	80	0	0	−4	3

Since $(Z_3 - C_3) = -4 < 0$, the current basic feasible solution is not optimal.

Also, since all $a_{ir} < 0$, it is not possible to find the positive ratio $\theta = \min\left\{\frac{X_{Bi}}{a_{ir}}, a_{ir} > 0\right\}$ *i.e.,* it is not possible to find the leaving variable.

The solution of this problem is ***unbounded.***

The above problem is represented graphically in the following figure to indicate that there is a unbounded optimal solution with unbounded solution space.

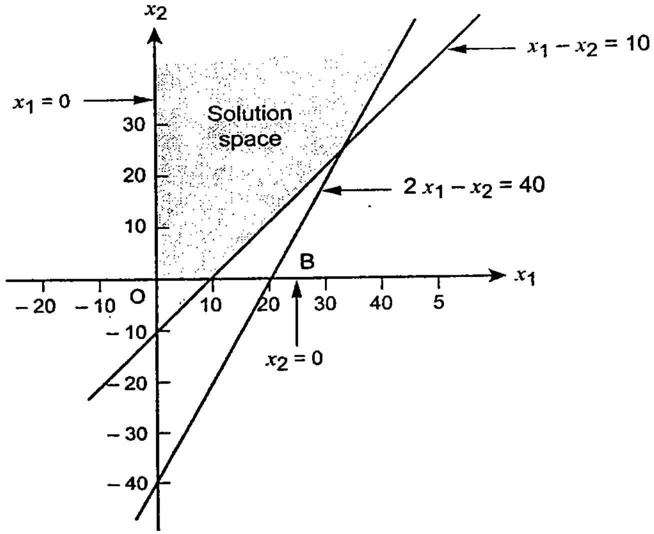


Fig. 3.2

Here the solution space is unbounded and the optimal solution is also unbounded.

1.4.8. Multiple solutions (or) Alternate optimal solutions

In some linear programming problems, the optimal solution need not be unique. There may be alternative or infinite number of solutions, *i.e.,* with different product mixes, we have the same value of the objective function.

While dealing with the graphical method, if the optimal solution occurs at a vertex of the solution space, then the problem is said to have a unique optimal solution. On the other hand, if the optimum solution occurs on an edge of the solution space, then the problem is said to have an alternative or infinite number of solutions.

While dealing with the simplex method, in the optimum simplex table, if the net evaluation $(Z_j - C_j) \neq 0$ for all non-basic variables, then the problem is said to have a unique optimal solution. On the otherhand, if the net evaluation $(Z_j - C_j) = 0$ for atleast one non-basic variable, then the problem is said to have an alternative or infinite number of solutions.

Example 1 **: Solve the LPP**

Maximize $Z = 6x_1 + 4x$ subject to the constraints

$$
\begin{aligned}
2x_1 + 3x_2 &\leq 30 \\
3x_1 + 2x_2 &\leq 24 \\
x_1 + x_2 &\geq 3 \\
\text{and } x_1, x_2 &\geq 0.
\end{aligned}
$$

Solution : By introducing the non-negative slack variables s_1, s_2, surplus variable s_3 and an artificial variable R_1, the standard form of the LPP becomes

$$
\begin{aligned}
\text{Max } Z &= 6x_1 + 4x_2 + 0s_1 + 0s_2 - MR_1 \\
\text{subject to} \quad 2x_1 + 3x_2 + s_1 + 0s_2 + 0s_3 &= 30 \\
3x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 &= 24 \\
x_1 + x_2 + 0s_1 + 0s_2 - s_3 + R_1 &= 3 \\
\text{and } x_1, x_2, s_1, s_2, s_3, R_1 &\geq 0.
\end{aligned}
$$

The initial basic feasible solution is given by

$s_1 = 30$, $s_2 = 24$, $R_1 = 3$ (basic) ($x_1 = x_2 = s_3 = 0$, non-basic)

Initial iteration :

C_B	Y_B	C_j → X_B	x_1 (6)	x_2 (4)	s_1 (0)	s_2 (0)	s_3 (0)	R_1 (–M)	θ
0	s_1	30	2	3	1	0	0	0	$\frac{30}{2}$
0	s_2	24	3	2	0	1	0	0	$\frac{24}{3}$
–M	R_1	3	(1)	1	0	0	–1	1	$\frac{3}{1}$*
$(Z_j - C_j)$		–3M	–M–6	–M–4	0	0	M	0	

First iteration : Introduce x_1 and drop R_1.

C_B	Y_B	C_j → X_B	x_1 (6)	x_2 (4)	s_1 (0)	s_2 (0)	s_3 (0)	θ
0	s_1	24	0	1	1	0	2	$\frac{24}{2}$
0	s_2	15	0	–1	0	1	(3)	$\frac{15}{3}$*
6	x_1	3	1	1	0	0	–1	–
$(Z_j - C_j)$		18	0	2	0	0	–6	

Second iteration : Introduce s_3 and drop s_2.

C_B	Y_B	C_j → X_B	x_1 (6)	x_2 (4)	s_1 (0)	s_2 (0)	s_3 (0)
0	s_1	14	0	$\frac{5}{3}$	1	$\frac{-2}{3}$	0
0	s_3	5	0	$\frac{-1}{3}$	0	$\frac{1}{3}$	1
6	x_1	8	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0
$(Z_j - C_j)$		48	0	0	0	2	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

The optimal solution is Max $z = 48, x_1 = 8, x_2 = 0$.

Alternative solutions :

Since $(z_2 - c_2) = 0$, corresponding to the non-basic variable x_2, the alternative solutions also exist. Therefore the solution as obtained above is not unique.

Thus we can bring x_2 into the basis in place of s_1 or s_3. So introducing x_2 into the basis in place of s_1, the new optimum simplex table is obtained as follows :

C_B	Y_B	C_j → X_B	x_1 (6)	x_2 (4)	s_1 (0)	s_2 (0)	s_3 (0)
4	x_2	$\frac{42}{5}$	0	1	$\frac{3}{5}$	$\frac{-2}{5}$	0
0	s_3	$\frac{39}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	1
6	x_1	$\frac{12}{5}$	1	0	$\frac{-2}{5}$	$\frac{3}{5}$	0
$(Z_j - C_j)$		48	0	0	0	2	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ Another optimal solution is Max $Z = 48, x_1 = \frac{12}{5}, x_2 = \frac{42}{5}$.

Thus, if two alternative optimal solutions can be obtained, then infinite number of optimum solutions can be obtained.

∴ The given LPP possess an **infinite number of optimum solutions or multiple optimal solutions.**

1.4.9 Non-existing feasible solution

In an LPP, when there is no point belonging to the solution space satisfying all the constraints, then the problem is said to have no feasible solution. In other words, in the optimum simplex table, if atleast one artificial variable appears in the basis at non–zero level (with positive value in X_B column) and the optimality condition is satisfied, then the problem is said to have no feasible solution.

Example (1) : See the example (1) under the Big – M method.

1.4.10 Unrestricted or unconstrained variables

In an LPP, if any variable is unconstrained (without specifying its sign) it can be expressed as the difference between two non-negative variables. The problem can be converted in to an equivalent one involving only non-negative variables.

Example 1 : **Solve the LPP :**

$$\text{Maximize } Z = 2x_1 + 3x_2$$

subject to

$$-x_1 + 2x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$x_1 + 3x_2 \le 9$$

and x_1, x_2 **are unrestricted.**

Solution : Since x_1, x_2 are unrestricted, we put $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$ so that $x_1', x_1'', x_2', x_2'' \ge 0$.

Then given LPP becomes

$$\text{Maximize Z} = 2x_1' - 2x_1'' + 3x_2' - 3x_2''$$

subject to

$$-x_1' + x_1'' + 2x_2' - 2x_2'' \le 4$$
$$x_1' - x_1'' + x_2' - x_2'' \le 6$$
$$x_1' - x_1'' + 3x_2' - 3x_2'' \le 9$$
$$x_1', x_1'', x_2', x_2'' \ge 0.$$

By introducing the non-negative slack variables s_1, s_2, s_3, the standard form of the LPP becomes

$$\text{Maximize Z} = 2x_1' - 2x_1'' + 3x_2' - 3x_2'' + 0s_1 + 0s_2 + 0s_3$$

subject to

$$-x_1' + x_1'' + 2x_2' - 2x_2'' + s_1 + 0s_2 + 0s_3 = 4$$
$$x_1' - x_1'' + x_2' - x_2'' + 0s_1 + s_2 + 0s_3 = 6$$
$$x_1' - x_1'' + 3x_2' - 3x_2'' + 0s_1 + 0s_2 + s_3 = 9$$
$$x_1', x_1'', x_2', x_2'', s_1, s_2, s_3 \ge 0.$$

The initial basic feasible solution is given by

$s_1 = 4,\ s_2 = 6, s_3 = 9$ (basic) $(x_1' = x_1'' = x_2' = x_2'' = 0$, non-basic)

Initial iteration :

	C_j	(2	−2	3	−3	0	0	0)	
C_B Y_B	X_B	x_1'	x_1''	x_2'	x_2''	s_1	s_2	s_3	θ
0 s_1	4	−1	1	(2)	−2	1	0	0	$\frac{4}{2} = 2$*
0 s_2	6	1	−1	1	−1	0	1	0	$\frac{6}{1} = 6$
0 s_3	9	1	−1	3	−3	0	0	1	$\frac{9}{3} = 3$
$(Z_j - C_j)$	0	−2	2	−3	3	0	0	0	

First iteration : Introduce x_2' and drop s_1.

	C_j	(2	−2	3	−3	0	0	0)	
C_B Y_B	X_B	x_1'	x_1''	x_2'	x_2''	s_1	s_2	s_3	θ
3 x_2'	2	$\frac{-1}{2}$	$\frac{1}{2}$	1	−1	$\frac{1}{2}$	0	0	–
0 s_2	4	$\frac{3}{2}$	$\frac{-3}{2}$	0	0	$\frac{-1}{2}$	1	0	$\frac{8}{3}$
0 s_3	3	$\left(\frac{5}{2}\right)$	$\frac{-5}{2}$	0	0	$\frac{-3}{2}$	0	1	$\frac{6}{5}$ *
$(Z_j - C_j)$	6	$\frac{-7}{2}$	$\frac{7}{2}$	0	0	$\frac{3}{2}$	0	0	

Second iteration : Introduce x_1' and drop s_3.

	C_j	(2	−2	3	−3	0	0	0)	
C_B Y_B	X_B	x_1'	x_1''	x_2'	x_2''	s_1	s_2	s_3	θ
3 x_2'	$\frac{13}{5}$	0	0	1	−1	$\frac{1}{5}$	0	$\frac{1}{5}$	13
0 s_2	$\frac{11}{5}$	0	0	0	0	$\left(\frac{2}{5}\right)$	1	$\frac{-3}{5}$	$\frac{11}{2}$ *
2 x_1'	$\frac{6}{5}$	1	−1	0	0	$\frac{-3}{5}$	0	$\frac{2}{5}$	–
$(Z_j - C_j)$	$\frac{51}{5}$	0	0	0	0	$\frac{-3}{5}$	0	$\frac{7}{5}$	

Third iteration : Introduce s_1 and drop s_2.

	C_j	(2	−2	3	−3	0	0	0)
C_B Y_B	X_B	x_1'	x_1''	x_2'	x_2''	s_1	s_2	s_3
3 x_2'	$\frac{3}{2}$	0	0	1	−1	0	$\frac{-1}{2}$	$\frac{1}{2}$
0 s_1	$\frac{11}{2}$	0	0	0	0	1	$\frac{5}{2}$	$\frac{-3}{2}$
2 x_1'	$\frac{9}{2}$	1	−1	0	0	0	$\frac{3}{2}$	$\frac{-1}{2}$
$(Z_j - C_j)$	$\frac{27}{2}$	0	0	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$

Since all $(Z_j - C_j) \ge 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = \frac{27}{2}$, $x_1' = \frac{9}{2}$, $x_2' = \frac{3}{2}$ $x_1'' = 0, x_2'' = 0$

But $x_1 = x_1' - x_1'' = \frac{9}{2} - 0 = \frac{9}{2}$; $x_2 = x_2' - x_2'' = \frac{3}{2} - 0 = \frac{3}{2}$

The optimal solution is Max $Z = \frac{27}{2}$, $x_1 = \frac{9}{2}$, $x_2 = \frac{3}{2}$.

Example 2 : Solve the LPP :

$$\text{Maximize } Z = 8x_2$$

subject to

$$x_1 - x_2 \geq 0$$
$$2x_1 + 3x_2 \leq -6$$

and x_1, x_2 are unrestricted.

Solution : Since x_1, x_2 are unrestricted, we put $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$ so that $x_1', x_1'', x_2', x_2'' \geq 0$.

$\therefore$ The given LPP becomes

$$\text{Maximize } Z = 8x_2' - 8x_2''$$

subject to

$$x_1' - x_1'' - x_2' + x_2'' \geq 0$$
$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' \geq 6$$
$$x_1', x_1'', x_2', x_2'' \geq 0.$$

By introducing the surplus variables s_1, s_2 and artificial variables R_1 and R_2 the standard form of LPP becomes

$$\text{Maximize } Z = 0x_1' + 0x_1'' + 8x_2' - 8x_2'' + 0s_1 + 0s_2 - MR_1 - MR_2$$

subject to

$$x_1' - x_1'' + x_2' + x_2'' - s_1 + 0s_2 + R_1 = 0$$
$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' + 0s_1 - s_2 + R_2 = 6$$
$$\text{and } x_1', x_1'', x_2', x_2'', s_1, s_2, R_1, R_2 \geq 0.$$

The initial basic feasible solution is given by $R_1 = 0$, $R_2 = 6$, (basic)

$(x_1' = x_1'' = x_2' = x_2'' = s_1 = s_2 = 0$, non-basic)

Initial iteration :

	C_j	(0	0	8	–8	0	0	–M	–M	
C_B Y_B	X_B	x_1'	x_1''	x_2'	x_2''	s_1	s_2	R_1	R_2	θ
–M R_1	0	1	–1	–1	(1)	–1	0	1	0	0*
–M R_2	6	–2	2	–3	3	0	–1	0	1	$\frac{6}{3} = 2$
$(Z_j - C_j)$	–6 M	M	–M	4M–8	–M + 8	M	M	0	0	

First iteration : Introduce x_2'' and drop R_1.

	C_j	(0	0	8	–8	0	0	–M	
C_B Y_B	X_B	x_1'	x_1''	x_2'	x_2''	s_1	s_2	R_2	θ
–8 x_2''	0	1	–1	–1	1	–1	0	0	–
–M R_2	6	–5	(5)	0	0	3	–1	1	$\frac{6}{5}$ *
$(Z_j - C_j)$	–6 M	5M–8	–5M+8	0	0	–3 M+8	M	0	

Second iteration : Introduce x_1'' and drop R_2.

	C_j	(0	0	8	–8	0	0
C_B Y_B	X_B	x_1'	x_1''	x_2'	x_2''	s_1	s_2
–8 x_2''	$\frac{6}{5}$	0	0	–1	1	$\frac{-2}{5}$	$\frac{-1}{5}$
0 x_1''	$\frac{6}{5}$	–1	1	0	0	$\frac{3}{5}$	$\frac{-1}{5}$
$(Z_j - C_j)$	$\frac{-48}{5}$	0	0	0	0	$\frac{16}{5}$	$\frac{8}{5}$

Since all $(Z_j - C_j) \geq 0$, and no artificial variable appears in the basis, the current basic feasible solution is optimal.

$\therefore$ The optimal solution Max $Z = \frac{-48}{5}$, $x_1' = 0$, $x_2' = 0$, $x_1'' = \frac{6}{5}$, $x_2'' = \frac{6}{5}$

But $x_1 = x_1' - x_1'' = 0 - \frac{6}{5} = \frac{-6}{5}$ and $x_2 = x_2' - x_2'' = 0 - \frac{6}{5} = \frac{-6}{5}$

The optimal solution is Max $Z = \frac{-48}{5}$, $x_1 = \frac{-6}{5}$, $x_2 = \frac{-6}{5}$.

EXERCISE

1. **How would you resolve the following situations in a LPP ?**
   (i) **minimization**
   (ii) **equalities in constraints**
   (iii) **tie for leaving basic variables.**
2. **What is degeneracy ? Discuss a method to resolve degeneracy in LPP.** ***[MU.M.Sc.79, MU.MBA.Apr.97]***
3. **Write a note on degeneracy and cycling in a LPP and outlining a method avoiding cycling.** ***[BRU. M.Sc. 86, MKU. B.Sc. 92]***

4. **Explain how to find the existence of multiple solutions of LPP while solving it by simplex method.** *[MU. MBA Nov 95]*

5. **What are the essential characteristics of a linear programming model ? What is an unbounded solution of a LPP ?** *[MU. MCA. Nov 95]*

6. **What do you understand from an unbounded solution? When does the simplex arithmetic indicate that the LPP has unbounded solution. ?** *[MU. MBA Apr 96]*

7. **What is infeasible solution ? How is it identified in the simplex table ?**

8. **Establish the difference between**
   (i) feasible solution
   (ii) basic feasible solution and
   (iii) degenerate basic feasible solution. *[MKU. BE. Apr 79]*

9. **Solve the LPP:** Maximize $Z = 3x_1 + 9x_2$

   subject to
$$x_1 + 4x_2 \le 8$$
$$x_1 + 2x_2 \le 4$$
and $x_1, x_2 \ge 0.$

10. **Solve the LPP :**
$$\text{Max } Z = 2x_1 + x_2$$
   subject to
$$4x_1 + 3x_2 \le 12$$
$$4x_1 + x_2 \le 8$$
$$4x_1 - x_2 \le 8$$
and $x_1, x_2 \ge 0.$

11. **Solve the LPP :**
$$\text{Min } Z = \tfrac{3}{4}x_1 + 20x_2 - \tfrac{1}{2}x_3 + 6x_4$$
   subject to
$$\tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \le 0$$
$$\tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 \le 0$$
$$x_3 \le 1$$
and $x_1, x_2, x_3, x_4 \ge 0.$

12. **Solve the LPP :** Max $Z = 107x_1 + x_2 + 2x_3$

   subject to
$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$
$$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5$$
$$3x_1 - x_2 - x_3 \le 0$$
and $x_1, x_2, x_3 \ge 0.$ *[MKU. BE. 89]*

13. **Solve the LPP :** Max $Z = 2x_1 + x_2$

   subject to
$$3x_1 + x_2 \ge 3$$
$$4x_1 + 3x_2 \ge 6$$
$$x_1 + 2x_2 \ge 2$$
and $x_1, x_2 \ge 0.$

14. **Solve the LPP :** Max $Z = 2x_1 + 4x_2 + x_3$

   subject to
$$x_1 - 2x_2 - x_3 \le 5$$
$$2x_1 - x_2 + 2x_3 = 2$$
$$-x_1 + 2x_2 + 2x_3 \ge 1$$
$$x_1, x_2, x_3 \ge 0.$$
[BNU BSc 82, MU. BSc 82]

15. **Solve the LPP :** Max $Z = 6x_1 - 2x_2$

   subject to
$$2x_1 - x_2 \le 2$$
$$x_1 \le 4$$
and $x_1, x_2 \ge 0.$

16. **Solve the LPP :** Max $Z = x_1 + 2x_2 + 3x_3 - x_4$

   subject to
$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
and $x_1, x_2, x_3, x_4 \ge 0.$

17. **Solve the LPP :** Max $Z = 3x_1 - x_2$

   subject to
$$x_1 + 2x_2 \le 5$$
$$x_1 + x_2 - x_3 \le 2$$
$$7x_1 + 3x_2 - 5x_3 \le 20$$
and $x_1, x_2, x_3 \ge 0.$

18. **Solve the LPP :** Max $Z = 2x_1 + x_2$

subject to

$$\frac{3}{2}x_1 + x_2 \le 6$$
$$x_1 \le 2$$
$$x_1 + x_2 \ge 7$$
$$-x_1 + x_2 \ge 4$$
and $x_1, x_2 \ge 0.$

19. **Solve the LPP :** Max $Z = 2x_1 + 3x_2 + 5x_3$

subject to

$$3x_1 + 10x_2 + 5x_3 \le 15$$
$$33x_1 - 10x_2 + 9x_3 \le 33$$
$$x_1 + 2x_2 + x_3 \ge 4$$
and $x_1, x_2, x_3 \ge 0.$

20. **Solve the LPP :** Max $Z = 3x_1 + 4x_2$

subject to

$$x_1 - x_2 \le -1$$
$$-x_1 + x_2 \le 0$$
and $x_1, x_2 \ge 0.$

21. **Solve the LPP :** Max $Z = 6x_1 - 3x_2 + 2x_3$

subject to

$$2x_1 + x_2 + x_3 \le 16$$
$$3x_1 + 2x_2 + x_3 \le 18$$
$$x_2 - 2x_3 \ge 8$$
and $x_1, x_2, x_3 \ge 0.$

22. **Solve the LPP :** Max $Z = 2x_1 + 3x_2$

subject to

$$x_1 - 2x_2 \le 0$$
$$-2x_1 + 3x_2 \le -6$$
and x_1, x_2 are unrestricted.

23. **Solve the LPP :** Max $Z = 3x_1 + 2x_2 + x_3$

subject to

$$2x_1 + 5x_2 + x_3 = 12$$
$$3x_1 + 4x_2 = 11$$
$x_2, x_3 \ge 0$ and x_1 unrestricted. *[BNU.MSc.85]*

24. **Solve the LPP :** Min $Z = x_1 + x_2 + x_3$

subject to

$$x_1 - 3x_2 + 4x_3 = 5$$
$$x_1 - 2x_2 \le 3$$
$$2x_2 - x_3 \ge 4$$
$x_1, x_2 \ge 0$ and x_3 is unrestricted.

ANSWERS

9. Max $Z = 18$, $x_1 = 0$, $x_2 = 2$
10. Max $Z = 5$, $x_1 = \frac{3}{2}$, $x_2 = 2$
11. Min $Z = -0.5$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$
12. Unbounded solution
13. Unbounded solution
14. Unbounded solution
15. Unbounded feasible region but bounded optimal solution. Max $Z = 12$, $x_1 = 4$, $x_2 = 6$
16. Multiple solutions. Max $Z = 15$, *(i)* $x_1 = x_2 = x_3 = \frac{5}{2}$, $x_4 = 0$ *etc.*
17. Multiple solutions: Max $Z = 15$, *(i)* $x_1 = 5$, $x_2 = 0$, $x_3 = 3$, *etc.*
18. No solution
19. No solution
20. Unbound solution
21. Max $Z = -20$, $x_1 = 0.667$, $x_2 = 8$, $x_3 = 0$
22. Unbounded solution
23. Max $Z = \frac{47}{3}$, $x_1 = \frac{11}{3}$ $x_2 = 0$, $x_3 = \frac{14}{3}$
24. Min $Z = \frac{43}{5}$, $x_1 = 0$, $x_2 = \frac{21}{5}$, $x_3 = \frac{22}{5}$

1.5 REVISED SIMPLEX METHOD

1.5.1 Revised Simplex Method

While solving a linear programming problem on a digital computer by regular simplex method, it requires storing the entire simplex tableau in the memory of the computer, which may not be feasible for very large problems. But it is not really necessary to calculate each tableau during each iteration. The revised simplex method which is a modification of the original method is more economical on the computer, as it computes and stores only the relevant information needed currently for testing and/or improving the current solution. *i.e.,* it needs only,

(i) The net evaluations row, $(Z_j - C_j)$ to determine the non-basic variable that enters the basis.

(ii) The pivot column.

(iii) The current basic variables and their values (*i.e.,* X_B column) to determine the minimum positive ratio and then to identify the basic variable to leave the basis.

The above information is directly obtained from the original equations by making use of the inverse of the current basis matrix at any iteration.

1.5.2 Revised Simplex Algorithm

Step 1: Introduce slack or surplus and artificial variables, if necessary, express the given LPP in standard form after converting it in to maximization type.

Step 2: Obtain an initial basic feasible solution with an initial basis $B = I_m$ and form the auxilliary matrix $\hat{B}$, such that

$$\hat{B} = \begin{pmatrix} B & 0 \\ C_B & 1 \end{pmatrix} \text{ and } \hat{B}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix}$$

Step 3: State the objective function $Z = CX$ as an additional constraint and form $\hat{A}$, such that

$$\hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix} \text{ and } \hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Step 4: Compute the net evaluations

$$(Z_j - C_j) = (C_B B^{-1}\ 1)\begin{pmatrix} A \\ -C \end{pmatrix} = (C_B B^{-1}\ 1)\hat{A}$$

Case (i) : If all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

Case (ii) : If atleast one $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal. If $(Z_k - C_k)$ is the most negative, then the variable x_k enters the basis, go to step (5).

Step 5: Compute $\hat{x}_k = \hat{B}^{-1}_{curr}\ \hat{a}_k$.

Case (i) : If all $x_{ik} \leq 0$, then there exists an unbounded solution to the given LPP.

Case (ii) : If atleast one $x_{ik} > 0$, consider the current X_B and determine the leaving variable, go to step (6).

Step 6: Write down the results obtained from step (2) to step (5) in the revised simplex table.

Step 7: Convert the leading element to unity and all other elements of the entering column to zero and update the current basic feasible solution.

Step 8: Go to step (4) and repeat the procedure until an optimum basic feasible solution is obtained or there is an indication of an unbounded solution.

Example 1 *Use revised simplex method to solve the LPP.*

$$\text{Maximize } Z = x_1 + x_2$$
$$\text{subject to } 3x_1 + 2x_2 \leq 6$$
$$x_1 + 4x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

[MSU. BE. Apr 97]

Solution: By introducing the non-negative slack variables s_1 and s_2, the standard form of the LPP becomes

$$\text{Max } Z = x_1 + x_2 + 0s_1 + 0s_2$$
$$\text{subject to } 3x_1 + 2x_2 + s_1 + 0s_2 = 6$$
$$x_1 + 4x_2 + 0s_1 + s_2 = 4$$
$$\text{and } x_1, x_2, s_1, s_2 \geq 0.$$

$$\text{Max } Z = (1\ \ 1\ \ 0\ \ 0)\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = CX$$

$$\text{subject to} \quad \underset{A}{\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix}} = \underset{b}{\begin{pmatrix} 6 \\ 4 \end{pmatrix}}$$

and $x_1, x_2, s_1, s_2 \geq 0$

where $C = (1 \ \ 1 \ \ 0 \ \ 0)$, $C_B = [0 \ \ 0]$

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}, \quad b = [6 \ \ 4], \quad \therefore \hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$$

The initial basic feasible solution is $s_1 = 6$, $s_1 = 4$,

with I_2 as the initial basis, $I_2 = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$

$$\text{Now } \hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 \end{array} \\ \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{pmatrix} \end{array} \text{ and}$$

$$\hat{B}^{-1}_{curr} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The net evaluations are given by

$$(Z_j - C_j) = (C_B B^{-1} \ \ 1)\begin{pmatrix} A \\ -C \end{pmatrix} = (C_B B^{-1} \ \ 1)\ (\hat{a}_1 \ \ \hat{a}_2)$$

$$= (0 \ \ 0 \ \ 1)\begin{pmatrix} 3 & 2 \\ 1 & 4 \\ -1 & -1 \end{pmatrix} = (-1, -1)$$

Since $(Z_1 - C_1)$ is the most negative, x_1 enters the basis.

$$\text{Now } \hat{x}_1 = \hat{B}^{-1}_{curr} \ \hat{a}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = [3 \ \ 1 \ \ -1]$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{curr} \ \hat{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} = [6 \ \ 4 \ \ 0]$$

The initial simplex table is

Y_B	$\hat{B}^{-1}_{curr}$	$\hat{x}$	$\hat{X}_B$	Ratio
s_1	1 0 0	(3)	6	$\frac{6}{3}$
s_2	0 1 0	1	4	$\frac{4}{1}$
	0 0 1	– 1	0	

$\therefore s_1$ leaves the basis.

First Iteration: Introduce x_1 and drop s_1. Convert the leading element unity and all other elements of the entering column to be zero. We shall have,

$$\hat{B}^{-1}_{curr} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix} \text{ and therefore}$$

$$(Z_j - C_j) = (C_B \ B^{-1} \ \ 1)\ (\hat{a}_2 \ \ \hat{a}_3)$$

$$= \begin{pmatrix} \frac{1}{3} & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 0 \\ -1 & 0 \end{pmatrix} = \left(\frac{-1}{3}, \frac{1}{3}\right)$$

Since $(Z_2 - C_2) = \frac{-1}{3}$ is most negative, x_2 enters the basis.

$$\therefore \hat{x}_2 = \hat{B}^{-1}_{next} \ \hat{a}_2 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{10}{3} \\ \frac{-1}{3} \end{pmatrix}$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{next} \ \hat{b} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{-1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Thus we have

Y_B	$\hat{B}^{-1}_{next}$			$\hat{x}_2$	$\hat{X}_B$	Ratio
x_1	$\frac{1}{3}$	0	0	$\frac{2}{3}$	2	$\frac{6}{3}$
s_2	$\frac{-1}{3}$	1	0	$\left(\frac{10}{3}\right)$	2	$\frac{6}{10}$
	$\frac{1}{3}$	0	1	$\frac{-1}{3}$	2	

$\therefore$ s_2 leaves the basis.

Second Iteration: Introduce x_2 and drop s_2. Convert the leading element unity and all other elements of the entering column to be zero. We have,

$$\hat{B}^{-1}_{curr} = \begin{pmatrix} \frac{2}{5} & \frac{-1}{5} & 0 \\ \frac{-1}{10} & \frac{3}{10} & 0 \\ \frac{3}{10} & \frac{1}{10} & 1 \end{pmatrix} \text{ and therefore,}$$

$$(Z_j - C_j) = (C_B B^{-1} \;\; 1)\,(\hat{a}_3 \;\; \hat{a}_4) = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \end{pmatrix}$$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$$\therefore \hat{X}_B = \hat{B}^{-1}_{curr}\,\hat{b} = \begin{pmatrix} \frac{2}{5} & \frac{-1}{5} & 0 \\ \frac{-1}{10} & \frac{3}{10} & 0 \\ \frac{3}{10} & \frac{1}{10} & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{5} \\ \frac{3}{5} \end{pmatrix} \Rightarrow x_1 = \frac{8}{5},\; x_2 = \frac{3}{5}$$

$$\text{and Max } Z = x_1 + x_2 = \frac{8}{5} + \frac{3}{5} = \frac{11}{5}$$

$\therefore$ The optimal solution is

$$\text{Max } Z = \frac{11}{5},\; x_1 = \frac{8}{5},\; x_2 = \frac{3}{5}$$

Example 2 ***Use revised simplex method to solve the LPP.***

$$\textit{Maximize } Z = 6x_1 - 2x_2 + 3x_3$$

$$\textit{subject to } \; 2x_1 - x_2 + 2x_3 \leq 2$$
$$x_1 + 4x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

[BRU, M.Sc., '88, MKU, M.Sc., '83]

Solution: By introducing the non-negative slack variables s_1 and s_2, the standard form of the LPP becomes

$$\text{Max } Z = 6x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2$$
$$\text{subject to } 2x_1 - x_2 + 2x_3 + s_1 + 0s_2 = 2$$
$$x_1 + 0x_2 + 4x_3 + 0s_1 + s_2 = 4$$
$$\text{and } x_1, x_2, x_3, s_1, s_2 \geq 0$$

$$\text{i.e., Max } Z = (6 \; -2 \; 3 \; 0 \; 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix} = CX$$

$$\text{subject to } \underbrace{\begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}_{b}$$

$$\text{and } x_1, x_2, x_3, s_1, s_2 \geq 0$$

$$\therefore C = (6 \; -2 \; 3 \; 0 \; 0),\; C_B = [0 \;\; 0]$$

$$A = \begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \end{pmatrix},\; b = [2 \;\; 4]$$

$$\therefore \hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

The initial basic feasible solution is $s_1 = 2,\; s_2 = 4$,

with I_2 as the initial basis, $I_2 = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$

$$\hat{a}_1 \quad \hat{a}_2 \quad \hat{a}_3 \quad \hat{a}_4 \quad \hat{a}_5$$

$$\text{Now } \hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 0 & 1 \\ -6 & 2 & -3 & 0 & 0 \end{pmatrix} \text{ and}$$

$$\hat{B}^{-1}_{\text{curr}} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The net evaluations are given by

$$(Z_j - C_j) = (C_B B^{-1} \;\; 1)\, \hat{A} = (C_B B^{-1} \;\; 1)\, (\hat{a}_1 \;\; \hat{a}_2 \;\; \hat{a}_3)$$

$$= (0 \;\; 0 \;\; 1) \begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 4 \\ -6 & 2 & -3 \end{pmatrix} = (-6, \; 2, \; -3)$$

Since $(Z_1 - C_1)$ is the most negative, x_1 enters the basis.

$$\text{Now} \quad \hat{x}_1 = \hat{B}^{-1}_{\text{curr}}\, \hat{a}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$$

$$\text{and} \quad \hat{X}_B = \hat{B}^{-1}_{\text{curr}}\, \hat{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

The initial revised simplex table is

Y_B	$\hat{B}^{-1}_{\text{curr}}$			$\hat{x}_1$	$\hat{X}_B$	Ratio
s_1	1	0	0	(2)	2	$\frac{2}{2} = 1$
s_2	0	1	0	1	4	$\frac{4}{1} = 4$
	0	0	1	−6	0	

$\therefore$ s_1 leaves the basis.

First Iteration: Introduce x_1 and drop s_1. Convert the leading element unity and all other elements of the entering column to be zero. We shall have

$$\hat{B}^{-1}_{\text{next}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \text{ and therefore,}$$

$$(Z_j - C_j) = (C_B B^{-1} \;\; 1)\, (\hat{a}_2 \;\; \hat{a}_3 \;\; \hat{a}_4)$$

$$= (3 \;\; 0 \;\; 1) \begin{pmatrix} -1 & 2 & 1 \\ 0 & 4 & 0 \\ 2 & -3 & 0 \end{pmatrix}$$

$$= (-1, \; 3, \; 3)$$

Since $(Z_2 - C_2) = -1$ is most negative, x_2 enters the basis.

$$\therefore \; \hat{x}_2 = \hat{B}^{-1}_{\text{next}}\, \hat{a}_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\text{and} \quad \hat{X}_B = \hat{B}^{-1}_{\text{next}}\, \hat{b} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

$\therefore$ The revised simplex table is

Y_B	$\hat{B}^{-1}_{\text{next}}$			$\hat{x}_2$	$\hat{X}_B$	Ratio
x_1	$\frac{1}{2}$	0	0	$\frac{-1}{2}$	1	–
s_2	$\frac{-1}{2}$	1	0	$\left(\frac{1}{2}\right)$	3	6
	3	0	1	−1	6	

$\therefore$ s_2 leaves the basis.

Second Iteration: Introduce x_2 and drop s_2. Convert the leading element unity and all other elements of the entering column to be zero. We have,

$$\hat{B}^{-1}_{\text{next}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix} \text{ and therefore,}$$

$$(Z_j - C_j) = (C_B B^{-1} \;\; 1)\, (\hat{a}_3 \;\; \hat{a}_4 \;\; \hat{a}_5)$$

$$= (2 \;\; 2 \;\; 1) \begin{pmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ -3 & 0 & 0 \end{pmatrix}$$

$$= (9, \; 2, \; 2)$$

Since all $(Z_j - C_j) \geq 0$, the current basis feasible solution is optimal.

$$\therefore \hat{X}_B = \hat{B}^{-1}_{next}\,\hat{b} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}$$

$$\Rightarrow x_1 = 4, \quad x_2 = 6, \quad x_3 = 0$$

$$\text{and Max } Z = 6x_1 - 2x_2 + 3x_3 = 24 - 12 + 0 = 12$$

$\therefore$ The optimal solution is

$$\text{Max } Z = 12, \; x_1 = 4, \; x_2 = 6, \; x_3 = 0.$$

Example 3 ***Use revised simplex method to solve the LPP:***

$$\textit{Maximize } Z = x_1 + x_2$$

$$\textit{subject to } 2x_1 + 5x_2 \leq 6$$

$$x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

[MU, MCA Nov. '95]

Solution: By introducing slack, surplus and artificial variables the standard form of the LPP becomes

$$\text{Max } Z = x_1 + x_2 + 0s_1 + 0s_2 - MR_1$$

subject to

$$2x_1 + 5x_2 + s_1 = 6$$

$$x_1 + x_2 - s_2 + R_1 = 2$$

$$x_1, x_2, s_1, s_2, R_1 \geq 0$$

Here $C = (1 \;\; 1 \;\; 0 \;\; 0 \;\; -M)$, $A = \begin{pmatrix} 2 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

The initial basic feasible solution is $s_1 = 6$, $R_1 = 2$,

with $I_2 = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$

$$\text{Now } \hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix} = \begin{array}{c} \begin{array}{ccccc} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 & \hat{a}_5 \end{array} \\ \begin{pmatrix} 2 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & M \end{pmatrix} \end{array} \text{ and}$$

$$\hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$$

$$\hat{B}^{-1}_{curr} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -M & 1 \end{pmatrix}, \quad C_B = [0, -M]$$

The net evaluations are given by

$$(Z_j - C_j) = (C_B B^{-1} \;\; 1)\,\hat{A} = (C_B B^{-1} \;\; 1)\,(\hat{a}_1 \;\; \hat{a}_2 \;\; \hat{a}_4)$$

$$= (0 \;\; -M \;\; 1) \begin{pmatrix} 2 & 5 & 0 \\ 1 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$= (-M - 1, -M - 1, M)$$

Since $(Z_1 - C_1)$ is the most negative, x_1 enters the basis.

$$\text{Now } \hat{x}_1 = \hat{B}^{-1}_{curr}\,\hat{a}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -M & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -M-1 \end{pmatrix}$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{curr}\,\hat{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -M & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -2M \end{pmatrix}$$

The initial simplex table is

Y_B	$\hat{B}^{-1}_{curr}$			$\hat{x}_1$	$\hat{X}_B$	Ratio
s_1	1	0	0	2	6	$\frac{6}{2}$
R_1	0	1	0	(1)	2	$\frac{2}{1}$
	0	$-M$	1	$-M-1$	$-2M$	

$\therefore R_1$ leaves the basis.

First Iteration: Introduce x_1 and drop R_1.

We shall have

$$\hat{B}^{-1}_{next} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Now the net evaluations are given by

$$(Z_j - C_j) = (C_B B^{-1} \ 1)\ (\hat{a}_2 \ \hat{a}_4 \ \hat{a}_5)$$

$$= (0 \ 1 \ 1)\begin{pmatrix} 5 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & M \end{pmatrix}$$

$$= (0, -1, M+1)$$

Since $(Z_4 - C_4)$ is most negative, s_2 enters the basis.

$$\therefore \ \hat{s}_2 = \hat{B}^{-1}_{next} \ \hat{a}_4 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

and $$\hat{X}_B = \hat{B}^{-1}_{next} \ \hat{b} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

The revised simplex table is

Y_B	$\hat{B}^{-1}_{next}$			$\hat{s}_2$	$\hat{X}_B$	Ratio
s_1	1	−2	0	(2)	2	$\frac{2}{2} = 1$
x_1	0	1	0	−1	2	–
	0	1	1	−1	2	

$\therefore$ s_1 leaves the basis.

Second Iteration: Introduce s_1 and drop s_1.

We have $$\hat{B}^{-1}_{curr} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

Now the net evaluations are given by

$$(Z_j - C_j) = (C_B B^{-1} \ 1)\ (\hat{a}_2 \ \hat{a}_3 \ \hat{a}_5)$$

$$= \left(\frac{1}{2} \ 0 \ 1\right)\begin{pmatrix} 5 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & M \end{pmatrix}\left(\frac{3}{2}, \frac{1}{2}, M\right)$$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$$\therefore \ \hat{X}_B = \hat{B}^{-1}_{curr} \ \hat{b} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

$$\Rightarrow s_2 = 1, \ x_1 = 3$$

and $\text{Max } Z = x_1 + x_2 = 3 + 0 = 3$

$\therefore$ $\text{Max } Z = 3, \ x_1 = 3, \ x_2 = 0$

Example 4 ***Use revised simplex method to solve the LPP:***

Minimize $Z = x_1 + 2x_2$

subject to $2x_1 + 5x_2 \geq 6$

$x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$ ***[MU.MCA. Nov. '95, MKU, ME '83]***

Solution: Given LPP is

$\text{Max } Z^* = -x_1 - 2x_2$ $[\because \text{Min } Z = -\text{Max}(-Z) = -\text{Max } Z^*]$

subject to $2x_1 + 5x_2 \geq 6$

$x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

By introducing the non-negative surplus variables s_1 and s_2 and artificial variables R_1 and R_2, the standard form of the LPP becomes

$$\text{Max } Z^* = -x_1 - 2x_2 + 0s_1 + 0s_2 - MR_1 - MR_2$$

subject to $$2x_1 + 5x_2 - s_1 + 0s_2 + R_1 + 0R_2 = 6$$

$$x_1 + x_2 + 0s_1 - s_2 + 0R_1 + R_2 = 2$$

$$x_1, x_2, s_1, s_2, R_1, R_2 \geq 0$$

i.e., $\text{Max } Z^* = CX$

subject to $AX = b$

$X \geq 0$

where $C = (-1 \ \ -2 \ \ 0 \ \ 0 \ \ -M \ \ -M)$, $A = \begin{pmatrix} 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$

$$b = \begin{pmatrix}6\\2\end{pmatrix}, \quad X = \begin{pmatrix}x_1\\x_2\\s_1\\s_2\\R_1\\R_2\end{pmatrix} \text{ and } C_B = [-M \ \ -M]$$

The initial basic feasible solution is $R_1 = 6$, $R_2 = 2$,

with the initial basis $I_2 = B = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = B^{-1}$

$$\text{Now } \hat{A} = \begin{pmatrix}A\\-C\end{pmatrix} = \begin{array}{c} \begin{array}{cccccc}\hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 & \hat{a}_5 & \hat{a}_6\end{array} \\ \begin{pmatrix}2 & 5 & -1 & 0 & 1 & 0\\1 & 1 & 0 & -1 & 0 & 1\\1 & 2 & 0 & 0 & M & M\end{pmatrix}\end{array},$$

$$\hat{b} = \begin{pmatrix}b\\0\end{pmatrix} = \begin{pmatrix}6\\2\\0\end{pmatrix} \text{ and}$$

$$\hat{B}^{-1}_{curr} = \begin{pmatrix}B^{-1} & 0\\C_BB^{-1} & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\-M & -M & 1\end{pmatrix}$$

The net evaluations are given by

$$(Z_j^* - C_j) = (C_BB^{-1} \ \ 1)\ \hat{A} = (C_BB^{-1} \ \ 1)\ (\hat{a}_1 \ \ \hat{a}_2 \ \ \hat{a}_3 \ \ \hat{a}_4)$$

$$= (-M \ \ -M \ \ 1)\begin{pmatrix}2 & 5 & -1 & 0\\1 & 1 & 0 & -1\\1 & 2 & 0 & 0\end{pmatrix}$$

$$= (-3M+1, \ -6M+2, \ M, \ M)$$

Since $(Z_2^* - C_2)$ is the most negative, x_2 enters the basis.

$$\text{Now } \hat{x}_2 = \hat{B}^{-1}_{curr}\ \hat{a}_2 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\-M & -M & 1\end{pmatrix}\begin{pmatrix}5\\1\\2\end{pmatrix} = \begin{pmatrix}5\\1\\-6M+2\end{pmatrix}$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{curr}\ \hat{b} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\-M & -M & 1\end{pmatrix}\begin{pmatrix}6\\2\\0\end{pmatrix} = \begin{pmatrix}6\\2\\-8M\end{pmatrix}$$

∴ The initial revised simplex table is

Y_B	$\hat{B}^{-1}_{curr}$			$\hat{x}_2$	$\hat{X}_B$	Ratio
R_1	1	0	0	(5)	6	$\frac{6}{5}$
R_2	0	1	0	1	2	$\frac{2}{1}$
	$-M$	$-M$	1	$-6M+2$	$-8M$	

∴ R_1 leaves the basis.

First Iteration: Introduce x_2 and drop R_1. We shall have

$$\hat{B}^{-1}_{next} = \begin{pmatrix}\frac{1}{5} & 0 & 0\\ \frac{-1}{5} & 1 & 0\\ \frac{M-2}{5} & -M & 1\end{pmatrix} \text{ and therefore}$$

$$(Z_j^* - C_j) = (C_BB^{-1} \ \ 1)\ (\hat{a}_1 \ \ \hat{a}_3 \ \ \hat{a}_4 \ \ \hat{a}_5)$$

$$= \left(\frac{M-2}{5} \ \ -M \ \ 1\right)\begin{pmatrix}2 & -1 & 0 & 1\\1 & 0 & -1 & 0\\1 & 0 & 0 & M\end{pmatrix}$$

$$= \left(\frac{-3M+1}{5}, \ \frac{-M+2}{5}, M, \ \frac{6M-2}{5}\right)$$

Since $(Z_1^* - C_1)$ is most negative, x_1 enters the basis.

$$\therefore\ \hat{x}_1 = \hat{B}^{-1}_{next}\ \hat{a}_1 = \begin{pmatrix}\frac{1}{5} & 0 & 0\\ \frac{-1}{5} & 1 & 0\\ \frac{M-2}{5} & -M & 1\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{2}{5}\\ \frac{3}{5}\\ \frac{-3M+1}{5}\end{pmatrix}$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{next}\ \hat{b} = \begin{pmatrix}\frac{1}{5} & 0 & 0\\ \frac{-1}{5} & 1 & 0\\ \frac{M-2}{5} & -M & 1\end{pmatrix}\begin{pmatrix}6\\2\\0\end{pmatrix} = \begin{pmatrix}\frac{6}{5}\\ \frac{4}{5}\\ \frac{-4M-12}{5}\end{pmatrix}$$

∴ The revised simplex table is

Y_B	$\hat{B}^{-1}_{next}$			$\hat{x}_1$	$\hat{X}_B$	Ratio
x_2	$\frac{1}{5}$	0	0	$\frac{2}{5}$	$\frac{6}{5}$	$\frac{6}{2} = 3$
R_2	$\frac{-1}{5}$	1	0	$\left(\frac{3}{5}\right)$	$\frac{4}{5}$	$\frac{4}{3}$
	$\frac{M-2}{5}$	$-M$	1	$\frac{-3M+1}{5}$	$\frac{-4M-12}{5}$	

$\therefore R_2$ leaves the basis.

Second Iteration: Introduce x_1 and drop R_2.

We have $$\hat{B}^{-1}_{curr} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & 0 \\ \frac{-1}{3} & \frac{5}{3} & 0 \\ \frac{-1}{3} & \frac{-1}{3} & 1 \end{pmatrix}$$ and therefore

$$(Z_j^* - C_j) = (C_B B^{-1}\ 1)\ (\hat{a}_3\ \ \hat{a}_4\ \ \hat{a}_5\ \ \hat{a}_6)$$

$$= \begin{pmatrix} \frac{-1}{3} & \frac{-1}{3} & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & M & M \end{pmatrix}$$

$$= \left(\frac{1}{3},\ \frac{1}{3},\ M - \frac{1}{3},\ M - \frac{1}{3}\right)$$

Since M is a very large quantity, all $(Z_j^* - C_j) \geq 0$, hence the current basic feasible solution is optimal.

$$\therefore \hat{X}_B = \hat{B}^{-1}_{curr}\ \hat{b} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & 0 \\ \frac{-1}{3} & \frac{5}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \end{pmatrix} \text{ and Min } Z = x_1 + 2x_2 = \frac{8}{3}$$

$\therefore$ The optimum solution is Min $Z = \frac{8}{3}$, $x_1 = \frac{4}{3}$, $x_2 = \frac{2}{3}$.

Example 5 *Use revised simplex method to solve the LPP.*

$$\begin{aligned} \textit{Minimize } Z &= -4x_1 + x_2 + 2x_3 \\ \textit{subject to } 2x_1 - 3x_2 + 2x_3 &\leq 12 \\ -5x_1 + 2x_2 + 3x_3 &\geq 4 \\ 3x_1 - 2x_3 &= -1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: Given LPP is

$$\begin{aligned} \text{Max } Z^* &= 4x_1 - x_2 - 2x_3 \\ \text{subject to } 2x_1 - 3x_2 + 2x_3 &\leq 12 \\ -5x_1 + 2x_2 + 3x_3 &\leq 4 \\ -3x_1 + 0x_2 + 2x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

By introducing the slack variable s_1, surplus variable s_2 and artificial variables R_1 and R_2, the standard form of the LPP becomes

$$\begin{aligned} \text{Max } Z^* &= 4x_1 - x_2 - 2x_3 + 0s_1 + 0s_2 - MR_1 - MR_2 \\ \text{subject to } 2x_1 - 3x_2 + 2x_3 + s_1 + 0s_2 + 0R_1 + 0R_2 &= 12 \\ -5x_1 + 2x_2 + 3x_3 + 0s_1 - s_2 + R_1 + 0R_2 &= 4 \\ -3x_1 + 0x_2 + 2x_3 + 0s_1 + 0s_2 + 0R_1 + R_2 &= 1 \\ x_1, x_2, x_3, s_1, s_2, R_1, R_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} \textit{i.e., } \text{Max } Z^* &= CX \\ \text{subject to } AX &= b \\ X &\geq 0 \\ \text{where } C &= (4\ -1\ -2\ 0\ 0\ -M\ -M) \end{aligned}$$

$$A = \begin{pmatrix} 2 & -3 & 2 & 1 & 0 & 0 & 0 \\ -5 & 2 & 3 & 0 & -1 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 12 \\ 4 \\ 1 \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ R_1 \\ R_2 \end{pmatrix}$$

The initial basic feasible solution is $s_1 = 12$, $R_1 = 4$, $R_2 = 1$.

with the initial basis $I_3 = B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B^{-1}, C_B = [0 \ -M \ -M]$

$$\begin{array}{c} \hat{a}_1 \ \ \hat{a}_2 \ \ \hat{a}_3 \ \ \hat{a}_4 \ \ \hat{a}_5 \ \ \hat{a}_6 \ \ \hat{a}_7 \end{array}$$

$$\text{Now } \hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix} = \begin{pmatrix} 2 & -3 & 2 & 1 & 0 & 0 & 0 \\ -5 & 2 & 3 & 0 & -1 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 0 & 1 \\ -4 & 1 & 2 & 0 & 0 & M & M \end{pmatrix},$$

$$\hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{and } \hat{B}^{-1}_{\text{curr}} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -M & -M & 1 \end{pmatrix}$$

The net evaluations are given by

$$(Z_j^* - C_j) = (C_B B^{-1} \ 1) \ \hat{A} = (C_B \ B^{-1} \ 1) \ (\hat{a}_1 \ \hat{a}_2 \ \hat{a}_3 \ \hat{a}_5)$$

$$= (0 \ -M \ -M \ 1) \begin{pmatrix} 2 & -3 & 2 & 0 \\ -5 & 2 & 3 & -1 \\ -3 & 0 & 2 & 0 \\ -4 & 1 & 2 & 0 \end{pmatrix}$$

$$= (8M - 4, \ 1 - 2M, \ 2 - 5M, \ M)$$

Since $(Z_3^* - C_3)$ is most negative, x_3 enters the basis.

$$\hat{x}_3 = \hat{B}^{-1}_{\text{curr}} \ \hat{a}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -M & -M & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ -5M + 2 \end{pmatrix}$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{\text{curr}} \ \hat{b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -M & -M & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ 1 \\ -5M \end{pmatrix}$$

$\therefore$ **The initial revised simplex table is**

Y_B	$\hat{B}^{-1}_{\text{curr}}$				$\hat{x}_3$	$\hat{X}_B$	Ratio
s_1	1	0	0	0	2	12	$\frac{12}{2}$
R_1	0	1	0	0	3	4	$\frac{4}{3}$
R_2	0	0	1	0	(2)	1	$\frac{1}{2}$
	0	$-M$	$-M$	1	$-5M + 2$	$-5M$	

$\therefore$ R_2 leaves the basis.

First Iteration: Introduce x_3 and drop R_2.

We have

$$\hat{B}^{-1}_{\text{next}} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{-3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & -M & \frac{3M-2}{2} & 1 \end{pmatrix} \text{ and therefore}$$

$$(Z_j^* - C_j) = (C_B B^{-1} \ 1) \ \hat{A} = \left(0 \ -M \ \frac{3M-2}{2} \ 1\right) \hat{A}$$

$$= \left(0 \ -M \ \frac{3M-2}{2} \ 1\right) (\hat{a}_1 \ \hat{a}_2 \ \hat{a}_5 \ \hat{a}_7)$$

$$= \left(\frac{M-2}{2}, \ -2M + 1, \ M, \ \frac{5M-2}{2}\right)$$

Since $(Z_2^* - C_2)$ is most negative, x_2 enters the basis.

$$\therefore \ \hat{x}_2 = \hat{B}^{-1}_{\text{curr}} \ \hat{a}_2 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ -2M + 1 \end{pmatrix}$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{\text{next}} \ \hat{b} = \begin{pmatrix} 11 \\ \frac{5}{2} \\ \frac{1}{2} \\ \frac{-5M-2}{2} \end{pmatrix}$$

The new revised simplex table is

Y_B	$\hat{B}^{-1}_{next}$				$\hat{x}_2$	$\hat{X}_B$	Ratio
s_1	1	0	-1	0	-3	11	–
R_1	0	1	$\frac{-3}{2}$	0	(2)	$\frac{5}{2}$	$\frac{5}{4}$
x_3	0	0	$\frac{1}{2}$	0	(2)	$\frac{1}{2}$	–
	0	$-M$	$\frac{-3M-2}{2}$	1	$-2M+1$	$\frac{-5M-2}{2}$	

$\therefore$ R_1 leaves the basis.

Second Iteration: Introduce x_2 and drop R_1.

We have $\hat{B}^{-1}_{curr} = \begin{pmatrix} 1 & \frac{3}{2} & \frac{-13}{4} & 0 \\ 0 & \frac{1}{2} & \frac{-3}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} & \frac{-1}{4} & 1 \end{pmatrix}$ and therefore

$$(Z_j^* - C_j) = (C_B B^{-1} \ 1)\ \hat{A} = \begin{pmatrix} 0 & \frac{-1}{2} & \frac{-1}{4} & 1 \end{pmatrix} (\hat{a}_1 \ \hat{a}_5 \ \hat{a}_6 \ \hat{a}_7)$$

$$= \left(\frac{-3}{4}, \frac{1}{2}, M - \frac{1}{2}, M - \frac{1}{4}\right)$$

Since $(Z_1^* - C_1)$ is the most negative, x_1 enters the basis.

$$\therefore \hat{x}_1 = \hat{B}^{-1}_{curr}\ \hat{a}_1 = \hat{B}^{-1}_{curr} \begin{pmatrix} 2 \\ -5 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{17}{4} \\ \frac{-1}{4} \\ \frac{-3}{2} \\ \frac{-3}{4} \end{pmatrix}$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{curr}\ \hat{b} = \hat{B}^{-1}_{curr} \begin{pmatrix} 12 \\ 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{59}{4} \\ \frac{5}{4} \\ \frac{1}{2} \\ \frac{-9}{4} \end{pmatrix}$$

The new revised simplex table is

Y_B	$\hat{B}^{-1}_{next}$				$\hat{x}_1$	$\hat{X}_B$	Ratio
s_1	1	$\frac{2}{3}$	$\frac{-13}{4}$	0	$\left(\frac{17}{4}\right)$	$\frac{59}{4}$	$\frac{59}{17}$
x_2	0	$\frac{1}{2}$	$\frac{-3}{4}$	0	$\frac{-1}{4}$	$\frac{5}{4}$	–
x_3	0	0	$\frac{1}{2}$	0	$\frac{-3}{2}$	$\frac{1}{2}$	–
	0	$\frac{-1}{2}$	$\frac{-1}{4}$	1	$\frac{-3}{4}$	$\frac{-9}{4}$	

$\therefore$ s_1 leaves the basis.

Third Iteration: Introduce x_1 and drop s_1, we have

$$\hat{B}^{-1}_{next} = \begin{pmatrix} \frac{4}{17} & \frac{6}{17} & \frac{-13}{17} & 0 \\ \frac{1}{7} & \frac{10}{17} & \frac{-16}{17} & 0 \\ \frac{6}{17} & \frac{9}{17} & \frac{-11}{17} & 0 \\ \frac{3}{17} & \frac{-4}{17} & \frac{-14}{17} & 1 \end{pmatrix}$$

$$\text{Therefore } (Z_j^* - C_j) = (C_B B^{-1} \ 1)\ \hat{A}$$

$$= \begin{pmatrix} \frac{3}{17} & \frac{-4}{17} & \frac{-14}{17} & 1 \end{pmatrix} (\hat{a}_4 \ \hat{a}_5 \ \hat{a}_6 \ \hat{a}_7)$$

$$= \left(\frac{3}{17}, \frac{4}{17}, M - \frac{4}{17}, M - \frac{14}{17}\right)$$

Since M is a very large quantity, all $(Z_j^* - C_j) \geq 0$. Hence the current basic feasible solution is optimal.

$$\hat{X}_B = \hat{B}^{-1}_{curr} = \begin{pmatrix} \frac{4}{17} & \frac{6}{17} & \frac{-13}{17} & 0 \\ \frac{1}{7} & \frac{10}{17} & \frac{-16}{17} & 0 \\ \frac{6}{17} & \frac{9}{17} & \frac{-11}{17} & 0 \\ \frac{3}{17} & \frac{-4}{17} & \frac{-14}{17} & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{59}{17} \\ \frac{36}{17} \\ \frac{97}{17} \end{pmatrix}$$

$$i.e., \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{59}{17} \\ \frac{36}{17} \\ \frac{97}{17} \end{pmatrix}$$

$$\text{Min } Z = -4x_1 + x_2 + 2x_3 = -4\left(\frac{59}{17}\right) + \left(\frac{36}{17}\right) + 2\left(\frac{97}{17}\right)$$

$$= \frac{-6}{17}$$

$\therefore$ The optimum solution is

$$\text{Min } Z = \frac{-6}{17}, \quad x_1 = \frac{59}{17}, \quad x_2 = \frac{36}{17}, \quad x_3 = \frac{97}{17}$$

Example 6 ***Use revised simplex method to solve the LPP.***

$$\textit{Maximize } Z = 2x_1 + 3x_2$$

$$\textit{subject to } \quad x_2 - x_1 \geq 0$$
$$x_1 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution: The given LPP is

$$\text{Max } Z = 2x_1 + 3x_2$$
$$\text{subject to } \quad x_1 - x_2 \leq 0$$
$$x_1 + 0x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

By introducing the non-negative slack variables s_1 and s_2 the standard form of the LPP becomes

$$\text{Max } Z = 2x_1 + 3x_2 + 0s_1 + 0s_2$$
$$\text{subject to } \quad x_1 - x_2 + s_1 + 0s_2 = 0$$
$$x_1 + 0x_2 + 0s_1 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \geq 0$$
$$i.e., \quad \text{Max } Z = CX$$
$$\text{subject to } \quad AX = b$$
$$X \geq 0$$

where $C = (2 \ 3 \ 0 \ 0)$, $A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$$b = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} \quad \text{and } C_B = [\,0 \ \ 0\,]$$

The initial basic feasible solution is $s_1 = 0$, $s_2 = 4$,

with the initial basis $I_2 = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$

$$\text{Now } \hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix} = \begin{array}{c} \begin{matrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 \end{matrix} \\ \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & -3 & 0 & 0 \end{pmatrix} \end{array}$$

$$\hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$\hat{B}^{-1}_{curr} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The net evaluations are given by

$$(Z_j - C_j) = (C_B \ \ B^{-1} \ \ 1)\ \hat{A} = (C_B\, B^{-1} \ \ 1)\ (\hat{a}_1 \ \ \hat{a}_2)$$

$$= (0 \ 0 \ 1) \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ -2 & -3 \end{pmatrix}$$

$$= (-2, -3)$$

$$\hat{X}_B = \hat{B}^{-1}_{\text{curr}} = \begin{pmatrix} \frac{4}{17} & \frac{6}{17} & \frac{-13}{17} & 0 \\ \frac{1}{7} & \frac{10}{17} & \frac{-16}{17} & 0 \\ \frac{6}{17} & \frac{9}{17} & \frac{-11}{17} & 0 \\ \frac{3}{17} & \frac{-4}{17} & \frac{-14}{17} & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{59}{17} \\ \frac{36}{17} \\ \frac{97}{17} \end{pmatrix}$$

$$i.e., \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{59}{17} \\ \frac{36}{17} \\ \frac{97}{17} \end{pmatrix}$$

$$\text{Min } Z = -4x_1 + x_2 + 2x_3 = -4\left(\frac{59}{17}\right) + \left(\frac{36}{17}\right) + 2\left(\frac{97}{17}\right)$$

$$= \frac{-6}{17}$$

$\therefore$ The optimum solution is

$$\text{Min } Z = \frac{-6}{17}, \quad x_1 = \frac{59}{17}, \quad x_2 = \frac{36}{17}, \quad x_3 = \frac{97}{17}$$

Example 6 ***Use revised simplex method to solve the LPP.***

$$\textit{Maximize } Z = 2x_1 + 3x_2$$

$$\textit{subject to } \quad x_2 - x_1 \geq 0$$
$$x_1 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution: The given LPP is

$$\text{Max } Z = 2x_1 + 3x_2$$
$$\text{subject to } x_1 - x_2 \leq 0$$
$$x_1 + 0x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

By introducing the non-negative slack variables s_1 and s_2 the standard form of the LPP becomes

$$\text{Max } Z = 2x_1 + 3x_2 + 0s_1 + 0s_2$$
$$\text{subject to } x_1 - x_2 + s_1 + 0s_2 = 0$$
$$x_1 + 0x_2 + 0s_1 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \geq 0$$
$$i.e., \quad \text{Max } Z = CX$$
$$\text{subject to AX} = b$$
$$X \geq 0$$

where $C = (2 \;\; 3 \;\; 0 \;\; 0)$, $A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$$b = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} \quad \text{and } C_B = [\,0 \;\; 0\,]$$

The initial basic feasible solution is $s_1 = 0$, $s_2 = 4$,

with the initial basis $I_2 = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$

$$\text{Now } \hat{A} = \begin{pmatrix} A \\ -C \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 \end{array} \\ \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & -3 & 0 & 0 \end{pmatrix} \end{array}$$

$$\hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$\hat{B}^{-1}_{\text{curr}} = \begin{pmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The net evaluations are given by

$$(Z_j - C_j) = (C_B \;\; B^{-1} \;\; 1)\,\hat{A} = (C_B B^{-1} \;\; 1)\,(\hat{a}_1 \;\; \hat{a}_2)$$

$$= (0 \;\; 0 \;\; 1) \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ -2 & -3 \end{pmatrix}$$

$$= (-2, -3)$$

Since $(Z_2 - C_2)$ is the most negative, x_2 enters the basis.

$$\text{Now } \hat{x}_2 = \hat{B}^{-1}_{\text{curr}} \, \hat{a}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$$

$$\text{and } \hat{X}_B = \hat{B}^{-1}_{\text{curr}} \, \hat{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$\therefore$ The initial revised simplex table is

Y_B	$\hat{B}^{-1}_{\text{curr}}$			$\hat{x}_2$	$\hat{X}_B$	Ratio
s_1	1	0	0	-1	0	–
s_2	0	1	0	0	4	–
	0	0	1	-3	0	

Here it is not possible to find the smallest ratio with positive denominator. Hence it is not possible to identify the leaving variable.

$\therefore$ The problem has an unbounded solution.

EXERCISE

1. **Discuss the advantages of revised simplex method over the ordinary simplex method.**

   [MU. MCA. Nov. '96, BRU. BE. Nov. '96]

2. **List the steps involved in the Revised simplex method.**

   [MU. MCA. May '95, Nov. '97]

3. **In what way do you consider the revised simplex method is superior to the usual simplex method.** ***[BRU. M.Sc., '81]***

4. **Given the standard model as, maximize Z = CX and AX = B derive the revised simplex method for the initial table. Indicate the criterions for determining the vector to leave the basis and the vector to enter the basis to perform the transformation to the new basic feasible solution.** ***[MU. MCA. May '91]***

5. **Use revised simplex method to solve the LPP.**

$$\begin{aligned} \text{Maximize } Z &= 2x_1 + x_2 \\ \text{subject to } 3x_1 + 4x_2 &\leq 6 \\ 6x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

[MU. M.Sc., '83, MKU, B.Sc., '84]

6. **Use revised simplex method to solve the LPP.**

$$\begin{aligned} \text{Maximize } Z &= 3x_1 + 5x_2 \\ \text{subject to } x_1 &\leq 4 \\ x_2 &\leq 6 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1, x_2 &\geq 0 \end{aligned}$$

[MKU, ME '84]

7. **Use revised simplex method to solve the LPP.**

$$\begin{aligned} \text{Maximize } Z &= x_1 + 2x_2 \\ \text{subject to } x_1 + x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 5 \\ 3x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

[MKU, B.Sc., '82]

8. **Use revised simplex method to solve the LPP.**

$$\begin{aligned} \text{Maximize } Z &= 3x_1 + 2x_2 + 5x_1 \\ \text{subject to } x_1 + 2x_2 + x_3 &\geq 430 \\ 3x_1 + 2x_3 &\leq 460 \\ x_1 + 4x_2 &\leq 420 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[MKU, M.Sc., '85, BRU, M.Sc., '81]

9. **Use revised simplex method to solve the LPP.**

$$\begin{aligned} \text{Maximize } Z &= 5x_1 + 3x_2 \\ \text{subject to } 4x_1 + 5x_2 &\geq 10 \\ 5x_1 + 2x_2 &\leq 10 \\ 3x_1 + 8x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

10. Use revised simplex method to solve the LPP.

$$\text{Maximize } Z = x_1 + 2x_2$$
$$\text{subject to } 3x_1 + 2x_2 \geq 6$$
$$x_1 + 6x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

11. Use revised simplex method to solve the LPP.

$$\text{Minimize } Z = 3x_1 + x_2$$
$$\text{subject to } x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

12. Use revised simplex method to solve the LPP.

$$\text{Minimize } Z = 4x_1 + 2x_2 + 3x_3$$
$$\text{subject to } 2x_1 + 4x_3 \geq 5$$
$$2x_1 + 3x_2 + x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

Use the revised simplex method to solve the following LPP's.

13.
$$\text{Maximize } Z = 6x_1 - 2x_2 + 3x_3$$
$$\text{subject to } 2x_1 - x_2 + 2x_3 \leq 2$$
$$x_1 + 4x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

14.
$$\text{Maximize } Z = 30x_1 + 23x_2 + 29x_3$$
$$\text{subject to } 6x_1 + 5x_2 + 3x_3 \leq 26$$
$$4x_1 + 2x_2 + 5x_3 \leq 7$$
$$x_1, x_2, x_3 \geq 0$$

15.
$$\text{Maximize } Z = 2x_1 - 6x_2$$
$$\text{subject to } x_1 - 3x_2 \leq 6$$
$$2x_1 + 4x_2 \geq 8$$
$$-x_1 + 3x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

16.
$$\text{Maximize } Z = 2x_1 + 4x_2 + 6x_3 - 2x_4$$
$$\text{subject to } x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$3x_1 + 6x_2 + 3x_3 + 3x_4 = 30$$
$$x_1, x_2, x_3, x_4 \geq 0$$

ANSWERS

5. Max $Z = \frac{13}{7}$, $x_1 = \frac{2}{7}$, $x_2 = \frac{9}{7}$.
6. Max $Z = 36$, $x_1 = 2$, $x_2 = 6$.
7. Max $Z = 5$, $x_1 = 0$, $x_2 = 2.5$
8. Max $Z = 1360$, $x_1 = 0$, $x_2 = 105$, $x_3 = 230$.
9. Max $Z = \frac{185}{17}$, $x_1 = \frac{28}{17}$, $x_2 = \frac{15}{17}$
10. Max $Z = 3$, $x_1 = 3$, $x_2 = 0$.
11. Min $Z = 1$, $x_1 = 0$, $x_2 = 1$.
12. Min $Z = \frac{67}{12}$, $x_1 = 0$, $x_2 = \frac{11}{12}$, $x_3 = \frac{5}{4}$
13. Max $Z = 12$, $x_1 = 4$, $x_2 = 6$, $x_3 = 0$.
14. Max $Z = \frac{161}{2}$, $x_1 = 0$, $x_2 = \frac{7}{2}$, $x_3 = 0$.
15. Max $Z = 12$, $x_1 = 6$, $x_2 = 0$.
16. Max $Z = 30$, $x_1 = 2.5$, $x_2 = 2.5$, $x_3 = 2.5$, $x_4 = 0$.

1.6. BOUNDED VARIABLE METHOD

1.6.1 Bounded Variable Method

A linear programming problem may have, in addition to the regular constraints, some or all variables with lower and upper limits. *i.e.*, constraints of the type $l_j \leq x_j \leq u_j$, where l_j and u_j are the lower and upper bounds of the variable x_j respectively. The standard form of such a problem may be expressed as

$$\begin{aligned} \text{Maximize } Z &= CX \\ \text{subject to } AX &= b \\ l_j \leq x_j &\leq u_j \ (j = 1, 2, \ldots n) \end{aligned}$$

where $C = (c_1, c_2, \ldots, c_n)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}$$

1.6.2 Bounded Variable Algorithm (or) Upper Bound Algorithm (or) Lower and Upper Bound Technique:

Step 1: Convert the minimization LPP into a maximization LPP.

Step 2: If any b_i is negative, convert it into positive by multiplying both sides of the corresponding constraint by -1.

Step 3: If any of the variable is at a positive lower bound substitute it out at its lower bound by $l_j = x_j - x_j'$ (or) $x_j = l_j + x_j'$ (or) $x_j' = x_j - l_j$ where $x_j' \geq 0$.

Step 4: By introducing slack and or surplus variables, express the given LPP in standard form and obtain an initial basic feasible solution.

Step 5: Compute the net evaluations $(Z_j - C_j)$ and examine their sign.

(*i*) If all $(Z_j - C_j) \geq 0$, then the current basic feasible solution is optimal.

(*ii*) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal and go to the next step.

Step 6: Choose the most negative of $(Z_j - C_j)$. Let it be $(Z_r - C_r)$ for some $j = r$. The corresponding non-basic variable x_r enters the basis.

Step 7: Compute the quantities

$$\theta_1 = \underset{i}{\text{Min}} \left\{ \frac{X_{B_i}}{a_{ir}}, \ a_{ir} > 0 \right\}, \quad \theta_2 = \underset{i}{\text{Min}} \left\{ \frac{u_j - X_{B_i}}{-a_{ir}}, \ a_{ir} > 0 \right\},$$

u_r – the upper bound of the entering variable x_r and $\theta = \text{Min}(\theta_1, \theta_2, u_r)$. It is to be noted that if all $a_{ir} \geq 0$, then $\theta_2 = \infty$.

Case (i): If $\theta = \theta_1$ which corresponds to X_{Bk}, then x_k leaves the basis. Use the regular row operations of the simplex method.

Case (ii): If $\theta = \theta_2$ which corresponds to X_{B_k}, then x_k leaves the basis, x_k being non-basic at its upper bound must be substituted by using

$(X_{B_k})_r = (X_{B_k})'_r - a_{kr} u_r$, where $0 \leq (X_{B_k})'_r \leq u_r$,

and $(X_{B_k})_r$ denotes that the variable x_r is x_k. The non-basic variable $x_r = u_r - x_r'$, $0 \leq x_r' \leq u_r$.

Case (iii): If $\theta = u_r$, x_r is substituted at its upper bound while remaining non-basic. *i.e.*, it can be put at zero level by using $x_r = u_r - x_r'$, $0 \leq x_r' \leq u_r$.

Step 8: Go to step (5) and repeat the procedure until an optimum basic feasible solution is obtained or there is an indication of an unbounded solution.

Note: If any variable is without its upper bound, then assume the upper bound as ∞.

$\boxed{\textit{Example 1}}$ ***Use bounded variable technique, solve the LPP.***

$$\begin{aligned} \textbf{Maximize } Z &= x_2 + 3x_3 \\ \textbf{subject to } x_1 + x_2 + x_3 &\leq 10 \\ x_1 - 2x_3 &\geq 0 \\ 2x_1 - x_3 &\leq 10 \\ 0 \leq x_1 \leq 8, \ 0 \leq x_2 \leq 4, \ x_3 &\geq 0 \end{aligned}$$

[BNU., B.Sc., '90]

Solution: Given LPP is maximize $Z = 0x_1 + x_2 + 3x_3$

$$\begin{aligned} \text{subject to } x_1 + x_2 + x_3 &\leq 10 \\ -x_1 + 0x_2 + 2x_3 &\leq 0 \\ 2x_1 + 0x_2 - x_3 &\leq 10 \\ 0 \leq x_1 \leq 8, \ 0 \leq x_2 \leq 4, \ 0 \leq x_3 &\leq \infty \end{aligned}$$

By introducing the non-negative slack variables x_4, x_5, x_6, the standard form of the LPP becomes

$$\text{Max } Z = 0x_1 + x_2 + 3x_3$$

subject to
$$x_1 + x_2 + x_3 + x_4 + 0x_5 + 0x_6 = 10$$
$$-x_1 + 0x_2 + 2x_3 + 0x_4 + x_5 + 0x_6 = 0$$
$$2x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 + x_6 = 10$$
$$0 \le x_1 \le 8,\ 0 \le x_2 \le 4,\ 0 \le x_3 \le \infty,$$
$$0 \le x_4 \le \infty,\ 0 \le x_5 \le \infty,\ 0 \le x_6 \le \infty$$

The initial basic feasible solution is given by

$x_4 = 10,\ x_5 = 0,\ x_6 = 10$ (basic) ; ($x_1 = x_2 = x_3 = 0$, non basic)

Initial iteration:

		C_j	(0	1	3	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	10	1	1	1	1	0	0
0	x_5	0	−1	0	(2)	0	1	0
0	x_6	10	2	0	−1	0	0	1
$(Z_j - C_j)$		0	0	−1	−3	0	0	0
		u_r	8	4	∞	∞	∞	∞

Since $(Z_3 - C_3) = -3$ is the most negative, the non-basic variable x_3 enters the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{ir}},\ a_{ir} > 0\right\} = \text{Min}\left\{\frac{X_{B_1}}{a_{13}},\ \frac{X_{B_2}}{a_{23}}\right\}$$
$$= \text{Min}\left\{\frac{10}{1},\ \frac{0}{2}\right\} = \text{Min}\ \{10, 0\} = 0 \text{ (corresponding to } x_5)$$
$$\text{Now } \theta_2 = \text{Min}\left\{\frac{u_i - X_{B_i}}{-a_{ir}},\ a_{ir} < 0\right\}$$
$$= \text{Min}\left\{\frac{u_6 - X_{B_3}}{-a_{33}},\ a_{33} < 0\right\}$$
$$= \text{Min}\left\{\frac{\infty - 10}{-(-1)}\right\} = \infty \text{ (corresponding to } x_6)$$

and $u_3 = \infty$ (upper bound of the entering variable x_3)

$$\therefore\ \theta = \text{Min}\ (\theta_1, \theta_2, u_3) = \{0, \infty, \infty\} = 0\ \ i.e.,\ \theta = \theta_1$$

∴ The basic variable x_5 leaves the basis.

First Iteration: Introduce x_3 and drop x_5.

		C_j	(0	1	3	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	10	$\left(\frac{3}{2}\right)$	1	0	1	$\frac{-1}{2}$	0
3	x_3	0	$\frac{-1}{2}$	0	1	0	$\frac{1}{2}$	0
0	x_6	10	$\frac{3}{2}$	0	0	0	$\frac{1}{2}$	1
$(Z_j - C_j)$		0	$\frac{-3}{2}$	−1	0	0	$\frac{3}{2}$	0
		u_r	8	4	∞	∞	∞	∞

Since $(Z_1 - C_1) = \frac{-3}{2}$ is the most negative, the non-basic variable x_1 enters the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{i1}},\ a_{i1} > 0\right\} = \text{Min}\left\{\frac{X_{B_1}}{a_{11}},\ \frac{X_{B_3}}{a_{31}}\right\}$$
$$= \text{Min}\left\{\frac{10}{\frac{3}{2}},\ \frac{10}{\frac{3}{2}}\right\} = \text{Min}\left\{\frac{20}{3},\ \frac{20}{3}\right\} = \frac{20}{3}$$

[corresponding to both x_4 and x_6]

$$\theta_2 = \text{Min}\left\{\frac{u_i - X_{B_i}}{-a_{i1}},\ a_{i1} < 0\right\}$$
$$= \text{Min}\left\{\frac{u_3 - X_{B_2}}{-a_{21}},\ a_{21} < 0\right\}$$
$$= \text{Min}\left\{\frac{\infty - 0}{-\left(-\frac{1}{2}\right)}\right\} = \infty \quad \text{[corresponding to } x_3]$$

and $u_1 = 8$ [corresponding to the entering variable x_1]

$$\therefore\ \theta = \text{Min}\ \{\theta_1, \theta_2, u_1\} = \text{Min}\left\{\frac{20}{3}, \infty, 8\right\}$$
$$= \frac{20}{3} = \theta_1$$

i.e., $\theta = \theta_1$ ∴ The basic variable x_4 leaves the basis.

Second Iteration: Introduce x_1 and drop x_4.

	C_j		(0	1	3	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	x_1	$\frac{20}{3}$	1	$\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{-1}{3}$	0
3	x_3	$\frac{10}{3}$	0	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0
0	x_6	0	0	–1	0	–1	1	1
$(Z_j - C_j)$		10	0	0	0	1	1	0
	u_r		8	4	∞	∞	∞	∞

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = 10,\ x_1 = \frac{20}{3}, x_2 = 0,\ x_3 = \frac{10}{3}$.

Example 2 ***Use Bounded variable algorithm to solve***

$$Max\ Z = x_1 + 3x_2 - 2x_3$$

$$subject\ to\quad x_2 - 2x_3 \leq 1$$

$$2x_1 + x_2 + 2x_3 \leq 8$$

$0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 3,\ 0 \leq x_3 \leq 2$ ***[MU., MCA., Nov. '94]***

Solution: By introducing the non-negative slack variables x_4 and x_5, the standard form of the LPP becomes

$$\text{Maximize } Z = x_1 + 3x_2 - 2x_3 + 0x_4 + 0x_5$$

$$\text{subject to}\quad 0x_1 + x_2 - 2x_3 + x_4 + 0x_5 = 1$$

$$2x_1 + x_2 + 2x_3 + 0x_4 + x_5 = 8$$

$$0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 3,\ 0 \leq x_3 \leq 2,$$

$$0 \leq x_4 \leq \infty,\ 0 \leq x_5 \leq \infty$$

The initial basic feasible solution is given by

$x_4 = 1,\ x_5 = 8$, (basic) $(x_1 = x_2 = x_3 = 0$, non basic)

Initial Iteration:

	C_j		(1	3	–2	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
0	x_4	1	0	(1)	–2	1	0
0	x_5	8	2	1	2	0	1
$(Z_j - C_j)$		0	–1	–3	2	0	0
	u_r		1	3	2	∞	∞

Since $(Z_2 - C_2) = -3$ is the most negative, the non-basic variable x_2 enters the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{ir}},\ a_{ir} > 0\right\} = \text{Min}\left\{\frac{X_{B_1}}{a_{12}}, \frac{X_{B_2}}{a_{22}}\right\}$$

$$= \text{Min}\left\{\frac{1}{1}, \frac{8}{1}\right\} = 1 \text{ (corresponding to } x_4)$$

$$\theta_2 = \infty,\ u_2 = 3 \text{ (corresponding to } x_2)$$

$$\theta = \text{Min}\{\theta_1, \theta_2, u_2\} = \text{Min}\{1, \infty, 3\} = 1 = \theta_1$$

$\therefore$ The basic variable x_4 leaves the basis.

First Iteration: Introduce x_2 and drop x_4.

	C_j		(1	3	–2	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
3	x_2	1	0	1	–2	1	0
0	x_5	7	2	0	4	–1	1
$(Z_j - C_j)$		3	–1	0	–4	3	0
	u_r		1	3	2	∞	∞

Since $(Z_3 - C_3) = -4$ is the most negative, the non-basic variable x_3 enters the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{i3}},\ a_{i3} > 0\right\} = \text{Min}\left\{\frac{X_{B_2}}{a_{23}}\right\}$$

$$= \text{Min}\left\{\frac{7}{4}\right\} = \frac{7}{4} \text{ [corresponding to } x_5]$$

$$\theta_2 = \text{Min}\left\{\frac{u_i - X_{B_i}}{-a_{ir}},\ a_{ir} < 0\right\} = \text{Min}\left\{\frac{u_2 - X_{B_1}}{-a_{13}}\right\}$$

$$= \text{Min}\left\{\frac{3-1}{-(-2)}\right\} = \text{Min}\left\{\frac{2}{2}\right\} = 1 \text{ [corresponding to } x_2]$$

$$u_3 = 2 \text{ [corresponding to } x_3]$$

$$\therefore\ \theta = \text{Min}\ \{\theta_1,\ \theta_2,\ u_3\}$$

$$= \text{Min}\left\{\frac{7}{4},\ 1,\ 2\right\} = 1$$

$$\theta = \theta_2$$

$\therefore$ The basic variable x_2 leaves the basis.

Second Iteration: Introduce x_3 and drop x_2.

	C_j		(1	3	−2	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
−2	x_3	$\frac{-1}{2}$	0	$\frac{-1}{2}$	1	$\frac{-1}{2}$	0
0	x_5	9	2	2	0	1	1
$(Z_j - C_j)$		1	−1	−2	0	1	0
		u_r	1	3	2	∞	∞

Since x_2 being non-basic, it must be substituted at its upper bound by using

$$x_2 = u_2 - x_2' = 3 - x_2' = 3 - x_2',\quad 0 \le x_2' \le u_2$$

$$\therefore\ X_{B_1} = X_{B_1}' - a_{kr}\,u_r = X_{B_1}' - a_{12}\,u_2 = \frac{-1}{2} - \left(-\frac{1}{2}\right)3 = 1$$

$$X_{B_2} = X_{B_2}' - a_{22}\,u_2 = 9 - 2\,(3) = 3$$

$$\text{and } Z_0 = Z_0' - (Z_2 - C_2)\,u_2 = 1 - (-2)\,(3) = 7$$

Using the updated values, the simplex table becomes

	C_j		(1	3	−2	0	0)
C_B	Y_B	X_B	x_1	x_2'	x_3	x_4	x_5
−2	x_3	1	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0
0	x_5	3	2	−2	0	1	1
$(Z_j - C_j)$		7	−1	2	0	1	0
		u_r	1	3	2	∞	∞

Since $(Z_1 - C_1) = -1$, the non-basic variable x_1 enters the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{ir}},\ a_{ir} > 0\right\} = \text{Min}\left\{\frac{X_{B_i}}{a_{i1}},\ a_{i1} > 0\right\}$$

$$= \text{Min}\left\{\frac{X_{B_2}}{a_{21}}\right\} = \text{Min}\left\{\frac{3}{2}\right\} = \frac{3}{2} \text{ [corresponding to } x_5]$$

$$\theta_2 = \infty \text{ (since all } a_{ir} \ge 0) \text{ and}$$

$$u_1 = 1 \text{ [corresponding to the entering variables } x_1]$$

$$\therefore\ \theta = \text{Min}\ \{\theta_1,\ \theta_2,\ u_1\}$$

$$= \text{Min}\left\{\frac{3}{2},\ \infty,\ 1\right\} = 1$$

$$\therefore\ \theta = u_1.$$

So x_1 is substituted at its upper bound and remains non-basic.

$$\text{Thus } x_1 = u_1 - x_1',\ 0 \le x_1' \le u_1$$

$$= 1 - x_1',\ 0 \le x_1' \le 1$$

The basic variables are updated by using

$$X_{B_1} = X_{B_1}' - a_{11}\,u_1 = 1 - 0\,(1) = 1$$

$$X_{B_2} = X_{B_2}' - a_{21}\,u_1 = 3 - 2\,(1) = 1$$

$$\text{and } Z_0 = Z_0' - (Z_1 - C_1)\,u_1 = 7 - (-1)\cdot 1 = 8$$

Using the updated values of the basic variables, the simplex table becomes

	C_j		(1	3	−2	0	0)
C_B	Y_B	X_B	x_1'	x_2'	x_3	x_4	x_5
−2	x_3	1	0	$\frac{1}{2}$	1	$\frac{-1}{2}$	0
0	x_5	1	−2	−2	0	1	1
$(Z_j - C_j)$		8	1	2	0	1	0
		u_r	1	3	2	∞	∞

Since all $(Z_j - C_j) \ge 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = 8,\ x_1' = 0,\ x_2' = 0,\ x_3 = 1.$

$$\text{but } x_1 = u_1 - x_1' = 1 - 0 = 1$$

$$x_2 = u_2 - x_2' = 3 - 0 = 3 \text{ and } x_3 = 1$$

$\therefore$ The optimal solution is Max Z = 8, $x_1 = 1$, $x_2 = 3$, $x_3 = 1$.

Example 3 ***Use the bounded variable algorithm to solve***

$$\textit{Max } Z = 3x_1 + 5x_2 + 2x_3$$

$$\textit{subject to } x_1 + 2x_2 + 2x_3 \leq 14$$

$$2x_1 + 4x_2 + 3x_3 \leq 23$$

$$0 \leq x_1 \leq 4,\ 2 \leq x_2 \leq 5,\ 0 \leq x_3 \leq 3$$

[BRU, M.Sc., '88]

Solution: Since x_2 has a positive lower bound,

we put $x_2' = x_2 - 2$ *i.e.,* $x_2 = x_2' + 2 \ \therefore\ 0 \leq x_2' \leq 3$

Now the given LPP becomes

$$\text{Maximize } Z = 3x_1 + 5(x_2' + 2) + 2x_3$$
$$= 3x_1 + 5x_2' + 10 + 2x_3$$

subject to

$$x_1 + 2(x_2' + 2) + 2x_3 \leq 14 \Rightarrow x_1 + 2x_2' + 2x_3 \leq 10$$
$$2x_1 + 4(x_2' + 2) + 3x_3 \leq 23 \Rightarrow 2x_1 + 4x_2' + 3x_3 \leq 15$$
$$0 \leq x_1 \leq 4,\ 0 \leq x_2' \leq 3,\ 0 \leq x_3 \leq 3,$$

By introducing the non-negative slack variables x_4 and x_5, the standard form of the LPP becomes,

$$\text{Maximize } Z = 3x_1 + 5x_2' + 2x_3 + 0x_4 + 0x_5 + 10$$

$$\text{subject to } x_1 + 2x_2' + 2x_3 + x_4 + 0x_5 = 10$$
$$2x_1 + 4x_2' + 3x_3 + 0x_4 + x_5 = 15$$
$$0 \leq x_1 \leq 4,\ 0 \leq x_2' \leq 3,\ 0 \leq x_3 \leq 3,\ 0 \leq x_4 \leq \infty,\ 0 \leq x_5 \leq \infty$$

The initial basic feasible solution is given by

$x_4 = 10$, $x_5 = 15$, (basic) ($x_1 = x_2 = x_3 = 0$, non basic)

Initial Iteration:

		C_j	(3	5	2	0	0)
C_B	Y_B	X_B	x_1	x_2'	x_3	x_4	x_5
0	x_4	10	1	2	2	1	0
0	x_5	15	2	4	3	0	1
$(Z_j - C_j)$		10	–3	–5	–2	0	0
		u_r	4	3	3	∞	∞

Since $(Z_2 - C_2) = -5$ is the most negative, x_2' enters the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{ir}},\ a_{ir} > 0\right\}$$
$$= \text{Min}\left\{\frac{10}{2}, \frac{15}{4}\right\} = \frac{15}{4} \quad \text{[corresponding to } x_5\text{]}$$
$$\theta_2 = \infty,\ u_2 = 3 \quad \text{[corresponding to } x_2'\text{]}$$
$$\therefore \theta = \text{Min}\{\theta_1, \theta_2, u_2\}$$
$$= \text{Min}\left\{\frac{15}{4}, \infty, 3\right\} = 3 = u_2$$
$$\therefore \theta = u_2$$

$\therefore$ x_2' is substituted at its upper bound and remains non-basic. Thus $x_2' = u_2 - x_2'' = 3 - x_2''$, where $0 \leq x_2'' \leq 3$.

The basic variables are updated by using $(X_{B_i})_r = (X_{B_i})_r' - a_{kr}\, u_r$

$$X_{B_1} = X_{B_1}' - a_{12}\, u_2 = 10 - 2 \times 3 = 4$$
$$X_{B_2} = X_{B_2}' - a_{22}\, u_2 = 15 - 4 \times 3 = 3$$
$$\text{and } Z_0 = Z_0' - (Z_2 - C_2)\, u_2 = 10 - (-5)(3) = 25$$

Using the updated values of the basic variables, the initial simplex table becomes,

		C_j	(3	5	2	0	0)
C_B	Y_B	X_B	x_1	x_2''	x_3	x_4	x_5
0	x_4	4	1	–2	2	1	0
0	x_5	3	(2)	–4	3	0	1
$(Z_j - C_j)$		25	–3	5	–2	0	0
		u_r	4	3	3	∞	∞

Since $(Z_1 - C_1) = -3$ is the most negative, the non-basic variable x_1 enters the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{i1}},\ a_{i1} > 0\right\}$$
$$= \text{Min}\left\{\frac{4}{1}, \frac{3}{2}\right\} = \frac{3}{2} \quad \text{[corresponding to } x_5\text{]}$$
$$\theta_2 = \infty,\ u_1 = 4 \quad \text{[corresponding to } x_1\text{]}$$
$$\therefore \theta = \text{Min}\{\theta_1, \theta_2, u_1\} = \text{Min}\left\{\frac{3}{2}, \infty, 4\right\} = \theta_1 = \frac{3}{2}$$

x_5 leaves the basis.

First Iteration: Introduce x_1 and drop x_5

	C_j	(3	5	2	0	0)
C_B Y_B	X_B	x_1	x_2''	x_3	x_4	x_5
0 x_4	$\frac{5}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$
3 x_1	$\frac{3}{2}$	1	-2	$\frac{3}{2}$	0	$\frac{1}{2}$
$(Z_j - C_j)$	$\frac{59}{2}$	0	-1	$\frac{5}{2}$	0	$\frac{3}{2}$
	u_r	4	3	3	∞	∞

Since $(Z_2 - C_2) = -1$ is the most negative, the non-basic variable x_2'' enters the basis.

$$\text{Now } \theta_1 = \infty,\ \theta_2 = \text{Min}\left\{\frac{4-\frac{3}{2}}{-(-2)}\right\} = \frac{5}{4},\ u_2 = 3$$

$$\theta = \text{Min}\{\theta_1, \theta_2, u_2\} = \text{Min}\left\{\infty, \frac{5}{4}, 3\right\} = \theta_2 = \frac{5}{4}$$

x_1 leaves the basis.

Second Iteration: Introduce x_2'' and drop x_1.

	C_j	(3	5	2	0	0)
C_B Y_B	X_B	x_1	x_2''	x_3	x_4	x_5
0 x_4	$\frac{5}{2}$	0	0	$\frac{1}{2}$	1	$\frac{-1}{2}$
0 x_2''	$\frac{-3}{4}$	$\frac{-1}{2}$	1	$\frac{-3}{4}$	0	$\frac{-1}{4}$
$(Z_j - C_j)$	$\frac{115}{4}$	$\frac{-1}{2}$	0	$\frac{7}{4}$	0	$\frac{5}{4}$
	u_r	4	3	3	∞	∞

Since x_1 being non-basic, it must be substituted at its upper bound by using $x_1 = u_1 - x_1' = 4 - x_1',\ 0 \le x_1' \le u_1$.

$$X_{B_1} = X_{B_1}' - a_{kr}\, u_r$$

$$= X_{B_1}' - a_{11}\, u_1 = \frac{5}{2} - 4\,(0) = \frac{5}{2}$$

$$X_{B_2} = X_{B_2}' - a_{21}\, u_1 = \frac{-3}{4} - 4\left(-\frac{1}{2}\right)$$

$$= \frac{-3}{4} + 2 = \frac{5}{4}$$

$$\text{and } Z_0 = Z_0' - (Z_1 - C_1)\, u_1 = \frac{115}{4} + 2 = \frac{123}{4}$$

Using the updated values, the simplex table becomes

	C_j	(3	5	2	0	0)
C_B Y_B	X_B	x_1'	x_2''	x_3	x_4	x_5
0 x_4	$\frac{5}{2}$	0	0	$\frac{1}{2}$	1	$\frac{-1}{2}$
5 x_2''	$\frac{5}{4}$	$\frac{1}{2}$	1	$\frac{-3}{4}$	0	$\frac{-1}{4}$
$(Z_j - C_j)$	$\frac{123}{4}$	$\frac{1}{2}$	0	$\frac{7}{4}$	0	$\frac{5}{4}$
	u_r	4	3	3	∞	∞

Since all $(Z_j - C_j) \ge 0$ the current feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = \frac{123}{4}$, $x_1' = 0$, $x_2'' = \frac{5}{4}$, $x_3 = 0$

$$\text{But } x_1 = u_1 - x_1' = 4 - 0 = 4$$

$$x_2 = x_2' + 2 = (u_2 - x_2'') + 2$$

$$= \left(3 - \frac{5}{4}\right) + 2$$

$$= \frac{7}{4} + 2 = \frac{15}{4}$$

$$\therefore \text{Max } Z = \frac{123}{4},\quad x_1 = 4,\quad x_2 = \frac{15}{4},\quad x_3 = 0$$

Example 4 ***Use bounded variable algorithm to solve***

$$\textit{Maximize } Z = 4x_1 + 2x_2 + 6x_3$$

$$\textit{subject to } 4x_1 - x_2 \le 9$$

$$-x_1 + x_2 + 2x_3 \le 8$$

$$-3x_1 + x_2 + 4x_3 \le 12$$

$$1 \le x_1 \le 3,\ 0 \le x_2 \le 5,\ 0 \le x_3 \le 2$$

Solution: Since x_1 has a positive lower bound,
we put $x_1 = x_1' + 1$ (or) $x_1' = x_1 - 1$ so that $0 \le x_1' \le 2$

$\therefore$ The given LPP becomes

$$\begin{aligned} \text{Max } Z &= 4x_1' + 2x_2 + 6x_3 + 4 \\ \text{subject to } 4x_1' - x_2 &\le 5 \\ -x_1' + x_2 + 2x_3 &\le 9 \\ -3x_1' + x_2 + 4x_3 &\le 15 \\ 0 \le x_1' \le 2, \ 0 \le x_2 &\le 5, \ 0 \le x_3 \le 2 \end{aligned}$$

By introducing the non-negative slack variables x_4, x_5 and x_6, the standard form of the LPP becomes

$$\begin{aligned} \text{Max } Z &= 4x_1' + 2x_2 + 6x_3 + 0x_4 + 0x_5 + 0x_6 + 4 \\ \text{subject to } 4x_1' - x_2 + 0x_3 + x_4 + 0x_5 + 0x_6 &= 5 \\ -x_1' + x_2 + 2x_3 + 0x_4 + x_5 + 0x_6 &= 9 \\ -3x_1' + x_2 + 4x_3 + 0x_4 + 0x_5 + x_6 &= 15 \\ 0 \le x_1' \le 2, \ 0 \le x_2 \le 5, \ 0 \le x_3 \le 2 & \\ 0 \le x_4 \le \infty, 0 \le x_5 \le \infty, \ 0 \le x_6 \le \infty & \end{aligned}$$

The initial basic feasible solution is given by

$x_4 = 5$, $x_5 = 9$, $x_6 = 15$, (basic) ($x_1' = x_2 = x_3 = 0$, non-basic)

Initial Iteration:

		C_j	(4	2	6	0	0	0)
C_B	Y_B	X_B	x_1'	x_2	x_3	x_4	x_5	x_6
0	x_4	5	4	–1	0	1	0	0
0	x_5	9	–1	1	2	0	1	0
0	x_6	15	–3	1	4	0	0	1
$(Z_j - C_j)$		4	–4	–2	–6	0	0	0
		u_r	2	5	2	∞	∞	∞

Since $(Z_3 - C_3) = -6$ is the most negative, the non-basic variable x_3 enter the basis.

$$\begin{aligned} \text{Now } \theta_1 &= \text{Min}\left\{\frac{X_{B_i}}{a_{i3}}, \ a_{i3} > 0\right\} \\ &= \text{Min}\left\{\frac{9}{2}, \frac{15}{4}\right\} = \frac{15}{4} && \text{[corresponding to } x_6\text{]} \\ \theta_2 &= \infty, \ u_3 = 2 && \text{[corresponding to } x_3\text{]} \\ \theta &= \text{Min}\ \{\theta_1, \theta_2, u_3\} = \text{Min}\left\{\frac{15}{4}, \infty, 2\right\} = 2 \end{aligned}$$

$\therefore$ $\theta = u_3$. So x_3 is substituted at its upper bound and remains non-basic. Thus $x_3 = u_3 - x_3'$, $0 \le x_3' \le 2$. The basic variables are updated by using,

$$\begin{aligned} (X_{B_k})_r &= (X_{B_k})_r' - a_{kr}\, u_r \\ X_{B_1} &= X_{B_1}' - a_{13}\, u_3 = 5 - (0)(2) = 5 \\ X_{B_2} &= X_{B_2}' - a_{23}\, u_3 = 9 - (2)(2) = 5 \\ X_{B_3} &= X_{B_3}' - a_{33}\, u_3 = 15 - 4(2) = 7 \\ \text{and } Z_0 &= Z_0' - (Z_3 - C_3)\, u_3 = 4 - (-6)(2) = 16 \end{aligned}$$

Using the updated values of the basic variables, the simplex table becomes

		C_j	(4	2	6	0	0	0)
C_B	Y_B	X_B	x_1'	x_2	x_3'	x_4	x_5	x_6
0	x_4	5	(4)	–1	0	1	0	0
0	x_5	5	–1	1	–2	0	1	0
0	x_6	7	–3	1	–4	0	0	1
$(Z_j - C_j)$		16	–4	–2	–6	0	0	0
		u_r	2	5	2	∞	∞	∞

Since $(Z_1 - C_1) = -4$ is the most negative, the non-basic variable x_1' enter the basis

$$\begin{aligned} \text{Now } \theta_1 &= \text{Min}\left\{\frac{X_{B_i}}{a_{ir}}, \ a_{ir} > 0\right\} \\ &= \text{Min}\left\{\frac{5}{4}\right\} = \frac{5}{4} && \text{[corresponding to } x_4\text{]} \\ \theta_2 &= \text{Min}\left\{\frac{u_i - X_{B_i}}{-a_{ir}}, \ a_{ir} < 0\right\} \\ &= \text{Min}\left\{\frac{\infty - 5}{-(-1)}, \frac{\infty - 16}{-(-3)}\right\} = \infty \\ \text{and } u_1 &= 2 \quad \text{[corresponding to } x_1'\text{]} \\ \therefore \theta &= \text{Min}\ \{\theta_1, \theta_2, u_1\} = \text{Min}\left\{\frac{5}{4}, \infty, 2\right\} = \frac{5}{4} \end{aligned}$$

$\therefore$ $\theta = \theta_1 = \dfrac{5}{4}$. The basic variable x_4 leaves the basis.

First Iteration: Introduce x_1' and drop x_4.

		C_j	(4	2	6	0	0	0)
C_B	Y_B	X_B	x_1'	x_2	x_3'	x_4	x_5	x_6
0	x_1'	$\frac{5}{4}$	1	$\left(\frac{-1}{4}\right)$	0	$\frac{1}{4}$	0	0
0	x_5	$\frac{25}{4}$	0	$\frac{3}{4}$	–2	$\frac{1}{4}$	1	0
0	x_6	$\frac{43}{4}$	0	$\frac{1}{4}$	–4	$\frac{3}{4}$	0	1
$(Z_j - C_j)$		21	0	–3	6	1	0	0
		u_r	2	5	2	∞	∞	∞

Since $(Z_2 - C_2) = -3$ is the most negative, the non-basic variable x_2 enter the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{ir}},\ a_{ir} > 0\right\}$$

$$= \text{Min}\left\{\frac{25}{3}, 43\right\} = \frac{25}{3} \qquad \text{[corresponding to } x_5\text{]}$$

$$\theta_2 = \text{Min}\left\{\frac{u_i - X_{B_i}}{-a_{ir}},\ a_{ir} < 0\right\}$$

$$= \text{Min}\left\{\frac{2-\frac{5}{4}}{-\left(-\frac{1}{4}\right)}\right\} = 3 \qquad \text{[corresponding to } x_1'\text{]}$$

$$\text{and } u_2 = 5$$

$$\therefore\ \theta = \text{Min}\{\theta_1, \theta_2, u_2\} = \text{Min}\left\{\frac{25}{3}, 3, 5\right\} = 3$$

$$\therefore\ \theta = \theta_2 \Rightarrow x_1' \text{ leaves the basis.}$$

Second Iteration: Introduce x_2 and drop x_1'.

		C_j	(4	2	6	0	0	0)
C_B	Y_B	X_B	x_1'	x_2	x_3'	x_4	x_5	x_6
2	x_2	–5	–4	1	0	–1	0	0
0	x_5	10	3	0	–2	1	1	0
0	x_6	12	1	0	–4	1	0	1
$(Z_j - C_j)$		6	–12	0	6	–2	0	0
		u_r	2	5	2	∞	∞	∞

Since x_1' becoming non-basic, it should be substituted at its upper bound by putting $x_1' = u_1 - x_1'' = 2 - x_1''$, $0 \le x_1'' \le 2$.

The basic variables are updated by using

$$(X_{B_k})_r = (X_{B_k})_r' - a_{kr}\, u_r$$

$$X_{B_1} = X_{B_1}' - a_{11}\, u_1 = -5 - (-4)(2) = 3$$

$$X_{B_2} = X_{B_2}' - a_{21}\, u_1 = 10 - 3(2) = 4$$

$$X_{B_3} = X_{B_3}' - a_{31}\, u_1 = 12 - 1(2) = 10$$

$$\text{and } Z_0 = Z_0' - (Z_1 - C_1)\, u_1 = 6 - (-12)(2) = 30$$

By using the updated values of the basic variables, the simplex table becomes

		C_j	(4	2	6	0	0	0)
C_B	Y_B	X_B	x_1''	x_2	x_3'	x_4	x_5	x_6
2	x_2	3	4	1	0	–1	0	0
0	x_5	4	–3	0	–2	1	1	0
0	x_6	10	–1	0	–4	1	0	1
$(Z_j - C_j)$		30	12	0	6	–2	0	0
		u_r	2	5	2	∞	∞	∞

Since $(Z_4 - C_4) = -2$ is the most negative, the non-basic variable x_4 enter the basis.

$$\text{Now } \theta_1 = \text{Min}\left\{\frac{X_{B_i}}{a_{ir}},\ a_{ir} > 0\right\}$$

$$= \text{Min}\left\{\frac{4}{1}, \frac{10}{1}\right\} = 4 \quad [\text{corresponding to } x_5]$$

$$\theta_2 = \text{Min}\left\{\frac{u_i - X_{B_i}}{-a_{ir}}, a_{i4} < 0\right\}$$

$$= \text{Min}\left\{\frac{5-3}{-(-1)}\right\} = 2 \quad [\text{corresponding to } x_2]$$

$$\text{and } u_2 = \infty$$

$$\therefore \theta = \text{Min}\{\theta_1, \theta_2, u_4\} = \{4, 2, \infty\} = 2$$

$$\therefore \theta = 2 = \theta_2 \Rightarrow x_2 \text{ leaves the basis.}$$

Third Iteration: Introduce x_4 and drop x_2.

		C_j	(4	2	6	0	0	0)
C_B	Y_B	X_B	x_1''	x_2	x_3'	x_4	x_5	x_6
0	x_4	–3	–4	–1	0	1	0	0
0	x_5	7	1	1	–2	0	1	0
0	x_6	13	3	1	–4	0	0	1
$(Z_j - C_j)$		24	4	–2	6	0	0	0
		u_r	2	5	2	∞	∞	∞

Since x_2 becoming non-basic, it should be substituted at its upper bound by putting $x_2 = u_2 - x_2' = 5 - x_2'$, $0 \le x_2' \le 5$.

The basic variables are updated by using $(X_{B_k})_r = (X_{B_k})_r' - a_{kr}\, u_r$

$$X_{B_1} = X_{B_1}' - a_{12}\, u_2 = -3 - (-1)(5) = 2$$

$$X_{B_2} = X_{B_2}' - a_{22}\, u_2 = 7 - (1)(5) = 2$$

$$X_{B_3} = X_{B_3}' - a_{32}\, u_2 = 13 - 1(5) = 8$$

$$\text{and } Z_0 = Z_0' - (Z_2 - C_2)\, u_2 = 24 - (-2)(5) = 34$$

By using the updated values of the basic variables, the simplex table becomes

		C_j	(4	2	6	0	0	0)
C_B	Y_B	X_B	x_1''	x_2'	x_3'	x_4	x_5	x_6
0	x_4	2	–4	1	0	1	0	0
0	x_5	2	1	–1	–2	0	1	0
0	x_6	8	3	–1	–4	0	0	1
$(Z_j - C_j)$		34	4	2	6	0	0	0
		u_r	2	5	2	∞	∞	∞

Since all $(Z_j - C_j) \ge 0$ the current basic feasible solution is optimal.

$\therefore$ The optimal solution is

$$\text{Max } Z = 34, \quad x_1'' = 0, \quad x_2' = 0, \quad x_3' = 0$$

$$\text{But } x_1 = x_1' + 1 = (2 - x_1'') + 1 = (2 - 0) + 1 = 3$$

$$x_2 = 5 - x_2' = 5 - 0 = 5$$

$$x_3 = 2 - x_3' = 2 - 0 = 2$$

$$\therefore \text{Max } Z = 34, \quad x_1 = 3, \quad x_2 = 5, \quad x_3 = 2$$

EXERCISE

1. Write an algorithm for the bounded variable problem.

[MU, MCA, Nov. '95]

2. Use bounded variable algorithm to solve

$$\textbf{Maximize } Z = 3x_1 + x_2 + x_3 + 7x_4$$

$$\text{subject to } 2x_1 + 3x_2 - x_3 + 4x_4 \le 40$$

$$-2x_1 + 2x_2 + 5x_3 - x_4 \le 35$$

$$x_1 + x_2 - 2x_3 + 3x_4 \le 100$$

$$x_1 \ge 2, \quad x_2 \ge 1, \quad x_3 \ge 3, \quad x_4 \ge 4$$

3. **Solve the following LPP by lower and upper bound technique.**

$$\text{Maximize } Z = 4x_1 + 4x_2 + 3x_3$$
$$\text{subject to } -x_1 + 2x_2 + 3x_3 \leq 15$$
$$-x_2 + x_3 \leq 4$$
$$2x_1 + x_2 - x_3 \leq 6$$
$$x_1 - x_2 + 2x_3 \leq 10$$
$$0 \leq x_1 \leq 8,\ 0 \leq x_2 \leq 4,\ 0 \leq x_3 \leq 4$$

4. **Use bounded variable technique to solve**

$$\text{Maximize } Z = 3x_1 + 2x_2$$
$$\text{subject to } x_1 - 3x_2 \leq 3$$
$$x_1 - 2x_2 \leq 3$$
$$2x_1 + x_2 \leq 20$$
$$x_1 + 3x_2 \leq 30$$
$$-x_1 + x_2 \leq 6$$
$$0 \leq x_1 \leq 8,\ 0 \leq x_2 \leq 6$$

5. **Use upper bound algorithm to solve**

$$\text{Maximize } Z = 3x_1 + 5x_2 + 2x_3$$
$$\text{subject to } x_1 + x_2 + 2x_3 \leq 14$$
$$2x_1 + 4x_2 + 3x_3 \leq 34$$
$$0 \leq x_1 \leq 4,\ 7 \leq x_2 \leq 10,\ 0 \leq x_3 \leq 3$$

ANSWERS

2. $\text{Max } Z = \frac{387}{4},\ x_1 = \frac{71}{4},\ x_2 = 1,\ x_3 = \frac{29}{2},\ x_4 = 4.$

3. $\text{Max } Z = \frac{199}{5},\ x_1 = \frac{17}{5},\ x_2 = \frac{16}{5},\ x_3 = 4.$

4. $\text{Max } Z = 33,\ x_1 = 7,\ x_2 = 6.$

5. $\text{Max } Z = 44,\ x_1 = 4,\ x_2 = 7,\ x_3 = 0.$

1.7 SENSITIVITY ANALYSIS

1.7.1 Introduction

After formulating mathematical model to linear programming problems and then attaining the optimal solution of the problem, it may be required to study the effect of changes (discrete or continuous) in the different parameters of the problem, on the optimum solution, that is it may be desirable to see the sensitiveness of the feasible optimal solution, corresponding to the variations in the parameters.

The investigations that deal with changes in the optimal solutions due to discrete variations in the parameters a_{ij}, b_i and c_j are called ***sensitivity analysis*** (or) ***post optimality analysis.***

The purpose of sensitive analysis is to find, how to preserve, to a minimum, the additional computational efforts which arise in solving the problem as a new one. In most of the cases it is not necessary to solve the problem again but a relatively small amount of calculations applied to the old optimal solution will be sufficient.

In this chapter we shall discuss the following changes (variations) in the optimal solution of an LPP.

(i) Variations in the cost vector C.

(ii) Variations in the requirement vector b.

(iii) Variations in the elements a_{ij} of the coefficient matrix A.

(iv) Addition and deletion of a new variable.

(v) Addition and deletion of a new constraint.

The above variations can affect either the optimality or feasibility or both of the current solution and will result in one of the following three cases.

1) The basic variables and their optimum values remain essentially unchanged, *i.e.,* the optimal solution remains unchanged.
2) The basic variables remain the same, but their optimum values change.
3) The basic variables and their values change completely.

1.7.2 Variations affecting feasibility:

There are two types of changes that could affect the feasibility of the current solution:

(i) changes in resources availability (or right side of the constraints or requirement vector b) and

(ii) addition of new constraints.

1.7.3 Variation in the right side of constraints

Since the optimality condition $(Z_j - C_j)$ does not involve any of b_i, if any component b_i of the requirement vector $b = [b_1, b_2, \ldots, b_m]$ is changed, then this change will not effect the conditions of optimality. Hence if any b_i is changed to $b_i + \Delta b_i$ then the new solution thus obtained will remain optimal.

But $X_B = B^{-1} b$ depends on b. Therefore any change in b may affect the feasibility of the current solution.

Example 1 **Consider the LPP**

$$\text{Max } Z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$x_1 + 2x_2 \le 5$$
$$5x_1 - x_2 + 2x_3 = 2$$
$$\text{and } x_1, x_2, x_3 \ge 0$$

(a) **Solve the LPP.**

(b) **Discuss the effect of changing the requirement vector from $\begin{pmatrix}5\\2\end{pmatrix}$ to $\begin{pmatrix}7\\2\end{pmatrix}$, on the optimal solution.**

Solution: *(a)* By using Big-M method, the optimal simplex table is displayed below:

	C_j	(5	12	4	0	−M)
C_B Y_B	X_B	x_1	x_2	x_3	s_1	R_1
12 x_2	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0
4 x_3	$\frac{9}{4}$	$\frac{11}{4}$	0	1	$\frac{1}{4}$	$\frac{1}{2}$
$Z_j - C_j$	39	12	0	0	7	M+2

$\therefore$ The optimal solution is

$$\text{Max } Z = 39, \quad x_1 = 0, \quad x_2 = \frac{5}{2}, \quad x_3 = \frac{9}{4}.$$

(b) If the requirement vector $\begin{pmatrix}5\\2\end{pmatrix}$ is changed to $\begin{pmatrix}7\\2\end{pmatrix}$,

$$\text{then } X_B = B^{-1}b = \begin{pmatrix}\frac{1}{2} & 0\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}7\\2\end{pmatrix} = \begin{pmatrix}\frac{7}{2}\\ \frac{11}{4}\end{pmatrix} = \begin{pmatrix}x_2\\x_3\end{pmatrix}$$

Since both $x_2 \ge 0$ and $x_3 \ge 0$, the current solution consisting x_2, x_3 remains feasible. But the value of x_2 and x_3 are changed.

New optimal solution is

$$\text{Max } Z = 5x_1 + 12x_2 + 4x_3$$
$$= 5(0) + 12\left(\frac{7}{2}\right) + 4\left(\frac{11}{4}\right) = 53$$

Example 2 **Consider the LPP**

$$\text{Max } Z = 2x_1 + x_2 + 4x_3 - x_4$$

subject to

$$x_1 + 2x_2 + x_3 - 3x_4 \le 8$$
$$-x_2 + x_3 + 2x_4 \le 0$$
$$2x_1 + 7x_2 - 5x_3 - 10x_4 \le 21$$
$$\text{and } x_1, x_2, x_3, x_4 \ge 0$$

(a) **Solve the LPP.**

(b) **Discuss the effect of change of b_2 to 11.**

(c) **Discuss the effect of change of b to [3 – 2, 4].**

Solution: *(a)* By using regular simplex method, the optimal simplex table is displayed below:

	C_j	(2	1	4	−1	0	0	0)
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1 x_2	0	0	1	−1	−2	0	−1	0
2 x_1	8	1	0	3	1	1	2	0
0 s_3	5	0	0	−4	2	−2	3	1
$Z_j - C_j$	16	0	0	1	1	2	3	0

$\therefore$ The optimal solution is

$$\text{Max } Z = 16, \quad x_1 = 8, \quad x_2 = x_3 = x_4 = 0.$$

(b) If b_2 is changed to 11, then $b = [8, 11, 21]$

$$\text{then} \quad X_B = B^{-1}b = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 11 \\ 21 \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \\ 38 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ s_3 \end{pmatrix}$$

$\Rightarrow$ the solution is infeasible.

That is, the change of b_2 from 0 to 11 affects the feasibility of the solution.

Th changes in the final table reduce to

		C_j	(2	1	4	−1	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	x_2	−11	0	1	−1	(−2)	0	−1	0
2	x_1	30	1	0	3	1	1	2	0
0	s_3	38	0	0	−4	2	−2	3	1
$Z_j - C_j$		49	0	0	1	1	2	3	0

Since the current solution is infeasible, we have to use dual simplex method.

First Iteration: Drop x_2 and introduce x_4.

		C_j	(2	1	4	−1	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
−1	x_4	$\frac{11}{2}$	0	$\frac{-1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0
2	x_1	$\frac{49}{2}$	1	$\frac{1}{2}$	$\frac{5}{2}$	0	1	$\frac{3}{2}$	0
0	s_3	27	0	1	−5	0	−2	2	1
$Z_j - C_j$		$\frac{87}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	2	$\frac{5}{2}$	0

Since all $(Z_j - C_j) \geq 0$ and all $X_{B_i} \geq 0$, the current solution is feasible and optimal.

$\therefore$ The optimal solution to the new problem is

$$\text{Max } Z = \frac{87}{2}, \quad x_1 = \frac{49}{2}, \quad x_2 = 0 = x_3, \quad x_4 = \frac{11}{2}$$

(c) If b is changed from [8, 0, 21] to [3, −2, 4]

$$\text{then} \quad X_B = B^{-1}b \Rightarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -8 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ s_3 \end{pmatrix}$$

$\Rightarrow$ the solution is infeasible.

That is, the change of b from [8, 0, 21] to [3, −2, 4] affects the feasibility of the solution.

The changes in the final table reduce to

		C_j	(2	1	4	−1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	x_2	2	0	1	−1	−2	0	−1	0
2	x_1	−1	1	0	3	1	1	2	0
0	s_3	−8	0	0	(−4)	2	−2	3	1
$Z_j - C_j$		0	0	0	1	1	2	3	0

Since the solution is infeasible, we have to use dual simplex method.

First Iteration: Drop s_3 and introduce x_3.

		C_j	(2	1	4	−1	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	x_2	4	0	1	0	$\frac{-5}{2}$	$\frac{1}{2}$	$\frac{-7}{4}$	$\frac{-1}{4}$
2	x_1	−7	1	0	0	$\frac{5}{2}$	$\left(-\frac{1}{2}\right)$	$\frac{17}{4}$	$\frac{3}{4}$
4	x_3	2	0	0	1	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{-3}{4}$	$\frac{-1}{4}$
$Z_j - C_j$		−2	0	0	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{15}{4}$	$\frac{1}{4}$

Second Iteration: Drop x_1 and introduce s_1.

C_B	Y_B	C_j → X_B	x_1 (2)	x_2 (1)	x_3 (4)	x_4 (−1)	s_1 (0)	s_2 (0)	s_3 (0)
1	x_2	−3	1	1	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$
0	s_1	14	−2	0	0	−5	1	$\frac{-17}{4}$	$\frac{-3}{2}$
4	x_3	−5	1	0	1	2	0	$\frac{7}{2}$	$\frac{1}{2}$
$Z_j - C_j$		−23	3	0	0	9	0	$\frac{33}{2}$	$\frac{5}{2}$

Since x_3 leaves the basis and all $a_{3j} \geq 0$, the new problem posses no feasible solution.

1.7.4 Addition of a New Constraint

Addition of a new constraint may or may not affect the feasibility of the current optimal solution. The addition of new constraint can result in one of two conditions.

(a) If the new constraint is satisfied by the current optimal solution, then this new constraint is said to be ***redundant*** and its addition will not change the solution *ie.,* the current solution remains feasible as well as optimal.

(b) If the new constraint is not satisfied by the current optimal solution the current optimal solution becomes infeasible. In this case the new solution is obtained by using the dual simplex method.

However, in general, whenever a new constraint is added to a linear programming problem, the old optimal value will always be better or atleast equal to the new optimal value. In other words, ***addition of a new constraint cannot improve the optimal value of any linear programming problem.***

Example 1 **Consider the LPP**

Max $Z = 3x_1 + 4x_2 + x_3 + 7x_4$

subject to

$$8x_1 + 3x_2 + 4x_3 + x_4 \leq 7$$
$$2x_1 + 6x_2 + x_3 + 5x_4 \leq 3$$
$$x_1 + 4x_2 + 5x_3 + 2x_4 \leq 8$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0$$

(a) **Solve the LPP.**

(b) **If a new constraint $2x_1 + 3x_2 + x_3 + 5x_4 \leq 4$ is added to the above LPP, discuss the effect of change in the optimum solution of the original problem.**

(c) **If a new constraint $2x_1 + 3x_2 + x_3 + 5x_4 \leq 2$ is added, (or) if the upper limit of the above constraint is reduced to 2, discuss the effect of change in the optimum solution of the original problem.**

Solution: *(a)* By introducing the non–negative slack variables s_1, s_2 and s_3 and then solving the problem by simplex method, the optimal simplex table is given by

C_B	Y_B	C_j → X_B	x_1 (3)	x_2 (4)	x_3 (1)	x_4 (7)	s_1 (0)	s_2 (0)	s_3 (0)
3	x_1	$\frac{16}{19}$	1	$\frac{9}{38}$	$\frac{1}{2}$	0	$\frac{5}{38}$	$\frac{-1}{38}$	0
7	x_4	$\frac{5}{19}$	0	$\frac{21}{19}$	0	1	$\frac{-1}{19}$	$\frac{4}{19}$	0
0	s_3	$\frac{126}{19}$	0	$\frac{59}{38}$	$\frac{9}{2}$	0	$\frac{-1}{38}$	$\frac{-15}{38}$	1
$Z_j - C_j$		$\frac{83}{19}$	0	$\frac{169}{38}$	$\frac{1}{2}$	0	$\frac{1}{38}$	$\frac{53}{38}$	0

The optimum solution is

$$\therefore \text{ Min } Z = \frac{83}{19}, \quad x_1 = \frac{16}{19}, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = \frac{5}{19}.$$

(b) Now consider the new constraint $2x_1 + 3x_2 + x_3 + 5x_4 \le 4$

As this constraint is satisfied by the optimal solution, the solution remains feasible and optimal for the modified problem also. Here the constraint $2x_1 + 3x_2 + x_3 + 5x_4 \le 4$ is ***redundant.***

(c) Consider the new constraint $2x_1 + 3x_2 + x_3 + 5x_4 \le 2$.

As this constraint is not satisfied by the current optimal solution, the current solution is no longer optimal for the modified problem. In order to find the new optimal solution, we add the new constraint as the fourth row in the optimum simplex table of *(a)*.

By adding the non–negative slack variable s_4, the constraint becomes $2x_1 + 3x_2 + x_3 + 5x_4 + s_4 = 2$.

Since x_1, x_4 and s_3 are in the basis of the optimum simplex table of *(a)*, they should be removed from this constraint.

From the optimum simplex table of *(a)*,

$$x_1 + \frac{9}{38}x_2 + \frac{1}{2}x_3 + \frac{5}{38}s_1 - \frac{1}{38}s_2 = \frac{16}{19}$$

$$\Rightarrow \quad x_1 = \frac{16}{19} - \frac{9}{38}x_2 - \frac{1}{2}x_3 - \frac{5}{38}s_1 + \frac{1}{38}s_2$$

$$\text{Also } \frac{21}{19}x_2 + x_4 - \frac{1}{19}s_1 + \frac{4}{19}s_2 = \frac{5}{19}$$

$$\Rightarrow \quad x_4 = \frac{5}{19} - \frac{21}{19}x_2 + \frac{1}{19}s_1 - \frac{4}{19}s_2$$

$\therefore$ The constraint $2x_1 + 3x_2 + x_3 + 5x_4 + s_4 = 2$ becomes

$$2\left(\frac{16}{19} - \frac{9}{38}x_2 - \frac{1}{2}x_3 - \frac{5}{38}s_1 + \frac{1}{38}s_2\right) + 3x_2 + x_3$$
$$+ 5\left(\frac{5}{19} - \frac{21}{19}x_2 + \frac{1}{19}s_1 - \frac{4}{19}s_2\right) + s_4 = 2$$

$$i.e., \left(\frac{-18}{38} - \frac{105}{19} + 3\right)x_2 + \left(\frac{-10}{38} + \frac{5}{19}\right)s_1 + \left(\frac{2}{38} - \frac{20}{19}\right)s_2 + s_4$$

$$i.e., \ -3x_2 - s_2 + s_4 = -1$$

Add this constraint at the bottom of the optimum simplex table of *(a)*

		C_j	(3	4	1	7	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4
3	x_1	$\frac{16}{19}$	1	$\frac{9}{38}$	$\frac{1}{2}$	0	$\frac{5}{38}$	$\frac{-1}{38}$	0	0
7	x_4	$\frac{5}{19}$	0	$\frac{21}{19}$	0	1	$\frac{-1}{19}$	$\frac{4}{19}$	0	0
0	s_3	$\frac{126}{19}$	0	$\frac{59}{38}$	$\frac{9}{2}$	0	$\frac{-1}{38}$	$\frac{-15}{38}$	1	0
0	s_4	-1	0	-3	0	0	0	-1	0	1
$Z_j - C_j$		$\frac{83}{19}$	0	$\frac{169}{38}$	$\frac{1}{2}$	0	$\frac{1}{38}$	$\frac{53}{38}$	0	0

Here the solution is optimal but infeasible. To obtain the feasible optimal solution, we have to use dual simplex method.

First Iteration: Drop s_4 and introduce s_2

		C_j	(3	4	1	7	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4
3	x_1	$\frac{33}{38}$	1	$\frac{12}{38}$	$\frac{1}{2}$	0	$\frac{5}{38}$	0	0	$-\frac{1}{38}$
7	x_4	$\frac{1}{19}$	0	$\frac{9}{19}$	0	1	$\frac{-1}{19}$	0	0	$\frac{4}{19}$
0	s_3	$\frac{267}{38}$	0	$\frac{52}{19}$	$\frac{9}{2}$	0	$\frac{-1}{38}$	0	1	$\frac{-15}{38}$
0	s_2	1	0	3	0	0	0	1	0	-1
$Z_j - C_j$		$\frac{113}{38}$	0	$\frac{5}{19}$	$\frac{1}{2}$	0	$\frac{1}{38}$	0	0	$\frac{53}{38}$

Since all $(Z_j - C_j) \ge 0$ and all $X_{Bi} \ge 0$, the current solution is feasible and optimal

The new optimal solution is

$$\text{Max } Z = \frac{113}{38}, \quad x_1 = \frac{33}{38}, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = \frac{1}{19}$$

From this we see that the additional constraint has decreased the optimum value of the objective function.

1.7.5. Changes Affecting optimality

(1) Changes in the objective function (or) changes in the cost coefficients C_j.

Changes in the coefficients of the objective function may take place due to a change in cost or profit of either basic variables or non–basic variables.

If X_B is the optimal basic feasible solution, then we have $X_B = B^{-1}b$.

It is clear that X_B is independent of C and therefore any change in some component C_j of C will not change X_B. *i.e.,* X_B will always remain basic feasible solution.

But, however, the optimal condition $(Z_j - C_j) \geq 0$ for all j which is satisfied for optimal solution C_j is changed. So, any change in some component C_j of C may affect the optimality of the current solution and will not affect the feasibility of the current solution.

The variation in the price vector C may be made in the following two ways.

(i) **Variation in $C_j \notin C_B$ (or) Variation in the co–efficient C_j of the non-basic variable x_j in the objective function.**

If $C_j \notin C_B$ changes to $(C_j + \Delta C_j)$ such that $\Delta C_j \leq Z_j - C_j$, then the value of the objective function and the optimal solution of the problem remain unchanged. The range over $C_j \notin C_B$ can vary maintaining the optimality of the solution is given by $-\infty \leq C_j \leq C_j + \Delta C_j$. Note that there is no lower bound to ΔC_j.

(ii) **Variation in $C_k \in C_B$ (or) Variation in the co-efficient C_k of the basic variable x_k in the objective function.**

If $C_k \in C_B$ changes to $(C_k + \Delta C_k)$, then the range over $C_k \in C_B$ can vary so that the solution remain optimal is given by

$$\underset{a_{kj} > 0}{\text{Max}} \left\{ \frac{-(Z_j - C_j)}{a_{kj}} \right\} \leq \Delta C_k \leq \underset{a_{kj} < 0}{\text{Min}} \left\{ \frac{-(Z_j - C_j)}{a_{kj}} \right\}$$

If there is no $a_{kj} > 0$, there is no lower bound to ΔC_k and if no $a_{kj} < 0$, there is no upper bound to ΔC_k. The value of the objective function will be further improved by an amount $X_{Bk} \cdot \Delta C_{Bk}$.

$$i.e., \; Z^* = Z + X_{Bk} \cdot \Delta C_{Bk}$$

Note:

1. Any change $C_j \notin C_B$ will affect only its net evaluation coefficients and not others.

2. Any change $C_j \in C_B$ will affect its net evaluation coefficients and the value of the objective function.

Example 1 **Consider the LPP**

$$\text{Max } Z = 5x_1 + 3x_2$$

subject to

$$3x_1 + 5x_2 \leq 15$$
$$5x_1 + 6x_2 \leq 10$$
$$\text{and } x_1, x_2 \geq 0$$

(a) **Solve the LPP.**

(b) **Find how far the component C_1 of C can be increased without affecting the optimality of the solution.**

Solution: *(a)* By introducing the non–negative slack variables s_1 and s_2 and using regular simplex method, the optimal simplex table is displayed below:

		C_j	(5	3	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
0	s_1	9	0	$\frac{7}{5}$	1	$\frac{-3}{5}$
5	x_1	2	1	$\frac{6}{5}$	0	$\frac{1}{5}$
$Z_j - C_j$		10	0	3	0	1

The optimal solution is

$\therefore$ Max $Z = 10, x_1 = 2, x_2 = 0.$

(b) Here $C_1 \in C_B$, *i.e.*, C_1 is the co-efficient of the basic variable x_1 and $C_1 = 5$.

$$\underset{a_{2j} > 0}{\text{Max}} \left\{ \frac{-(Z_j - C_j)}{a_{2j}} \right\} \leq \Delta C_1 \leq \underset{a_{2j} < 0}{\text{Min}} \left\{ \frac{-(Z_j - C_j)}{a_{2j}} \right\}$$

$$\text{Max} \left\{ \frac{0}{1}, \frac{-3}{\frac{6}{5}}, \frac{-1}{\frac{1}{5}} \right\} \leq \Delta C_1$$

[Since no $a_{2k} < 0$, no upper bound to ΔC_1]

$\Rightarrow$ $\text{Max} \left\{ 0, \frac{-5}{2}, -5 \right\} \leq \Delta C_1 \leq \infty$

$\Rightarrow$ $0 \leq \Delta C_1 \leq \infty$

$\Rightarrow$ $C_1 - 0 \leq C_1 \leq C_1 + \infty$

$\Rightarrow$ $5 - 0 \leq C_1 \leq 5 + \infty$

$\Rightarrow$ $5 \leq C_1 \leq \infty$, which is the range over C_1 can increase without destroying the optimality of the solution.

Example 2 **Consider the LPP**

Max $Z = 2x_1 + x_2 + 4x_3 - x_4$

subject to

$$x_1 + 2x_2 + x_3 - 3x_4 \leq 8$$
$$-x_2 + x_3 + 2x_4 \leq 0$$
$$2x_1 + 7x_2 - 5x_3 - 10x_4 \leq 21$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0$$

(a) **Solve the LPP.**

(b) **Discuss the effect of change of C_1 to 1.**

(c) **Discuss the effect of change of (C_3, C_4) to (3, 4).**

(d) **Discuss the effect of change of (C_1, C_2, C_3, C_4) to (1, 2, 3, 4).**

Solution: *(a)* By introducing the non–negative slack variables s_1, s_2 and s_3 and using regular simplex method, the optimal simplex table is given by

		C_j	(2	1	4	−1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
2	x_1	8	1	0	3	1	1	2	0
1	x_2	0	0	1	−1	−2	0	−1	0
0	s_3	5	0	0	−4	2	−2	3	1
$Z_j - C_j$		16	0	0	1	1	2	3	0

The optimum solution is

$\therefore$ Max $Z = 16, x_1 = 8, x_2 = 0, x_3 = 0, x_4 = 0.$

(b) If C_1 is changed to 1, then the optimal simplex table of *(a)* reduce to

		C_j	(1	1	4	−1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	x_1	8	1	0	(3)	1	1	2	0
1	x_2	0	0	1	−1	−2	0	−1	0
0	s_3	5	0	0	−4	2	−2	3	1
$Z_j - C_j$		8	0	0	−2	0	1	1	0

Here the solution is not optimal.

First Iteration: Introduce x_3 and drop x_1.

		C_j	(1	1	4	−1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
4	x_3	$\frac{8}{3}$	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0
1	x_2	$\frac{8}{3}$	$\frac{1}{3}$	1	0	$\frac{-5}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$	0
0	s_3	$\frac{47}{3}$	$\frac{4}{3}$	0	0	$\frac{10}{3}$	$\frac{-2}{3}$	$\frac{17}{3}$	1
$Z_j - C_j$		$\frac{40}{3}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{5}{3}$	$\frac{7}{3}$	0

Since all $(Z_j - C_j) \geq 0$, the current solution is optimal.

$\therefore$ The optimal solution to the new problem is

$$\text{Max } Z = \frac{40}{3}, \quad x_1 = 0, \quad x_2 = \frac{8}{3}, \quad x_3 = \frac{8}{3}, \quad x_4 = 0.$$

(*c*) If (C_3, C_4) is changed to (3,4), then the optimal simplex table of (*a*) reduce to

		C_j	(2	1	3	4	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
2	x_1	8	1	0	3	1	1	2	0
1	x_2	0	0	1	–1	–2	0	–1	0
0	s_3	5	0	0	–4	(2)	–2	3	1
$Z_j - C_j$		16	0	0	2	–4	2	3	0

First Iteration: Introduce x_4 and drop s_3.

		C_j	(2	1	3	4	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
2	x_1	$\frac{11}{2}$	1	0	(5)	0	2	$\frac{1}{2}$	$\frac{-1}{2}$
1	x_2	5	0	1	–5	0	–2	2	1
4	x_4	$\frac{5}{2}$	0	0	–2	1	–1	$\frac{3}{2}$	$\frac{1}{2}$
$Z_j - C_j$		26	0	0	–6	0	–2	9	2

Second Iteration: Introduce x_3 and drop x_1.

		C_j	(2	1	3	4	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
3	x_3	$\frac{11}{10}$	$\frac{1}{5}$	0	1	0	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{-1}{10}$
1	x_2	$\frac{21}{2}$	1	1	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$
4	x_4	$\frac{47}{10}$	$\frac{2}{5}$	0	0	1	$\frac{-1}{5}$	$\frac{17}{10}$	$\frac{3}{10}$
$Z_j - C_j$		$\frac{163}{5}$	$\frac{6}{5}$	0	0	0	$\frac{2}{5}$	$\frac{48}{5}$	$\frac{7}{5}$

Since all $(Z_j - C_j) \geq 0$, the current solution is optimal.

$\therefore$ The optimal solution to the new problem is

$$\text{Max } Z = \frac{163}{5}, \quad x_1 = 0, \quad x_2 = \frac{21}{2}, \quad x_3 = \frac{11}{10}, \quad x_4 = \frac{47}{10}.$$

(*d*) If (C_1, C_2, C_3, C_4) is changed to (1, 2, 3, 4), then the optimal simplex table of (*a*) reduce to

		C_j	(1	2	3	4	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	x_1	8	1	0	3	1	1	2	0
2	x_2	0	0	1	–1	–2	0	–1	0
0	s_3	5	0	0	–4	(2)	–2	3	1
$Z_j - C_j$		8	0	0	–2	–7	1	0	0

First Iteration: Introduce x_4 and drop s_3.

		C_j	(1	2	3	4	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	x_1	$\frac{11}{2}$	1	0	(5)	0	2	$\frac{1}{2}$	$\frac{-1}{2}$
2	x_2	5	0	1	–5	0	–2	2	1
4	x_4	$\frac{5}{2}$	0	0	–2	1	–1	$\frac{3}{2}$	$\frac{1}{2}$
$Z_j - C_j$		$\frac{51}{2}$	0	0	–16	0	–6	$\frac{21}{2}$	$\frac{7}{2}$

Second Iteration: Introduce x_3 and drop x_1.

		C_j	(1	2	3	4	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
3	x_3	$\frac{11}{10}$	$\frac{1}{5}$	0	1	0	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{-1}{10}$
2	x_2	$\frac{21}{2}$	1	1	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$
4	x_4	$\frac{47}{10}$	$\frac{2}{5}$	0	0	1	$\frac{-1}{5}$	$\frac{17}{10}$	$\frac{3}{10}$
$Z_j - C_j$		$\frac{431}{10}$	$\frac{16}{5}$	0	0	0	$\frac{2}{5}$	$\frac{121}{10}$	$\frac{19}{10}$

Since all $(Z_j - C_j) \geq 0$, the current solution is optimal.

$\therefore$ The optimal solution to the new problem is

$$\text{Max } Z = \frac{431}{10}, x_1 = 0, \; x_2 = \frac{21}{2}, \; x_3 = \frac{11}{10}, \; x_4 = \frac{47}{10}.$$

1.7.6 Variation in the co-efficients a_{ij} of the constraints (or) Variation in the component a_{ij} of the co-efficient matrix A:

Case (i): If the coefficients a_{ij} of a ***non–basic variable*** x_j ($a_{ij} \notin$ B) get changed in a current optimal solution, then this change will not affect the feasibility of the current solution but may affect the optimality of the current solution.

Case (ii): If the coefficients a_{ij} of a ***basic variable*** x_j ($a_{ij} \in$ B) get changed in a current optimal solution, then this change may effect both the feasibility and optimality of the current solution. As the basic matrix is affected, it may affect all the quantities given in the current optimal table. Under this situation, it may be better to solve the problem a new again.

Example 1 **Consider the LPP**

$$\text{Max } Z = 5x_1 + 12x_2 + 4x_3$$

$$\text{subject to} \quad \begin{aligned} x_1 + 2x_2 + x_3 &\le 5 \\ 2x_1 - x_2 + 3x_3 &= 2 \\ \text{and } x_1, x_2, x_3 &\ge 0. \end{aligned}$$

(a) **Discuss the effect of changing a_3 to $\begin{pmatrix}2\\5\end{pmatrix}$ from $\begin{pmatrix}1\\3\end{pmatrix}$.**

(b) **Discuss the effect of changing a_3 to $\begin{pmatrix}-5\\2\end{pmatrix}$ from $\begin{pmatrix}1\\3\end{pmatrix}$.**

(c) **Discuss the effect of changing a_3 to $\begin{pmatrix}-1\\2\end{pmatrix}$ from $\begin{pmatrix}1\\3\end{pmatrix}$.**

Solution: The optimum simplex table of this problem is displayed below:

		C_j	(5	12	4	0	–M)
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	R_1
12	x_2	$\frac{8}{5}$	0	1	$\frac{-1}{5}$	$\frac{2}{5}$	$\frac{-1}{5}$
5	x_1	$\frac{9}{5}$	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
$Z_j - C_j$		$\frac{141}{5}$	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$\frac{5M-2}{5}$

(*a*) x_3 is a non–basic variable in the optimal solution.

when the coefficients $a_3 = \begin{pmatrix}1\\3\end{pmatrix}$ of x_3 changes to $\begin{pmatrix}2\\5\end{pmatrix}$, new calculations for the column are

$$\begin{pmatrix}a_{13}\\a_{23}\end{pmatrix} = \begin{pmatrix}\frac{2}{5} & \frac{-1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}\frac{-1}{5}\\ \frac{12}{5}\end{pmatrix} = \bar{a}_3$$

$$\therefore \text{ and } (Z_3 - C_3) = C_B\,\bar{a}_3 - C_3 = (12,5)\begin{pmatrix}\frac{-1}{5}\\ \frac{12}{5}\end{pmatrix} - 4 = \frac{48}{5} - 4 = \frac{28}{5}.$$

Since $(Z_3 - C_3)$ remains non–negative, the original optimal solution remains optimum for the new problem also.

(*b*) When the coefficients $a_3 = \begin{pmatrix}1\\3\end{pmatrix}$ of a non–basic variable x_3 changes to $\begin{pmatrix}-5\\2\end{pmatrix}$, new calculation for the column are:

$$\begin{pmatrix}a_{13}\\a_{23}\end{pmatrix} = \begin{pmatrix}\frac{2}{5} & \frac{-1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}\begin{pmatrix}-5\\2\end{pmatrix} = \begin{pmatrix}\frac{-12}{5}\\ \frac{-1}{5}\end{pmatrix} = \bar{a}_3$$

$$\text{and } \quad (Z_3 - C_3) = C_B\,\bar{a}_3 - C_3$$

$$= (12, 5)\begin{pmatrix}\frac{-12}{5}\\ \frac{-1}{5}\end{pmatrix} - 4 = \frac{149}{5} - 4 = \frac{-169}{5}.$$

$$\text{i.e.,} \quad (Z_3 - C_3) = \frac{-169}{5} < 0.$$

$\therefore$ The optimality of the current solution is affected.

The changes in the optimal table reduce to

		C_j	(5	12	4	0	–M)
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	R_1
12	x_2	$\frac{8}{5}$	0	1	$\frac{-12}{5}$	$\frac{2}{5}$	$\frac{-1}{5}$
5	x_1	$\frac{9}{5}$	1	0	$\frac{-1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
$Z_j - C_j$		$\frac{141}{5}$	0	0	$\frac{-169}{5}$	$\frac{29}{5}$	$\frac{M-2}{5}$

Since x_3 enters the basis and all $a_{i3} < 0$, there is an unbounded solution to the new problem.

(*c*) when the coefficients $a_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ of a non-basic variable x_3 changes to $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, new calculations for the column are:

$$\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{-4}{5} \\ \frac{3}{5} \end{pmatrix} = \bar{a}_3$$

and $$(Z_3 - C_3) = C_B \bar{a}_3 - C_3 = (12, 5) \begin{pmatrix} \frac{-4}{5} \\ \frac{3}{5} \end{pmatrix} - 4 = \frac{-53}{5}$$

i.e., $$(Z_3 - C_3) = \frac{-53}{5} < 0.$$

$\therefore$ The optimality of the current solution is affected.

The changes in the optimal table reduce to

		C_j	(5	12	4	0	–M)	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	R_1	θ
12	x_2	$\frac{8}{5}$	0	1	$\frac{-4}{5}$	$\frac{2}{5}$	$\frac{-1}{5}$	–
5	x_1	$\frac{9}{5}$	1	0	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	3
$Z_j - C_j$		$\frac{141}{5}$	0	0	$\frac{-53}{5}$	$\frac{29}{5}$	$\frac{M-2}{5}$	

First iteration: Introduce x_3 and drop x_1.

		C_j	(5	12	4	0	–M)
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	R_1
12	x_2	4	$\frac{4}{3}$	1	0	$\frac{2}{3}$	$\frac{1}{3}$
4	x_3	3	$\frac{5}{3}$	0	1	$\frac{1}{3}$	$\frac{2}{3}$
$Z_j - C_j$		60	$\frac{53}{3}$	0	0	$\frac{28}{3}$	$\frac{3M+20}{3}$

Since all $(Z_j - C_j) \geq 0$, the current solution is optimal.

$\therefore$ The optimal solution to the new problem is

Max Z $= 60, x_1 = 0, x_2 = 4, x_3 = 3.$

1.7.7 Addition of a new activity (or variable):

If a new variable is introduced in a LPP whose optimal solution has been obtained, then the solution of the problem will remain feasible. Addition of an extra variable x_{n+1} to the problem will introduce an extra column say a_{n+1} to the coefficient matrix A and an extra cost C_{n+1} will be introduced in the price vector C. Thus the addition of this extra variable may effect the optimality of the problem.

If $Z_{n+1} - C_{n+1} \geq 0$ then the solution of the original problem will remain optimal for the new problem also. Otherwise use simplex method to obtain the optimal solution to the new problem.

Further, it can be noted that the ***addition of a new variable can never worsen the value of the objective function.***

Example 1 **Consider the LPP**

Maximize $Z = 3x_1 + 5x_2$

subject to $x_1 \leq 4$

$3x_1 + 2x_2 \leq 18$

and $x_1, x_2 \geq 0$

(a) **Solve this LPP.**

(b) If a new variable x_5 is added to this problem with a column $\begin{pmatrix}1\\2\end{pmatrix}$ and $C_5 = 7$, find the change in the optimal solution.

Solution: *(a)* By using simplex method, the optimal solution of the original problem is

		C_j	(3	5	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
0	x_3	4	1	0	1	0
5	x_2	9	$\frac{3}{2}$	1	0	$\frac{1}{2}$
$Z_j - C_j$		45	$\frac{9}{2}$	0	0	$\frac{5}{2}$

i.e., Max $Z = 45$, $x_1 = 0$, $x_2 = 9$.

(b) If a new variable x_5 is introduced with $a_5 = \begin{pmatrix}1\\2\end{pmatrix}$ and $C_5 = 7$, then

$$\bar{a}_5 = \begin{pmatrix}1 & 0\\0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \bar{a}_5$$

and $Z_5 - C_5 = C_B\,\bar{a}_5 \quad - C_5 = (0, 5)\begin{pmatrix}1\\1\end{pmatrix} - 7 = -2.$

$\therefore$ The optimality of the solution is affected

The change in the optimal table reduce to

		C_j	(3	5	0	0	7)	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	θ
0	x_3	4	1	0	1	0	(1)	4
5	x_2	9	$\frac{3}{2}$	1	0	$\frac{1}{2}$	1	9
$Z_j - C_j$		45	$\frac{9}{2}$	0	0	$\frac{5}{2}$	-2	

First iteration: Introduce x_5 and drop x_3.

		C_j	(3	5	0	0	7)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
7	x_5	4	1	0	1	0	1
5	x_2	5	$\frac{1}{2}$	1	-1	$\frac{1}{2}$	0
$Z_j - C_j$		53	$\frac{13}{2}$	0	2	$\frac{5}{2}$	0

Since all $(Z_j - C_j) \geq 0$, the current solution is optimal. The optimal solution to the new problem is

Max $Z = 53$, $x_1 = 0$, $x_2 = 5$, $x_5 = 4$.

1.7.8 Deletion of a Variable:

Deletion of a non-basic variable is a totally superfluous operation and does not affect the feasibility and/or optimality of the current optimal solution. But the deletion of a basic variable may affect the optimality of the current solution. To find the new optimum solution assigning a large penalty – M (+M for minimization problems) to the variables under consideration and then use the regular simplex method to the modified simplex table.

Example 1 **Consider the optimal table of a maximzation problem.**

		C_j	(3	5	0	0	7)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
7	x_5	4	1	0	1	0	1
5	x_2	5	$\frac{1}{2}$	1	-1	$\frac{1}{2}$	0
$Z_j - C_j$		53	$\frac{13}{2}$	0	2	$\frac{5}{2}$	0

Find the change in the optimal solution, when the basic variable x_2 is deleted.

Solution: Since this problem is of maximization type we assign –M to the variable x_2. The modified simplex table becomes

		C_j	(3	–M	0	0	7)	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	θ
7	x_5	4	1	0	1	0	1	–
–M	x_2	5	$\frac{1}{2}$	1	–1	$\left(\frac{1}{2}\right)$	0	10
$Z_j - C_j$		–5M + 28	$\frac{-M+8}{2}$	0	$\frac{M+}{7}$	$\frac{-M}{2}$	0	

First iteration: Introduce x_4 and drop x_2.

		C_j	(3	–M	0	0	7)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
7	x_5	4	1	0	1	0	1
0	x_4	10	1	2	–2	1	0
$Z_j - C_j$		28	4	M	7	0	0

Since all $(Z_j - C_j) \geq 0$, the current solution is optimal.

∴ The optimal solution to the new problem is

Max Z = 28, $x_1 = 0$, $x_2 = 0$ and $x_5 = 4$.

1.7.9 Deletion of a Constraint:

The constraint to be deleted may be either binding or unbinding (redundant) on the optimal solution. The deletion of an unbinding constraint can only enlarge the feasible region but will not affect the optimal solution. Moreover, if the constraint under consideration has a slack or surplus variable of zero value in the basis matrix, it cannot be binding and hence will not affect the optimal solution.

The deletion of a binding constraint will, however, cause post–optimality problem. The simplest way to proceed in this case is via the addition of one or two new variables.

EXERCISE

1. **What do you understand by the term Sensitivity Analysis ?**
2. **What is sensitivity analysis in an LP problem ? Discuss its significance fully.** ***[MU. MCA. Nov 97]***
3. **Discuss the effect of *(i)* variation of b_i *(ii)* Variation of C_j *(iii)* Variation of a_{ir}**
4. **Discuss the effect of *(i)* addition and deletion of a constraint *(ii)* addition and deletion of a variable.**
5. **Consider the LPP**

$$\text{Max } Z = 2x_2 - 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ 2x_1 + x_2 + 6x_3 &\leq 6 \\ x_1 - x_2 + 3x_3 &= 0 \\ \text{and } x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(*a*) Solve the LPP

(*b*) Discuss the effect of changing *b* from [2, 6, 0] to [2, 10, 5]

[*Ans* (*a*) Max Z = 2, $x_1 = 1$, $x_2 = 1$, $x_3 = 0$

(*b*) Max Z = – 5, $x_1 = 2$, $x_2 = 0$, $x_3 = 1$]

6. **Consider the LPP**

$$\text{Max } Z = 5x_1 + 3x_2 + 7x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\leq 22 \\ 3x_1 + 2x_2 + x_3 &\leq 26 \\ x_1 + x_2 + x_3 &\leq 18 \\ \text{and } x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(*a*) Solve the LPP

(*b*) What will be the solution if the first constraint changes to $x_1 + x_2 + 2x_3 \leq 26$.

[*Ans* (*a*) Max Z = 86, $x_1 = 6$, $x_2 = 0$, $x_3 = 8$

(*b*) Max $Z = \frac{494}{5}$, $x_1 = \frac{26}{5}$, $x_2 = 0$, $x_3 = \frac{52}{5}$]

7. *(a)* Solve the LPP

$$\text{Max } Z = 3x_1 + 5x_2$$

subject to
$$x_1 \le 4$$
$$3x_1 + 2x_2 \le 18$$
$$\text{and } x_1, x_2 \ge 0$$

(b) Let a linear constraint $x_2 \le 10$ be added to the constraints of the problem. Check whether there is any change in the optimum solution of the original problem.

(c) Also discuss the case when the upper limit of the above constraint is reduced to 6.

[*Ans:* *(a)* Max $Z = 45$, $x_1 = 0$, $x_2 = 9$
(b) No change in the optimal solution.
(c) Max $Z = 36$, $x_1 = 2$, $x_2 = 6$]

8. Consider the following tableau which presents an optimum solution to some linear programming problem:

		C_j	4	6	2	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
4	x_1	1	1	0	−1	$\frac{4}{3}$	$\frac{-1}{3}$
6	x_2	2	0	1	2	$\frac{-1}{3}$	$\frac{1}{3}$
$Z_j - C_j$		16	0	0	6	$\frac{10}{3}$	$\frac{2}{3}$

(a) If the additional constraint $2x_1 + 3x_2 + 2x_3 \le 10$ where annexed to the system, would there be any change in the optimal solution ?. Justify your answer.

(b) If the upper limit of the above constraint is reduced to 4, would there be any change in the optimum solution ? Determine the new optimal solution.

[*Ans:* *(a)* No change in the optimum solution, because the new constraint is redundant. *(b)* Max $Z = 12, x_1 = \frac{5}{3}, x_2 = \frac{2}{3}, x_3 = \frac{2}{3}$]

9. *(a)* Solve the following LPP

$$\text{Max } Z = 4x_1 + 6x_2 + 2x_3$$

subject to
$$x_1 + x_2 + x_3 \le 3$$
$$x_1 + 4x_2 + 7x_3 \le 9$$
$$\text{and } x_1, x_2, x_3 \ge 0$$

(b) Find the range on the values of non–basic variable coefficient C_3 such that the current solution remains optimal.

(c) What happens if C_3 is increased to 12? What is the new optimal solution in this case ?

(d) Find the range on basic variable coefficient C_1 such that the current solution remains optimal.

(e) Find the effect of changing C_1 to 8 on the optimal solution.

(f) Find the effect of changing the objective function to $Z = 2x_1 + 8x_2 + 4x_3$ on the current optimal solution.

[*Ans:* *(a)* Max $Z = 16, x_1 = 1, x_2 = 2, x_3 = 0,$
(b) $C_3 \le 18$ *(c)* Max $Z = 20, x_1 = 2, x_2 = 0, x_3 = 1$
(d) $\frac{3}{2} \le C_1 \le 6$ *(e)* Max $Z = 24, x_1 = 3, x_2 = 0, x_3 = 0$
(f) No change in the optimal solution]

10. *(a)* Solve the following LPP

$$\text{Max } Z = x_1 + 5x_2 + 3x_3$$

subject to
$$x_1 + 2x_2 + x_2 = 3$$
$$2x_1 - x_2 = 4$$
$$\text{and } x_1, x_2, x_3 \ge 0$$

(b) If the objective function is changed to Max Z $= 2x_1 + 5x_2 + 2x_3$, find the new optimal solution.

[*Ans:* *(a)* Max $Z = 5$, $x_1 = 2$, $x_2 = 0$, $x_3 = 1$
(b) Max $Z = \frac{32}{5}$, $x_1 = \frac{11}{5}$, $x_2 = \frac{2}{5}$, $x_3 = 0$]

11. *(a)* **Solve the following LPP.**

$$\text{Max } Z = 45x_1 + 100x_2 + 30x_3 + 50x_4$$

subject to
$$7x_1 + 10x_2 + 4x_2 + 9x_4 \le 1200$$
$$3x_1 + 40x_2 + x_3 + x_4 \le 800$$
$$\text{and } x_1, x_2, x_3, x_4 \ge 0$$

(b) **Discuss the effect of changing a_1 to $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ from $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$**

(c) **Discuss the effect of changing a_1 to $\begin{pmatrix} 5 \\ 8 \end{pmatrix}$ from $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$**

[*Ans:* (*a*) Max Z = $\frac{28000}{3}$, $x_1 = 0$, $x_2 = \frac{40}{3}$, $x_3 = \frac{800}{3}$, $x_4 = 0$

(*b*) No change in the optimal solution.

(*c*) Max Z = $\frac{86000}{9}$, $x_1 = \frac{2000}{27}$, $x_2 = 0$, $x_3 = \frac{5600}{27}$, $x_4 = 0$]

12. *(a)* **Solve the following LPP**

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3$$

subject to
$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_2 \le 460$$
$$x_1 + 4x_2 \le 420$$
$$\text{and } x_1, x_2, x_3 \ge 0$$

(b) **Suppose that a_1 is changing from $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$ and c_2, c_3 are changed from (2, 5) to (1,3), find the new optimal solution.**

[*Ans* (*a*) Max Z = 2150, $x_1 = 0$, $x_2 = 0$, $x_3 = 430$.

(*b*) Max Z = 1290, $x_1 = 70$, $x_2 = 0$, $x_3 = 360$.]

13. *(a)* **Solve the following LPP**

$$\text{Max } Z = 4x_1 + 6x_2 + 2x_3$$

subject to
$$x_1 + x_2 + x_3 \le 3$$
$$x_1 + 4x_2 + 7x_3 \le 9$$
$$\text{and } x_1, x_2, x_3 \ge 0$$

(b) **If a new variable x_6 is added to this problem with a column $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $c_6 = 3$, find the change in the optimal solution.**

[*Ans* (*a*) Max Z = 16, $x_1 = 1$, $x_2 = 2$ $x_3 = 0$

(*b*) No change in the optimal solution]

14. *(a)* **Solve the following LPP**

$$\text{Max } Z = x_1 + 5x_2 + 3x_3$$

subject to
$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - x_3 = 4$$
$$\text{and } x_1, x_2, x_3 \ge 0$$

(b) **If a new variable x_4 is added to this problem with a column $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $c_4 = 5$, find the change in the optimal solution.**

(c) **If a new variable x_4 is added to this problem with a column $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ and $c_4 = 3$, find the change in the optimal solution.**

[*Ans:* (*a*) Max Z = 4.5, $x_1 = 2$, $x_2 = 0.5$, $x_3 = 0$

(*b*) Max Z = 6, $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$

(*c*) No, solution remains unchanged and $x_4 = 0$]

Unit 2

Duality and Networks

2.1. DUALITY

2.1.1 Introduction

For every linear programming problem there is a unique linear programming problem associated with it, involving the same data and closely related optimal solutions. The original (given) problem is then called the ***Primal*** problem while the other is called its ***dual*** problem. But in general, the two problems are said to be ***duals*** of each other.

The importance of the duality concept is due to two main reasons. Firstly,if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting in to the dual problem and then solving it. Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions in the activities being programmed.

2.1.2. Formulation of dual problems

There are two important forms of primal – dual pairs, namely, ***symmetric form*** and ***unsymmetric form.***

Symmetric form:

Consider the following LPP.

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots c_nx_n$

subject to the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \leq b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \leq b_m \\ \text{and} \quad x_1, x_2, \ldots\ldots\ldots\ldots x_n \geq 0 \end{array}\right| \rightarrow (1)$$

i.e., Max $Z = \text{CX}$

subject to $\text{AX} \leq b$

and $\quad X \geq 0$

where $C = (c_1\ c_2 \ldots\ldots c_n)$

$$A = \begin{pmatrix} a_{11}\, a_{12} & \cdots\ \cdots\ \cdots\ a_{1n} \\ a_{12}\, a_{22} & \cdots\ \cdots\ \cdots\ a_{2n} \\ \cdots & \cdots\cdots\cdots\cdots \\ \cdots & \cdots\cdots\cdots\cdots \\ a_{m1}\, a_{m2} & \cdots\ \cdots\ \cdots a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}$$

To construct the dual problem, we adopt the following guidelines:

(i) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(ii) The maximization problem has ($\leq$) constraints while the minimization problem has ($\geq$) constraints.

(iii) If the primal contains m constraints and n variables, then the dual will contain n constraints and m variables. *i.e.,* the transpose of the body matrix of the primal problem gives the body matrix of the dual and vice versa.

(iv) The constants $c_1, c_2, c_3 \ldots\ldots c_n$ in the objective function of the primal appear in the constraints of the dual.

(v) The constants $b_1, b_2, \ldots\ldots b_m$ in the constraints of the primal appear in the objective function of the dual.

(vi) The variables in both problems are non–negative.

The primal – dual relationships can be conveniently displayed as below:

Primal variables

$$\begin{array}{ccc} & & x_1 \quad x_2 \quad x_3 \ \cdots\cdots\ x_n & & \\ & y_1 & \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13}\cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23}\cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3}\cdots & \cdots & a_{mn} \end{array}\right] & \begin{array}{c} \leq b_1 \\ \leq b_2 \\ \cdots \\ \cdots \\ \leq b_m \end{array} & \text{R.H.S of Primal constraints} \\ \text{Dual Variables} & \begin{array}{c} y_2 \\ \cdots \\ \cdots \\ y_m \end{array} & & & \\ & & \geq \quad \geq \quad \geq \ \cdots\cdots \geq & & \\ & & c_1 \quad c_2 \quad c_3 \ \cdots\cdots\ c_n & & \end{array}$$

R.H.S of dual constraints

Then the dual problem of (1) will be
Minimize $W = b^T Y$ $\quad \left(\text{where } Y = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ y_m \end{pmatrix} \right)$

subject to the constraints $\quad A^T Y \geq C^T$

and $\quad Y \geq 0$

i.e., Minimize $W = b_1 y_1 + b_2 y_2 + \ldots\ldots b_m y_m$

subject to the constraints

$$\left.\begin{array}{l} a_{11}y_1 + a_{21}y_2 + \ldots\ldots\ldots\ldots + a_{m1}y_m \geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots\ldots\ldots\ldots + a_{m2}y_m \geq c_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}y_1 + a_{2n}y_2 + \ldots\ldots\ldots\ldots + a_{mn}y_m \geq c_n \\ \quad \text{and } y_1, y_2 \cdots y_m \geq 0 \end{array}\right\} \rightarrow (2)$$

Equations (1) and (2) are called ***symmetric primal–daul pairs.***

Example 1 **Write the dual of the following primal LPP.**

$$\textbf{Maximize F} = x_1 + 2x_2 + x_3$$

$$\text{subject to } 2x_1 + x_2 - x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_1, x_3, x_3 \ge 0$$

[MU. MCA Nov 94]

Solution: Given primal LPP Maximize $F = x_1 + 2x_2 + x_3$

subject to
$$2x_1 + x_2 - x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$\text{and } x_1, x_2, x_3 \ge 0.$$

That is, Maximize $F = x_1 + 2x_2 + x_3$

subject to
$$2x_1 + x_2 - x_3 \le 2$$
$$2x_1 - x_2 + 5x_3 \le 6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$\text{and } x_1, x_2, x_3 \ge 0.$$

That is Max $F = CX$

subject to $AX \le b$

and $X \ge 0$

Where $C = (1\ 2\ 1)$, $A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 5 \\ 4 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal problem is maximization type with ($\le$) type constraints, with 3 constraints and 3 variables, the dual problem is mini–mization type with ($\ge$) type constraints, with 3 constraints and 3 dual variables y_1, y_2, y_3.

The dual problem is Minimize $W = b^T Y$

subject to the constraints $A^T Y \ge C^T$

and $Y \ge 0$

$$\text{i.e., Min W} = (2\ 6\ 6)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\text{subject to } \begin{pmatrix} 2 & 2 & 4 \\ 1 & -1 & 1 \\ -1 & 5 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \ge \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \ge 0$$

$$\text{i.e., Min W} = 2y_1 + 6y_2 + 6y_3$$

subject to
$$2y_1 + 2y_2 + 4y_3 \ge 1$$
$$y_1 - y_2 + y_3 \ge 2$$
$$-y_1 + 5y_2 + y_3 \ge 1$$
$$y_1, y_2, y_3 \ge 0$$

Example 2 **Find the dual of the LPP**

$$\textbf{Max Z} = 3x_1 - x_2 + x_3$$

subject to
$$4x_1 - x_2 \le 8$$
$$8x_1 + x_2 + 3x_3 \ge 12$$
$$5x_1 - 6x_3 \le 13$$
$$x_1, x_2, x_3 \ge 0$$

[MU. BE Apr 90]

Solution: Given primal LPP is

Max $Z = 3x_1 - x_2 + x_3$

subject to
$$4x_1 - x_2 + 0x_3 \le 8$$
$$-8x_1 - x_2 - 3x_3 \le -12$$
$$5x_1 + 0x_2 - 6x_3 \le 13$$
$$\text{and } x_1, x_2, x_3 \ge 0.$$

That is Max $Z = CX$

subject to $AX \le b$

and $X \ge 0$

Where $C = (3\ -1\ 1)$, $A = \begin{pmatrix} 4 & -1 & 0 \\ -8 & -1 & -3 \\ 5 & 0 & -6 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} 8 \\ -12 \\ 13 \end{pmatrix}$

The dual problem is Minimize $W = b^T Y$

subject to the constraints $A^T Y \geq C^T$

and $Y \geq 0$

i.e., $\text{Min } W = (8\ -12\ 13)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

subject to $\begin{pmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & -6 \end{pmatrix} \geq \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq 0$

$$\begin{aligned} \textit{i.e.,}\ \text{Min } W &= 8y_1 - 12y_2 + 13y_3 \\ \text{subject to}\quad 4y_1 - 8y_2 + 5y_3 &\geq 3 \\ -y_1 - y_2 + 0y_3 &\geq -1 \\ 0y_1 - 3y_2 - 6y_3 &\geq 1 \\ y_1,\ y_2,\ y_3 &\geq 0 \end{aligned}$$

i.e., The dual problem is

$$\begin{aligned} \text{Min } W &= 8y_1 - 12y_2 + 13y_3 \\ \text{subject to}\quad 4y_1 - 8y_2 + 5y_3 &\geq 3 \\ y_1 + y_2 &\leq 1 \\ -3y_2 - 6y_3 &\geq 1 \\ \text{and}\quad y_1,\ y_2,\ y_3 &\geq 0 \end{aligned}$$

Example 3 **Construct the dual of the LPP**

$$\begin{aligned} \textbf{Min } Z &= 4x_1 + 6x_2 + 18x_3 \\ \textbf{subject to}\quad x_1 + 3x_2 &\geq 3 \\ x_2 + 2x_3 &\geq 5 \\ \textbf{and}\quad x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

Solution: Given primal LPP is

$$\begin{aligned} \text{Min } Z &= 4x_1 + 6x_2 + 18x_3 \\ \text{subject to}\quad x_1 + 3x_2 + 0x_3 &\geq 3 \\ 0x_1 + x_2 + 2x_3 &\geq 5 \\ \text{and}\quad x_1, x_2, x_3 &\geq 0. \end{aligned}$$

That is

$$\begin{aligned} \text{Min } Z &= CX \\ \text{subject to } AX &\geq b \\ \text{and } X &\geq 0. \end{aligned}$$

Where $C = (4\ 6\ 18)$, $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal contains 2 constraints and 3 varaibles, the dual will contain 3 constraints and 2 variabels y_1, y_2

The dual LPP is Max $W = b^T Y$

subject to the constraints $A^T Y \leq C^T$

and $Y \geq 0.$

i.e., $\text{Max } W = (3\ \ 5)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ subject to $\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} 4 \\ 6 \\ 18 \end{pmatrix}$

and $y_1,\ y_2,\ y_3 \geq 0$

$$\begin{aligned} \textit{i.e.,}\ \text{The dual problem is Max } W &= 3y_1 + 5y_2 \\ \text{subject to}\quad y_1 &\leq 4 \\ 3y_1 + y_2 &\leq 6 \\ 2y_2 &\leq 18 \\ \text{and}\quad y_1,\ y_2 &\geq 0. \end{aligned}$$

Example 4 **Write the dual of the LPP**

$$\begin{aligned} \textbf{Min } Z &= 3x_1 - 2x_2 + 4x_3 \\ \textbf{subject to}\quad 3x_1 + 5x_2 + 4x_3 &\geq 7 \\ 6x_1 + x_2 + 3x_3 &\geq 4 \\ 7x_1 - 2x_2 - x_3 &\leq 10 \\ x_1 - 2x_2 + 5x_3 &\geq 3 \\ 4x_1 + 7x_2 - 2x_3 &\geq 2 \\ \textbf{and}\quad x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

Solution: Given primal LPP is

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

subject to

$$\begin{aligned} 3x_1 + 5x_2 + 4x_3 &\geq 7 \\ 6x_1 + x_2 + 3x_3 &\geq 4 \\ -7x_1 + 2x_2 + x_3 &\geq -10 \\ x_1 - 2x_2 + 5x_3 &\geq 3 \\ 4x_1 + 7x_2 - 2x_3 &\geq 2 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

i.e.,

$$\begin{aligned} \text{Min } Z &= CX \\ \text{subject to } AX &\geq b \\ \text{and } X &\geq 0. \end{aligned}$$

where $C = (3\ -2\ 4)$, $A = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 1 & 3 \\ -7 & 2 & 1 \\ 1 & -2 & 5 \\ 4 & 7 & -2 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 4 \\ -10 \\ 3 \\ 2 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the primal contains 5 constraints and 3 variables, the dual will contain 3 constraints and 5 variables y_1, y_2, y_3, y_4 and y_5.

$$\begin{aligned} \text{The dual LPP is Max } W &= b^T Y \\ \text{subject to the constraints } A^T Y &\leq C^T \\ \text{and } Y &\geq 0. \end{aligned}$$

$$\text{Max } W = (7\ \ 4\ \ -10\ \ 3\ \ 2), \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

$$\text{subject to } \begin{pmatrix} 3 & 6 & -7 & 1 & 4 \\ 5 & 1 & 2 & -2 & 7 \\ 4 & 3 & 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \leq \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \geq 0.$$

i.e.,

$$\begin{aligned} \text{Max } W &= 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5 \\ \text{subject to } 3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 &\leq 3 \\ 5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 &\leq -2 \\ 4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 &\leq 4 \\ y_1, y_2, y_3, y_4, y_5 &\geq 0. \end{aligned}$$

2.1.3. Unsymmetric Form

S.No	Primal / Dual	Dual / Primal
1.	Max $Z = CX$ subject to $AX = b$ $X \geq 0$	Min $W = b^T Y$ subject to $A^T Y \geq C^T$ Variables are unrestricted
2.	Min $Z = CX$ subject to $AX = b$ $X \geq 0$	Max $W = b^T Y$ subject to $A^T Y \leq C^T$ Variables are unrestricted

Remark 1: If the k^{th} constraint of the primal problem is an equality, then the corresponding dual variable y_k is unrestricted in sign and vice versa.

Remark 2: If any variable of the primal problem is unrestricted in sign, the corresponding constraint in the dual problem will be an equality and vice versa.

Example 1 **Write the dual of the LPP**

$$\begin{aligned} \textbf{Max } Z &= 3x_1 + 10x_2 + 2x_3 \\ \textbf{subject to} \quad 2x_1 + 3x_2 + 2x_3 &\leq 7 \\ 3x_1 - 2x_2 + 4x_3 &= 3 \\ \textbf{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Solution: Given primal LPP is

$$\begin{aligned} \text{Max } Z &= 3x_1 + 10x_2 + 2x_3 \\ \text{subject to} \quad 2x_1 + 3x_2 + 2x_3 &\leq 7 \\ 3x_1 - 2x_2 + 4x_3 &= 3 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Since the primal problem contains 2 constraints and 3 variables, the dual problem will contain 3 constraints and 2 dual variables y_1, y_2.

Also, since the second constraint in the primal problem is an equality, the corresponding second dual variable y_2 is unrestricted in sign.

$\therefore$ The dual problem is

$$\text{Min } W = 7y_1 + 3y_2$$

subject to
$$2y_1 + 3y_2 \geq 3$$
$$3y_1 - 2y_2 \geq 10$$
$$2y_1 + 4y_2 \geq 2$$
and $y_1 \geq 0$, y_2 is unrestricted.

Example 2 **Write the dual of the LPP.**

$$\textbf{Min Z} = x_2 + 3x_3$$

subject to
$$2x_1 + x_2 \leq 3$$
$$x_1 + 2x_2 + 6x_3 \geq 5$$
$$-x_1 + x_2 + x_3 = 2$$
$$x_1, x_2, x_3 \geq 0.$$

Solution: Given primal LPP is

$$\text{Min } Z = 0x_1 + x_2 + 3x_3$$

subject to
$$-2x_1 - x_2 + 0x_3 \geq -3$$
$$x_1 + 2x_2 + 6x_3 \geq 5$$
$$-x_1 + x_2 + x_3 = 3$$
and $x_1, x_2, x_3 \geq 0.$

Since the primal problem contains 3 constraints and 3 variables, the dual problem will contain 3 constraints and 3 dual variables y_1, y_2 , y_3.

Also, since the third constraint in the primal problem is an equality, the corresponding third dual variable y_3 is unrestricted in sign.

The dual LPP is
$$\text{Max } W = -3y_1 + 5y_2 + 2y_3$$

subject to
$$-2y_1 + y_2 - y_3 \leq 0$$
$$-y_1 + 2y_2 + y_3 \leq 1$$
$$6y_2 + y_3 \leq 2$$
and $y_1, y_2 \geq 0$, y_3 is unrestricted.

Example 3 **Write the dual of the following primal LPP**

$$\textbf{Min Z} = 4x_1 + 5x_2 - 3x_3$$

subject to
$$x_1 + x_2 + x_3 = 22$$
$$3x_1 + 5x_2 - 2x_3 \leq 65$$
$$x_1 + 7x_2 + 4x_3 \geq 120$$
$x_1 \geq 0$, $x_2 \geq 0$ **and** x_3 **unrestricted.**

[MU. MCA. Nov 94]

Solution: Given primal LPP is

$$\text{Min } Z = 4x_1 + 5x_2 - 3x_3$$

subject to
$$x_1 + x_2 + x_3 = 22$$
$$-3x_1 - 5x_2 + 2x_3 \geq -65$$
$$x_1 + 7x_2 + 4x_3 \geq 120$$
$x_1 \geq 0$, $x_2 \geq 0$ and x_3 unrestricted.

Since the primal problem contains 3 constraints and 3 variables, the dual problem will contain 3 constraints and 3 dual variables y_1, y_2 y_3. Since the first primal constraint is an equality, the corresponding first dual variable y_1 is unrestricted in sign. Also, since the third primal variable x_3 is unrestricted in sign, the corresponding third dual constraint will be an equality.

The dual LPP is
$$\text{Max } W = 22y_1 - 65y_2 + 120y_3$$

subject to
$$y_1 - 3y_2 + y_3 \leq 4$$
$$y_1 - 5y_2 + 7y_3 \leq 5$$
$$y_1 + 2y_2 + 4y_3 = -3$$
and y_2, $y_3 \geq 0$, y_1 is unrestricted.

Example 4 **Write the dual of the primal:**

$$\textbf{Max Z} = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$$

subject to
$$3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$$
$$2x_1 + x_2 + 3x_4 + 9x_5 = 6$$
$x_1, x_2, x_3, x_4 \geq 0$ **and** x_5 **unrestricted.**

Solution: Given primal LPP is

$$\text{Max } Z = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$$

subject to
$$3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$$
$$2x_1 + x_2 + 0x_3 + 3x_4 + 9x_5 = 6$$

$x_1, x_2, x_3, x_4 \geq 0$ and x_5 unrestricted.

Since the primal problem contains 2 constraints and 5 variables, the dual problem will contain 5 constraints and 2 dual variables y_1, y_2. Since all the constraints in the primal are equality, all the dual variable are unrestricted in sign. Also, since the primal variable x_5 is unrestricted in sign, the corresponding fifth dual constraint is an equality.

The dual LPP is $\quad \text{Min } W = 2y_1 + 6y_2$

subject to
$$3y_1 + 2y_2 \geq 6$$
$$7y_1 + y_2 \geq 6$$
$$8y_1 \geq 1$$
$$5y_1 + 3y_2 \geq 7$$
$$y_1 + 9y_2 = 5$$

and y_1, y_2 are unrestricted.

2.1.4. Some important Results in duality

Result 1: The dual of the dual is the primal.

Result 2: If one is a maximization problem then the other is a minimization problem.

Result 3: The necessary and sufficient condition for any LPP and its dual to have an optimal solution is that both have feasible solutions.

Result 4: ***Fundamental theorem of Duality [MU. BE. Oct 96].*** If either the primal or the dual problem has a finite optimal solution, then the other problem also has finite optimal solution and the values of the objective functions are equal *i.e.,* Max Z = Min W. The solution of the other problem can be read from the $(Z_j - C_j)$ row below the columns of slack, surplus variables. The values of the dual variables are called ***shadow prices.***

Result 5: ***(Existence theorem):*** If either the primal or the dual problem has an unbounded solution, then the other problem has no feasible solution.

Result 6: If i^{th} dual constraint is multiplied by -1, then ith primal variable computed from the $(Z_j - C_j)$ row of the dual problem must be multiplied by -1.

Result 7: If dual has no feasible solution, then primal also admits no feasible solution.

Result 8: The value of the objective function z for any feasible solution of the primal is less than or equal to the value of the objective function w for any feasible solution of the dual.

Result 9: ***(complementary slackness theorem)***

(i) If a primal variable is positive, then the corresponding dual constraint is an equation at the optimum and vice versa.

(ii) If a primal constraint is a strict inequality, then the corresponding dual variable is zero at the optimum and vice versa.

Example 1 **Write down the dual of the following LPP and solve it.**

$$\text{Max } Z = 4x_1 + 2x_2$$

subject to the constraints
$$-x_1 - x_2 \leq -3$$
$$-x_1 + x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

Hence or otherwise write down the solution of the primal.

Solution: Given primal LPP is Max $Z = 4x_1 + 2x_2$

subject to
$$-x_1 - x_2 \leq -3$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Its dual problem is $\text{Min } W = -3y_1 + 2y_2$

subject to
$$-y_1 + y_2 \geq 4$$
$$-y_1 - y_2 \geq 2$$
$$y_1, y_2 \geq 0$$

$$\therefore \text{Max } W^* = 3y_1 - 2y_2$$

subject to
$$-y_1 + y_2 \geq 4$$
$$-y_1 - y_2 \geq 2$$
$$y_1, y_2 \geq 0$$

By introducing the surplus variables s_1, s_2 and the artificial variables R_1, R_2, we have

$$\text{Max } W^* = 3y_1 - 2y_2 + 0s_1 + 0s_2 - MR_1 - MR_2$$

subject to

$$-y_1 + y_2 - s_1 + 0s_2 + R_1 + 0R_2 = 4$$
$$-y_1 - y_2 + 0s_1 - s_2 + 0R_1 + R_2 = 2$$
$$y_1, y_2, s_1, s_2, R_1, R_2 \geq 0$$

The initial basic feasible solution is given by $R_1 = 4$, $R_2 = 2$(basic)

($y_1 = y_2 = s_1 = s_2 = 0$, non-basic)

Initial iteration

b_j (3 −2 0 0 −M −M)

C_B Y_B	X_B	y_1	y_2	s_1	s_2	R_1	R_2
−M R_1	4	−1	(1)	−1	0	1	0
−M R_2	2	−1	−1	0	−1	0	1
$(W_j^* - b_j)$	−6M	2M − 3	2	M	M	0	0

Since all $(W_j^* - b_j) \geq 0$ and the artificial variables R_1 and R_2 appears in the basis at non–zero level, the dual problem does not possess any optimum basic feasible solution.

∴ There exists no finite optimum solution to the given primal LPP.

Example 2 **Use duality to solve the following LPP.**

$$\textbf{Minimize } Z = 2x_1 + 2x_2$$

subject to

$$2x_1 + 4x_2 \geq 1$$
$$-x_1 - 2x_2 \leq -1$$
$$2x_1 + x_2 \geq 1$$
$$\textbf{and } x_1, x_2 \geq 0.$$

Solution: Given primal LPP is Minimize $Z = 2x_1 + 2x_2$

subject to

$$2x_1 + 4x_2 \geq 1$$
$$x_1 + 2x_2 \geq 1$$
$$2x_1 + x_2 \geq 1$$
$$\text{and } x_1, x_2 \geq 0.$$

Its dual problem is Max $W = y_1 + y_2 + y_3$

subject to

$$2y_1 + y_2 + 2y_3 \leq 2$$
$$4y_1 + 2y_2 + y_3 \leq 2$$
$$\text{and } y_1, y_2, y_3 \geq 0.$$

By introducing the non-negative slack variables s_1 and s_2, the standard form of the dual LPP becomes

$$\text{Max } W = y_1 + y_2 + y_3 + 0s_1 + 0s_2$$

subject to

$$2y_1 + y_2 + 2y_3 + s_1 + 0s_2 = 2$$
$$4y_1 + 2y_2 + y_3 + 0s_1 + s_2 = 2$$
$$\text{and } y_1, y_2, y_3, s_1, s_2 \geq 0.$$

The initial basic feasible solution is $s_1 = 2$, $s_2 = 2$.

Initial iteration:

b_j (1 1 1 0 0)

C_B Y_B	X_B	y_1	y_2	y_3	s_1	s_2	θ
0 s_1	2	2	1	2	1	0	$\frac{2}{2}$
0 s_2	2	(4)	2	1	0	1	$\frac{2}{4}$
$(W_j - b_j)$	0	−1	−1	−1	0	0	

First iteration: Introduce y_1 and drop s_2.

b_j (1 1 1 0 0)

C_B Y_B	X_B	y_1	y_2	y_3	s_1	s_2	θ
0 s_1	1	0	0	$\left(\frac{3}{2}\right)$	1	$\frac{-1}{2}$	$\frac{2}{3}$
1 y_1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	2
$(W_j - b_j)$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-3}{4}$	0	$\frac{1}{4}$	

Second iteration: Introduce y_3 and drop s_1.

	b_j	(1	1	1	0	0)	
C_B Y_B	X_B	y_1	y_2	y_3	s_1	s_2	θ
1 y_3	$\frac{2}{3}$	0	0	1	$\frac{2}{3}$	$\frac{-1}{3}$	–
1 y_1	$\frac{1}{3}$	1	$\left(\frac{1}{2}\right)$	0	$\frac{-1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$*
$(W_j - b_j)$	1	0	$\frac{-1}{2}$	0	$\frac{1}{2}$	0	

Third iteration: Introduce y_2 and drop y_1.

	b_j	(1	1	1	0	0)
C_B Y_B	X_B	y_1	y_2	y_3	s_1	s_2
1 y_3	$\frac{2}{3}$	0	0	1	$\frac{2}{3}$	$\frac{-1}{3}$
1 y_2	$\frac{2}{3}$	2	1	0	$\frac{-1}{3}$	$\frac{2}{3}$
$(W_j - b_j)$	$\frac{4}{3}$	1	0	0	$\frac{1}{3}$	$\frac{1}{3}$

Since all $(W_j - b_j) \geq 0$, the current basic feasible solution is optimal.

The optimal solution to the dual LPP is Max $w = \frac{4}{3}$, $y_1 = 0$, $y_2 = \frac{2}{3}$, $y_3 = \frac{2}{3}$. Here it is observed that the primal variables x_1 and x_2 respectively correspond to the slack variables s_1 and s_2 of the dual problem.

As seen from above table, the net evaluations $(W_j - b_j)$ corresponding to the slack variables s_1 and s_2 are $\frac{1}{3}$ and $\frac{1}{3}$ respectively.

$\therefore$ The optimum solution to the original primal LPP is Min $Z = \frac{4}{3}$, $x_1 = \frac{1}{3}$ and $x_2 = \frac{1}{3}$.

Example 3 **Apply the principle of duality to solve the LPP.**

$$\text{Maximize } Z = 3x_1 + 2x_2$$

subject to
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$x_2 \leq 3 \text{ and } x_1, x_2 \geq 0.$$

[MU. BE. Oct 96]

Solution: Given primal LPP is

$$\text{Max } Z = 3x_1 + 2x_2$$

subject to
$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$x_2 \leq 3 \text{ and } x_1, x_2 \geq 0.$$

Its dual problem is Min $W = -y_1 + 7y_2 + 10y_3 + 3y_4$

subject to
$$-y_1 + y_2 + y_3 \geq 3$$
$$-y_1 + y_2 + 2y_3 + y_4 \geq 2$$
$$\text{and } y_1, y_2, y_3, y_4 \geq 0.$$

i.e., $\text{Max } W^* = y_1 - 7y_2 - 10y_3 - 3y_4$

subject to
$$-y_1 + y_2 + y_3 \geq 3$$
$$-y_1 + y_2 + 2y_3 + y_4 \geq 2$$
$$\text{and } y_1, y_2, y_3, y_4 \geq 0$$

By introducing the surplus variables s_1, s_2 and the artificial variables R_1, R_2, the standard form of the dual LPP is

$$\text{Max } W^* = y_1 - 7y_2 - 10y_3 - 3y_4 + 0s_1 + 0s_2 - MR_1 - MR_2$$

subject to
$$-y_1 + y_2 + y_3 + 0y_4 - s_1 + 0s_2 + R_1 = 3$$
$$-y_1 + y_2 + 2y_3 + y_4 + 0s_1 - s_2 + R_2 = 2$$
$$\text{and } y_1, y_2, y_3, y_4, s_1, s_2, R_1, R_2 \geq 0.$$

The initial basic feasible solution is given by
$R_1 = 3$, $R_2 = 2$ (basic) $(y_1 = y_2 = y_3 = y_4 = s_1 = s_2 = 0$, non-basic)

Initial iteration:

b_j (1 −7 −10 −3 0 0 −M −M)

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	s_1	s_2	R_1	R_2	θ
−M	R_1	3	−1	1	1	0	−1	0	1	0	3
−M	R_2	2	−1	1	(2)	1	0	−1	0	1	1
$(W_j^* - b_j)$		−5M	2M−1	−2M+7	−3M+10	−M+3	M	M	0	0	

First iteration: Introduce y_3 and drop R_2.

b_j (1 −7 −10 −3 0 0 −M)

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	s_1	s_2	R_1	θ
−M	R_1	2	$\frac{-1}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	−1	$\frac{1}{2}$	1	4
−10	y_3	1	$\frac{-1}{2}$	$\left(\frac{1}{2}\right)$	1	$\frac{1}{2}$	0	$\frac{-1}{2}$	0	2
$(W_j^* - b_j)$		−2M−10	$\frac{M+8}{2}$	$\frac{-M+4}{2}$	0	$\frac{M-4}{2}$	M	$\frac{-M+10}{2}$	0	

Second iteration: Introduce y_2 and drop y_3.

b_j (1 −7 −10 −3 0 0 −M)

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	s_1	s_2	R_1	θ
−M	R_1	1	0	0	−1	−1	−1	(1)	1	1
−7	y_2	2	−1	1	2	1	0	−1	0	–
$(W_j^* - b_j)$		−M−14	6	0	M − 4	M − 4	M	− M + 7	0	

Third iteration: Introduce s_2 and drop R_1.

b_j (1 −7 −10 −3 0 0)

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	s_1	s_2
0	s_2	1	0	0	−1	−1	−1	1
−7	y_2	3	−1	1	1	0	−1	0
$(W_j^* - b_j)$		−21	6	0	3	3	7	0

Since all $(W_j^* - b_j) \geq 0$, the current basic feasible solution is optimal.

The optimal solution of the dual problem is

$$\text{Max } W^* = -21, \quad y_2 = 3, \quad y_1 = y_3 = y_4 = 0.$$

i.e., $$\text{Min } W = 21, \quad y_2 = 3, \quad y_1 = y_3 = y_4 = 0.$$

Also, from the above optimum simplex table of the dual problem, the optimum solution of the original (primal) problem is given by

$$\text{Max } Z = 21, \quad x_1 = 7, \quad x_2 = 0.$$

$\boxed{\text{Example 4}}$ **Prove using duality theory that the following linear program is feasible but has no optimal solution.**

$$\textbf{Minimize } Z = x_1 - x_2 + x_3$$

subject to

$$x_1 - x_3 \geq 4$$
$$x_1 - x_2 + 2x_3 \geq 3$$
$$\textbf{and } x_1, x_2, x_3 \geq 0.$$

[MU. MCA. Nov 93]

Solution: Given primal LPP is

$$\text{Min } Z = x_1 - x_2 + x_3$$

subject to

$$x_1 + 0x_2 - x_3 \geq 4$$
$$x_1 - x_2 + 2x_3 \geq 3$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

Its dual problem is

$$\text{Max } W = 4y_1 + 3y_2$$

subject to

$$y_1 + y_2 \leq 1$$
$$0y_1 + y_2 \geq 1$$
$$-y_1 + 2y_2 \leq 1$$
$$\text{and } y_1, y_2 \geq 0.$$

By introducing the slack variables s_1, s_3 and surplus variable s_2 and an artificial variable R_1, the standard form of the dual LPP is

$$\text{Max } W = 4y_1 + 3y_2 + 0s_1 + 0s_2 + 0s_3 - MR_1$$

subject to

$$\begin{aligned} y_1 + y_2 + s_1 + 0s_2 + 0s_3 &= 1 \\ 0y_1 + y_2 + 0s_1 - s_2 + 0s_3 + R_1 &= 1 \\ -y_1 + 2y_2 + 0s_1 + 0s_2 + s_3 &= 1 \\ \text{and } y_1, y_2, s_1, s_2, s_3, R_1 &\geq 0. \end{aligned}$$

The initial basic feasible solution is given by

$s_1 = 1$, $R_1 = 1$, $s_3 = 1$ (basic) ($y_1 = y_2 = s_2 = 0$, non-basic)

Initial iteration

	b_j	(4	3	0	0	$-M$	0)	
C_B Y_B	X_B	y_1	y_2	s_2	s_1	R_1	s_3	θ
0 s_1	1	1	1	0	1	0	0	1
$-M$ R_1	1	0	1	-1	0	1	0	1
0 s_3	1	-1	(2)	0	0	0	1	$\frac{1}{2}$*
$(W_j - b_j)$	$-M$	-4	$-M-3$	M	0	0	0	

First iteration: Introduce y_2 and drop s_3.

	b_j	(4	3	0	0	$-M$	0)	
C_B Y_B	X_B	y_1	y_2	s_2	s_1	R_1	s_3	θ
0 s_1	$\frac{1}{2}$	$(\frac{3}{2})$	0	0	1	0	$-\frac{1}{2}$	$\frac{1}{3}$
$-M$ R_1	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	0	1	$-\frac{1}{2}$	1
3 y_2	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	–
$(W_j - b_j)$	$\frac{-M+3}{2}$	$\frac{-M-11}{2}$	0	M	0	0	$\frac{M+3}{2}$	

Second iteration: Introduce y_1 and drop s_1.

	b_j	(4	3	0	0	$-M$	0)
C_B Y_B	X_B	y_1	y_2	s_2	s_1	R_1	s_3
4 y_1	$(\frac{1}{3})$	1	0	0	$\frac{2}{3}$	0	$-\frac{1}{3}$
$-M$ R_1	$\frac{1}{3}$	0	0	-1	$-\frac{1}{3}$	1	$-\frac{1}{3}$
3 y_2	$\frac{2}{3}$	0	1	0	$\frac{1}{3}$	0	$\frac{1}{3}$
$(W_j - b_j)$	$\frac{10-M}{3}$	0	0	M	$\frac{11+M}{3}$	0	$\frac{M-1}{3}$

Since all $(W_j - b_j) \geq 0$, and an artificial variable R_1 appears in the basis at non–zero level, the dual problem has no optimal basic feasible solution.

∴ The exists no finite optimum solution to the given primal LPP.

2.1.5. Dual Simplex Method

The dual simplex method is used to solve problems which start dual feasible *i.e.*, whose primal is optimal but infeasible. In this method the solution starts optimum but infeasible and remains infeasible until the true optimum is reached at which the solution becomes feasible.

The regular simplex method starts with a basic feasible but non–optimal solution and works towards optimality, whereas the dual simplex method starts with a basic infeasible but optimal solution and works towards feasibility. Also in regular simplex method we first determine the entering variable and then the leaving variable while in the case of dual simplex method we first determine the leaving variable and then the entering variable.

Working procedure for dual simplex method:

Step 1: Convert the problem to maximization form if it is initially in the minimization form.

Step 2: Convert ($\geq$) type constraints, if any to ($\leq$) type by multiplying both sides by -1.

Step 3: Convert the inequality constraints into equalities by introducing slack variables. Obtain the initial basic solution and express this information in the simplex table.

Step 4: ***(Optimal condition)*** Test the nature of $(Z_j - C_j)$ and X_{Bi}.

Case *(i)*: If all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, then the current solution is an optimum feasible solution.

Case *(ii)*: If all $(Z_j - C_j) \geq 0$ and atleast one $X_{Bi} < 0$, then the current solution is not an optimum basic feasible solution and go to the next step.

Case *(iii)*: If any $(Z_j - C_j) < 0$, then this method fails.

Step 5: ***(Feasibility condition)***

(i) **(Leaving variable):** The leaving variable is the basic variable corresponding to the most negative value of X_{Bi}. Let x_k be the leaving variable.

(ii) ***(Entering variable)***: Compute the ratio between $(Z_j - C_j)$ row and the key row. *i.e.,* compute $\theta = \max \left\{ \frac{(Z_j - C_j)}{a_{ik}}, a_{ik} < 0 \right\}$. (consider the ratios with –ve denominators alone). The entering variable is the non-basic variable corresponding to the maximum ratio θ. If there is no such ratio with –ve denominator, then the problem doesnot have a feasible solution.

Step 6: Carry out the row operations as in the regular simplex method and repeat the procedure until either an optimum feasible solution is obtained or there is an indication of non–existence of a feasible solution.

Example 1 **Using dual simplex method solve the LPP**

$$\text{Minimize } Z = 2x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 &\geq 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\geq 3 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

[MU. BE Apr 90, MKU B.Sc 81]

Solution: After converting the objective function in to maximization type and all the constraints in ≤ type, the given LPP becomes

$$\text{Max } Z^* = -2x_1 - x_2$$

subject to

$$\begin{aligned} -3x_1 - x_2 &\leq -3 \\ -4x_1 - 3x_2 &\leq -6 \\ -x_1 - 2x_2 &\leq -3 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

By introducing the non–negative slack variables s_1, s_2 and s_3, the LPP becomes

$$\text{Max } Z^* = -2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$\begin{aligned} -3x_1 - x_2 + s_1 &= -3 \\ -4x_1 - 3x_2 + s_2 &= -6 \\ -x_1 - 2x_2 + s_3 &= -3 \\ \text{and } x_1, x_2, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

The initial basic solution is given by

$s_1 = -3, s_2 = -6, s_3 = -3$ (basic) $(x_1 = x_2 = 0$, non-basic)

Initial iteration

		C_j	(–2	–1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	–3	–3	–1	1	0	0
0	s_2	–6	–4	(–3)	0	1	0
0	s_3	–3	–1	–2	0	0	1
$(Z_j^* - C_j)$		0	2	1	0	0	0

Since all $(Z_j^* - C_j) \geq 0$ and all $X_{Bi} < 0$, the current solution is not an optimum basic feasible solution.

Since $X_{B2} = -6$ is the most negative, the corresponding basic variable s_2 leaves the basis.

$$\text{Now } \theta = \text{Max} \left\{ \frac{(Z_j^* - C_j)}{a_{ik}}, a_{ik} < 0 \right\} \text{ where } x_k \text{ is the leaving variable.}$$

$$= \text{Max} \left\{ \frac{2}{-4}, \frac{1}{-3} \right\} = \text{Max} \left\{ \frac{-1}{2}, \frac{-1}{3} \right\} = \frac{-1}{3}$$

∴ The corresponding non–basic variable x_2 enters into the basis.

First iteration: Drop s_2 and introduce x_2.

C_B	Y_B	C_j → X_B	x_1 (−2)	x_2 (−1)	s_1 (0)	s_2 (0)	s_3 (0)
0	s_1	−1	$\left(\frac{-5}{3}\right)$	0	1	$\frac{-1}{3}$	0
−1	x_2	2	$\frac{4}{3}$	1	0	$\frac{-1}{3}$	0
0	s_3	1	$\frac{5}{3}$	0	0	$\frac{-2}{3}$	1
$(Z_j^* - C_j)$		−2	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0

Since all $(Z_j^* - C_j) \geq 0$, and $X_{B1} = -1 < 0$, the current solution is not optimum basic feasible solution.

Since $X_{B1} = -1$ is negative, the corresponding basic variable s_1 leaves the basis.

$$\text{Now } \theta = \text{Max}\left\{\frac{(Z_j^* - C_j)}{a_{ik}}, a_{ik} < 0\right\}$$

$$= \text{Max}\left\{\frac{2/3}{-5/3}, \frac{1/3}{-1/3}\right\} = \text{Max}\left\{\frac{-2}{5}, -1\right\} = \frac{-2}{5}$$

∴ The corresponding non-basic variable x_1 enters the basis

Second iteration: Drop s_1 and introduce x_1.

C_B	Y_B	C_j → X_B	x_1 (−2)	x_2 (−1)	s_1 (0)	s_2 (0)	s_3 (0)
−2	x_1	$\frac{3}{5}$	1	0	$\frac{-3}{5}$	$\frac{1}{5}$	0
−1	x_2	$\frac{6}{5}$	0	1	$\frac{4}{5}$	$\frac{-3}{5}$	0
0	s_3	0	0	0	1	−1	1
$(Z_j^* - C_j)$		$\frac{-12}{5}$	0	0	$\frac{2}{5}$	$\frac{1}{5}$	0

Since all $(Z_j^* - C_j) \geq 0$, and all $X_{Bi} \geq 0$, the current solution is an optimum basic feasible solution.

∴ The optimum solution is Max $Z^* = \frac{-12}{5}$, $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$

$$\text{But Min } Z = -\text{Max } Z^* = -\left(\frac{-12}{5}\right) = \frac{12}{5}$$

$$\therefore \text{Min } Z = \frac{12}{5}, \quad x_1 = \frac{3}{5}, \quad x_2 = \frac{6}{5}.$$

$\boxed{\textbf{Example 2}}$ **Using dual simplex method solve the LPP**

$$\textbf{Maximize } Z = 6x_1 + 4x_2 + 4x_3$$

subject to

$$3x_1 + x_2 + 2x_3 \geq 2$$
$$2x_1 + x_2 - x_3 \geq 1$$
$$-x_1 + x_2 + 2x_3 \geq 1$$
$$\textbf{and } x_1, x_2, x_3 \geq 0.$$

[MU. BE Nov 91]

Solution: The given LPP is

$$\text{Max } Z = 6x_1 + 4x_2 + 4x_3$$

subject to

$$-3x_1 - x_2 - 2x_3 \leq -2$$
$$-2x_1 - x_2 + x_3 \leq -1$$
$$x_1 - x_2 - 2x_3 \leq -1$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

By introducing the non-negative slack variables s_1, s_2 and s_3, the LPP becomes

$$\text{Max } Z = 6x_1 + 4x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$-3x_1 - x_2 - 2x_3 + s_1 = -2$$
$$-2x_1 - x_2 + x_3 + s_2 = -1$$
$$x_1 - x_2 - 2x_3 + s_3 = -1$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

The initial basic solution is given by
$s_1 = -2, s_2 = -1, s_3 = -1$ (basic) $(x_1 = x_2 = x_3 = 0$, non–basic)

Initial iteration

C_B	Y_B	C_j / X_B	x_1 (6)	x_2 (4)	x_3 (4)	s_1 (0)	s_2 (0)	s_3 (0)
0	s_1	–2	–3	–1	–2	1	0	0
0	s_2	–1	–2	–1	1	0	1	0
0	s_3	–1	1	–1	–2	0	0	1
$(Z_j - C_j)$		0	–6	–4	–4	0	0	0

Since there are some $(Z_j - C_j) < 0$, this method fails. *i.e.*, we cannot solve this problem by this dual simplex method.

Example 3 **Using dual simplex method solve the LPP.**

$$\text{Minimize } Z = x_1 + x_2$$

subject to

$$2x_1 + x_2 \geq 2$$
$$-x_1 - x_2 \geq 1$$
$$\text{and } x_1, x_2 \geq 0.$$

Solution: The given LPP is Max $Z^* = -x_1 - x_2$

subject to

$$-2x_1 - x_2 \leq -2$$
$$x_1 + x_2 \leq -1$$
$$\text{and } x_1, x_2 \geq 0.$$

By introducing the non-negative slack variables s_1, and s_2, the LPP becomes

$$\text{Max } Z^* = -x_1 - x_2 + 0s_1 + 0s_2$$

subject to

$$-2x_1 - x_2 + s_1 = -2$$
$$x_1 + x_2 + s_2 = -1$$
$$\text{and } x_1, x_2, s_1, s_2 \geq 0.$$

The initial basic solution is given by

$s_1 = -2$, $s_2 = -1$, (basic) $(x_1 = x_2 = 0$, non-basic)

Initial iteration

C_B	Y_B	C_j / X_B	x_1 (–1)	x_2 (–1)	s_1 (0)	s_2 (0)
0	s_1	–2	(–2)	–1	1	0
0	s_2	–1	1	1	0	1
$(Z_j^* - C_j)$		0	1	1	0	0

Since all $(Z_j^* - C_j) \geq 0$, and all $X_{Bi} < 0$, the current solution is not an optimum basic feasible solution.

Since $X_{B1} = -2$ is most negative, the corresponding basic variable s_1 leaves the basis.

$$\text{Now } \theta = \text{Max}\left\{\frac{(Z_j^* - C_j)}{a_{ik}}, a_{ik} < 0\right\}$$
$$= \text{Max}\left\{\frac{1}{-2}, \frac{1}{-1}\right\} = \text{Max}\left\{\frac{-1}{2}, -1\right\} = \frac{-1}{2}.$$

$\therefore$ The corresponding non-basic variable x_1 enters the basis.

First iteration: Drop s_1 and introduce x_1

C_B	Y_B	C_j / X_B	x_1 (–1)	x_2 (–1)	s_1 (0)	s_2 (0)
–1	x_1	1	1	$\frac{1}{2}$	$\frac{-1}{2}$	0
0	s_2	–2	0	$\frac{1}{2}$	$\frac{1}{2}$	1
$(Z_j^* - C_j)$		–1	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Since all $(Z_j^* - C_j) \geq 0$, and $X_{B2} = -2 < 0$, the current solution is not an optimum basic feasible solution.

Since $X_{B2} = -2$, corresponding basic variable s_2 leaves the basis.

$$\text{Now } \theta = \text{Max}\left\{\frac{(Z_j - C_j)}{a_{ik}}, a_{ik} < 0\right\} \text{ where } x_k \text{ is the leaving variable.}$$

Since all the entries in the key row are positive, we cannot find the ratio θ with negative denominators. So, there is no feasible solution to the given LPP.

Example 4 **Use dual simplex method to solve the LPP.**

$$\textbf{Maximize } Z = -3x_1 - 2x_2$$

$$\textbf{subject to} \quad \begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 &\leq 7 \\ x_1 + 2x_2 &\geq 10 \\ x_2 &\leq 3 \\ \textbf{and } x_1, x_2 &\geq 0. \end{aligned}$$

[MU. BE. Apr 93]

Solution: The given LPP is Maximize $Z = -3x_1 - 2x_2$

$$\text{subject to} \quad \begin{aligned} -x_1 - x_2 &\leq -1 \\ x_1 + x_2 &\leq 7 \\ -x_1 - 2x_2 &\leq -10 \\ 0x_1 + x_2 &\leq 3 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

By introducing the non–negative slack variables s_1, s_2, s_3 and s_4, the LPP becomes

$$\text{Max } Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

$$\text{subject to} \quad \begin{aligned} -x_1 - x_2 + s_1 &= -1 \\ x_1 + x_2 + s_2 &= 7 \\ -x_1 - 2x_2 + s_3 &= -10 \\ 0x_1 + x_2 + s_4 &= 3 \\ \text{and } x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0. \end{aligned}$$

The initial basic solution is given by
$s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3$ (basic) ($x_1 = x_2 = 0$, non-basic)

Initial iteration

		C_j	$(-3$	-2	0	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	-1	-1	-1	1	0	0	0
0	s_2	7	1	1	0	1	0	0
0	s_3	-10	-1	(-2)	0	0	1	0
0	s_4	3	0	1	0	0	0	1
$(Z_j - C_j)$		0	3	2	0	0	0	0

First iteration: Drop s_3 and introduce x_2

		c_j	$(-3$	-2	0	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	4	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0
0	s_2	2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0
-2	x_2	5	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0
0	s_4	-2	$\left(-\frac{1}{2}\right)$	0	0	0	$\frac{1}{2}$	1
$(Z_j - C_j)$		-10	2	0	0	0	1	0

Second iteration: Drop s_4 and introduce x_1

		C_j	$(-3$	-2	0	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	2	0	0	1	0	-1	-1
0	s_2	0	0	0	0	1	1	1
-2	x_2	3	0	1	0	0	0	1
-3	x_1	4	1	0	0	0	-1	-2
$(Z_j - C_j)$		-18	0	0	0	0	3	4

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is an optimum basic feasible solution.

$\therefore$ The optimum solution is Max $Z = -18$, $x_1 = 4$, $x_2 = 3$.

EXERCISE

1. **What is duality ? Explain.** *[MU. MCA May 89]*
2. **What is duality in LPP? Explain its applications.** *[MU. MCA. May 92]*
3. **Explain the relationship of the dual and primal problems.** *[MU. MCA May 89]*
4. **What is the essential difference between regular simplex method and dual simplex method ?** *[BRU MSc 83]*

5. Develop the theory of dual simplex algorithm.
6. Explain the importance of duality in LPP.
7. Write the dual of the LPP.

$$\text{Maximize } Z = 2x_1 + 5x_2 + 6x_3$$
$$\text{subject to} \quad \begin{aligned} x_1 + 6x_2 - x_3 &\leq 3 \\ -2x_1 + x_2 + 4x_3 &\leq 4 \\ x_1 - 5x_2 + 3x_3 &\leq 1 \\ -3x_1 - 3x_2 + 7x_3 &\leq 6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Also verify that the dual of the dual problem is the primal problem. *[MU. BE.Nov 91]*

8. Write the dual of the LPP.

$$\text{Max } Z = 2x_1 + 5x_2 + 3x_3$$
$$\text{subject to} \quad \begin{aligned} 2x_1 + 4x_2 - x_3 &\leq 8 \\ -2x_1 - 2x_2 + 3x_3 &\geq -7 \\ x_1 + 3x_2 - 5x_3 &\geq -2 \\ 4x_1 + x_2 + 3x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

9. Write the dual of the LPP.

$$\text{Minimize } Z = 7x_1 + 3x_2 + 8x_3$$
$$\text{subject to} \quad \begin{aligned} 8x_1 + 2x_2 + x_3 &\geq 3 \\ 3x_1 + 6x_2 + 4x_3 &\geq 4 \\ 4x_1 + x_2 + 5x_3 &\geq 1 \\ x_1 + 5x_2 + 2x_3 &\geq 7 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

10. Obtain the dual of the given LPP.

$$\text{Minimize } Z = 4x_1 + x_2$$
$$\text{subject to} \quad \begin{aligned} x_1 + x_2 &\geq 10 \\ 2x_1 + 3x_2 &\geq 24 \\ x_1, x_2 &\geq 0. \end{aligned}$$

[BNU. BE. Nov 96]

11. Write the dual of

$$\text{Max } Z = 3x_1 + 17x_2 + 9x_3$$
$$\text{subject to} \quad \begin{aligned} x_1 - x_2 + x_3 &\geq 3 \\ -3x_1 + 2x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

[MU. B.Tech. Oct 96]

12. Write down the dual of the following LPP.

$$\text{Max } Z = 5x_1 + 12x_2 + 4x_3$$
$$\text{subject to} \quad \begin{aligned} x_1 + 2x_2 - x_3 &\leq 5 \\ 2x_1 - x_2 + 3x_3 &= 2 \\ x_1 - x_2 + 3x_3 &\geq 2 \\ x_i &\geq 0. \end{aligned}$$

[BRU. BE. Nov 96]

13. Write down the dual of the following Problem.

$$\text{Max } Z = 2x_1 + 4x_2 + x_3$$
$$\text{subject to} \quad \begin{aligned} x_1 + 2x_2 - x_3 &\leq 5 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ -x_1 + 2x_2 + 2x_3 &\geq 1 \end{aligned}$$

$x_1 \geq 0$, $x_2 \geq 0$ and x_3 unrestricted. *[BRU. BE.Apr 96]*

14. Write the dual of the LPP.

$$\text{Minimize } Z = 3x_1 - 2x_2 + 4x_3$$
$$\text{subject to} \quad \begin{aligned} 3x_1 + 5x_2 + 4x_3 &\geq 7 \\ -6x_1 - x_2 - 3x_3 &\leq -4 \\ 7x_1 - 2x_2 - x_3 &\leq 10 \\ x_1 - 2x_2 + 5x_3 &\geq 10 \\ 4x_1 + 7x_2 - 2x_3 &\geq 2 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

15. Write the dual of the LPP.

$$\text{Max } Z = 4x_1 + 5x_2 + 12x_3$$
$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 + x_3 &\leq 4 \\ 3x_1 - 2x_2 + x_3 &= 3 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

16. Write the dual of the LPP.

$$\begin{aligned} \text{Min } Z &= 10x_1 - 6x_2 - 8x_3 \\ \text{subject to} \quad x_1 - 3x_2 + x_3 &= 5 \\ -2x_1 + x_2 + 3x_3 &= 8 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

17. Write the dual of the LPP.

$$\begin{aligned} \text{Max } Z &= 4x_1 + 2x_2 \\ \text{subject to} \quad x_1 - 2x_2 &\geq 2 \\ x_1 + 2x_2 &= 8 \\ x_1 - x_2 &\leq 10 \end{aligned}$$

$x_1 \geq 0$, x_2 unrestricted in sign.

18. Write the dual of the LPP.

$$\begin{aligned} \text{Minimize } Z &= 21x_1 + 48x_2 \\ \text{subject to} \quad 14x_1 + 13x_2 &\geq 16 \\ -4x_1 - 17x_2 &\leq -14 \\ 14x_1 + 80x_2 &= 36 \\ 8x_1 + 2x_2 &= 6 \end{aligned}$$

$x_2 \geq 0$, x_1 unrestricted in sign.

19. Use duality theory to solve the following LPP.

$$\begin{aligned} \text{Max } Z &= 3x_1 + 4x_2 \\ \text{subject to} \quad x_1 - x_2 &\leq 1 \\ x_1 + x_2 &\geq 4 \\ x_1 - 3x_2 &\leq 3 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

20. Write down the dual of the following of LPP.

$$\begin{aligned} \text{Min } Z &= -x_1 + 7x_2 + 10x_3 + 3x_4 \\ \text{subject to} \quad -x_1 + x_2 + x_3 &\geq 3 \\ -x_1 + x_2 + 2x_3 + x_4 &\geq 2 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0 \end{aligned}$$

Solve the dual problem graphically. *[BRU BE Nov 96]*

21. Solve the LPP by formulating its dual.

$$\begin{aligned} \text{Min } Z &= 50x_1 - 80x_2 - 140x_3 \\ \text{subject to} \quad x_1 - x_2 - 3x_3 &\geq 4 \\ x_1 - 2x_2 - 2x_3 &\geq 3 \\ \text{and } x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

22. Use duality to solve the LPP.

$$\begin{aligned} \text{Min } Z &= 8x_1 - 2x_2 - 4x_3 \\ \text{subject to} \quad x_1 - 4x_2 - 2x_3 &\geq 2 \\ x_1 + x_2 - 3x_3 &\geq -1 \\ -3x_1 - x_2 + x_3 &\geq 1 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

23. Use duality to solve the LPP.

$$\begin{aligned} \text{Min } Z &= 4x_1 + 2x_2 + 3x_3 \\ \text{subject to} \quad 2x_1 + 4x_3 &\geq 5 \\ 2x_1 + 3x_2 + x_3 &\geq 4 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

24. Use duality to solve the LPP.

$$\begin{aligned} \text{Max } Z &= 5x_1 + 12x_2 + 4x_3 \\ \text{subject to} \quad x_1 + 2x_2 + x_3 &\leq 5 \\ 2x_1 - x_2 + 3x_3 &= 2 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

25. Using dual simplex method solve the LPP.

$$\begin{aligned} \text{Min } Z &= 3x_1 + 2x_2 + x_3 + 4x_4 \\ \text{subject to} \quad 2x_1 + 4x_2 + 5x_3 + x_4 &\geq 10 \\ 3x_1 - x_2 + 7x_3 - 2x_4 &\geq 2 \\ 5x_1 + 2x_2 + x_3 + 6x_4 &\geq 15 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0. \end{aligned}$$

[MU. BE. Nov 92, MKU M.Sc. 85]

26. **Use dual simplex method to solve the LPP.**

$$\text{Max } Z = 2x_1 + 3x_2$$

subject to
$$2x_1 - x_2 - x_3 \geq 3$$
$$x_1 - x_2 + x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0.$$

[MKU. ME. 83]

27. **Use dual simplex method of solve the LPP.**

$$\text{Min } Z = 80x_1 + 60x_2 + 80x_3$$

subject to
$$x_1 + 2x_2 + 3x_3 \geq 4$$
$$2x_1 + 3x_3 \geq 3$$
$$2x_1 + 2x_2 + x_3 \geq 4$$
$$4x_1 + x_2 + x_3 \geq 6$$
$$x_1, x_2, x_3 \geq 0.$$

[MU. B.Sc. 80]

28. **Use dual simplex method to solve.**

$$\text{Max } Z = -3x_1 - x_2$$

subject to
$$x_1 + x_2 \geq 1$$
$$x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0.$$

29. **Use dual simplex method to solve.**

$$\text{Max } Z = -2x_1 - 2x_2 - 4x_3$$

subject to
$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$x_2 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0.$$

[BNU. M.Sc. 86]

30. **Use dual simplex method to solve.**

$$\text{Max } Z = -4x_1 - 6x_2 - 18x_3$$

subject to
$$x_1 + 3x_3 \geq 3$$
$$x_2 + 2x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0.$$

31. **Show that the following LPP has a feasible solution but no finite optimal solution.**

$$\text{Max } Z = 3x_1 + 2x_2$$

subject to
$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0.$$

32. **Solve the following problem by dual simplex method.**

$$\text{Min } Z = 2x_1 + 2x_2 + 4x_3$$

subject to
$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0.$$

[BNU. BE. Nov 96, MSU. BE. Nov 97]

ANSWERS

7. $$\text{Min } W = 3y_1 + 4y_2 + y_3 + 6y_4$$

subject to
$$y_1 - 2y_2 + y_3 - 3y_4 \geq 2$$
$$6y_1 + y_2 - 5y_3 - 3y_4 \geq 5$$
$$-y_1 + 4y_2 + 3y_3 + 7y_4 \geq 6$$
$$y_1, y_2, y_3, y_4 \geq 0.$$

8. $$\text{Min } W = 8y_1 + 7y_2 + 2y_3 + 4y_4$$

subject to
$$2y_1 + 2y_2 - y_3 + 4y_4 \geq 2$$
$$4y_1 + 2y_2 - 3y_3 + y_4 \geq 5$$
$$-y_1 - 3y_2 + 5y_3 + 3y_4 \geq 3$$
$$y_1, y_2, y_3, y_4 \geq 0.$$

9. $$\text{Max } W = 3y_1 + 4y_2 + y_3 + 7y_4$$

subject to
$$8y_1 + 3y_2 + 4y_3 + y_4 \leq 7$$
$$2y_1 + 6y_2 + y_3 + 5y_4 \leq 3$$
$$y_1 + 4y_2 + 5y_3 + 2y_4 \leq 8$$
$$y_1, y_2, y_3, y_4 \geq 0.$$

10. Max W $= 10y_1 + 24y_2$

subject to
$$y_1 + 2y_2 \le 4$$
$$y_1 + 3y_2 \le 1$$
$$y_1, y_2 \ge 0.$$

11. Min W $= -3y_1 + y_2$

subject to
$$-y_1 - 3y_2 \ge 3$$
$$y_1 \ge 17$$
$$-y_1 + 2y_2 \ge 9$$
$$y_1, y_2 \ge 0.$$

12. Min W $= 5y_1 + 2y_2 - 2y_3$

subject to
$$y_1 + 2y_2 - y_3 \ge 5$$
$$2y_1 - y_2 + y_3 \ge 12$$
$$-y_1 + 3y_2 - 3y_3 \ge 4$$
$y_1, y_3 \ge 0,\ \ y_2$ unrestricted.

13. Min W $= 5y_1 + 2y_2 - y_3$

subject to
$$y_1 + 2y_2 + y_3 \ge 2$$
$$2y_1 - y_2 - 2y_3 \ge 4$$
$$-y_1 + 2y_2 - 2y_3 = 1$$
$y_1, y_3 \ge 0$ and y_2 unrestricted.

14. Max W $= 7y_1 + 4y_2 - 10y_3 + 10y_4 + 2y_5$

subject to
$$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \le 3$$
$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \le -2$$
$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \le 4$$
$$y_1, y_2, y_3, y_4, y_5 \ge 0.$$

15. Min W $= 4y_1 + 3y_2$

subject to
$$2y_1 + 3y_2 \ge 4$$
$$y_1 - 2y_2 \ge 5$$
$$y_1 + y_2 \ge 12$$
$y_1 \ge 0,\ y_2$ unrestricted.

16. Max W $= 5y_1 + 8y_2$

subject to
$$y_1 - 2y_2 \le 10$$
$$-3y_1 + y_2 \le -6$$
$$y_1 + 3y_2 \le -8$$
y_1, y_2 are unrestricted.

17. Min W $= -2y_1 + 8y_2 + 10y_3$

subject to
$$-y_1 + y_2 + y_3 \ge 4$$
$$2y_1 + 2y_2 - y_3 = 2$$
$y_1, y_3 \ge 0,\ y_2$ unrestricted in sign.

18. Max W $= 16y_1 + 14y_2 + 36y_3 + 6y_4$

subject to
$$14y_1 + 4y_2 + 14y_3 + 8y_4 = 21$$
$$13y_1 + 17y_2 + 80y_3 + 2y_4 \le 48$$
$y_1, y_2 \ge 0,\ y_3, y_4$ unrestricted.

19. No feasible solution exists for the dual problem and hence primal has no finite optimum solution.

20. Max W $= 3y_1 + 2y_2$

subject to
$$y_1 + y_2 \ge 1$$
$$y_1 + y_2 \le 7$$
$$y_1 + 2y_2 \le 10$$
$$y_2 \le 3$$
$$y_1, y_2 \ge 0.$$

Also, Max $W = 21,\ y_1 = 7,\ y_2 = 0.$

21. Dual has no feasible solution and hence primal has no feasible solution (or) unbounded solution.

22. Dual has an unbounded solution and hence primal has either an unbounded solution or no solution.

23. Primal: Min $Z = \frac{67}{12},\ x_1 = 0,\ x_2 = \frac{11}{12},\ x_3 = \frac{5}{4}$.

Dual : Max $W = \frac{67}{12},\ y_1 = \frac{7}{12},\ y_2 = \frac{2}{3}$.

24. Primal: Max Z $= \frac{141}{5}$, $x_1 = \frac{9}{5}$, $x_2 = \frac{8}{5}$, $x_3 = 0$.

    Dual : Min W $= \frac{141}{5}$, $y_1 = \frac{29}{5}$, $y_2 = \frac{-2}{5}$.

25. Min Z $= \frac{215}{23}$, $x_1 = \frac{65}{23}$, $x_2 = 0$, $x_3 = \frac{20}{23}$, $x_4 = 0$.

26. Cannot be solved by dual simplex method.

27. Min Z = 175.38, $x_1 = \frac{16}{13}$, $x_2 = \frac{6}{13}$, $x_3 = \frac{8}{13}$.

28. Max Z = –1, $x_1 = 0$, $x_2 = 1$.

29. Max Z $= \frac{-4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$.

30. Max Z = –36, $x_1 = 0$, $x_2 = 3$, $x_3 = 1$.

32. Min Z $= \frac{4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$.

2.2 TRANSPORTATION MODEL

2.2.1 Introduction

Transportation deals with the transportation of a commodity (single product) from 'm' sources (origins or supply or *capacity* centres) to 'n' destinations (sinks or demand or requirement centres). It is assumed that

i) Level of supply at each source and the amount of demand at each destination and

ii) The unit transportation cost of commodity from each source to each destination are known [given].

It is also assumed that the cost of transportation is linear.

The objective is to determine the amount to be shifted from each source to each destination such that the total transportation cost is minimum.

Note: The transportation model also can be modified to account for multiple commodities.

2.2.2 Mathematical Formulation of a Transportation Problem:

Let us assume that there are m sources and n destinations.

Let a_i be the supply (capacity) at source i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from source i to destination j and x_{ij} be the number of units shifted from source i to destination j.

Then the transportation problem can be expressed mathematically as

$$\text{Minimize Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, 3 \ldots m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, 3 \ldots n.$$

$$\text{and } x_{ij} \geq 0, \text{ for all } i \text{ and } j.$$

24. Primal: Max Z $= \frac{141}{5}$, $x_1 = \frac{9}{5}$, $x_2 = \frac{8}{5}$, $x_3 = 0$.

Dual : Min W $= \frac{141}{5}$, $y_1 = \frac{29}{5}$, $y_2 = \frac{-2}{5}$.

25. Min Z $= \frac{215}{23}$, $x_1 = \frac{65}{23}$, $x_2 = 0$, $x_3 = \frac{20}{23}$, $x_4 = 0$.

26. Cannot be solved by dual simplex method.

27. Min Z = 175.38, $x_1 = \frac{16}{13}$, $x_2 = \frac{6}{13}$, $x_3 = \frac{8}{13}$.

28. Max Z = -1, $x_1 = 0$, $x_2 = 1$.

29. Max Z $= \frac{-4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$.

30. Max Z = -36, $x_1 = 0$, $x_2 = 3$, $x_3 = 1$.

32. Min Z $= \frac{4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$.

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Then the transportation problem can be expressed mathematically as

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, 3 \ldots m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, 3 \ldots n.$$

and $x_{ij} \geq 0$, for all i and j.

Note 1: The two sets of constraints will be ***consistent*** if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(total supply) *(total demand)*

which is the necessary and sufficient condition for a transportation problem to have a feasible solution. Problems satisfying this condition are called ***balanced transportation problems***.

Note 2: If $\sum a_i \neq \sum b_j$, then the transportation problem is said to be ***unbalanced.***

Note 3: For any transportation problem, the coefficients of all x_{ij} in the constraints are unity.

Note 4: The objective function and the constraints being all linear, the transportation problem is a special class of linear programming problem. Therefore it can be solved by simplex method. But the number of variables being large, there will be too many calculations. So we can look for some other technique which would be simpler than the usual simplex method.

Standard transportation table:

Transportation problem is explicitly represented by the following transportation table.

Destination

		D_1	D_2	D_3	...	D_j	...	D_n	Supply
	S_1	c_{11}	c_{12}	c_{13}		c_{1j}		c_{1n}	a_1
Source	S_2	c_{21}	c_{22}	c_{23}		c_{2j}		c_{2n}	a_2
									:
	S_i	c_{i1}	c_{i2}			c_{ij}		c_{in}	:
	S_m	c_{m1}	c_{m2}			c_{mj}		c_{mn}	a_m
Demand		b_1	b_2	b_3	...	...	...	b_n	$\Sigma a_i = \Sigma b_j$

The *mn* squares are called ***cells***. The unit transportation cost c_{ij} from the i^{th} source to the j^{th} destination is displayed in the ***upper left side of the (i, j)th cell.*** Any feasible solution is shown in the table by entering the value of x_{ij} ***in the centre of the*** $(i,j)^{th}$ cell. The various *a*'s and *b*'s are called ***rim requirements***. The feasibility of a solution can be verified by summing the values of x_{ij} along the rows and down the columns.

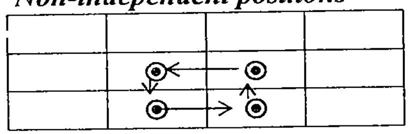
Definition 1: A set of non-negative values x_{ij}, $i = 1,2, \dots m$; $j = 1, 2\dots n$. that satisfies the constraints (rim conditions and also the non-negativity restrictions) is called a ***feasible solution*** to the transportation problem.

Note: A balanced transportation problem will always have a feasible solution.

Definition 2: A feasible solution to a $(m \times n)$ transportation problem that contains no more than $m + n - 1$ non-negative allocations is called a ***basic feasible solution*** (BFS) to the transportation problem.

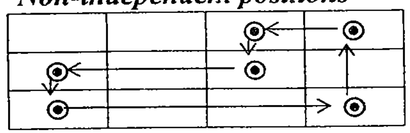
The allocations are said to be in ***independent positions*** if it is impossible to increase or decrease any allocation without either changing the position of the allocation or violating the rim requirements. A simple rule for allocations to be in independent positions is that it is impossible to travel from any allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of the route. Example

Non-independent positions



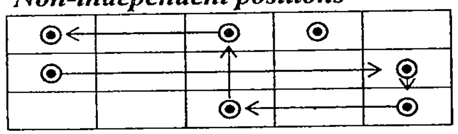
(i)

Non-independent positions



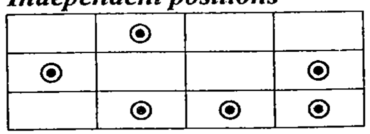
(ii)

Non-independent positions



(i)

Independent positions



(ii)

Definition 3: A basic feasible solution to a $(m \times n)$ transportation problem is said to be a ***non-degenerate basic feasible solution*** if it contains exactly $m + n - 1$ non-negative allocations in independent positions.

Definition 4: A basic feasible solution that contains less than $m + n - 1$ non-negative allocations is said to be a ***degenerate basic*** feasible solution.

Definition 5: A feasible solution (not necessarily basic) is said to be an ***optimal solution*** if it minimizes the total transportation cost.

Note: The number of basic variables in an $m \times n$ balanced transportation problem is atmost $m + n - 1$.

Note: The number of non–basic variables in an $m \times n$ balanced transportation problem is atleast $mn - (m + n - 1)$

2.2.3 Methods for finding initial basic feasible solution

The transportation problem has a solution if and only if the problem is balanced. Therefore before starting to find the initial basic feasible solution, check whether the given transportation problem is balanced. If not one has to balance the transportation problem first. The way of doing this is discussed in section 7.4 page 7.40 In this section all the given transportation problems are balanced.

Method 1: North west Corner Rule:

Step 1: The first assignment is made in the cell occupying the upper left–hand (north–west) corner of the transportation table. The maximum possible amount is allocated there. That is $x_{11} = \min\ \{a_1, b_1\}$.

Case (i) : If min $\{a_1, b_1\} = a_1$, then put $x_{11} = a_1$, decrease b_1 by a_1 and move vertically to the 2nd row (*i.e.,*) to the cell (2,1) cross out the first row.

Case (ii) : If min $\{a_1, b_1\} = b_1$, then put $x_{11} = b_1$, and decrease a_1 by b_1 and move horizontally right (*i.e.,*) to the cell (1,2) cross out the first column

Case (iii) : If min $\{a_1, b_1\} = a_1 = b_1$ then put $x_{11} = a_1 = b_1$ and move diagonally to the cell (2,2) cross out the first row and the first column.

Step 2: Repeat the procedure until all the rim requirements are satisfied.

Method 2: Least Cost method (or) Matrix minima method (or) Lowest cost entry method:

Step 1: Identify the cell with smallest cost and allocate $x_{ij} = \text{Min}\ \{a_i, b_j\}$.

Case (i) : If min $\{a_i, b_j\} = a_i$, then put $x_{ij} = a_i$, cross out the i^{th} row and decrease b_j by a_i, Go to step (2).

Case (ii) : If min $\{a_i, b_j\} = b_j$ then put $x_{ij} = b_j$ cross out the j^{th} column and decrease a_i by b_j Go to step (2).

Case (iii) : If min $\{a_i, b_j\} = a_i = b_j$, then put $x_{ij} = a_i = b_j$, cross out either i^{th} row or j^{th} column but not both, Go to step (2).

Step 2: Repeat step (1) for the resulting reduced transportation table until all the rim requirements are satisfied.

Method 3: Vogel's approximation method (VAM) (or) Unit cost penalty method: ***[MU. MBA. Nov 96, Apr 95, Apr 97]***

Step 1: Find the difference (penalty) between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

Step 2: Identify the row (or) column with largest penalty. If a tie occurs, break the tie arbitrarily. Choose the cell with smallest cost in that selected row or column and allocate as much as possible to this cell and cross out the satisfied row or column and go to step (3).

Step 3: Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Example 1 **Determine basic feasible solution to the following transportation problem using North West Corner Rule:**

Sink

		A	B	C	D	E	Supply
	P	2	11	10	3	7	4
Origin	**Q**	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

[MU. BE. Apr 94]

Solution: Since $a_i = b_j = 21$, the given problem is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

2 (3)	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9
3	3	4	5	6	

Following North West Corner rule, the first allocation is made in the cell (1,1).

Here $x_{11} = \min \{a_1, b_1\} = \min \{4, 3\} = 3$

$\therefore$ Allocate 3 to the cell (1,1) and decrease 4 by 3 *i.e.,* $4 - 3 = 1$

As the first column is satisfied, we cross out the first column and the resulting reduced Transportation table is

11 (1)	10	3	7	1
4	7	2	1	8
9	4	8	12	9
3	4	5	6	

Here the North West Corner cell is (1,2).

So Allocate $x_{12} = \min \{1,3\} = 1$ to the cell (1,2) and move vertically to cell (2,2). The resulting reduced transportation table is

4 (2)	7	2	1	8
9	4	8	12	9
2	4	5	6	

Allocate $x_{22} = \min \{8, 2\} = 2$ to the cell (2, 2) and move horizontally to the cell (2,3). The resulting transportation table is

7 (4)	2	1	6
4	8	12	9
4	5	6	

Allocate $x_{23} = \min \{6,4\} = 4$ and move horizontally to the cell (2,4). The resulting reduced transportation table is

2 (2)	1	2
8	12	9
5	6	

Allocate $x_{24} = \{2,5\} = 2$ and move vertically to the cell (3,4). The resulting reduced transportation table is

8 (3)	12	9
3	6	

Allocate $x_{34} = \min \{9, 3\} = 3$ and move horizontally to the cell (3, 5), which is

12 (6)	6
6	

Allocate $x_{35} = \min\{6,6\} = 6$

Finally the initial basic feasible solution is as shown in the following table.

2 (3)	11 (1)	10	3	7
1	4 (2)	7 (4)	2 (2)	1
3	9	4	8 (3)	12 (6)

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 5 - 1 = 7$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $= \text{Rs. } 2 \times 3 + 11 \times 1 + 4 \times 2 + 7 \times 4$
$+ 2 \times 2 + 8 \times 3 + 12 \times 6$
$= \text{Rs.}153/-$

Example 2 **Find the initial basic feasible solution for the following transportation problem by Least Cost Method.**

	To				Supply
From	1	2	1	4	30
	3	3	2	1	50
	4	2	5	9	20
Demand	20	40	30	10	

[MU. BE. Apr 95, MSU. BE. Nov 96]

Solution: Since $\Sigma a_i = \Sigma b_j = 100$, the given TPP is balanced.
$\therefore$ There exists a feasible solution to the transportation problem.

1 (20)	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	10	

By least cost method, min $c_{ij} = c_{11} = c_{13} = c_{24} = 1$

$\therefore$ Since more than one cell having the same minimum c_{ij}, break the tie.

Let us choose the cell (1,1) and allocate $x_{11} = \min\{a_1, b_1\} = \min\{30, 20\} = 20$ and cross out the satisfied column and decrease 30 by 20.

The resulting reduced transportation table is

2	1 (10)	4	10
3	2	1	50
2	5	9	20
40	30	10	

Here min $c_{ij} = c_{13} = c_{24} = 1$

Choose the cell (1,3) and allocate $x_{13} = \min\{a_1, b_3\} = \min\{10,30\} = 10$ and cross out the satisfied row.

The resulting reduced transportation table is

3	2	1 (10)	50
2	5	9	20
40	20	10	

Here min $c_{ij} = c_{24} = 1$,

$\therefore$ Allocate $x_{24} = \min\{a_2, b_4\} = \min(50,10) = 10$ and cross out the satisfied column.

The resulting transportation table is

3	2 **20**	40
2	5	20
40	20	

Here min $c_{ij} = c_{23} = c_{32} = 2$. Choose the cell (2,3) and allocate $x_{23} = \min\{a_2, b_3\} = \min(40,20) = 20$ and cross out the satisfied column.

The resulting reduced transportation table is

3	20
2 20	20
40	

Here min $c_{ij} = c_{32} = 2$. Choose the cell (3, 2) and allocate $x_{32} = \min\{a_3, b_2\} = \min(20,40) = 20$ and cross out the satisfied row.
The resulting reduced transportation table is

3 **20**	20
20	

Finally the initial basic feasible solution is as shown in the following table.

1 **20**	2	1 **10**	4
3	3 **20**	2 **20**	1 **10**
4	2 **20**	5	9

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $= \text{Rs. } 1 \times 20 + 1 \times 10 + 3 \times 20 + 2 \times 20 + 1 \times 10 + 2 \times 20$

$= 20 + 10 + 60 + 40 + 10 + 40$

$= \text{Rs.}180/-$

Example 3 **Find the initial basic feasible solution for the following transportation problem by VAM.**

Distribution Centres

		D_1	D_2	D_3	D_4	Availability
	S_1	11	13	17	14	250
Origin	S_2	16	18	14	10	300
	S_3	21	24	13	10	400
Requirements		200	225	275	250	

Solution: Since $\Sigma a_i = \Sigma b_j = 950$, the given problem is balanced.

$\therefore$ There exists a feasible solution to this problem.

11 200	13	17	14	250	(2)
16	18	14	10	300	(4)
21	24	13	10	400	(3)
200	225	275	250		
(5)	(5)	(1)	(0)		

First let us find the difference (penalty) between the smallest and next smallest costs in each row and column and write them in brackets against the respective rows and columns.

The largest of these differences is (5) and is associated with the first two columns of the transportation table. We choose the first column arbitrarily.

In this selected column, the cell (1,1) has the minimum unit transportation cost $c_{11} = 11$.

$\therefore$ Allocate $x_{11} = \min\{250, 200\} = 200$ to this cell (1,1) and decrease 250 by 200 and cross out the satisfied column.

The resulting reduced transportation table is

13 50	17	14	50	(1)
18	14	10	300	(4)
24	13	10	400	(3)
225	275	250		
(5)	(1)	(0)		

The row and column differences are now computed for this reduced transportation table. The largest of these is (5) which is associated with the second column. Since $c_{12} = 13$ is the minimum cost, we allocate $x_{12} = \min\{50,225\} = 50$ to the cell (1,2) and decrease 225 by 50 and cross out the satisfied row.

Continuing in this manner, the subsequent reduced transportation tables and the differences for the surviving rows and columns are shown below:

18 175	14	10	300	(4)
24	13	10	400	(3)
175	275	250		
(6)	(1)	(0)		

(i)

14	10 125	125	(4)
13	10	400	(3)
	250		
(1)	(0)		

(ii)

13	10 125	400
275	125	

(iii)

13 275	275
275	

(iv)

Finally the initial basic feasible solution is as shown in the following table.

11 **200**	13 **50**	17	14
16	18 **175**	14	10 **125**
21	24	13 **275**	10 **125**

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $=$ Rs. $11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125$

$=$ Rs. 12075/-

Example 4 **Find the starting solution of the following transportation model**

1	**2**	**6**	**7**
0	**4**	**2**	**12**
3	**1**	**5**	**11**
10	**10**	**10**	

using ***(i)*** ***North West Corner rule***

(ii) ***Least Cost method***

(iii) ***Vogel's approximation method.***

Solution: Since $\Sigma a_i = \Sigma b_j = 30$, the given Transportation problem is balanced. Hence there exists a basic feasible solution to this problem.

(i) North West Corner rule: Using this method, the allocations are shown in the tables below:

1 7	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

(i)

0 3	4	2	12
3	1	5	11
3	10	10	

(ii)

4 **8**	2	9
1	5	11
10	10	

(iii)

1 **1**	5	11
1	10	

(iv)

5 **10**	10
10	

(v)

The starting solution is as shown in the following table:

1 **7**	2	6
0 **3**	4 **9**	2
3	1 **1**	5 **10**

∴ The initial transportation cost $= \text{Rs. } 1 \times 7 + 0 \times 3 + 4 \times 9 + 1 \times 1 + 5 \times 10$

$= \text{Rs. } 94/-$

(ii) Least Cost Method: Using this method, the allocations are as shown in the table below:

1	2	6 7	7
0 **10**	4	2	12
3	1	5	11
10	10	10	

(i)

2	6	7
4	2	2
1 **10**	5	11
10	10	

(ii)

6	7
2 **2**	2
5	1
10	

(iii)

6	7
5 **1**	1
8	

(iv)

6 **7**	7
7	

(v)

The starting solution is as shown in the following table:

1	2	6 **7**
0 **10**	4	2 **2**
3	1 **10**	5 **1**

∴ The initial transportation cost $= \text{Rs. } 6 \times 7 + 0 \times 10 + 2 \times 2 + 1 \times 10 + 5 \times 1$

$= \text{Rs. } 61/-$

(iii) Vogel's Approximation Method: Using this method, the allocations are shown in the tables below:

1	2	6	7	(1)
0	4	2 10	12	(2)
3	1	5	11	(2)
10	10	10		
(1)	(1)	**(3)**		

(i)

1	2	7	(1)
0 2	4	2	**(4)**
3	1	11	(2)
10	10		
(1)	(1)		

(ii)

1	2	7	(1)
3	1 10	11	(2)
8	10		
(2)	(1)		

(iii)

1 7	7
3	1
8	

(iv)

3 1	1
1	

(v)

The starting solution is as shown in the following table:

1 **7**	2	6
0 **2**	4	2 **10**
3 **1**	1 **10**	5

$\therefore$ The initial transportation cost $=$ Rs. $1 \times 7 + 0 \times 2 + 2 \times 10 + 3 \times 1 + 1 \times 10$

$=$ Rs. 40/-

Note: For the above problem, the number of positive allocations in independent positions is $(m + n - 1)$ (*i.e.*, $m + n - 1 = 3 + 3 - 1 = 5$). This ensures that the given problem has a non-degenerate basic feasible solution by using all the three methods. This need not be the case in all the problems.

2.2.4 Transportation Algorithm (or) MODI Method (modified distribution method) (Test for optimal solution).

[MU. MBA. Apr 96, Apr 97]

Step 1: Find the initial basic feasible solution of the given problem by Northwest Corner rule (or) Least Cost method or VAM.

Step 2: Check the number of occupied cells. If these are less than $m + n - 1$, there exists degeneracy and we introduce a very small positive assignment of $\in$ (≈ 0) in suitable independent positions, so that the number of occupied cells is exactly equal to $m + n - 1$.

Step 3: Find the set of values u_i, v_j ($i = 1,2,3, \ldots m$; $j = 1, 2, 3 \ldots n$) from the relation $c_{ij} = u_i + v_j$ for each occupied cell (i, j), by starting initially with $u_i = 0$ or $v_j = 0$ preferably for which the corresponding row or column has maximum number of individual allocations.

Step 4: Find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding cell (i, j).

Step 5: Find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (d_{ij} = upper left – upper right) for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding cell (i, j).

Step 6: Examine the cell evaluations d_{ij} for all unoccupied cells (i, j) and conclude that

(i) if all $d_{ij} > 0$, then the solution under the test is optimal and unique.

(ii) if all $d_{ij} > 0$, with atleast one $d_{ij} = 0$, then the solution under the test is optimal and an alternative optimal solution exists.

(iii) if atleast one $d_{ij} < 0$, then the solution is not optimal. Go to the next step.

Step 7: Form a new B.F.S by giving maximum allocation to the cell for which d_{ij} is most negative by making an occupied cell empty. For that draw a closed path consisting of horizontal and vertical lines beginning and ending at the cell for which d_{ij} is most negative and having its ***other corners at some allocated cells.*** Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. Choose minimum of the allocations from the cells having $-\theta$. Add this minimum allocation to the cells with $+\theta$ and subtract this minimum allocation from the allocation to the cells with $-\theta$.

Step 8: Repeat steps (2) to (6) to test the optimality of this new basic feasible solution.

Step 9: Continue the above procedure till an optimum solution is attained.

Note: The Vogels approximation method (VAM) takes into account not only the least cost c_{ij} but also the costs that just exceed the least cost c_{ij} and therefore yields better initial solution than obtained from other methods in general. This can be justified by the above example (4). So to find the initial solution, give preference to VAM unless otherwise specified.

Example 1 **Solve the transportation problem:**

	1	2	3	4	Supply
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Demand	6	10	12	15	

[MU. BE. Apr 91, Apr 92, Apr 93, Apr 97, MSU. BE. Apr 97]

Solution: Since $\Sigma a_i = \Sigma b_j = 43$, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:3

21	16	25	13 **11**	(3)	–	–	–
17 **6**	18 **3**	14	23 **4**	(3)	(3)	(3)	(4)
32	27 **7**	18 **12**	41	(9)	(9)	(9)	(9)
(4)	(2)	(4)	(10)				
(15)	(9)	(4)	(18)				
(15)	(9)	(4)	–				
–	(9)	(4)	–				

That is

21	16	25	13 **11**
17 **6**	18 **3**	14	23 **4**
32	27 **7**	18 **12**	41

From this table, we see that the number of non-negative independent allocations is $(m + n - 1) = (3 + 4 - 1) = 6$. Hence the solution is non-degenerate basic feasible.

$\therefore$ The initial transportation cost

= Rs. $13 \times 11 + 17 \times 6 + 18 \times 3 + 23 \times 4 + 27 \times 7 + 18 \times 12$

= Rs. 796/-

To find the optimal solution

Consider the above transportation table. Since $m + n - 1 = 6$, we apply MODI method,

Now we determine a set of values u_i and v_j for each occupied cell (i, j) by using the relation $c_{ij} = u_i + v_j$. As the 2nd row contains maximum number of allocations, we choose $u_2 = 0$.

Therefore

$$c_{21} = u_2 + v_1 \Rightarrow 17 = 0 + v_1 \Rightarrow v_1 = 17$$
$$c_{22} = u_2 + v_2 \Rightarrow 18 = 0 + v_2 \Rightarrow v_2 = 18$$
$$c_{24} = u_2 + v_4 \Rightarrow 23 = 0 + v_4 \Rightarrow v_4 = 23$$
$$c_{14} = u_1 + v_4 \Rightarrow 13 = u_1 + 23 \Rightarrow u_1 = -10$$
$$c_{32} = u_3 + v_2 \Rightarrow 27 = u_3 + 18 \Rightarrow u_3 = 9$$
$$c_{33} = u_3 + v_3 \Rightarrow 18 = 9 + v_3 \Rightarrow v_3 = 9$$

Thus we have the following transportation table:

21	16	25	13 **11**	$u_1 = -10$
17 **6**	18 **3**	14	23 **4**	$u_2 = 0$
32	27 **7**	18 **12**	41	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Now we find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding unoccupied cell (i, j).

Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (*ie.,* upper left corner – upper right corner) for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell (i, j)

Thus we get the following table:

21 7 14	16 8 8	25 −1 26	13 **11**	$u_1 = -10$
17 **6**	18 **3**	14 9 5	23 **4**	$u_2 = 0$
32 26 6	27 **7**	18 **12**	41 32 9	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Since all $d_{ij} > 0$, the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is given by $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$ and the optimum (minimum) transportation cost

= Rs. $13 \times 11 + 17 \times 6 + 18 \times 3 + 23 \times 4 + 27 \times 7 + 18 \times 12$

= Rs. 796/-

Example 2 **Obtain on optimum basic feasible solution to the following transportation problem:**

	To			Available
From	**7**	**3**	**2**	**2**
	2	**1**	**3**	**3**
	3	**4**	**6**	**5**
Demand	**4**	**1**	**5**	**10**

[MU. MCA. May 93]

Solution: Since $\Sigma a_i = \Sigma b_j = 10$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:

7	3	2 **2**	(1)	**(5)**	–
2	1 **1**	3 **2**	(1)	(1)	(1)
3 **4**	4	6 **1**	(1)	(3)	**(3)**
(1)	**(2)**	(1)			
(1)	–	(1)			
(1)	–	(3)			

That is

7	3	2 **2**
2	1 **1**	3 **2**
3 **4**	4	6 **1**

From this table we see that the number of non-negative allocations is $m + n - 1 = (3 + 3 - 1) = 5$.

Hence the solution is non-degenerate basic feasible

$\therefore$ The initial transportation cost

$$= \text{Rs. } 2 \times 2 + 1 \times 1 + 3 \times 2 + 3 \times 4 + 6 \times 1$$
$$= \text{Rs. } 29/-$$

For optimality: Since the number of non-negative independent allocations is $m + n - 1$, we apply MODI method.

Since the third column contains maximum number of allocations, we choose $v_3 = 0$.

Now we determine a set of values u_i and v_j by using the occupied cells and the relation $c_{ij} = u_i + v_j$.

That is,

7 –1	3 0	2 **2**	$u_1 = 2$
2	1 **1**	3 **2**	$u_2 = 3$
3 **4**	4	6 **1**	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Now we find $u_i + v_j$ for each unoccupied cell (i,j) and enter at the upper right corner of the corresponding unoccupied cell (i,j).

Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding unoccupied cell (i,j).

Thus we get the following table

7 –1 / 8	3 0 / 3	2 **2**	$u_1 = 2$
2 0 / 2	1 **1**	3 **2**	$u_2 = 3$
3 **4**	4 4 / 0	6 **1**	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given by $x_{13} = 2$, $x_{22} = 1$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 1$ and the optimum (minimum) transportation cost

$= \text{Rs. } 2 \times 2 + 1 \times 1 + 3 \times 2 + 3 \times 4 + 6 \times 1 = \text{Rs. } 29/-$

$\boxed{\text{Example 3}}$ **Find the optimal transportation cost of the following matrix using least cost method for finding the critical solution.**

Market

	A	B	C	D	E	Available
P	4	1	2	6	9	100
Factory Q	6	4	3	5	7	120
R	5	2	6	4	8	120
Demand	40	50	70	90	90	

[MU. MCA. Apr 93]

Solution:

Since $\Sigma a_i = \Sigma b_j = 340$, the given transportation problem is balanced.

$\therefore$ There exists a basic feasible solution to this problem.

By using Least cost method, the initial solution is as shown in the following table:

4	1 50	2 50	6	9
6 10	4	3 20	5	7 90
5 30	2	6	4 90	8

$\therefore$ The initial transportation cost

$= \text{Rs. } 1 \times 50 + 2 \times 50 + 6 \times 10 + 3 \times 20 + 7 \times 90 + 5 \times 30 + 4 \times 90$

$= \text{Rs. } 1410/-$

For optimality: Since the number of non-negative independent allocations is $(m + n - 1)$, we apply MODI method:

That is

4 5 −1	1 50	2 50	6 4 2	9 6 3	$u_1 = -1$
6 10	4 2 2	3 20	5 5 0	7 90	$u_2 = 0$
5 30	2 1 1	6 2 4	4 90	8 6 2	$u_3 = -1$
$v_1 = 6$	$v_2 = 2$	$v_3 = 3$	$v_4 = 5$	$v_5 = 7$	

Since $d_{11} = -1 < 0$, the current solution is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (i,j) for which d_{ij} is most negative by making an occupied cell empty. Here the cell $(1,1)$ having the negative value $d_{11} = -1$. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell $(1,1)$ and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. we have

4 $+\theta$	1 50	2 50 $-\theta$	6	9
6 10 $-\theta$	4	3 20 $+\theta$	5	7 90
5 30	2	6	4 90	8

From the two cells (1,3), (2,1) having $-\theta$, we find that the minimum of the allocations 50,10 is 10. Add this 10 to the cells with $+\theta$ and subtract this 10 to the cells with $-\theta$.

Hence the new basic feasible solution is displayed in the following table:

4 **10**	1 **50**	2 **40**	6	9
6	4	3 **30**	5	7 **90**
5 **30**	2	6	4 **90**	8

We see that the above table satisfies the rim conditions with $(m + n - 1)$ non-negative allocations at independent positions. So we apply MODI method.

4 **10**	1 **50**	2 **40**	6 3 3	9 6 3	$u_1 = 0$
6 5 1	4 2 2	3 **30**	5 4 1	7 **90**	$u_2 = 1$
5 **30**	2 2 0	6 3 3	4 **90**	8 7 1	$u_3 = 1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 2$	$v_4 = 3$	$v_5 = 6$	

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given by $x_{11} = 10$, $x_{12} = 50$, $x_{13} = 40$, $x_{23} = 30$, $x_{25} = 90$, $x_{31} = 30$, $x_{34} = 90$ and the optimum (minimum) transportation cost

$= \text{Rs. } 4 \times 10 + 1 \times 50 + 2 \times 40 + 3 \times 30 + 7 \times 90 + 5 \times 30 + 4 \times 90$

$= \text{Rs. } 1400/-$

2.2.5 Degeneracy in Transportation Problems

In a transportation problem, whenever the number of non-negative independent allocations is less than $m + n - 1$, the transportation problem is said to be a ***degenerate*** one. Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

To resolve degeneracy, we allocate an extremely small amount (close to zero) to one or more empty cells of the transportation table (generally minimum cost cells if possible), so that the total number of occupied cells becomes $(m + n - 1)$ at independent positions. We denote this small amount by $\in$ (epsilon) satisfying the following conditions:

(i) $0 < \in < x_{ij}$, for all $x_{ij} > 0$

(ii) $x_{ij} \pm \in = x_{ij}$, for all $x_{ij} > 0$

The cells containing $\in$ are then treated like other occupied cells and the problem is solved in the usual way. The $\in$'s are kept till the optimum solution is attained. Then we let each $\in \to 0$.

Example 1 **Find the non-degenerate basic feasible solution for the following transportation problem using**

(i) North west corner rule

(ii) Least cost method

(iii) Vogel's approximation method.

	To				Supply
From	**10**	**20**	**5**	**7**	**10**
	13	**9**	**12**	**8**	**20**
	4	**5**	**7**	**9**	**30**
	14	**7**	**1**	**0**	**40**
	3	**12**	**5**	**19**	**50**
Demand	**60**	**60**	**20**	**10**	

[MU. MCA. Apr 93]

Solution: Since $\Sigma a_i = \Sigma b_j = 150$, the given transportation problem is balanced.

$\therefore$ There exists a basic feasible solution to this problem.

***(i)* The starting solution by NWC rule is as shown in the following table.**

10 **10**	20	5	7
13 **20**	9	12	8
4 **30**	5	7	9
14	7 **40**	1	0
3	12 **20**	5 **20**	19 **10**

Since the number of non-negative allocations at independent positions is 7 which is less than $(m + n - 1) = (5 + 4 - 1) = 8$, this basic feasible solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the unoccupied cell (5,1) so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is as shown in the following table.

10 **10**	20	5	7
13 **20**	9	12	8
4 **30**	5	7	9
14	7 **40**	1	0
3 $\in$	12 **20**	5 **20**	19 **10**

$$\begin{aligned}\text{The initial transportation cost} &= \text{Rs. } 10 \times 10 + 13 \times 20 + 4 \times 30 \\ &\quad + 7 \times 40 + 3 \times \in + 12 \times 20 \\ &\quad + 5 \times 20 + 19 \times 10 \\ &= \text{Rs.}(1290 + 3\in) \\ &= \text{Rs. } 1290/- \text{, as } \in \to 0.\end{aligned}$$

***(ii)* Least cost method:** Using this method the starting solution is as shown in the following table:

10	20 **10**	5	7
13	9 **20**	12	8
4 **10**	5 **20**	7	9
14	7 **10**	1 **20**	0 **10**
3 **50**	12	5	19

Since the number of non-negative allocations at independent positions is $(m + n - 1) = 8$, the solution is non-degenerate basic feasible.

$$\begin{aligned}\text{The initial transportation cost} &= \text{Rs. } 20 \times 10 + 9 \times 20 + 4 \times 10 \\ &\quad + 5 \times 20 + 7 \times 10 + 1 \times 20 + 0 \times 10 + 3 \times 50 \\ &= \text{Rs. } 760/-\end{aligned}$$

***(iii)* Vogel's approximation Method:** The starting solution by this method is as shown in the following table:

10 **10**	20	5	7
13	9 **20**	12	8
4	5 **30**	7	9
14	7 **10**	1 **20**	0 **10**
3 **50**	12	5	19

Since the number of non-negative allocations is 7 which is less than $(m + n - 1) = (5 + 4 - 1) = 8$, this basic solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the unoccupied cell (5, 2) so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is as shown in the following table.

10 **10**	20	5	7
13	9 **20**	12	8
4	5 **30**	7	9
14	7 **10**	1 **20**	0 **10**
3 **50**	12 $\in$	5	19

$\therefore$ The initial transportation cost

$= \text{Rs. } 10 \times 10 + 9 \times 20 + 5 \times 30 + 7 \times 10 + 1 \times 20 + 0 \times 10 + 3 \times 50 + 12 \times \in$

$= \text{Rs. } (670 + 12 \in)$

$= \text{Rs. } 670/- = \text{as } \in \to 0.$ *[Please refer note in page 7.17]*

Example 2 **Solve the following transportation problem using Vogel's method.**

Warehouse

	A	B	C	D	E	F	Available
Factory 1	**9**	**12**	**9**	**6**	**9**	**10**	**5**
2	**7**	**3**	**7**	**7**	**5**	**5**	**6**
3	**6**	**5**	**9**	**11**	**3**	**11**	**2**
4	**6**	**8**	**11**	**2**	**2**	**10**	**9**
Requirement	**4**	**4**	**6**	**2**	**4**	**2**	

[MU. MCA. Apr 92]

Solution: Since $\sum a_i = \sum b_j = 22$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution to this problem. By Vogel's approximation method, the initial solution is as shown in the following table:

9	12	9 **5**	6	9	10
7	3 **4**	7	7	5	5 **2**
6 **1**	5	9 **1**	11	3	11
6 **3**	8	11	2 **2**	2 **4**	10

Since the number of non-negative allocations is 8 which is less than $(m + n - 1) = (4 + 6 - 1) = 9$, this basic solution is degenerate one.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (3,2), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is as shown in the following table.

9	12	9 **5**	6	9	10
7	3 **4**	7	7	5	5 **2**
6 **1**	5 $\in$	9 **1**	11	3	11
6 **3**	8	11	2 **2**	2 **4**	10

The initial transportation cost $= \text{Rs. } 9 \times 5 + 3 \times 4 + 5 \times 2 + 6 \times 1$

$+ 5 \times \in + 9 \times 1 + 6 \times 3 + 2 \times 2 + 2 \times 4$

$= \text{Rs. } (112 + 5 \in) = \text{Rs. } 112/-, \in \to 0.$

To find the optimal solution

Now the number of non-negative allocations at independent positions is $(m + n - 1)$. We apply the MODI method.

9 6 3	12 5 7	9 **5**	6 2 4	9 2 7	10 7 3	$u_1 = 0$
7 4 3	3 **4**	7 7 0	7 0 7	5 0 5	5 **2**	$u_2 = -2$
6 **1**	5 $\in$	9 **1**	11 2 9	3 2 1	11 7 4	$u_3 = 0$
6 **3**	8 5 3	11 9 2	2 **2**	2 **4**	10 7 3	$u_4 = 0$
$v_1 = 6$	$v_2 = 5$	$v_3 = 9$	$v_4 = 2$	$v_5 = 2$	$v_6 = 7$	

Since all $d_{ij} > 0$ with $d_{23} = 0$, the solution under the test is optimal and an alternative optimal solution is also exists.

$\therefore$ The optimum allocation schedule is given by $x_{13} = 5$, $x_{22} = 4$, $x_{26} = 2$, $x_{31} = 1$, $x_{32} = \in$, $x_{33} = 1$, $x_{41} = 3$, $x_{44} = 2$, $x_{45} = 4$ and the optimum (minimum) transportation cost is

$= \text{Rs. } 9 \times 5 + 3 \times 4 + 5 \times 2 + 6 \times 1 + 5 \times \in + 9 \times 1 + 6 \times 3$

$+ 2 \times 2 + 2 \times 4$

$= \text{Rs. } (112 + 5 \in)$

$= \text{Rs. } 112$ as $\in \to 0$.

Example 3 **Solve the transportation problem:**

	To				
					Supply
	1	**2**	**3**	**4**	**6**
From	**4**	**3**	**2**	**0**	**8**
	0	**2**	**2**	**1**	**10**
Demand	**4**	**6**	**8**	**6**	

[MU. MCA. Apr 87]

Solution: Since $\Sigma a_i = \Sigma b_j = 24$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution.

By using Vogel's approximation method, the initial solution is as shown in the following table:

1	2 **6**	3	4
4	3	2 **2**	0 **6**
0 **4**	2	2 **6**	1
	4		

Since the number of non-negative allocations at independent positions is 5, which is less than $(m + n - 1) = (3 + 4 - 1) = 6$, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (3,2) so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate initial basic feasible solution is given by

1	2 **6**	3	4
4	3	2 **2**	0 **6**
0 **4**	2 $\in$	2 **6**	1

The initial transportation cost

$= \text{Rs. } 2 \times 6 + 2 \times 2 + 0 \times 6 + 0 \times 4 + 2 \times \in \; + 2 \times 6$

$= \text{Rs. } (28 + 2 \in)$

$= \text{Rs. } 28/-$, as $\in \to 0$.

To find the optimal solution

Now the number non-negative allocations at independent positions is $(m + n - 1)$. We apply the MODI method.

1 0 1	2 **6**	3 2 1	4 0 4	$u_1 = 0$
4 0 4	3 2 1	2 **2**	0 **6**	$u_2 = 0$
0 **4**	2 $\in$	2 **6**	1 0 1	$u_3 = 0$
$v_1 = 0$	$v_2 = 2$	$v_3 = 2$	$v_4 = 0$	

Since all $d_{ij} > 0$ the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is given by $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = \in$, $x_{33} = 6$ and the optimum (minimum) transportation cost.

$= \text{Rs. } 2 \times 6 + 2 \times 2 + 0 \times 6 + 0 \times 4 + 2 \times \in + 2 \times 6$

$= \text{Rs. } (28 + 2 \in) = \text{Rs. } 28$, as $\in \to 0$.

Example 4 **Find the optimal solution of the following problem**

Destination

		X	Y	Z	Supply
	P	1	2	0	30
Origin	Q	2	3	4	35
	R	1	5	6	35
Demand		30	40	30	

[MU. BE. Apr 95]

Solution: Since $\Sigma a_i = \Sigma b_j = 100$, the given transportation problem is balanced.

By using the Vogel's approximation method, the basic feasible solution is displayed in the following table.

1	2	0 **30**
2	3 **35**	4
1 **30**	5 **5**	6

Since the number of non-negative allocations at independent positions is 4 which is less than $(m + n - 1) = 3 + 3 - 1 = 5$, this initial solution is degenerate.

To resolve degeneracy we allocate a very small quantity $\in$ to the cell (3, 3) , so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is given by

1	2	0 30
2	3 35	4
1 30	5 5	6 $\in$

Now the number of non-negative allocations at independent positions is $(m + n - 1) = 5$. We apply MODI method.

1 (−5) 6	2 (−1) 3	0 30	$u_1 = -6$
2 (−1) 3	3 35	4 (4) 0	$u_2 = -2$
1 30	5 5	6 $\in$	$u_3 = 0$
$v_1 = 1$	$v_2 = 5$	$v_3 = 6$	

Since all $d_{ij} > 0$ with $d_{23} = 0$, the solution under the test is optimal and there exists an alternative optimal solution.

$\therefore$ The optimal allocation schedule is given by $x_{13} = 30$, $x_{22} = 35$, $x_{31} = 30$, $x_{32} = 5$, $x_{33} = \in$ and the optimum (minimum) transportation cost.

$= \text{Rs. } 0 \times 30 + 3 \times 35 + 1 \times 30 + 5 \times 5 + 6 \times \in$

$= \text{Rs. } (160 + 6\in)$

$= \text{Rs. } 160/-$ as $\in \to 0$.

Example 5 Solve the following transportation problem to minimize the total cost of transportation.

Destination

		1	2	3	4	Supply
	1	14	56	48	27	70
Origin	2	82	35	21	81	47
	3	99	31	71	63	93
Demand		70	35	45	60	210

[BNU. BE. Nov 96]

Solution: Since $\sum a_i = \sum b_j = 210$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table:

14 70	56	48	27
82	35	21 45	81 2
99	31 35	71	63 58

Since the number of non-negative allocations is 5, which is less than $(m + n - 1) = (3 + 4 - 1) = 6$, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (1,4), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is given in the following table

14 **70**	56	48	27 ∈
82	35	21 **45**	81 **2**
99	31 **35**	71	63 **58**

To find the optimal solution:

Now the number of non-negative allocations at independent positions is $(m+n-1) = 6$. We apply MODI method.

14 **70**	56 −5 61	48 −33 81	27 ∈	$u_1 = 27$
82 68 14	35 49 −14	21 **45**	81 **2**	$u_2 = 81$
99 50 49	31 **35**	71 3 68	63 **58**	$u_3 = 63$
$v_1 = -13$	$v_2 = -32$	$v_3 = -60$	$v_4 = 0$	

Since $d_{22} = -14 < 0$, the solution under the test is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (2,2) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (2,2) and having its other corners at some occupied cells. Along this closed loop, indicate $+\theta$ and $-\theta$ alternatively at the corners.

14 **70**	56	48	27 ∈
82	35 $+\theta$	21 **45**	81 **2** $-\theta$
99	31 **35** $-\theta$	71	63 $+\theta$ **58**

From the two cells (2,4), (3,2) having $-\theta$ we find that the minimum of the allocations 2,35 is 2. Add this 2 to the cells with $+\theta$ and subtract this 2 to the cells with $-\theta$. Hence the new basic feasible solution is given by

14 **70**	56	48	27 ∈
82	35 **2**	21 **45**	81
99	31 **33**	71	63 **60**

We see that the above table satisfies the rim conditions with $(m+n-1)$ non-negative allocations at independent positions. We apply MODI method for optimality.

14 **70**	56 −5 61	48 −19 67	27 $\in$	$u_1 = -40$
82 54 28	35 **2**	21 **45**	81 67 14	$u_2 = 0$
99 50 49	31 **33**	71 17 54	63 **60**	$u_3 = -4$
$v_1 = 54$	$v_2 = 35$	$v_3 = 21$	$v_4 = 67$	

Since all $d_{ij} > 0$, the solution under the test is optimal.

$\therefore$ The optimal allocation schedule is given by $x_{11} = 70$, $x_{14} = \in$, $x_{22} = 2$, $x_{23} = 45$, $x_{32} = 33$, $x_{34} = 60$ and the optimum (minimum) transportation cost

$= \text{Rs. } 14 \times 70 + 27 \times \in + 35 \times 2 + 21 \times 45 + 31 \times 33 + 63 \times 60$

$= \text{Rs. } 6798/-$ as $\in \to 0$.

Example 6 **Solve the following transportation problem, in which a_i is the availability at origin O_i and b_j is the requirement at the destination D_j and cell entries are unit costs of transportation from any origin to any destination:**

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
b_j	8	3	7	2	2	

[BRU. BE. Nov 96]

Solution: Since $\Sigma a_i = \Sigma b_j = 22$, the given problem is balanced. $\therefore$ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table:

4 **1**	7	3 **1**	8	2 **2**
1 **7**	4	7	3	8
7	2 **3**	4 **6**	7	7
4	8	2 $\in$	4 **2**	7

Since the number of non-negative allocations is 7, which is less than $(m + n - 1) = (4 + 5 - 1) = 8$, this basic feasible solution is degenerate.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the cell (4,3), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is given in the following table

4 **1**	7	3 **1**	8	2 **2**
1 **7**	4	7	3	8
7	2 **3**	4 **6**	7	7
4	8	2 $\in$	4 **2**	7

To find the optimal solution: Now the number of non-negative allocations at independent positions is $(m + n - 1) = 8$. We apply MODI method.

4 **1**	7 1 6	3 **1**	8 5 3	2 **2**	$u_1 = 0$
1 **7**	4 −2 6	7 0 7	3 2 1	8 −1 9	$u_2 = -3$
7 5 2	2 **3**	4 **6**	7 6 1	7 3 4	$u_3 = 1$
4 3 1	8 0 8	2 **∈**	4 **2**	7 1 6	$u_4 = -1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 3$	$v_4 = 5$	$v_5 = 2$	

Since all $d_{ij} > 0$, the solution under the test is optimal.

$\therefore$ The optimal allocation schedule is given by $x_{11} = 1$, $x_{13} = 1$, $x_{15} = 2$, $x_{21} = 7$, $x_{32} = 3$, $x_{33} = 6$, $x_{43} = \in$, $x_{44} = 2$ and the optimum (minimum) transportation cost

$= \text{Rs. } 4 \times 1 + 3 \times 1 + 2 \times 2 + 1 \times 7 + 2 \times 3 + 4 \times 6 + 2 \times \in + 4 \times 2$

$= \text{Rs. } (56 + 2\in)$

$= \text{Rs. } 56/\text{-} \text{ as } \in \to 0.$

2.2.6 Unbalanced Transportation Problems

If the given transportation problem is unbalanced one, *i.e.*, if $\Sigma a_i \neq \Sigma b_j$, then convert this into a balanced one by introducing a dummy source or dummy destination with zero cost vectors (zero unit transportation costs) as the case may be and then solve by usual method.

When the total supply is greater than the total demand, a dummy destination is included in the matrix with zero cost vectors. The excess supply is entered as a rim requirement for the dummy destination.

When the total demand is greater than the total supply, a dummy source is included in the matrix with zero cost vectors. The excess demand is entered as a rim requirement for the dummy source.

Example 1 **Solve the transportation problem**

Destination

	A	B	C	D	Supply
1	11	20	7	8	50
Source 2	21	16	20	12	40
3	8	12	18	9	70
Demand:	30	25	35	40	

Solution: Since the total supply ($\Sigma a_i = 160$) is greater than the total demand ($\Sigma b_j = 130$), the given problem is an unbalanced transportation problem. To convert this into a balanced one, we introduce a dummy destination E with zero unit transportation costs and having demand equal to $160 - 130 = 30$ units.

$\therefore$ The given problem becomes

Destination

	A	B	C	D	E	Supply
1	11	20	7	8	0	50
Source 2	21	16	20	12	0	40
3	8	12	18	9	0	70
	30	25	35	40	30	160

By using VAM the initial solution is as shown in the following table

11	20	7 **35**	8 **15**	0
21	16	20	12 **10**	0 **30**
8 **30**	12 **25**	18	9 **15**	0

$\therefore$ The initial transportation cost

$= \text{Rs. } 7 \times 35 + 8 \times 15 + 12 \times 10 + 0 \times 30 + 8 \times 30 + 12 \times 25 + 9 \times 15$

$= \text{Rs. } 1160/-$

For optimality: Since the number non-negative allocations at independent positions is $(m + n - 1)$, we apply the MODI method.

11 7 4	20 11 9	7 **35**	8 **15**	0 −4 4	$u_1 = 8$
21 11 10	16 15 1	20 11 9	12 **10**	0 **30**	$u_2 = 12$
8 **30**	12 **25**	18 8 10	9 **15**	0 −3 3	$u_3 = 9$
$v_1 = -1$	$v_2 = 3$	$v_3 = -1$	$v_4 = 0$	$v_5 = -12$	

Since all $d_{ij} > 0$, the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is

$x_{13} = 35, x_{14} = 15, x_{24} = 10, x_{25} = 30, x_{31} = 30, x_{32} = 25, x_{34} = 15$

It can be noted that $x_{25} = 30$ means that 30 units are despatched from source 2 to the dummy destination E. In other words, 30 units are left undespatched from source 2.

The optimum (minimum) transportation cost

$= \text{Rs. } 7 \times 35 + 8 \times 15 + 12 \times 10 + 0 \times 30 + 8 \times 30 + 12 \times 25 + 9 \times 15$

$= \text{Rs. } 1160/-$

Example 2 **Solve the transportation problem with unit transportation costs, demands and supplies as given below:**

Destination

	D_1	D_2	D_3	D_4	Supply
S_1	6	1	9	3	70
Source S_2	11	5	2	8	55
S_3	10	12	4	7	70
Demand	85	35	50	45	

[MU MBA Apr 95]

Solution: Since the total demand $(\Sigma b_i = 215)$ is greater than the total supply $(\Sigma a_j = 195)$, the given problem is unbalanced transportation problem. To convert this into a balanced one, we introduce a dummy source S_4 with zero unit transportation costs and having supply equal to $215 - 195 = 20$ units. $\therefore$ The given problem becomes

Destination

	D_1	D_2	D_3	D_4	Supply
S_1	6	1	9	3	70
Source S_2	11	5	2	8	55
S_3	10	12	4	7	70
S_4	0	0	0	0	20
	85	35	50	45	215

As this problem is balanced, there exists a basic feasible solution to this problem. By using Vogel's approximation method, the initial solution is as shown in the following table.

6 **65**	1 **5**	9	3
11	5 **30**	2 **25**	8
10	12	4 **25**	7 **45**
0 **20**	0	0	0

∴ The initial transportation cost

= Rs. $6 \times 65 + 1 \times 5 + 5 \times 30 + 2 \times 25 + 4 \times 25 + 7 \times 45 + 0 \times 20$

= Rs. 1010/-

For optimality: Since number of non-negative allocations at independent positions is $(m + n - 1)$, we apply the MODI method:

6 **65**	1 **5**	9 −2 11	3 1 2	$u_1 = 6$
11 10 1	5 **30**	2 **25**	8 5 3	$u_2 = 10$
10 12 −2	12 7 5	4 **25**	7 **45**	$u_3 = 12$
0 **20**	0 −5 5	0 −8 8	0 −5 5	$u_4 = 0$
$v_1 = 0$	$v_2 = -5$	$v_3 = -8$	$v_4 = -5$	

Since $d_{31} = -2 < 0$, the solution under the test is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (3,1) (since d_{31} is $-ve$) by making an occupied cell empty. For this, we draw a closed path consisting of horizontal and vertical lines beginning and ending at this cell (3,1) and having its other corners at some occupied cells. Along this closed loop, indicate $+\theta$ and $-\theta$ alternatively at the corners.

We have

6 **65** $-\theta$	1 **5** $+\theta$	9	3
11	5 **30** $-\theta$	2 **25** $+\theta$	8
10 $+\theta$	12	4 **25** $-\theta$	7 **45**
0 **20**	0	0	0

From the three cells (1,1), (2,2), (3,3) having $-\theta$, we find that the minimum of the allocations 65, 30, 25 is 25. Add this 25 to the cells with $+\theta$ and subtract this 25 to the cells with $-\theta$. Finally, the new basic feasible solution is displayed in the following table.

6 **40**	1 **30**	9	3
11	5 **5**	2 **50**	8
10 **25**	12	4	7 **45**
0 **20**	0	0	0

We see that the above table satisfies the rim conditions with $(m+n-1)$ non-negative allocations at independent positions. Now we check for optimality

6 **40**	1 **30**	9 −2 11	3 3 0	$u_1 = 6$
11 10 1	5 **5**	2 **50**	8 7 1	$u_2 = 10$
10 **25**	12 5 7	4 2 2	7 **45**	$u_3 = 10$
0 **20**	0 −5 5	0 −8 8	0 −3 3	$u_4 = 0$
$v_1 = 0$	$v_2 = -5$	$v_3 = -8$	$v_4 = -3$	

Since all $d_{ij} > 0$ with $d_{14} = 0$, the solution under the test is optimal and an alternative optimal solution exists.

$\therefore$ The optimum allocation schedule is given by

$x_{11} = 40, x_{12} = 30, x_{22} = 5, x_{23} = 50, x_{31} = 25, x_{34} = 45, x_{41} = 20.$

It can be noted that $x_{41} = 20$ means that 20 units are despatched from the dummy source S_4 to the destination D_1. In other words, 20 units are not fulfilled for the destination D_1.

The optimum (minimum) transportation cost

$= \text{Rs. } 6 \times 40 + 1 \times 30 + 5 \times 5 + 2 \times 50 + 10 \times 25 + 7 \times 45 + 0 \times 20$

$= \text{Rs. } 960/-$

Example 3 **Solve the transportation problem with unit transportation costs in rupees, demands and supplies as given below:**

Destination

		D_1	D_2	D_3	Supply (units)
	A	5	6	9	100
Origin	B	3	5	10	75
	C	6	7	6	50
	D	6	4	10	75
Demand (units)		70	80	120	

[MU MBA Apr 98]

Solution: Since the total supply ($\Sigma a_i = 300$) is greater than the total demand ($\Sigma b_j = 270$), the given transportation problem is unbalanced.

To convert this into a balanced one, we introduce a dummy source D_4 with zero unit transportation costs and having demand equal to $300 - 270 = 30$ units. $\therefore$ The given problem becomes

Destination

		D_1	D_2	D_3	D_4	Supply
	A	5	6	9	0	100
Origin	B	3	5	10	0	75
	C	6	7	6	0	50
	D	6	4	10	0	75
Demand		70	80	120	30	300

By using VAM the initial solution is given by

5	6	9 **100**	0
3 **70**	5 **5**	10	0
6	7	6 **20**	0 **30**
6	4 **75**	10	0

Since the number of non-negative allocations is 6, which is less than $(m + n - 1) = 4 + 4 - 1 = 7$, this basic feasible solution is degenerate.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the cell (2, 4), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is given in the following table

5	6	9 **100**	0
3 **70**	5 **5**	10	0 $\in$
6	7	6 **20**	0 **30**
6	4 **75**	10	0

Now the number of non-negative allocations at independent positions is $(m + n - 1)$. We apply MODI method.

5 6 −1	6 8 −2	9 **100**	0 3 −3	$u_1 = 3$
3 **70**	5 **5**	10 6 4	0 $\in$	$u_2 = 0$
6 3 3	7 5 2	6 **20**	0 **30**	$u_3 = 0$
6 2 4	4 **75**	10 5 5	0 −1 1	$u_4 = -1$
$v_1 = 3$	$v_2 = 5$	$v_3 = 6$	$v_4 = 0$	

Since there are some $d_{ij} < 0$, the current solution is not optimal.

Since $d_{14} = -3$ is the most negative, let us form a new basic feasible solution by giving maximum allocation to the corresponding cell (1,4) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1,4) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners.

We have

5	6	9 $-\theta$ **100**	0 $+\theta$
3 **70**	5 **5**	10	0 $\in$
6	7	6 **20** $+\theta$	0 **30** $-\theta$
6	4 **75**	10	0

From the two cells (1,3), (3,4) having $-\theta$, we find that the minimum of the allocations 100,30 is 30. Add this 30 to the cells with $+\theta$ and subtract this 30 to the cells with $-\theta$. Hence the new basic feasible solution is given in the following table.

5	6	9 **70**	0 **30**
3 **70**	5 **5**	10	0 $\in$
6	7	6 **50**	0
6	4 **75**	10	0

We see that the above table satisfies the rim conditions with $(m + n - 1)$ non-negative allocations at independent positions. So we apply MODI method.

5 3 2	6 5 1	9 **70**	0 **30**	$u_1 = 0$
3 **70**	5 **5**	10 9 1	0 $\in$	$u_2 = 0$
6 0 6	7 2 5	6 **50**	0 -3 3	$u_3 = -3$
6 2 4	4 **75**	10 8 2	0 -1 1	$u_4 = -1$
$v_1 = 3$	$v_2 = 5$	$v_3 = 9$	$v_4 = 0$	

Since all $d_{ij} > 0$, the current solution is optimal and unique.

The optimum allocation schedule is given by

$x_{13} = 70, x_{14} = 30, x_{21} = 70, x_{22} = 5, x_{24} = \in, x_{33} = 50, x_{42} = 75$ and the optimum (minimum) transportation cost

$= \text{Rs. } 9 \times 70 + 0 \times 30 + 3 \times 70 + 5 \times 5 + 0 \times \in + 6 \times 50 + 4 \times 75$

$= \text{Rs. } 1465/-$

2.2.7 Maximization case in Transportation Problems

So far we have discussed the transportation problems in which the objective has been to minimize the total transportation cost and algorithms have been designed accordingly.

If we have a transportation problem where the objective is to maximize the total profit, first we have to convert the maximization problem into a minimization problem by multiplying all the entries by -1 (or) by subtracting all the entries from the highest entry in the given transportation table. The modified minimization problem can be solved in the usual manner.

Example 1 Solve the following transportation problem to maximize profit

Profits (Rs)/Unit

Destination

Source	A	B	C	D	Supply
1	40	25	22	33	100
2	44	35	30	30	30
3	38	38	28	30	70
Demand	40	20	60	30	

Solution: Since the given problem is of maximization type, first convert this into a minimization problem by subtracting the cost elements (entries or c_{ij}) from the highest cost element (c_{ij} = 44) in the given transportation problem. Then the given problem becomes.

Destination

Source	A	B	C	D	Supply
1	4	19	22	11	100
2	0	9	14	14	30
3	6	6	16	14	70
Demand	40	20	60	30	

This modified minimization problem is unbalanced (Σa_i = 200, Σb_j = 150 and $\Sigma a_i \neq \Sigma b_j$). To make it balanced, we introduce a dummy destination E with demand (200 – 150) = 50 units with zero costs c_{ij}. Hence the balanced minimization transportation problem becomes

Destination

Source	A	B	C	D	E	Supply
1	4	19	22	11	0	100
2	0	9	14	14	0	30
3	6	6	16	14	0	70
Demand	40	20	60	30	50	200

Since $\Sigma a_i = \Sigma b_j = 200$, there exists a basic feasible solution to this problem and is displayed in the following table by using VAM. [Try least cost method]

4 **10**	19	22 **60**	11 **30**	0
0 **30**	9	14	14	0
6	6 **20**	16	14	0 **50**

Since the number of non-negative allocations at independent position is 6, which is less than $(m + n - 1) = (3 + 5 - 1) = 7$, this initial solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (3, 3), so that the number of occupied cells becomes $(m + n - 1)$. Hence the initial solution is given by

4 **10**	19	22 **60**	11 **30**	0
0 **30**	9	14	14	0
6	6 **20**	16 $\in$	14	0 **50**

Now the number of non-negative allocations at independent positions is $(m + n - 1)$. We apply MODI method for optimal solution.

4 **10**	19 12 7	22 **60**	11 **30**	0 6 –6	$u_1 = 0$
0 **30**	9 8 1	14 18 –4	14 7 7	0 2 –2	$u_2 = -4$
6 –2 8	6 **20**	16 ∈	14 5 9	0 **50**	$u_3 = -6$
$v_1 = 4$	$v_2 = 12$	$v_3 = 22$	$v_4 = 11$	$v_5 = 6$	

Since d_{15}, d_{23}, d_{25} are less than zero, the current solution under the test is not optimal. Here $d_{15} = -6$ is the most negative value of d_{ij}.

Let us form a new basic feasible solution by giving maximum allocation to the cell (1,5) by making an occupied cell empty. For this, we draw a closed path consisting of horizontal and vertical lines beginning and ending at this cell (1,5) and having its other corners at some occupied cells. Along this closed loop, indicate +θ and –θ alternatively at the corners.

4 **10**	19	22 **60** –θ	11 **30**	0 +θ
0 **30**	9	14	14	0
6	6 **20**	16 +θ ∈	14	0 –θ **50**

From the two cells (1,3), (3,5) having – θ, we find that the minimum of 60, 50 is 50. Add this 50 to the cells with + θ and subtract this 50 to the cells with – θ. Hence the new basic feasible solution is displayed in the following table.

4 **10**	19	22 **10**	11 **30**	0 **50**
0 **30**	9	14	14	0
6	6 **20**	16 **50**	14	0

We see that the above table satisfies the rim conditions with $(m + n - 1)$ non-negative allocations at independent positions.

Now we apply the MODI method for optimality.

4 **10**	19 12 7	22 **10**	11 **30**	0 **50**	$u_1 = 0$
0 **30**	9 8 1	14 18 –4	14 7 7	0 –4 4	$u_2 = -4$
6 –2 8	6 **20**	16 **50**	14 5 9	0 –6 6	$u_3 = -6$
$v_1 = 4$	$v_2 = 12$	$v_3 = 22$	$v_4 = 11$	$v_5 = 0$	

Since $d_{23} = -4 < 0$ the current solution is not optimal.

Let us form a new basic feasible solution by giving maximum allocation to the cell (2,3) by making an occupied cell empty. For this, we draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (2,3) and having its other corner at some occupied cells. Along this closed loop, indicate +θ and –θ alternatively at the corners.

4 **20** +θ	19	22 **10** −θ	11 **30**	0 **50**
0 **30** −θ	9	14 +θ	14	0
6	6 **20**	16 **50**	14	0

From the two cells (1,3), (2,1) having −θ, we find that the minimum of the allocations 10, 30 is 10. Add this 10 to the cells with +θ and subtract this 10 to the cells with −θ. Hence the new basic feasible solution is displayed in the following table.

4 **20**	19	22	11 **30**	0 **50**
0 **20**	9	14 **10**	14	0
6	6 **20**	16 **50**	14	0

Now the number non-negative allocations at independent positions is $(m + n - 1)$. We apply MODI method for the optimality

4 **20**	19 8 11	22 18 4	11 **30**	0 **50**	$u_1 = 0$
0 **20**	9 4 5	14 **10**	14 7 7	0 −4 4	$u_2 = -4$
6 2 4	6 **20**	16 **50**	14 9 5	0 −2 2	$u_3 = -2$
$v_1 = 4$	$v_2 = 8$	$v_3 = 18$	$v_4 = 11$	$v_5 = 0$	

Since all $d_{ij} > 0$, the current solution is optimal and unique.

$\therefore$ The optimum allocation schedule is given by

$x_{11} = 20, x_{14} = 30, x_{15} = 50, x_{21} = 20, x_{23} = 10, x_{32} = 20, x_{33} = 50.$

The optimum profit

$= \text{Rs. } 40 \times 20 + 33 \times 30 + 0 \times 50 + 44 \times 20 + 30 \times 10 + 38 \times 20 + 28 \times 50$

$= \text{Rs. } 5130/-.$

Example 2 **Solve the following transportation problem to maximize profit.**

Destination

		A	B	C	D	Supply
Source	**1**	**15**	**51**	**42**	**33**	**23**
	2	**80**	**42**	**26**	**81**	**44**
	3	**90**	**40**	**66**	**60**	**33**
Demand		**23**	**31**	**16**	**30**	**100**

Solution: Since the given problem is of maximization type, we convert this in to minimization problem by multiplying the profit costs c_{ij} by -1.

Destination

	A	B	C	D	Supply
Source 1	–15	–51	–42	–33	23
2	–80	–42	–26	–81	44
3	–90	–40	–66	–60	33
Demand	23	31	16	30	100

Since $\Sigma a_i = \Sigma b_j = 100$, there exists a basic feasible solution to this problem and is displayed in the following table by using VAM.

–15	–51 **23**	–42	–33
–80 **6**	–42 **8**	–26	–81 **30**
–90 **17**	–40	–66 **16**	–60

Since the number of non-negative allocations at independent positions is $(m + n - 1) = 6$, we apply MODI method for optimal solution.

–15 –89 74	–51 **23**	–42 –65 23	–33 –90 57	$u_1 = -9$
–80 **6**	–42 **8**	–26 –56 30	–81 **30**	$u_2 = 0$
–90 **17**	–40 –52 12	–66 **16**	–60 –91 31	$u_3 = -10$
$v_1 = -80$	$v_2 = -42$	$v_3 = -56$	$v_4 = -81$	

Since all $d_{ij} > 0$, the current solution is optimal and unique.

$\therefore$ The optimum allocations are given by $x_{12} = 23$, $x_{21} = 6$, $x_{22} = 8$, $x_{24} = 30$, $x_{31} = 17$, $x_{33} = 16$

$\therefore$ The optimum profit

= Rs. $51 \times 23 + 80 \times 6 + 42 \times 8 + 81 \times 30 + 90 \times 17 + 66 \times 16$

= Rs. 7005 /-

EXERCISE

1. **What do you mean by transportation model ?**
2. **Define: Feasible solution, Basic feasible solution, Degenerate basic feasible solution, Non-degenerate basic feasible solution and optimal solution of a transportation problem.**
3. **Explain, in brief, with examples**
   (i) **North West Corner rule.** ***(ii)*** **Lowest Cost entry method.** ***(iii)*** **Vogel's approximation method.**
   [MU. MBA. Nov. 96, Apr 95]
4. **What do you mean by balanced and unbalanced transportation problems? Explain how would you convert the unbalanced problem into a balanced one?**
5. **State all the constraints in a transportation problem and how they are different from linear programming problem.**
   [MU. BE. Apr 94]
6. **Write down the dual of a transportation problem. Explain how this helps us in identifying whether the current solution is optimal or not.** ***[MU. MCA. Nov 96]***
7. **Explain an algorithm for solving a transportation problem.**
   [BRU. BE. Apr 95]
8. **Describe the method of solving unbalanced transportation problem.*****[MU. MCA. Apr 95]***
9. **State the classical transportation problem and write down its mathematical model.**
10. **Give mathematical formulation of a transportation problem.**
    [MU. BE. Nov 93, Nov 89]
11. **How the problem of degeneracy arises in a transportation problem ? Explain how does one over come it.**
    [MU. BE. Apr 92, BRU. BE. Apr 96]
12. **Describe a transportation problem and give a method of finding an initial feasible solution .** ***[MU. BE. Nov 91]***

13. **Obtain the initial (starting) solution for the following transportation problem.**

Destination

	A	B	C	Supply
Source 1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	34

(i) North West Corner rule *(ii)* Least Cost method
(iii) Vogel's approximation method

14. Solve the following transportation problem

To

	A	B	C	Availability
From I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	

[MU. BE. Nov 89]

15. Obtain an optimum basic feasible solution to the transportation problem:

Warehouse

	W_1	W_2	W_3	W_4	Capacity
Factory F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

[MU. BE. Apr 90, B. Tech. Leather. Oct 96]

16. Obtain an optimum basic feasible solution to the following transportation problem:

To

From	7	3	4	2
	2	1	3	3 Available
	3	4	6	5
Demand	4	1	5	10

[MU. BE. Nov 91]

17. **Solve the transportation problem:**

Destinations

	1	2	3	
Source 1	2	2	3	10
2	4	1	2	15 Capacities
3	1	3	1	40
Demands	20	15	30	

[MU. BE. Nov 92]

18. **Solve the following transportation problem where the cell entries denote the unit transportation costs.**

Destination

	A	B	C	D	Available
Origin P	5	4	2	6	20
Q	8	3	5	7	30
R	5	9	4	6	50
Required	10	40	20	30	

[MU. BE. Nov 91]

19. **A company has 4 warehouses and 6 stores, the cost of shipping one unit from warehouse *i* to store *j* is c_{ij}**

$$C_{ij} = \begin{matrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & 8 \end{matrix}$$

and the requirements of six stores are 4, 4, 6, 2, 4, 2 and quantities at warehouses are 5, 6, 2, 9, find the minimum cost solution. *[MU. BE. Nov 93]*

20. **A company has four warehouses *a, b, c, d*. It is required to deliver a product from these warehouses to three customers A, B and C. The warehouses have the following amounts in stock:**

Ware house : *a* *b* *c* *d*

No. of Units : 15 16 12 13

and the customer's requirements are:

Customer : *A* *B* *C*

No. of Units : 18 20 18

The table below shows the costs of transporting one unit from warehouse to the customer:

		Warehouse			
		a	*b*	*c*	*d*
	A	8	9	6	3
Customer	B	6	11	5	10
	C	3	8	7	9

Find the optimal transportation routes. *[MU. BE. Nov 93]*

21. Explain the concept of degeneracy in transportation problems:

Obtain an optimum basic feasible solution to the following transportation problem

	To			Supply
	7	3	4	2
From	2	1	3	3
	3	4	6	3
Demand	4	1	3	8

[MU. MCA. May 92]

22. Solve the transportation problem with unit transportation costs, demands and supplies as given below:

		Destination Centre				
		D_1	D_2	D_3	D_4	Supply
	F_1	3	3	4	1	100
Factory	F_2	4	2	4	2	125
	F_3	1	5	3	2	75
Demand		120	80	75	25	

[MU. MBA Nov 95]

23. Find the minimum cost of transportation, given

	Warehouses				
	D_1	D_2	D_3	D_4	Supply
Factory F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Demand	5	8	7	14	

[MU. BE. Apr 90, Nov 94]

24. Compute the initial feasible solution to the following transportation problem, given the cost of transportation in rupees.

		To			
		P	Q	R	Supply
	A	5	1	7	10
From	B	6	4	6	80
	C	3	2	5	15
Demand		75	20	50	

[MU. BE. Oct 95]

25. Solve the transportation problem where the cell entries are transportation costs.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	4	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Required	4	6	8	6	

[MU. BE. Apr 88]

26. Solve the transportation problem

		Demand point				
		1	2	3	4	Supply
	1	2	3	11	7	6
Source	2	1	0	6	1	1
	3	5	8	15	9	10
Demand		7	5	3	2	

[MU. BE. Apr 89]

27. **An automobile dealer is faced with the problem of determining the minimum cost policy for supplying dealers with the desired number of automobiles. The relevant data are given below. Obtain the minimum total cost of transportation**

	Dealers 1	2	3	4	5	Supply
Plant A	1.2	1.7	1.6	1.8	2.4	300
Plant B	1.8	1.5	2.2	1.2	1.6	400
Plant C	1.5	1.4	1.2	1.5	1.0	100
Requirement	100	50	300	150	200	

The cost unit is in 100 rupees. *[MU. BE. Apr 89]*

28. **Food packets have to be air lifted by three aircrafts from an airport and air–dropped to five villages. The quantities that can be carried in one trip by these aircrafts to the village are given below. The total number of trips per day an aircraft can make to the villages are also given. Find the number of trips each aircraft should make to each village so that the total quantity of food transported is maximum.**

	V_1	V_2	V_3	V_4	V_5	Trips/day by air crafts
A_1	10	8	6	9	12	50
A_2	5	3	8	4	10	90
A_3	7	9	6	10	4	60
Trips/day to village	100	80	70	40	20	

29. **A company produces a small component for an industrial products and distributes it to five wholesalers at a fixed delivered price of Rs. 2.50 per unit. Sales forecasts indicate that monthly deliveries will be 3000, 3000, 10,000, 5000 and 4000 units to wholesalers 1,2,3,4, and 5 respectively. The monthly production capacities are 5000, 10000 and 12500 at plants 1,2 and 3 respectively. The direct costs of production of each unit are Rs.1.00, Rs.0.90 and Rs.0.80 at plants 1,2 and 3 respectively. The transportation cost of shipping a unit from a plant to a wholesaler are given below:**

	wholesaler 1	2	3	4	5
Plant 1	0.05	0.07	0.10	0.15	0.15
Plant 2	0.08	0.06	0.09	0.12	0.14
Plant 3	0.10	0.09	0.08	0.10	0.15

Find how many components each plant supplies to each wholesaler in order to maximize its profit.

30. **A company has three plants A, B, C and three warehouses X, Y, Z. The number of units available at the plants is 60,70, 80 and the demand at X, Y, Z are 50, 80, 80 respectively. The unit costs of transportation is given by the following table.**

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

Find the allocation so that the total transportation cost is minimum. *[MU. BE. Oct 96]*

31. **An oil corporation has got three refineries R_1, R_2, R_3 and it has to send petrol to four different depots D_1, D_2, D_3 and D_4. The cost of supplying of one unit of petrol from each refinery to each depot is given below. The requirements of the depot and the available petrol at the refineries are also given. Find the minimum cost of shipping after obtaining the initial solution by Vogel's Approximation method.**

	Depot D_1	D_2	D_3	D_4	Available
Refinery R_1	10	12	15	8	130
Refinery R_2	14	11	9	10	150
Refinery R_3	20	5	7	18	170
Required:	90	100	140	120	

[MU. BE. Oct 96]

32. Find the optimum solution to the following transportation problem in which cells contain the transportation cost in rupees.

Table

	W_1	W_2	W_3	W_4	W_5	Available
F_1	7	6	4	5	9	40
F_2	8	5	6	7	8	30
F_3	6	8	9	6	5	20
F_4	5	7	7	8	6	10
Required:	30	30	15	20	5	100 Total

[MU. B. Tech. Leather. Oct 96]

33. The following is a transportation problem relating three warehouses (A, B and C) and four customers (1,2,3 and 4). The capacities at the warehouses and the demands from the customers are shown around the perimeter. Per unit transportation costs are shown in the cells. Cost minimization is the objective.

	1	2	3	4	Cap
A	7	8	11	10	30
B	10	12	5	4	45
C	6	11	10	9	35
Dem	20	28	19	33	–

Find the optimal solution and the total cost of transportation.

[BRU. BE. Apr 94]

34. A fan manufacturing company has its plants at Calcutta and Delhi. The company is having warehouses in Nagpur, Patna and Baroda. The following table indicates the maximum capacities of the plants and the demands of the warehouses. The cost in rupees of shipping one unit from the particular plant to the given warehouses are shown on the corner of the cells. Find the least cost shipping assignment.

	Warehouses			
Plants	Nagpur	Patna	Baroda	Capacity
Delhi	3	4	2	600
Calcutta	3	2	5	700
Demand	400	500	400	–

[BRU. BE. Nov 95]

35. The projects X, Y, Z require truck loads of 45,50 and 20 respectively per week. The availabilities in plants A,B,C are 40, 40 and 40 of truck loads respectively per week. The cost of transport per unit of truck load from plant to project is given below:

		Project		
		X	Y	Z
	A	5	20	5
Plant	B	10	30	8
	C	10	20	12

(i) Determine an initial feasible solution by VAM.

(ii) Obtain an optimal solution by MODI method. The objective is to minimize the total cost of transportation.

[BRU. BE. Nov 94, MSU. BE. Apr 97]

36. Production shops P, Q, R can produce 5 new products A, B, C, D, E with their excess production capacity. The unit costs are given below with sale potential and availability of the capacity. It is given that the production shop R cannot produce the fifth product E. Find the optimal production schedule. Start the solution by VAM.

		New Products					
		A	B	C	D	E	Availability
	P	20	19	14	21	16	40
Production shop	Q	15	20	13	19	16	60
	R	18	15	18	20	–	90
Sales potential		30	40	70	40	60	

Find the allocation so that the total transportation cost is minimum. *[BRU. BE. Nov 96]*

37. Solve the transportation problem with unit transportation costs in rupees, requirements and availability as given below:

Distribution centre

		D_1	D_2	D_3	D_4	Availability
	F_1	10	15	12	12	200
Factory	F_2	8	10	11	9	150
	F_3	11	12	13	10	120
Requirement		140	120	80	220	

[MU. MBA. Apr. 97]

38. Solve the transportation problem with unit transportation costs in rupees and units of demand and supply as given below:

Destination

		D_1	D_2	D_3	D_4	Availability
	S_1	10	13	11	8	65
Source	S_2	9	12	12	10	44
	S_3	13	9	11	9	41
Demand		60	40	55	45	

[MU. MBA. Nov. 97]

39. Solve the transportation problem with unit transportation costs, demands and supplies as given below:

Destination

		D_1	D_2	D_3	Supply
	S_1	4	1	7	80
Source	S_2	3	2	2	20
	S_3	5	3	4	50
Demand		60	40	35	

[MU. MBA. Apr. 96]

ANSWERS

13. (i) $x_{11} = 5, x_{21} = 2, x_{22} = 6, x_{32} = 3, x_{33} = 4, x_{41} = 14.$
and the transportation cost = Rs. 102/-
(ii) $x_{11} = 5, x_{21} = 2, x_{22} = 6, x_{32} = 3, x_{33} = 4, x_{41} = 14.$
and the transportation cost = Rs. 102/-
(iii) $x_{11} = 5, x_{21} = 2, x_{22} = 6, x_{32} = 3, x_{33} = 4, x_{41} = 14.$
and the transportation cost = Rs. 102/-
14. $x_{11} = 1, x_{21} = 3, x_{31} = \epsilon, x_{32} = 2$, and $x_{33} = 2$
and minimum T.P. cost = Rs. 743.
15. $x_{12} = 5, x_{14} = 2, x_{22} = 2, x_{23} = 7, x_{32} = 6$ and $x_{34} = 12.$
and minimum T.P. cost = Rs. 743.
16. $x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4, x_{33} = 1$ and the optimum transportation cost is Rs. 33.
17. $x_{11} = 10, x_{22} = 15, x_{31} = 10, x_{33} = 30$, and the optimum transportation cost is Rs. 75/-.
18. $x_{12} = 10, x_{13} = 10, x_{22} = 30, x_{31} = 10, x_{33} = 10, x_{34} = 30$ and the optimum transportation cost is Rs. 420/-.
19. $x_{13} = 5, x_{22} = 4, x_{26} = 2, x_{31} = 1, x_{32} = \epsilon, x_{33} = 1, x_{41} = 3,$ $x_{44} = 2, x_{45} = 4$, and the optimal transportation cost = Rs. $(68 + 3\epsilon)$ = Rs. 68/- , as $\epsilon \to 0$.
20. $x_{12} = 5, x_{14} = 13, x_{22} = 8, x_{23} = 12, x_{31} = 15, x_{32} = 3,$
and the optimum transportation cost = Rs. 301/-.
21. $x_{13} = 2, x_{21} = 1, x_{22} = 1, x_{23} = 1, x_{31} = 3$
and the optimum transportation cost = Rs.23/-.
22. $x_{11} = 45, x_{13} = 30, x_{14} = 25, x_{22} = 80, x_{23} = 45, x_{31} = 75,$
and the optimum transportation cost = Rs.695/-.
23. $x_{11} = 5, x_{14} = 2, x_{22} = 2, x_{23} = 7, x_{32} = 6, x_{34} = 12,$
and the optimum transportation cost = Rs.743/-.
24. $x_{12} = 10, x_{21} = 60, x_{22} = 10, x_{23} = 10, x_{31} = 15, x_{43} = 40,$
and the optimum transportation cost = Rs. 515/-
25. $x_{12} = 6, x_{23} = 2, x_{24} = 6, x_{31} = 4, x_{33} = 6$
and the optimum transportation cost = Rs. 28/-
26. $x_{12} = 5, x_{23} = 1, x_{33} = 1, x_{13} = 1, x_{31} = 7, x_{34} = 2,$
and the optimum transportation cost = Rs. 100/-
27. $x_{11} = 100, x_{13} = 200, x_{22} = 50, x_{24} = 150, x_{25} = 200, x_{33} = 100,$
and the optimum transportation cost = Rs. 1,13,500/-

28. $x_{11} = 50$, $x_{23} = 70$, $x_{25} = 20$, $x_{32} = 20$, $x_{34} = 40$, $x_{41} = 50$, $x_{42} = 60$ and the maximum quantity of transportation is 1840 units.
29. $x_{11} = 2500$, $x_{16} = 2500$, $x_{21} = 500$, $x_{22} = 3000$, $x_{23} = 2500$, $x_{25} = 4000$, $x_{33} = 7500$, $x_{34} = 5000$ units respectively. The total transportation cost = Rs. 23,730/-. Total sale = Rs. 62,500/-. Total production cost = Rs. 23,600/-.
    Therefore, the net maximum profit
    = Total sale – (Total transportation cost + Total production cost)
    = 62,500 – (23,730 + 23,600)
    = Rs. 15,170/-
30. $x_{13} = 60, x_{21} = 50, x_{23} = 20, x_{32} = 80$
    and the optimum transportation cost is Rs. 750/-
31. $x_{11} = 90, x_{14} = 40, x_{23} = 70, x_{24} = 80, x_{32} = 100, x_{33} = 70$
    and the minimum (optimum) shipping cost is Rs. 3640/-
32. $x_{11} = 5, x_{13} = 15, x_{14} = 20, x_{21} = \in, x_{22} = 30, x_{31} = 15, x_{35} = 5, x_{41} = 10$, and the optimum solution is Rs.510/- as $\in \to 0$.
33. $x_{12} = 28, x_{15} = 2, x_{23} = 19, x_{24} = 26, x_{31} = 20, x_{34} = 7, x_{35} = 8$, and the optimum (minimum) transportation cost is Rs. 606/-.
34. $x_{11} = 200, x_{13} = 400, x_{21} = 200, x_{22} = 500$,
    and the least shipping cost is Rs. 3000/-.
35. *(i)* $x_{11} = 40$, $x_{21} = 10$, $x_{22} = 15$, $x_{23} = 20$, $x_{32} = 35$, $x_{34} = 5$, and the initial transportation cost is Rs.1560/-.
    (ii) $x_{11} = 30$, $x_{12} = 10$, $x_{21} = 15$, $x_{23} = 20$, $x_{24} = 5$, $x_{32} = 40$, and the optimum (minimum) transportation cost is Rs.1460/-
36. $x_{13} = 10$, $x_{15} = 30$, $x_{23} = 60$, $x_{31} = 30$, $x_{32} = 40$, $x_{34} = 20$, $x_{44} = 20$, $x_{45} = 30$ and the optimum production cost is Rs. 2940/-. Note that an alternative optimum production schedule also exists.
37. $x_{11} = 140$, $x_{13} = 60$, $x_{22} = 50$, $x_{24} = 100$, $x_{34} = 120$, $x_{42} = 70$, $x_{43} = 20$. The minimum transportation cost is Rs. 4720/-.
38. $x_{11} = 16$, $x_{13} = 4$, $x_{14} = 45$, $x_{21} = 44$, $x_{32} = 40$, $x_{33} = 1$, $x_{43} = 50$. The minimum transportation cost = Rs. 1331/-
39. $x_{11} = 40$, $x_{12} = 40$, $x_{23} = 20$, $x_{31} = 20$, $x_{33} = 15$, $x_{34} = 15$. The Minimum transportation cost = Rs. 400/-.

2.3 ASSIGNMENT MODEL

2.3.1 Introduction

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (Jobs or origins or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost (or maximum profit).

Suppose that we have 'n' jobs to be performed on 'm' machines (one Job to one machine) and our objective is to assign the jobs to the machines at the minimum cost (or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be stated in the form of $m \times n$ matrix (c_{ij}) called a ***Cost matrix*** (or) ***Effectiveness matrix*** where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

		Jobs				
		1	2	3	....	n
	1	c_{11}	c_{12}	c_{13}	....	c_{1n}
	2	c_{21}	c_{22}	c_{23}	....	c_{2n}
Machines	3	c_{31}	c_{32}	c_{33}	....	c_{3n}
	:	...	...	...	....	...
	:	...	...	...	....	...
	:	...	...	...	....	...
	m	c_{m1}	c_{m2}	c_{m3}	....	c_{mn}

2.3.2 Mathematical formulation of an assignment problem.

[MU. BE. Apr 93, Oct 96, MU. MBA. Nov 96]

Consider an assignment problem of assigning n jobs to n machines (one job to one machine). Let c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and

$$\text{let } x_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine} \end{cases}$$

The assignment model is then given by the following LPP

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \; i = 1, 2, \ldots n$$

$$\text{and } x_{ij} = 0 \text{ (or) } 1.$$

2.3.3 Comparison with Transportation Model

[MU. MBA Nov 95, Apr. 97]

The assignment problem may be considered as a special case of the transportation problem. Consider a transportation problem with 'n' sources and 'n' destinations.

Destination

		1	2	3	....	n	*Supply* (a_i)
	1	c_{11}	c_{12}	c_{13}	....	c_{1n}	a_1
	2	c_{21}	c_{22}	c_{23}	....	c_{2n}	a_2
Source	3	c_{31}	c_{32}	c_{33}	....	c_{3n}	a_3
	:	...	...	...	....	...	...
	:	...	...	...	....	...	...
	:	...	...	...	....	...	...
	n	c_{n1}	c_{n2}	c_{n3}	....	c_{nn}	a_n
Demand (b_j)		b_1	b_2	b_3	....	b_n	

We have to find x_{ij} $(i, j = 1, 2, 3 \ldots n)$ for which the total transportation cost

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

is minimized.

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots n$$

$$\sum_{i=1}^{n} x_{ij} = b_j, \quad j = 1, 2, \ldots n$$

$$\sum a_i = \sum b_j, \quad i, j = 1, 2, \ldots n$$

$$\text{and } x_{ij} \geq 0, \quad i, j = 1, 2, 3, \ldots n$$

Here the 'sources' represent 'facilities' or 'machines' and 'destinations' represent 'jobs'.

Suppose that the supply available at each source is 1 *i.e.*, $a_i = 1$ and the demand required at each destination is 1 *i.e.*, $b_j = 1$.

Let c_{ij} be the unit transportation cost from the i^{th} source to the j^{th} destination. Here it means the cost of assigning the i^{th} machine to the j^{th} job.

Let x_{ij} be the amount to be shipped from i^{th} source to the j^{th} destination. Here it means the assignment of the i^{th} machine to the j^{th} job. We can restrict the value of x_{ij} to be either 0 (or) 1. $x_{ij} = 0$ means that the i^{th} machine does not get the j^{th} job and $x_{ij} = 1$ means that the i^{th} machine gets the j^{th} job.

Since each machine should be assigned to only one job and each job requires only one machine, the total assignment value of the i^{th} machine is 1, (*i.e.,*) $\sum x_{ij} = 1$ and the total assignment value of the j^{th} job is 1, (*i.e.,*) $\sum x_{ij} = 1$.

Hence the assignment problem can be expressed as

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job subject to the constraints.

$$x_{ij} = \begin{cases} 1, \text{ if } i^{th} \text{ machine is assigned to the } j^{th} \text{ job} \\ 0, \text{ if } i^{th} \text{ machine is not assigned to the } j^{th} \text{ job} \end{cases}$$

$$\text{i.e., } x_{ij} = 0 \text{ or } 1 \Rightarrow x_{ij}(x_{ij} - 1) = 0 \Rightarrow x_{ij}^2 = x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \; i = 1, 2, \ldots n \text{ and}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, 2, \; \dots \; n.$$

From this we see that assignment problem represents a transportation problem with all demands and supplies equal to 1.

The units available at each source and units demanded at each destination are equal to 1. It means exactly that there is only one occupied cell in each row and each column of the transportation table. *i.e.*, only 'n' occupied cells in place of the required $n + n - 1 = 2n - 1$ occupied cells. Hence ***an assignment problem is always a degenerate form of a transportation problem.***

But the transportation technique (or) simplex method can not be used to solve the assignment problem because of degeneracy. In fact a very convenient iterative procedure is available for solving an assignment problem.

The technique used for solving assignment problem makes use of the following two theorems.

Theorem 1: The optimum assignment schedule remains unaltered if we add or subtract a constant from all the elements of the row or column of the assignment cost matrix.

Theorem 2: If for an assignment problem all $c_{ij} > 0$, then an assignment schedule (x_{ij}) which satisfies $\sum c_{ij} x_{ij} = 0$, must be optimal.

DIFFERENCE BETWEEN THE TRANSPORTATION PROBLEM AND THE ASSIGNMENT PROBLEM. *[MU. BE. Nov 94]*

	Transportation Problem	*Assignment Problem*
(a)	Supply at any source may be any positive quantity a_i	Supply at any source (machine) will be 1 *i.e.*, $a_i = 1$.
(b)	Demand at any destination may be any positive quantity b_j	Demand at any destination (job) will be 1. *i.e.*, $b_j = 1$
(c)	One or more source to any number of destinations	One source (machine) to only one destination (job).

2.3.4 Assignment Algorithm (or) Hungarian Method

[MU. MBA. Apr 95, BRU. BE. Nov 96, MU. BE. Apr 95, MU. MCA. Nov 95, Nov 98]

First check whether the number of rows is equal to the number of columns, If it is so, the assignment problem is said to be **balanced**. Then proceed to step 1. If it is not balanced, then it should be balanced before applying the algorithm. The method of balancing is discussed in sec 8.6 page 8.15.

Step 1: Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

Step 2: Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1.

Step 3: (Assigning the zeros)

(a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zeros in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

(b) Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

Step 4: (Apply optimal Test)

(a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

(b) If atleast one row/column is without an assignment (i.e., if there is atleast one row/column is without one encircled zero), then the current assignment is not optimal. Go to step 5.

Step 5: Cover all the zeros by drawing a minimum number of straight lines as follows.

(a) Mark (3) the rows that do not have assignment.

(b) Mark (3) the columns (not already marked) that have zeros in marked rows.

(c) Mark (3) the rows (not already marked) that have assignments in marked columns.

(d) Repeat *(b)* and *(c)* until no more marking is required.

(e) Draw lines through all unmarked rows and marked columns. If the number of these lines is equal to the order of the matrix then it is an optimum solution otherwise not.

Step 6: Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

Step 7: Repeat steps (1) to (6), until an optimum assignment is attained.

Note 1: In case some rows or columns contain more than one zero, encircle any unmarked zero arbitrarily and cross all other zeros in its column or row. Proceed in this way until no zero is left unmarked or encircled.

Note 2: The above assignment algorithm is only for minimization problems.

Note 3: If the given assignment problem is of maximization type, convert it to a minimization assignment problem by max Z = – min (–Z) and multiply all the given cost elements by –1 in the cost matrix and then solve by assignment algorithm.

Note 4: Some times, a final cost matrix contains more than required number of zeros at independent positions. This implies that there is more than one optimal solution (multiple optimal solutions) with the same optimum assignment cost.

Example 1 **Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:**

$$\text{Person}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \overset{\textbf{Job}}{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{pmatrix} \end{array}}$$

Determine the optimum assignment schedule.

[MU. BE. Apr 90, Apr 91]

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Step 1: Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix

$$\begin{pmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{pmatrix}$$

Step 2: Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the reduced matrix.

$$\begin{pmatrix} 7 & 3 & 0 & 5 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignments in the reduced matrix.

Step 3: Examine the rows successively until a row with exactly one unmarked zero is found. Since the 2nd row contains a single zero, encircle this zero and cross all other zeros of its column. The 3rd row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 4th row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 1st row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Finally the last row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Like wise examine the columns successively. The assignments in rows and columns in the reduced matrix is given by

$$\begin{pmatrix} 7 & 3 & \cancel{0} & 5 & (0) \\ (0) & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & (0) & 4 \\ 4 & 3 & (0) & \cancel{0} & 3 \\ 4 & (0) & 2 & 4 & \cancel{0} \end{pmatrix}$$

Step 4: Since each row and each column contains exactly one assignment (*i.e.,* exactly one encircled zero) the current assignment is optimal.

∴ The optimum assignment schedule is given by $A \to 5$, $B \to 1$, $C \to 4$, $D \to 3$, $E \to 2$.

The optimum (minimum) assignment cost = (1 + 0 + 2 + 1 + 5) cost units = 9 units of cost.

Example 2 **The processing time in hours for the jobs when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum**

		Machines				
		M_1	**M_2**	**M_3**	**M_4**	**M_5**
	J_1	**9**	**22**	**58**	**11**	**19**
	J_2	**43**	**78**	**72**	**50**	**63**
Jobs	**J_3**	**41**	**28**	**91**	**37**	**45**
	J_4	**74**	**42**	**27**	**49**	**39**
	J_5	**36**	**11**	**57**	**22**	**25**

[MU. MBA. Nov 95]

Solution: The cost matrix of the given problem is

9	22	58	11	19
43	78	72	50	63
41	28	91	37	45
74	42	27	49	39
36	11	57	22	25

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Step 1: Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{pmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{pmatrix}$$

Step 2: Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the following reduced matrix

$$\begin{pmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{pmatrix}$$

Step 3: Now we shall examine the rows successively. Second row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Third row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Fourth row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. After this no row is with exactly one unmarked zero. So go for columns.

Examine the columns successively. Fourth column contains exactly one unmarked zero, encircle this zero and cross all other zeros in its row. After examining all the rows and columns, we get

$$\begin{pmatrix} \cancel{0} & 13 & 49 & (0) & \cancel{0} \\ (0) & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & (0) & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{pmatrix}$$

Step 4: Since the 5th row and 5th column do not have any assignment, the current assignment is not optimal.

Step 5: Cover all the zeros by drawing a minimum number of straight lines as follows:

(a) Mark (3) the rows that do not have assignment. The row 5 is marked.

(b) Mark (3) the columns (not already marked) that have zeros in marked rows. Thus column 2 is marked.

(c) Mark (3) the rows (not already marked) that have assignments in marked columns. Thus row 3 is marked.

(d) Repeat *(b)* and *(c)* until no more marking is required. In the present case this repetition is not necessary.

(e) Draw lines through all unmarked rows (rows 1, 2 and 4). and marked columns (column 2). We get

$$\begin{pmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & (4) \end{pmatrix}$$

Step 6: Here 4 is the smallest element not covered by these straight lines. Subtract this 4 from all the uncovered elements and add this 4 to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines, we get the following matrix.

$$\begin{pmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we examine the rows and columns successively, *i.e.*, repeat step 3 above, we get

$$\begin{pmatrix} \cancel{0} & 17 & 49 & (0) & \cancel{0} \\ (0) & 39 & 29 & 5 & 10 \\ 9 & (0) & 59 & 3 & 3 \\ 47 & 19 & (0) & 20 & 2 \\ 21 & \cancel{0} & 42 & 5 & (0) \end{pmatrix}$$

In the above matrix, each row and each column contains exactly one assignment (*i.e.*, exactly one encircled zero), therefore the current assignment is optimal.

∴ The optimum assignment schedule is $J_1 \rightarrow M_4$, $J_2 \rightarrow M_1$, $J_3 \rightarrow M_2$, $J_4 \rightarrow M_3$, $J_5 \rightarrow M_5$ and the optimum (minimum) processing time

$= 11 + 43 + 28 + 27 + 25$ hours

$= 134$ hours.

Example 3 **Four different jobs can be done on four different machines. The set up and take down time costs are assumed to be prohibitively high for changeovers. The matrix below gives the cost in rupees of processing job *i* on machine *j*.**

		Machines			
		M_1	M_2	M_3	M_4
Jobs	J_1	5	7	11	6
	J_2	8	5	9	6
	J_3	4	7	10	7
	J_4	10	4	8	3

How should the jobs be assigned to the various machines so that the total cost is minimized? *[MU. BE. Nov 92]*

Solution: The assignment problem is given by the cost matrix

$$\begin{pmatrix} 5 & 7 & 11 & 6 \\ 8 & 5 & 9 & 6 \\ 4 & 7 & 10 & 7 \\ 10 & 4 & 8 & 3 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix

$$\begin{pmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the following reduced matrix

$$\begin{pmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns of this reduced matrix.

Examine the rows successively until a row with exactly one unmarked zero is found. The first row contains exactly one unmarked zero, encircle this zero and cross all other zeros of its column. The fourth row contains exactly one unmarked zero, encircle this zero and cross all other zeros in its column. The 2nd column contains exactly one unmarked zero, encircle this zero and cross all other zeros in its row. We get

$$\begin{pmatrix} (0) & 2 & 2 & 1 \\ 3 & (0) & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & (0) \end{pmatrix}$$

Since there are some rows and columns without assignment (*i.e.*, without encircled zero), the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

We get

$$\begin{pmatrix} 0 & 2 & 2 & (1) \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{pmatrix}$$

Here 1 is the smallest cost element not covered by these straight lines. Add this 1 to those elements which lie in the intersection of these straight lines, subtract this 1 from all the uncovered elements and do not change the remaining elements which lie on the straight lines we get

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 8 & 1 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignment in the rows and columns of this reduced matrix. We get

$$\begin{pmatrix} \cancel{0} & 1 & 1 & \cancel{0} \\ 4 & (0) & \cancel{0} & 1 \\ (0) & 2 & 1 & 2 \\ 8 & 1 & 1 & (0) \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

We get

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 8 & (1) & 1 & 0 \end{pmatrix}$$

Here 1 is the smallest cost element not covered by these straight lines. Subtract this 1 from all the uncovered elements, add this 1 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 8 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in the rows and columns of this reduced matrix. We get

$$\begin{pmatrix} (0) & 0 & 0 & 0 \\ 5 & (0) & 0 & 2 \\ 0 & 1 & (0) & 2 \\ 8 & 0 & 0 & (0) \end{pmatrix}$$

Since each row and each column contains exactly one assignment (*i.e.*, exactly one encircled zero) the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $J_1 \to M_1$, $J_2 \to M_2$, $J_3 \to M_3$, $J_4 \to M_4$ and the optimum (minimum) assignment cost

$= \text{Rs. } (5 + 5 + 10 + 3) = \text{Rs. } 23$

Example 4 **The assignment cost of assigning any one operator to any one machine is given in the following table**

		Operators			
		I	**II**	**III**	**IV**
	A	**10**	**5**	**13**	**15**
Machine	**B**	**3**	**9**	**18**	**3**
	C	**10**	**7**	**3**	**2**
	D	**5**	**11**	**9**	**7**

Find the optimal assignment by Hungarian method.

[BNU. BE. Nov 96]

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} 10 & 5 & 13 & 15 \\ 3 & 9 & 18 & 3 \\ 10 & 7 & 3 & 2 \\ 5 & 11 & 9 & 7 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{pmatrix} 5 & 0 & 8 & 10 \\ 0 & 6 & 15 & 0 \\ 8 & 5 & 1 & 0 \\ 0 & 6 & 4 & 2 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the reduced matrix

$$\begin{pmatrix} 5 & 0 & 7 & 10 \\ 0 & 6 & 14 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 6 & 3 & 2 \end{pmatrix}$$

Since each row and each column contain atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix.

$$\begin{pmatrix} 5 & (0) & 7 & 10 \\ 0 & 6 & 14 & (0) \\ 8 & 5 & (0) & 0 \\ (0) & 6 & 3 & 2 \end{pmatrix}$$

Since each row and each column contain exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

The optimum assignment schedule is

$A \to II$, $B \to IV$, $C \to III$, $D \to I$

and the optimum (minimum) assignment cost

$= \text{Rs. } (5 + 3 + 3 + 5) = \text{Rs. } 16/-$

2.3.5 Unbalanced Assignment Models

If the number of rows is not equal to the number columns in the cost matrix of the given assignment problem, then the given assignment problem is said to be unbalanced.

First convert the unbalanced assignment problem in to a balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depending upon whether $m < n$ or $m > n$ and then solve by the usual method.

Example 1 **A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.**

Machines

		1	2	3	4
	A	**18**	**24**	**28**	**32**
Jobs	**B**	**8**	**13**	**17**	**19**
	C	**10**	**15**	**19**	**22**

What are job assignments which will minimize the cost?

[MU. BE. 1977]

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job D (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column), we get the reduced matrix

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this reduced matrix, we shall make the assignment in rows and columns having single zero. We have

$$\begin{pmatrix} (0) & 6 & 10 & 14 \\ \cancel{0} & 5 & 9 & 11 \\ \cancel{0} & 5 & 9 & 12 \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover the all zeros by drawing a minimum number of straight lines. Choose the smallest cost element not covered by these straight lines.

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & (5) & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here 5 is the smallest cost element not covered by these straight lines. Subtract this 5 from all the uncovered elements, add this 5 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignment in the rows and columns having single zero. We get

$$\begin{pmatrix} (0) & 1 & 5 & 9 \\ \cancel{0} & (0) & 4 & 6 \\ \cancel{0} & \cancel{0} & 4 & 7 \\ 5 & \cancel{0} & (0) & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & (4) & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

Choose the smallest cost element not covered by these straight lines, subtract this from all the uncovered elements, add this to those elements which are in the intersection of the lines and do not change the remaining elements which lie on these straight lines. Thus we get

$$\begin{pmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in the rows and columns having single zero. We get

$$\begin{pmatrix} (0) & 1 & 1 & 5 \\ \cancel{0} & (0) & \cancel{0} & 2 \\ \cancel{0} & \cancel{0} & (0) & 3 \\ 9 & 4 & \cancel{0} & (0) \end{pmatrix}$$

Since each row and each column contains exactly one assignment (*i.e.,* exactly one encircled zero) the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \to 1$, $B \to 2$, $C \to 3$, $D \to 4$ and the optimum (minimum) assignment cost

$= (18 + 13 + 19 + 0)$ cost units $= 50$/- units of cost

Note 1: For this problem, the alternative optimum schedule is $A \to 1$, $B \to 3$, $C \to 2$, $D \to 4$, with the same optimum assignment cost = Rs. $(18 + 17 + 15 + 0) = 50$/- units of cost.

Note 2: Here the assignment $D \to 4$ means that the dummy Job D is assigned to the 4th Machine. It means that machine 4 is left without any assignment.

Example 2 **Assign four trucks 1,2,3 and 4 to vacant spaces A, B, C, D, E and F so that the distance travelled is minimized. The matrix below shows the distance.**

	1	2	3	4
A	4	7	3	7
B	8	2	5	5
C	4	9	6	9
D	7	5	4	8
E	6	3	5	4
F	6	8	7	3

Solution: The matrix of the assignment problem is

$$\begin{pmatrix} 4 & 7 & 3 & 7 \\ 8 & 2 & 5 & 5 \\ 4 & 9 & 6 & 9 \\ 7 & 5 & 4 & 8 \\ 6 & 3 & 5 & 4 \\ 6 & 8 & 7 & 3 \end{pmatrix}$$

Since the number of rows is more than the number of columns, the given assignment problem is unbalanced. To make it balanced, let us introduce two dummy trucks (columns) with zero costs. We get

$$\begin{pmatrix} 4 & 7 & 3 & 7 & 0 & 0 \\ 8 & 2 & 5 & 5 & 0 & 0 \\ 4 & 9 & 6 & 9 & 0 & 0 \\ 7 & 5 & 4 & 8 & 0 & 0 \\ 6 & 3 & 5 & 4 & 0 & 0 \\ 6 & 8 & 7 & 3 & 0 & 0 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column). We get

$$\begin{pmatrix} 0 & 5 & 0 & 4 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 \\ 0 & 7 & 3 & 6 & 0 & 0 \\ 3 & 3 & 1 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} \cancel{0} & 5 & (0) & 4 & \cancel{0} & \cancel{0} \\ 4 & (0) & 2 & 2 & \cancel{0} & \cancel{0} \\ (0) & 7 & 3 & 6 & \cancel{0} & \cancel{0} \\ 3 & 3 & 1 & 5 & (0) & \cancel{0} \\ 2 & 1 & 2 & 1 & \cancel{0} & (0) \\ 2 & 6 & 4 & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since each row and each column contains exactly one assignment (*i.e.,* exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \to 3$, $B \to 2$, $C \to 1$, $D \to 5$, $E \to 6$, $F \to 4$, and the optimum (minimum) distance.

$$= (3 + 2 + 4 + 0 + 0 + 3)$$

units of distance = 12/- units of distance.

Example 3 **A batch of 4 jobs can be assigned to 5 different machines. The set up time (in hours) for each job on various machines is given below:**

		Machine				
		1	**2**	**3**	**4**	**5**
	1	**10**	**11**	**4**	**2**	**8**
	2	**7**	**11**	**10**	**14**	**12**
Job	**3**	**5**	**6**	**9**	**12**	**14**
	4	**13**	**15**	**11**	**10**	**7**

Find an optimal assignment of jobs to machines which will minimize the total set up time.

[BRU. BE. Nov 96, BNU. BE. Nov 96, MSU. BE. Nov 97]

Solution: The matrix of the given assignment problem is

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job 5 (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column), we get the reduced cost matrix.

$$\begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each columns contains atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix

$$\begin{pmatrix} 8 & 9 & 2 & (0) & 6 \\ (0) & 4 & 3 & 7 & 5 \\ \cancel{0} & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & (0) \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns with out assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & (1) & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \checkmark \\ \checkmark \\ \\ \\ \end{matrix}$$

Here 1 is the smallest cost element not covered by these straight lines. Subtract this 1 from all the uncovered elements, add this 1 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$\begin{pmatrix} 9 & 9 & 2 & 0 & 6 \\ 0 & 3 & 2 & 6 & 4 \\ 0 & 0 & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix

$$\begin{pmatrix} 9 & 9 & 2 & (0) & 6 \\ (0) & 3 & 2 & 6 & 4 \\ \cancel{0} & (0) & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & (0) \\ 1 & \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by Job 1 $\rightarrow$ M/c 4, Job 2 $\rightarrow$ M/c 1, Job 3 $\rightarrow$ M/c 2, Job 4 $\rightarrow$ M/c 5. M/c 3 is left without any assignment.

The optimum (minimum) total set up time

$$= 2 + 7 + 6 + 7 \text{ hours}$$
$$= 22 \text{ hours.}$$

2.3.6 Maximization case in Assignment Problems

In an assignment problem, we may have to deal with maximization of an objective function. For example, we may have to assign persons to jobs in such a way that the total profit is maximized. The maximization problem has to be converted into an equivalent minimization problem and then solved by the usual Hungarian Method.

The conversion of the maximization problem into an equivalent minimization problem can be done by any one of the following methods:

(i) Since max Z = – min (–Z), multiply all the cost elements c_{ij} of the cost matrix by –1.

(ii) Subtract all the cost elements c_{ij} of the cost matrix from the highest cost element in that cost matrix.

$\boxed{\text{Example 1}}$ **A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below:**

		Districts			
		1	2	3	4
	A	16	10	14	11
Salesmen	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesmen to various districts which will yield maximum profit. ***[MU. BE. Nov 93]***

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} (16) & 10 & 14 & 11 \\ 14 & 11 & 15 & 15 \\ 15 & 15 & 13 & 12 \\ 13 & 12 & 14 & 15 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost element 16 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced cost matrix

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} (0) & 6 & 2 & 5 \\ 1 & 4 & (0) & \cancel{0} \\ \cancel{0} & (0) & 2 & 3 \\ 2 & 3 & 1 & (0) \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by A $\rightarrow$ 1, B $\rightarrow$ 3, C $\rightarrow$ 2, D $\rightarrow$ 4 and the optimum (maximum) profit

$$= \text{Rs. } (16 + 15 + 15 + 15)$$
$$= \text{Rs. } 61/\text{-}.$$

$\boxed{\textbf{Example 2}}$ **Solve the assignment problem for maximization given the profit matrix (profit in rupees).**

		Machines			
		P	Q	R	S
	A	51	53	54	50
Job	B	47	50	48	50
	C	49	50	60	61
	D	63	64	60	60

[MU. BE. Apr 95]

Solution: The profit matrix of the given assignment problem is

$$\begin{pmatrix} 51 & 53 & 54 & 50 \\ 47 & 50 & 48 & 50 \\ 49 & 50 & 60 & 61 \\ 63 & (64) & 60 & 60 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the profit elements in the profit matrix from the highest profit element 64 of this profit matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 13 & 11 & 10 & 14 \\ 17 & 14 & 16 & 14 \\ 15 & 14 & 4 & 3 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost in each row and subtract this from all the cost elements of the corresponding row. We get

$$\begin{pmatrix} 3 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 \\ 12 & 11 & 1 & 0 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the cost elements of the corresponding column. We get

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 2 & 0 & 2 & 0 \\ 11 & 11 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 2 & 1 & (0) & 4 \\ 2 & (0) & 2 & \cancel{0} \\ 11 & 11 & 1 & (0) \\ (0) & \cancel{0} & 4 & 4 \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by A $\to$ R, B $\to$ Q, C $\to$ S, D $\to$ P and the optimum (maximum) profit

$$= \text{Rs. } (54 + 50 + 61 + 63)$$

$$= \text{Rs. } 228/-$$

Example 3 **A company is faced with the problem of assigning four different salesman to four territories for promoting its sales. Territories are not equally rich in their sales potential and the salesman also differ in their ability to promote sales. The following table gives the expected annual sales (in thousands of Rs) for each salesman if assigned to various territories. Find the assignment of salesman so as to maximize the annual sales.**

		Territories			
		1	**2**	**3**	**4**
	1	**60**	**50**	**40**	**30**
Salesmen	**2**	**40**	**30**	**20**	**15**
	3	**40**	**20**	**35**	**10**
	4	**30**	**30**	**25**	**20**

[BRU. BE. Apr 95]

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} (60) & 50 & 40 & 30 \\ 40 & 30 & 20 & 15 \\ 40 & 20 & 35 & 10 \\ 30 & 30 & 25 & 20 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost element 60 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 0 & 10 & 20 & 30 \\ 20 & 30 & 40 & 45 \\ 20 & 40 & 25 & 50 \\ 30 & 30 & 35 & 40 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced cost matrix

$$\begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & 10 & 15 & 15 \\ 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns of this reduced cost matrix

$$\begin{pmatrix} (0) & 10 & 15 & 20 \\ \cancel{0} & 10 & 15 & 15 \\ \cancel{0} & 20 & (0) & 20 \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines

$$\begin{array}{c} \begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & (10) & 15 & 15 \\ 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \checkmark \\ \checkmark \\ \\ \\ \end{array} \\ \checkmark \end{array}$$

Here 10 is the smallest cost element not covered by these straight lines. Subtract this 10 from all the uncovered elements, add this 10 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 0 & 0 & 5 & 10 \\ 0 & 0 & 5 & 5 \\ 10 & 20 & 0 & 20 \\ 10 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns of this reduced cost matrix.

$$\begin{pmatrix} (0) & \cancel{0} & 5 & 10 \\ \cancel{0} & (0) & 5 & 5 \\ 10 & 20 & (0) & 20 \\ 10 & \cancel{0} & \cancel{0} & (0) \end{pmatrix}$$

Since each row and each column contains exactly one assignment (*i.e.*, exactly one encircled zero), the current assignment is optimal.

∴ The optimum assignment schedule is given by Salesman 1 → Territory 1, Salesman 2 → Territory 2, Salesman 3 → Territory 3, Salesman 4 → Territory 4.

The optimum (maximum) annual sales

$= 60 + 30 + 35 + 20$ (in thousand of rupees)

$= 145$ (in thousand of rupees)

= Rs. 1,45,000/-.

Note: For this problem, there exists alternative optimal assignment schedule with the same maximum sales Rs. 1,45,000/-.

2.3.7 Restrictions in Assignments

The assignment technique assumes that the problem is free from practical restrictions and any task could be assigned to any facility. But in some cases, it may not be possible to assign a particular task to a particular facility due to space, size of the task, process capability of the facility, technical difficulties or other restrictions. This can be overcome by assigning a very high processing time or cost element (it can be ∞) to the corresponding cell. This cell will be automatically excluded in the assignment because of the unused high time cost associated with it.

$\boxed{\textbf{Example 1}}$ **A machine shop purchased a drilling machine and two lathes of different capacities. The positioning of the machines among 4 possible locations on the shop floor is important from the standard of materials handling. Given the cost estimate per unit time of materials below, determine the optimal location of the machines**

	Location			
	1	**2**	**3**	**4**
Lathe 1	**12**	**9**	**12**	**9**
Drill	**15**	**not suitable**	**13**	**20**
Lathe 2	**4**	**8**	**10**	**6**

[MU. BE. Apr 93]

Solution: Since the drilling machine is not suitable for location 2, the corresponding cost element should be taken as ∞. Thus the cost matrix of the given assignment problem is

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \end{pmatrix}$$

Since the number of rows is less than the number of columns, we add a dummy row (a dummy drilling machine or a dummy lathe 3) with zero cost elements. The cost matrix for the balanced assignment problem is

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Select the smallest cost in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced matrix

$$\begin{pmatrix} 3 & 0 & 3 & 0 \\ 2 & \infty & 0 & 7 \\ 0 & 4 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 3 & (0) & 3 & \cancel{0} \\ 2 & \infty & (0) & 7 \\ (0) & 4 & 6 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & (0) \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by

Lathe 1 $\rightarrow$ Location 2, Drill $\rightarrow$ Location 3,

Lathe 2 $\rightarrow$ Location 1, Dummy drill $\rightarrow$ Location 4

and the optimum (minimum) assignment cost

$= (9 + 13 + 4 + 0)$ unit of cost

$=$ 26/- units of cost.

Note: For this the alternate optimum assignment is

Lathe 1 $\rightarrow$ Location 4, Drill $\rightarrow$ Location 3,

Lathe 2 $\rightarrow$ Location 1, Dummy drill $\rightarrow$ Location 2.

with the same optimum (minimum) assignment cost

$= (9 + 13 + 4 + 0)$ units of cost

$=$ 26/- units of cost.

Example 2 **Five workers are available to work with the machines and the respective costs (in rupees) associated with each worker–machine assignment is given below. A sixth machine is available to replace one of the existing machines and the associated costs are also given below:**

		Machines					
		M_1	M_2	M_3	M_4	M_5	M_6
	W_1	12	3	6	–	5	8
	W_2	4	11	–	5	–	3
Workers	W_3	8	2	10	9	7	5
	W_4	–	7	8	6	12	10
	W_5	5	8	9	4	6	–

(i) **Determine whether the new machine can be accepted ?**

(ii) **Determine also optimal assignment and the associated saving in cost.**

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} 12 & 3 & 6 & \infty & 5 & 8 \\ 4 & 11 & \infty & 5 & \infty & 3 \\ 8 & 2 & 10 & 9 & 7 & 5 \\ \infty & 7 & 8 & 6 & 12 & 10 \\ 5 & 8 & 9 & 4 & 6 & \infty \end{pmatrix}$$

Since the number of rows is less than the number of columns, the given assignment problem is unbalanced. Add a dummy worker W_6 (dummy row) with zero cost elements.

Thus the cost matrix of the balanced assignment problem is

$$\begin{pmatrix} 12 & 3 & 6 & \infty & 5 & 8 \\ 4 & 11 & \infty & 5 & \infty & 3 \\ 8 & 2 & 10 & 9 & 7 & 5 \\ \infty & 7 & 8 & 6 & 12 & 10 \\ 5 & 8 & 9 & 4 & 6 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Select the smallest cost in each row and column and subtract this from all the cost elements of the corresponding row and column of the cost matrix. We get

$$\begin{pmatrix} 9 & 0 & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & 0 \\ 6 & 0 & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & 0 & 6 & 4 \\ 1 & 4 & 5 & 0 & 2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 9 & (0) & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & (0) \\ 6 & \cancel{0} & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & (0) & 6 & 4 \\ 1 & 4 & 5 & \cancel{0} & 2 & \infty \\ (0) & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing minimum number of straight lines.

$$\begin{array}{c c} \begin{pmatrix} 9 & 0 & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & 0 \\ 6 & 0 & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & 0 & 6 & 4 \\ (1) & 4 & 5 & 0 & 2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} \checkmark \\ \\ \checkmark \\ \checkmark \\ \checkmark \\ \\ \end{matrix} \\ \begin{matrix} \quad & \quad & \checkmark & \quad & \checkmark & \quad & \quad \end{matrix} & \end{array}$$

Choose the smallest cost element not covered by these straight lines. Here 1 is such element. Subtract this 1 from all the uncovered elements, add this 1 to those elements which are in the intersection of the straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 8 & 0 & 2 & \infty & 1 & 4 \\ 1 & 9 & \infty & 3 & \infty & 0 \\ 5 & 0 & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & 0 & 5 & 3 \\ 0 & 4 & 4 & 0 & 1 & \infty \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zero.

$$\begin{pmatrix} 8 & (0) & 2 & \infty & 1 & 4 \\ 1 & 9 & \infty & 3 & \infty & (0) \\ 5 & \cancel{0} & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & (0) & 5 & 3 \\ (0) & 4 & 4 & \cancel{0} & 1 & \infty \\ 0 & 1 & (0) & 1 & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing minimum number of straight lines.

$$\begin{array}{c}\begin{pmatrix} 8 & 0 & 2 & \infty & (1) & 4 \\ 1 & 9 & \infty & 3 & \infty & 0 \\ 5 & 0 & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & 0 & 5 & 3 \\ 0 & 4 & 4 & 0 & 1 & \infty \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}\end{array}\begin{array}{c} \checkmark \\ \\ \checkmark \\ \\ \\ \\ \end{array}$$

Subtract the smallest uncovered element 1 from all the uncovered elements, add this 1 to those elements which are in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 7 & 0 & 1 & \infty & 0 & 3 \\ 1 & 10 & \infty & 3 & \infty & 0 \\ 4 & 0 & 6 & 6 & 3 & 1 \\ \infty & 2 & 1 & 0 & 5 & 3 \\ 0 & 5 & 4 & 0 & 1 & \infty \\ 0 & 2 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros.

$$\begin{pmatrix} 7 & \cancel{0} & 1 & \infty & (0) & 3 \\ 1 & 10 & \infty & 3 & \infty & (0) \\ 4 & (0) & 6 & 6 & 3 & 1 \\ \infty & 2 & 1 & (0) & 5 & 3 \\ (0) & 5 & 4 & \cancel{0} & 1 & \infty \\ \cancel{0} & 2 & (0) & 1 & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by

$W_1 \to M_5,\ W_2 \to M_6,\ W_3 \to M_2,\ W_4 \to M_4, W_5 \to M_1, W_6 \to M_3,$

and the optimum (minimum) assignment cost according to this schedule is

$$= \text{Rs. } (5+3+2+6+5+0)$$
$$= \text{Rs. } 21/-$$

Now, if the sixth machine M_6 is not assigned to any of the workers, the given problem reduces to balanced one (deleting the sixth column). Applying the assignment algorithm to this balanced problem (reduced problem),the optimal assignment schedule is given by

$W_1 \to M_5,\ W_2 \to M_1,\ W_3 \to M_2,\ W_4 \to M_3,\ W_5 \to M_4,$

and the optimum (minimum) assignment cost according to this schedule is

$$= \text{Rs. } (5+4+2+8+4)$$
$$= \text{Rs. } 23/-$$

It is clear from the above that the minimum cost is more when there are only five machines. Hence, the sixth machine should be accepted. By accepting this sixth machine the associated saving cost will be Rs. (23 – 21) = Rs. 2.

2.3.8 Travelling Salesman Problem

A salesman normally must visit a number of cities starting from his head quarters. The distance (or time or cost) between every pair of cities are assumed to be known. The problem of finding the shortest distance (or minimum time or minimum cost) if the salesman starts from his headquarters and passes through each city under his Jurisdiction exactly once and returns to the headquarters is called the ***Travelling salesman problem or A Travelling salesperson problem.***

A travelling salesman problem is very similar to the assignment problem with the additional constraints.

(a) The salesman should go through every city exactly once except the starting city (headquarters).

(b) The salesman starts from one city (head quarters) and comes back to that city (headquarters).

(c) Obviously going from any city to the same city directly is not allowed (*i.e.*, no assignments should be made along the diagonal line).

Note 1: Conditions *(a)* and *(b)* are usually called ***route conditions.***

Note 2: If a salesman has to visit n cities, then he will have a total of $(n-1)!$ possible round trips.

Therefore, the necessary basic steps to solve a travelling salesman problem are:

(i) Assigning an infinitely large element (∞) in each of the squares along the diagonal line in the cost matrix.

(ii) Solving the problem as a routine assignment problem.

(iii) Scrutinizing the solution obtained under *(ii)* to see if the "route" conditions are satisfied.

(iv) If not, making adjustments in assignments to satisfy the condition with minimum increase in total cost (*i.e.,* to satisfy route condition "next best solution" may require to be considered).

Example 1 **Solve the following travelling salesman problem**

		To			
		A	**B**	**C**	**D**
	A	–	**46**	**16**	**40**
From	**B**	**41**	–	**50**	**40**
	C	**82**	**32**	–	**60**
	D	**40**	**40**	**36**	–

[MU. BE. Apr 93]

Solution: The cost matrix of the given travelling salesman problem is

$$\begin{pmatrix} \infty & 46 & 16 & 40 \\ 41 & \infty & 50 & 40 \\ 82 & 32 & \infty & 60 \\ 40 & 40 & 36 & \infty \end{pmatrix}$$

Solve this as a routine assignment problem

Subtract the smallest cost element in each row from all the elements of the corresponding row. We get.

$$\begin{pmatrix} \infty & 30 & 0 & 24 \\ 1 & \infty & 10 & 0 \\ 50 & 0 & \infty & 28 \\ 4 & 4 & 0 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each column from all the elements of the corresponding column. We get

$$\begin{pmatrix} \infty & 30 & 0 & 24 \\ 0 & \infty & 10 & 0 \\ 49 & 0 & \infty & 28 \\ 3 & 4 & 0 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} \infty & 30 & (0) & 24 \\ (0) & \infty & 10 & \cancel{0} \\ 49 & (0) & \infty & 28 \\ 3 & 4 & \cancel{0} & \infty \end{pmatrix}$$

Since some rows and columns are with out assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{array}{cc} \begin{pmatrix} \infty & 30 & 0 & 24 \\ 0 & \infty & 10 & 0 \\ 49 & 0 & \infty & 28 \\ (3) & 4 & \cancel{0} & \infty \end{pmatrix} & \begin{array}{c} \checkmark \\ \\ \\ \checkmark \end{array} \\ \begin{array}{cccc} \quad & \quad & \checkmark & \quad \end{array} & \end{array}$$

Subtract the smallest uncovered cost element 3 from all uncovered elements, add this 3 to those elements which are in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We have

$$\begin{pmatrix} \infty & 27 & 0 & 21 \\ 0 & \infty & 13 & 0 \\ 49 & 0 & \infty & 28 \\ 0 & 1 & 0 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros. We get

$$\begin{pmatrix} \infty & 27 & (0) & 21 \\ \cancel{0} & \infty & 13 & (0) \\ 49 & (0) & \infty & 28 \\ (0) & 1 & \cancel{0} & \infty \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal for the assignment problem.

∴ The optimum assignment schedule is given by

$A \to C, \quad B \to D, \quad C \to B, \quad D \to A,$

i.e., $A \to C, \quad C \to B, \quad B \to D, \quad D \to A,$

i.e., $A \to C \to B \to D \to A$

Check whether the route conditions are satisfied.

$A \to C \to B \to D \to A$ satisfies the route condition.

∴ The required minimum costs.

$= (16 + 32 + 40 + 40)$ units of cost.

$=$ 128/- units of cost.

Example 2 **Solve the following travelling salesman problem so as to minimize the cost per cycle.**

			To		
	A	**B**	**C**	**D**	**E**
A	–	3	6	2	3
From B	3	–	5	2	3
C	6	5	–	6	4
D	2	2	6	–	6
E	3	3	4	6	–

[MU. BE. 85, Nov 93]

Solution: The cost matrix of the given travelling salesman problem is

$$\begin{pmatrix} \infty & 3 & 6 & 2 & 3 \\ 3 & \infty & 5 & 2 & 3 \\ 6 & 5 & \infty & 6 & 4 \\ 2 & 2 & 6 & \infty & 6 \\ 3 & 3 & 4 & 6 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each row from all the elements of the corresponding row. We get

$$\begin{pmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each column from all the elements of the corresponding column. We get

$$\begin{pmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros. We get

$$\begin{pmatrix} \infty & 1 & 3 & (0) & 1 \\ 1 & \infty & 2 & \cancel{0} & 1 \\ 2 & 1 & \infty & 2 & (0) \\ \cancel{0} & (0) & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 3 & \infty \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{array}{cc}
\begin{pmatrix}
\infty & (1) & 3 & 0 & 1 \\
1 & \infty & 2 & 0 & 1 \\
2 & 1 & \infty & 2 & 0 \\
0 & 0 & 3 & \infty & 4 \\
0 & 0 & 0 & 3 & \infty
\end{pmatrix} & \begin{matrix} \checkmark \\ \checkmark \\ \\ \\ \\ \end{matrix} \\
\begin{matrix} & & & \checkmark & \end{matrix} &
\end{array}$$

Subtract the smallest uncovered cost element 1 from all uncovered elements, add this 1 to those elements which are in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We have

$$\begin{pmatrix}
\infty & 0 & 2 & 0 & 0 \\
0 & \infty & 1 & 0 & 0 \\
2 & 1 & \infty & 3 & 0 \\
0 & 0 & 3 & \infty & 4 \\
0 & 0 & 0 & 4 & \infty
\end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix}
\infty & \cancel{0} & 2 & (0) & \cancel{0} \\
(0) & \infty & 1 & \cancel{0} & \cancel{0} \\
2 & 1 & \infty & 3 & (0) \\
\cancel{0} & (0) & 3 & \infty & 4 \\
\cancel{0} & \cancel{0} & (0) & 4 & \infty
\end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimal assignment schedule is given by

$$A \to D, \quad B \to A, \quad C \to E, \quad D \to B, \quad E \to C$$

i.e., $A \to D \to B \to A, \quad C \to E \to C$

and the corresponding optimum (minimum) assignment cost

$= (2 + 3 + 4 + 2 + 4)$ units of cost

$=$ 15/- units of cost.

But this assignment schedule does not provide the solution of this travelling salesman problem, because it does not satisfy the 'route' condition.

We try to find the next best solution which satisfies the route condition also. The next minimum (non-zero) cost element in the cost matrix is 1. So we try to bring 1 in to the solution. But the 1 occurs at two places. We shall consider all the cases separately until the acceptable solution is reached.

We start with making an assignment at (2, 3) instead of zero assignment at (2, 1). The resulting feasible solution will then be

$$\begin{pmatrix}
\infty & \cancel{0} & 2 & (0) & \cancel{0} \\
\cancel{0} & \infty & (1) & \cancel{0} & \cancel{0} \\
2 & 1 & \infty & 3 & (0) \\
\cancel{0} & (0) & 3 & \infty & 4 \\
(0) & \cancel{0} & \cancel{0} & 4 & \infty
\end{pmatrix}$$

$\therefore$ The optimum assignment is given by

$$A \to D, \quad B \to C, \quad C \to E, \quad D \to B, \quad E \to A,$$

i.e., $A \to D \to B \to C \to E \to A$

Also, when an assignment is made at (3, 2) instead of zero assignment at (3, 5), the resulting feasible solution will be

$$\begin{pmatrix}
\infty & \cancel{0} & 2 & \cancel{0} & (0) \\
\cancel{0} & \infty & 1 & (0) & \cancel{0} \\
2 & (1) & \infty & 3 & \cancel{0} \\
(0) & \cancel{0} & 3 & \infty & 4 \\
\cancel{0} & \cancel{0} & (0) & 4 & \infty
\end{pmatrix}$$

$\therefore$ The optimum assignment is given by

$$A \to E, \quad B \to D, \quad C \to B, \quad D \to A, \quad E \to C,$$

i.e., $A \to E \to C \to B \to D \to A$

$\therefore$ For the given travelling salesman problem, the optimum assignment schedule is given by

$$A \to D \to B \to C \to E \to A, \text{ (or)}$$

$$A \to E \to C \to B \to D \to A$$

In both cases, the optimum (minimum) assignment cost is 16/- units of cost.

EXERCISE

1. What are assignment problems? Describe Mathematical formulation of an assignment problem?

   [MU. BE. Apr 93, Oct 96, BRU. BE. Nov 96, MU. MBA. Nov 96,]

2. Distinguish between transportation model and assignment model. *[MU. BE. Nov 94]*

3. Explain how the assignment problem can be treated as a particular case of transportation problem? Why this method is not preferred ? *[MU. MBA Nov. 95, Apr.97]*

4. Explain the steps in the Hungarian Method used for solving assignment problems.

   [MU. MBA. Apr 95, BRU. BE. Nov 96, MU. MCA. Nov 95]

5. Define an unbalanced assignment problem and describe the steps involved in solving it.

6. Explain how maximization problems are solved using assignment model technique ?

7. What do you understand by restricted assignments? Explain how should one overcome it ?

8. Enumerate the steps to solve an unbalanced profit maximization problem containing one or more restricted assignments?

9. Is it possible to have more than one optimal solution to an assignment problem? How is the presence of an alternate solution established?

10. What is the difference between assignment problem and travelling salesman problem?

11. What is the objective of the travelling salesman problem ?

    [MU. BE. Oct 96]

12. What is travelling salesman problem ?*[BRU. BE. Nov 96]*

13. **Solve the assignment problem**

	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

[MU. BE. Apr 91]

14. Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:

Jobs

Person	1	2	3	4	5
A	8	5	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

Determine the optimum assignment schedule.

[MU. BE. Apr 90, Apr 91]

15. A department has four subordinates and four tasks are to performed. The subordinates differ in efficiency and tasks differ in their intrinsic difficulties. The estimate of time (in hours) each man would take to perform each task is given by

Tasks

Subordinate	I	II	III	IV
1	8	26	17	11
2	13	28	4	26
3	38	19	18	15
4	19	26	24	10

Find out how the tasks be allotted to each subordinate so as to optimize the total man-hours. *[MU. BE. Nov 91]*

16. Solve the assignment problem

		Job			
		P	Q	R	S
	A	18	26	17	11
Machine	B	13	28	14	26
	C	38	19	18	15
	D	19	26	24	10

[MU. BE. Oct 95]

17. A department head has four tasks to be performed by three subordinates, the subordinates differing in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix. How should he allocate the tasks one to each man, so as to minimize the total man-hour ?

		Men	
	1	2	3
I	9	26	15
Tasks II	13	27	6
III	35	20	15
IV	18	30	20

[MU. BE. Nov 92]

18. A company has four machines on which to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

		Machine			
		P	Q	R	S
	A	18	24	28	32
Job	B	8	13	17	19
	C	10	15	19	22

What are the job assignments which will minimize the cost ? *[MU. BE. Apr 87, MBA. Apr 95]*

19. Solve the following unbalanced problem of assigning four jobs to three different men (only one job to each man). The time to perform the job by different men is given in the following table:

		Job			
		J_1	J_2	J_3	J_4
Men	M_1	7	5	8	4
	M_2	5	6	7	4
	M_3	8	7	9	8

[MU. BE. Nov 92]

20. Solve the following assignment problem to find the maximum total expected sale.

		Area			
		I	II	III	IV
	A	42	35	28	21
Salesman	B	30	25	20	15
	C	30	25	20	15
	D	24	20	16	12

[MU. BE. Nov 90]

21. A sales manager has to assign salesmen to four territories. He has 4 candidates of varying experience and capabilities and assesses the possible profit for each salesman in each territory as given below. Find the assignment which maximises the profit.

		Territory			
		A	B	C	D
	1	35	27	28	37
Salesman	2	28	34	29	40
	3	35	24	32	28
	4	24	32	25	28

[MU. BE. Nov 92]

22. A company has 5 jobs to be done. The following data shows the return (in rupees) by assigning the i^{th} machine to the j^{th} job. Using Hungarian method, assign the 5 jobs to the 5 machines so as to maximize the total expected profit.

Job

Machine	1	2	3	4	5
A	62	78	50	101	82
B	71	84	61	73	59
C	87	92	111	70	81
D	45	64	87	77	80
E	60	70	98	66	83

[MU. BE. Apr 91, Apr 98, BRU. ME. 81, MKU. BE. Nov 97]

23. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as follows:

Machine

Job	A	B	C	D
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

Solve the problem to maximize the total profit.

[MU. BE. Apr 90, Nov 94]

24. Five operators have to be assigned to five machines. The assignment costs are given in the table below:

Machine

Operator	I	II	III	IV	V
A	5	5	–	2	6
B	7	4	2	3	4
C	9	3	5	–	3
D	7	2	6	7	2
E	6	5	7	9	1

Operator A cannot operate machine III and operator C cannot operate machine IV. Find the optimal assignment schedule.

25. Solve the following assignment problem

Task

Machine	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	Not suitable	5	4
M_3	Not suitable	6	9	6	2
M_4	9	3	7	2	3

26. Solve the following assignment problem

Machine

Task	1	2	3	4	5
A	7	7	∞	4	8
B	9	6	4	5	6
C	11	5	7	∞	5
D	9	4	8	9	4
E	8	7	9	11	3

27. Given the following matrix of setup costs show how to sequence production so as to minimize setup cost per cycle.

To

From	A	B	C	D	E
A	–	2	5	7	1
B	6	–	3	8	2
C	8	7	–	4	7
D	12	4	6	–	5

E	1	3	2	8	–

28. Solve the following travelling salesman problem so as to minimize cost per cycle:

To city

From city	1	2	3	4	5
1	∞	10	25	25	10
2	1	∞	10	15	2
3	8	9	∞	20	10
4	14	10	24	∞	15
5	10	8	25	27	∞

29. A salesman has to visit five cities A, B, C, D, and E. The distances (in hundred km) between the five cities are as follows:

To

From	A	B	C	D	E
A	–	7	6	8	4
B	7	–	8	5	6
C	6	8	–	9	7
D	8	5	9	–	8
E	4	6	7	8	–

If the salesman starts from city A and has to come back to city A, which route he should select so that the total distance travelled by him is minimized.

30. Solve the following travelling salesman problem:

	A	B	C	D	E	F
A	∞	5	12	6	4	8
B	6	∞	10	5	4	3
C	8	7	∞	6	3	11
D	5	4	11	∞	5	8
E	5	2	7	8	∞	4

F	6	3	11	5	4	∞

31. Solve the travelling salesman problem given by the following data.

$c_{12} = 20,\quad c_{13} = 4,\quad c_{14} = 10,\quad c_{23} = 5,\quad c_{34} = 6,$

$c_{25} = 10,\quad c_{35} = 6,\quad c_{45} = 20,\quad \text{where}\quad c_{ij} = c_{ji}$

and there is no route between cities i and j if a value for c_{ij} is not shown above.

32. A company has five jobs to be done one five machines ; any job can be done on any machine. The costs of doing the jobs in different machines are given below. Assign the jobs for different machines so as to minimize the total cost.

Machines

Jobs	A	B	C	D	E
I	13	8	16	18	19
II	9	15	24	9	12
III	12	9	4	4	4
IV	6	12	10	8	13
V	15	17	18	12	20

[MU. BE. Oct 96]

33. The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows:

Jobs

Machinist	A	B	C	D	E
M_1	12	28	0	51	32
M_2	12	34	11	23	9
M_3	37	42	61	21	31
M_4	0	14	37	27	30

Assign machinists to jobs which results in overall maximum profit. Which job should be declined ?

[MU. MBA. Nov 96]

34. **A machine tool company decides to make four sub assemblies through four contractors. Each contractor is to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in table given below in hundreds of rupees. Assign the different subassemblies to contractors so as to minimize the total cost.**

Table

		Contractors			
		1	2	3	4
	1	15	13	14	17
Subassemblies	2	11	12	15	13
	3	13	12	10	11
	4	15	17	14	16

Assign machinists to jobs which results in overall maximum profit. Which job should be declined ?

[MU. B. Tech. Leather. Oct 96]

35. **The R and D company has recently requested the skill testing agency to test four applicants for the three jobs that are available at this time. Each job has a primary skill and R and D's objective is to pick the three applicants whose aptitude test scores will maximize R and D's total performance. Only one applicant can be assigned to each job. Their aptitude test scores is listed below:**

		Job		
		A	B	C
	1	95	110	103
Applicant	2	89	95	100
	3	120	132	118
	4	107	119	112

Determine the three best applicants for the three jobs. What are their total aptitude test scores ? *[BRU. BE. Apr 96]*

ANSWERS

13. $1 \longrightarrow A, 2 \longrightarrow C, 3 \longrightarrow B, 4 \longrightarrow D$,
minimum cost Rs. 21.
14. $A \longrightarrow 5, B \longrightarrow 1, C \longrightarrow 4, D \longrightarrow 3, E \longrightarrow 2$,
minimum cost Rs. 9.
15. $1 \longrightarrow I, 2 \longrightarrow III, 3 \longrightarrow II, 4 \longrightarrow IV$,
minimum time 41 hours.
16. $A \longrightarrow R, B \longrightarrow P, C \longrightarrow Q, D \longrightarrow S$,
minimum cost Rs. 59.
17. $I \longrightarrow 1, II \longrightarrow 3, III \longrightarrow 2, IV \longrightarrow 4$
minimum time 35 hours.
18. $A \longrightarrow P, B \longrightarrow Q, C \longrightarrow R, D \longrightarrow S$,
minimum cost Rs. 50.
19. $M_1 \longrightarrow J_4, M_2 \longrightarrow J_1, M_3 \longrightarrow J_2, M_4 \longrightarrow J_3$,
minimum cost Rs. 16.
20. $A \longrightarrow I, B \longrightarrow II, C \longrightarrow III, D \longrightarrow IV$, (or)
$A \longrightarrow I, B \longrightarrow III, C \longrightarrow II, D \longrightarrow IV$,
maximum total expected sale = Rs. 99000.
21. $1 \longrightarrow A, 2 \longrightarrow D, 3 \longrightarrow C, 4 \longrightarrow B$,
maximum profit Rs. 139.
22. $A \longrightarrow 4, B \longrightarrow 2, C \longrightarrow 1, D \longrightarrow 5, E \longrightarrow 3$,
maximum profit is Rs. 450.
23. $2 \longrightarrow A, 3 \longrightarrow B, 4 \longrightarrow D, 6 \longrightarrow C$
maximum profit is Rs. 28.
24. $A \longrightarrow IV, B \longrightarrow III, C \longrightarrow II, D \longrightarrow I, E \longrightarrow V$, (or)
$A \longrightarrow IV, B \longrightarrow III, C \longrightarrow V, D \longrightarrow II, E \longrightarrow I$,
minimum cost Rs. 15.

25. $M_1 \longrightarrow A, M_2 \longrightarrow B, M_3 \longrightarrow E, M_4 \longrightarrow D, M_5 \longrightarrow C$

minimum assignment cost is Rs. 12.

26. $A \longrightarrow 4, B \longrightarrow 3, C \longrightarrow 2, D \longrightarrow 1, E \longrightarrow 5$

minimum assignment cost is Rs. 25.

27. $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow A$

minimum cost is Rs. 15.

28. $1 \longrightarrow 5 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$

minimum cost is Rs. 62.

29. $A \longrightarrow E \longrightarrow B \longrightarrow D \longrightarrow C \longrightarrow A$

minimum distance is Rs. 30 (in hundred km).

30. $A \longrightarrow B \longrightarrow F \longrightarrow E \longrightarrow C \longrightarrow D \longrightarrow A$

minimum cost is Rs. 30.

31. $A \longrightarrow C \longrightarrow B \longrightarrow D \longrightarrow A$

minimum cost is Rs. 19.

32. I $\longrightarrow$ B, II $\longrightarrow$ E, III $\longrightarrow$ C, IV $\longrightarrow$ A, V $\longrightarrow$ D

minimum cost is Rs. 42/-.

33. $M_1 \longrightarrow D, M_2 \longrightarrow B, M_3 \longrightarrow C, M_4 \longrightarrow E, M_5 \longrightarrow A$

The maximum profit is Rs. 176/-. Job A should be declined.

34. $1 \longrightarrow 2,\ 2 \longrightarrow 1,\ 3 \longrightarrow 4,\ 4 \longrightarrow 3$, and the minimum assignment cost is Rs. 49/-.

35. Applicant 1 $\rightarrow$ Job 3, Applicant 3 $\rightarrow$ Job 1, Applicant 4 $\rightarrow$ Job 2, and their total aptitude test scores = 103 + 120 + 119 = 342.

2.4. NETWORKS

2.4.1 SHORTEST ROUTE PROBLEMS

The problem if determming the shortest route be turen a source and a destination is called THE SHORTEST ROUTE PROBLEM. Many practical situations like transportation, equipment replacement, communication network, production planning can be represented by the same model.

Example 1 **Consider the details of a distance network an shown below:**

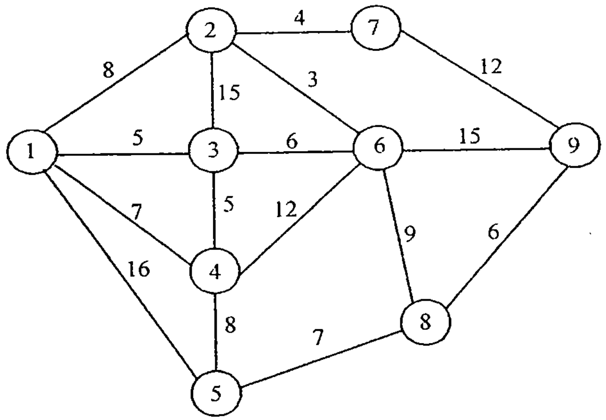
Are	1 – 2	1 – 3	1 – 4	1 – 5	2 – 3	2 – 6	2 – 7	3 – 4
Distance	8	5	7	16	15	3	4	5
Are	3 – 6	4 – 5	4 – 6	5 – 8	6 – 8	6 – 9	7 – 9	8 – 9
Distance	6	8	12	7	9	15	12	6

(i) **Construct the distant network.**

(ii) **Find the shortest path from node 1 to the node 9 and its length**

Solution:

(*i*)



(iii) Shortest path in 1 – 2 – 7 – 9 by inspection. The length of the shortest path = 24 units.

There are standard algorithms for solving the shortest route problems. Two of them will be considered now. They are

(1) Dijksra's algorithm

(2) Floyd's algorithm

(3) (3) Systematic method algorithm

2.4.2 Dijkstra's Shortest Path Algorithm :

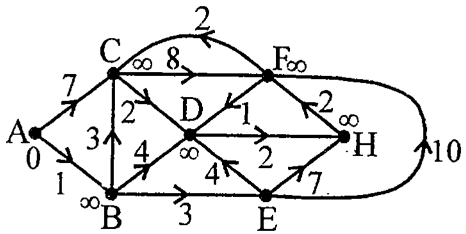
Dijkstra's Algorithm seems to be the most efficient one among several algorithms proposed for the shortest path between a specified vertex pair.

Let the problem be to find the shortest path from vertex a to vertex b in a connected weighted graph.

Let $w(i,j) > 0$ denote the weight of the edge (i,j) of the graph and let the label of any vertex $x \in G$ be denoted by $L(x)$

1. Initialisation. Set $L(a) = 0$. For all vertices $x \neq a$, set $L(x) = \infty$. Let T be the set of vertices.
2. If $b \in T$ stop. $L(b)$ is the length of the shortest path from a to b.
3. Get new vertex. Choose $v \in T$ with the smallest value of $L(v)$ Set$T := T - \{v\}$.
4. Revise Labels. For each vertex $v \in$ T adjacent to v, set $L(x) = \min \{L(x), L(v) + w(v,x)\}$. Go to step 2.

Example 2 **Use Dijkstra's shortest path algorithm to determine the shortest path between A and H in the following graph.**



Solution:

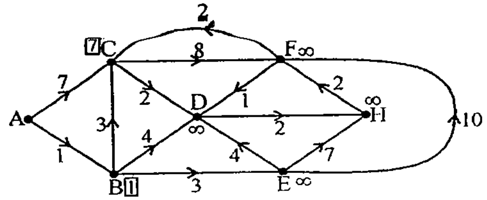
$L(A) = 0$

$L(B) = \min (\infty, 0 + 1)$

$= 1$

$L(C) = \min (\infty, 0 + 7)$

Since H is not circled, select the vertex B and circle it.

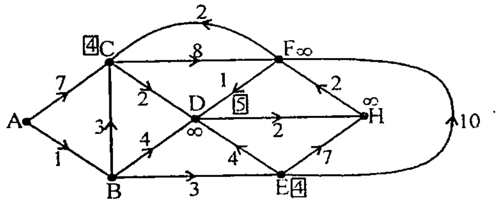


$L(E) = \min (\infty, 1 + 3) = 4$

$L(C) = \min (7, 1 + 3) = 4$

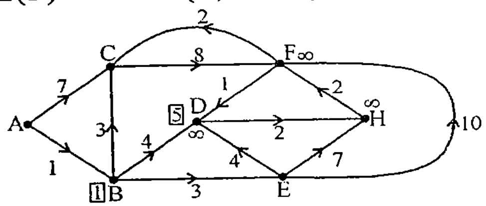
$L(D) = \min (\infty, 1 + 4) = 5$

Select E or C, say C



$L(D) = \text{Min } \{5, 6\} = 5$ Select D.

$L(F) = \text{Mm } \{4, 4 + 8\}$

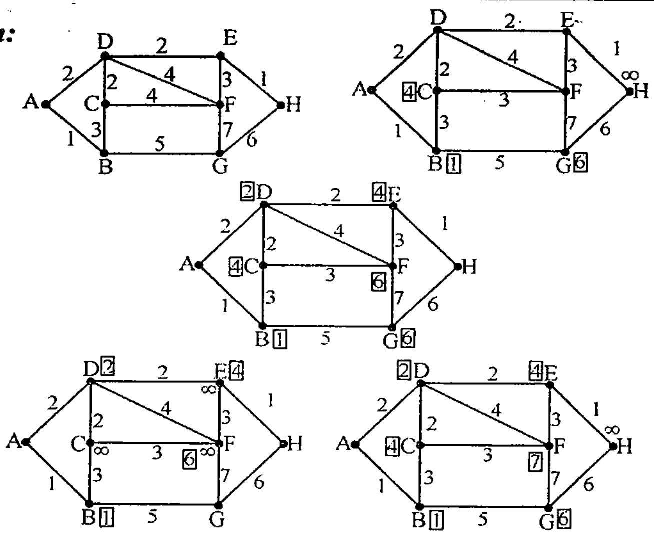


$L(H) = \text{Min } (\infty, 5 + 2) = 7$ We stop at step 2. the process terminates.

$\therefore$ Length of the shortest path = 7 and the shortest path $A \to B \to D \to H$

Example 3 **Find the length of the shortest path from the verte A to H by using Dijkstra's shortest Path Algorithm.**

Solution:



The length of the shortest path is 5.The path is $A \to D \to E \to H$.

2.4.3 Floyd's Algorithm

This algorithm is more general than Dijkstha's algorithm and is used to find the shortest path and the corresponding distance from any source node to any destination node.

We form the initial distance matrix $D^{(O)}$ and initial precedence matrix $P^{(D)}$. Then we perform 'n' iterations where n is the number of nodes in the distance matrix. Also we generate the final distance matrix $D^{(n)}$ and the final precedence matrix $P^{(n)}$. Now we can get the shortest distance between any two nodes, and the corresponding path from these final matrices,

Floyd's algorithm

Step 1 Set $\lambda = 0$

Step 2 From the distance network construct the initial distance matrix $D^{(O)}$ and the initial precedence matrix $P^{(O)}$ as follows:

(i) The leading diagonal values of the initial matrix are assumed as zero.

(ii) The distance between any two nodes is assumed to be ∞ if there is no direct arc.

(iii) Other entries are the distances different nodes between as given in the network.

(iv) The leading diagonal values are assumed as '–'. Other entries in the i^{th} row are assumed to be i, for I = 1, 2, 4 ... , n.

Step 3 Set $\lambda = \lambda + 1$

Step 4 Using the following formula obtain the values of the distance matrix $D^{(\lambda)}$ for all the cells where $i \neq j$,

$$D_{ij}^{(\lambda)} = \min\left[D_{ij}^{(\lambda-1)}, \left(D_{i\lambda}^{(\lambda-1)} + D_{\lambda j}^{(\lambda-1)}\right)\right]$$

Step 5 Compute the values of all the cells of the precedence matrix $P^{(\lambda)}$ when $i \neq j$ using the formula

$$P_{ij}^{(\lambda)} = \min P_{ij}^{(\lambda-1)} \;\; if \left(D_{ij}^{(\lambda)} \neq D_{ij}^{(\lambda-1)}\right)$$

$$= P_{ij}^{(\lambda-1)} \text{ otherwise}$$

Step 6 If $\lambda = n$ go to step 7, otherwise set $\lambda = \lambda + 1$ and go to step 4.

Step 7 To trace the shortest path for a given combination of source node and destination mole proceed as follows using the final distance and final precedence matrices.

7.1 Let the source mode be A and the destination mode be ƶ.

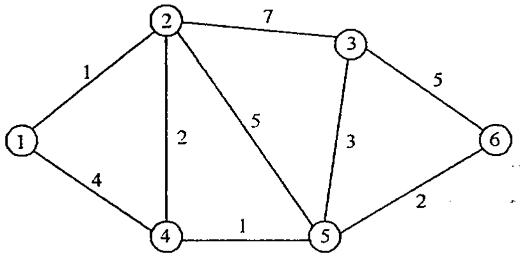
7.2 Fix node ƶ as the last node in the partially formed shortest path.

7.3 Find the value from the final precedence matrix $P^{(n)}$ for the row corresponding the node A and the column corresponding to node ƶ. Let it be W. prefix node W to the partially formed shortest path.

7.4 Check whether W is equal to A If not so, set ƶ = Wand go to step 7.3, otherwise go to step 7.5

7.5 The path constructed is the required path from the source node a to the destination node ƶ.

Example 4 **Consider the distance network as shown below**



(i) Apply Floyd's algorithm to it and generate the final distance matrix and precedence matrix.

(ii) Find the shortest path from the source mode 1 to the destination node 6 as well as the shortest path from node 5 to node 1.

Solution:

Using Floyd's algorithm we get the following tables:

Iteration D

$$D^{(0)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&1&\infty&4&\infty&\infty\\1&0&7&2&5&\infty\\\infty&7&0&\infty&3&5\\4&2&\infty&0&1&\infty\\\infty&5&3&1&0&2\\\infty&\infty&5&\infty&2&0\end{bmatrix}\end{array} \qquad P^{(0)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}-&1&1&1&1&1\\2&-&2&2&2&2\\3&3&-&3&3&3\\4&4&4&-&4&4\\5&5&5&5&-&5\\6&6&6&6&6&-\end{bmatrix}\end{array}$$

Iteration 1

$$D^{(1)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&1&\infty&4&\infty&\infty\\1&0&7&2&5&\infty\\\infty&7&0&\infty&3&5\\4&2&\infty&0&1&\infty\\\infty&5&3&1&0&2\\\infty&\infty&5&\infty&2&0\end{bmatrix}\end{array} \qquad P^{(1)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}-&1&1&1&1&1\\2&-&2&2&2&2\\3&3&-&3&3&3\\4&4&4&-&4&4\\5&5&5&5&-&5\\6&6&6&6&6&-\end{bmatrix}\end{array}$$

Iteration 2

$$D^{(2)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&1&8&3&6&\infty\\1&0&7&2&5&\infty\\8&7&0&9&3&5\\3&2&9&0&1&\infty\\6&5&3&1&0&2\\\infty&\infty&5&\infty&2&0\end{bmatrix}\end{array} \qquad P^{(2)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}-&1&2&2&2&1\\2&-&2&2&2&2\\3&3&-&2&3&3\\4&4&4&-&4&4\\2&5&5&5&-&5\\6&6&6&6&6&-\end{bmatrix}\end{array}$$

Iteration 3

$$D^{(3)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&1&8&3&6&13\\1&0&7&2&5&12\\8&7&0&9&3&5\\3&2&9&0&1&14\\6&5&3&1&0&2\\13&12&5&14&2&0\end{bmatrix}\end{array} \qquad P^{(3)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}-&1&2&2&2&3\\2&-&2&2&2&3\\3&3&-&2&3&3\\4&4&4&-&4&3\\2&5&5&5&-&5\\3&3&6&2&2&-\end{bmatrix}\end{array}$$

Iteration 4

$$D^{(4)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}0&1&8&3&6&13\\1&0&7&2&3&13\\8&7&0&9&3&5\\3&2&9&0&1&14\\4&3&3&1&0&2\\13&12&5&14&2&0\end{bmatrix}\end{array} \qquad P^{(4)} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}-&1&2&2&4&3\\2&-&2&2&4&3\\3&3&-&2&3&3\\4&4&4&-&4&3\\4&4&5&5&-&5\\6&6&6&6&6&-\end{bmatrix}\end{array}$$

Iteration 5

$$D^{(5)} \qquad\qquad\qquad\qquad P^{(5)}$$

$$\begin{array}{cc} & \begin{matrix}1&2&3&4&5&6\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} & \begin{bmatrix}0&1&7&3&4&6\\1&0&6&2&3&5\\7&6&0&4&3&5\\3&2&4&0&1&3\\4&3&3&1&0&2\\6&5&5&3&2&0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}1&2&3&4&5&6\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} & \begin{bmatrix}-&1&5&2&4&5\\2&-&5&2&2&5\\4&4&-&5&3&3\\4&4&5&-&4&5\\4&4&5&5&-&5\\4&4&6&5&6&-\end{bmatrix}\end{array}$$

Iteration 6

$$D^{(6)} \qquad\qquad\qquad\qquad P^{(6)}$$

$$\begin{array}{cc} & \begin{matrix}1&2&3&4&5&6\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} & \begin{bmatrix}0&1&7&3&4&6\\1&0&7&2&5&5\\7&6&0&4&3&5\\3&2&4&0&1&3\\4&3&3&1&0&2\\6&5&5&3&2&0\end{bmatrix}\end{array} \qquad \begin{array}{cc} & \begin{matrix}1&2&3&4&5&6\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} & \begin{bmatrix}-&1&5&2&4&5\\2&-&6&2&2&5\\4&4&-&5&3&3\\4&4&5&-&4&5\\4&4&5&5&-&5\\4&4&6&5&6&-\end{bmatrix}\end{array}$$

Final Distance Matrix Final Precedence Matrix

(ii) The shorts path from model to mode 6 is

1 – 2 – 4 – 5 – 6

Length is 6 units.

The shortest path from mode 5 to mode 1 is 5 – 4 – 2 – 1

Its length is 4 units

2.4.4 Systematic Method Algorithm

The algorithm of the systematic method determining the shortest path between two given nodes is as follows:

Step 1 : Provide a column for each of the nodes in the distance network.

Step 2 : Set the cumulative distance covered up to source node say 1 as zero.

Step 3 : Delete all the arcs in the table that are pointing towards 1.

Step 4 : Include the recently selected node in List L of another table.

Step 5 : For each node in List L of the table in step 4 calculate the cumulative distance upto the nearest node as shown in the last column of the second table. Then select the nearest node which has the least cumulative distance and mark a circle around it in the third column of the second table.

Step 6 : Write the cumulative distance covered upto the selected node at the top of the respective column of the first table.

Step 7 : If the recently node is the same as the required destination, go to step 9. Otherwise goto step 8.

Step 8 : Delete all arcs in the first table that are pointing towards the recently selected node. Go to step 4.

Step 9 : The selected node is the last node in the shortest path.

Step 10: Find the node in List L, corresponding to the recently selected prefixed node and prefix the node in the partially formed shortest path.

Step 11: If the prefixed node is the required source node, stop. If not, go to step 12. Otherwise go to step 13.

Step 12: Go to the iteration in which the recently prefixed node is selected (circled) in the third column of the second table. Then go to step 10.

Step 13: The path which is constructed based on the above guidelines is the shortest path. The shortest distance is equal to the cumulative distance in the last iteration of the second table.

Example

Consider the details of the network as shown below :

Arc	Distance
1-2	8
1-3	5
1-4	7

1-5	16
2-3	15
2-6	3
2-7	4
3-4	5
3-6	6
4-5	8
4-6	12
5-8	7
6-8	9
6-9	15
7-9	12
8-9	6

(i) Construct the distance network

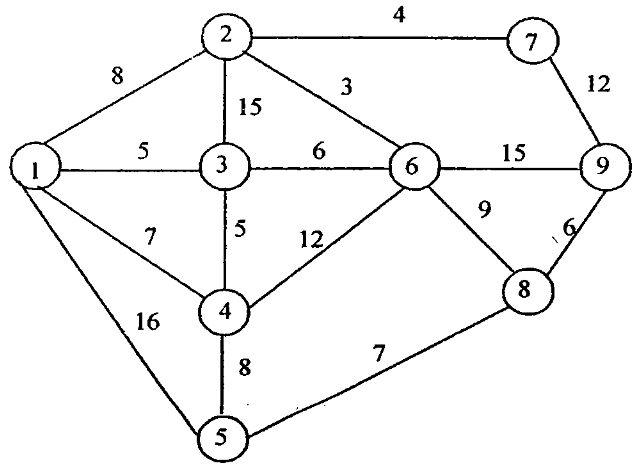
(ii) Find the shortest path from the node 1 to node 9 using the systematic method. (B.E./B.Tech.May/June 2007Anna University)

Solution

Source node : 1

Destination node : 9

(i)



(ii) Step 1

Iteration 1 **Distance table 1.1**

1		2		3		4		5	
1 – 3	5	2 – 6	3	3 – 1	5	4 – 3	5	5 – 8	7
1 – 4	7	2 – 7	4	3 – 4	5	4 – 1	7	5 – 4	8
1 – 2	8	2 – 1	8	3 – 6	6	4 – 5	8	5 – 1	16
1 – 15	16	2 – 3	15	3 – 2	15	4 – 6	12		

6		7		8		9	
6 – 2	3	7 – 2	4	8 – 9	6	9 – 8	6
6 – 3	6	7 – 9	12	8 – 5	7	9 – 7	12
6 – 8	9			8 – 6	9	9 – 6	15
6 – 4	12						
6 – 9	15						

Table 1.2

Selection of node in iteration 1

Iteration No.	List L Nodes included	Nearest nodes	Cumulative distance
1	–	1	0

Step 2 : Node 1 is selected cumulative distance travelled upto node 1 is set as zero as shown in the last column of table 1.2 above

Step 3 : All the arcs in table 1.1 that are pointing towards node 1 (2-1, 3-1, 4-1, 5-1) are deleted in the table 1.1 and the new distance table 1.3 is formed as follows :

Iteration2 **Table 1.3**

1		2		3		4		5	
1 – 3	5	2 – 6	3	3 – 4	5	4 – 3	5	5 – 8	7
1 – 4	7	2 – 7	4	3 – 6	6	4 – 5	8	5 – 4	8
1 – 2	8	2 – 3	15	3 – 2	15	4 – 8	12		
1 – 15	16								

6		7		8		9	
6 – 2	3	7 – 2	4	8 – 9	6	9 – 8	6
6 – 3	6	7 – 9	12	8 – 5	7	9 – 7	12
6 – 8	9			8 – 6	9	9 – 6	15
6 – 4	12						
6 – 9	15						

Steps 4, 5 and 6

Table 1.4

Selection of node in iteration 2

Iteration No.	List L nodes included	Nearest nodes	Cumulative distance travelled
1	–	1	0
2	3	3	0 + 5 = 5

Step 7 : The recently selected node is 3 and it is not the required destination node. Go to step 8.

Step 8 : All arcs pointing towards node 3 are deleted in table 1.3 and the new distance table 1.5 is formed as follows :

Iteration 3 **Table 1.5**

1		2		3		4		5	
1 – 4	7	2 – 6	3	3 – 4	5	4 – 5	8	5 – 8	7
1 – 2	8	2 – 7	4	3 – 6	6	4 – 6	12	5 – 4	8
1 – 15	16			3 – 2	15				

6		7		8		9	
6 – 2	3	7 – 2	4	8 – 9	6	9 – 8	6
6 – 8	9	7 – 9	12	8 – 5	7	9 – 7	12
6 – 4	12			8 – 6	9	9 – 6	15
6 – 9	15						

Steps 4, 5 and 6

Table 1.6

Selection of node in iteration 3

Iteration No.	List L nodes included	Nearest nodes	Cumulative distance travelled
1	–	1	0
2	3	3	0 + 5 = 5
3	1, 3	4 ,4	0 + 7 = 7 5 + 5 = 10

Step 7 : The recently selected node is 4 and it is not the required destination node. Go to step 8.

Step 8 : All arcs pointing towards node 4 are deleted in table 1.5 and the new table 1.6 is formed as follows:

Iteration 4 **Table 1.6**

1		2		3		4		5	
1 – 2	8	2 – 6	3	3 – 6	6	4 – 5	8	5 – 8	7
1 – 15	16	2 – 7	4	3 – 2	15	4 – 6	12		

6		7		8		9	
6 – 2	3	7 – 2	4	8 – 9	6	9 – 8	6
6 – 8	9	7 – 9	12	8 – 5	7	9 – 7	12
6 – 9	15			8 – 6	9	9 – 6	15

Steps 4, 5 and 6

Table 1.7

Selection of node in iteration 4

Iteration No.	List L nodes included	Nearest nodes	Cumulative distance travelled
1	–	1	0
2	3	3	0 + 5 = 5
3	1, 3	4, 4	0 + 7 = 7 5 + 5 = 10
4	1,3,4	2 , 6, 5	0 + 8 = 8 5 + 6 = 11 7 + 8 = 15

Step 7 : The recently selected node is 2 and it is not the destination node. Go to step 8

Step 8 : All arcs pointing towards node 2 are deleted in table 1.6 and the new table 1.8 informed as follows :

Iteration 5

Table 1.8

1	2	3	4	5
1 – 15 16	2 – 6 3 2 – 7 4	3 – 6 6	4 – 5 8 4 – 6 12	5 – 8 7

6	7	8	9
6 – 8 9 6 – 9 15	7 – 9 12	8 – 9 6 8 – 5 7 8 – 6 9	9 – 8 6 9 – 7 12 9 – 6 15

Steps 4, 5 and 6

Table 1.9

Selection of node in iteration 5

Iteration No.	List L nodes included	Nearest nodes	Cumulative distance travelled
1	–	1	0
2	3	3	0 + 5 = 5
3	1, 3	4, 4	0 + 7 = 7 5 + 5 = 10
4	1,3,4	2, 6, 5	0 + 8 = 8 5 + 6 = 11 7 + 8 = 15
5	1, 2, 3, 4	5, 6, 6, 5	0 + 16 = 16 Tie $\left\{ \begin{array}{l} 8 + 3 = 11 \\ 5 + 6 = 11 \end{array} \right.$ 7 + 8 = 15

Step 7 : The recently selected node is 6 and it is not the destination node. Go to step 8. Tie is broken r and only in the last row of the table 1.9

Step 8 : All arcs pointing towards node 6 are deleted in table 1.8 and the new table 1.10 informed as follows :

Iteration 6

Table 1.10

1	2	3	4	5
1 – 15 16	2 – 7 4	–	4 – 5 8	5 – 8 7

6	7	8	9
6 – 8 9 6 – 9 15	7 – 9 12	8 – 9 6 8 – 5 7	9 – 8 6 9 – 7 12

Steps 4, 5 and 6

Table 1.11

Selection of node in iteration 6

Iteration No.	List L nodes included	Nearest nodes	Cumulative distance travelled
1	–	1	0
2	3	3	5
3	1, 3	4, 4	7, 10
4	1,3,4	2, 6, 5	8, 11, 15
5	1,2,4	5, 6, 5	16, 11, 15
6	1, 2, 4, 6	5, 7, 6, 9	0 + 16 = 16 8 + 4 = 12 7 + 12 = 19 11 + 15 = 26

Step 7 : The recently selected node is 7 and it is not the destination node. Go to step 8.

Step 8 : All arcs pointing towards node 7 are deleted in table 1.10 and the new table 1.12 informed as follows :

Iteration 7

Table 1.12

1	2	3	4	5
1 – 15 16	–	–	4 – 5 8	5 – 8 7

6	7	8	9
6 – 8 9 6 – 9 15	7 – 9 12	8 – 9 6 8 – 5 7	9 – 8 6

Steps 4, 5 and 6

Table 1.13

Selection of node in iteration 7

Iteration No.	List L nodes included	Nearest nodes	Cumulative distance travelled
1	–	1	0
2	3	3	5
3	1, 3	4, 4	7, 10
4	1, 3, 4	2, 6, 5	8, 11, 15
5	1, 2, 4	5, 6, 5	16, 11, 15
6	1, 2, 4, 6	5, 7, 6, 9	16, 12, 19, 16
7	1, 4, 6, 7	5, 5, 8, 9	0 + 16 = 16 8 + 7 = 15 11 + 9 = 20 12 + 12 = 24

Step 7: The recently selected node is 5 and it is not the destination node. Go to step 8.

Step 8 : All arcs pointing towards 5 are deleted in table 1.13. The new table 1.14 is informed as follows :

Iteration 8

Table 1.14

1	2	3	4	5
–	–	–	–	5 – 8 7

6	7	8	9
6 – 8 9 6 – 9 15	7 – 9 12	8 – 9 6	9 – 8 6

Steps 4, 5 and 6

Table 1.15

Selection of node in iteration 8

Iteration No.	List L nodes included	Nearest nodes	Cumulative distance travelled
1	–	0	0
2	3	3	5
3	1, 3	4	7, 10
4	1, 3, 4	2, 6, 5	8, 11, 15
5	1, 2, 4	5, 6, 5	16, 11, 15
6	1, 2, 4, 6	5, 7, 6, 9	16, 12, 19, 16
7	1, 4, 6, 7	5, 5, 8, 9	7 + 8 = 15 5 + 6 + 9 = 20 8 + 4 + 12 = 24
8	6, 7, 5	8, 9, 8	5 + 6 + 9 = 20 8 + 4 + 12 = 24 16 + 7 = 23

Step 7 : The recently selected node is 8 and it is not the destination node. Go to step 8.

Step 8 : All arcs pointing towards the node 8 are deleted in table 1.14. The new table 1.16 is informed as follows :

Iteration 9

Table 1.16

1	2	3	4	5
–	–	–	–	–

6	7	8	9
6 – 9 15	7 – 9 12	8 – 9 6	–

Steps 4, 5 and 6

Table 1.17

Selection of node in iteration 9

Iteration No.	List L nodes included	Nearest nodes	Cumulative distance travelled
1	–	1	0
2	3	3	5
3	1, 3	4	7, 10
4	1, 3, 4	2, 6, 5	8, 11, 15
5	1, 2, 4	5, 6, 5	16, 11, 15
6	1, 2, 4, 6	5, 7, 6, 9	16, 12, 19, 16
7	1, 4, 6, 7	5, 5, 8, 9	16, 15, 20, 16
8	6, 7, 5	8, 9, 5	20, 24, 23
9	6, 7, 8	9, 9, 9	26, 24, 26

Step 7 : The recently selected node is 9 and it is the destination node. Go to step 9.

Step 9 : The selected node is the last node in the shortest path and the partial path : 9

Step 10 : The node in list L corresponding to the node 9 is 7.50 prefix the partial path becomes 7-9

Step 11 : The recently prefixed node 7 is not the source node. So go to step 12.

Step 12 : Move to the iteration 6 where the node 7 is found circled and go to step 10

Step 10 : The node in list L corresponding to the node 7 is 2. So prefix the node 2 to the partial path. The partial path is becomes 2 – 7 – 9

Step 11 : The recently prefixed node is not the source node. So go to step 12

Step 12 : Move to iteration 4 where the node 2 is circled and go to step 10

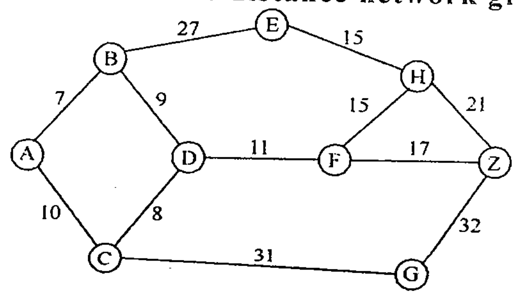
Step 10 : The node in the list L corresponding to node 2 is 1 and the node 1 is the source node. So prefix node 1 in the partial path new the partial path becomes 1–2–7–9

Step 11 : The recently prefixed mode is the source node. So go to step 13.

Step 13 : The shortest path (route) is 1–2–7–9 covering the distance of 24 units.

EXERCISE

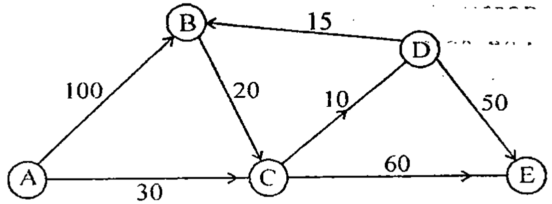
1. Using Dijkstra's algorithm find the shortest path from node A to node ~~Z~~ of the distance network given below:



Ans: A – B – D – F – ~~Z~~

Length = 44 units

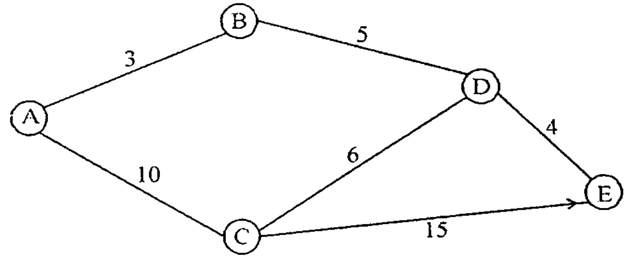
2. Using Dijkstra's algorithm determine the shortest route from node A to node B for the following network:



Ans: A → C → D → B

Length is 55 units.

3. Using Floyd's algorithm determine the shortest route from node A to node E for the following network.



Ans: A – B – D – E

Length is 12 units

4. The final distance matrix and the final precedence of a certain network are as given below:

$$D^{(n)} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 3 & 5 & 6 & 7 & 8 \\ 3 & 0 & 5 & 4 & 6 & 7 \\ 5 & 5 & 0 & 1 & 2 & 3 \\ 6 & 4 & 1 & 0 & 2 & 3 \\ 7 & 7 & 2 & 3 & 0 & 2 \\ 8 & 8 & 3 & 4 & 1 & 0 \end{bmatrix} \end{array} \qquad P^{(n)} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} - & 1 & 1 & 3 & 3 & 5 \\ 2 & - & 4 & 2 & 4 & 5 \\ 3 & 4 & - & 3 & 3 & 5 \\ 3 & 4 & 4 & - & 4 & 5 \\ 3 & 4 & 5 & 3 & - & 5 \\ 3 & 6 & 5 & 3 & 6 & - \end{bmatrix} \end{array}$$

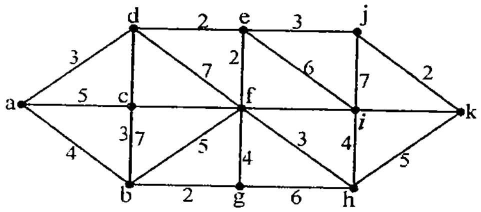
Find (i) The shortest path from model to node 6

(ii) Find he shortest path from mode 4 W node 1

Ans : (i) 1 – 3 – 5 – 6 Length = 8 units

(ii) 4 – 3 – 1 Length = 6 units

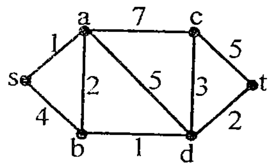
5. **Find the length of the shortest path and the shortest path between the pair of vertices given below in the following weighted graph.**



(i) a, k *(ii)* a, f *(iii)* b, j

6. ***(a)*** **Set up Dijkstra algorithm to find the shortest path in a two terminal weighted graph.** *(MU. MCA Nov '91)*

(b) **Find the shortest path from** s **to t in the graph shown in the figure.**



7. **Solve problems 1, 2 and 3 in this exercise by systematic algorithm.**

Unit 3

Integer Programming

3.1.1 INTRODUCTION

A linear programming problem in which some or all of the variables in the optimal solution are restricted to assume non-negative integer values is called an ***integer programming problem*** [or ***I.P.P*** or ***integer linear programming***].

In a linear programming problem, if all the variables in the optimal solution are restricted to assume non-negative integer values, then it is called the ***Pure (all) integer programming problem [Pure I.P.P].***

In a linear programming problem, if only some of the variables in the optimal solution are restricted to assume non-negative integer values, while the remaining variables are free to take any non-negative values, then it is called a ***Mixed integer programming problem [Mixed I.P.P]***

Further, if all the variables in the optimum solution are allowed to take values either 0 or 1 as in 'do' or 'not to do' type decisions, then the problem is called the ***Zero–one programming problem (or) standard discrete programming problem.***

The general integer programming problem is given by

$$\text{Maximize } Z = CX$$

subject to the constraints

$$AX \leq b,$$

$$X \geq 0 \text{ and some or all variables are integers.}$$

3.1.2 Importance of integer Programming:

In linear programming problem, all the decision variables were allowed to take any non-negative real (continuous or fractional) values, as it is quite possible and appropriate to have fractional values in many situations. For example, it is quite possible to use 6.38 kg of raw material, or 5.62 machine hours etc. However in many situations, especially in business and industry, these decision variables make sense only if they have integer values in the optimal solution. For example, it is meaningless to produce 8.13 chairs or 6.85 tables, or to open 3.83 branches of a bank or to run 9.6 cars etc. Hence a new procedure has been developed in this direction for the case of LPP subjected to the additional restriction that the decision variables must have integer values.

3.1.3 Applications of Integer Programming:

Integer programming problems occur quite frequently in business and industry.

All transportation, assignment and travelling salesman problems are integer programming problems, since the decision variables are either zero or one. *i.e.,* $x_{ij} = 0$ or 1.

All sequencing and routing decisions are integer programming problems as it requires the integer values of the decision variables.

Capital budgeting and production scheduling problems are integer programming problems. In fact, any situation involving decisions of the type "either to do a job or not to do" (either – or) can be treated as an integer programming problem. In these situations,

$$x_j = \begin{cases} 1, & \text{if } j^{th} \text{ job is performed} \\ 0, & \text{if } j^{th} \text{ job is not performed} \end{cases}$$

All allocation problems involving the allocation of goods, men, machines, give rise to integer programming problems, since such commodities can be assigned only integer and not fractional values.

3.1.4 Pitfalls in rounding the optimum solution of an I.P.P.

We may think to solve such problems by the usual simplex method (ignoring the integrality restriction) and then rounding off the non–integer values to integers in the optimal solution obtained by the simplex method. But there is no guarantee that the integer valued solution thus obtained will satisfy all the constraints *i.e.,* it may not satisfy one or more constraints and as such the new solution may not be feasible. For example, consider the problem

$$\begin{aligned} \text{Max } Z &= 20x_1 + 8x_2 \\ \text{subject to } 3x_1 + 4x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \text{ and integers} \end{aligned}$$

By using graphical method (or simplex method), the optimal non–integer solution (ignoring the integrality condition) of this problem is given by Max Z = 53.40, $x_1 = 2.67$, $x_2 = 0$.

Now rounding the solution to $x_1 = 3$, $x_2 = 0$, it does not satisfy the constraint $3x_1 + 4x_2 \leq 8$. Hence the solution $x_1 = 3$, $x_2 = 0$ is not feasible. Thus a new rounded solution may not be feasible.

Further if we rounding the solution to $x_1 = 2$, $x_2 = 0$, then the solution is feasible but gives Max Z = 40, which is far away from the optimum solution Max Z = 53.4. There is no guarantee that the rounded down solution will be optimum also.

Due to these difficulties, there is a need for developing a systematic and efficient algorithm for obtaining the exact optimum integer solution to an integer programming problem.

3.1.5 Methods of Integer Programming

Integer programming methods can be categorized as (1) cutting methods and (2) search methods.

Cutting Methods: A systematic procedure for solving pure integer programming problem was first developed by R.E. Gomory in 1958. Later on, he extended the procedure to solve mixed I.P.P. named as ***Cutting plane algorithm,*** the method consists in first solving the I.P.P as ordinary L.P.P. by ignoring the integrality restriction and then introducing additional constraints one after the other to cut (eliminate) certain part of the solution space until an integral solution is obtained.

Search Method: It is an enumeration method in which all feasible integer points are enumerated. The widely used search method is the ***branch and bound technique.*** It also starts with the continuous optimum, but systematically partitions the solution space into sub problems that eliminate parts that contain no feasible integer solution. It was originally developed by A.H Land and A.G Doig.

3.1.6 Gomory's Fractional Cut algorithm (or) Cutting Plane Method for pure (all) I.P.P:

Step 1: Convert the minimization I.P.P. in to an equivalent maximization I.P.P. and all the coefficients and constants should be integers. Ignore the integerality condition.

Step 2: Find the optimum solution of the resulting maximization L.P.P. by using simplex method.

Step 3: Test the integerality of the optimum solution.

(i) If all $X_{Bi} \geq 0$ and are integers, an optimum integer solution is obtained.

(ii) If all $X_{Bi} \geq 0$ and atleast one X_{Bi} is not an integer, then go to the next step.

Step 4: Rewrite each X_{Bi} as $X_{Bi} = [X_{Bi}] + f_i$, where $[X_{Bi}]$ is the integral part of X_{Bi} and f_i is the positive fractional part of X_{Bi}, $0 \leq f_i < 1$. Choose the largest fraction of X_{Bi}, s, *ie.,* choose Max $\{f_i\}$. In case of a tie, select arbitrarily. Let Max $\{f_i\} = f_k$ corresponding to X_{Bk} (the k^{th} row corresponding to this f_k is called ***source row***).

Step 5: Express each of the negative fractions if any, in the k^{th} row (source row) of the optimum simplex table as the sum of a negative integer and a non–negative fraction.

Step 6: Find the fractional cut constraint (Gomorian constraint or secondary constraint)

From the source row $\quad \sum_{j=1}^{n} a_{kj} x_j = X_{Bk}$

ie., $\quad \sum_{j=1}^{n} ([a_{kj}] + f_{kj}) x_j = [X_{Bk}] + f_k$

in the form $\quad \sum_{j=1}^{n} f_{kj} x_j \geq f_k$

(or) $\quad -\sum_{j=1}^{n} f_{kj} x_j \leq -f_k$ or $-\sum_{j=1}^{n} f_{kj} + s_1 = -f_k$

where s_1 is the Gomorian slack.

Step 7: Add the fractional cut constraint obtained in step 6 at the bottom of the optimum simplex table obtained in step 2. Find the new feasible optimum solution using dual simplex method.

Step 8: Go to step 3 and repeat the procedure until an optimum integer solution is obtained.

Note: In this cutting plane method, the fractional cut constraints cut the unuseful area of the feasible region in the graphical solution of the problem. *ie.,* cut that area which has no integer – valued feasible solution. Thus these Gomorian constraints eliminate all the non–integral solutions without loosing any integer – valued solution.

Example 1 **Find the optimum integer solution to the following L.P.P.**

$$\text{Max } Z = x_1 + x_2$$

subject to constraints

$$3x_1 + 2x_2 \leq 5$$
$$x_2 \leq 2$$

and $x_1 > 0$, $x_2 \geq 0$ and are integers.

[BRU. BE. 84, MU. BE. 82 Nov 92, Oct 96]

Solution: Ignoring the integrality condition and introducing the non–negative slack variables x_3 and x_4, the standard form of the continuous LPP becomes

$$\text{Max } Z = x_1 + x_2 + 0x_3 + 0x_4$$

subject to
$$3x_1 + 2x_2 + x_3 + 0x_4 = 5$$
$$0x_1 + x_2 + 0x_3 + x_4 = 2$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0$$

The initial basic feasible solution is given by $x_3 = 5$, $x_4 = 2$, ($x_1 = x_2 = 0$, non-basic).

Initial iteration:

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	5	(3)	2	1	0	$\frac{5}{3}$
0	x_4	2	0	1	0	1	–
$(Z_j - C_j)$		0	–1	–1	0	0	

Since some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration: Introduce x_1 and drop x_3.

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
1	x_1	$\frac{5}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{5}{2}$
0	x_4	2	0	(1)	0	1	2*
$(Z_j - C_j)$		$\frac{5}{3}$	0	$\frac{-1}{3}$	$\frac{1}{3}$	0	

Since some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration: Introduce x_2 and drop x_4.

		C_j	(1	1	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{-2}{3}$
1	x_2	2	0	1	0	1
$(Z_j - C_j)$		$\frac{7}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal and non-integer. *i.e.,* Max $Z = \frac{7}{3}$, $x_1 = \frac{1}{3}$, $x_2 = 2$.

To obtain the optimum integer solution, we have to add a fractional cut constraint in the optimum simplex table.

Since $x_1 = \frac{1}{3}$, from the source row (first row)

we have $$\frac{1}{3} = x_1 + \frac{1}{3} x_3 - \frac{2}{3} x_4$$

Expressing the negative fraction as a sum of a negative integer and non–negative fraction, we have

$$\frac{1}{3} = x_1 + \frac{1}{3} x_3 + \left(-1 + \frac{1}{3}\right) x_4$$

The fractional cut (Gomorian) constraint is given by

$$\frac{1}{3} x_3 + \frac{1}{3} x_4 \geq \frac{1}{3} \Rightarrow \frac{-1}{3} x_3 - \frac{1}{3} x_4 \leq \frac{-1}{3}$$

$$\Rightarrow -\frac{1}{3} x_3 - \frac{1}{3} x_4 + s_1 = -\frac{1}{3}$$

where s_1 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have the new simplex table,

		C_j	(1	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{-2}{3}$	0
1	x_2	2	0	1	0	1	0
0	s_1	$-\frac{1}{3}$	0	0	$\frac{-1}{3}$	$\frac{-1}{3}$	1
$(Z_j - C_j)$		$\frac{7}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0

Here the solution is optimal but infeasible.

To obtain the feasible optimal solution, we have to use the ***dual simplex method.***

Since $s_1 = -\frac{1}{3}$, s_1 leaves the basis.

To find the entering variable: Let Max $\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\}$

$$= \text{Max}\left\{\frac{\frac{1}{3}}{-\frac{1}{3}}, \frac{\frac{1}{3}}{-\frac{1}{3}}\right\} = \text{Max}\{-1, -1\} = -1 \text{ which corresponds to both}$$

x_3 and x_4. We choose x_3 as the entering variable arbitrarily.

Third iteration: Drop s_1 and introduce x_3.

		C_j	(1	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_1	0	1	0	0	-1	1
1	x_2	2	0	1	0	1	1
0	x_3	1	0	0	1	1	-3
$(Z_j - C_j)$		2	0	0	0	0	1

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and optimal and integer.

$\therefore$ The optimum integer solution is

$$\text{Max } Z = 2, \quad x_1 = 0, \quad x_2 = 2.$$

Geometrical interpretation of cutting plane method:

For the above I.P.P, the feasible regions OABC, is shown shaded in the following figure.

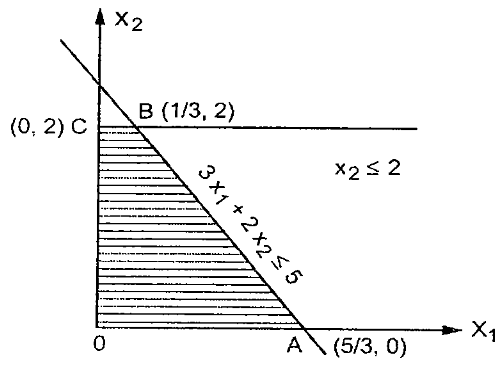


Fig 9.1

The optimum solution is

$$\text{Max } Z = \frac{7}{3}, \quad x_1 = \frac{1}{3}, \quad x_2 = 2$$

Since this solution is not an integer optimum solution, we introduce the secondary (Gomorian) Constraint

$$\frac{x_3}{3} + \frac{x_4}{3} \geq \frac{1}{3}$$

The express this in terms of x_1 and x_2, we know that

$$3x_1 + 2x_2 + x_3 = 5 \Rightarrow x_3 = 5 - 3x_1 - 2x_2$$

$$\text{and } x_2 + x_4 = 2 \Rightarrow x_4 = 2 - x_2$$

Substituting in the Gomory Constraints, we have

$$\frac{1}{3}(5 - 3x_1 - 2x_2) + \frac{1}{3}(2 - x_2) \geq \frac{1}{3}$$

$$\Rightarrow 5 - 3x_1 - 2x_2 + 2 - x_2 \geq 1$$

$$\Rightarrow -3x_1 - 3x_2 + 7 \geq 1 \Rightarrow -3x_1 - 3x_2 \geq -6$$

$$\Rightarrow 3x_1 + 3x_2 \leq 6 \Rightarrow x_1 + x_2 \leq 2$$

Drawing the line $x_1 + x_2 = 2$, the above feasible region is cut off to the shaded region shown below:

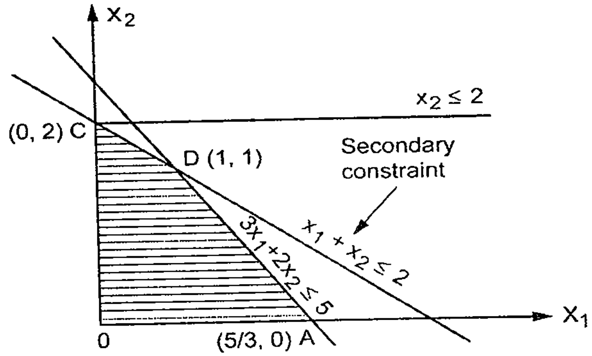


Fig. 9.2

Thus the required optimal integer valued solution is

$$\text{Max } Z = 2, \quad x_1 = 0, \quad x_2 = 2 \text{ (or)}$$

$$\text{Max } Z = 2, \quad x_1 = 1, \quad x_2 = 1.$$

$\boxed{\textbf{Example 2}}$ **Using Gomory's cutting plane method**

$$\textbf{Maximize } Z = 2x_1 + 2x_2$$

$$\begin{aligned} \textbf{subject to} \quad 5x_1 + 3x_2 &\leq 8 \\ 2x_1 + 4x_2 &\leq 8 \\ \textbf{and } x_1, x_2 &\geq 0 \textbf{ and are all integers.} \end{aligned}$$

[MU. BE. Nov 89]

Solution: Ignore the integrality condition and introducing the non–negative slack variables x_3. x_4, the standard form of the continuous LPP becomes

$$\text{Maximize } Z = 2x_1 + 2x_2 + 0x_3 + 0x_4$$

$$\begin{aligned} \text{subject to} \quad 5x_1 + 3x_2 + x_3 + 0x_4 &= 8 \\ 2x_1 + 4x_2 + 0x_3 + x_4 &= 8 \\ \text{and } x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The initial basic feasible solution is given by

$$x_3 = 8, \quad x_4 = 8 \text{ (basic)}$$

$$(x_1 = x_2 = 0, \text{ non basic})$$

Initial iteration:

C_j (2 2 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	8	(5)	3	1	0	$\frac{8}{5}$*
0	x_4	8	2	4	0	1	$\frac{8}{2}$
$Z_j - C_j$		0	–2	–2	0	0	

First iteration: Introduce x_1 and drop x_3.

C_j (2 2 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
2	x_1	$\frac{8}{5}$	1	$\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{8}{3}$
0	x_4	$\frac{24}{5}$	0	$\left(\frac{14}{5}\right)$	$\frac{-2}{5}$	1	$\frac{24}{14} = \frac{12}{7}$*
$Z_j - C_j$		$\frac{16}{5}$	0	$\frac{-4}{5}$	$\frac{2}{5}$	0	

Second iteration: Introduce x_2 and drop x_4.

C_j (2 2 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4
2	x_1	$\frac{4}{7}$	1	0	$\frac{2}{7}$	$\frac{-3}{4}$
2	x_2	$\frac{12}{7}$	0	1	$\frac{-1}{7}$	$\frac{5}{14}$
$Z_j - C_j$		$\frac{32}{7}$	0	0	$\frac{2}{7}$	$\frac{2}{7}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal but non-integer.

To obtain the optimum integer solution, we have to construct a fractional cut constraint.

$$\text{Now} \quad x_1 = \frac{4}{7} = 0 + \frac{4}{7} = [X_{B1}] + f_1$$

$$x_2 = \frac{12}{7} = 1 + \frac{5}{7} = [X_{B2}] + f_2$$

$\therefore$ Max $\{f_1, f_2\}$ = Max $\left\{\frac{4}{7}, \frac{5}{7}\right\} = \frac{5}{7}$ which corresponds to the second row (called the source row). Then from this source row, we have

$$\frac{12}{7} = x_2 - \frac{1}{7}x_3 + \frac{5}{14}x_4$$

Express the negative fraction as a sum of a negative integer and a non-negative fraction, we have

$$1 + \frac{5}{7} = x_2 + \left(-1 + \frac{6}{7}\right)x_3 + \frac{5}{14}x_4$$

$\therefore$ The fractional cut (Gomorian) constraint is given by

$$\frac{6}{7}x_3 + \frac{5}{14}x_4 \geq \frac{5}{7}$$

$$\Rightarrow \quad \frac{-6}{7}x_3 - \frac{5}{14}x_4 \leq -\frac{5}{7}$$

$$\Rightarrow \quad -\frac{6}{7}x_3 - \frac{5}{14}x_4 + s_1 = -\frac{5}{7}$$

where s_1 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have

C_j (2 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
2	x_1	$\frac{4}{7}$	1	0	$\frac{2}{7}$	$\frac{-3}{4}$	0
2	x_2	$\frac{12}{7}$	0	1	$\frac{-1}{7}$	$\frac{5}{14}$	0
0	s_1	$\frac{-5}{7}$	0	0	$\left(\frac{-6}{7}\right)$	$\frac{-5}{14}$	1
$Z_j - C_j$		$\frac{32}{7}$	0	0	$\frac{2}{7}$	$\frac{2}{7}$	0

Here the solution is optimal but infeasible $\left(\text{because } s_1 = \frac{-5}{7}\right)$. To find the feasible optimal solution, we have to use the ***dual simplex*** method.

Since $s_1 = -\frac{5}{7}$, s_1 leaves the basis.

$$\text{Also Max}\left\{\frac{Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{2}{7}}{\frac{-6}{7}},\ \frac{\frac{2}{7}}{\frac{-5}{14}}\right\}$$

$= \text{Max}\left\{\frac{-1}{3}, \frac{-4}{5}\right\} = \frac{-1}{3}$ which corresponds to x_3. So x_3 enters the basis.

Third iteration: Drop s_1 and introduce x_3.

		C_j	(2	2	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
2	x_1	$\frac{1}{3}$	1	0	0	$\frac{-1}{3}$	$\frac{1}{3}$
2	x_2	$\frac{11}{6}$	0	1	0	$\frac{5}{12}$	$\frac{-1}{6}$
0	x_3	$\frac{5}{6}$	0	0	1	$\frac{5}{12}$	$\frac{-7}{6}$
$Z_j - C_j$		$\frac{13}{3}$	0	0	0	$\frac{1}{6}$	$\frac{1}{3}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bj} \geq 0$, the current solution is feasible and optimal but non-integer.

$\therefore$ The obtain the optimum integer solution, we have to construct a fractional cut constraint.

$$\text{Max } Z = 2,\ x_1 = 0,\ x_2 = 2.$$

$$\text{Now } x_1 = \frac{1}{3} = 0 + \frac{1}{3} = [X_{B1}] + f_1$$

$$x_2 = \frac{11}{6} = 1 + \frac{5}{6} = [X_{B2}] + f_2$$

$$\text{Now } x_3 = \frac{5}{6} = 0 + \frac{5}{6} = [X_{B3}] + f_3$$

$\therefore \text{Max}\{f_1, f_2, f_3\} = \text{Max}\left(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right) = \frac{5}{6}$ which corresponds to both second and third rows, we select the second row arbitrarily as the source row.

From this source row, we have

$$\frac{11}{6} = x_2 + \frac{5}{12} x_4 + \frac{-1}{6} s_1$$

$$1 + \frac{5}{6} = x_2 + \frac{5}{12} x_4 + \left(-1 + \frac{5}{6}\right) s_1$$

$\therefore$ The fractional cut (Gomorian) construct is given by

$$\frac{5}{12} x_4 + \frac{5}{6} s_1 \geq \frac{5}{6}$$

$$\Rightarrow \frac{-5}{12} x_4 - \frac{5}{6} s_1 \leq -\frac{5}{6}$$

$$\Rightarrow -\frac{5}{12} x_4 - \frac{5}{6} s_1 + s_2 = -\frac{5}{6}$$

where s_2 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have

		C_j	(2	2	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2
2	x_1	$\frac{1}{3}$	1	0	0	$\frac{-1}{3}$	$\frac{1}{3}$	0
2	x_2	$\frac{11}{6}$	0	1	0	$\frac{5}{12}$	$\frac{-1}{6}$	0
0	x_3	$\frac{5}{6}$	0	0	1	$\frac{5}{12}$	$\frac{-7}{6}$	0
0	s_2	$\frac{-5}{6}$	0	0	0	$\frac{-5}{12}$	$\frac{-5}{6}$	1
$Z_j - C_j$		$\frac{13}{3}$	0	0	0	$\frac{1}{6}$	$\frac{1}{3}$	0

Here the solution is optimal but infeasible. so we have to use dual simplex method to obtain the feasible optimal solution.

Since $s_2 = \frac{-5}{6}$, s_2 leaves the basis.

Also Max $\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{6}}{-\frac{-5}{12}}, \frac{\frac{1}{3}}{-\frac{-5}{6}}\right\}$

$= \text{Max}\left\{\frac{-2}{5}, \frac{-2}{5}\right\} = \frac{-2}{5}$ which corresponds to both x_4 and s_1.

We select x_4 arbitrarily as the entering variable.

Fourth Iteration: Introduce x_4 and drop s_2.

C_B	Y_B	X_B	x_1 (2)	x_2 (2)	x_3 (0)	x_4 (0)	s_1 (0)	s_2 (0)
2	x_1	1	1	0	0	0	1	$\frac{-4}{5}$
2	x_2	1	0	1	0	0	–1	1
0	x_3	0	0	0	1	0	–2	1
0	x_4	2	0	0	0	1	2	$\frac{-12}{5}$
$Z_j - C_j$		4	0	0	0	0	0	$\frac{2}{5}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{B_i} \geq 0$, the current solution is feasible integer optimal.

∴ The optimal solution to the new problem is

$$\text{Max } Z = 4, \quad x_1 = 1, \quad x_2 = 1.$$

Note: The alternative integer optimal solution to this problem is

$$\text{Max } Z = 4, \quad x_1 = 0, \quad x_2 = 2.$$

Example 3 **Solve the following IPP**

$$\text{Minimize } Z = -2x_1 - 3x_2$$

subject to

$$2x_1 + 2x_2 \leq 7$$
$$x_1 \leq 2$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0 \text{ and integers.}$$

[MU. BE. Apr. 95]

Solution: Given I.P.P is

$$\text{Minimize } Z = -2x_1 - 3x_2$$

subject to

$$2x_1 + 2x_2 \leq 7$$
$$x_1 \leq 2$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0 \text{ and integers.}$$

i.e., Maximize $Z^* = 2x_1 + 3x_2$

subject to

$$2x_1 + 2x_2 \leq 7$$
$$x_1 \leq 2$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0 \text{ and integers.}$$

Ignoring the integrality condition and introducing the non–negative slack variables x_3, x_4 and x_5, the standard form of the continuous L.P.P becomes.

$$\text{Maximize } Z^* = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to

$$2x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 7$$
$$x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 2$$
$$0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 2$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The initial basic feasible solution is given by

$x_3 = 7$, $x_4 = 2$, $x_5 = 2$ (basic) $(x_1 = x_2 = 0$, non basic)

Initial iteration:

C_B	Y_B	X_B	x_1 (2)	x_2 (3)	x_3 (0)	x_4 (0)	x_5 (0)	θ
0	x_3	7	2	2	1	0	0	$\frac{7}{2}$
0	x_4	2	1	0	0	1	0	–
0	x_5	2	0	(1)	0	0	1	2*
$Z_j^* - C_j$		0	–2	–3	0	0	0	

First iteration: Introduce x_2 and drop x_5.

C_B	Y_B	X_B	x_1 (2)	x_2 (3)	x_3 (0)	x_4 (0)	x_5 (0)	θ
0	x_3	3	(2)	0	1	0	–2	$\frac{3}{2}$*
0	x_4	2	1	0	0	1	0	2
3	x_2	2	0	1	0	0	1	–
$Z_j^* - C_j$		6	–2	0	0	0	3	

Second iteration: Introduce x_1 and drop x_3.

C_B	Y_B	X_B	x_1 (2)	x_2 (3)	x_3 (0)	x_4 (0)	x_5 (0)
2	x_1	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	–1
0	x_4	$\frac{1}{2}$	0	0	$\frac{-1}{2}$	1	1
3	x_2	2	0	1	0	0	1
$Z_j - C_j$		9	0	0	1	0	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal but non-integer.

To obtain the optimum integer solution, we have to construct a fractional cut constraint.

$$\text{Now } x_1 = \frac{3}{2} = 1 + \frac{1}{2} = [X_{B1}] + f_1$$

$$x_4 = \frac{1}{2} = 0 + \frac{1}{2} = [X_{B2}] + f_2$$

$\therefore$ Max $\{f_1, f_2\}$ = Max $\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$ which corresponds to both first and second rows. We select the first row arbitrarily as the source row.

Now from this source row we have,

$$\frac{3}{2} = x_1 + \frac{1}{2}x_3 - x_5$$

$$1 + \frac{1}{2} = x_1 + \frac{1}{2}x_3 - x_5$$

$\therefore$ The fractional cut (Gomorian) constraint is given by

$$\frac{1}{2}x_3 \geq \frac{1}{2} \Rightarrow \frac{-1}{2}x_3 \leq \frac{-1}{2}$$

$$\Rightarrow \frac{-1}{2}x_3 + s_1 = -\frac{1}{2}$$

where s_1 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table and using the dual simplex method, we have

C_B	Y_B	X_B	x_1 (2)	x_2 (3)	x_3 (0)	x_4 (0)	x_5 (0)	s_1 (0)
2	x_1	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	–1	0
0	x_4	$\frac{1}{2}$	0	0	$\frac{-1}{2}$	1	1	0
3	x_2	2	0	1	0	0	1	0
0	s_1	$\frac{-1}{2}$	0	0	$\left(-\frac{1}{2}\right)$	0	0	1
$Z_j^* - C_j$		9	0	0	1	0	1	0

Since $s_1 = -\frac{1}{2}$, s_1 leaves the basis. Further

$$\text{Max}\left\{\frac{Z_j^* - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{1}{-\frac{1}{2}}\right\} = -2 \text{ which corresponds to } x_3.$$

So x_3 enters the basis.

Third iteration: Drop s_1 and introduce x_3.

C_B	Y_B	X_B	x_1 (2)	x_2 (3)	x_3 (0)	x_4 (0)	x_5 (0)	s_1 (0)
2	x_1	1	1	0	0	0	–1	1
0	x_4	1	0	0	0	1	1	–1
3	x_2	2	0	1	0	0	1	0
0	x_3	1	0	0	1	0	0	–2
$Z_j^* - C_j$		8	0	0	0	0	1	2

Since all $(Z_j^* - C_j) \geq 0$ and all $X_{B_i} \geq 0$, the current solution is feasible and integer optimal.

$\therefore$ The optimal integer solution is

$$\text{Max } Z^* = 8, x_1 = 1, x_2 = 2.$$

$$\text{But Min } Z = -\text{Max}(-Z) = -\text{Max } Z^* = -8$$

$$\therefore \text{Min } Z = -8, x_1 = 1, x_2 = 2.$$

Example 4 **A manufacturer of baby – dolls makes two types of dolls, doll X and doll Y. Processing of these two dolls is done on two machines, A and B. Doll X requires two hours on machine A and six hours on machine B. Doll Y requires five hours on machine A and also five hours on machine B. There are sixteen hours of time per day available on machine A and thirty hours on machine B. The profit gained on both the dolls is same, ie., one rupee per doll. What should be the daily production of each of the two dolls ?**

(a) **Set up and solve the I.P.P**

(b) **If the optimal solution is not integer valued, use the Gomory technique to derive the optimal solution.**

[Delhi. MBA. 73, BRU.M.Sc 86]

Solution: Let the manufacturer decide to manufacture x_1 number of Doll X and x_2 number of Doll Y so as to maximize his profit. Then the complete formulation of the I.P.P. is given by.

$$\text{Maximize } Z = x_1 + x_2$$

subject to

$$2x_1 + 5x_2 \leq 16$$
$$6x_1 + 5x_2 \leq 30$$
$$\text{and } x_1, x_2 \geq 0 \text{ and are integers}$$

Ignoring the integrality condition and introducing the non–negative slack variables x_3 and x_4, the standard form of the continuous L.P.P. becomes

$$\text{Maximize } Z = x_1 + x_2 + 0x_3 + 0x_4$$

subject to

$$2x_1 + 5x_2 + 0x_3 + 0x_4 = 16$$
$$6x_1 + 5x_2 + 0x_3 + 0x_4 = 30$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0$$

The initial basic feasible solution is given by

$$x_3 = 16, \quad x_4 = 30 \text{ (basic)}$$

$$(x_1 = x_2 = 0, \text{ non basic})$$

Initial iteration:

		C_j	1	1	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	16	2	5	1	0	8
0	x_4	30	(6)	5	0	1	5*
$Z_j - C_j$		0	–1	–1	0	0	

First iteration: Introduce x_1 and drop x_4.

		C_j	1	1	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	6	0	$\left(\frac{10}{3}\right)$	1	$\frac{-1}{3}$	$\frac{18}{10} = \frac{9}{5}$*
1	x_1	5	1	$\frac{5}{6}$	0	$\frac{1}{6}$	$\frac{30}{5} = 6$
$Z_j - C_j$		5	0	$\frac{-1}{6}$	0	$\frac{1}{6}$	

Second iteration: Introduce x_2 and drop x_3.

		C_j	1	1	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
1	x_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$\frac{-1}{10}$
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{4}$	$\frac{1}{4}$
$Z_j - C_j$		$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal and non-integer.

i.e., $$\text{Max } Z = \frac{53}{10}, \quad x_1 = \frac{7}{2}, \quad x_2 = \frac{9}{5}.$$

In order to obtain the optimum integer solution, we have to construct a fractional cut constraint.

$$\text{Now } x_1 = \frac{7}{2} = 3+\frac{1}{2} = [X_{B1}] + f_1$$

$$x_2 = \frac{9}{5} = 1+\frac{4}{5} = [X_{B2}] + f_2$$

$\therefore$ Max $\{f_1, f_2\}$ = Max $\left\{\frac{1}{2}, \frac{4}{5}\right\} = \frac{4}{5}$ which corresponds to the first row (called the source row). Then from this source row, we have

$$1+\frac{4}{5} = 0x_1 + x_2 + \frac{3}{10}x_3 - \frac{1}{10}x_4$$

Expressing the negative fraction as a sum of a negative integer and a non–negative fraction, we have

$$1+\frac{4}{5} = x_2 + \frac{3}{10}x_3 + \left(-1+\frac{9}{10}\right)x_4$$

$\therefore$ The fractional cut (Gomorian) construct is given by

$$\frac{3}{10}x_3 + \frac{9}{10}x_4 \geq \frac{4}{5}$$

$$\Rightarrow \frac{-3}{10}x_3 - \frac{9}{10}x_4 \leq -\frac{4}{5}$$

$$\Rightarrow -\frac{3}{10}x_3 - \frac{9}{10}x_4 + s_1 = -\frac{4}{5}$$

where s_1 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have

		C_j	(1	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$\frac{-1}{10}$	0
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{4}$	$\frac{1}{4}$	0
0	s_1	$\frac{-4}{5}$	0	0	$\frac{-3}{10}$	$\frac{-9}{10}$	1
$Z_j - C_j$		$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$	0

Here the solution is optimal but infeasible

To find the feasible optimal solution, we have to use the ***dual simplex*** method.

Since $s_1 = -\frac{4}{5}$, s_1 leaves the basis.

To find the entering variable :

$$\text{Also Max}\left\{\frac{Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{20}}{\frac{-3}{10}}, \frac{\frac{3}{20}}{\frac{-9}{10}}\right\}$$

$= \text{Max}\left\{\frac{-1}{6}, \frac{-1}{6}\right\} = \frac{-1}{6}$ which corresponds to x_3 and x_4. We shall choose x_3 arbitrarily as the entering variable,

Third iteration: Drop s_1 and introduce x_3.

		C_j	(1	1	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_2	1	0	1	0	–1	1
1	x_1	$\frac{25}{6}$	1	0	0	1	$\frac{-5}{6}$
0	x_3	$\frac{8}{3}$	0	0	1	3	$\frac{-10}{3}$
$Z_j - C_j$		$\frac{31}{6}$	0	0	0	0	$\frac{1}{6}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and optimal but non-integer.

$\therefore$ The obtain the optimum integer solution, we have to construct a fractional cut constraint.

$$\text{Now } x_1 = \frac{25}{6} = 4+\frac{1}{6} = [X_{B1}] + f_1$$

$$x_3 = \frac{8}{3} = 2+\frac{2}{3} = [X_{B2}] + f_3$$

$\therefore$ Max $\{f_1, f_3\}$ = Max $\left[\frac{1}{6}, \frac{2}{3}\right] = \frac{2}{3}$ which corresponds to the third row.

From this source row, we have

$$2+\frac{2}{3} = x_3 + 3x_4 - \frac{10}{3}s_1$$

$$= x_3 + 3x_4 + \left(-4+\frac{2}{3}\right)s_1$$

∴ The fractional cut (Gomorian) construct is given by

$$\frac{2}{3}s_1 \geq \frac{2}{3} \Rightarrow \frac{-2}{3}s_1 \leq \frac{-2}{3}$$

$$\Rightarrow \frac{-2}{3}s_1 + s_2 = \frac{-2}{3}$$

where s_2 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table and using the dual simplex method, we have

C_B	Y_B	C_j → / X_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)	s_1 (0)	s_2 (0)
1	x_2	1	0	1	0	–1	1	0
1	x_1	$\frac{25}{6}$	1	0	0	1	$\frac{-5}{6}$	0
0	x_3	$\frac{8}{3}$	0	0	1	3	$\frac{-10}{3}$	0
0	s_2	$\frac{-2}{3}$	0	0	0	0	$\left(\frac{-2}{3}\right)$	1
$Z_j - C_j$		$\frac{31}{6}$	0	0	0	3	$\frac{1}{6}$	0

Since $s_2 = \frac{-2}{3}$, s_2 leaves the basis.

Now $\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{6}}{\frac{-2}{3}}\right\} = \frac{-1}{4}$ which corresponds to s_1. So, s_1 enters the basis.

Fourth Iteration: Introduce s_1 and drop s_2.

C_B	Y_B	C_j → / X_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)	s_1 (0)	s_2 (0)
1	x_2	0	0	1	0	–1	0	$\frac{3}{2}$
1	x_1	5	1	0	0	1	0	$\frac{-5}{4}$
0	x_3	6	0	0	1	3	0	–5
0	s_1	1	0	0	0	0	1	$\frac{-3}{2}$
$Z_j - C_j$		5	0	0	0	0	0	$\frac{1}{4}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, and integers, the current solution is feasible and integer optimal.

∴ The optimal solution to the new problem is

Max $Z = 5$, $x_1 = 5$, $x_2 = 0$.

i.e., the manufacturer should produce 5 number of doll X alone in order to get the maximum profit Rs. 5.

Note: For this problem, since $(Z_4 - C_4) = 0$ corresponding to the non–basic variable x_4, there exists alternative optimal solutions. Such solutions are $x_1 = 3, x_2 = 2$, and $x_1 = 4, x_2 = 1$ with same Max $Z = 5$.

Example 5 **Find the optimum integer solution to the following linear programming problem:**

$$\textbf{Maximize } Z = x_1 + 2x_2$$

subject to

$$2x_2 \leq 7$$
$$x_1 + x_2 \leq 7$$
$$2x_1 \leq 11$$

and $x_1, x_2 \geq 0$ **and are integers.**

[MU. BE. Apr 91, BRU. BE. Nov 96]

Solution: Ignoring the integrality condition and introducing the non-negative slack variables x_3, x_4, and x_5, the standard form of the continuous L.P.P. becomes

$$\text{Maximize } Z = x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to

$$0x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 7$$
$$x_1 + x_2 + 0x_3 + x_4 + 0x_5 = 7$$
$$2x_1 + 0x_2 + 0x_3 + 0x_4 + x_5 = 11$$
$$\text{and } x_i \geq 0, \quad i = 1, 2, 3, 4, 5.$$

The initial basic feasible solution is given by

$x_3 = 7$, $x_4 = 7$, $x_5 = 11$ (basic)

($x_1 = x_2 = 0$, non basic)

Initial iteration:

C_j (1 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	θ
0	x_3	7	0	(2)	1	0	0	$\frac{7}{2}$
0	x_4	7	1	1	0	1	0	7
0	x_5	11	2	0	0	0	1	–
$Z_j - C_j$		0	–1	–2	0	0	0	

First iteration: Introduce x_2 and drop x_3.

C_j (1 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	θ
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	–
0	x_4	$\frac{7}{2}$	(1)	0	$\frac{-1}{2}$	1	0	$\frac{7}{2}$
0	x_5	11	2	0	0	0	1	$\frac{11}{2}$
$Z_j - C_j$		7	–1	0	1	0	0	

Second iteration: Introduce x_1 and drop x_4.

C_j (1 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_3	x_4
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{2}$	1	0
0	x_5	4	0	0	1	–2	1
$Z_j - C_j$		$\frac{21}{2}$	0	0	$\frac{1}{2}$	1	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal but non-integer.

$$\text{i.e., Max } Z = \frac{21}{2}, \; x_1 = \frac{7}{2}, \; x_2 = \frac{7}{2}$$

To obtain the integer optimal solution, we have to construct a fractional cut constraint.

$$\text{Now } x_1 = \frac{7}{2} = 3 + \frac{1}{2} = [X_{B1}] + f_1$$
$$x_2 = \frac{7}{2} = 3 + \frac{1}{2} = [X_{B2}] + f_2$$

$\therefore$ Max $\{f_1, f_2\}$ = Max $\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$ which corresponds to both first and second rows. We choose the first row arbitrarily as the source row.

From this source row, we have

$$\frac{7}{2} = x_2 + \frac{1}{2} x_3$$
$$\text{i.e., } 3 + \frac{1}{2} = x_2 + \frac{1}{2} x_3$$

$\therefore$ The fractional cut (Gomorian) construct is given by

$$\frac{1}{2} x_3 \geq \frac{1}{2}$$
$$\Rightarrow \frac{-1}{2} x_3 \leq -\frac{1}{2} \Rightarrow -\frac{1}{2} x_3 + s_1 = -\frac{1}{2}$$

where s_1 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have

C_B	Y_B	X_B	x_1 (1)	x_2 (2)	x_3 (0)	x_4 (0)	x_5 (0)	s_1 (0)
2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	0
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{2}$	1	0	0
0	x_5	4	0	0	1	-2	1	0
0	s_1	$\frac{-1}{2}$	0	0	$\left(\frac{-1}{2}\right)$	0	0	1
$Z_j - C_j$		$\frac{21}{2}$	0	0	$\frac{1}{2}$	1	0	0

Since $s_1 = -\frac{1}{2}$, the solution is infeasible. To find the feasible optimal solution, we have to use the dual simplex method.

Since $s_1 = \frac{-1}{2}$, s_1 leaves the basis.

Also $\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{2}}{\frac{-1}{2}}\right\} = -1$ which corresponds to x_3. So, x_3 enters the basis.

Third Iteration: Introduce x_3 and drop s_1.

C_B	Y_B	X_B	x_1 (1)	x_2 (2)	x_3 (0)	x_4 (0)	x_5 (0)	s_2 (0)
2	x_2	3	0	1	0	0	0	1
1	x_1	4	1	0	0	1	0	-1
0	x_5	3	0	0	0	-2	1	2
0	x_3	1	0	0	1	0	0	-2
$Z_j - C_j$		10	0	0	0	1	0	1

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible integer optimal.

$\therefore$ The optimal integer solution is

Max $Z = 10$, $x_1 = 4$, $x_2 = 3$.

3.1.7 Gomory's Mixed Integer Method:

In mixed integer programming problem only some of the variables are integer constrained, while the other variables may take integer or other real values. Like the pure integer problem, the mixed integer problem should be of the maximization type and all the coefficients and constants should be integers.

The problem is first solved as a continuous LPP by ignoring the integrality condition. If the values of the integer constrained variables are integers, then the current solution is an optimum solution to the given mixed IPP. Otherwise, select the source row which corresponds to the largest fractional part f_k among those basic variables which are constrained to be integers. Then construct the Gomorian constraint (secondary constraint) from the source row.

From the source row $\sum_{j=1}^{n} a_{kj}\, x_j = X_{Bk}$

i.e., $\sum_{j=1}^{n} ([a_{kj}] + f_{kj})x_j = [X_{Bk}] + f_k$

in the form of $\sum_{j=1}^{n} f_{kj}\, x_j \geq f_k$

i.e., $\sum_{j\in J^+} f_{kj}\, x_j + \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj}\, x_j \geq f_k$

i.e., $-\sum_{j\in J^+} f_{kj}\, x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj}\, x_j \leq -f_k$

i.e., $-\sum_{j\in J^+} f_{kj}\, x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj}\, x_j + s_k = -f_k$

where s_k : Gomorian slack

$J^+ = \{J/f_{kj} \geq 0\}$

$J^- = \{J/f_{kj} < 0\}$

Add this secondary constraint at the bottom of the optimum simplex table and use dual simplex method to obtain the new feasible optimal solution. Repeat the procedure until the values of the integer restricted variables are integers in the optimum solution obtained.

$\boxed{\text{Example 1}}$ **Solve the following mixed integer programming problem:**

$$\text{Max } Z = x_1 + x_2$$

subject to constraints

$$2x_1 + 5x_2 \le 16$$

$$6x_1 + 5x_2 \le 30$$

$$x_2 \ge 0\; x_1, \text{ non-negative integer.}$$

Solution: Ignoring the integrality condition and introducing the non–negative slack variables x_3 and x_4, the standard form of the continuous LPP becomes

$$\text{Max } Z = x_1 + x_2 + 0x_3 + 0x_4$$

subject to

$$2x_1 + 5x_2 + x_3 + 0x_4 = 16$$

$$6x_1 + 5x_2 + 0x_3 + x_4 = 30$$

$$\text{and } x_1, x_2, x_3, x_4 \ge 0$$

The initial basic feasible solution is given by $x_3 = 16$, $x_4 = 30$, $(x_1 = x_2 = 0$, non-basic).

Initial iteration:

		C_j	(1	1	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	16	2	5	1	0	8
0	x_4	30	(6)	5	0	1	5
$(Z_j - C_j)$		0	–1	–1	0	0	

First iteration: Introduce x_1 and drop x_4.

		C_j	(1	1	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	6	0	$\left(\frac{10}{3}\right)$	1	$\frac{-1}{3}$	$\frac{9}{5}$
1	x_1	5	1	$\frac{5}{6}$	0	$\frac{1}{6}$	6
$(Z_j - C_j)$		5	0	$\frac{-1}{6}$	0	$\frac{1}{6}$	

Second iteration: Introduce x_2 and drop x_3.

		C_j	(1	1	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
1	x_2	$\frac{18}{10}$	0	1	$\frac{3}{10}$	$\frac{-1}{10}$
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{4}$	$\frac{1}{4}$
$(Z_j - C_j)$		$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$

Since all $(Z_j - C_j) \ge 0$, the current basic feasible solution is optimal.

Since the integer constrained variable x_1 is non–integer, we have from the second (source) row

$$\frac{7}{2} = x_1 + 0x_2 - \frac{1}{4}x_3 + \frac{1}{4}x_4$$

$$3 + \frac{1}{2} = x_1 + 0x_2 - \frac{1}{4}x_3 + \frac{1}{4}x_4$$

The Gomorian constraint is given by

$$\left(\frac{\frac{1}{2}}{\frac{1}{2}-1}\right)\left(\frac{-1}{4}\right)x_3 + \frac{1}{4}x_4 \ge \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{4}x_3 + \frac{1}{4}x_4 \ge \frac{1}{2}$$

$$\Rightarrow \qquad -\frac{1}{4}x_3 - \frac{1}{4}x_4 \leq -\frac{1}{2}$$

$$\Rightarrow \qquad -\frac{1}{4}x_3 - \frac{1}{4}x_4 + s_1 = -\frac{1}{2}$$

where s_1 is the Gomorian slack.

Add this Gomorian constraint at the bottom of the above optimum simplex table, we have

C_j (1 1 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$\frac{-1}{10}$	0
1	x_1	$\frac{7}{2}$	1	0	$\frac{-1}{4}$	$\frac{1}{4}$	0
0	s_1	$\frac{-1}{2}$	0	0	$\frac{-1}{4}$	$\frac{-1}{4}$	1
$(Z_j - C_j)$		$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$	0

Here the solution is optimal but infeasible.

So, we have to use the dual simplex method.

Since $s_1 = -\frac{1}{2}$, s_1 leaves the basis.

$$\text{Now Max}\left\{\frac{Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{20}}{\frac{-1}{4}}, \frac{\frac{3}{20}}{\frac{-1}{4}}\right\}$$

$= \text{Max}\left\{\frac{-4}{20}, \frac{-12}{20}\right\} = \text{Max}\left\{\frac{-1}{5}, \frac{-3}{5}\right\} = \frac{-1}{5}$ which corresponds to the variable x_3, so x_3 enters the basis.

Third iteration: Drop s_1 and introduce x_3.

C_j (1 1 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_2	$\frac{6}{5}$	0	1	0	$\frac{-2}{5}$	$\frac{6}{5}$
1	x_1	4	1	0	0	$\frac{1}{2}$	−1
0	x_3	2	0	0	1	1	−4
$(Z_j - C_j)$		$\frac{26}{5}$	0	0	0	$\frac{1}{10}$	$\frac{1}{5}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{B_i} \geq 0$, the current solution is feasible and optimal.

Also, since the integer constrained variable x_1 is integer, the required optimum solution is

$$\text{Max } Z = \frac{26}{5},\ x_1 = 4,\ x_2 = \frac{6}{5}$$

Example 2 **Solve the following Mixed integer Programming problem by Gomory's cutting plane algorithm:**

$$\textbf{Maximize Z} = x_1 + x_2$$

subject to

$$3x_1 + 2x_2 \leq 5$$
$$x_2 \leq 2$$

and $x_1, x_2 \geq 0$ **and** x_1 **an integer.**

[MU. BE. Apr 90]

Solution: Ignoring the integrality condition and introducing the non-negative slack variables x_3, x_4, the standard form of the continuous LPP becomes

$$\text{Maximize } Z = x_1 + x_2 + 0x_3 + 0x_4$$

subject to

$$3x_1 + 2x_2 + x_3 + 0x_4 = 5$$
$$0x_1 + x_2 + 0x_3 + x_4 = 2$$

and $x_1, x_2, x_3, x_4 \geq 0$

The initial basic feasible solution is given by

$$x_3 = 5, \quad x_4 = 2 \text{ (basic)}$$
$$(x_1 = x_2 = 0, \text{ non basic})$$

Initial iteration:

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	5	(3)	2	1	0	$\frac{5}{3}$
0	x_4	2	0	1	0	1	–
$Z_j - C_j$		0	–1	–1	0	0	

First iteration: Introduce x_1 and drop x_3.

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
1	x_1	$\frac{5}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{5}{2}$
0	x_4	2	0	(1)	0	1	2
$Z_j - C_j$		$\frac{5}{3}$	0	$\frac{-1}{3}$	$\frac{1}{3}$	0	

Second iteration: Introduce x_2 and drop x_4.

C_j (1 1 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4
1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{2}{3}$
1	x_2	2	0	1	0	1
$Z_j - C_j$		$\frac{7}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal But x_1 is non-integer.

From the source row (first row), we have

$$\frac{1}{3} = x_1 + 0x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4$$

$\therefore$ The Gomorian construct is given by

$$\frac{1}{3}x_3 + \left(\frac{\frac{1}{3}}{1/3 - 1}\right)\left(\frac{-2}{3}\right)x_4 \geq \frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{1}{3} \Rightarrow -\frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{1}{3}$$

$$\Rightarrow \quad -\frac{1}{3}x_3 - \frac{1}{3}x_4 + s_1 = -\frac{1}{3}$$

where s_1 is the Gomorian slack.

Add this Gomarian constraint at the bottom of the above optimum simplex table, we have

C_j (1 1 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{-2}{3}$	0
1	x_2	2	0	1	0	1	0
0	s_1	$\frac{-1}{3}$	0	0	$\frac{-1}{3}$	$\frac{-1}{3}$	1
$Z_j - C_j$		$\frac{7}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0

Here the solution is optimal but infeasible. So, we have to use the ***dual simplex*** method.

Since $s_1 = -\frac{1}{3}$, s_1 leaves the basis.

$$\text{Also Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{3}}{\frac{-1}{3}}, \frac{\frac{1}{3}}{\frac{-1}{3}}\right\}$$

$= \text{Max}\{-1, -1\} = -1$ which corresponds to both x_3 and x_4. We choose x_3 arbitrarily as the entering variable.

Third iteration: Drop s_1 and introduce x_3.

	C_j		1	1	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
1	x_1	0	1	0	0	–1	1
1	x_2	2	0	1	0	1	0
0	x_3	1	0	0	1	1	–3
$Z_j - C_j$		2	0	0	0	0	1

Since all $(Z_j - C_j) \geq 0$ and all $X_{B_i} \geq 0$, the current solution is feasible and optimal.

∴ The required solution is

$$\text{Max Z} = 2, \quad x_1 = 0, \quad x_2 = 2.$$

Example 3 **Solve the following Mixed integer programming problem.**

$$\textbf{Minimize Z} = x_1 - 3x_2$$

subject to

$$x_1 + x_2 \leq 5$$
$$-2x_1 + 4x_2 \leq 11$$
$$x_1, x_2 \geq 0 \text{ and } x_2 \text{ is an integer.}$$

Solution: Given Mixed I.P.P be

$$\text{Min } Z = x_1 - 3x_2$$

subject to

$$x_1 + x_2 \leq 5$$
$$-2x_1 + 4x_2 \leq 11$$
$$x_1, x_2 \geq 0 \text{ and } x_2 \text{ is an integer.}$$

i.e., Maximize $Z^* = -x_1 + 3x_2$

subject to

$$x_1 + x_2 \leq 5$$
$$-2x_1 + 4x_2 \leq 11$$
$$x_1, x_2 \geq 0 \text{ and } x_2 \text{ is an integer.}$$

Ignoring the integrality condition and introducing the non-negative slack variables x_3 and x_4, the standard form of the continuous L.P.P becomes.

$$\text{Maximize Z}^* = -x_1 + 3x_2 + 0x_3 + 0x_4$$

subject to

$$x_1 + x_2 + x_3 + 0x_4 = 5$$
$$-2x_1 + 4x_2 + 0x_3 + x_4 = 11$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The initial basic feasible solution is given by

$$x_3 = 5, \quad x_4 = 11, \text{ (basic)}$$
$$(x_1 = x_2 = 0, \text{ non-basic)}$$

Initial iteration:

	C_j		–1	3	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	5	1	1	1	0	$\frac{5}{1}$
0	x_4	11	–2	(4)	0	1	$\frac{11}{4}$
$Z_j^* - C_j$		0	1	–3	0	0	

First iteration: Introduce x_2 and drop x_4.

	C_j		–1	3	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	$\frac{9}{4}$	$\left(\frac{3}{2}\right)$	0	1	$\frac{-1}{4}$	$\frac{3}{2}$
3	x_2	$\frac{11}{4}$	$\frac{-1}{2}$	1	0	$\frac{1}{4}$	–
$Z_j^* - C_j$		$\frac{33}{4}$	$\frac{-1}{2}$	0	0	$\frac{3}{4}$	

Second iteration: Introduce x_1 and drop x_3.

		C_j	$(-1$	3	0	$0)$
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
-1	x_1	$\frac{3}{2}$	1	0	$\frac{2}{3}$	$\frac{-1}{6}$
3	x_2	$\frac{7}{2}$	0	1	$\frac{1}{3}$	$\frac{1}{6}$
$Z_j^* - C_j$		9	0	0	$\frac{1}{3}$	$\frac{2}{3}$

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

But the integer constrained variable $x_2 = \frac{7}{2}$ is non-integer.

To obtain the integer value for x_2, we have to construct the Gomorian constraint. Now from the source row, we have

$$\frac{7}{2} = x_2 + \frac{1}{3} x_3 + \frac{1}{6} x_4$$

$$3 + \frac{1}{2} = x_2 + \frac{1}{3} x_3 + \frac{1}{6} x_4$$

$\therefore$ The Gomorian constraint is given by

$$\frac{1}{3} x_3 + \frac{1}{6} x_4 \geq \frac{1}{2} \quad \Rightarrow \quad \frac{-1}{3} x_3 - \frac{1}{6} x_4 \leq -\frac{1}{2}$$

$$\Rightarrow \frac{-1}{3} x_3 - \frac{1}{6} x_4 + s_1 = -\frac{1}{2}$$

where s_1 is the Gomorian slack.

Add this secondary (Gomorian) constraint at the bottom of the above optimum simplex table, we have

		C_j	$(-1$	3	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
-1	x_1	$\frac{3}{2}$	1	0	$\frac{2}{3}$	$\frac{-1}{6}$	0
3	x_2	$\frac{7}{2}$	0	1	$\frac{1}{3}$	$\frac{1}{6}$	0
0	s_1	$\frac{-1}{2}$	0	0	$\left(\frac{-1}{3}\right)$	$\frac{-1}{6}$	1
$Z_j^* - C_j$		9	0	0	$\frac{1}{3}$	$\frac{2}{3}$	0

Here the solution is optimal but infeasible. So we have to use dual simplex method.

Since $s_1 = -\frac{1}{2}$, s_1 leaves the basis.

Also, $\text{Max}\left\{\frac{Z_j^* - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{1}{3}}{\frac{-1}{3}}, \frac{\frac{2}{3}}{\frac{-1}{6}}\right\}$

$= \text{Max} \ \{-1, -4\} = -1$ which corresponds to x_3. So x_3. enters the basis.

Third iteration: Drop s_1 and introduce x_3.

		C_j	$(-1$	3	0	0	$0)$
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1
-1	x_1	$\frac{1}{2}$	1	0	0	$\frac{-1}{2}$	2
3	x_2	3	0	1	0	0	1
0	x_3	$\frac{3}{2}$	0	0	1	$\frac{1}{2}$	-3
$Z_j^* - C_j$		$\frac{17}{2}$	0	0	0	$\frac{1}{2}$	1

Since all $(Z^*_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible optimal.

$\therefore$ The optimal integer solution is

$$\text{Max } Z^* = \frac{17}{2}, x_1 = \frac{1}{2}, x_2 = 3$$

$$\text{But Min } Z = -\text{Max } Z^* = -\frac{17}{2}$$

$$\therefore \text{Min } Z = \frac{-17}{2}, \quad x_1 = \frac{1}{2}, \quad x_2 = 3.$$

Example 4 **Solve the following mixed integer problem**

$$\textbf{Minimize } Z = 10x_1 + 9x_2$$

subject to

$$x_1 \leq 8$$
$$x_2 \leq 10$$
$$5x_1 + 3x_2 \geq 45$$
$$x_1, x_2 \geq 0, \; x_1 \textbf{ integer.}$$

Solution: Given Mixed I.P.P. be

$$\text{Minimize } Z = 10x_1 + 9x_2$$

subject to

$$x_1 \leq 8$$
$$x_2 \leq 10$$
$$5x_1 + 3x_2 \geq 45$$
$$x_1, x_2 \geq 0, \; x_1 \text{ integer.}$$

i.e.,

$$\text{Max } Z^* = -10x_1 - 9x_2$$

subject to

$$x_1 \leq 8$$
$$x_2 \leq 10$$
$$5x_1 + 3x_2 \geq 45$$
$$x_1, x_2 \geq 0, \; x_1 \text{ integer.}$$

Ignoring the integrality condition and introducing the non–negative slack variables x_3, x_4, surplus variable x_5 and artificial variable R_1, the standard form of the continuous LPP becomes:

$$\text{Max } Z^* = -10x_1 - 9x_2 + 0x_3 + 0x_4 + 0x_5 - MR_1$$

subject to

$$x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 + 0R_1 = 8$$
$$0x_1 + x_2 + 0x_3 + x_4 + 0x_5 + 0R_1 = 10$$
$$5x_1 + 3x_2 + 0x_3 + 0x_4 - x_5 + R_1 = 45$$
$$\text{and } x_1, x_2, x_3, x_4, x_5, R_1 \geq 0$$

The initial basic feasible solution is given by

$$x_3 = 8, \; x_4 = 10, \; R_1 = 45 \text{ (basic)}$$
$$(x_1 = x_2 = 0, \; x_5 = 0, \text{ non basic})$$

Initial iteration:

		C_j	$(-10$	-9	0	0	0	$-M)$	
C_B	Y_B	X_B	x_1	x_2	x_5	x_3	x_4	R_1	θ
0	x_3	8	(1)	0	0	1	0	0	8
0	x_4	10	0	1	0	0	1	0	–
$-M$	R_1	45	5	3	-1	0	0	1	9
$Z_j^* - C_j$		$-45M$	$-5M+10$	$-3M+9$	M	0	0	0	

First iteration: Introduce x_1 and drop x_3.

		C_j	$(-10$	-9	0	0	0	$-M)$	
C_B	Y_B	X_B	x_1	x_2	x_5	x_3	x_4	R_1	θ
-10	x_1	8	1	0	0	1	0	0	–
0	x_4	10	0	1	0	0	1	0	10
$-M$	R_1	5	0	(3)	-1	-5	0	1	$\frac{5}{3}$
$Z_j^* - C_j$		$-5M-80$	0	$-3M+9$	M	$5M-10$	0	0	

Second iteration: Introduce x_2 and drop R_1.

C_j (–10 –9 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_5	x_3	x_4
–10	x_1	8	1	0	0	1	0
0	x_4	$\frac{25}{3}$	0	0	$\frac{1}{3}$	$\frac{5}{3}$	1
–9	x_2	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	$\frac{-5}{3}$	0
$Z_j^* - C_j$		–95	0	0	3	5	0

Since all $(Z_j - C_j) \geq 0$, and the integer constrained variable x_1 is an integer, the current basic feasible solution is optimal.

$$\text{Max } Z^* = -95, \quad x_1 = 8, \quad x_2 = \frac{5}{3}$$

$$\text{But Min } Z = -\text{Max}(-Z) = -\text{Max } Z^*$$
$$= -(-95) = 95$$

∴ The optimal solution is

$$\therefore \text{Min Z} = 95, \quad x_1 = 8, \quad x_2 = \frac{5}{3}$$

Example 5 **Solve the following mixed integer problem:**

$$\textbf{Maximize } Z = -3x_1 + x_2 + 3x_3$$

subject to

$$-x_1 + 2x_2 + x_3 \leq 4$$
$$2x_2 - \frac{3}{2}x_3 \leq 1$$
$$x_1 - 3x_2 + 2x_3 \leq 3$$
$$\text{and } x_1, x_2 \geq 0, \; x_3 \text{ non-negative integer}$$

Solution: The given Mixed I.P.P is

$$\text{Maximize } Z = -3x_1 + x_2 + 3x_3$$

subject to

$$-x_1 + 2x_2 + x_3 \leq 4$$
$$4x_2 - 3x_3 \leq 2$$
$$x_1 - 3x_2 + 2x_3 \leq 3$$
$$\text{and } x_1, x_2 \geq 0, \; x_3 \text{ non-negative integer}$$

Ignoring the integrality condition and introducing the non–negative slack variables x_4, x_5 and x_6, the standard form of the continuous L.P.P. becomes

$$\text{Maximize } Z = -3x_1 + x_2 + 3x_3$$

subject to

$$-x_1 + 2x_2 + x_3 + x_4 + 0x_5 + 0x_6 = 4$$
$$0x_1 + 4x_2 - 3x_3 + 0x_4 + x_5 + 0x_6 = 2$$
$$x_1 - 3x_2 + 2x_3 + 0x_4 + 0x_5 + x_6 = 3$$
$$\text{and } x_i \geq 0, \quad i = 1, 2, 3, 4, 5, 6$$

The initial basic feasible solution is given by

$$x_4 = 4, \; x_5 = 2, \; x_6 = 3$$
$$(x_1 = x_2 = x_3 = 0, \text{ non basic})$$

Initial iteration:

C_j (–3 1 3 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
0	x_4	4	–1	2	1	1	0	0	4
0	x_5	2	0	4	–3	0	1	0	–
0	x_6	3	1	–3	(2)	0	0	1	$\frac{3}{2}$
$Z_j - C_j$		0	3	–1	–3	0	0	0	

First iteration: Introduce x_3 and drop x_6.

C_j (–3 1 3 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
0	x_4	$\frac{5}{2}$	$\frac{-3}{2}$	$\left(\frac{7}{2}\right)$	0	1	0	$\frac{-1}{2}$	$\frac{5}{7}$
0	x_5	$\frac{13}{2}$	$\frac{3}{2}$	$\frac{-1}{2}$	0	0	1	$\frac{3}{2}$	–
3	x_3	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{-3}{2}$	1	0	0	$\frac{1}{2}$	–
$Z_j - C_j$		$\frac{9}{2}$	$\frac{9}{2}$	$\frac{-11}{2}$	0	0	0	$\frac{3}{2}$	

Second iteration: Introduce x_2 and drop x_4.

C_B	Y_B	C_j → X_B	x_1 (−3)	x_2 (1)	x_3 (3)	x_4 (0)	x_5 (0)	x_6 (0)
1	x_2	$\frac{5}{7}$	$\frac{-3}{7}$	1	0	$\frac{2}{7}$	0	$\frac{-1}{7}$
0	x_5	$\frac{48}{7}$	$\frac{9}{7}$	0	0	$\frac{1}{7}$	1	$\frac{10}{7}$
3	x_3	$\frac{18}{7}$	$\frac{-1}{7}$	0	1	$\frac{3}{7}$	0	$\frac{2}{7}$
$Z_j - C_j$		$\frac{59}{7}$	$\frac{15}{7}$	0	0	$\frac{11}{7}$	0	$\frac{5}{7}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal. Since the integer constrained variable x_3 is non–integer, from the source row (third row) we have

$$\frac{18}{7} = \frac{-1}{7}x_1 + 0x_2 + x_3 + \frac{3}{7}x_4 + 0x_5 + \frac{2}{7}x_6$$

$$i.e.,\ 2 + \frac{4}{7} = \frac{-1}{7}x_1 + x_3 + \frac{3}{7}x_4 + \frac{2}{7}x_6$$

∴ The Gomorian constraint is given by

$$\left(\frac{\frac{4}{7}}{4/7 - 1}\right)\left(\frac{-1}{7}\right)x_1 + \frac{3}{7}x_4 + \frac{2}{7}x_6 \geq \frac{4}{7}$$

$$\left(\frac{\frac{4}{7}}{-3/7}\right)\left(\frac{-1}{7}\right)x_1 + \frac{3}{7}x_4 + \frac{2}{7}x_6 \geq \frac{4}{7}$$

$$\Rightarrow \quad \frac{4}{21}x_1 + \frac{3}{7}x_4 + \frac{2}{7}x_6 \geq \frac{4}{7}$$

$$\Rightarrow \quad \frac{-4}{21}x_1 - \frac{3}{7}x_4 - \frac{2}{7}x_6 + s_1 = \frac{-4}{7}$$

where s_1 is the Gomorian slack.

Add this Gomorian cut constraint at the bottom of the above optimum simplex table, we have

C_B	Y_B	C_j → X_B	x_1 (−3)	x_2 (1)	x_3 (3)	x_4 (0)	x_5 (0)	x_6 (0)	s_1 (0)
1	x_2	$\frac{5}{7}$	$\frac{-3}{7}$	1	0	$\frac{2}{7}$	0	$\frac{-1}{7}$	0
0	x_5	$\frac{48}{7}$	$\frac{9}{7}$	0	0	$\frac{1}{7}$	1	$\frac{10}{7}$	0
3	x_3	$\frac{18}{7}$	$\frac{-1}{7}$	0	1	$\frac{3}{7}$	0	$\frac{2}{7}$	0
0	s_1	$\frac{-4}{7}$	$\frac{-4}{21}$	0	0	$\frac{-3}{7}$	0	$\left(\frac{-2}{7}\right)$	1
$Z_j - C_j$		$\frac{59}{7}$	$\frac{15}{7}$	0	0	$\frac{11}{7}$	0	$\frac{5}{7}$	0

Here the solution is optimal but infeasible. So we have to use the dual simplex method.

Since $s_1 = -\frac{4}{7}$, s_1 leaves the basis.

$$\text{Also Max}\left\{\frac{Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\} = \text{Max}\left\{\frac{\frac{15}{7}}{\frac{-4}{21}}, \frac{\frac{11}{7}}{\frac{-3}{7}}, \frac{\frac{5}{7}}{\frac{-2}{7}}\right\}$$

$= \frac{-5}{2}$, which corresponds to x_6. So, x_6 enters the basis.

Third Iteration: Introduce x_6 and drop s_1.

C_B	Y_B	C_j → X_B	x_1 (−3)	x_2 (1)	x_3 (3)	x_4 (0)	x_5 (0)	x_6 (0)	s_1 (0)
1	x_2	1	$\frac{-1}{3}$	1	0	$\frac{1}{2}$	0	0	$\frac{-1}{2}$
0	x_5	4	$\frac{1}{3}$	0	0	−2	1	0	5
3	x_3	2	$\frac{-1}{3}$	0	1	0	0	0	1
0	x_6	2	$\frac{2}{3}$	0	0	$\frac{3}{2}$	0	1	$\frac{-7}{2}$
$Z_j - C_j$		7	$\frac{5}{3}$	0	0	$\frac{1}{2}$	0	0	$\frac{5}{2}$

Since all $(Z_j - C_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is optimal and feasible.

$\therefore$ The integer optimal solution is

$$\text{Max } Z = 7, \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 2$$

Example 6 **Solve the following mixed integer programming problem:**

$$\begin{aligned} \textbf{Maximize } Z &= 4x_1 + 6x_2 + 2x_3 \\ \textbf{subject to} \quad 4x_1 - 4x_2 &\leq 5 \\ -x_1 + 6x_2 &\leq 5 \\ -x_1 + x_2 + x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0 \textbf{ and } x_1, x_3 \textbf{ are integers.} \end{aligned}$$

Solution: Ignoring the integrality condition and introducing the non-negative slack variables x_4, x_5 and x_6, the standard form of the continuous L.P.P. becomes

$$\begin{aligned} \text{Maximize } Z &= 4x_1 + 6x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6 \\ \text{subject to} \quad 4x_1 - 4x_2 + 0x_3 + x_4 + 0x_5 + 0x_6 &= 5 \\ -x_1 + 6x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 &= 5 \\ -x_1 + x_2 + x_3 + 0x_4 + 0x_5 + x_6 &= 5 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

The initial basic feasible solution is given by

$$x_4 = 5, \quad x_5 = 5, \quad x_6 = 5 \text{ (basic)}$$

$$(x_1 = x_2 = x_3 = 0, \quad \text{non basic)}$$

Initial iteration:

C_j (4 6 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
0	x_4	5	4	–4	0	1	0	0	–
0	x_5	5	–1	(6)	0	0	1	0	$\frac{5}{6}$
0	x_6	5	–1	1	1	0	0	1	5
$Z_j - C_j$		0	–4	–6	–2	0	0	0	

First iteration: Introduce x_2 and drop x_5.

C_j (4 6 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
0	x_4	$\frac{25}{3}$	$\left(\frac{10}{3}\right)$	0	0	1	$\frac{2}{3}$	0	$\frac{25}{10}$
6	x_2	$\frac{5}{6}$	$-\frac{1}{6}$	1	0	0	$\frac{1}{6}$	0	–
0	x_6	$\frac{25}{6}$	$\frac{-5}{6}$	0	1	0	$\frac{-1}{6}$	1	–
$Z_j - C_j$		5	–5	0	–2	0	1	0	

Second iteration: Introduce x_1 and drop x_4.

C_j (4 6 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
4	x_1	$\frac{5}{2}$	1	0	0	$\frac{3}{10}$	$\frac{1}{5}$	0	–
6	x_2	$\frac{5}{4}$	0	1	0	$\frac{1}{20}$	$\frac{1}{5}$	0	–
0	x_6	$\frac{25}{4}$	0	0	(1)	$\frac{1}{4}$	0	1	$\frac{25}{4}$
$Z_j - C_j$		$\frac{70}{4}$	0	0	–2	$\frac{3}{2}$	2	0	

Third iteration: Introduce x_3 and drop x_6.

C_j (4 6 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6
4	x_1	$\frac{5}{2}$	1	0	0	$\frac{3}{10}$	$\frac{1}{5}$	0
6	x_2	$\frac{5}{4}$	0	1	0	$\frac{1}{20}$	$\frac{1}{5}$	0
2	x_3	$\frac{25}{4}$	0	0	1	$\frac{1}{4}$	0	1
$Z_j - C_j$		30	0	0	0	2	2	2

Since all $(Z_j - C_j) \geq 0$, the current solution is optimal.

But the integer constrained variables x_1 and 3 have non-integer values.

i.e.,
$$x_1 = \frac{5}{2} = 2 + \frac{1}{2} = [X_{B1}] + f_1$$
$$x_3 = \frac{25}{4} = 6 + \frac{1}{4} = [X_{B3}] + f_3$$
$$\therefore \max\{f_1, f_3\} = \text{Max}\left\{\frac{1}{2}, \frac{1}{4}\right\} = \frac{1}{2}$$

So, from the first row (source row) we have
$$\frac{5}{2} = 2 + \frac{1}{2} = x_1 + \frac{3}{10} x_4 + \frac{1}{5} x_5$$

$\therefore$ The Gomorian constraint is given by
$$\frac{3}{10} x_4 + \frac{1}{5} x_5 \geq \frac{1}{2} \Rightarrow \frac{-3}{10} x_4 - \frac{1}{5} x_5 \leq -\frac{1}{2}$$
$$\Rightarrow \frac{-3}{10} x_4 - \frac{1}{5} x_5 + s_1 = -\frac{1}{2}$$

where s_1 is the Gomorian slack.

Add this secondary constraint at the bottom of the above optimum simplex table, we have

	C_j		(4	6	2	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	s_1
4	x_1	$\frac{5}{2}$	1	0	0	$\frac{3}{10}$	$\frac{1}{5}$	0	0
6	x_2	$\frac{5}{4}$	0	1	0	$\frac{1}{20}$	$\frac{1}{5}$	0	0
2	x_3	$\frac{25}{4}$	0	0	1	$\frac{1}{4}$	0	1	0
0	s_1	$\frac{-1}{2}$	0	0	0	$\left(\frac{-3}{10}\right)$	$\frac{-1}{5}$	0	1
$Z_j - C_j$		30	0	0	0	2	2	2	0

Here the solution is optimal but infeasible. So we have to use the dual simplex method.

Since $s_1 = -\frac{1}{2}$, s_1 leaves the basis.

Also $\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{2}{\frac{-3}{10}}, \frac{2}{\frac{-1}{5}}\right\}$

$= \text{Max}\left\{\frac{-20}{3}, -10\right\} = \frac{-20}{3}$, which corresponds to x_4. So, x_4 enters the basis.

Fourth Iteration: Introduce x_4 and drop s_1.

	C_j		(4	6	2	0	0	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	s_1
4	x_1	2	1	0	0	0	0	0	1
6	x_2	$\frac{7}{6}$	0	1	0	0	$\frac{1}{6}$	0	$\frac{1}{6}$
2	x_3	$\frac{35}{6}$	0	0	1	0	$\frac{-1}{6}$	1	$\frac{5}{6}$
0	x_4	$\frac{5}{3}$	0	0	0	1	$\frac{2}{3}$	0	$\frac{-10}{3}$
$Z_j - C_j$		$\frac{80}{3}$	0	0	0	0	$\frac{2}{3}$	2	$\frac{20}{3}$

Here the solution is feasible and optimal. But the integer restricted variable x_3 has non-integer value.

$$i.e., \quad x_3 = \frac{35}{6} = 5 + \frac{5}{6} = [X_{B3}] + f_3$$

So, from the third row we have
$$\frac{35}{6} = 5 + \frac{5}{6} = x_3 - \frac{1}{6} x_5 + x_6 + \frac{5}{6} s_1$$

$\therefore$ The Gomorian constraint is given by
$$\left(\frac{\frac{5}{6}}{\frac{5}{6} - 1}\right)\left(\frac{-1}{6}\right) x_5 + \frac{5}{6} s_1 \geq \frac{5}{6}$$
$$\Rightarrow \frac{5}{6} x_5 + \frac{5}{6} s_1 \geq \frac{5}{6} \Rightarrow -\frac{5}{6} x_5 - \frac{5}{6} s_1 \leq \frac{-5}{6}$$
$$\Rightarrow \frac{-5}{6} x_5 + \frac{-5}{6} s_1 + s_2 = -\frac{5}{6}$$

where s_2 is the Gomorian slack.

Add this secondary constraint at the bottom of the above optimum simplex table, we have

C_B	Y_B	C_j → X_B	x_1 (4)	x_2 (6)	x_3 (2)	x_4 (0)	x_5 (0)	x_6 (0)	s_1 (0)	s_2 (0)
4	x_1	2	1	0	0	0	0	0	1	0
6	x_2	$\frac{7}{6}$	0	1	0	0	$\frac{1}{6}$	0	$\frac{1}{6}$	0
2	x_3	$\frac{35}{6}$	0	0	1	0	$-\frac{1}{6}$	1	$\frac{5}{6}$	0
0	x_4	$\frac{5}{3}$	0	0	0	1	$\frac{2}{3}$	0	$\frac{-10}{3}$	0
0	s_2	$\frac{-5}{6}$	0	0	0	0	$\left(\frac{-5}{6}\right)$	0	$\frac{-5}{6}$	1
$Z_j - C_j$		$\frac{80}{3}$	0	0	0	0	$\frac{2}{3}$	2	$\frac{20}{3}$	0

Here the solution is optimal but infeasible. So we have to use the dual simplex method.

Since $s_2 = -\frac{5}{6}$, s_2 leaves the basis.

$$\text{Also Max}\left\{\frac{Z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\}$$

$$= \text{Max}\left\{\frac{2/3}{\frac{-5}{6}},\ \frac{20/3}{\frac{-5}{6}}\right\}$$

$$= \text{Max}\left\{\frac{-4}{5},\ -8\right\}$$

$$= \frac{-4}{5}, \text{ which corresponds to } x_5.$$

So, x_5 enters the basis.

Fifth Iteration: Introduce x_5 and drop s_2.

C_B	Y_B	C_j → X_B	x_1 (4)	x_2 (6)	x_3 (2)	x_4 (0)	x_5 (0)	x_6 (0)	s_1 (0)	s_2 (0)
4	x_1	2	1	0	0	0	0	0	1	0
6	x_2	1	0	1	0	0	0	0	0	$\frac{1}{5}$
2	x_3	6	0	0	1	0	0	1	1	$\frac{-1}{5}$
0	x_4	1	0	0	0	1	0	0	−4	$\frac{4}{5}$
0	x_5	1	0	0	0	0	1	0	1	$\frac{-6}{5}$
$Z_j - C_j$		26	0	0	0	0	0	2	6	$\frac{4}{5}$

∴ The optimum integer solution is

Max $Z = 26$, $x_1 = 2$, $x_2 = 1$, $x_3 = 6$.

3.1.8 Branch and Bound Method

This method is applicable to both pure (all) as well as mixed integer programming problems and involves the continuous version of the problem.

Let the given IPP be

$$\begin{aligned} \text{Maximize } Z &= CX \\ \text{subject to } AX &\le b \\ X &\ge 0 \text{ and integers.} \end{aligned}$$

In this method also, the given problem is first solved as a continuous LPP by ignoring the integrality condition. If in the optimal solution some one of the variables say x_r is not an integer, then

$x_r^* < x_r < x_r^* + 1$, where x_r^* and $x_r^* + 1$ are consecutive non-negative integers.

Hence any feasible integer value of x_r must satisfy one of the two conditions.

$$x_r \le x_r^* \quad \text{or} \quad x_r \ge x_r^* + 1.$$

Note that these two conditions are mutually exclusive (both cannot be true simultaneously) and hence both can not be amended in the LPP simultaneously. By adding these two conditions separately to the continuous LPP. we form two different sub-problems.

Sub problem 1

$$\begin{aligned} \text{Max } Z &= CX \\ \text{subject to } AX &\le b \\ x_r &\le x_r^* \\ \text{and } X &\ge 0 \end{aligned}$$

Sub - problem 2

$$\begin{aligned} \text{Max } Z &= CX \\ \text{subject to } AX &\le b \\ x_r &\ge x_r^* + 1 \\ \text{and } X &\ge 0 \end{aligned}$$

Thus we have ***branched*** or ***partitioned*** the original problem in to two sub-problems. Geometrically it means that the branching process eliminates that portion of the feasible region that contains no feasible – integer solution. Each of these sub–problems is then solved separately as a LPP

If any sub-problem yields an optimum integer solution, it is not further brached. But, if any sub-problem yields a non-integer solution, it is further branched in to two sub-problems. This branching process is continued, until each problem terminates with either integer optimal solution or there is evidence that it can not yield a better solution. The integer valued solution among all the sub-problems which gives the most optimum value of the objective function is then selected as the optimum solution.

Main disadvantage of this method is that it requires the optimum solution of each sub-problem. In large problems this could be very tedious job. But the computational efficiency of this method is increased by using the concept of ***bounding.*** By this concept whenever the continuous optimum solution of a sub–problem yields a value of the objective function lower than that of the best available integer solution (for maximization problem) it is useless to explore the problem any further. This sub–problem is said to be ***fathomed*** and is dropped from further consideration. Thus once a feasible integer solution is obtained, its associated objective function can be taken as a ***lower bound*** (for maximization problem) to delete inferior sub–problems. Hence efficiency of a branch and bound method depends upon how soon the successive sub–problems are fathomed.

Note: For minimization problems, the procedure is the same except that ***upper bounds*** are used.

Example 1 Use Branch and Bound technique to solve the following:

$$\begin{aligned} \text{Maximize } Z &= x_1 + 4x_2 \\ \text{subject to constraints} \quad 2x_1 + 4x_2 &\le 7 \\ 5x_1 + 3x_2 &\le 15 \\ x_1,\ x_2 &\ge 0 \text{ and integers.} \end{aligned}$$

[MU. MCA. Nov 95, MSU. BE. Nov 96]

Solution: Ignoring the integrality condition, the continuous LPP becomes

$$\begin{aligned} \text{Max } Z &= x_1 + 4x_2 \\ \text{subject to} \quad 2x_1 + 4x_2 &\le 7 \\ 5x_1 + 3x_2 &\le 15 \\ \text{and } x_1,\ x_2 &\ge 0 \end{aligned}$$

By using graphical method, the solution space is given by the region OABC.

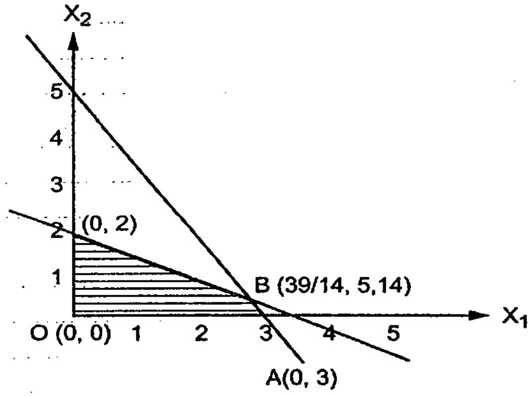


Fig. 9.3

The optimum solution of this problem is

$$\text{Max } Z = 7, \quad x_1 = 0, \quad x_2 = \frac{7}{4}$$

Since $x_2 = \frac{7}{4}$, this problem should be branched in to two sub-problems.

$$\text{For, } x_2 = \frac{7}{4} \Rightarrow 1 < x_2 < 2$$

$$\Rightarrow x_2 \le 1 \text{ or } x_2 \ge 2$$

Applying these two conditions separately in the continuous LPP we have two sub problems

Sub–problem (1)

$$\text{Max } Z = x_1 + 4x_2$$

subject to

$$2x_1 + 4x_2 \leq 7$$
$$5x_1 + 3x_2 \leq 15$$
$$x_2 \leq 1$$
$$\text{and } x_1, x_2 \geq 0$$

Its solution space is given by the region OABDE.

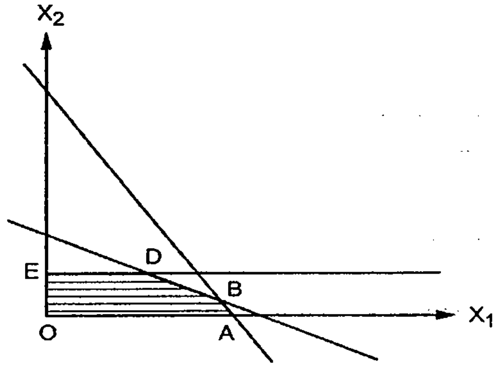


Fig. 9.4

and its optimal solution is

$$\text{Max Z} = \frac{11}{2}, \quad x_1 = \frac{3}{2}, \quad x_2 = 1$$

Since $x_1 = \frac{3}{2}$, this sub-problem is branched again.

Sub–problems (2)

$$\text{Max } Z = x_1 + 4x_2$$

subject to

$$2x_1 + 4x_2 \leq 7$$
$$5x_1 + 3x_2 \leq 15$$
$$x_2 \geq 2$$
$$\text{and } x_1, x_2 \geq 0$$

Its solution space is given by the region OABC and FGH

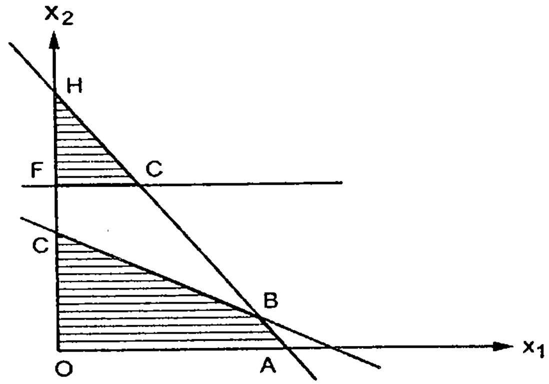


Fig. 9.5

and it has no feasible solution.

Hence this sub-problem is fathomed.

In sub-problem (1), Since $x_1 = \frac{3}{2}$, we have $1 \leq x_1 \leq 2$.

$\Rightarrow \quad x_1 \leq 1$ or $x_1 \geq 2$

Applying these two conditions separately in the sub–problem (1), we have two sub-problems.

Sub-problems (3)

$$\text{Max } Z = x_1 + 4x_2$$

subject to

$$2x_1 + 4x_2 \leq 7$$
$$5x_1 + 3x_2 \leq 15$$
$$x_2 \leq 1$$
$$x_1 \leq 1$$
$$\text{and } x_1, x_2 \geq 0$$

and its solution space is given by

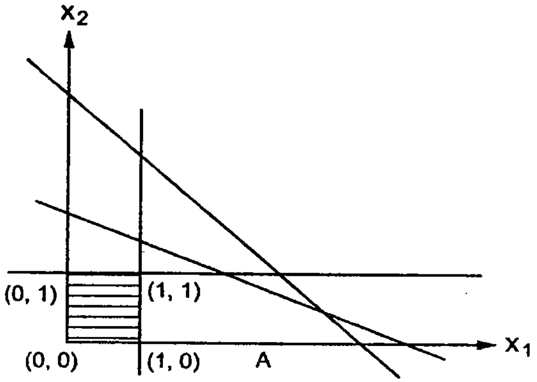


Fig. 9.6

Its optimal solution is given by Max Z = 5, $x_1 = 1$, $x_2 = 1$

Since this solution, is integer valued, this sub–problem can not be further branched and the lower bound of the objective function is 5

Sub–problems (4)

$$\text{Max } Z = x_1 + 4x_2$$

subject to

$$\begin{aligned} 2x_1 + 4x_2 &\le 7 \\ 5x_1 + 3x_2 &\le 15 \\ x_2 &\le 1 \\ x_1 &\ge 2 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

and its solution space is given by

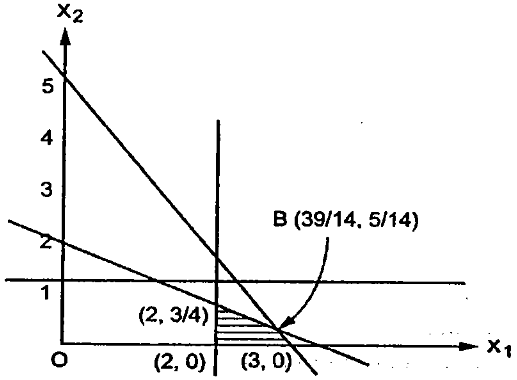


Fig. 9.7

Its optimal solution is given by

$$\text{Max Z} = 5, \ x_1 = 2, \ x_2 = \frac{3}{4}$$

Since $x_2 = \frac{3}{4}$, this sub-problem is branched further.

In sub-problem (4), since $x_2 = \frac{3}{4}$, we have $0 \le x_2 \le 1$

$$\Rightarrow x_2 \le 0 \quad \text{or} \quad x_2 \ge 1$$

Applying these two conditions one by one in the sub-problem (4), we have two sub-problems.

Sub-problem (5)

$$\text{Max } Z = x_1 + 4x_2$$

subject to

$$\begin{aligned} 2x_1 + 4x_2 &\le 7 \\ 5x_1 + 3x_2 &\le 15 \\ x_2 &\le 1 \\ x_1 &\ge 2 \\ x_2 &\le 0 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

and its optimal solution is given by

$$\text{Max Z} = 3, \ x_1 = 3, \ x_2 = 0$$

This sub-problem is fathomed.

Sub-problem (6)

$$\text{Max } Z = x_1 + 4x_2$$

subject to

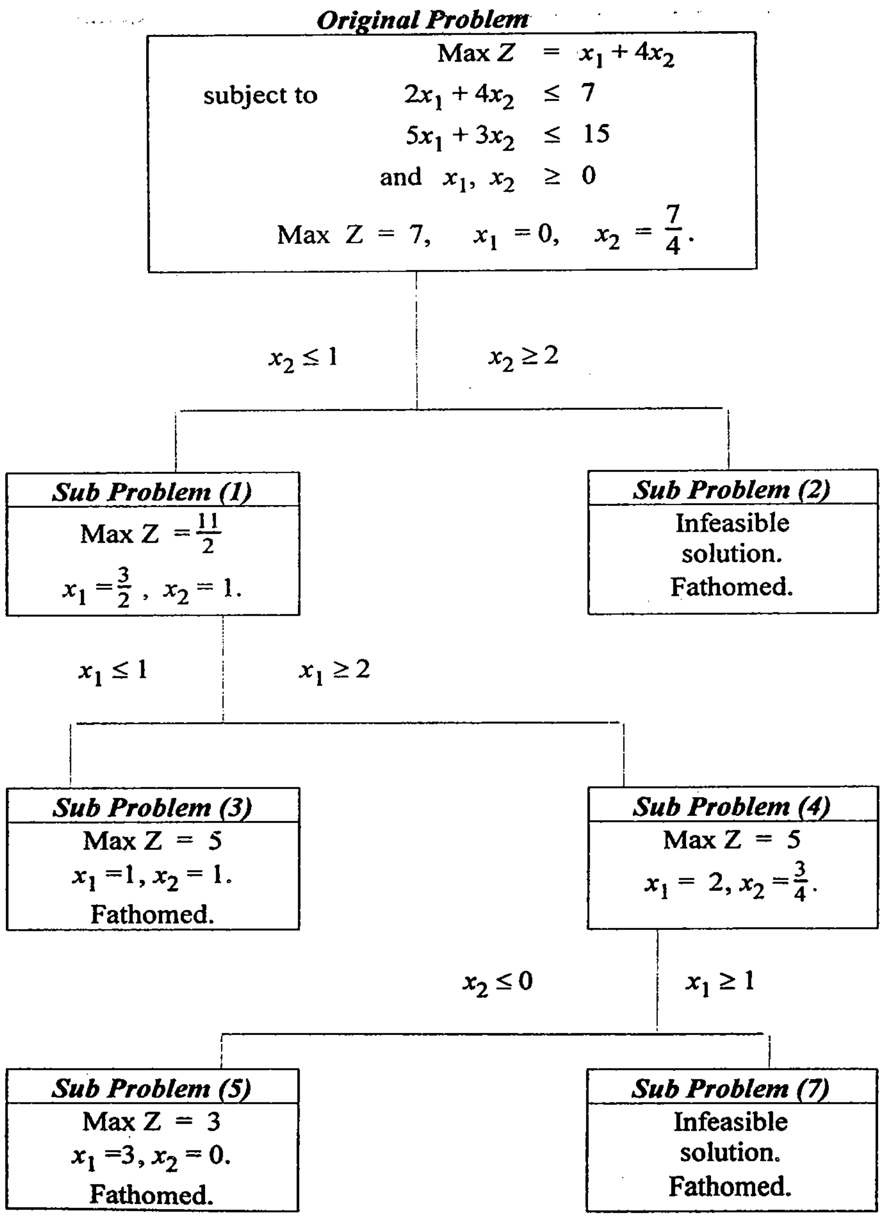
$$\begin{aligned} 2x_1 + 4x_2 &\le 7 \\ 5x_1 + 3x_2 &\le 15 \\ x_2 &\le 1 \\ x_1 &\ge 2 \\ x_2 &\ge 1 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

This sub-problem has no feasible solution. Hence this sub-problem is also fathomed.

Among the available integer valued solutions, the best integer solution is given by sub-problem (3).

∴ The optimum integer solution is

$$\text{Max Z} = 5, \ x_1 = 1, \ x_2 = 1.$$



The best available integer optimal solution is

$$\text{Max } Z = 5, \quad x_1 = 1, \quad x_2 = 1$$

Example 2 **Use Branch and Bound method to solve the following:**

$$\textbf{Maximize } Z = 2x_1 + 2x_2$$

subject to

$$5x_1 + 3x_2 \le 8$$
$$x_1 + 2x_2 \le 4$$
$$\textbf{and } x_1, x_2 \ge 0 \textbf{ and integer.}$$

[MU. MCA. May 95, MKU. M.Sc. 83]

Solution: Ignoring the integrality condition and introducing the non-negative slack variables x_3, x_4, the standard form of the continuous LPP becomes

$$\text{Maximize } Z = 2x_1 + 2x_2 + 0x_3 + 0x_4$$

subject to

$$5x_1 + 3x_2 + x_3 + 0x_4 = 8$$
$$x_1 + 2x_2 + 0x_3 + x_4 = 4$$
$$\text{and } x_1, x_2, x_3, x_4 \ge 0$$

The initial basic feasible solution is given by

$$x_3 = 8, \quad x_4 = 4 \text{ (basic)}$$
$$(x_1 = x_2 = 0, \text{ non-basic})$$

Initial iteration:

C_j (2 2 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	8	(5)	3	1	0	$\frac{8}{5}$
0	x_4	4	1	2	0	1	4
$Z_j - C_j$		0	-2	-2	0	0	

First iteration: Introduce x_1 and drop x_3.

C_j (2 2 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
2	x_1	$\frac{8}{5}$	1	$\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{8}{3}$
0	x_4	$\frac{12}{5}$	0	$\left(\frac{7}{5}\right)$	$\frac{-1}{5}$	1	$\frac{12}{7}$
$Z_j - C_j$		$\frac{16}{5}$	0	$\frac{-4}{5}$	$\frac{2}{5}$	0	

Second iteration: Introduce x_2 and drop x_4.

		C_j	(2	2	0	0)
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
2	x_1	$\frac{4}{7}$	1	0	$\frac{2}{7}$	$\frac{-3}{7}$
2	x_2	$\frac{12}{7}$	0	1	$\frac{-1}{7}$	$\frac{5}{7}$
$Z_j - C_j$		$\frac{32}{7}$	0	0	$\frac{2}{7}$	$\frac{4}{7}$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal, but non-integer.

$$\text{Max Z} = \frac{32}{7}$$
$$x_1 = \frac{4}{7}$$
$$x_2 = \frac{12}{7}$$

In order to obtain the integer optimal solution, we have to branch this problem in to two sub-problem.

Now from $x_2 = \frac{12}{7} \Rightarrow 1 < x_2 < 2$

$$\Rightarrow x_2 \leq 1 \text{ or } x_2 \geq 2$$

Applying these two conditions separately in the continuous LPP, we have two sub-problems.

Sub-problem (1)

$$\begin{aligned} \text{Max Z} &= 2x_1 + 2x_2 \\ \text{subject to } 5x_1 + 3x_2 &\leq 8 \\ x_1 + 2x_2 &\leq 4 \\ x_2 &\leq 1 \\ \text{and } x_1, x_2 &\geq 0 \end{aligned}$$

Its optimal solution is Max Z = 4, $x_1 = 1$, $x_2 = 1$.

So, this sub-problem is fathomed. The lower bound of the objective function is 4.

Sub-problem (2)

$$\begin{aligned} \text{Max Z} &= 2x_1 + 2x_2 \\ \text{subject to } 5x_1 + 3x_2 &\leq 8 \\ x_1 + 2x_2 &\leq 4 \\ x_2 &\geq 2 \\ \text{and } x_1, x_2 &\geq 0 \end{aligned}$$

Its optimal solution is Max Z = 4, $x_1 = 0$, $x_2 = 2$.

So, this sub-problem is also fathomed.

Hence from both the sub-problems (1) and (2), the integer optimum solution is given by

$$\text{Max Z} = 4$$

with $x_1 = 1$, $x_2 = 1$ Or $x_1 = 0$, $x_2 = 2$.

Original Problem

$$\begin{aligned} \text{Max Z} &= 2x_1 + 2x_2 \\ \text{subject to} \quad 5x_1 + 3x_2 &\leq 8 \\ x_1 + 2x_2 &\leq 4 \\ \text{and } x_1, x_2 &\geq 0 \end{aligned}$$

Max Z $= \frac{32}{7}$, $x_1 = \frac{4}{7}$, $x_2 = \frac{12}{7}$.

$x_2 \leq 1$ $x_2 \geq 2$

Sub Problem (1)	***Sub Problem (2)***
Max Z = 4	Max Z = 4
$x_1 = 1$, $x_2 = 1$.	$x_1 = 0$, $x_2 = 2$.
Fathomed.	Fathomed.

Hence the integer optimum solution is

$$\text{Max Z} = 4$$

with $x_1 = 1$, $x_2 = 1$ (or) $x_1 = 0$, $x_2 = 2$.

Example 3 Use Branch and Bound Method to solve the following integer programming problem:

$$\text{Maximize } Z = 3x_1 + 4x_2$$

subject to
$$7x_1 + 16x_2 \le 52$$
$$3x_1 - 2x_2 \le 18$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$ *[MU. MCA. Nov.96]*

Solution: Ignoring the integrality condition and introducing the non-negative slack variables x_3, x_4, the standard form of the continuous LPP becomes.

$$\text{Maximize } Z = 3x_1 + 4x_2 + 0x_3 + 0x_4$$

subject to
$$7x_1 + 16x_2 + x_3 + 0x_4 = 52$$
$$3x_1 - 2x_2 + 0x_3 + x_4 = 18$$
$$x_1, x_2, x_3, x_4 \ge 0$$

The initial basic feasible solution is given by

$$x_3 = 52,\ x_4 = 18, \text{ (basic)}$$
$$(x_1 = x_2 = 0, \text{ non-basic)}$$

Initial iteration:

C_j (3 4 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	52	7	(16)	1	0	$\frac{52}{16}$
0	x_4	18	3	–2	0	1	–
$Z_j - C_j$		0	–3	–4	0	0	

First iteration: Introduce x_2 and drop x_3.

C_j (3 4 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
4	x_2	$\frac{13}{4}$	$\frac{7}{16}$	1	$\frac{1}{16}$	0	$\frac{52}{7}$
0	x_4	$\frac{49}{2}$	$\left(\frac{31}{8}\right)$	0	$\frac{1}{8}$	1	$\frac{196}{31}$ *
$Z_j - C_j$		13	$\frac{-5}{4}$	0	$\frac{1}{4}$	0	

Second iteration: Introduce x_1 and drop x_4.

C_j (3 4 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4
4	x_2	$\frac{15}{31}$	0	1	$\frac{-1}{12}$	$\frac{7}{62}$
3	x_1	$\frac{196}{31}$	1	0	$\frac{1}{3}$	$\frac{8}{31}$
$Z_j - C_j$		$\frac{648}{31}$	0	0	$\frac{2}{3}$	$\frac{19}{43}$

Since all $(Z_j - C_j) \ge 0$, the current basic feasible solution is optimal, but non-integer.

$$i.e.,\quad x_2 = \frac{15}{31} = 0 + \frac{15}{31} \quad\text{and}$$
$$x_1 = \frac{196}{31} = 6 + \frac{10}{31}$$

The maximum fractional part is $\frac{15}{31}$ which corresponds to x_2

$$\therefore\ 0 < x_2 < 1 \quad \Rightarrow x_2 \le 0 \text{ (or) } x_2 \ge 1$$

Applying these two conditions separately in the continuous LPP, we have the two sub-problems.

Sub-Problem 1

Maximize $Z = 3x_1 + 4x_2$

subject to
$$7x_1 + 16x_2 \le 52$$
$$3x_1 - 2x_2 \le 18$$
$$x_2 \le 0$$
$$\text{and } x_1, x_2 \ge 0$$

Its optimal solution is given by
$$\text{Max } Z = 18$$
$$x_1 = 6,\ x_2 = 0$$

Since this solution is in integers, this sub-problem is fathomed. The lower limit of the objective function is 18.

Sub-Problem 2

Maximize $Z = 3x_1 + 4x_2$

subject to
$$7x_1 + 16x_2 \le 52$$
$$3x_1 - 2x_2 \le 18$$
$$x_2 \ge 1$$
$$\text{and } x_1, x_2 \ge 0$$

Its optimal solution is given by
$$\text{Max } Z = 19.43$$
$$x_1 = 5.14,\ x_2 = 1$$

Also in sub-problem (2), since $x_1 = 5.14$

$$\Rightarrow\ 5 < x_1 < 6 \Rightarrow\ x_1 \le 5 \text{ or } x_1 \ge 6$$

Applying these two conditions separately in sub problem (2), we have two sub problems.

Sub-Problem 3

Maximize $Z = 3x_1 + 4x_2$

subject to

$$\begin{aligned} 7x_1 + 16x_2 &\le 52 \\ 3x_1 - 2x_2 &\le 18 \\ x_2 &\ge 1 \\ x_1 &\le 5 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

Its optimal solution is given by

$$\text{Max } Z = 19.25$$
$$x_1 = 5, \quad x_2 = 1.063$$

Sub-Problem 4

Maximize $Z = 3x_1 + 4x_2$

subject to

$$\begin{aligned} 7x_1 + 16x_2 &\le 52 \\ 3x_1 - 2x_2 &\le 18 \\ x_2 &\ge 1 \\ x_1 &\ge 6 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

and has no feasible solution. So this sub problem is also fathomed.

In sub-problem (3), since $x_2 = 1.063$

$$\Rightarrow \quad 1 < x_2 < 2 \quad \Rightarrow \quad x_2 \le 1 \text{ or } x_2 \ge 2.$$

Applying these two conditions separately in sub problem (3), we have two sub problems.

Sub-Problem 5

Maximize $Z = 3x_1 + 4x_2$

subject to

$$\begin{aligned} 7x_1 + 16x_2 &\le 52 \\ 3x_1 - 2x_2 &\le 18 \\ x_2 &\ge 1 \\ x_1 &\le 5 \\ x_2 &\le 1 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

Its optimal solution is given by

$$\text{Max } Z = 19$$
$$x_1 = 5, \quad x_2 = 1$$

Since the solution is in integers, this sub-problem is also fathomed. The new lower bound of the objective function is 19.

Sub-Problem 6

Maximize $Z = 3x_1 + 4x_2$

subject to

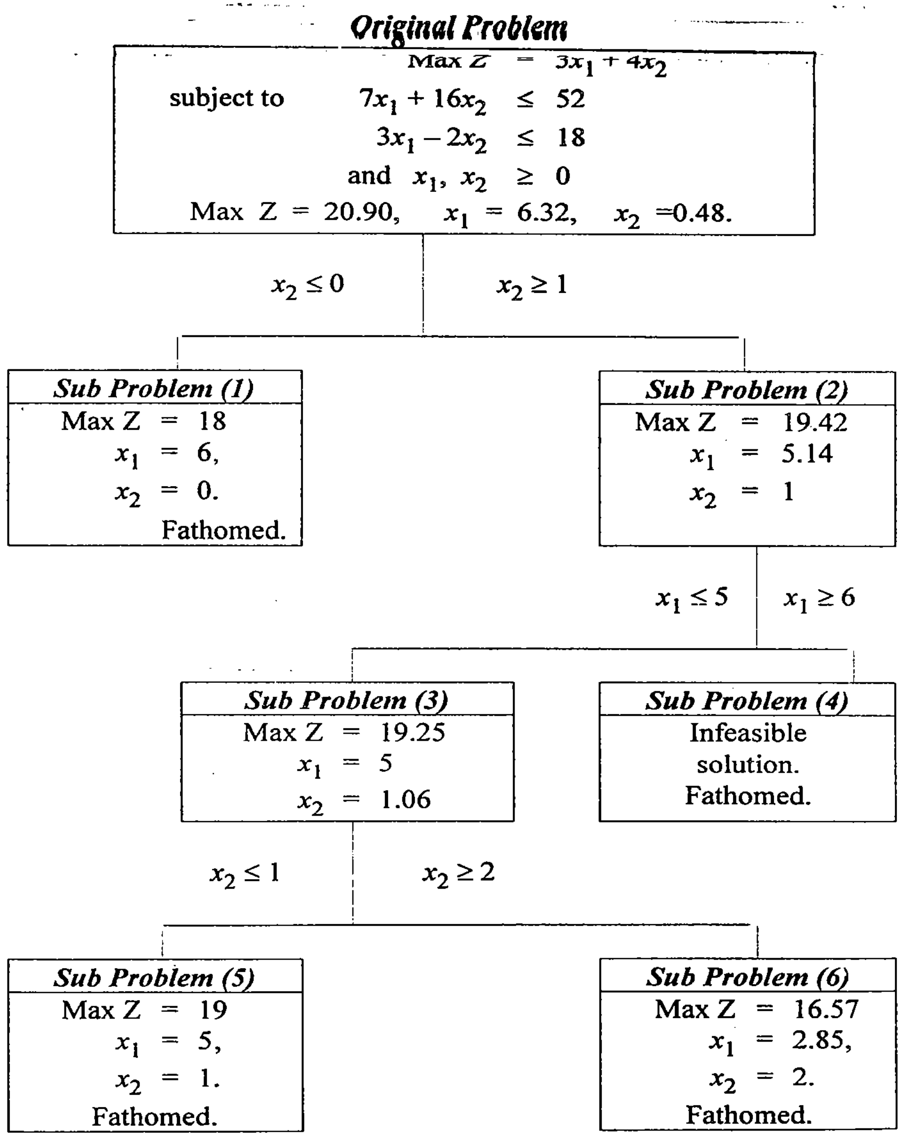
$$\begin{aligned} 7x_1 + 16x_2 &\le 52 \\ 3x_1 - 2x_2 &\le 18 \\ x_2 &\ge 1 \\ x_1 &\le 5 \\ x_2 &\ge 2 \\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

Its optimal solution is given by

$$\text{Max } Z = 16.57$$
$$x_1 = 2.85, \quad x_2 = 2$$

Since 16.57 is less than the new lower bound of the objective function, this sub-problem can also be fathomed.

From the available integer optimal solutions, the best optimal solution is Max $Z = 19$, $x_1 = 5$, $x_2 = 1$, which is the required integer optimal solution.



The best available integer optimal solution is

Max $Z = 19$, $x_1 = 5$, $x_2 = 1$.

EXERCISE

1. What is integer programming.
[MU.MCA. May 89, BNU. BE. Nov 96]
2. Explain whether an integer programming problem can be solved by rounding off he corresponding simplex solution . *[MU. BE.81]*
3. Why not round off the optimum values instead of resorting to integer programming ? Explain. *[MU. MCA. May 89]*
4. Differentiate between pure and Mixed integer programming problem. *[MU. BE. Oct 96]*
5. Write a brief note on cutting plane method.
[MU. MCA. Nov 98, MU. BE. Apr 95]
6. Describe Gomory's all integer programming problem method and its algorithm. *[MU. BE.Nov 89, MKU.M.Sc. 79]*
7. Describe any one method of solving mixed integer programming problem. *[MU. B.Sc. 84]*
8. Explain the importance of integer programming problems and their applications. *[MU. BE. 80]*
9. Explain some of the practical applications of integer programming problems. *[MU. MCA. Nov 97, MU. BE. 79]*
10. Explain the Branch and Bound method in integer programming problem. *[BNU. M.Sc. 81, MU. MCA. Nov 97]*
11. Write down the Branch and Bound algorithm for solving a mixed integer programming problem.
[MU. MCA. Nov. 94, Nov 98]

Find the optimum integer solution of the following all integer programming problems:

12.
$$\begin{aligned} \text{Max } Z &= 3x_1 + 4x_2 \\ \text{subject to } 3x_1 + 2x_2 &\le 8 \\ x_1 + 4x_2 &\le 10 \\ \text{and } x_1, x_2 &\ge 0 \text{ and are integers.} \end{aligned}$$
[MU. BE. Nov. 93, Annamalai BE 82]

13.
$$\begin{aligned} \text{Max } Z &= x_1 - 2x_2 \\ \text{subject to } 4x_1 + 2x_2 &\le 15 \\ \text{and } x_1, x_2 &\ge 0 \text{ and are integers.} \end{aligned}$$

14.
$$\begin{aligned} \text{Max } Z &= 7x_1 + 9x_2 \\ \text{subject to } -x_1 + 3x_2 &\le 6 \\ 7x_1 + x_2 &\le 35 \end{aligned}$$
and x_1, x_2 are non – negative integers.

15.
$$\begin{aligned} \text{Max } Z &= 3x_1 + 4x_2 \\ \text{subject to } 3x_1 + 2x_2 &\le 8 \\ x_1 + 4x_2 &\ge 10 \\ \text{and } x_1, x_2 &\ge 0 \text{ and are integers.} \end{aligned}$$

16.
$$\begin{aligned} \text{Min } Z &= 9x_1 + 10x_2 \\ \text{subject to } x_1 &\le 9 \\ x_2 &\le 8 \\ 4x_1 + 3x_2 &\ge 40 \\ \text{and } x_1, x_2 &\ge 0 \text{ are non-negative integers.} \end{aligned}$$

17.
$$\begin{aligned} \text{Max } Z &= 3x_1 - 2x_2 + 5x_3 \\ \text{subject to } 5x_1 + 2x_2 + 7x_3 &\le 28 \\ 4x_1 + 5x_2 + 5x_3 &\le 30 \\ \text{and } x_1, x_2, x_3 &\ge 0 \text{ and are integers.} \end{aligned}$$
[MKU. BE. 76]

18.
$$\begin{aligned} \text{Max } Z &= 2x_1 + 20x_2 - 10x_3 \\ \text{subject to } 5x_1 + 20x_2 + 4x_3 &\le 15 \\ 6x_1 + 20x_2 + 4x_3 &= 20 \\ x_1, x_2, x_3 &\ge 0 \text{ and are integers.} \end{aligned}$$

19. The owner of a ready-made garments store two types of shirts known as Zee – Shirts and Button – Down – Shirts. He makes a profit of Rs. 1 and Rs. 4 per shirt on Zee – Shirt and Button –Down – Shirts respectively. He has two tailors (A and B) at his disposal to stitch the shirts. Tailor A and Tailor B can devote at the most 7 hours and 15 hours per day respectively. Both these shirts are to be stitched by both the tailors. Tailor A and Tailor B spend two hours and five hours respectively in stitching Zee – Shirts and four and three hours respectively in stitching a Button–Down – shirt. How many shirts of both the types should be stitched in order to maximize daily profit ?

(a) Set up and solve the linear programming problem.

(b) If the optimal solution is not integer – valued, use Gomory technique to derive the optimal integer solution.

Solve the following Mixed integer programming problems:

20. $$\text{Max } Z = 7x_1 + 9x_2$$
$$\text{subject to } -x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$
$$\text{and } x_1, x_2 \ge 0 \; x_1 \text{ is an integer.}$$

[BNU. BE. Apr 97]

21. $$\text{Max } Z = 3x_1 + x_2 + 3x_3$$
$$\text{subject to } -x_1 + 2x_2 + x_3 \le 4$$
$$4x_2 - 3x_3 \le 2$$
$$x_1 - 3x_2 + 2x_3 \le 3$$
$x_2 > 0$ and x_1, x_3 are non-negative integers

22. $$\text{Max } Z = 1.5x_1 + 3x_2 + 4x_3$$
$$\text{subject to } 2.5x_1 + 2x_2 + 4x_3 \le 12$$
$$2x_1 + 4x_2 - x_3 \le 7$$
$x_1, x_2, x_3 \ge 0$ and x_3 is an integer.

Use Branch and Bound technique to solve the following integer programming problems:

23. $$\text{Max } Z = x_1 + x_2$$
$$\text{subject to } 3x_1 + 2x_2 \le 12$$
$$x_2 \le 2$$
$$\text{and } x_1, x_2 \ge 0 \text{ and are integer.}$$

24. $$\text{Max } Z = x_1 + x_2$$
$$\text{subject to } 4x_1 - x_2 \le 10$$
$$2x_1 + 5x_2 \le 10$$
$$2x_1 - 3x_2 \le 6$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

25. $$\text{Max } Z = 3x_1 + 3x_2 + 13x_3$$
$$\text{subject to } -3x_1 + 6x_2 + 7x_3 \le 8$$
$$5x_1 - 3x_2 + 7x_3 \le 8$$
$$0 \le x_j \le 5$$
and all x_j are integers for $j = 1, 2, 3$.

26. $$\text{Max } Z = 3x_1 + 4x_2$$
$$\text{subject to } 3x_1 - x_2 + x_3 = 12$$
$$3x_1 + 11x_2 + x_4 = 66$$
$x_j \ge 0$ and integers for $j = 1, 2, 3, 4$

27. $$\text{Max } Z = 2x_1 + 20x_2 - 10x_3$$
$$\text{subject to } 2x_1 + 20x_2 + 4x_3 \le 15$$
$$6x_1 + 20x_2 + 4x_3 = 20$$
$x_1, x_2, x_3, \ge 0$ and integers.

28. Solve the following integer LP problem by Gomory's cutting plane method.
$$\text{Maximize } Z = x_1 + x_2$$
subject to constraints:
$$2x_1 + x_2 \le 6$$
$$4x_1 + 5x_2 \le 20$$
$x_1 \ge 0, x_2 \ge 0$ and integers. *[MU. BE. Oct 97]*

29. Solve the following integer programming problem by Gomory's cutting plane algorithm:
$$\text{Maximize } Z = x_1 + 2x_2$$
$$\text{subject to } 2x_2 \le 7$$
$$x_1 + x_2 \le 7$$
$x_1, x_2 \ge 0$ and integers. *[MU. BE. Oct 97]*

30. Solve the following integer programming problem:
$$\text{Maximize } Z = 2x_1 + x_2$$
$$\text{subject to } 15x_1 + 7x_2 \le 105$$
$$11x_1 + 15x_2 \le 165$$
$x_1, x_2 \ge 0$ and integers. *[MU. MCA. Nov 97]*

31. **Use the cutting-plane method, to solve the following pure-integer programming problem:**

$$\begin{aligned} \text{Maximize } Z &= 8x_1 + 6x_2 \\ \text{subject to } 4x_1 + 6x_2 &\le 16 \\ 15x_1 + 3x_2 &\le 27 \\ x_1, x_2 &\ge 0 \text{ and are integers.} \end{aligned}$$

[BRU. BE. Apr 97]

32. **Solve the following integer programming problem:**

$$\text{Maximize } Z = 6x_1 + 4x_2$$

subject to the constraints:

$$\begin{aligned} 3x_1 + 4x_2 &\le 20 \\ 6x_1 + 5x_2 &\le 25 \\ 3x_1 + 3x_2 &\le 10 \end{aligned}$$

and x_1, x_2 are non-negative integers. *[BRU. BE. Nov 97]*

33. **Use Branch and Bound method to solve the integer programming problem:**

$$\text{Maximize } Z = 3x_1 + 3x_2 + 13x_3$$

subject to the constraints:

$$\begin{aligned} -3x_1 + 6x_2 + 7x_3 &\le 8 \\ 5x_1 - 3x_2 + 7x_3 &\le 8 \\ 0 \le x_i &\le 5 \end{aligned}$$

All x_i are integers for $i = 1, 2, 3$. *[MKU. MBA. Nov 97]*

34. **Solve the following integer programming problem by Gomory's method:**

$$\begin{aligned} \text{Maximize } Z &= 7x_1 + 5x_2 \\ \text{subject to } \quad x_1 + x_2 &\le 4 \\ 5x_1 + 3x_2 &\le 15 \end{aligned}$$

$x_1, x_2 \ge 0$ and integers. *[MU. BE. Oct 98]*

ANSWERS

12. Max Z = 11, $x_1 = 1$, $x_2 = 2$.
13. Max Z = 3, $x_1 = 3$, $x_2 = 0$.
14. Max Z = 55, $x_1 = 4$, $x_2 = 3$.
15. Max Z = 16, $x_1 = 0$, $x_2 = 4$.

16. Min Z = 101, $x_1 = 9$, $x_2 = 2$.
17. Max Z = 20, $x_1 = 0$, $x_2 = 0$, $x_3 = 4$.
18. This integer program has no feasible solution.
19. Max Z = $x_1 + 4x_2$ subject to $2x_1 + 4x_2 \le 7$, $5x_1 + 3x_2 \le 15$, $x_1, x_2 \ge 0$ and are integers. Also Max Z = 5, $x_1 = 1$, $x_2 = 1$.
20. Max Z = 58, $x_1 = 4$, $x_2 = \frac{10}{3}$.
21. Max Z = 26.75, $x_1 = 5$, $x_2 = 2.75$, $x_3 = 3$.
22. Max Z = 14, $x_1 = 0$, $x_2 = 2$, $x_3 = 2$.
23. Max Z = 4, with *(i)* $x_1 = 2, x_2 = 2$, *(ii)* $x_1 = 3, x_2 = 1$, *(iii)* $x_1 = 4, x_2 = 0$.
24. Max Z = 3, $x_1 = 2$, $x_2 = 1$.
25. Max Z = 9, $x_1 = 2$, $x_2 = 1$, $x_3 = 0$
26. Max Z = 31, $x_1 = 5$, $x_2 = 4$, $x_3 = 1$, $x_4 = 7$.
27. Max Z = −16, $x_1 = 2$, $x_2 = 0$, $x_3 = 2$.
28. Max Z = 4, $x_1 = 2$, $x_2 = 2$.
29. Max Z = 10, $x_1 = 4$, $x_2 = 3$.
30. Max Z = 14, $x_1 = 3$, $x_2 = 8$.
31. Max Z = 20, $x_1 = 1$, $x_2 = 2$.
32. Max Z = 18, $x_1 = 3$, $x_2 = 0$.
33. Max Z = 12, $x_1 = 2$, $x_2 = 2$, $x_3 = 0$.
34. Max Z = 22, $x_1 = 1$, $x_2 = 3$.

3.2 DYNAMIC PROGRAMMING

3.2.1 Introduction

Dynamic programming is a mathematical technique of optimization using multistage decision process. That is, the process in which a sequence of interrelated decisions has to be made. It provides a systematic procedure for determining the combination of decisions which maximize overall effectiveness. Classical mathematics has proved inadequate in handling many optimization problems that involve large number of decision variables and/or large number of inequality constraints. Such type of problems are handled by the dynamic programming technique. The dynamic programming technique decomposes the original problem in n-variables in to n-sub problems (stage) each in one variable. The solution is obtained in an orderly manner by starting from one stage to the next and is completed after the final stage is reached. This dynamic programming technique was developed by ***Richard Bellman*** in the early 1950.

3.2.2 Need of Dynamic Programming:

In many situations we observed that the decision making process consists of selecting a combination of plans from a large number of alternative combinations.

Before making a decision, it is required that

(a) all the decisions of a combination are specified

(b) the optimal policy can be selected only after all the combinations are evaluated.

While enumerating the problems a whole, the following are the important drawbacks:

(i) Lot of computational work and too much time is involved.

(ii) All combinations may not satisfy the limitations and thus may be infeasible.

(iii) The number of combinations is so large.

The dynamic programming technique deals with such situations by dividing the given problem in to sub problems or stages. Only one stage is considered at a time and the various infeasible combinations are eliminated with the objective of reducing the volume of computations. The solution is obtained by moving from one stage to the next and is completed when the final stage is reached.

3.2.3 Bellman's Principle of optinality:

It states that, **"An optimal policy (set of decisions) has the property that whatever be the initial state and initial decisions, the remaining decisions must constitute an optimal policy for the state resulting from the first decision."**.

It implies that given the initial state of a system, an optimal policy for the subsequent stages does not depend upon the policy adopted at the proceeding stages.

Note: A problem which does not satisfy the principle of optimality can not be solved be dynamic programming.

3.2.4 Characteristics of Dynamic Programming problems:

[M.U.MCA Nov.97]

The characteristics of dynamic programming problem may be outlined as follows:

1) The problem can be divided in to stages, with a policy decision required at each stage.
2) Each stage has a number of stages associated with it. The states are various possible conditions in which the system may find itself at that stage of the problem. The number of states may be finite or infinite.
3) The effect of the policy decision at each stage is to transform the current state into a state associated with the next stage.
4) The current situation (state) of the system at a stage is described by a set of variables, called ***state variables.*** It is defined to reflect the status of the constraints that bind all stages together.
5) Given the current state, an optimal policy for the remaining stages is independent of the policy adopted in previous stages.
   For dynamic programming problem, in general, knowledge of the current state of the system conveys all of the informations about its previous behavior necessary for determining the optimal policy hence forth. This is the ***Markovian property.***
6) The solution procedure begins by finding the optimal policy for each state of the last stage.
7) A recursive relationship (recurrence relation or functional equation) which identifies the optimal policy for each state with n stages remaining, given the optimal policy for each state with $(n - 1)$ stages left.

8) Using this recursive relationship, the solution procedure moves backward stage-by-stage, each time finding the policy when starting at the initial stage.

Note: A stage may be defined as the portion of the problem that possesses a set of mutually exclusive alternatives from which the best alternative is to be selected. If we are to take six sequential decisions then we have six stages.

3.2.5 Applications of Dynamic Programming

1) In the production area, this technique has been used for production, scheduling and employment smoothening problems
2) It has been used to determine the inventory level and for formulating the inventory recording.
3) It can be applied for allocating the scarce resources to different alternative uses such as, allocating the salesmen to different sales districts etc.
4) It is used to determine the optimal combination of advertising media (TV, Radio, News papers) and the frequency of advertising.
5) It can be applied in Replacement theory to determine at which age the equipment is to be replaced for optimal return from the facilities.
6) Spare part level determination to guarantee high efficiency utilization of expensive equipment.
7) Other areas: Scheduling methods, Markovian decision models, infinite stage system, probabilistic decision problems etc.

3.2.6 Dynamic ProgrammingAlgorithm

The solution of a multi-stage problem by dynamic programming involves the following steps:

Step 1: Identify the decision variables and specify objective function to be optimized.

Step 2: Decompose the given problem in to a number of smaller sub problems. Identify the state variables at each stage.

Step 3: Write down a general recursive relationship for computing the optimal policy. Decide either forward or backward method is to follow to solve the problem.

Step 4: Write the relation giving the optimal decision function for one stage sub-problem and solve it.

Step 5: Solve the optimal decision function for 2-stage, 3-stage, ... $(n-1)$ stage and n – stage problem.

Remarks: 1) Generally the solution of recursive equation involves two types of computations, according as the system is continuous or discrete. In case of continuous system, the optimal decisions at each stage are obtained by using usual classical techniques such as differentiation etc. In case of discrete system a tabulator computational scheme is followed at each stage. In each table the number of rows and columns are equal to the number of corresponding feasible state values and the number of possible decisions respectively.

2) If the dynamic programming problem is solved by using the recursive equation starting from the first through the last stage, i.e., obtaining the sequence $f_1 \rightarrow f_2 \rightarrow f_3 \rightarrow \ldots.. f_n$ of the optimal solutions. This computation is called the ***forward computational procedure.*** If the recursive equations are formulated to obtain a sequence $f_n \rightarrow f_{n-1} \rightarrow \ldots \ldots \rightarrow f_2 \rightarrow f_1$, then the computation is known as the ***backward computational procedure.***

3) In calculus, the function y = f(x) will attain its maximum,

when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0.$

4) The function y = f(x) will attain its minimum

when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0.$

Example 1 : **(Optimal Sub-division Problems)**

Divide a positive quantity c in n-parts in such a way that their product is maximum (or)

Maximize $Z = y_1 y_2 \ldots y_n$ subject to the constraints

$$y_1 + y_2 + \ldots + y_n = c$$
$$\text{and } y_1 \geq 0, i = 1,2,3, \ldots. n$$

Solution: *To develop th;e recursive equation:*

First we shall develop a recursive equation connecting the optimal decision function for the n-stage problem with the optimal decision function for the $(n-1)$ stage sub-problem.

Let y_i be the i^{th} part of c and each i may be regarded as a stage. Since y_i may assume any non-negative value which satisfies $y_1 + y_2 + \ldots + y_n = c$, the alternative at each stage are infinite. This means y_i is continuous. Hence the optimal decisions at each stage are obtained by using usual classical method (differentiation).

Let $f_n(c)$ be the maximum attainable product $y_1 . y_2 . y_3 \ldots\ldots y_n$ when c is divided into n parts $y_1, y_2, \ldots y_n$.

Thus $f_n(c)$ becomes a function of n.

For $n = 1$ (One stage problem)

If c is divided into one poart only, they $y_1 = c$

$$\therefore f_n(c) = c \text{ (trival case)} \qquad \ldots(1)$$

For $n = 2$ (*Two stage problem*)

Here c is divided into two parts $y_1 = x$ and $y_2 = c - x$ such that $y_1 + y_2 = c$. Then

$$f_2(c) = \underset{0 \le x \le c}{\text{Max}} \{y_1 . y_2\} = \underset{0 \le x \le c}{\text{Max}} \{x(c-x)\}$$

$$= \underset{0 \le x \le c}{\text{Max}} \{x f_1(c-x)\} \quad (\because f_1(c) = c) \qquad \ldots(2)$$

For $n = 3$ (*Three stage problem*)

Here c is divided into three parts y_1, y_2, y_3.

Let $y_1 = x$ and $y_2 + y_3 = c - x$ so that $y_1 + y_2 + y_3 = c$.

i.e., $(c - x)$ is further divided into two parts whose maximum attainable product $y_2 y_3$ is $f_2(c - x)$.

Then $$f_3(c) = \underset{0 \le x \le c}{\text{Max}} \{y_1 . y_2 . y_3\} = \underset{0 \le x \le c}{\text{Max}} \{x f_2(c-x)\}$$

In general, the recursive equation for the n-stage problem is

$$f_n(c) = \underset{0 \le x \le c}{\text{Max}} \{x f_{n-1}(c-x)\} \qquad \ldots(3)$$

To solve the recursive equation

For $n = 2$, equation (3) becomes

$$f_2(c) = \underset{0 \le x \le c}{\text{Max}} \{x f_1(c-x)\}$$

$$f_2(c) = \underset{0 \le x \le c}{\text{Max}} \{x(c-x)\}$$

The function $x(c - x)$ will the maximum if

$$\frac{d}{dx}(x(c-x)) = 0 \Rightarrow x(-1) + (c-x) = 0$$

$$\Rightarrow -x + c - x = 0 \Rightarrow c - 2x = 0 \Rightarrow x = \frac{c}{2}$$

$$\therefore f_2(c) = \underset{0 \le x \le c}{\text{Max}} \{x(c-x)\} = \frac{c}{2}\left(c - \frac{c}{2}\right) = \left(\frac{c}{2}\right)^2$$

The optimal policy is $\left(\frac{c}{2}, \frac{c}{2}\right)$ and $f_2(c) = \left(\frac{c}{2}\right)^2$

For $n = 3$, equation (3) becomes

$$f_3(c) = \underset{0 \le x \le c}{\text{Max}} \{x f_2(c-x) = \underset{0 \le x \le c}{\text{Max}} \left\{x\left(\frac{c-x}{2}\right)^2\right\}$$

The function $x\left(\frac{c-x}{2}\right)^2$ will attain its maximum value at $x = \left(\frac{c}{3}\right)^3$

$$\therefore f_3(c) = \underset{0 \le x \le c}{\text{Max}} \left\{x\left(\frac{c-x}{2}\right)\right\} = \left(\frac{c}{3}\right)\left(\frac{c - \frac{c}{3}}{2}\right) = \left(\frac{c}{3}\right)^2$$

$\therefore$ The optimal policy is $\left(\frac{c}{3}, \frac{c}{3}, \frac{c}{3}\right)$ and $f_3(c) = \left(\frac{c}{3}\right)^3$

Let us assume that the optimal policy for $n = m$

Is $\left(\frac{c}{m}, \frac{c}{m} \ldots \frac{c}{m}\right)$ and $f_m(c) = \left(\frac{c}{m}\right)^m$

Now, for $n = m + 1$, equation (3) becomes

$$f_{m+1}(c) = \underset{0 \le x \le c}{\text{Max}} \{x f_m(c-x)\} = \underset{0 \le x \le c}{\text{Max}} \left\{x\left(\frac{c-x}{m}\right)^m\right\}$$

The function $x\left(\frac{c-x}{m}\right)^m$ attains its maximum value at $x = \frac{c}{m+1}$

$$\therefore \; f_{m+1}(c) = \underset{0 \le x \le c}{\text{Max}} \left\{ x\left(\frac{c-x}{m}\right)^m \right\}$$

$$= \left(\frac{c}{m+1}\right)\left(\frac{c - \dfrac{c}{m+1}}{m}\right)^m = \left(\frac{c}{m+1}\right)^{m+1}$$

$\therefore$ The result is also true for n = m +1.

Hence by mathematical induction, the optimal policy is

$\left(\frac{c}{n}, \frac{c}{n} \ldots \frac{c}{n}\right)$ and $f_n(c) = \left(\frac{c}{n}\right)^n$

Example 2 **Solve the following by dynamic programming.**

$$\text{Min } Z = y_1 + y_2 + \ldots + y_n$$

Subject to the constraints

$$y_1 \, y_2, \ldots . \, y_n = b$$

$$\text{and } y_1, y_2, \ldots . \, y_n \ge 0 \text{ (or)}$$

Factorize a positive quantity b into n factors in such a way so that their sum is a minimum.

Solution: *To develop the recursive equation:*

Let f_n(b) be the minimum attainable sum $y_1 + y_2 + \ldots + y_n$ when the positive quantity b is factorized into z factors $y_1, y_2, \ldots \; y_n$.

For n = 1 (One stage problem)

Here b is factorized into one factor $y_1 = b$ only.

$$\therefore \; f_1(b) = \underset{y_1 = b}{\text{Max}} \{y_1\} = b \text{ (trivial case)} \qquad \ldots (1)$$

For n = 2 (Two stage problem)

Here b is factorized into one factor $y_1 = x$ and $y_2 = \frac{b}{x}$ so that $y_1 \cdot y_2 = b$.

$$\text{Then} \quad f_2(b) = \underset{0 \le x \le b}{\text{Max}} \{y_1 + y_2\} = \underset{0 \le x \le b}{\text{Max}} \left\{x + \frac{b}{x}\right\}$$

$$= \underset{0 \le x \le b}{\text{Max}} \left\{x + f_1\left(\frac{b}{x}\right)\right\} \rightarrow (2) \quad \text{(by (1))}$$

For n = 3 (Three stage problem)

Here b is factorized into three factors $y_1 = x$ and $y_2 \; y_3 = \frac{b}{x}$ so that $y_1 \cdot y_2 \; y_3 = b$. *i.e.*, $\frac{b}{x}$ is further factorized into two factors whose minimum attainable sum is $f_2 = \left(\frac{b}{x}\right)$

$$\text{Then} \quad f_3(b) = \underset{0 \le x \le b}{\text{Min}} \{y_1 + y_2 + y_3\}$$

$$= \underset{0 \le x \le b}{\text{Min}} \left\{x + f_2\left(\frac{b}{x}\right)\right\} \qquad \text{(by (2))}$$

In general, the recursive equation for the n-stage problem is

$$f_n(b) = \underset{0 \le x \le b}{\text{Min}} \left\{x + f_{n-1}\left(\frac{b}{x}\right)\right\} \qquad \ldots (3)$$

To solve the recursive equation

For $n = 2$, equation (3) becomes

$$f_2(b) = \underset{0 \le x \le b}{\text{Min}} \left\{x + f_1\left(\frac{b}{x}\right)\right\} = \underset{0 \le x \le b}{\text{Min}} \left\{x + \frac{b}{x}\right\}$$

The function $x + \frac{b}{x}$ will attain its minimum when $x = \sqrt{b}$

$$\therefore \; f_2(b) = \underset{0 \le x \le b}{\text{Min}} \left\{x + \frac{b}{x}\right\}$$

$$= \sqrt{b} + \frac{b}{\sqrt{b}} = 2\sqrt{b} = 2(b)^{\frac{1}{2}}$$

The optimal policy $(b^{\frac{1}{2}}, b^{\frac{1}{2}})$ is and $f_2(b) = 2b^{\frac{1}{2}}$

For $n = 3$, equation (3) becomes

$$f_3(b) = \underset{0 \le x \le b}{\text{Min}} \left\{ x + f_2\left(\frac{b}{x}\right) \right\} = \underset{0 \le x \le b}{\text{Min}} \left\{ x + 2\left(\frac{b}{x}\right)^{\frac{1}{2}} \right\}$$

The function $x + 2\left(\frac{b}{x}\right)^{\frac{1}{2}}$ will attain its minimum when $= 3b^{\frac{1}{3}}$

$$f_3(b) = \underset{0 \le x \le b}{\text{Min}} \left\{ x + 2\left(\frac{b}{x}\right)^{\frac{1}{2}} = b^{\frac{1}{3}} + 2\left(\frac{b}{b^{1/3}}\right)^{\frac{1}{2}} = 3b^{\frac{1}{3}} \right\}$$

$\therefore$ The optimal policy is $(b^{\frac{1}{3}}, b^{\frac{1}{3}}, b^{\frac{1}{3}})$ and $f_3(b) = 3b^{\frac{1}{3}}$

Let us assume that the optimal policy for $n = m$ is

$(b^{\frac{1}{m}}, b^{\frac{1}{m}}, b^{\frac{1}{m}}, \dots b^{\frac{1}{m}})$ and $f_m(b) = mb^{\frac{1}{m}}$.

Now, for $n = m + 1$, equation (3) becomes

$$f_{m+1}(b) = \underset{0 \le x \le b}{\text{Min}} \left\{ x + f_x\left(\frac{b}{x}\right) \right\}$$

$$= \underset{0 \le x \le b}{\text{Min}} \left\{ x + m\left(\frac{b}{x}\right)^{\frac{1}{m}} \right\}$$

The function $x + m\left(\frac{b}{x}\right)^{\frac{1}{m}}$ will attain its minimum when $x = b^{\frac{1}{m+1}}$

$$\therefore f_{m+1}(b) = \underset{0 \le x \le b}{\text{Min}} \left\{ x + m\left(\frac{b}{x}\right)^{\frac{1}{m}} \right\}$$

$$= b^{\frac{1}{m+1}} + m\left(\frac{b}{b^{\frac{1}{m+1}}}\right)^{\frac{1}{m+1}} = (m+1)b^{\frac{1}{m+1}}$$

i.e, the result is also true for n = m + 1. Hence by mathematical induction, the optimal policy is $(b^{\frac{1}{n}}, b^{\frac{1}{n}}, b^{\frac{1}{n}}, \dots b^{\frac{1}{n}})$ and $f_n(b) = nb^{\frac{1}{n}}$.

$\boxed{\textbf{Example 3}}$ **Solve by DPP:**

$$\text{Min } Z = y_1^2 + y_2^2 + \dots + y_n^2$$

Subject to

$$y_1 \cdot y_2 \cdot y_3 \cdots y_n = b$$

$$\text{and} \qquad y_i \ge 0, i = 1, 2, 3 \dots n \text{ (or)}$$

Factorize a positive quantity b into n factors in such a way so that the sum of their squares is a minimum.

Solution: *to develop the recursive equation:*

Let $f_n(b)$ be the minimum attainable sum $y_1^2 + y_2^2 + y_3^2 + \dots + y_n^2$ when the positive quantity b is factorized into n factors $y_1, y_2, y_3, \dots y_n$.

For n = 1 (One stage problem)

Here b is factorized in to one factor $y_1 = b$ only.

$$\therefore f_1(b) = \underset{y_1 = b}{Min} = \{y_1^2\} = b^2 \text{ (trivial case)} \qquad \dots (1)$$

For n = 2 (Two stage problem)

Here b is factorized in to one factor $y_1 = x$ and $y_2 = \frac{b}{x}$ so that $y_1 . y_2 = b$

$$\text{Then } f_2(b) = \underset{0 \le x \le b}{\text{Max}} \{y_1^2 + y_1^2\} = \underset{0 \le x \le b}{\text{Max}} \left\{ x^2 + \left(\frac{b}{x}\right)^2 \right\}$$

$$= \underset{0 \le x \le b}{\text{Max}} \left\{ x^2 + f_1\left(\frac{b}{x}\right) \right\} \rightarrow (2) \qquad (\because f_1(b) = b^2)$$

For n = 3 (Three stage problem)

Here b is factorized in to one factors $y_1 . y_2 . y_3 = b$

Let $y_1 = x$ and $y_2 . y_3 = \frac{b}{x}$ so that $y_1 . y_2 . y_3 = b$

i.e., $\frac{b}{x}$ is further factorized into two factors whose minimum attainable sum $y_2^2 + y_3^2$ is $f_2\left(\frac{b}{x}\right)$.

Then
$$f_3(b) = \underset{0 \le x \le b}{\text{Min}} \{y_1^2 + y_2^2 + y_3^2\}$$

$$= \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + f_2\left(\frac{b}{x}\right)\right\} \qquad \text{(by (2))}$$

In general, the recursive equation for the n-stage problem is

$$f_n(b) = \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + f_{n-1}\left(\frac{b}{x}\right)\right\} \qquad \dots (3)$$

To solve the recursive equation

For $n = 2$, equation (3) becomes

$$f_2(b) = \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + f_1\left(\frac{b}{x}\right)\right\}$$

$$= \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + \left(\frac{b}{x}\right)^2\right\}$$

The function $x^2 + \left(\frac{b}{x}\right)^2$ will attain its minimum when $x = b^{\frac{1}{2}}$.

$$\therefore\ f_2(b) = \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + \left(\frac{b}{x}\right)^2\right\}$$

$$= \left(b^{\frac{1}{2}}\right)^2 + \left(\frac{b}{\frac{1}{b^{m+1}}}\right)^2 = 2\left(b^{\frac{1}{2}}\right)^2$$

The optimal policy is $(b^{\frac{1}{2}}, b^{\frac{1}{2}})$ and $f_2(b) = 2(b^{\frac{1}{2}})^2$

For $n = 3$, equation (3) becomes

$$f_3(c) = \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + f_2\left(\frac{b}{x}\right)\right\}$$

$$= \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + 2\left[\left(\frac{b}{x}\right)^{\frac{1}{2}}\right]^2\right\} = \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + 2\left(\frac{b}{x}\right)\right\}$$

The function $x^2 + 2\left(\frac{b}{x}\right)$ will attain its minimum when $x = b^{\frac{1}{3}}$.

$$f_3(b) = \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + 2\left(\frac{b}{x}\right)\right\}$$

$$= \left(b^{\frac{1}{3}}\right)^2 + 2\left(\frac{b}{b^{1/3}}\right) = 3\,b^{\frac{2}{3}}$$

$\therefore$ The optimal policy is $(b^{\frac{1}{3}}, b^{\frac{1}{3}}, b^{\frac{1}{3}}$ and $f_3(b) = 3b^{\frac{2}{3}}$.

Let us assume that the optimal policy for $n = m$ is

$(b^{\frac{1}{m}}, b^{\frac{1}{m}}, b^{\frac{1}{m}}, \dots b^{\frac{1}{m}})$ and $f_m(b) = mb^{\frac{2}{m}}$.

Now, for $n = m + 1$, equation (3) becomes

$$f_{m+1}(b) = \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + f_m\left(\frac{b}{x}\right)\right\}$$

$$= \underset{0 \le x \le b}{\text{Min}} \left\{x^2 + m\left(\frac{b}{x}\right)^{\frac{2}{m}}\right\}$$

The function $x^2 + m\left(\frac{b}{x}\right)^{\frac{2}{m}}$ will attain its minimum when $x = b^{\frac{1}{m+1}}$.

$$\therefore\ f_{m+1}(b) = \underset{0 \le x \le b}{\text{Min}} \left\{ x^2 + m\left(\frac{b}{x}\right)^{\frac{2}{m}} \right\}$$

$$= \left(b^{\frac{1}{m+1}}\right)^2 + m\left(\frac{b}{b^{\frac{1}{m+1}}}\right)^{\frac{2}{m}} = (m+1)b^{\frac{2}{m+1}}$$

i.e., the result is also true for n = m + 1. Hence by mathematical induction, the optimal policy is $(b^{\frac{1}{n}}, b^{\frac{1}{n}}, b^{\frac{1}{n}}, \ldots b^{\frac{1}{n}})$ and $f_n(b) = nb^{\frac{2}{n}}$.

Example 4 **Used dynamic programming to show that $p_1 \log p_1 + p_2 \log p_2 + \ldots + p_n \log p_n$ subject to the constraints $p_1 + p_2 + \ldots + p_n = 1$ and is $p_1 \ge 0$ ($i = 1,2,\ldots n$) is minimum when $p_1 = p_2 = p_3 = \ldots = p_n = \frac{1}{n}$**

(or)

Divide unity into n parts so as to minimize the quantity $\Sigma p_i \log P_i$.

Solution: *To develop the recursive equation:*

Let $f_n(1)$ be he minimum attainable sum $\Sigma p_i \log p_i$ when the unity 1 is divided into n parts $p_1, p_2, p_3, \ldots p_n$.

For $n = 1$ (One stage problem)

Let $p_1 = 1$, then

$$\therefore\ f_1(1) = \underset{p_1 = 1}{\text{Min}} \{p_1 \log p_1\} = 1 \log 1 \text{ (trivial case)} \rightarrow (1)$$

For $n = 2$ (Two stage problem)

Let $p_1 = x$ and $p_2 = 1 - x$ so that $p_1 + p_2 = 1$.

$$\text{Then}\quad f_2(1) = \underset{0 \le x \le 1}{\text{Min}} \{p_1 \log p_1 + p_2 \log p_2\}$$

$$= \underset{0 \le x \le 1}{\text{Min}} \{x \log x + (1-x) \log (1-x)\}$$

$$= \underset{0 \le x \le 1}{\text{Min}} \{x \log x + f_1(1-x)\} \rightarrow (2)$$

$$(\because f_1(1) = 1 \log 1)$$

In general, the recursive equation for the n-stage problem is

$$f_n(1) = \underset{0 \le x \le 1}{\text{Min}} \{p_1 \log p_1 + p_2 \log p_2 + \ldots + p_n \log p_n\}$$

$$= \underset{0 \le x \le 1}{\text{Min}} \{x \log x + f_{n-1}(1-x)\} \rightarrow (3)$$

To solve the recursive equation

For $n = 2$, equation (3) becomes

$$f_2(1) = \underset{0 \le x \le 1}{\text{Min}} \{x \log x + f_1(1-x)\}$$

$$= \underset{0 \le x \le 1}{\text{Min}} \{x \log x + (1-x) \log (1-x)\}$$

The function $x \log x + (1-x) \log (1-x)$ will attain its minimum when $x = \frac{1}{2}$.

$$\therefore\ f_2(1) = \underset{0 \le x \le 1}{\text{Min}} \{x \log x + (1-x) \log (1-x)\}$$

$$= \frac{1}{2} \log \frac{1}{2} + \left(1 - \frac{1}{2}\right) \log \left(1 - \frac{1}{2}\right)$$

$$= 2\left(\frac{1}{2} \log \frac{1}{2}\right)$$

$\therefore$ The Optimal policy is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $f_2(1) = 2\left(\frac{1}{2} \log \frac{1}{2}\right)$

For $n = 3$, equation (3) becomes

$$f_3(1) = \underset{0 \le x \le 1}{\text{Min}} \{x \log x + f_2(1-x)\}$$

$$= \underset{0 \le x \le 1}{\text{Min}} \left\{x \log x + 2\left(\frac{1-x}{2}\right) \log\left(\frac{1-x}{2}\right)\right\}$$

The function $x \log x + 2\left(\frac{1-x}{2}\right)\log\left(\frac{1-x}{2}\right)$ will attain its minimum when $x = \frac{1}{3}$.

$$\therefore\ f_3(1) = \underset{0 \le x \le 1}{\text{Min}} \left\{x \log x + 2\left(\frac{1-x}{2}\right)\log\left(\frac{1-x}{2}\right)\right\}$$

$$= \frac{1}{3}\log\frac{1}{3} + 2\left(\frac{1-1/3}{2}\right)\log\left(\frac{1-1/3}{2}\right)$$

$$= \frac{1}{3}\log\frac{1}{3} + 2\left(\frac{1}{3}\log\frac{1}{3}\right) \quad 3\left(\frac{1}{3}\log\frac{1}{3}\right)$$

$\therefore$ The Optimal policy is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $f_3(1) = 3\left(\frac{1}{3}\log\frac{1}{3}\right)$

Let us assume that the optimal policy for $n = m$ is $\left(\frac{1}{m}, \frac{1}{m}, \ldots \frac{1}{m}\right)$ and

$$f_m(1) = m\left(\frac{1}{m}\log\frac{1}{m}\right)$$

Now for $n = m + 1$, equation (3) becomes

$$f_{m+1}(1) = \underset{0 \le x \le 1}{\text{Min}} \{x \log x + f_m(1-x)\}$$

$$= \underset{0 \le x \le 1}{\text{Min}} \left\{x \log x + m\,(m) \log\left(\frac{1-x}{m}\right)\right\}$$

The function $x \log x + m\left(\frac{1-x}{m}\right)\log\left(\frac{1-x}{m}\right)$ will attain its minimum when $x = \frac{1}{m+1}$.

$$\therefore\ f_{m+1}(1) = \underset{0 \le x \le 1}{\text{Min}} \left\{x \log x + m\left(\frac{1-x}{m}\right)\log\left(\frac{1-x}{m}\right)\right\}$$

$$= \left(\frac{1}{m+1}\right)\log\left(\frac{1}{m+1}\right) + m\left(\frac{1-\frac{1}{m+1}}{m}\right)\log\left(\frac{1-\frac{1}{m+1}}{m}\right)$$

$$= (m+1)\left(\frac{1}{m+1}\right)\log\left(\frac{1}{m+1}\right)$$

i.e., the result is also true for $n = m + 1$. Hence by mathematical induction, the optimal policy is $\left(\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n}\right)$ and $f_n(1) = n\left(\frac{1}{n}\log\frac{1}{n}\right)$

Example 5 **Use Bellman's principle of optimality to maximize $b_1x_1 + b_2x_2 + \ldots + b_nx_n$ where $x_1 + x_2 + \ldots + x_n = c$ (positive constant) $x_1, x_2, \ldots x_n \ge 0$.** ***[MU.MCA May 96]***

Solution: *To develop the recursive equation:*

Let $fn(c)$ be the maximum attainable sum $b_1x_1 + \ldots + b_nx_n$ when the positive constant c is divided into parts $x_1, x_2, \ldots xn$.

For $n = 1$, let $x_1 = c$.

Then $$\therefore f_1(c) = \underset{x_1 = c}{\text{Max}} \{b_1\, x_1\} = b_1 c \qquad \ldots (1) \text{ (trivial case)}$$

For $n = 2$ Here c is divided into two parts $x2 = z$ and $x_1 = c - z$

Such that $x_1 + x_2 = c$.

Then $$f_2(c) = \underset{0 \le x \le c}{\text{Max}} \{b_1\, x_1 + b_2\, x_2\}$$

$$= \underset{0 \le x \le c}{\text{Max}} \{b_1\,(c - z) + b_2 z\}$$

$$= \underset{0 \le x \le c}{\text{Max}} \{b_2 z + f_1\,(c - z)\} \rightarrow (2)\ (\because f_1(c) = b_1 c)$$

For $n = 3$, Here c is divided into three parts $x_3 = z$ and $x_1 + x_2 = c - z$ such that $x_1 + x_2 + x_3 = c$. i.e, c –z is divided into two parts whose maximum attainable sum is $f_2(c - z)$

Then $$f_3(c) = \max_{0 \le x \le c} \{b_1 x_1 + b_2 x_2 + b_3 x_3\}$$

$$= \max_{0 \le x \le c} \{b_3 z + f_{21}(c - z)\}$$

In general, the recursive equation for the n-stage problem is

$$f_n(c) = \max_{0 \le x \le c} \{b_1 x_1 + b_2 x_2 + \ldots + b_n x_n\}$$

$$= \max_{0 \le x \le c} \{b_n z + f_{n-1}(c - z)\}$$

To solve the recursive equation

For $n = 2$, equation (3) becomes

$$f_2(c) = \max_{0 \le x \le c} \{b_2 z + f_1(c - z)\}$$

$$= \max_{0 \le x \le c} \{b_2 z + b_1(c - z)\}$$

$$= \max_{0 \le x \le c} \{(b_2 - b_1) z + b_1 c\}$$

If $(b_2 - b_1)$ is positive, then this is maximum for $z = c$, Otherwise it will be minimum.

$$\therefore f_2(c) = \max_{0 \le x \le c} \{(b_2 - b_1) z + b_1 c\} = b_2 c$$

The optimum policy is $(x_1 = 0, x_2 = c)$ and

$$f_2(c) = b_2 c.$$

For $n = 3$, equation (3) becomes

$$f_3(c) = \max_{0 \le x \le c} \{b_3 z + f_2(c - z)\}$$

$$= \max_{0 \le x \le c} \{b_3 z + b_2(c - z)\}$$

$$= \max_{0 \le x \le c} \{(b_3 - b_2) z + b_2 c\}$$

If $(b_3 - b_2)$ is positive, then this is Maximum for z = c, Otherwise it will be minimum.

$$\therefore f_3(c) = \max_{0 \le x \le c} \{(b_3 - b_2) z + b_2 c\}$$

$$= b_3 c$$

$\therefore$ The optimum policy is $(x_1 = 0, x_2 = 0, x_3 = c)$ and

$$f_3(c) = b_3 c.$$

Let us assume that the optimal policy for $n = m$ is

$(x_1 = 0, x_2 = 0, \ldots x_m = c)$ and $f_m(c) = b_m c$.

Now, for $n = m + 1$, equation (3) becomes

$$f_{m+1}(c) = \max_{0 \le x \le c} \{b_{m+1} z + f_m(c - z)\}$$

$$= \max_{0 \le x \le c} \{b_{m+1} z + b_m(c - z)\}$$

$$= \max_{0 \le x \le c} \{(b_{m+1} - b_m) z + b_m c\}$$

If $(b_m + 1 - b_m)$ is positive, then this is maximum for $z = c$, otherwise it will be minimum.

$$\therefore f_{m+1}(c) = \max_{0 \le x \le c} \{(b_{m+1} - b_m) z + b_m c\} = b_{m+1} c.$$

i.e., the result is also true for $n = m + 1$. Hence by mathematical induction, the optimal policy for the given problem is

$(x_1 = 0, x_2 = 0, \ldots x_{n-1} = 0, x_n = c)$ and $f_n(c) = b_n c$.

Example 6 **By dynamic programming technique, solve the problem**

$$\text{Min } Z = x_1^2 + x_2^2 + x_3^2$$

Subject to the constraints

$$x_1 + x_2 + x_3 \ge 15$$

$$\text{and } x_1, x_2, x_3 \ge 0$$

Solution: To develop the recursive equation:

It is a three stage problem. The decision variables are x_1, x_2, x_3 and the state variables, s_1, s_2 and s_3 are defined as

$$s_3 = x_1 + x_2 + x_3 \ge 15$$

$$s_2 = x_1 + x_2 = s_3 - x_3$$

$$\text{and } s_1 = x_1 = s_2 - x_2$$

Let $f_i(s_i)$ be the minimum value of Z at the i^{th} stage

where $s_i = x_1 + x_2 \dots + xi \ (i = 1,2,3)$

Now the recursive equations are

$$f_1(s_1) = \underset{0 \le x_1 \le s_1}{\text{Max}} \{x_1^2\} = s_1^2 = (s_2 - x_2)^2$$

$$f_2(s_2) = \underset{0 \le x_2 \le s_2}{\text{Max}} \{x_1^2 + x_2^2\}$$

$$= \underset{0 \le x_2 \le s_2}{\text{Max}} \{x_2^2 + f_1(s_1)\}$$

and $$f_3(s_3) = \underset{0 \le x_3 \le s_3}{\text{Max}} \{x_1^2 + x_2^2 + x_3^2\}$$

$$= \underset{0 \le x_3 \le s_3}{\text{Max}} \{x_3^2 + f_2(s_2)\}$$

To solve the recursive equations

From (1), $f_1(s_1) = (s_2 - x_2)^2$ ...(1)

From (2), $$f_2(s_2) = \underset{0 \le x_2 \le s_2}{\text{Min}} \{x_2^2 + f_1(s_1)\}$$

$$= \underset{0 \le x_2 \le s_2}{\text{Min}} \{x_2^2 + f_1(s_1 - x_2)\}$$

$$= \underset{0 \le x_2 \le s_2}{\text{Min}} \{x_2^2 + (s_2 - x_2)^2\} \quad (\because s_1 = s_2 - x_2)$$

$$= \underset{0 \le x_2 \le s_2}{\text{Min}} \{2x_2^2 - 2s_2 x_2 + s_2^2\}$$

The function $\{2x_2^2 - 2s_2 x_2 + s_2^2\}$ will attain its minimum when

$x_2 = \frac{s_2}{2}$.

$$\therefore f_2(s_2) = \underset{0 \le x_2 \le s_2}{\text{Min}} \{2x_2^2 - 2s_2x_2 + s_2^2\}$$

$$= 2\left(\frac{s_2}{2}\right)^2 - 2s_2\left(\frac{s_2}{2}\right)^2 + s_2^2 = \frac{s_2^2}{2}$$

From (3), $$f_3(s_3) = \underset{0 \le x_3 \le s_3}{\text{Min}} \{x_3^2 + f_2(s_2)\}$$

$$= \underset{0 \le x_3 \le s_3}{\text{Min}} \{x_3^2 + f_2(s_3 - x_3)\} \ (\because s_2 = s_3 - x_3)$$

$$= \underset{0 \le x_3 \le s_3}{\text{Min}} \left\{x_3^2 + \frac{(s_3 - x_3)^2}{2}\right\}$$

The function $x_3^2 + \frac{(s_3 x_3)^2}{2}$ will attain its minimum when $x_3 = \frac{s_3}{3}$.

$$\therefore f_3(s_3) = \underset{0 \le x_3 \le s_3}{\text{Min}} \left\{x_3^2 + \frac{(s_3 - x_3)^2}{2}\right\}$$

$$= \left(\frac{s_3}{3}\right)^2 + \frac{\left(s_3 - \frac{s_3}{3}\right)^2}{2}$$

$$= \frac{s_3^2}{3}$$

But $s_3 \ge 15$, i.e., minimum $s_3 = 15$.

$\therefore$ Z is minimum when $x_3 = \frac{s_3}{3} = \frac{15}{3} = 5$

But $s_2 = s_3 - x_3 = 15 - 5 = 10$

$$\therefore x_2 = \frac{s_2}{2} = \frac{10}{2} = 5$$

Also $s_1 = s_2 - x_2 = 10 - 5 = 5$

$$\therefore x_1 = s_1 = 5$$

$$\therefore f_3(s_3) = \frac{s_3^2}{3} = \frac{15^2}{3} = 75 = f_3(15)$$

$\therefore$ The optimal policy is (5, 5, 5) and Min Z = 75 = $f_3(15)$

Example 7 **Use dynamic programming, solve**

Maximum $Z = y_1 \cdot y_2 \cdot y_3$

Subject to the constraints

$$y_1 + y_2 + y_3 = 5$$

and $y_1, y_2, y_3 \geq 0.$

Solution: *To develop the recursive equation:*

The decision variables are y_1, y_2, y_3 and the state variables s_1, s_2 and s_3 and defined as

$$s_3 = y_1 + y_2 + y_3 = 5$$
$$s_3 = y_1 + y_2 = s_3 - y_3$$
$$\text{and } s_1 = y_1 = s_2 - y_2$$

The recursive equations are

$$f_1(s_1) = \max_{0 \leq y_1 \leq s_1} \{y_1\} = s_1 = (s_2 - y_2) \quad \ldots (1)$$

$$f_2(s_2) = \max_{0 \leq y_2 \leq s_2} \{y_1 \cdot y_2\}$$

$$= \max_{0 \leq y_2 \leq s_2} \{y_1 \cdot f_1(s_1)\} \quad \ldots (2)$$

$$\text{and} \quad f_3(s_3) = \max_{0 \leq y_3 \leq s_3} \{y_1 \cdot y_2 y_3\}$$

$$= \max_{0 \leq y_3 \leq s_3} \{y_3 \cdot f_2(s_2)\} \quad \ldots (3)$$

To solve the recursive equations:

From (1), $f_1(s_1) = (s_2 - y_2) = s_1 \quad \ldots (1)$

From (2), $$f_2(s_2) = \max_{0 \leq y_2 \leq s_2} \{y_2\, f_1(s_1)\}$$

$$= \max_{0 \leq y_2 \leq s_2} \{y_2 \cdot f_1(s_2 - y_2)\} \quad (\because s_1 = s_2 - y_2)$$

$$= \max_{0 \leq y_2 \leq s_2} \{y_2 (s_2 - y_2)\}$$

The function $y_2 (s_2 - y_2)$ will attain its maximum when $y_2 = \frac{s_2}{2}$.

$$\therefore \quad f_2(s_2) = \max_{0 \leq y_2 \leq s_2} \{y_2 (s_2 - y_2)\}$$

$$= \left(\frac{s_2}{2}\right)\left(s_2 - \frac{s_2}{2}\right) = \left(\frac{s_2}{2}\right)^2$$

From (3), $$f_3(s_3) = \max_{0 \leq y_3 \leq s_3} \{y_1 \cdot f_2(s_2)\}$$

$$= \max_{0 \leq y_3 \leq s_3} \{y_3\, f_2(s_3 - y_3)\} \quad (\because s_2 = s_3 - y_3)$$

$$= \max_{0 \leq y_3 \leq s_3} \left\{y_3 \left(\frac{s_3 - y_3}{2}\right)^2\right\}$$

The function $y_3 \left(\frac{s_3 - y_3}{2}\right)$ will attain its maximum when $y_3 = \frac{s_3}{3}$.

$$\therefore f_3(s_3) = \left(\frac{s_3}{3}\right)\left(\frac{s_3 - \frac{s_3}{3}}{2}\right)^2 = \left(\frac{s_3}{3}\right)^3$$

But $s_3 = 5 \Rightarrow y_3 = \frac{s_3}{3} = \frac{5}{3}$

and $s_2 = s_3 - y_3 = 5 - \frac{5}{3} = \frac{10}{3} \Rightarrow y_2 = \frac{s_2}{2} = \frac{5}{3}$

and $s_2 = s_3 - y_2 = \frac{10}{3} - \frac{5}{3} = \frac{5}{3} \Rightarrow y_1 = s_1 = \frac{5}{3}$

and $f_3(s_3) = f_3(5) = \left(\frac{5}{3}\right)^3 = \frac{125}{27} = \text{Max } Z.$

$\therefore$ The optimal policy is $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$ and Max $Z = \frac{125}{27}$

Let us now consider an example in which the decision variables can taken only integer values. Such a problem has to be solved by tabulation method not by calculus method.

Example 8 **A student has to take examinations in three courses A,B and C. He has three days available for study. He feels it would be best to devote a whole day to the study of the same course, so that he may study a course for one day, two days, or three days or not at all. His estimates of grades he may get by study are as follows:**

Course/ Study days	A	B	C
0	0	1	0
1	1	1	1
2	1	3	3
3	3	4	3

How should he plan to study so that he maximizes the sum of the grades? ***[MU. BE. Apr 95, Apr 99]***

Solution: Let n_1, n_2 and n_3 be the number of days he should study the courses A, B and C respectively. If $f_1(n_1)$, $f_2(n_2)$, $f_3(n_3)$ be the grads earned by such a study, then the problem becomes:

$$\text{Max } Z = f_1(n_1) + f_2(n_2) + f_3(n_3)$$

$$\text{Subject to } n_1 + n_2 + n_3 \leq 3$$

$$n_1, n_2, n_3 \geq 0 \text{ and integers.}$$

Here n_j are the decision variables and $f_i(n_j)$ are the corresponding return functions for j = 1, 2, 3.

Now, let us define the state variables s_j as follows:

$$s_3 = n_1 + n_2 + n_3 \leq 3$$

$$s_2 = n_1 + n_2 = s_3 - n_3$$

$$s_1 = n_1 = s_2 - n_2$$

Thus, state transformation functions are defined as

$$s_{j-1} = T_j\,(s_j, n_j), j\; 2, 3$$

The recursive equations applicable here are:

$$F_i(s_j) = \underset{nj}{\text{Max}} \{f_j(n_j) + F_{j-1}(s_{j-1})\}, j = 2,3.$$

$$F_1(s_1) = f_1(n_1)$$

where $F_3(s_3) = \underset{n_1, n_2, n_3}{\text{Max}} \{f_1(n_t) + f_2(n_2) + f_3(n_3)\}$ for any feasible value of s_3. Then the required solution becomes $\underset{s_3}{\text{Max}}\ F_3(s_3)$.

Recursive operations leading to the answer are tabulated as follows:

Stage returns $f_j(n_j)$

j \ n_j	0	1*	2	3
1	0	(1)	1	3
2	1	1	3	4
3	0	1	3	3

Stage transformation $s_{j-1}, j = 2,3$

s_j \ n_j	0	1	2	3
0	0	-	-	-
1	1	0	0	0
2	2	1	0	-
3	3	2	1	0

Recursive operations

	$f_2(n_2)$	$F_1(s_2) = f_1(n_1)$	$f_1(n_2) + F_1(s_1)$	$F_2(s_2)$
s_2/n_2	0 1 2 3	0 1 2 3	0* 1 2 3	
0	1 - - -	0 - - -	1 - - -	1
1	(1) 1 - -	1 0 - -	2 1 - -	(2)
2	1 1 3 -	1 1 0 -	2 2 3 -	3
3	1 1 3 4	3 1 1 0	4 2 4 4	4

s_3/n_3	$F_3(n_3)$	$F_2(s_2) = f_2(n_2)$	$f_3(n_2)+ F_2(s_2)$	$F_3(s_3)$
	0 1 2 3	0 1 2 3	0 1 2* 3	
0	0 - - -	1 - - -	1 - - -	1
1	0 1 - -	2 1 - -	2 2 - -	2
2	1 1 3 -	3 2 1 -	3 3 4 -	4
3	1 1 3 4	4 3 (2) 1	4 4 (5) 4	(5)

Proceeding backwards through enclosed type numbers, the optimal policy is obtained as $n_3 = 2, n_2 = 0, n_1 = 1$ keeping in view $n_1 + n_2 + n_3 \leq 3$. The required maximum return is 5.

Example 9 **A contractor has to supply the following number OI items at the end of each month.**

Month	Jan	Feb	Mar	Apr	May	June	Total
No. of Items:	85	180	300	375	375	285	1600

Production during a moth is available for supply at the end of the month. The stock holding cost per item per month is Rs.1.00. The set up cost is Rs. 900.00 per set up and Rs.2.00 per item.

In which months a batch should be made, and of what size, so that total cost may be minimum.

Solution: Here the production cost Rs. 2.00/item is incurred always, whether they are produced in the beginning or at any other time. Thus there will always be a fixed cost 1600 × 2 = Rs. 3200.00

Now if a single (batch of 1600 items is) produced in the starting of January, then the total cost for 6 months period is

$$(1515 \times 1 + 1335 \times 1 + 1035 \times 1 + 285 \times 1) + 900 = \text{Rs. } 5730.00$$

Where 1515, 1335, 1035, 660, 285 are the stock levels in the following 5 months respectively and 900 is the setup cost.

It is one of the alternatives. We shall optimize the cost starting from the back of the 6 months period. Thus we shall first optimize for the 6th month (June) then for May etc.

For June : There will be no stock left at the end of the month May because any such items could be made during June with no extra cost and with a saving in stock holding costs. Also we don't want any unit after June. Hence the optimal policy for June is produce 285 items, costing Rs.900.00.

For May : In the beginning of May also there will be not stock left at the end of April. Hence here we have two alternatives.

(1) Produce 660 (demand of May and June) in beginning of May. It will cost Rs. (900 + 285) = Rs. 1185.00. Note that the 285 units will be carried in inventory through out the month of June.

(2) Produce 375 in this month and hence 285 in June. This will cost Rs. (900 + 900) = Rs. 1800.00. Note that no stock is carried for any month. Hence the optimal policy for May is produce 660 in May, none later, costs Rs. 1185.00.

For April: In this month there are three alternatives

(1) Produce 1035 non and none later.

It will cost Rs (900 + 660 × 1 + 285 × 1) = Rs.1845.00

(2) Produce 750 now, some more in June and none in May.

It will cost Rs (900 + 900 + 375 × 1) = Rs.2175.00.

Here we have added Rs.900 two times as during this three months period, we are producing two times.

(3) Produce 375 now and 660 in May.

It will cost Rs. (900 + 1185) = Rs.2085.00

Note the when we have decided to produce in May, we shall produce 660 items in May, as this is the optimal policy, This is why we have not considered the case when 375 are produced in April, 375 in May and 285 in June.

∴ The optimal policy for April is:

Produce 1035 now and none later costing Rs. 1845.00

For March : In this month there are four alternatives

(1) Produce 1335 now, none later.

It will cost Rs (900 + 1 × 1035 + 1 × 660 + 1 × 285) = Rs.2880.00

(2) Produce 1050 now, and 285 in June. Costs

Rs (900 + 900 + 750 × 1 375 × 1) Rs. 2925.00

s_3/n_3	$F_3(n_3)$	$F_2(s_2)=f_2(n_2)$	$f_3(n_2)+F_2(s_2)$	$F_3(s_3)$
	0 1 2 3	0 1 2 3	0 1 2* 3	
0	0 - - -	1 - - -	1 - - -	1
1	0 1 - -	2 1 - -	2 2 - -	2
2	1 1 3 -	3 2 1 -	3 3 4 -	4
3	1 1 3 4	4 3 (2) 1	4 4 (5) 4	(5)

Proceeding backwards through enclosed type numbers, the optimal policy is obtained as $n_3 = 2, n_2 = 0, n_1 = 1$ keeping in view $n_1 + n_2 + n_3 \leq 3$. The required maximum return is 5.

Example 9 **A contractor has to supply the following number OI items at the end of each month.**

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In which months a batch should be made, and of what size, so that total cost may be minimum.

Solution: Here the production cost Rs. 2.00/item is incurred always, whether they are produced in the beginning or at any other time. Thus there will always be a fixed cost 1600 × 2 = Rs. 3200.00

Now if a single (batch of 1600 items is) produced in the starting of January, then the total cost for 6 months period is

(1515 × 1 + 1335 × 1 + 1035 × 1 + 285 × 1) + 900 = Rs. 5730.00

Where 1515, 1335, 1035, 660, 285 are the stock levels in the following 5 months respectively and 900 is the setup cost.

It is one of the alternatives. We shall optimize the cost starting from the back of the 6 months period. Thus we shall first optimize for the 6th month (June) then for May etc.

For June : There will be no stock left at the end of the month May because any such items could be made during June with no extra cost and with a saving in stock holding costs. Also we don't want any unit after June. Hence the optimal policy for June is produce 285 items, costing Rs.900.00.

For May : In the beginning of May also there will be not stock left at the end of April. Hence here we have two alternatives.

(1) Produce 660 (demand of May and June) in beginning of May. It will cost Rs. (900 + 285) = Rs. 1185.00. Note that the 285 units will be carried in inventory through out the month of June.

(2) Produce 375 in this month and hence 285 in June. This will cost Rs. (900 + 900) = Rs. 1800.00. Note that no stock is carried for any month. Hence the optimal policy for May is produce 660 in May, none later, costs Rs. 1185.00.

For April: In this month there are three alternatives

(1) Produce 1035 non and none later.

It will cost Rs (900 + 660 × 1 + 285 × 1) = Rs.1845.00

(2) Produce 750 now, some more in June and none in May.

It will cost Rs (900 + 900 + 375 × 1) = Rs.2175.00.

Here we have added Rs.900 two times as during this three months period, we are producing two times.

(3) Produce 375 now and 660 in May.

It will cost Rs. (900 + 1185) = Rs.2085.00

Note the when we have decided to produce in May, we shall produce 660 items in May, as this is the optimal policy, This is why we have not considered the case when 375 are produced in April, 375 in May and 285 in June.

∴ The optimal policy for April is:

Produce 1035 now and none later costing Rs. 1845.00

For March : In this month there are four alternatives

(1) Produce 1335 now, none later.

It will cost Rs (900 + 1 × 1035 + 1 × 660 + 1 × 285) = Rs.2880.00

(2) Produce 1050 now, and 285 in June. Costs

Rs (900 + 900 + 750 × 1 375 × 1) Rs. 2925.00

(3) Produce 675 now and 660 in May. Costs

Rs (900 + 1 × 375 + 1185) = Rs.2460.00

(4) Produce 300 now and 1035 in April. Costs

Rs (900 + 1845) = Rs. 2745.00

∴ The optimal policy is : Produce 675 now and 660 in May, cost is Rs. 2460.00

For February: There are five alternatives.

(1) Produce 1515 now and none later.

It will Rs (900 + 1335 × 1 + 1035 × 1 + 660 × 1 + 285 × 1) = Rs.4215.00

(2) Produce 1230 now and 285 in June. Costs

Rs (900 + 900 + 1050 × 1 + 750 × 1 + 375 × 1) = Rs. 3975.00

(3) Produce 855 now and 660 in May. Costs

Rs (900 + 1 × 675 + 1 × 375 + 1185) = Rs.3135.00

(4) Produced 480 now and some move in April i.e., 1035 in April. Costs Rs (900 + 1 × 300 + 1845) = Rs. 3045.00

(5) Produce 180 now and some more 675 in March and 660 in May, at this gives he minimum cost. Costs Rs. (900 + 2460) = Rs.3360.00

Hence the optimal policy is: Produce 480 now and 1035 in April. Cost is Rs. 3045.00.

For January : There are six alternatives.

(1) Produce 1600 now and none later. Costs Rs. 5730.00

(2) Produce 1315 now and 285 in June Costs

Rs (900 + 1 × 1230 + 1 × 1050 + 1 × 375 + 1 × 375 + 900) = Rs. 5205.00

(3) Produce 940 now and 660 in May. Costs

Rs (900 + 1 × 855 + 1 × 675 + 1 × 375 + 1185) = Rs. 3990.00

(4) Produce 565 now and 1035 in April. Costs

Rs (900 + 1 × 480 + 1 × 300 + 1845) = Rs.3525.

(5) Produce 265 now and some more in March (Produce 675 in March and 660 in May). Costs.

Rs (900 + 1 × 180 + 2460) = Rs. 3540.00

(6) Produce 85 now and some more Feb (480 in Feb, 1035 in April). Cost of Rs. (900 + 3045) = Rs.3945.00

Hence the optimal policy is that produce 565 in January, 1035 in April: Costs Rs. 3525.00.

Hence the best policy is that produce 565 in January, 1035 in April. The Minimum cost is Rs. 3525.00

3.2.7 Solution of LPP by DPP:

Consider the general LPP:

$$\text{Max } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\begin{aligned} \text{Subject to } & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\ & \text{and } x_1, x_2, \ldots x_n \ge 0. \end{aligned}$$

This problem can be considered as a multistage problem with each activity j (j = 1, 2, ... n) as individual stage. Hence it is an n – stage problem.

Let x_j be decision variable at stage j. We know that in a LPP, each constraint describes the allocation of limited resources to various activities. Define the states as the amount of resources $(b_1, b_2, \ldots b_m)$. Let $(B_{1j}, B_{2j}, B_{3j}, \ldots, B_{mj})$ be the state of the system at stage j and $f_j(B_{1j}, B_{2j}, \ldots, B_{mj})$ be the optimum value of the objective function for the stage j given the states $B_{1j}, B_{2j}, B_{3j}, \ldots, B_{mj}$.

Here we shall use backward computational procedure. i.e., we shall optimize the last stage first and then last but one etc.

$$\text{Thus } f_n(B_{1n}, B_{2n}, \ldots B_{mn}) = \underset{0 \le a_{in}x_n \le B_{in}}{\text{Max}} \{c_nx_n\}$$

and $f_j(B_{1j}, B_{2j}, B_{3j}, \ldots, B_{mj})$

$$= \underset{0 \le a_{ij}x_j \le B_{ij}}{\text{Max}} \{c_jx_j + f_{j+1}B_{1j} - a_{1j}x_1, B_{2j} - a_{2j}x_2, \ldots B_{mj} - a_{mj}x_j)\}$$

$$i = 1, 2, \ldots m, \; j = 1, 2 \ldots (n-1) \rightarrow (1)$$

Here $0 \le B_{ij} \le b_i$ for all i and j. The above recursive equation (1) is used to solve a LPP by dynamic programming approach.

Example 1 **Solve the following LPP using dynamic programming principles:**

$$\text{Max } Z = 2x_1 + 5x_2$$

$$\text{Subject to } 2x_1 + x_2 \le 43$$

$$2x_2 \le 46$$

$$\text{and } x_1 + x_2 \le 0 \qquad \textit{[BRU.MSc 1986]}$$

Solution: The problem consists of two decision variables and two resources (constraints). Hence the problem has two stages and two state variables:

Let (B_{1j}, B_{2j}) be the state of the system at stage j (j = 1,2) and $f_i(B_{1j}, B_{2j})$.

Using backward computational procedure, we have

$$f_2(B_{12}, B_{22}) = \underset{\substack{0 \le x_2 \le B_{12} \\ 0 \le 2x_2 \le B_{22}}}{\text{Max}} \{5x_2\} = 5 \underset{\substack{0 \le x_2 \le B_{12} \\ 0 \le x_2 \le \frac{B_{22}}{2}}}{\text{Max}} \{x_2\}$$

Since Max $\{x_2\}$ which satisfies $0 \le x_2 \le B_{12}, 0 \le x_2 \le \frac{B_{22}}{2}$ is the minimum of $B_{12}, \frac{B_{22}}{2}$.

i.e., $$x_2{}^* = \text{Max}\{x_2\} = \text{Min}\left\{B_{12}, \frac{B_{22}}{2}\right\} \qquad \dots (1)$$

$$\therefore f_2(B_{12}, B_{22}) = 5 \text{ Min}\left\{B_{12}, \frac{B_{22}}{2}\right\} \qquad \dots (2)$$

Also

$$f_1(B_{11}, B_{21}) = \underset{0 \le 2x_1 \le B_{11}}{\text{Max}} \{2x_1 + f_2(B_{11} - 2x_1, B_{21} - 0)\} ..$$

From (2),

$$f_2(B_{11} - 2x_1, B_{21}) = 5 \text{ Min}\left\{B_{11} - 2x_1, \frac{B_{21}}{2}\right\}$$

$$\therefore f_1(B_{11}, B_{21}) = \underset{0 \le x_1 \le \frac{B_{11}}{2}}{\text{Max}} 5 \text{ Min}\left\{B_{12}, \frac{B_{22}}{2}\right\}$$

Since it is a two stage problem, at the first stage $B_{11} = 43, B_{21} = 46$

$$\therefore f_1(B_{11}, B_{21}) = f_1(43, 46)$$

$$= \underset{0 \le x_1 \le \frac{43}{2}}{\text{Max}} \left\{2x_1 + 5 \text{ Min}\left(43 - 2x_1, \frac{46}{2}\right)\right\}$$

$$= \underset{0 \le x_1 \le \frac{43}{2}}{\text{Max}} \{2x_1 + 5 \text{ Min}(43 - 2x_1, 23)\}$$

$$\text{Now, Min } (43 - 2x_1, 23) = \begin{cases} 23, \text{ if } 0 \le x_1 \le 10 \\ 43 - 2x_1, \text{ if } 10 \le x_1 \le \frac{43}{2} \end{cases}$$

From (3),

$$f_1(B_{11}, B_{21}) = \text{Max}\left\{2x_1 + 5\begin{pmatrix} 23, \text{ if } 0 \le x_1 \le 10 \\ 43 - 2x_1, \text{ if } 10 \le x_1 \le \frac{43}{2} \end{pmatrix}\right\}$$

$$= \text{Max}\begin{cases} 2x_1 + 115 \text{ if } 0 \le x_1 \le 10 \\ 215 - 8x_1, \text{ if } 10 \le x_1 \le \frac{43}{2} \end{cases}$$

Since Max of $2x_1 + 115, 0 \le x_1 \le 10$ occurs at x_1 = 10 and

Max of $215 - 8x_1, 10 \le x_1 \le \frac{43}{2}$ also occurs at x_1 = 10.

$$f_1(B_{11}, B_{21}) = 2(10) + 155 = 135$$

$$\text{Now } x_2 = \text{Min}\left\{B_{11} - 2x_1, \frac{B_{21}}{2}\right\}$$

$$= \text{Min}\{43 - 2x_1, 23\}$$

$$= \text{Min}\{23, 23\} = 23.$$

The optimal solution is

$\therefore$ The optimal solution is

$$\text{Max } Z = 135, x_1 = 10, x_2 = 23.$$

Example 2 **Solve the following LPP using dynamic programming approach:**

$$\text{Max } Z = 3x_1 + 5x_2$$

subject to
$$x_1 \le 4$$
$$x_2 \le 6$$
$$3x_1 + 2x_2 \le 18$$
and
$$x_1, x_2 \ge 0$$

Solution: Here we have two decision variables and three resources (constraints). Hence the problem has two stages and three state variables.

Let (B_{1j}, B_{2j}, B_{3j}) be the state of the system at stage j ($j = 1,2$) and $f_i(B_{1j}, B_{2j}, B_{3j})$ be the optimal (maximum) value of the objective function for stage $j = 1,2$, given the state (B_{1j}, B_{2j}, B_{3j})

Using backward computational procedure, we have

$$f_2(B_{12}, B_{22}, B_{32}) = \max_{\substack{0 \le x_2 \le B_{22} \\ 0 \le 2x_2 \le B_{32}}} \{5x_2\} = 5 \max_{\substack{0 \le x_2 \le B_{22} \\ 0 \le x_2 \le \frac{B_{32}}{2}}} \{x_2\}$$

Since Max $\{x_2\}$ which satisfies $0 \le x_2 \le B_{12}, 0 \le x_2 \le \frac{B_{22}}{2}$ is the minimum of $B_{12}, \frac{B_{22}}{2}$.

i.e.,
$$\text{Max }\{x_2\} = x_2{}^* = \text{Min}\left\{B_{22}, \frac{B_{32}}{2}\right\} \qquad \ldots (1)$$

$$\therefore f_2(B_{12}, B_{22}, B_{32}) = 5 \text{ Min}\left\{B_{22}, \frac{B_{32}}{2}\right\} \qquad \ldots (2)$$

Also

$$f_1(B_{11}, B_{21}, B_{31}) = \max_{\substack{0 \le 2x_1 \le B_{11} \\ 0 \le 3x_1 \le B_{11}}} \{3x_1 + f_2(B_{11} - x_1, B_{21} - 0, B_{31} - 3x_1$$

From (2),

$$f_2(B_{11} - x_1, B_{2i}, B_{31} - 3x_1) = 5 \text{ Min}\left\{B_{21}, \frac{B_{31} - 3x_1}{2}\right\}$$

$$\therefore f_1(B_{11}, B_{21}, B_{31}) = \max_{\substack{0 \le x_1 \le B_{11} \\ 0 \le x_1 \le \frac{B_{31}}{3}}} \left\{3x_1 + 5 \text{ Min}\left(B_{21}, \frac{B_{31} - 3x}{2}\right)\right\}$$

Since it is a two stage problem, at the first stage $B_{11} = 4$, $B_{21} = 6$, $B_{31} = 18$.

$$\therefore f_1(B_{11}, B_{21}, B_{31})$$

$$= \max_{\substack{0 \le x_1 \le 4 \\ 0 \le x_1 \le 6}} \left\{3x_1 + 5 \text{ Min}\left(6, \frac{18 - 3x_1}{2}\right)\right\}$$

$$= \max_{0 \le x_1 \le 4} \left\{3x_1 + 5 \text{ Min}\left(6, \frac{18 - 3x_1}{2}\right)\right\} \qquad \ldots (3)$$

Now, $$\text{Min}\left(6, \frac{18 - 3x_1}{2}\right) = \begin{cases} 6, \text{ if } 0 \le x_1 \le 2 \\ \frac{18 - 3x_1}{2}, \text{ if } 2 \le x_1 \le 4 \end{cases}$$

From (3),

$$f_1(B_{11}, B_{21}, B_{31}) = \text{Max}\begin{cases} 3x_1 + 5(6) & \text{if } 0 \le x_1 \le 2 \\ 3x_1 + 5\left(\frac{18 - 3x_1}{2}\right) & \text{if } 0 \le x_1 \le 2 \end{cases}$$

$$= \text{Max}\begin{cases} 3x_1 + 30, & \text{if } 0 \le x_1 \le 2 \\ \frac{90 - 9x_1}{2} & \text{if } 2 \le x_1 \le 4 \end{cases}$$

Since Max of $3x_1 + 30$, $0 \le x_1 \le 2$ occurs at $x_1 = 2$ and

Max of $\frac{90 - 9x_1}{2}$, $2 \le x_1 \le 4$ also occurs at $x_1 = 2$

$$f_1(B_{11}, B_{21}, B_{31}) = 3(2) + 30 = 36$$

Now $$x_2 = \text{Min}\left\{B_{21}, \frac{B_{31} - 3x_1}{2}\right\}$$

$$= \text{Min}\left\{6, \frac{18 - 3x_1}{2}\right\} = \text{Min }\{6,6\} = 6$$

$\therefore$ The optimal solution is Max $Z = 36$, $x_1 = 2$, $x_2 = 6$.

Example 3 **Solve the following LPP by dynamic programming approach:**

$$\text{Max } Z = 4x_1 + 14x_2$$

$$\text{subject to } 2x_1 + 7x_2 \le 21,$$

$$7x_1 + 2x_2 \le 21$$

$$7x_1 + 2x_2 \le 21$$

$$\text{and } x_1, x_2 \ge 0$$

Solution: This problem consists of two decision variables and two resources (constraints). Therefore, the problem has two stages and two state variables.

Let (B_{1j}, B_{2j}) be the state of the system at stage j (j = 1,2) and $f_i(B_{1j}, B_{2j})$ be the optimal (maximum) value of the objective function for stage j = 1,2, given the state (B_{1j}, B_{2j})

Using backward computational procedure, we have

$$f_2(B_{12}, B_{22}) = \underset{\substack{0 \le 7x_2 \le B_{12} \\ 0 \le 2x_2 \le B_{22}}}{\text{Max}} \{14x_2\}$$

$$= 14 \underset{\substack{0 \le x_2 \le \frac{B_{12}}{7} \\ 0 \le x_2 \le \frac{B_{22}}{2}}}{\text{Max}} \{x_2\}$$

$$= 14 \text{ Min}\left\{\frac{B_{12}}{7}, \frac{B_{22}}{2}\right\},$$

because Max $\{x_2\}$ which satisfies $0 \le x_2 \le \frac{B_{12}}{7}, 0 \le x_2 \le \frac{B_{22}}{2}$ is the minimum of $\frac{B_{12}}{7}, \frac{B_{22}}{2}$.

Also

$$f_1(B_{11}, B_{21}) = \underset{\substack{0 \le x_1 \le \frac{B_{11}}{2} \\ 0 \le x_1 \le \frac{B_{21}}{7}}}{\text{Max}} (4x_1 + f_2(B_{11} - 2x_1, B_{21} - 7x_1))$$

$$= \underset{\substack{0 \le x_1 \le \frac{B_{11}}{2} \\ 0 \le x_1 \le \frac{B_{21}}{7}}}{\text{Max}} \left\{4x_1 + 14 \text{ Min}\left(\frac{B_{11} - 2x_1}{7}, \frac{B_{21} - 7x_1}{2}\right)\right\} \quad \text{[By (1)]}$$

Now at first stage, $B_{11} = 21$, $B_{21} = 21$

$$\therefore f_1(B_{11}, B_{21}) = f_1(21, 21)$$

$$= \underset{\substack{0 \le x_1 \le \frac{21}{2} \\ 0 \le x_1 \le 3}}{\text{Max}} \left\{4x_1 + 14 \text{ Min}\left(\frac{21 - 2x_1}{7}, \frac{21 - 7x_1}{2}\right)\right\}$$

$$= \underset{0 \le x_1 \le 3}{\text{Max}} \left\{4x_1 + 14 \text{ Min}\left(\frac{21 - 2x_1}{7}, \frac{21 - 7x_1}{2}\right)\right\} \quad \ldots (2)$$

$$\text{Now, Min}\left(\frac{21 - x_1}{7}, \frac{21 - 7x_1}{2}\right) = \begin{cases} \frac{21 - 2x_1}{7}, & \text{if } 0 \le x_1 \le \frac{7}{3} \\ \frac{21 - 7x_1}{2}, & \text{if } \frac{7}{3} \le x_1 \le 3 \end{cases}$$

Therefore from (2),

$$f_1(B_{11}, B_{21}) = \underset{0 \le x_1 \le 2}{\text{Max}} \left\{4x_1 + 14 \text{ Min}\left(\frac{21 - 2x_1}{7}, \frac{21 - 7x_1}{2}\right)\right\}$$

$$f_1(B_{11}, B_{21}) = \text{Max}\begin{cases} 42, & \text{if } 0 \le x_1 \le \dfrac{7}{3} \\ 147 - 45x_1, & \text{if } \dfrac{7}{3} \le x_1 \le 3 \end{cases}$$

The maximum value of the above function occurs at each value of x_1 in (0, 7/3) and the maximum value is 42.

∴ There are infinitely many solutions.

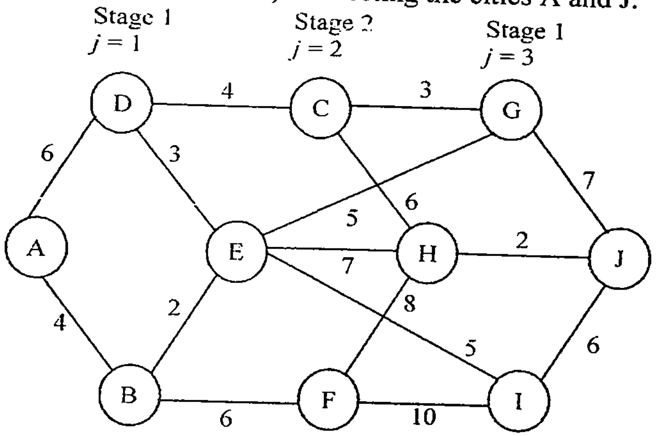
Let $x_1 = \lambda$,

$$x_2 = \text{Min}\left\{\frac{21-2x_1}{7}, \frac{21-7x_1}{2}\right\} = \text{Min}\left\{\frac{21-2\lambda}{7}, \frac{21-7\lambda}{2}\right\}$$

$$= \frac{21-2\lambda}{2} \text{ where } 0 \le \lambda \le \frac{7}{3} \text{ and Max } Z = 42.$$

3.2.8 Solving a least cost route problem by DPP

A company has to transport goods from city A to city J. The cost of transportation between the different cities is given along the lines connecting the nodes. Where a node represents a city. Determine the optimal route (least cost route) connecting the cities A and J.



The problem can be divided into 4 stages. Each stage decision will consist of selecting a path out of a number of alternatives. For example, at stage 1 one of the two paths AB and AC is to be selected and at stage 2 one of the four paths CD, CE, BE, BF is to be selected. The optimal policy will consist of a set of paths connecting A with J.

If s_j represents the state, then s_0 = A at $j = 0$.

and s_1 will have two values B and C at $j = 1$.

The paths from one stage to the next will be referred to as decision variables. Let d represent the decision variable which takes from state $j-1$ to state j thereby the state changing from s_0 to s_j will be denoted by $F_j(s_j)$. The return from a decision d_j will be represented by $f_i(d_j)$

$$\text{Here} \quad f_j(d_j) = d_j$$

Start with node J at stage 4 at $j = 4$, d_4 has three values 7, 2 and 6 leading back to state s_3 which has three values G, H and I. The smallest of the values $f_4(s_4)$ is to be selected.

$$f_1(s_j) = f_4(s_4) = \text{Min}\begin{cases} 7 + F_3(G) \\ 2 + F_3(H) \\ 6 + F_3(I) \end{cases}$$

$$= \text{Min}\,[d_4 + f_3(s_3)]$$

Similarly at stage 3 ($j = 3$)

$$F_3(G) = \text{Min}\begin{matrix} 3 + F_2(D) \\ 5 + F_2(E) \end{matrix}$$

$$F_3(H) = \text{Min}\begin{cases} 6 + F_2(G) \\ 7 + F2(H) \\ 8 + F2(I) \end{cases}$$

$$F_3(I) = \text{Min}\begin{cases} 5 + F_2(E) \\ 10 + F_2(F) \end{cases}$$

In general

$$F_3(s_3) = \underset{d_3}{\text{Min}}\,[d_3 + F_2(s_2)]$$

But

$$F_2(s_2) = \underset{d_3}{\text{Min}}\,[d_3 + F_1(s_1)]$$

$$F_1(s_1) = d_1$$

The whole process can be expressed by the recursion equation

$$F_j(s_j) = \underset{d_3}{\text{Min}}\,[d_3 + F_{j-1}(s_j - 1)],\ j = 4, 3, 2$$

$$F_1(s_1) = d_j$$

This equation helps us in determining the optimal policy consisting of a set of decisions such that the cost $F_4(s_4)$ is minimum. Now, starting with Stage 1, we have the following table

State s_1	Decision Variable d_1	$F_1(s_1)$
B	4	4
C	6	6

Since s_1 has two values B and C s_3 has three values D, E and F. The value of $F_1(d_1)$ are transferred from stage 1 to stage 1 to stage 2 to which the value of d_2 are added as given in the following table.

Stage 2

d_2 / s_2	$F_1(d_1)$ 2 3 4 6	$D2 + F_1(d_1)$ 2 3 4 6	optimal $F_2(d2)$
D	- - 6 -	- - 10 -	10
E	4 6 - -	6 9 - -	6
F	- - - 4	- - - 10	10

(- indicates s_j d_j not feasible)

For various values of the stage optimal returns is tabulated in the table. The procedure is carried out for stages 3 and 4 as given in the following tables.

Stage 3

d_3 / s_3	$F_2(d_2)$ 3 5 6 7 7 10	$D2 + F_1(d_1)$ 3 5 6 7 7 10	optimal $F_2(d2)$
G	10 6 - - - -	13 11 - - - -	11
H	- 10 10 - - -	- 15 16 13 - -	13
I	- 6 - - - 10	- 11 - - - 20	1

Stage 4

d_4 / s_4	$F_3(d_3)$ 2 6 7	$d_4 + F_3(d_3)$ 2 6 7	Optimal $F_4(d_4)$
j	13 11 11	15 17 18	15

The minimum cost of the route connected A to j is 15. We follow the backward pass to determine the route.

$$F_4(d_4) = 15 = d_4 + F_3(d_3)$$
$$= 2 + 13 \Rightarrow \text{path HJ}$$
$$F_3(s_3) = 13 = d_3 + F_2(d_2)$$
$$= 7 + 6$$
$$d_3 = 7 \Rightarrow \text{path EH}$$

Similarly $d_2 = 2 \Rightarrow$ path B E

$$d_1 = 4 \Rightarrow \text{path A B}$$

Then the optimal route connecting A and J is A – B – E – H – J.

EXERCISE

1. **What is dynamic programming?** *[MU. BE. Oct 96]*
2. **Explain the concept of dynamic programming.**
3. **Define the following terms in dynamic programming:**
   (a) Stage (b) Stage (c) State variable (d) decision variable (e) State transformation function.
4. **State the Bellman's principle of optimality and give an example of a dynamic programming.** *[MU. BE. Nov 93]*
5. **State Bellman's principle of optimality. Describe the backward induction method of discrete dynamic programming for solving any problem in which the objective function can be represented as an additive recursion function.** *[MU. MCA May 91]*
6. **What are the essential characteristics of dynamic programming problems?** *[MU. BE. Apr 80, BRU, BE. 84, MU, MCA. Nov 97]*
7. **State and advantage of dynamic programming.** *[MU. BE. Apr 97]*
8. **Write the advantage of dynamic programming.** *[MU. BE. Apr 97]*
9. **State Bellman's Principle of optimality. Explain the forward and backward induction methods.**
10. **Set up the recursive relation, using dynamic programming approach, when an n-stage objective function is to be maximized.** *[BRU. M.Sc. 81]*
11. **What difficulties you overcome when dynamic programming is designed?** *[BRU. M.Sc. 92]*
12. **State the applications of dynamic programming.**

13. Describe the method of solution of a LPP by dynamic programming? *[MU. BE. Apr 91]*
14. Discuss the relation between linear programming and dynamic programming.
15. Obtain the functional equation for maximizing
$Z = g_1(x_1) + g_2(x_2) + ... + g_n(x_n)$
subject to $x_1 + x_2 + x_3 + ... + x_n = c$
and $x_j \geq 0,\ j = 1, 2, 3 ... n$.
16. Solve the following problems using dynamic programming
$$\text{Max } Z = y_1^2 + y_2^2 + y_3^2$$
subject to $y_1, y_2, y_3 \leq 4$,
Where y_1, y_2 and y_3 are positive integers.
17. Using dynamic programming to solve
$$\text{Min } Z = u_1^2 + u_2^2 + u_3^2$$
$$\text{Subject to } u_1 + u_2 + u_3 \geq 10,$$
$$\text{And } u_1, u_2, u_3 \geq 0.$$
18. Use dynamic programming to show that $\Sigma - p_i \log p_i$ subject to
$\sum_{i=1}^{n} p_i = 1$, is maximum
When $p_1 = p_2 = p_3 = ... p_n = \left(\frac{1}{n}\right)$.
19. Find the maximum value of $Z = x_1^2 + 2x_2^2 + 4x_3$ subject to constraint $x_1 + 2x_2 + x_3 \leq 8$, and $x_1, x_2, x_3 \geq 0$.
20. Find the minimum value of $x_1^2 + 2x_2^2 + 3x_2 + x_3$ subject to the constraint $x_1 + 2x_2 + x_3 \leq 8$, and $x_1, x_2, x_3 \geq 0$.
21. Find the maximum value of $Z = -x_1^2 + 2x_2^2 + 3x_2 + x_3$ subject to constraint $x_1 + x_2 + x_3 \leq 1$, and $x_1, x_2, x_3 \geq 0$.
22. Using dynamic programming, Minimize xyz subject to constraint $x + y + z = 5$, and $x, y, z \geq 0$. *[MU.BE. Apr 95]*
23. Maximize $Y = x_1 x_2 x_3 x_4$ subject to the constraint $x_1 + x_2 + x_3 + x_4 = 15$ $x_1, x_2, x_3, x_4 \geq 0$.
[BRU. BE. Apr 98]

24. Solve the following dynamic programming problem.
$$\text{Minimize } Z = x_1 + x_2 + x_3$$
$$x_1 . x_2 . x_3 = 27$$
$$x_1, x_2, x_3 \geq 0$$
25. Use dynamic programming technique to solve the following problem.
$$\text{Maximize } Z = x_1\ x_2\ x_3\ x_4$$
subject to the constraint $x_1 + x_2 + x_3 + x_4 = 12$
$x_1,\ x_2,\ x_3,\ x_4, \geq 0$ *[BRU. BE. Apr 97]*
26. Solve by dynamic programming technique.
$$x_1 + x_2 + x_3 = 6,$$
$$x_1\ x_2\ x_3 \geq 0$$
$\text{Maximize } Z = 2x_1\ x_2\ x_3\ x_4$ *[BNU. B.E. Nov 96]*
27. A student has to take examination in three courses X, Y and Z. He has three days available for study. He feels it would be best to devote a whole day to the study of the same course, so that he may study a course for one day, two days or three days or not at all. His estimates of grades he may get by study are as follows:

Course/ Study days	X	Y	Z
0	1	2	1
1	2	2	2
2	2	4	4
3	4	5	4

How should be plan to study so that he maximizes the sum of his graders?
28. A man is engaged in buying and selling identical items. He operates form a warehouse that can hold 500 items. Each month he can any quantity that he looses up to the stock at the beginning of the month. Each month, he can buy as much as he wishes for delivery at the end of the month so long as his stock does not exceed 500 items. For the next four months he has the following error-free forecasts of cost and sales prices:

Month (i)	:	1	2	3	4
Cost (c_i)	:	27	24	26	28
Sales price (Pi)	:	28	25	25	27

If the currently has a stock of 200 units, what quantities should he sell and buy in the next four months? Find the solution using dynamic programming

29. Solve the following linear Programming problems by dynamic programming.

(i) Maximize $Z = 3x + 2y$

subject to $x + y \le 300$

$2x + 3y \le 800$

and $x, y \ge 0$ *[MU.BE. Apr 91]*

(ii) Maximize $Z = x_1 + 9x_2$

subject to $2x_1 + x_2 \le 25$

$x_2 \le 11$

and $x_1, x_2 \ge 0$

(iii) Maximize $Z = 8x_1 + 7x$

subject to $2x_1 + x_2 \le 8$

$5x_1 + 2x_2 \le 15$

and $x_1, x_2 \ge 0$ *[MU. MCA. Nov 98]*

(iv) Maximize $Z = 3x_1 + x_2$

subject to $2x_1 + x_2 \le 6$

$x_1 \le 2$

$x_2 \le 4$

and $x_1, x_2 \ge 4$ *[MSU.BE. Nov 97]*

30. Apply the technique of dynamic programming. Problem given below.

Maximize $Z = 4x + 5y$

subject to $x \le 4$

$y \le 6$

$3x + x_2 \le 18$

and $x, y \ge 0$

ANSWERS

15. $f_1(c) = \max\limits_{0 \le z \le c} \{g_1(z) = g_1(c)$ and

$f_n(c) = \max\limits_{0 \le z \le c} \{g_1(z) = f_{n-1}(c-z)\}$

16. $y_3 = 1,\ y_2 = 1,\ y_1 = 4$ and Max $z = 18$

17. $u_1 = u_2 = u_3 = \frac{10}{3}$ and Min $z = \frac{100}{2}$

18. The optimal policy is $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}\right)$ and $f_n(1) = -n\left(\frac{1}{n}\log\frac{1}{n}\right)$

19. The optimal policy is $x_1 = 8,\ x_2 = 0,\ x_3 = 0$ and $f_3(8) = 64$.

20. (2, 2, 2) with $f_3(8) = 20$.

21. $\left(0, \frac{1}{2}, \frac{1}{2}\right)$ with $f_3(1) = \frac{3}{2}$

22. $x = 0,\ y = 0,\ z = 5$ and Max $z = 0$

23. $\left(\frac{15}{4}, \frac{15}{4}, \frac{15}{4}, \frac{15}{4}\right)$ with Max Y $= \left(\frac{15}{4}\right)^4$

24. (3, 3, 3) with Min Z = 9.

25. (3, 3, 3, 3) with Max Z = 81.

26. (2, 2, 2) with Max Z = 16.

27. $X \to 1,\ Y \to 0,\ Z \to 2$ and Max Ret = 8

28. The solution is shown in the table below:

Month	:	1	2	3	4
Purchase	:	0	500	0	0
Sale	:	200	0	0	500

Maximum possible return = 28 × 200 + 1500 = 7100

29. (i) $x_1 = 300,\ x_2 = 11$ and Max Z = 900.

(ii) $x_1 = 7,\ x_2 = 11$ and Max Z = 160.

(iii) $x_1 = 0,\ x_2 = 7.5$ and Max Z = 52.5.

(iv) $x_1 = 2,\ x_2 = 2$ and Max Z = 8.

30. Max Z = 38, $x_1 = 2,\ x_2 = 6$

Unit 4

Classical Optimisation Theory

4.1 UNCONSTRAINED EXTEMAL POROPLEMS

4.1.1 Maxima and minima for a function of one variable

Let $y = f(x)$ be a differentiable function in (a, b) We have studied in easier semesters that the necessary condition for $y = f(x)$ to have an extremum is

$$\frac{dy}{dx} = f'(x) = 0 \qquad \dots (1)$$

Let $x_1, x_2, \dots x_n$ re the roots of $f'(x) = 0$

Then

(i) $f(x)$ is said to have a maximum at x_1 if $f''(x_1) < 0$.

(ii) $f(x)$ is said to have a minimum at x_1 if $f''(x_1) > 0$.

(iii) It $f''(x_1) = 0$ then x_1 gives a point of inflexion $(x_1, f(x_1))$ if $f'''(x_1) \neq 0$.

Similar conclusions for the other roots $x_2, x_3 \dots x_4$ of (1)

Facts : If x_0 is a stationary point of $f(x)$ i.e, if $f'(x_0) = 0$ then if the first (n – 1) derivatives are zero and $f^{(n)}(x_0) \neq D$ then

(a) If n is odd, x_0 gives an inflexion point $(x_0, f(x_0))$

(b) If n is even then x_0 makes $f(x_0)$ a minimum if $f^{(n)}(x_0) > 0$ and a maximum if $f^{(n)}(x_0) < 0$

Newton Raphson Method

The necessary condition for $y = f(x)$ to have an extremum is $f'(x) = 0$. Solving this equation may be very difficult and we should be satisfied with a reasonably approximate value of the roots of the equation $f'(x) = 0$. There are many numerical methods for solving $f'(x) = 0$. One standard method studied in earlier semesters in numerical methods is Newton Raphson method.

If x_0 is an initial approximation of a root of $f'(x) = 0$ chosen properly in the vicinity of the root α of $f'(x) = 0$, $a \le \alpha \le b$. so as to ensure the convergence of the approximations then the Newton - Raphson formula is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \ k = 0, 1, 2$$

When the successive iterations x_k and x_{k+1} are approximately equal with in a specified degree of accuracy then the convergence occurs (ie, when x_k and x_{k+1} agree with desired number of decimal places we may r top iterating further.

Example 1

Find the stationary points of $f(x) = 4x^4 - x^2 + 5$ and determine the nature of the stationary points.

Solution:

$$f(x) = 4x^4 - x^2 + 5 \qquad \ldots (1)$$

$$f'(x) = 16x^3 - 2x \qquad \ldots (2)$$

Stationary points are given lay

$$f'(x) = 0$$

ie, $16x^3 - 2x = 0$

i.e. $2x(8x^2 - 1) = 0$

$$x = 0 \text{ or } 8x^2 - 1 = 0$$

$$x = 0 \text{ or } x = \pm\frac{1}{2\sqrt{2}}$$

There are 3 stationary values

$$x = 0, \ x = \frac{1}{2\sqrt{2}} \text{ and } x = \frac{-1}{2\sqrt{2}}$$

To determine the nature of these values:

Now $f''(x) = 48x^2 - 2$ from (2) ... (3)

Case (i)

Consider $x = 0$

$f''(0) = -2$ from (3)

$\therefore x = 0$ maximises $f(x)$

and the maximum value of is $f(x) = 5$

Case (ii)

Consider $x = \frac{1}{2\sqrt{2}}$

From (3) $f''\left(\frac{1}{2\sqrt{2}}\right) = 48 \times \frac{1}{8} - 2 = 4 > 0$

$x = \frac{1}{2\sqrt{2}}$ minimises f(x) and the minimum value of f(x) is

$$f\left(\frac{1}{2\sqrt{2}}\right) = 4 \times \frac{1}{64} - \frac{1}{8} + 5 = \frac{79}{16} \text{ from (1)}$$

Case (iii)

Consider $x = -\frac{1}{2\sqrt{2}}$

From (3) $f''\left(-\frac{1}{2\sqrt{2}}\right) = 48 \times \frac{1}{8} - 2 = 4 > 0$

$\therefore x = -\frac{1}{2\sqrt{2}}$ also minimises $f(x)$ and the minimum value of

$$f(x) = \frac{79}{16}$$

Note:

The stationary value $x = 0$

lies between the other stationary values $x = \frac{\pm 1}{2\sqrt{2}}$

For any continuous differentiable function there is a minimum between any two maxima and there is a maximum between any two minima.

Example 2

Investigate the functions $f(x) = 6x^5 - 43x^3 + 12$ for maxima and maxima

Solution:

$$f(x) = 6x^5 - 4x^3 + 12 \quad \text{... (1)}$$

$$f'(x) = 30x^4 - 12x^2 \quad \text{... (2)}$$

At an extremum $f'(x) = 0$

$$f'(x) = 0 \text{ gives } 30x^4 - 12x^2 = 0$$

$$3x^2 (10x^2 - 4) = 0$$

$$x = 0 \text{ or } 10x^2 - 4 = 0$$

$$x = 0 \text{ or } x = \pm\frac{2}{\sqrt{10}}$$

$\therefore \; x = 0 \; x = \frac{2}{\sqrt{10}}$ and $x = -\frac{2}{\sqrt{10}}$ are the stationary values.

To determine the nature of these

From (2),

$$f''(x) = 120\,x^3 - 24x \quad \text{... (3)}$$

Case (i)

Consider $x = 0$

$$f''(0) = 0$$

$$f'''(0) = -24 \neq 0$$

$\therefore \; x = 0$ gives a point of inflexion ie, (0,0) in a point of inflexion for the function $f(x)$.

Case (ii)

Consider $x = \frac{2}{\sqrt{10}}$

$$f''\left(\frac{2}{\sqrt{10}}\right) = \frac{48}{\sqrt{10}}\left(5 \times \frac{4}{10} - 1\right) = \frac{48}{\sqrt{10}} > 0$$

$$\therefore \; x = \frac{2}{\sqrt{10}} \text{ minimise } f(x)$$

Minimum value of $f(x)$ in

$$f\left(\frac{2}{\sqrt{10}}\right) = 6 \times \frac{16}{100} \times \frac{2}{\sqrt{10}} - \frac{4}{3} \times \frac{4}{10} \times \frac{2}{\sqrt{10}} + 12$$

$$= \frac{300\sqrt{10} - 32}{25\sqrt{10}}$$

Case (iii)

Consider $x = -\frac{2}{\sqrt{10}}$

$$f''\left(-\frac{2}{\sqrt{10}}\right) = -\frac{48}{\sqrt{10}}\left(5 \times \frac{4}{10} - 1\right)$$

$$= -\frac{48}{\sqrt{10}} < 0$$

$$\therefore \; x = -\frac{2}{\sqrt{10}} \text{ maximises } f(x)$$

$$\text{Max } f(x) = f\left(-\frac{2}{\sqrt{10}}\right) = \frac{300\sqrt{10} + 32}{25\sqrt{10}}$$

Example 3

Determine the maximum and minimum value of the function $f(x) = (3x - 4)^2 (2x - 3)^2$

Solution:

$$f(x) = (3x - 4)^2 (2x - 3)^2 \quad \text{... (1)}$$

$$f'(x) = 6(3x-4)(2x-3)^2 + 4(2x-3)(3x-4)^2$$

$$= (2x-3)(3x-4)\{6(2x-3)+4(3x-4)\}$$

$$= 2(2x-3)(3x-4)(24x-34) \quad \text{... (2)}$$

Stationary values are given by

$$f'(x) = 0$$

ie $(2x-3)(3x-4)(24x-34) = 0$

$$\therefore x = \frac{3}{2}, \frac{4}{5}, \frac{17}{12}$$

From (2),

$$f''(x) = 2(3x-4)(24x-34) + 3(2x-3)(24x-34) + 24(2x-3)(3x-4) \quad \text{... (3)}$$

Case (i)

Consider $x = \frac{3}{2}$

From (3)

$$f''\left(\frac{3}{2}\right) = 2\left(\frac{9}{2}-4\right)(36-34)$$

$$= 2 > 0$$

$$x = \frac{3}{2} \text{ Minimises } f(x)$$

$$\text{Min } f(x) = 0 \text{ from (1)}$$

Case (ii)

Consider $x = \frac{4}{3}$

From (4)

$$f''\left(\frac{4}{3}\right) = 3\left(\frac{8}{3}-3\right)(32-34) = 2 > 0$$

$$\therefore x = \frac{4}{3} \text{ minimies } f(x)$$

From (1), Minimum $f(x) = f\left(\frac{4}{3}\right) = 0$

Case (iii)

Consider $x = \frac{17}{12}$

From (3),

$$f''\left(\frac{17}{12}\right) = 24\left(\frac{34}{12}-3\right)\left(\frac{51}{12}-4\right) < 0$$

$$x = \frac{17}{12} \text{ maximises } f(x)$$

From (1),

$$\text{Min } f(x) = f\left(\frac{17}{12}\right) = \left(\frac{51}{12}-4\right)^2\left(\frac{34}{12}-3\right)^2$$

$$= \frac{36}{144^2} = \frac{1}{576}$$

Note:

That minimum values need not be the smallest and the maximum value need not be the greatest. Recall the definition of maxima and minima of a function.

Example 4

Investigate $f(x) = x^4 + 4x^2$ ***for maxima and minima.***

Solution:

$$f(x) = x^4 + 4x^2 \quad \text{... (1)}$$

$$f'(x) = 4x^3 + 8x \quad \text{... (2)}$$

$$f'(x) = 0 \text{ gives}$$

$$4x^3 + 8x = 0$$

ie, $4x(x^2+2) = 0$

$$x = 0 \text{ or } x^2 + 2 = 0$$

$x^2 + 2 = 0$ does not have real solution

$\therefore x = 0$ is the only stationary or extreme value.

Now $f''(x) = 12x^2 + 8$ from (2)

When $x = 0$, $f''(0) = 8 > 0$

$x = 0$ minimises $f(x)$ and

Minimum value of $f(x) = f(0) = 0$

Example 5

Investigate $f(x) = x^4 - 2x^2 - 16x + 1$ for maxima and minima, use Newton – Raphton method to determine the extreme value corrent to 3 decimal placeS.

Solution:

$$f(x) = x^4 - 2x^2 - 16x + 1 \quad \dots (1)$$

$$f'(x) = 4x^3 - 4x - 16$$

$$f'(x) = 0 \text{ gives}$$

$$4x^3 - 4x - 16 = 0$$

$$x^3 - x - 4 = 0 \quad \dots (2)$$

Using Descarte's rule of signs there is at most one positive root and no negative root. Two other roots are complex The positive root his between 1 and 2 since

$$f'(1) < 0 \text{ and } f'(2) > 0$$

This is a real root between 1 and 2. Take the initial approximation as $x_0 = \frac{1+2}{2} = 1.5$ and use the Newton - Raphson formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \text{ where } f(x) = x^3 - x - 4$$

$$k = 0, 1, 2...$$

$$x_{k+1} = x_k - \frac{(x_k^3 - x_k - 4)}{3x_{12}^2 - 1} = \frac{2x_k^3 + x_k + 4}{3x_k^2 - 1}$$

x_k	$x_{k+1} = \frac{2x_k^3 + x_k + 4}{3x_k^2 - 1}$
x_0	1.5
x_4	1.87
x_2	1.80
x_3	1.796
x_4	1.796

The only extreme value is 1.796 correct to is 3 decimal places since two consecutive iterations agree at k = 3, 4

$$f''(x) = 12x^2 - 4$$

$$\therefore f''(1.796) = 12 \times (1.796)^2 - 4 > 0$$

$$\therefore x = 1.796 \text{ minimies } f(x)$$

$$\text{Min} \quad f(x) = (1.796)^4 - 2(1.796)^2 - 16 \times (1.796) + 1$$

$$= -23.783$$

Exercise 1

1. Investigate the following functions for maxima and **minima**

   (i) $f(x) = 6x^5 - 15x^2 + 20$

   (ii) $f(x) = x^5 + x$

   Ans: (i) $x = 0$ in a point of inflexion

   $x = 1$ minimises $f(x)$

   Min $f = 11$

   (ii) No maximum, no minimum.

2. Investigate the function $f(x) = x^4 + 6x^2 - 4x + 5$

   for extrema (Newton – Raphson method may be used to determine the extveme value)

   Ans: Only one minimum, no maximum

   $x = 0.32218$

   minimises $f(x)$

   Min $f(x) = f(0.32218)$

   $= 4.34485$

3. Investigate $f(x) = e^{\frac{-x^2}{2}}$ for extrema

   Ans: $x = 0$ maximises $f(x)$; Max $f(x) = 1$

   $x = \pm 1$ give points of inflexion

   The points to inflexion are $(1, e^{-\frac{1}{2}})$ and $(-1, e^{-\frac{1}{2}})$.

4. Investigate for extrema the function

   $f(x) = x^3 - 6x^2 + 15x - 18$

   Ans: No Minima, no maxima, (2, 0) is a point of is inflexion

4.1.2 Maxima and Minima for a function of two variables

Let, $z = f(x, y)$

$f(x, y)$ is said to have

1. a relative maximum at a point (a, b) if $f(a, b) > f(a + h, b + k)$ for small positive or negative values of h and k

Extremum is a point which is either maximum or minimum. The values of the function f at an extreme point is known as the extreme value of the function f.

$z = f(x, y)$ represents a surface geometrically. The maximum point on the surface in the point from which the surface descends in every direction towards the xy plane (ex hill top)

The minimum point in the bottom point of the surface from which the surface ascends in every direction. In both these cases the tangent plane to the surface at a maximum of minimum point is horizontal (parallel to the xy plane)

Saddle point is a point where the function is neither maximum nor minimum. The function at such a point is maximum in one direction and minimum in another direction. Such a surface looks like the leather seat on the back of a horse.

Necessary conditions for the extrema of a function of two variables $z = f(x, y)$

Notation: Let $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x\, \partial g}$ and $t = \dfrac{\partial^2 z}{\partial y^2}$

The necessary conditions for a point (a, b) to be an extremum is

$\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$ at (a, b)

Note :

The necessary conditions for a function of n variables

$$z = f(x_1, x_2, \dots x_n) \text{ is}$$

$$\nabla f = 0$$

i.e., $\dfrac{\partial f}{\partial x_i} = 0, i = 1, 2 \dots, n$ at an extreme point

Sufficient conditions

i. f attains a maximum at an extreme point (a, b) if $rt - s^2 > 0$ and $r < 0$ at that point.

ii. f attains a minimum of (a, b) if $rt - s^2 > 0$ and $r > 0$ at that point.

iii. f has a saddle point at (a, b) If $rt - s^2 < 0$.

iv. If $rt - s^2 = 0$ further investigation is required to determine the nature of the extreme point

Steps for finding the extrema of $z = f(x, y)$

Step 1:

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

Step 2:

Solve $\dfrac{\partial z}{\partial y} = 0$ and $\dfrac{\partial z}{\partial y} = 0$

The solution (s) gives the critical points or stationary points of $z = f(x, y)$

Step 3:

Calculate r, s and t at the critical points.

Step 4:

(i) If $rt - s^2 > 0$ and $r < 0$ then f has a maximum at the critical point

(ii) If $rt - s^2 > 0$ and $r > 0$ then f has a minimum at the critical point.

(iii) If $rt - s^2 < 0$ then f has neither a maximum nor a minimum It has a saddle point.

(iv) If $rt - s^2 = 0$ further investigation is required.

Note:

Extrema occur only at critical points. But critical points need not be extrema.

Example 6

Investigate, for maxima, minima and saddle point the function

$z = x^4 + y^4 - x^2 - y^2 + 1$

Solution:

$$z = x^4 + y^4 - x^2 - y^2 + 1 \quad \ldots (1)$$

$$\frac{\partial z}{\partial x} = 4x^3 - 2x$$

$$\frac{\partial z}{\partial y} = 4y^3 - 2y$$

$$\frac{\partial z}{\partial x} = 0 \text{ give, } 4x^3 - 2x = 0$$

i.e.

$$2x(2x^2 - 1) = 0$$

$$x = 0 \text{ or } \pm\frac{1}{\sqrt{2}}$$

$$\frac{\partial z}{\partial y} = 0 \text{ give, } 4y^3 - 2y = 0$$

$$2y(2y^2 - 1) = 0$$

$$y = 0 \text{ or } \pm\frac{1}{\sqrt{2}}$$

$$r = \frac{\partial^2 z}{\partial x^2} = 12x^2 - 2; \quad \ldots (2)$$

$$t = \frac{\partial^2 z}{\partial y^2} = 12y^2 - 2 \quad \ldots (3)$$

$$s = \frac{\partial^2 z}{\partial x\, \partial y} = 0$$

$$\therefore rt - s^2 = (12x^2 - 2)(12y^2 - 2)$$

$$= 4(6x^2 - 1)(6y^2 - 1) \quad \ldots (4)$$

The extreme or critical points are $(0, 0)$, $\left(0, \frac{1}{\sqrt{2}}\right)$, $\left(0, -\frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, 0\right)$, $\left(\frac{1}{\sqrt{2}}, 0\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Case (i): At (0,0), $\left.\begin{aligned} rt - s^2 &= 4 > 0 \\ r &= -2 < 0 \end{aligned}\right\}$ from (2), (3) and (4)

$\therefore z$ is maximum at (0, 0) and max $z = 1$

Case (ii): $\left(0, \frac{1}{\sqrt{2}}\right)$, $rt - s^2 = -4 \times 2 = -8 < 0$

$\therefore \left(0, \frac{1}{\sqrt{2}}\right)$ is a saddle point

Similarly $\left(0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, 0\right)$

are all saddle points

Case (iii): At $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$,

$$rt - s^2 = 4 \times 3 \times 2 = 16 > 0$$

$$r = 2 > 0$$

z is minimum at $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

Minimum Value of $z = 1$ from (1)

Example 7

Find the shortest distance from the origin to the surface $xyz^2 = 2$.

Solution:

Distance of any point (x, y, z) on the surface from the origin is given by $\sqrt{x^2 + y^2 + z^2}$

$$\text{Let} \quad d = \sqrt{x^2 + y^2 + z^2}$$

$$d^2 = x^2 + y^2 + z^2 \quad \ldots (1)$$

If d² is minimum, d is minimum. But (x, y, z) is an point on the surface

$$\therefore \quad xyz^2 = 2 \qquad \dots (2)$$

using (2) in (1), $d^2 = x^2 + y^2 + \frac{2}{xy} = f\,(say) \qquad \dots (3)$

At extrema

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = 2x - \frac{2}{x^2 y} \text{ from (3)}$$

$$\frac{\partial f}{\partial y} = 2y - \frac{2}{xy^2} \text{ from (3)}$$

$$\frac{\partial f}{\partial x} = 0 \text{ gives } 2x - \frac{2}{x^3 y} = 0$$

ie, $\quad x^3 y - 1 = 0 \qquad \dots (4)$

$\frac{\partial f}{\partial y} = 0$ gives $2y - \frac{2}{xy^2} = 0$ ie, $xy^3 - 1 = 0 \qquad \dots (5)$

$$\therefore \quad x^3 y = 1 = xy^3$$

$$xy\,(x^2 - y^2) = 0$$

$\therefore \; x = \pm y$ since $x \neq 0, y \neq 0\,(x, y, z)$ being a point on the surface

$\therefore$ x and y should have the same sign since $z^2 = \frac{2}{xy} > 0$

$\therefore$ The stationaty points are (1, 1) and (–1,–1)

i.e, $(1, 1, \sqrt{2})$ and $(-1, -1, \sqrt{2})$ in space

From (3) and (4),

$$r = \frac{\partial^2 f}{\partial x^2} = 2 + \frac{4}{x^3 y} = 6 > 0 \text{ at } (1,1) \text{ and } (-1,-1)$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{2}{x^2 y^2} = 2 \text{ at } (1,1) \text{ and } (-1,-1)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2 + \frac{4}{xy^3} = 6 \text{ at } (1,1) \text{ and } (-1,-1)$$

$$rt - s^2 = 36 - 4 = 32 > 0, r > 0$$

$\therefore \; x = 1, \; y = 1$ minimises f

$x = -1, \; y = -1$ also minimises f

Min $f = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+1+2} = \sqrt{4} = 2$ since $d > 0$

Shortest distance $x = 2$ units

Example 8

Given that the perimeter of a triangle is constant show that the area of this triangle is maximum when the triangle in equilateral.

Solution:

With in the usual notation

$2s = a + b + c$ where a, b, c are the sides

We know that the area of the triangle A in given by

$$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

Let $\quad f = A^2 = s\,(s-a)\,(s-b)\,(s-c) \qquad \dots (1)$

If A^2 is maximum then A in maximum

$$\frac{\partial f}{\partial a} = -s\,(s-b)\,(s-c) = 0 \Rightarrow s = 0 \text{ or } s = b \text{ or } s = c$$

$$\frac{\partial t}{\partial b} = -s\,(s-a)\,(s-c) = 0 \Rightarrow s = 0 \text{ or } s = a \text{ or } s = c$$

$$\frac{\partial t}{\partial c} = -s\,(s-a)\,(s-b) = 0 \Rightarrow s = 0 \text{ or } s = a \text{ or } s = b$$

clearly $s \neq 0$

$$\therefore \; s = a = b = c$$

$\therefore$ Given $2s = a+b+c =$ a constant k say

$$a = b = c = \frac{2s}{3} = \frac{K}{3}$$

These values can only maximise A obviously.

$\therefore$ Area is maximum when the tiangle is equilateral.

Note: Area is minimum when a or b or $c = 0$

Exercise 2

1. **Investigate the following functions for extrema:**

(i) $z = x^2 + y^2$

(ii) $z = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

(iii) $z = 1 - x^2y^2$

Ans: (i) (0, 0) is the minimum point min $z = 0$

(ii) (0, 0), (4, 0), (5, 1) and (5, 1) are stationary points.

Point	Nature of the extreme point	optimum Value of z
(6, 0)	Min	108
(4, 0)	Max	112
(5, 1)	Saddle point	Neither max nor min
(5, –1)	Saddle point	Neither max nor min

(iii) Note that $rt - s^2 = D$

(0, 0) is the maximum point and Max $z = 1$ by inspection.

(2). **Find the shortest distance from the origin to the plane $x - 2y - 2z = 3$ using differential calculus.**

Ans: Shortest distance is 1

From the pt (0, 0, 1) to $\left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$

3. **Investigate $z = x^3 y^2 (1 - x - y)$ for maxima and minima**

Ans: $\left(\frac{1}{2}, \frac{1}{3}\right)$ maximises z and Max $z = \frac{1}{432}$

At (0, 0), $rt - s^2 = 0$ (Test in not conlusive further invertigation requirel)

4. Determine the extrema of $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Ans: (i) f is maximum at (0, 0)

Max $f = 4$

(ii) f is minimum at (2, 0)

Max $f = 0$

(iii) (1, 1) and (–1, –1) give saddle points. The saddle points are (1, 1, 1) and (–1, –1, –6).

4.2 CONSTRAINED EXTREMAL PROBLENMS

4.2.1 Introduction

The characterestic assumption in Linear Programming in the linearity of both the objective function and constraints. We have effective methods of solving such problems by the simplex methods. Even though this assumption of linearity holds in many practical situations we shall have many other practical situations in which either the objective function or some or all of the constraints or both are nonlinear.

The general nonlinear programming problem (NLPP) is to determine X = ($x_1, x_2, \ldots, xn$) so as to optimise.

$$z = f(X)$$

subject to

$$g_i(X) \geq = \leq b_i; \; i = 1, 2, 3, \ldots m \text{ with } X \geq 0$$

Where $f(X)$ or some or all of $g_i(X)$ or both are nonlinear.

4.2.2 Lagrangean Method

The most common method of solving extremal problems having continuous differentiable objective function as well as constraint functions with respect to the decision variables is the Lagrangean nultiplier method.

The Lagrangean Multiplier method can be illustrated by the following simple two variable problem with one constraint

Maximise or Minimise $Z = f(x_1, x_2)$

subject to $g(x_1, x_2) \leq b$

$$x_1, x_2 \geq 0$$

Step 1:

The constraint is replaced by another function $h(x_1, x_2)$ such that

$$h(x_1, x_2) = g(x_1, x_2) - b = 0$$

The problem now becomes

Maximise or Minimise $Z = f(x_1, x_2)$

subject to $h(x_1, x_2) = 0$

$$x_1, x_2 \geq 0$$

Step 2:

The Lagrangean function L can be constructed as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2)$$

where λ is called the Lagrange multiplier, a constant.

Step 3:

The necessary conditions for optimization of $f(x_1, x_2)$ subject to $h(x_1, x_2) = 0$ are clearly

$$\frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = 0$$

and $\frac{\partial L}{\partial \lambda} = 0$ where $L = L(x_1, x_2, \lambda)$

Let $f = f(x_1, x_2)$, $h = h(x_1, x_2)$

Then conditions become

$$\frac{\partial f}{\partial x_1} = \lambda \frac{\partial f}{\partial x_1}, - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 - h = 0$$

$$ie., \quad \frac{\partial L}{\partial \lambda} = -h = 0$$

$$ie., \quad \frac{\partial f}{\partial x_1} = \lambda \frac{\partial h}{\partial x_1}, \frac{\partial f}{\partial x_2} = \lambda \frac{\partial h}{\partial x_2}, \ h = 0$$

Thus the necessary conditions are given by

$$f_1 = \lambda h_1$$

$$f_2 = \lambda h_2$$

$$h = 0$$

Where $f_1 = \frac{\partial f}{\partial x_1}$, $f_2 = \frac{\partial f}{\partial x_2}$, $h_1 = \frac{\partial h}{\partial x_1}$, $h_2 = \frac{\partial h}{\partial x_2}$

Note:

These necessary conditions are also suffinent conditions when the objective function in concave in a maximisation problem and the objective function is convex in a minimisation problem.

4.2.3 Extermal problem with Equality constraints

The necessary and sufficient conditions for the problem in n variables and m equality type constraints for the optimization of the objective functions.

A general problem with n variables and m equality constants (n > m) can be expressed as

Optimise $\cancel{z} = f(X)$, ... (1)

$X = (x_1, x_2, \ldots x_3)$

subject to

$g^i(X) = b_i$ ... (2)

$i = 1, 2, 3, \ldots m$

$X \geq 0$ ... (3)

(2) can be rewritten as

$$h\,i\,(X) = g^i\,(X) - b_i = 0\ \forall i \quad \ldots (4)$$

The Lagrangean function becomes

$$L(X, \lambda) = f(X) - \sum_{I=1}^{m} \lambda_i\, h^i(X)$$

The necessary conditions for the objective function to hare an optimum are

$$\frac{\partial L}{\partial xj} = \frac{\partial f}{\partial xj} - \sum_{I=1}^{m} \lambda_i \frac{\partial h^i}{\partial xj} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = O - h^i$$

and $-h^i = 0$ where $j = 1, 2, 3, \ldots, n$

$i = 1, 2, 3, \ldots, m$

assuming that L, f and h^i are differentiable partially with respect to $x_1, x_2, \ldots x_n$ and $\lambda_1, \lambda_2, \ldots \lambda_m$

Fact 1:

These m + n necessary conditions also become sufficient conditions when in a maximisation problem the objection function is concave and in a minimisation problem the objective function in convex and the constraints are of equality type.

Fact 2:

For the problem with n variables having a single equality constraint the necessary conditions for an extemum are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0, \; j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda} = -h(x) = 0$$

$$\lambda = \frac{\partial f}{\partial x_j} / \frac{\partial f}{\partial x_j} \quad j = 1, 2, \ldots, n$$

from the first equation

These n + 1 equations in n + 1 unknowns can be solved to obtain the optimal solution.

But for determining whether the solution results in maximisation or minimisation of the objective function find the first n − 1 principal minors of the following determinant

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial n_1\, \partial x_2} - \lambda \frac{\partial' h}{\partial x_1\, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\, \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1\, \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\, x_n} - \lambda \frac{\partial^2 h}{\partial x_2\, \partial x_n} \\ \vdots & & & & \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_1 \partial y_{xn}} - \lambda \frac{\partial^2 h}{\partial x_n\, \partial x_1} & \frac{\partial^2 f}{\partial x_n\, \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \frac{\partial^4 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

If the signs of the principal minors $\lambda\, \Delta_3, \Delta_4, \Delta_5, \ldots$ are alternatively positive and negative the stationary point provides a local maxima.

It all the principal minors are negative the stationary point provides a local minima.

Example 9

Using Lagrange multiplier

Minimise $Z = x_2^1 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

subject to $x_1 + x_2 + x_3 = 7$

$$x_1, x_2, x_3 \geq 0$$

Solution:

The Lagrangean function is

$$L = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3 - \lambda(x_1 + x_2 + x_3 - 7)$$

The necessary conditions for are given by

$$\frac{\partial L}{\partial x_1} = 2x_1 - 10 - \lambda = 0 \quad \ldots (1)$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 6 - \lambda = 0 \quad \ldots (2)$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda = 0 \quad \ldots (3)$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 7) = 0 \quad \ldots (4)$$

From (1) and (3)

$$2x_1 - 10 = 2x_2 - 6 = 2x_3 - 4 = \lambda$$

ie., $x_1 - x_2 = 2$ ... (5)

and $x_2 - x_3 = 1$ ... (6)

From (4) $x_1 + x_2 - x_3 = 7$ ... (7)

using (5) and (6) in (7)

$$2 + x_2 + x_2 + x_2 - 1 = 7$$

$$3x_2 = 6 \;\; ie \; x_2 = 2$$

Solution is $x_1 = 4$, $x_2 = 2, x_3 = 1$

To test whether these values of x_1, x_2 of x_3 maximises or minimises Z consider the n – 1 ie 3 – 1 = 2 principal mirors Δ_3 and Δ_4

$$\Delta_{n+1} = \Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 t}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 t}{\partial x_1 \partial x_2} & \frac{\partial^2 t}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 t}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 t}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 t}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 t}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

where $f = f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

$h(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 7 = 0$

$$\therefore \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$$

and have $\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$

Thus

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = 12, \Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -4$$

Boths Δ_3 and Δ_4 are negative $\therefore x_1 = 4$, $x_2 = 2$ and $x_3 = 1$ minimises Z

$$Z_{min} = 35$$

Example 10

Using Lagrangean Multiplier, Maximise

$Z = 10x_1 + 4x_2 + 4\,x_1x_2 - x_1^2 - 5x_2^2$

Rewriting the problem,

Max $Z = 10x_1 + 4x_2 + 4\,x_1x_2 - x_1^2 - 5x_2^2$

subject to

$x_1 + x_2 - 6 = 0$

$x_1,\ x_2 \geq 0$

The Langrangean function can be taken as

$$L = 10x_1 + 4x_2 + 4\,x_1x_2 - x_1^2 - 5x_2^2 - \lambda\,(x_1 + x_2 - 6)$$

The necessary conditions for extrema and given by

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 + 4x_2 - \lambda = 0 \quad \ldots (1)$$

$$\frac{\partial L}{\partial x_2} = 4 + 4x_1 - 10x_2 - \lambda = 0 \quad \ldots (2)$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 6) = 0 \quad \ldots (3)$$

From (1), and (3)

$$10 - 2x_1 + 4x_2 = \lambda \quad \ldots (4)$$

$$4 + 4x_1 - 10x_2 = \lambda \quad \ldots (5)$$

$$x_1 + x_2 = 6 \quad \ldots (6)$$

From (4) & (5)

$10 - 2x_1 + 4x_2 = 4 + 4x_1 - 10x_2$

i.e, $3x_1 - 7x_2 = 3 \quad \ldots (7)$

(6) is $x_1 + x_2 = 6$

$3x_1 + 3x_2 = 18 \quad \ldots (8)$

(8) – (7) is $10x_2 = 15$

$$x_2 = \frac{3}{2}$$

$$x_1 = 6 - 3/2 = \frac{9}{2}$$

Solution is $x_1 = \frac{9}{2}, x_2 = \frac{3}{2}$

$$\text{Max } Z = 10 \times \frac{9}{2} + 4 \times \frac{2}{2} - \frac{87}{4} + 4 \times \frac{27}{4} - 5 \times \frac{9}{4}$$

$$= 78 - \frac{63}{2} = \frac{93}{2}$$

New $n = 2$

$$\therefore \Delta_{n+1} = \Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 4 & -10 \end{vmatrix} = 20 > 0$$

$\therefore x_1 = \frac{9}{2}, x_2 = \frac{3}{2}$ maximises Z

$$\frac{d^2 Z}{dx_1^2} = -20 < 0$$

$\therefore x_1 = \frac{9}{2}, x_2 = \frac{3}{2}$ Maximises Z

Max $Z = \frac{93}{2}$

Another Method (Using Maxima & Minima of a function ofone variables)

Max $Z = 10x_1 + 4x_2 + 4x_1\,x_2 - x_1^2 - 5x_2^2$.

subject to $x_1 + x_2 = 6,\ x_1, x_2 \geq 0$

From the scond equation $x_2 = 6 - x_1$

Using this in the firut equation

$$Z = 10x_1 + 4(6 - x_1) + 4x_1(6 - x_1) - x_1^2 - 5(6 - x_1)^2$$

$$ie., Z = -10x_1^2 + 90x_1 - 156$$

$$\frac{\partial Z}{\partial x_1} = -20x_1 + 90 \qquad \ldots (A)$$

For extrema $\frac{dZ}{\partial x_1} = 0$. This gives $-20x_1 + 90 = 0$

$$x_1 = \frac{9}{2}$$

$$\therefore x_2 = 6 - \frac{9}{2} = 3/2$$

To test whether $x_1 = \frac{9}{2}, x_2 = \frac{3}{2}$ maximises Z

$$\frac{\partial^2 Z}{\partial xy^2} = -20 < 0 \therefore x_1 = 2, x_2 = 3/2 \text{ maximises } Z$$

In this method and for this particular problem, the objective function is converted to a function of one variable by incorporating the constaint in the objective function and using the usual method for the function of one variable.

Thus constrained optimization problem is converted to an unconstrained optimization problem and then solved.

Example 11

Determine the optimal subsdivision of a positive quantity C into n parts in such a way that the product of the n parts is maximum using Lagrangean multifliers.

Solution:

Let $x_1, x_2, x_3 \ldots x_n$ be the n parts in such a way that

$$x_1 + x_2 + x_3 + \ldots + x_n = C$$

and $Z = x_1, x_2, x_3 \ldots x_n$

Now the problem is to maximsise Z.

ie, Max $Z = x_1 x_2 \ldots x_n$

subject to

$$x_1 + x_2 + \ldots + x_n = C$$

$$x_1 x_2, \ldots x_n \geq 0$$

The Lagrangean function is

$$L = x_1 x_2 \ldots x_n - \lambda(x_1 + x_2 + \ldots + x_n - C)$$

The necessary conditions for extremum are

$$\frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial x_n} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0$$

ie.,

$$x_2 x_3 \ldots x_n - \lambda = 0 \qquad \ldots (1)$$

$$x_1 x_3 \ldots x_n - \lambda = 0 \qquad \ldots (2)$$

$$x_1 x_2 x_4 \ldots x_n - \lambda = 0 \qquad \ldots (3)$$

$$x_1 x_2 \ldots x_{n-1} - \lambda = 0 \qquad \ldots (n)$$

$$-(x_1 + x_2 + \ldots + x_n - C) = 0 \qquad \ldots (n+1)$$

Multiplying the first n equations above by $x_1, x_2 \ldots x_n$ respectively and adding we have

$$n\, x_1 x_2 \ldots x_n - \lambda(x_1 + x_3 + \ldots + x_n) = 0 \qquad \ldots (A)$$

From (n + 1)th equation $x_1 + x_2 + \ldots + x_n = C \qquad \ldots (B)$

Using (B) in (A) we have

$$n(x_1 x_2 \ldots x_n) - \lambda C = 0$$

$$\lambda = \frac{n(x_1 x_2 \ldots x_n)}{C}$$

Substituting this value of λ in each of the n equations given by $\frac{\partial L}{\partial x_i} = 0$

We get $x_1 = x_2 = \ldots = x_n = \frac{C}{n}$

$$\therefore \text{ Max } z = \left(\frac{C}{n}\right)\left(\frac{C}{n}\right)\cdots\left(\frac{C}{n}\right), \; n \text{ factors}$$

$$ie, \text{ Max } z = \left(\frac{C}{n}\right)^n$$

$\therefore$ The optimal subdivision of C into n partsis given by

$$x_1 = x_2 = \ldots = x_n = \frac{C}{n}$$

Example 12

Minimise $z = 3e^{2x_1+3} + e^{2x_2+5}$

subject to

$$x_1 + x_2 = 4$$

$x_1, x_2 \geq 0$ using Lagrange's Multiplier

The Lagrangean function L can be taken as

$$L(x, \lambda) = 3e^{2x_1+3} + e^{2x_2+5} - \lambda(x_1 + x_2 - 4)$$

The necessary conditions for extrema are

$$\frac{\partial L}{\partial x_1} = 62^{2x_1+3} - \lambda = 0 \quad \ldots (1)$$

$$\frac{\partial L}{\partial x_2} = 2e^{2x_2+5} - \lambda = 0 \quad \ldots (2)$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 4) = 0 \quad \ldots (3)$$

From (1) and (2)

$$6e^{2x_1+3} = 2e^{2x_2+5}$$

$$ie., \quad 3e^{2x_1+3} = e^{2x_2+5} \quad \ldots (4)$$

From (3), $\quad x_2 = 4 - x_1 \quad \ldots (5)$

using (5) in (4),

$$3e^{2x_1+3} = e^{2(4-x_1)+5}$$

$$ie., \quad 3e^{2x_1+3} = e^{8-2x_1+5} = e^{13-2x_1}$$

$$ie, \log 3 + 2x_1 + 3 = 13 - 2x_1$$

$$4x_1 = 10 - \log_e^3$$

$$x_2 = 4 - \frac{1}{4}(10 - \log_e^3)$$

$$= \frac{6}{4} + \frac{1}{4}\log 3$$

$$= \frac{3}{2} + \frac{1}{4}\log_e^3$$

$$x_1 = \frac{1}{4}(10 - \log_e^3), \; x_2 = \frac{3}{2} + \frac{1}{4}\log_e^3$$

Minimises z since the objective function is obviously convex.

$\therefore$ Solution is

$$x_1 = \frac{1}{4}(10 - \log e^3), \; x_2 = \frac{3}{2} + \frac{1}{4}\log e^3$$

$$\text{Min} = 3'e^{2\times\frac{1}{4}(10-\log e^3)+3} + e^{2\left(\frac{3}{2}+\frac{1}{4}\log_e^3\right)+5}$$

$$\text{Min} = 3e^{8-\frac{1}{2}\log_e^3} + e^{8+\frac{1}{2}\log_e^3}$$

Example 13

Find the dimensions of a rectangular parallelepiped with largest volume whose side are parallel to the coordinate planes to be inscribed in the ellipsoid.

$$G(x_1, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

(M.E. M.Tech A.U.2010)

Solution:

Let $2x$, $2y$ and $2z$ be the sides of the inscribed parallelepiped then the volume of such a parallelepiped $V = 8xyz \quad \ldots(1)$

Consider the Lagrangean function

$$L = V - \lambda G(x, y, z)$$

ie, $$L = 8xyz - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Extreme points are given by

$$\frac{\partial L}{\partial x} = 8yz - \lambda\left(\frac{2x}{a^2}\right) = 0 \quad \text{... (1)}$$

$$\frac{\partial L}{\partial x} = 8xz - \lambda\left(\frac{2y}{b^2}\right) = 0 \quad \text{... (2)}$$

$$\frac{\partial L}{\partial z} = 8xy - \lambda\left(\frac{2z}{c^2}\right) = 0 \quad \text{... (3)}$$

and $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

(1) × x + (2) × y + (3) × z gives

$$24xyz - 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{r^2} + \frac{z^2}{c^2}\right) = 0 \quad \text{... (5)}$$

using (4) in (5)

$$24\,xyz - 2\lambda = 0$$

$$\lambda = 12xyz$$

using (6), (1) becomes $4yz = \dfrac{\lambda}{3x}$ ... (6)

$$\frac{2\lambda}{3x} - \frac{2\lambda x}{a^2} = 0, \; x \neq 0$$

ie, $$2\lambda\left(1 - \frac{3x^2}{a^2}\right) = 0$$

ie, $$1 - \frac{3x^2}{a^2} = 0 \; ie, \; x = \frac{a}{\sqrt{3}}$$

similarly $y = \dfrac{b}{\sqrt{3}}$ and $z = \dfrac{c}{\sqrt{3}}$

That values obviously maximise

∴ The required dimensions are

$$\frac{29}{\sqrt{3}}, \frac{2b}{\sqrt{3}} \text{ and } \frac{2c}{\sqrt{3}} \text{ units}$$

Max $$V = \frac{8\,abc}{3\,3} \text{ on units}$$

Exercise 3

1. Using Lagrangean Multiplier solve:

   Maximsise $Z = x_1 x_2$

   subject to

   $$2x_1 + x_2 = 10$$

   $$x_1, x_2 \geq 0$$

   Rewrite this problem as an unconstrained optimization problem and solve.

   Ans: $[x_1 = 2.5, x_2 = 5 \text{ Max } Z = 75]$

2. Maximise $f(x_1 x_2) = 4x_1 + 6x_2 - 2x_1 x_2 - 2x_1^2 - 2x_2^2$ subject to $x_1 + 2x_2 = 1. x_1 x_2 \geq 0$ using Lagrangean Method

   Ans: $\left[x_1 = \frac{1}{3}, x_2 = \frac{5}{6} \text{ Max } f = \frac{25}{6}\right]$

3. Using Lagrangean Method

   Maximise $f(x_1, x_2) = 5x_1 + x_2 - (x_1 - x_2)^2$

   subject to

   $$x_1 + x_2 = 4, \; x_1, x_2 \geq 0$$

   Ans: $\left[x_1 = \frac{5}{2}, x_2 = \frac{5}{6} \; x_2 = \frac{3}{2}, \text{Max } f = 12\right]$

4. Optimise

$$Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 180$$

subject to $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 \geq 0$ using

Lagrangem Method

Ans: $[x_1 = 5, x_2 = 11, x_3 = 4 \text{ Min } Z = 281]$

5. Minimise $f(x_1 x_2) = 6x_1^2 + 5x_2^2$

subject to

$x_1 + 5x_2 = 3, \; x_1, x_2 \geq 0$

Ans: $\left[x_1 = \dfrac{3}{31}, x_2 = \dfrac{18}{31} \min f = \dfrac{54}{31}\right]$

6. Find the greatest rectangle inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

using Lagrangean Multiplier method.
(Hint: the problem can be expressed as

Max $Z = 4xy$ subject to $\dfrac{x^2}{x^2} + \dfrac{y^2}{b^2} = 1$

Table $L(x, y, \lambda) = 4xy - \lambda\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1\right)$

Ans: Max Area = 2ab; $x = \dfrac{a}{\sqrt{2}}, y = \dfrac{b}{\sqrt{2}}$)

7. Find the maximum value of $x^m y^n Z^p$ when $x + y + Z = a$

Ans: Max $f = \left(\dfrac{am}{m+n+p}\right)^m \left(\dfrac{an}{m+n+p}\right)^n \left(\dfrac{ap}{m+n+p}\right)^p$

where $f = x^m y^n Z^p$; $x = \dfrac{am}{m+n+p}, y = \dfrac{an}{m+n+p}$

$Z = \dfrac{ap}{m+n+p}$

8. A retangular tank open at the top in to have volume of 32 cubic meters. Find the dimensions of the tank requiring least material for its construction.

(Hinf: Min $f = xy + 2yz + 2zx + \lambda(xyz - 32)$)

Ans: x = 4 meters, y = 4 meters, $z = 2$ meters

4.2.4 Constrained Extremal problems with more than one equality constraint

The general form of the non linear programming problem having n variables and m constraints (n > m) can be taken as

Optimise $Z = f(X), X = (x_1, x_2, \dots x_n)$

subject to

$h^i(X) = 0$, i = 1, 2, 3, ... , m

$X \geq 0$

The Lagrangean function can be taken as

$$L(X, \lambda) = f(X) - \sum_{i=1}^{m} h^i \lambda\, h^i(X)$$

Where $\lambda_i, i = 1, 2, 3, \dots, n$ are Lagrangean multipliers

Assuming that the functions $L(X, \lambda)$, $f(X)$ and $h^i(X)$ are partially differentiable with respect fo X and λ, the necessary conditions for optimum solution are

$$\frac{\partial L}{\partial x_j} = 0, \quad j = 1, 2, \dots, n$$

and $$\frac{\partial L}{\partial \lambda_i} = 0, \quad i = 1, 2, 3, \dots, m$$

The sufficiency conditions for the stationary point to be a maximum or minimum are obtained by evaluating the principal minors of the Bordered Hessian Matrix

$$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}_{(m+n)\times(m+n)}$$

Where O is an mxm null matrix and

$$\text{and } Q = \begin{bmatrix} \dfrac{\partial^2 Z}{\partial x_1^2} & \dfrac{\partial^2 Z}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 Z}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 Z}{\partial x_2 \partial x_1} & \dfrac{\partial^2 Z}{\partial x_2^2} & \cdots & \dfrac{\partial^2 Z}{\partial x_2 \partial x_n} \\ \hline \dfrac{\partial^2 Z}{\partial x_n \partial x_1} & \dfrac{\partial^2 Z}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 Z}{\partial x_n^2} \end{bmatrix}, \quad P = \begin{bmatrix} h_1^1(x) & h_2^1(x) & \cdots & h_1^n(x) \\ h_1^2(x) & h_2^2(x) & \cdots & h_2^n(x) \\ \cdots & \cdots & \cdots & \cdots \\ h_1^m(x) & h_2^m(x) & \cdots & h_n^m(x) \end{bmatrix}$$

Let (X^*, λ^*) be the stationary point for the function $L(X, \lambda)$ and H^{B*} be the corresponding Bordered Hessian Matrix. The sufficient but not necessary condition for the maxima or minima is determined by the signs of the last (n – m) principal minuses of H^{B*}, starting with the principal minor of order 2m + 1.

Now

(i) X^* maximises L if the last (n – m) principal minors form an alternate sign pattern with $(-1)^{m+n}$ and.

(ii) X^* minimises L if the signs of last (n – m) principal minors have the sign $(-1)^m$.

Example 14

Minimise $Z = x_1^2 + x_2^2 + x_3^2$

subject to

$$x_1 + x_2 + 3x_3 = 2$$
$$5x_1 + 2x_2 + x_3 = 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

Let $f(X) = x_1^2 + x_2^2 + x_3^2$, $X = (x_1, x_2, x_3)$

$$h^1(X) = x_1 + x_2 + 3x_3 - 2$$
$$h^2(X) = 5x_1 + 2x_2 + x_3 - 5$$
$$x_1, x_2, x_3 \geq 0$$

The Lagrangean function

$$L(X, \lambda) = f(X) - \lambda_1 h^1(X) - \lambda_2 h^2(X)$$

$$\text{ie., } L = x_1^2 + x_2^2 + x_3^2 - \lambda_1 (x_1 + x_2 + 3x_3 - 2) - \lambda_2 (5x_1 + 2x_2 + x_3 - 5)$$

The stationary point (X^*, λ^*) is given by the following necessary conditions:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0 \quad \ldots (1)$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 = 0 \quad \ldots (2)$$

$$\frac{\partial L}{\partial x_3} = 2x_2 - 3\lambda_1 - \lambda_2 = 0 \quad \ldots (3)$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) - 0 \quad \ldots (4)$$

$$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0 \quad \ldots (5)$$

(1) is $2x_1 = \lambda_1 + 5\lambda_2$

$$ie, \quad x_1 = \frac{\lambda_1 + 5\lambda_2}{2} \quad \ldots (6)$$

(2) is $2x_2 = \lambda_1 + 2\lambda_2$

$$ie, \quad x_2 = \frac{\lambda_1 + 2\lambda_2}{2} \quad \ldots (7)$$

(3) is $2x_3 = 3\lambda_1 + \lambda_2$

$$ie, \quad x_3 = \frac{3\lambda_1 + \lambda_2}{2} \quad \ldots (8)$$

(4) is $x_1 + x_2 + 3x_3 = 2$

Using (6), (7) & (8) in the equation above

$$\frac{\lambda_1 + 5\lambda_2}{2} + \frac{\lambda_1 + 2\lambda_2}{2} + 3x \frac{(3\lambda_1 + \lambda_2)}{2} = 2$$

ie, $11\lambda_1 + 10\lambda_2 = 4$ ... (9)

(5) is $5x_1 + 2x_2 + x_3 = 5$

$$ie,\ 5\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + 2\left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + \frac{3\lambda_1 + \lambda_2}{2} = 5$$

ie., $10\lambda_1 + 30\lambda_2 = 10$

ie, $\lambda_1 + 3\lambda_2 = 1$

$11\lambda_1 + 33\lambda_2 = 11$ ... (10)

(10) – (9) gives

$$23\lambda_2 = 7$$

$$\lambda_2 = \frac{7}{23}$$

$$\lambda_1 = 1 - 3\lambda_2 = 1 - \frac{21}{23} = \frac{2}{23}$$

Using these in(6), (7) & (8)

we have $x_1 = \frac{37}{46}, x_2 = \frac{16}{46}$ and $x_3 = \frac{2}{46}$.

$$\text{Now}\quad H^{B*} = \begin{bmatrix} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

n = 3, m = 2 gives n - m = 1 and 2m + 1 = 5

The only principal minor of H^{B*}
of order 5 is to be considered

$H^{B*} = 460 > 0$

For minimisation the sign

should be $(-1)^m = (-1)^2$ = +ve which is true

∴ Solution is

$$\therefore \text{Min } Z = 0.857, x_1 = \frac{37}{46}, x_2 = \frac{8}{23}, x_3 = \frac{13}{46}$$

Example 15

Optimise $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1 x_2$

subject to

$$x_1 + x_2 + x_3 = 25$$

$$2x_1 - x_2 + 2x_3 = 15$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

The Lagrangean function can be taken as

$$L = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1 x_2 - \lambda_1 (x_1 + x_2 + x_3 - 25) - \lambda_2 (2x_1 + x_2 + 2x_3 - 15)$$

The stationary point in given by

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \quad \text{... (1)}$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{... (2)}$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \quad \text{... (3)}$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 25) = 0 \quad \text{... (4)}$$

$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 15) = 0 \quad \text{... (5)}$$

(1) – (2) gives

$$12x_1 - 8x_2 - 3\lambda_2 = 0$$

$$ie,\quad \lambda_2 = \frac{12x_1 - 8x_2}{3} \quad \text{... (6)}$$

(1) + (2) × 2 gives

$$4x_2 - 3\lambda_1 = 0$$

$$\lambda_1 = \frac{4}{3} \quad \text{... (7)}$$

Using (6) & (7) in (3)

$$2x_3 - \frac{4}{3}x_2 - \frac{2}{3}(12x_1 - 8x_2) = 0$$

i.e.,	$-24x_1 + 12x_2 + 6x_3 = 0$	
i.e.,	$-4x_1 + 2x_2 + x_3 = 0$	... (8)
(4) is	$x_1 + x_2 + x_3 = 25$	... (9)
(5) is	$2x_1 - x_2 + 2x_3 = 15$	... (10)
(8) – (9) is	$-5x_1 + x_2 = -25$	
	$5x_1 - x_2 = 25$	
(9) × 2 is	$2x_1 + 2x_2 + 2x_3 = 50$	... (11)
(11) – (10) 5	$3x_2 = 35$	
	$x_2 = \frac{35}{3}$	

$$x_1 = \frac{x_2 + 25}{5} = \frac{\frac{35}{3} + 25}{5} = \frac{110}{15} = \frac{22}{3}; x_3 = 6 \text{ from (9)}$$

n = 3, m = 2, n – m = 1 2m + 1 = 5

Only one principal miner of order 5 of H^{B*} 5 is to be considered.

For maximisation sign should be $(-1)^{m+m} = (-1)^{2+3}$

$= -$ ve

For minimisation the

sign should be $(-1)^m = (-1)^2 = +ve$

$$H^{B*} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

The principal minor of said of H^{B*} has the value 90 which is +ve

$$\therefore x_1 = \frac{22}{3}, x_2 = \frac{35}{3}, x_3 = 6 \text{ minimiser } Z$$

$$Z_{min} = 4\left(\frac{22}{3}\right)^2 + 2 \times \left(\frac{35}{3}\right)^2 + 6^2 - 4 \times \frac{22}{3} \times \frac{35}{3}$$

$$= \frac{1624}{9}$$

Exercise 4

1. Using Lagrangean Method optimise

$$f(x_1, x_2) = 4x_1 + 9x_2 - x_1^2 - x_2^2$$

subject to

$$4x_1 + 3x_2 = 15$$

$$3x_1 + 5x_2 = 14, x_1, x_2 \geq 0$$

Ans: $(x_1 = 3, x_2 = 1, Min\ f = 11)$

2. Optimise

$$f(x_1, x_2, x_3) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1 x_2$$

subject to

$$x_1 + x_2 + x_3 = 15$$

$$2x_1 - x_2 + 2x_3 = 20$$

$$x_1, x_2, x_3 \geq 0$$

Ans: $\left[x_1 = \frac{11}{3}, x_2 = \frac{10}{3}, x_3 = 8 \text{ Min } f = \frac{820}{9}\right]$

3. Minimise $f(x_1, x_2) = 6x_1 + 8x_2 - x_1^2 - x_2^2$

subject to

$$3x_1 + 5x_2 = 15$$

$$4x_1 + 3x_3 = 16$$

$$x_1, x_2, \geq 0$$

Ans: $\left[x_1 = \frac{35}{11}, x_2 = \frac{12}{11}, x_3 = 8 \text{ Min } f = 281\right]$

4.2.5 Costrained extremal problems with inequality constraints (KUHN – TUCKER CONDITIONS)

MAXIMISATION PROBLEM

Maximise $Z = f(x)$

subject to

$$g(X) \leq b \qquad \dots (A)$$

$$X \geq O, X = (x_1\ x_2, \dots x_n)$$

Let $h(X) = g(X) - b$ Then $h(X) \leq 0$ from (A)

First the inequality constraint is changed to equality type by introducing a slack variable S in the form of S^2 to ensure the non negativity.

Thus the constraint can be expressed as $h(X) + S^2 = 0$ and the NLPP can be expressed in the form

Max $Z = f(X)$

subject to

$$h(X) + S^2 = 0$$

$$X \geq 0$$

We have now n + 1 variables with single inequality constraint.

Secondly construct the Lagrangean function.

$$L(X, S, \lambda) = f(X) - \lambda [h(X) + S^2]$$

The necessary conditions for the stationary point are given by

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \qquad \dots (1)$$

$$j = 1, 2, \dots, n$$

$$\frac{\partial L}{\partial \lambda} = -[h(X) + S^2] = 0 \qquad \dots (2)$$

and $$\frac{\partial L}{\partial S} = -2S\lambda = 0 \qquad \dots (3)$$

From (3) we have

either $S = 0$ or $\lambda = 0$

If $S = 0$ then from (2) we have

$h(X) = 0$

$\therefore$ either $\lambda = 0$ or $h(X) = 0$

$$ie, \quad \lambda h(X) = 0 \qquad \dots (4)$$

From (2) again we have

$h(X) = -S^2 = -$ Ve

$$\therefore h(X) \leq 0 \qquad \dots (5)$$

If $h(X) < 0$ then $\lambda = 0$ from (4)

If $\lambda > 0$ then $h(X) = 0$

Thus the necessary condition a are summarized as

$$\frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0, j = 1, 2, \dots, n \qquad \dots (I)$$

$$\lambda h(X) = 0 \qquad \dots (II)$$

$$h(X) \leq 0 \qquad \dots (III)$$

$$\lambda \geq 0 \qquad \dots (IV)$$

These necessary conditions are called KUHN - TUCKER Conditions or KRUSH - KUHN - TUCKER conditions

MINIMISATION PROBLEM

Minimise $Z = f(X)$, $X = (x_1, x_2, \dots, x_n)$

subject to

$$g(X) \geq b$$

$$X \geq 0$$

This is rewritten as

Minimise $Z = f(X)$

subject to

$$h(X) = g(X) - b \geq 0$$

$$X \geq 0$$

Introducing slack variable in the form of S^2 we have the problem as

Minimise $Z = f(X)$

subject to

$$h(X) - S^2 = 0$$

Following the analysis similar to the one used for maximisation problem the KUHN - TUCKER conditions become

$$\frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0, \quad j = 1, 2, \ldots, n$$

$$\lambda\, h(X) = 0$$

$$h(X) \geq 0$$

$$\lambda \geq 0$$

For a single constraint nonlinear programming problem the Kuhn - Tucker conditions are also sufficent conditions if

(1) $f(X)$ is concave and $h(X)$ is concave in the maximisation problem and

(2) both $f(X)$ and $h(X)$ are concave in the minimisation problem.

4.2.6 Kuhn - Tucker conditions for a general non linear programming problem

(i) For maximisation problem

Maximise $Z = f(X)$

subject to

$$g^i(X) \leq b_i, i = 1, 2, 3, \ldots, m$$

$$X \geq 0, \ X = (x_1, x_2, \ldots, x_3)$$

The constraint inequality can be rewritten as

$$h^i(X) = g^i(X) - b_i \leq 0, \qquad i = 1, 2, \ldots, m$$

The modified equality constraint becomes

$$h^i(X) + S_i^2 = 0, \ i = 1, 2, \ldots, m$$

The Lagrangean function in

$$L(X, S, \lambda) = f(X) - \sum_{i=1}^{m} \lambda_i \left[h^i(X) + S_i^2\right]$$

The necessary conditions for maximisation are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f(X)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i(X)}{\partial x_j} = 0, \qquad j = 1, 2, \ldots n$$

$$\frac{\partial L}{\partial \lambda_j} = \left[h^i(X) + S_i^2\right] = 0, \qquad i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial S_i} = 2\,S_i\,\lambda_i = 0, \qquad i = 1, 2, \ldots, m$$

The last two conditions above become

$$\lambda_i\, h^i(X) = 0$$

$$h^i(X) \leq 0, \quad \lambda_i \geq 0$$

Thus the Kuhn - Tucker conditions for a non-linear programming problem of maximising $f(X)$ subject to $h^i(X) \leq 0$ are given by

$$f_j(X) - \sum_{i=1}^{m} \lambda_i\, h_j^i(X) = 0$$

$$\lambda_1\, h^i(X) = 0$$

$$h^i(X) \leq 0$$

$$\lambda_i \geq 0, \qquad i = 1, 2, \ldots m$$

$$j = 1, 2, \ldots, n$$

These K-T conditions are also sufficisnt if $f(X)$ is concave and all $h^i(X)$ are convex

Similarly

The Kuhn - Tucker conditions for a minimisation problem are

$$f_j(X) - \sum_{0=1}^{m} \lambda_i\, h_j^i(X) = 0$$

$$\lambda_i\, h^i(X) = 0$$

$$h^i(X) \geq 0$$

$$\lambda_i \geq 0$$

$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

These Kuhn – Tucker conditions become the sufficient conditions if f(X) is convex and all $h^i(X)$ are concave in X.

Note:

(i). The $\leq$ constraints may be changed to $\geq$ by the usual method used in linear programming problem.

(ii). The Lagrange multipliers corresponding to equality constraints must be unrestricted in sign.

Example 16

Maximise $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to

$$3x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Solution:

Let $f(x) = 8x_1 + 10x_2 - x_1^2 - x_2^2$

$$h(x) = 3x_1 + 2x_2 - 6$$

Kuhn – Tucker conditions for the maximisation problem become

$$f(X) - \lambda\, h(X) = 0, \; j = 1, 2$$

$$\lambda\, h(X) \leq 0$$

$$\lambda \geq 0$$

ie., $8 - 2x_1 - 3\lambda = 0$ ... (1)

$10 - 2x_2 - 2\lambda = 0$ ... (2)

$\lambda(3x_1 + 2x_2 - 6) = 0$ ... (3)

$3x_1 + 2x_2 - 6 \leq 0$ and $\lambda \geq 0$ ... (4)

Case (i) $\lambda = 0$

The equations above become

$8 - 2x_1 = 0$ from (1)

$10 - 2x_2 = 0$ from (2)

$\therefore x_1 = 4, \quad x_2 = 5$

This solution is not feasible since $3x_1 + 2x_2 \neq 6$ when $x_1 = 4$ and $x_2 = 5$.

Case (ii)

$\lambda \neq 0$

Then the equations (1), (2) and (3) become

$8 - 2x_1 - 3\lambda = 0$ ... (4)

$10 - 2x_2 - 2\lambda = 0$ ... (5)

$3x_1 - 2x_2 - 6 = 0$ ... (6)

From (4) and (5),

$$x_1 = \frac{8 - 3\lambda}{2}, \; x_2 = \frac{10 - 2\lambda}{2} = 5 - \lambda$$

Using these in (6)

$$3\left(\frac{8 - 3\lambda}{2}\right) + 2(5 - \lambda) - 6 = 0$$

$$13\lambda - 32 = 0$$

$$\lambda = \frac{32}{13}$$

$$\therefore x_1 = \frac{8 - 3\lambda}{2} = \frac{4}{13}; \; x_2 = 5 - \lambda = \frac{33}{13}$$

$$\text{Max } Z = f(x_1, x_2) = 8 \times \frac{4}{13} + 10 \times \frac{33}{13} - \left(\frac{4}{13}\right) - 5^2$$

Max $Z = 21.3$

Solution:

$$x_1 = \frac{4}{13}, \; x_2 = \frac{33}{13}, \text{ Max } Z = 21.3$$

Example 17

Minimise $Z = 0.3x_1^2 - 2x_1 + 0.4x_2^2 - 2.4x_2 + 0.6x_1x_2 + 100$

subject to

$2x_1 + x_2 \geq 4$, , $\quad x_1, x_2 \geq 0$

Solution:

Let $f(X) = 0.3x_1^2 - 2x_1 + 0.4x_2^2 - 2.4x_2 + 0.6x_1x_2 + 100$

and $h(X) = 2x_4 + x_2 - 4$

Kuhn – Tucker conditions for a minimisation problem are

$$f_j(X) - h_j(X) = 0$$

$$\lambda\, h(X) = 0$$

$$h(X) \geq 0$$

$\lambda \geq 0$ with the usual notation

Thus we have

$$0.6\,x_1 - 2 + 0.6\,x_2 - 2\lambda = 0 \qquad \dots (1)$$

$$0.8\,x_1 - 2.4 + 0.6\,x_1 - \lambda = 0 \qquad \dots (2)$$

$$\lambda\,(2x_1 + x_2 - 4) = 0 \qquad \dots (3)$$

$$\text{and } 2x_1 + x_2 - 4 \geq 0, \qquad \dots (4)$$

$$x_1, x_2 \geq 0, \lambda \geq 0$$

Case (i):

$\lambda = 0$

The equations (1) and (2) above become

$$0.6x_1 + 0.6x_2 = 2$$

$$0.8x_2 + 0.6x_1 = 2.4$$

i.e., $3x_1 + 3x_2 = 10$

and $3x_1 + 4x_2 = 12$

$$\therefore\ x_2 = 2 \text{ and } x_1 = \frac{4}{3}$$

But $2x_1 + x_2 = \frac{8}{3} + 2 = \frac{14}{3} \geq 4$

This solution is feasible

Case (ii)

$\lambda \neq 0$

The Kuhn Tucker - conditions become from (1) & (2),

$$0.6\,x_1 + 0.6\,x_2 = 2 + 2\lambda$$

$$0.6\,x_1 + 0.8\,x_2 = \lambda + 2.4$$

$$\cdot\, 2x_2 = 0.4 - \lambda$$

$$\lambda = 0.4 - 0.2\,x_2$$

From (3)

$$2x_1 + x_2 - 4 = 0$$

$$\therefore\ 0.6\,x_1 + 0.8\,x_2 = 0.4 + 0.2\,x_2 + 2.4$$

$$0.6x_1 + 10\,x_2 = 28$$

$$ie,\ 3x_1 + 5x_2 = 14$$

But $2x_1 + x_2 = 4$

$$\therefore\ x_1 = -\frac{6}{7} < 0,\ x_2 = \frac{40}{7}$$

This solution is not feasible since $x_1 < 0$

$\therefore$ The optimal solution is given by case (i)

ie, $x_1 = \frac{4}{3}, x_2 = 2$

$$Z_{min} = 0.3 \times \frac{16}{9} - \frac{8}{3} + 0.4 \times 4 - 2.4 \times 2 + 0.6 \times \frac{4}{3} \times 2 + 100$$

$$= \frac{292}{3} = 97.33$$

Solution is $x_1 = \frac{4}{3}, x_2 = 2, Z_{min} = 97.33$

Example 18

Maximise $Z = 7x_1^2 + 6x_1 + 5x_2$

subject to

$$x_1 + 2x_2 \leq 10$$

$$x_1 - 3x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

Solution:

Let $f(X) = 7x_1^2 + 6x_1 + 5x_2$

$$h^1(X) = x_1 + 2x_2 - 10$$

$$h^2(X) = x_1 - 3x_2 - 9$$

The Kuhn – Tucker conditions for the maximation problem become

$$f_j(X) - \sum_{i=1}^{2} \lambda_i h^i(X) = 0, \qquad i = 1, 2$$

$$\lambda_1 h^1(x) = 0 \qquad j = 1, 2$$

$$\lambda_2 h^2(x) = 0$$

$$h^i(X) \le 0, \; i = 1, 2$$

$$\lambda_1, \lambda_2, \ge 0, \; x_1, x_2 \ge 0$$

Thus

$$14x_1 + 6 - \lambda_1 = 0 \qquad \text{... (1)}$$

$$5 - 2\lambda_1 + 3\lambda_2 = 0 \qquad \text{... (2)}$$

$$\lambda_1 (x_1 + 2x_2 - 10) = 0 \qquad \text{... (3)}$$

$$\lambda_2 (x_1 - 3x_2 - 9) = 0 \qquad \text{... (4)}$$

$$x_1 + 2x_2 - 10 \le 0 \qquad \text{... (5)}$$

$$x_1 - 3x_2 - 9 \le 0 \qquad \text{... (6)}$$

$$x_1, x_2 \ge 0, \; \lambda_1, \lambda_2 \ge 0$$

Case (i)

$$\lambda_1 = 0, \quad \lambda_2 = 0$$

(2) gives absurd result namely $5 = 0$ NO feasible solution in this case

Case (ii)

$$\lambda_1 = 0, \quad \lambda_2 \ne 0$$

(1) becomes $14x_1 + 6 = 0$

$$x_1 = \frac{-6}{14} < 0$$

Not feasible in this case

Case (iii): $\lambda_1 \ne 0, \quad \lambda_2 = 0$

$$14x_1 + 6 - \lambda_1 = 0 \qquad \text{... (7)}$$

$$5 - 2\lambda_1 = 0 \qquad \text{... (8)}$$

$$x_1 + 2x_2 - 10 = 0 \qquad \text{... (9)}$$

$$\lambda_1 = 14x_1 + 6 \quad \text{from (3)}$$

$$\therefore \; 5 - 2(14x_1 + 6) = 0$$

$$x_1 = -\frac{7}{28} < 0$$

No feasible solution in this case

Case (iv) :

$$\lambda_1 \ne 0, \quad \lambda_2 \ne 0$$

(3) and (4) become

$$x_1 + 2x_2 - 10 = 0$$

$$x_1 - 3x_2 - 9 = 0$$

$$5x_2 - 1 = 0, \; x_2 = \frac{1}{5}$$

$$x_1 = 10 - 2x_2 = \frac{48}{5}, \; \lambda_1 = 14x_1 + 6 = \frac{702}{5}, \lambda_2 = \frac{1399}{15}$$

$$\text{Max } Z = 7\left(\frac{48}{5}\right)^2 + 6 \times \frac{48}{5} + 1 = 703.72$$

$\therefore$ The optimal solution

$$x_1 = \frac{48}{5}, x_2 = \frac{1}{5}, \text{Max } Z = 703.72$$

Example 19

Maximise $Z = 3x_1 + x_2$

subject to

$$x_1^2 + x_2^2 \le 5$$

$$x_1 - x_2 \le 1$$

$$x_1, x_2 \ge 0$$

Solution:

Let $f(X) = 3x_1 + x_2$

$$h^1(X) = x_1^2 + x_2^2 - 5$$

$$h^2(X) = x_1 - x_2 - 1$$

Kuhn – Tucker conditions for maximisation are given by

$$f_j(X) - \sum_{i=1}^{2} \lambda_1 h_j^i(X) = 0, \; i = 1,2 \,;\, j = 1,2$$

$$\lambda_1 (x_1^2 + x_2^2 - 5) = 0$$

$$\lambda_2 (x_1 - x_2 - 1) = 0$$

$$x_1^2 + x_2^2 - 5 \leq 0$$

$$x_1 - x_2 - 1 \leq 0$$

$$x_1, x_2 \geq 0, \; \lambda_1, \lambda_2 \geq 0$$

Thus we have

$$3 - 2\lambda_1 x_1 - \lambda_2 = 0 \quad \text{... (1)}$$

$$1 - 2\lambda_1 x_2 + \lambda_2 = 0 \quad \text{... (2)}$$

$$\lambda_1 (x_1^2 + x_2^2 - 5) = 0 \quad \text{... (3)}$$

$$\lambda_2 (x_1 - x_2 - 1) = 0 \quad \text{... (4)}$$

$$x_1^2 + x_2^2 - 5 \leq 0 \quad \text{... (5)}$$

$$x_1 - x_2 - 1 \leq 0 \quad \text{... (6)}$$

$$x_1, x_2, \lambda_1, \lambda_2 \geq 0$$

Case (i):

$$\lambda_1 = 0, \; \lambda_2 = 0$$

(2) becomes $1 = 0$. $\therefore$ No feasible solution

Case (ii):

$$\lambda_1 = 0, \; \lambda_2 \neq 0,$$

(1) is $3 - \lambda_2 = 0$

$$\lambda_2 = 3$$

(ii) is $\lambda_2 + 1 = 0, \; \lambda_2 = -1 < 0$

No feasible solution

Case (iii)

$$\lambda_1 \neq 0, \; \lambda_2 = 0$$

(1) is $3 - 2\lambda_1, x_1 = 0$

(2) is $1 - 2\lambda_1 x_2 = 0$

(3) is $x_1^2 + x_2^2 - 5 = -0$

$$\therefore x_1 = \frac{3}{2\lambda_1}, \; x_2 = \frac{1}{2\lambda_1}$$

and $\dfrac{9}{4\lambda_1^2} + \dfrac{1}{4\lambda_1^2} - 5 = 0$

$$10 = 20\lambda_1^2$$

$$\lambda_1^2 = \frac{1}{2}, \; \lambda_1 = \pm \frac{1}{\sqrt{2}}$$

Take $\lambda_1 = \dfrac{1}{\sqrt{2}} > 0$

$$\therefore x_1 = \frac{3\sqrt{2}}{2} \text{ and } x_2 = \frac{1}{\sqrt{2}}$$

But (6) is $x_1 - x_2 - 1 \leq 0$

$$x_1 - x_2 - 1 = \frac{3\sqrt{2}}{2} - \frac{1}{\sqrt{2}} - 1 = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1 > 0$$

$\therefore$ (6) is violated

No feasible solution in this lase

Case (iv):

$$\lambda_1 \neq 0, \; \lambda_2 \neq 0$$

$$\therefore x_1^2 + x_2^2 - 5 = 0$$

$x_1 - x_2 - 1 = 0$ from (3) and (4)

$$x_1 = x_2 + 1$$

$$\therefore (x_2 + 1)^2 + x_2^2 - 5 = 0$$

$$2x_2^2 + 2x_2 - 4 = 0$$

$$x_2^2 + x_2 - 2 = 0$$

$$x_2 = 1, -2$$
$$x_2 = -2 \text{ inadmissible}$$
$$x_2 = 1 \text{ gives } x_1 = 2$$

(1) gives $3 - 4\lambda_1 - \lambda_2 = 0$

(2) gives $1 - 2\lambda_1 + \lambda_2 = 0$

$$-6\lambda_1 + 4 = 0, \lambda_1 = \frac{2}{3} > 0$$
$$\lambda_2 = 3 - 4\lambda_1 = 3 - \frac{8}{3} = \frac{1}{3} > 0$$

$\therefore$ The optimal solution is

$x_1 = 2, x_2 = 1 \quad \text{Max } Z = 7$

Exercise 5

1. Solve, using Kuhn – Tucker conditions,

Maximise $Z = 7x_1^2 - 6x_1 + 5x_2^2$

subject to

$$x_1 + 2x_2 \le 10$$
$$x_1 - 3x_2 \le 9$$
$$x_1, x_2, \ge 0$$

Ans: $\left[x_1 = \frac{48}{5}, x_2 = \frac{2}{5}, \text{Min } Z = 587.72\right]$

2. Write down the Kuhn – Tucker conditions for the following problem and solve

Minimise $Z = x_1^2 + x_2^2$

subject to

$$x_1 + x_2 \ge 8$$
$$x_1 + 2x_2 \ge 10$$
$$2x_1 + x_2 \ge 10$$
$$x_1, x_2, \ge 0$$

Ans: $[x_1 = 4, x_2 = 4, \text{Min } Z = 37]$

3. Minimise $Z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

subject to

$$x_1 + 3x_2 \le 6$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2, \ge 0$$

Ans: $\left[x_1 = 1, x_2 = \frac{3}{4}, \text{Min } Z = \frac{17}{8}\right]$

4. Solve Max $Z = 2x_1 - x_1^2 + x_2$

subject to

$$2x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 4$$
$$x_1, x_2, \ge 0$$

Ans: $\left[x_1 = \frac{2}{3}, x_2 = \frac{14}{9}, \text{Min } Z = \frac{22}{9}\right]$

5. Solve Max $Z = 8x_1^2 + 2x_2^2$

subject to

$$x_1^2 + x_2^2 \le 9$$
$$x_1 \le 2;\ x_1, x_2, \ge 0$$

Ans: $[x_1 = 2, x_2 = \sqrt{5}, \text{Min } Z = 42]$

Unit 5

Project Scheduling

5.1 INTRODUCTION

A ***project*** is defined as a combination of interrelated activities all of which must be executed in a certain order to achieve a set goal. A large and complex project involves usually a number of interrelated activities requiring men, machines and materials. It is impossible for the management to make and execute an optimum schedule for such a project just by intuition, based on the organisational capabilities and work experience. A systematic scientific approach has become a necessity for such projects. So a number of methods applying network scheduling techniques has been developed. ***Programme Evaluation Review Technique*** (PERT) and ***Critical Path*** Method (CPM) are two of the many network techniques which are widely used for planning, scheduling and controlling large complex projects.

The three main managerial functions for any project are

1. Planning
2. Scheduling
3. Control

Planning

This phase involves a listing of tasks or jobs that must be performed to complete a project under consideration. In this phase, men, machines and materials required for the project in addition to the estimates of costs and durations of various activities of the project are also determined.

Scheduling

This phase involves the laying out of the actual activities of the projects in a ***logical sequence*** of time in which they have to be performed.

Men and material requirements as well as the ***expected completion time*** of each activity at each stage of the project are also determined.

Control

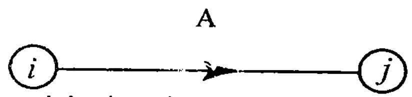
This phase consists of reviewing the progress of the project whether the actual performance is according to the planned schedule and finding the reasons for difference, if any, between the schedule and performance. The basic aspect of control is to analyse and correct this difference by taking remedial action wherever possible.

PERT and CPM are especially useful for scheduling and controlling.

5.2 BASIC TERMINOLOGIES

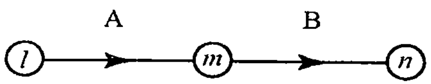
Activity is a task or an item of work to be done in a project. An activity consumes resources like time, money, labour etc.

An activity is represented by an arrow with a node (event) at the beginning and a node (event) at the end indicating the start and termination (finish) of the activity. Nodes are denoted by circles. Since this is a logical diagram length or shape of the arrow has no meaning. The direction indicates the progress of the activity. Initial node and the terminal node are numbered as $i - j$ $(j > i)$ respectively. For example If A is the activity whose initial node is i and the terminal node is j then it is denoted diagrammatically by



The name of the activity is written over the arrow, ***not inside the circle.*** The diagram in which arrow represents an activity is called ***arrow diagram.*** The initial and terminal nodes of activities are also called tail and head events.

If an activity B can start immediately after an activity A then it is denoted by



A is called the ***immediate predecessor*** of B and B is called the ***immediate successor*** of A. If C can start only after completing activities A and B then it is diagrammatically represented as follows:

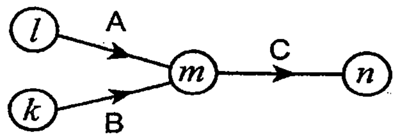


Fig. 1

Notation: "A is a predecessor of B" is denoted as "A < B". "B is a successor of A" is denoted by "B > A".

If the project contains two or more activities which have some of their immediate predecessors in common then there is a need for introducing what is called ***dummy activity.*** Dummy activity is an imaginary activity which does not consume any resource and which serves the purpose of indicating the predecessor or successor relationship clearly in any activity on arrow diagram. The need for a dummy activity is illustrated by the following usual example.

Let P, Q be the predecessors of R and Q be the only predecessor of S.

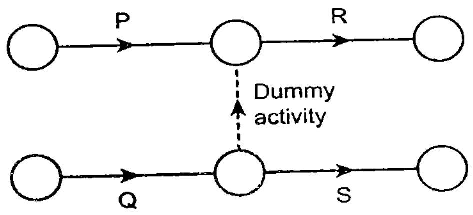


Fig. 2

Activities which have no ***predecessors*** are called ***start activities*** of the project. All the ***start activities*** can be made to have the ***same initial node.*** Activities which have ***no successors*** are called ***terminal activities*** of the project. These can be made to have ***the same*** terminal node (end node) of the project.

A project consists of a number of activities to be performed in some technological sequence. For example while constructing a building the activity of laying the foundation should be done before the activity of erecting the walls for the building. The diagram denoting all the activities of a project by arrows taking into account the technological sequence of the activities is called the ***project network*** represented by ***activity on arrow diagram*** or simply ***arrow diagram.***

Note: There is another representation of a project network representing activities on nodes called AON diagram. To avoid confusion we use only activity on arrow diagram throughout the text.

5.3 RULES FOR CONSTRUCTING A PROJECT NETWORK

1. There must be no loops. For example, the activities F, D, E.

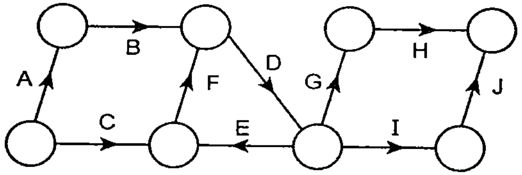


Fig. 3

Obviously form a loop which is obviously not possible in any real project network.

2. Only one activity should connect any two nodes.

3. No dangling should appear in a project network *i.e.,* no node of any activity except the terminal node of the project should be left without any activity emanating from it. Such a node can be joined to the terminal node of the project to avoid.

Nodes may be numbered using the rule given below:

(Ford and Fulkerson's Rule)

1. Number the start node which has no predecessor activity, as 1.
2. Delete all the activities emanating from this node 1.
3. Number all the resulting start nodes without any predecessor as 2, 3, ...
4. Delete all the activities originating from the start nodes 2, 3, ... in step 3.
5. Number all the resulting new start nodes without any predecessor next to the last number used in step (3).
6. Repeat the process until the terminal node without any successor activity is reached and number this terminal node suitably.

Immediate predecessor (successor) will be simply called as predecessor (successor) unless otherwise stated.

Example 1 **If there are five activities P, Q, R, S and T such that P, Q, R have no immediate predecessors but S and T have immediate predecessors P, Q and Q, R respectively. Represent this situation by a network.**

Solution:

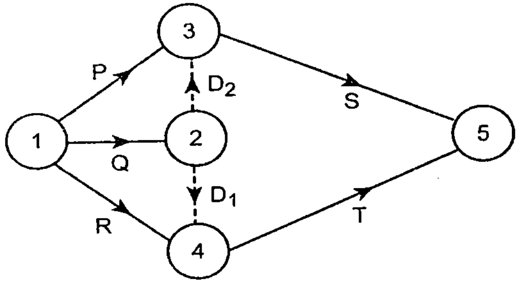


Fig. 4

D_1 and D_2 are dummy activities.

Example 2: **Draw the network for the project whose activities and their precedence relationships are given below:**

Activity	:	**P**	**Q**	**R**	**S**	**T**	**U**
Predecessor	:	–	–	–	**P, Q**	**P, R**	**Q, R**

Solution:

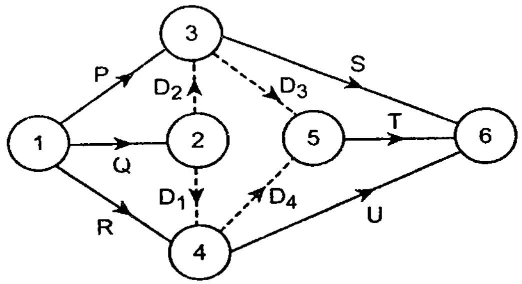


Fig. 5

D_1, D_2, D_3, D_4 are dummy activities.

Example 3 **Draw the network for the project whose activities with their predecessor relationships are given below:**

A,C,D can start simultaneously ; E > B, C ; F, G > D ; H, I > E, F ; J > I, G ; K > H ; B > A.

Solution: Identify the start activities *i.e.,* activities which have no predecessors. They are A, C and D as given. These three activities should start with the same start node. Also identify the terminal activities – activities which have no successors. They are J and K. These two activities should end with the same end node, the last terminal node indicating the completion of the project. Taking into account the predecessor relationships given, the required network is as follows:

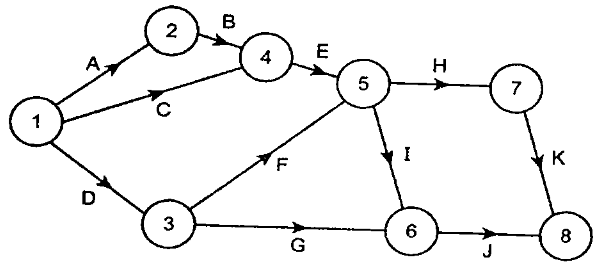


Fig. 6

Example 4 **: Construct the network for the project whose activities and their relationships are as given below:**

Activities : A, D, E can start simultaneously.

Activities : B, C > A ; G, F > D, C ; H > E, F.

Solution: Start activities are A, D, E.

End activities are H, G, B.

The required network is

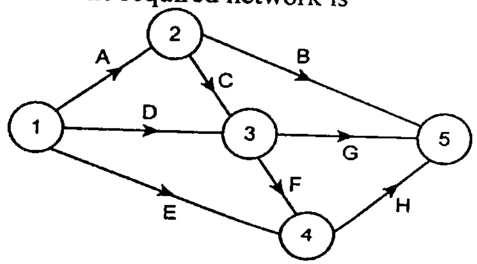


Fig. 7

Note: See how the nodes of the activity F are numbered. Can we number C as 2 – 4 and F as 4 – 3 ?.

Example 5 **: Draw the network for the project whose activities and their precedence relationships are as given below:**

Activities :	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**
Immediate Predecessor :	**–**	**A**	**A**	**–**	**D**	**B,C,E**	**F**	**E**	**G,H**

[BE. Apr 95]

Solution: Start activities: A, D, Terminal activities: I only.. Activities B and C starting with the same node are both the predecessors of the activity F. Also the activity E has to be the predecessor of both F and H. Therefore dummy activities are necessary.

Thus the required network is

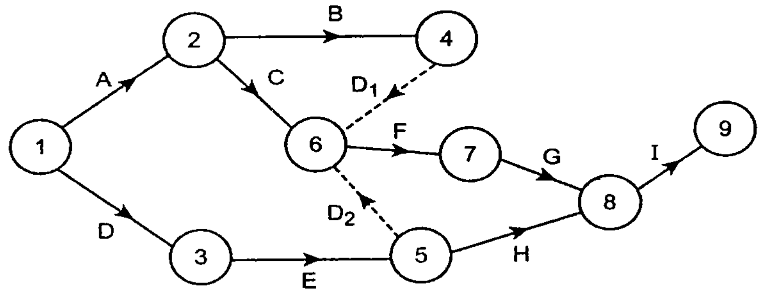


Fig. 8

D_1 and D_2 are dummy activities.

Note: Sometimes while constructing a network you may introduce more dummy activities than necessary. Redundant dummy activities can always be found out when one checks whether all the given precedence relationships given in the problem are satisfied exactly. (Nothing more, nothing less).

Example 6 : **Construct the network for the project whose precedence relationships are as given below:**

B < E, F ; C < G, L ; E, G < H ; L, H < I ; L < M ; H, M < N ; A < J ; I, J < P ; P < Q.

Solution: Start activities B, C. : End Activities: N, Q

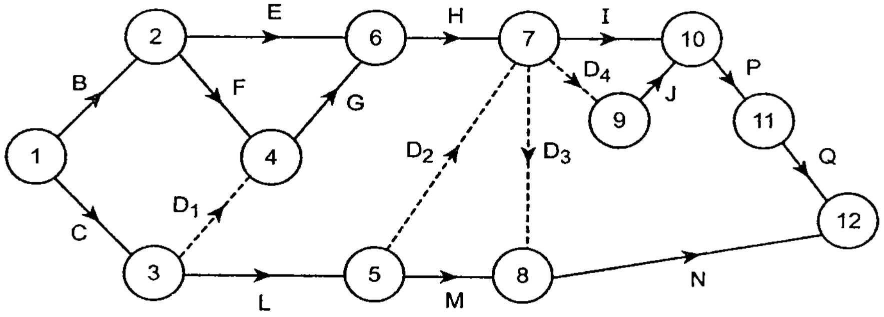


Fig. 9

D_1, D_2, D_3 and D_4 are dummy activities.

Example 7 : **Draw the event oriented network for the following data:**

Event No:	1	2	3	4	5	6	7
Immediate Predecessors	–	1	1	2, 3	3	4, 5	5, 6

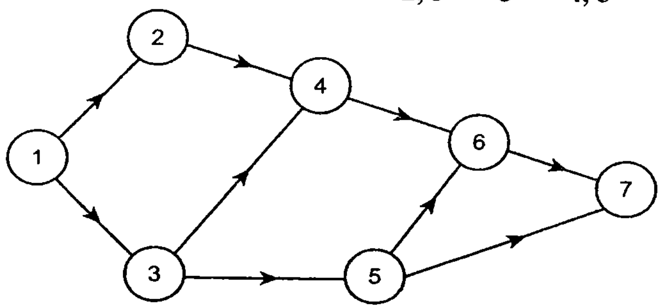


Fig. 10

EXERCISE

I. 1. What is a project ?

2. What are the three main phases of a Project ?

3. Explain the following terms:

(i) **Activity** ***(ii)*** **Dangling**

(iii) **Dummy activity** ***(iv)*** **Activity on arrow diagram**

(v) **Activity on node diagram**

(vi) **Network of a project**

4. What are the characteristics of a good network representing a project ?

5. Explain the method of numbering the nodes of an Activity on Arrow diagram.

II. Construct the network for each of the projects whose activities and their precedence relationships are as given below:

1.

Activity	A	B	C	D	E	F	G	H	I	J	K
Immediate Predecessor	–	–	–	A	B	B	C	D	E	H,I	F,G

2. A < C, D, I ; B < G, F ; D < G, F ; F < H, K ; G, H < J ; I, J, K < E *[MU. MCA, BE. Apr 96]*

3. A, B, C can start simultaneously A < F, E; B < D, C, E, D < G.

4.

Activity	A	B	C	D	E
Immediate Predecessor	–	A	A	A	B, C, D

5. A < C, D ; B < C, D ; C < E ; D, E < F.

[MU. MBA Nov 95]

6. A < C, D ; B < E ; C, E < F, G ; D < H ; G < I ; H, I < J.

7.

Activity	A	B	C	D	E	F	G	H	I
Immediate Predecessor	–	–	A,B	B	B	A,B	F,D	F,D	C,G

8. Draw the network for the following data and number the events.

A < C, B; B < D, E; C < F; E < G; F < I, J; J < K;
G < L; K, L < M *[M.U., M.B.A., Apr '98]*

ANSWERS – EXERCISE

II. 1.

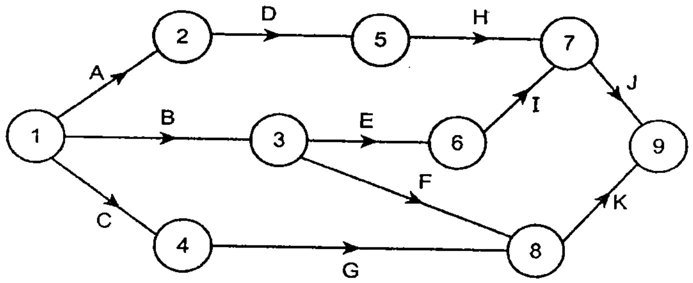


Fig. 11

2.

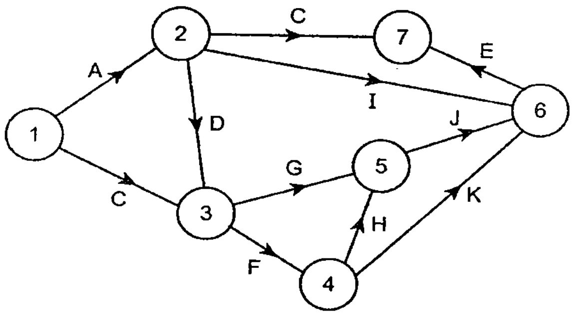


Fig. 12

3.

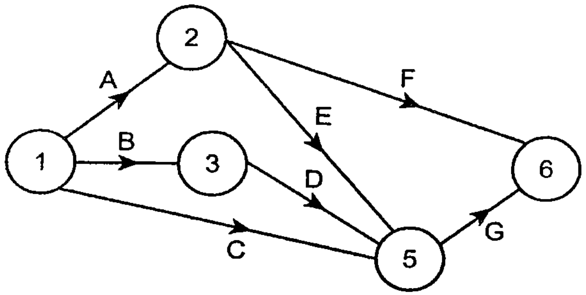


Fig. 13

4.

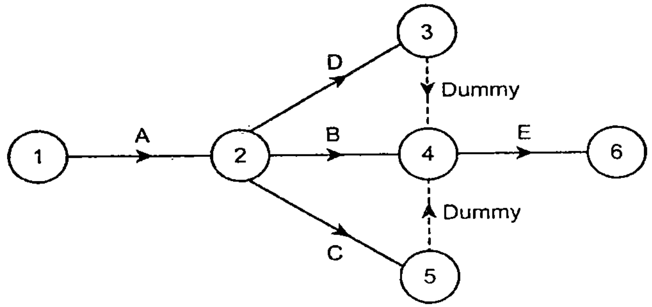


Fig. 14

5.

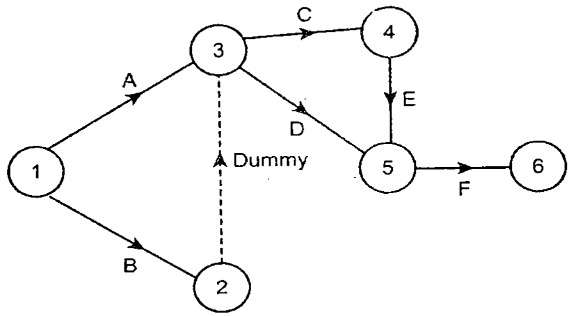


Fig. 15

6.

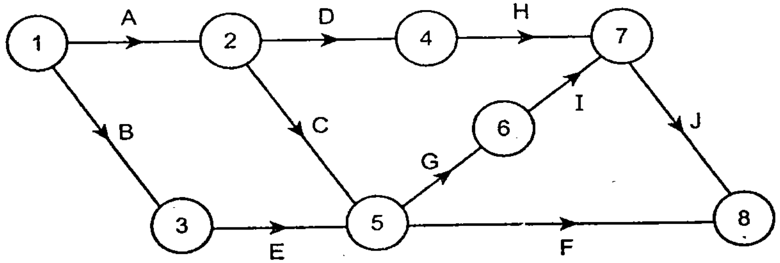


Fig. 16

7.

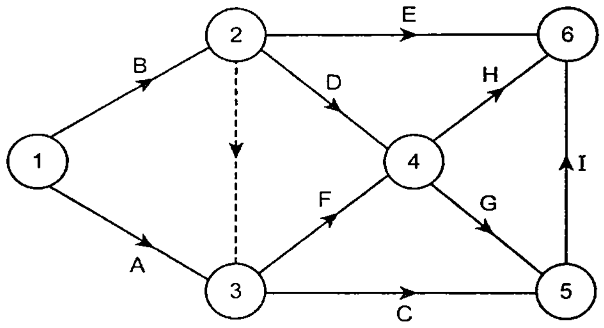


Fig. 17

8.

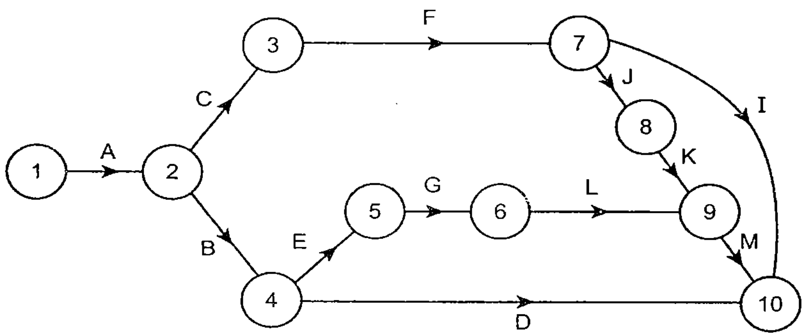
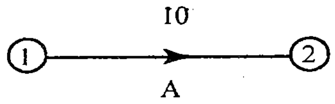


Fig. 18

5.4 NETWORK COMPUTATIONS
(Earliest Completion time of a Project and Critical path)

It is obvious that the completion time of the project is one of the very important things to be calculated knowing the durations of each activity. In real world situation the duration of any activity has an element of uncertainty because of sudden unexpected shortage of labour, machines, materials etc. Hence the completion time of the project also has an element of uncertainty. We first consider the situation where the duration of each activity is deterministic without taking the uncertainty into account.



The above diagram represents an activity whose direction is 10 time units (hours or days or weeks or months etc)

The first net work calculation one does is the computation of earliest start and earliest finish (completion) time of each activity given the duration of each activity. The method used is called **forward pass calculation** and it is best illustrated by means of the following example.

Example 1 : Compute the earliest start, earliest finish latest start and latest finish of each activity of the project given below:

Activity	**1–2**	**1–3**	**2–4**	**2–5**	**3–4**	**4–5**
Duration (in days)	**8**	**4**	**10**	**2**	**5**	**3**

First draw the network.

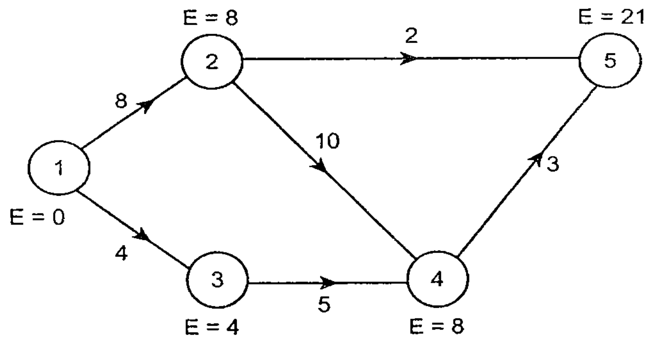


Fig. 19

We take the earliest time of all the start activities as zero.

So earliest starts of 1 – 2 and 1 – 3 are zero.

To find earliest start of 2–4.

The activity 2 – 4 can start only after finishing the only preceding activity 1 – 2 i.e., after 8 days.

∴ Earliest start of 2 – 4 is 8 days. Similarly earliest start of 2 – 5 is also 8 days.

Similarly earliest start of 3 – 4 is 4 days

To find the earliest start of 4–5 we first notice that the activity ***4 – 5 has more than one predecessor*** and also the activity 4 – 5 can start only after finishing all its preceding activities.

There are two paths leading to the activity 4 – 5: namely 1 – 2 – 4 which takes 18 days and 1 – 3 – 4 which takes 9 days. Obviously after 18 days all the activities 1 – 2, 1 – 3, 2 – 4, 3 – 4 can be finished but not earlier than that.

∴ Earliest start of 4 – 5 is 18 days.

Note: Earliest start of an activity $i - j$ can be denoted as ES_i or ES_{ij}. It can also be called the ***earliest occurrence of the event i.***

Earliest finish of any activity $i - j$ is got by adding the duration of the activity denoted by t_{ij} to the earliest start of $i - j$.

Hence the earliest finish of 1 – 2, 1 – 3, 2 – 4, 2 – 5, 3 – 4, 4 – 5 are 8,4, 18, 10, 9, 21 respectively

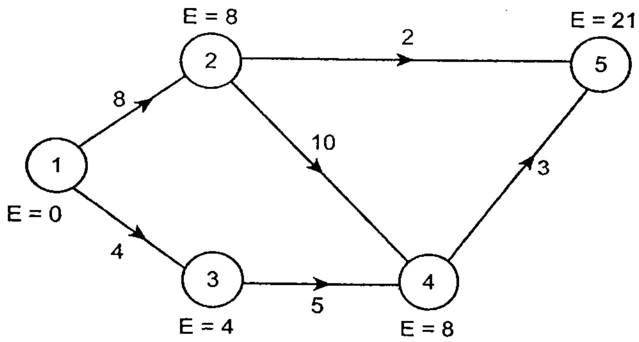


Fig. 20

Obviously ***earliest completion time of the project is 21 days, the greatest number among all these*** since all the activities can be finished only after 21 days.

Formula for Earliest Start of an activity $i - j$ in a project network is given by

$$ES_j = \text{Max}\,[ES_i + t_{ij}] \text{ where}$$

ES_i denotes the earliest start time of all the activities emanating from node i and t_{ij} is the estimated duration of the activity $i - j$.

To compute the latest finish and latest start of each activity

The method used here is called ***backward pass calculation*** since we start with the terminal activity and go back to the very first node.

We first calculate the latest finish of each activity as follows:

Latest finish of all the terminating (end) activities is taken as the earliest completion time of the project. Similarly latest finish of all the start activities is obviously taken as the same as the earliest start of these start activities.

Thus the latest finish of the terminal activities 2 –5 and 4 – 5 are 21 days which is the earliest completion time of the project.

Latest finish of the activity 2 – 4 and 3 – 4 are 21 – 3 = 18 days.

Latest finish of 1 – 3 is 18 – 5 = 13 days

To find the latest finish of the activity 1 – 2, we observe that the activity 1 – 2 has ***more than one successor activity.*** Therefore the latest finish of the activity 1 – 2 is the smaller of the two numbers 21 – 2 = 19 and 18 – 10 = 8. *i.e.,* 8 days.

Note: Latest finish of an activity can be denoted by LF_j or LF_{ij}. It can also be called the ***latest occurrence of the event j.***

Latest start of each activity is the latest finish of that activity minus the duration of that activity.

The latest start of the activities 4 – 5, 2 – 5, 2 – 4, 3 – 4, 1 – 3, 1 – 2 are 21, 21, 18, 18, 13, 8 respectively.

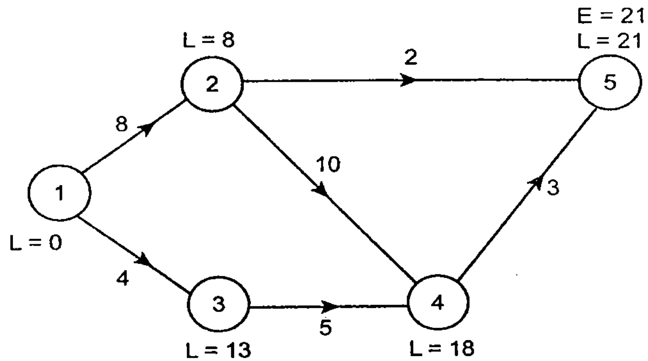


Fig. 21

Formula for the latest start time of all the activities emanating from the event i of the activity $i - j$, $LS_i = \text{Min}\,[LS_j - t_{ij}]$ for all defined $i - j$ activities where t_{ij} is the estimated duration of the activity $i - j$.

We can tabulate the results and represent these earliest and latest occurrence of the events in the network diagram as follows:

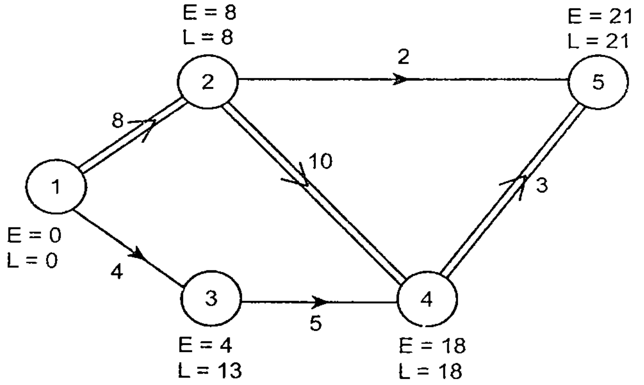


Fig. 22

Activity	Duration days	Earliest Start ES	Earliest Finish EF EF= ES + t_{ij}	Latest Start LS LF – t_{ij}	Latest Finish LF
1 – 2	8	0	8	0	8
1 – 3	4	0	4	9	13
2 – 4	10	8	18	8	18
2 – 5	2	8	10	19	21
3 – 4	5	4	9	13	18
4 – 5	3	18	**21**	18	**21**

Note: For small networks, it is not difficult to draw the network with E and L values calculated directly by looking at the diagram itself and constructing the table given above.

Critical path: **Path,** connecting the first initial node to the very last terminal node, ***of longest duration*** in any project network is called the ***Critical path.***

All the activities in any critical path are called ***Critical activities.*** Critical path is 1 – 2 – 4 – 5, usually denoted by double lines (Ref fig. 22)

Critical path plays a very important role in project scheduling problems.

Example 2: Calculate the earliest start, earliest finish, latest start and latest finish of each activity of the project given below and determine the Critical path of the project.

Activity	**1 – 2**	**1 – 3**	**1 – 5**	**2 – 3**	**2 – 4**
Duration (in weeks)	**8**	**7**	**12**	**4**	**10**
Activity	**3 – 4**	**3 – 5**	**3 – 6**	**4 – 6**	**5 – 6**
Duration (in weeks)	**3**	**5**	**10**	**7**	**4**

Solution:

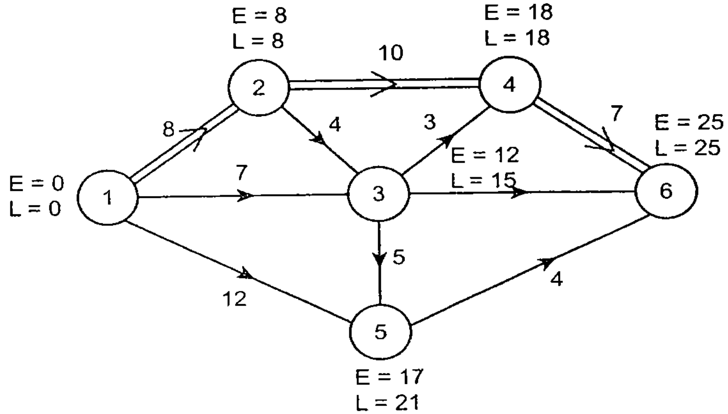


Fig. 23

Activity	Duration (in weeks)	Earliest Start	Earliest Finish	Latest Start	Latest Finish
1 – 2	8	0	8	0	8
1 – 3	7	0	7	8	15
1 – 5	12	0	12	9	21
2 – 3	4	8	12	11	15
2 – 4	10	8	18	8	18
3 – 4	3	12	15	15	18
3 – 5	5	12	17	16	21
3 – 6	10	12	22	15	25
4 – 6	7	18	**25**	18	25
5 – 6	4	17	21	21	**25**

5.5 FLOATS

Total float of an activity (T.F) is defined as the ***difference*** between the ***latest finish*** and the ***earliest finish of the activity*** or the difference between the ***latest start*** and the ***earliest start*** of the activity.

$$\text{Total float of an activity } i-j \;=\; (LF)_{ij}-(EF)_{ij}$$
$$\text{or} \;=\; (LS)_{ij}-(ES)_{ij}.$$

Total float of an activity is the amount of time by which that particular activity may be delayed without affecting the duration of the project. If the total float is positive then it may indicate that the resources for the activity are more than adequate. If the total float of an activity is zero it may indicate that the resources are just adequate for that activity. If the total float is negative, it may indicate that the resources for that activity are inadequate.

Note: (L – E) of an event of $i-j$ is called the ***slack*** of the event j.

There are three other types of floats for an activity, namely, Free float, Independent float and interference (interfering) float.

Free Float of an activity (F.F.) is that ***portion of the total float*** which can be used for rescheduling that activity without affecting the succeeding activity. It can be calculated as follows:

Free float of an activity $i-j$ = Total float of $i-j$ – (L – E) of the event j

= Total float of $i-j$ – Slack of the head event j

= Total float of I - J – Slack of the head event j

where L = Latest occurrence

E = Earliest occurrence

Obviously Free Float ≤ Total float for any activity.

Independent float (I.F) of an activity is the amount of time by which the activity can be rescheduled without affecting the preceding or succeeding activities of that activity.

Independent float of an activity $i-j$ = Free float of $i-j$ – (L –E) of event i.

= Free float of $i-j$ – Slack of the tail event i.

Clearly,

Independent float ≤ Free float for any activity

Thus I.F ≤ F.F ≤ T.F.

Interfering Float or Interference Float of an activity $i-j$ is nothing but the slack of the head event j.

Obviously,

Interfering Float of $i-j$ =Total Float of $i-j$ – Free Float of $i-j$.

Example 3: Calculate the total float, free float and independent float for the project whose activities are given below:

Activity	**1 – 2**	**1 –3**	**1 – 5**	**2 – 3**	**2 –4**
Duration (in weeks)	**8**	**7**	**12**	**4**	**10**
Activity	**3 – 4**	**3 – 5**	**3 – 6**	**4 – 6**	**5 – 6**
Duration (in weeks)	**3**	**5**	**10**	**7**	**4**

The data is the same as given in example 2 above.

The network with L and E of every event is given by

Solution:

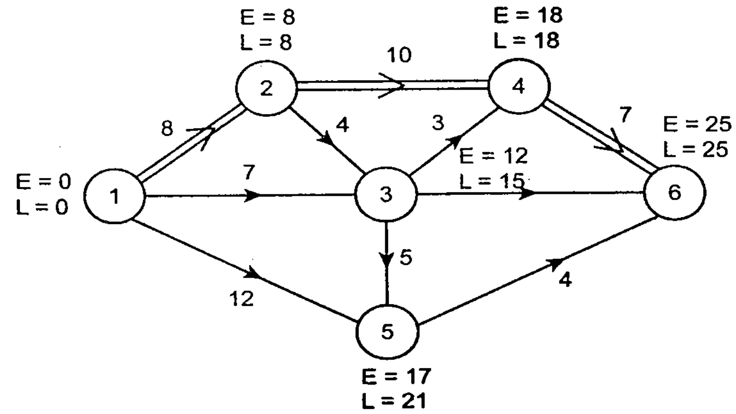


Fig. 24

Activity	Duration (in weeks)	Earliest Start	Earliest Finish	Latest Start	Latest Finish	Floats TF	Floats FF	Floats IF
1 – 2	8	0	8	0	8	0	0	0
1 – 3	7	0	7	8	15	8	5	5
1 – 5	12	0	12	9	21	9	5	5
2 – 3	4	8	12	11	15	3	0	0
2 – 4	10	8	18	8	18	0	0	0
3 – 4	3	12	15	15	18	3	3	0
3 – 5	5	12	17	16	21	4	0	– 3
3 – 6	10	12	22	15	25	3	3	0
4 – 6	7	18	**25**	18	25	0	0	0
5 – 6	4	17	21	21	**25**	4	4	0

Explanation: To find the total float of 2 – 3.

Total float of (2 – 3) = (LF – EF) of (2 – 3) = 15 – 12 = 3 from the table against the activity 2 – 3.

Free Float of (2 – 3) = Total float of (2 – 3) – (L – E) of event 3
= 3 – (15 – 12) from the figure for event 3 = 0

Free Float of (1 – 5) = Total float of (1 – 5) – (L – E) of event 5
= (21 – 12) – (21 – 17) from the figure for event 5
= 9 – 4 = 5

Independent float of (1 – 5) = Free float of (1 – 5) – (L–E) of event 1
= 5 – (0 – 0) = 5

Important Note: Note that all the critical activities have their total float as zero. ***In fact the critical path can also be defined as the path of least (zero) total float.*** As we have noticed total float is 3 for the activity 2 – 3. This means that the activity 2 – 3 can be delayed by 3 weeks without delaying the duration (completion date) of the project.

Free float of 3 – 4 is 3. This means that the activity 3 – 4 can be delayed by 3 weeks without affecting its succeeding activity 4 – 6.

Independent float of 1 – 5 is 5 means that the activity 1 – 5 can be delayed by 5 weeks without affecting its preceding or succeeding activity. Of course 1 – 5 has no preceding activity.

Uses of floats: Floats are useful in resource levelling and resource allocation problems which will be discussed in the last section of this chapter. Floats give some flexibility in rescheduling some activities so as to smoothen the level of resources or allocate the limited resources as best as possible.

Example 4 : Construct the network for the project whose activities are given below and compute the total, free and independent float of each activity and hence determine the critical path and the project duration.

Activity	**0 – 1**	**1 2**	**1 – 3**	**2 – 4**	**2 –5**
Duration (in weeks)	**3**	**8**	**12**	**6**	**3**

Activity	**3 – 4**	**3 6**	**4 – 7**	**5 – 7**	**6 –7**
Duration (in weeks)	**3**	**8**	**5**	**3**	**8**

Solution:

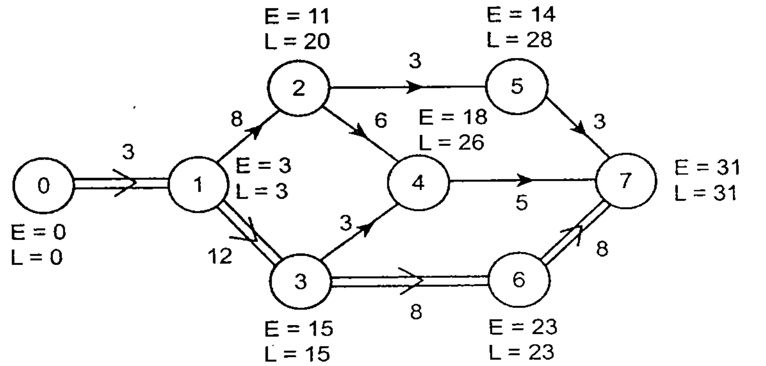


Fig. 25

Activity	Duration (in weeks)	Earliest Start	Earliest Finish	Latest Start	Latest Finish	Floats TF	Floats FF	Floats IF
0 – 1	3	0	3	0	3	0	0	0
1 – 2	8	3	11	12	20	9	0	0
1 – 3	12	3	15	3	15	0	0	0
2 – 4	6	11	17	20	26	9	1	–8
2 – 5	3	11	14	25	28	14	0	–9
3 – 4	3	15	18	23	26	8	0	0
3 – 6	8	15	23	15	23	0	. 0	0
4 – 7	5	18	23	26	31	8	8	0
5 – 7	3	14	17	28	31	14	14	0
6 – 7	8	23	31	23	31	0	0	0

Critical path is 0 –1 – 3 – 6 – 7. Project duration = 31 weeks.

Example 5: Using CPM find the critical path and the minimum time for completion of the project whose details are given below. Activities A and B can start simultaneously each taking 15 days. Activity C can start after 7 days and activity D after 5 days of starting the activity A. Activity D can start after 4 days of starting activity C and 7 days of starting activity B. Activity E can start after activity A is one third finished and activity B is completely finished. Activities C, D and E can take 10,8 and 11 days respectively.

Solution: Draw the network on time scale as follows:

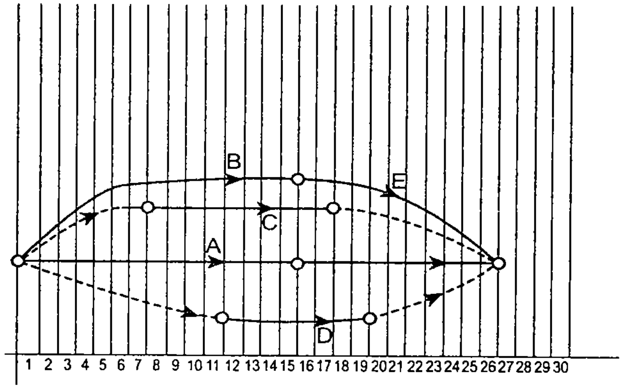


Fig. 26

The diagram is self-explanatory.

Critical path B – E.

Minimum time for completion = 26 days.

EXERCISE

I. 1. Write down the formulae for computing (a) Earlier start (b) Earliest finish (c) Latest start and (d) Latest finish of an activity of a project.

2. What is critical path in PERT/CPM ? Explain its importance. *[M.U. MBA, Nov. '96]*

3. Write down the stepwise procedure for determining the critical path of a project.

4. What is the significance of the total float with regard to the resources available for a project ?

5. Write short notes on (a) Total float (b) Free float (c) Independent float and (d) Interfering float of our activity of a project explaining their significance as well.

6. Define (a) total float (b) free float and (c) independent float of activities in CPM and give their interpretations. *[M.U. MBA., April '96]*

II. 1. A project schedule has the following characteristics:

Activity	1–2	1–3	2–4	3–4	3–5	4–9	5–6
Time	4	1	1	1	6	5	4
Activity	5–7	6–8	7–8	8–10	9–10		
Time	8	1	2	5	7		

Construct PERT network and find the critical path. *[MU. BE. Apr 97]*

2. What are different types of floats associated with an activity in a CPM model ? What are their uses ? *[MU. BE. Apr 97]*

3. A maintenance activity consists of the following jobs. Draw the network for the project and calculate the total float and free float for each activity. What can you say about the slacks of the events of the project ?

jobs	1–2	2–3	3–4	3–7	4–5	4–7	5–6	6–7
Duration (days)	3	4	4	4	2	2	3	2

4. Draw the network and determine the critical path for the given data:

jobs	1–2	1–3	2–4	3–4	3–5	4–5	4–6	5–6
Duration	6	5	10	3	4	6	2	9

Find the total float Free Float and Independent float of each activity.

5. A project consists of a series of tasks labelled A, B, ... H, I with the following relationships: (W < X, Y means X and Y cannot start until W is completed; X, Y < W means W can not start until both X and Y are completed) with this notation construct the network diagram having the following constraints:

A < D, E ; B, D < F ; C < G ; B < H ; F, G < I.

Find also the minimum time of completion of the project, when the time (in days) of completion of each task is as follows:

Task	A	B	C	D	E	F	G	H	I
Time	23	8	20	16	24	18	19	4	10

Determine all the four floats for each activity.

6. The following refers to a project network:

Activity	A	B	C	D	E	F	G	H	I	J
Predecessor	–	A	A	A	A	E	D	G,F	C,H	B
Duration in days	1	4	2	3	2	3	2	1	3	2
Crew required per day	7	1	5	4	3	6	2	9	10	8

(i) Draw the network *(ii)* determine the critical path and project completion time *(iii)* Draw the graph showing optimum requirements versus time. *[BE. BrU. Nov 96]*

7. A small CPM project consist of 11 activities A, B, ..., J, K. The precedence relationships are: A, B can start immediately. A < C, D, I; B < G, F; D < G, F; F < H, K; G, H < J; I, J, K < E. The durations of the activities are as below:

Activity	A	B	C	D	E	F	G	H	I	J	K
Duration (days)	5	3	10	2	8	4	5	6	12	8	9

Draw the network of the project. Summarize the CPM calculations in a tabular form, computing total and free floats of activities and hence determine critical path. *[MU. MBA. Apr 96]*

ANSWERS – EXERCISE

II. 1.

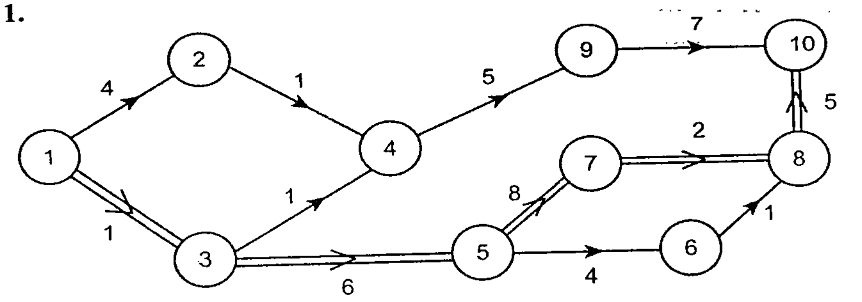


Fig 27

	Paths	Duration
(i)	1 – 2 – 4 – 9 – 10	17 time units
(ii)	1 – 3 – 4 – 9 –10	14
(iii)	**1 – 3 – 5 – 7 – 8 – 10**	**22**
(iv)	1 – 3 – 5 – 6 – 8 – 10	17

Critical path is 1 – 3 – 5 – 7 – 8 – 10.

3.

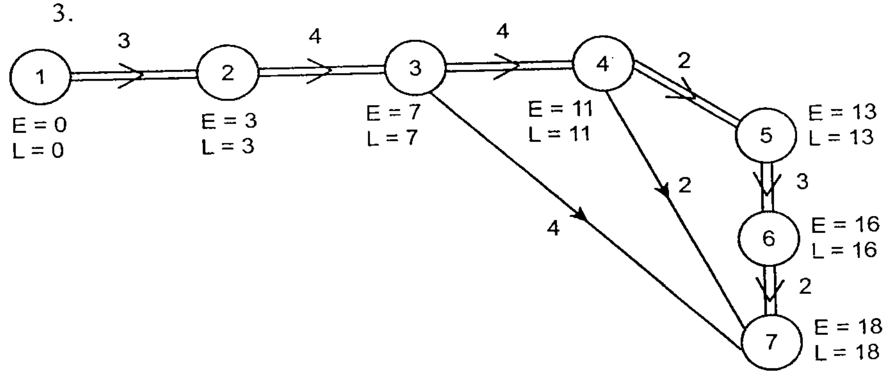


Fig 28

Slack of each event is zero.

Job	Duration (in weeks)	Earliest Start	Earliest Finish	Latest Start	Latest Finish	TF	FF
1 – 2	3	0	3	0	3	0	0
2 – 3	4	3	7	3	7	0	0
3 – 4	4	7	11	7	11	0	0

3 – 7	4	7	11	14	18	7	7
4 – 5	2	11	13	11	13	0	0
4 – 7	2	11	13	16	18	5	5
5 – 6	3	13	16	13	16	0	0
6 – 7	2	16	18	16	18	0	0

4.

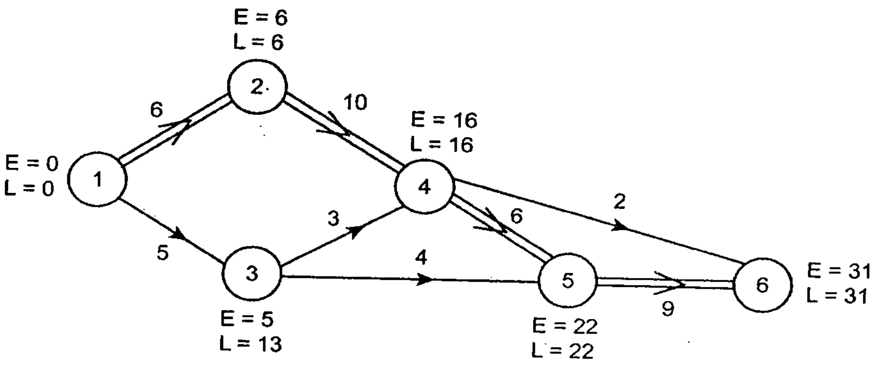


Fig 29

Activity	1 –2	1 –3	2 –4	3 –4	3 –5	4 –5	4 –6	5 –6
TF	0	8	0	8	13	0	13	0
FF	0	0	0	8	13	0	13	0
IF	0	0	0	8	5	0	13	0

	Paths	***Duration***
(i)	1 – 2 – 4 – 6	18 days
(ii)	**1 – 2 – 4 – 5 –6**	**31**
(iii)	1 – 3 – 4 – 5 – 6	23
(iv)	1 – 3 – 5 – 6	18
(v)	1 – 3 – 4 – 6	10

Critical path is 1 – 2 – 4 – 5 – 6

5.

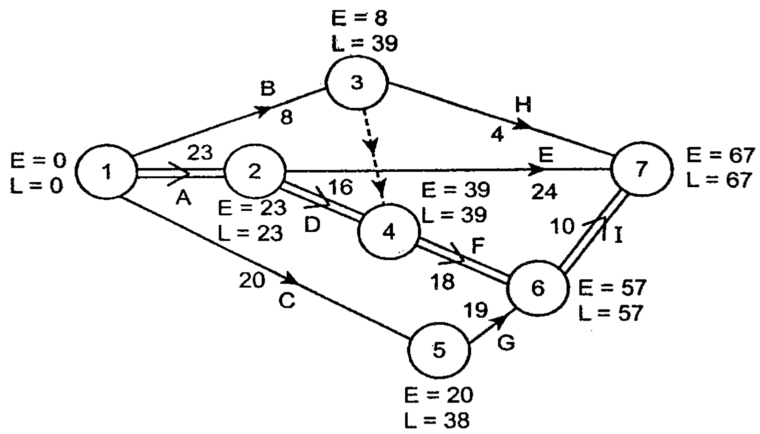


Fig 30

	Paths	***Duration***
(i)	1 – 3 – 7	12 days
(ii)	1 – 2 – 7	47
(iii)	**1 – 2 – 4 – 6 – 7**	**67**
(iv)	1 – 3 – 4 – 6 – 7	36
(iv)	1 – 5 – 6 – 7	49

Minimum time of completion = length of the critical path

= 67 days

	A	B	C	D		E	F	G	H	I
Acti vity	1 –2	1 –3	1 –5	2 –4	3 –4	2 –7	4 –6	5 –6	3 –7	6 –7
TF	0	31	18	0	31	20	0	18	55	0
FF	0	0	0	0	31	20	0	18	55	0
IF	0	0	0	0	0	20	0	0	24	0
ITF	0	31	18	0	0	0	0	0	0	0

ITF: Interfering Float.

6 (i)

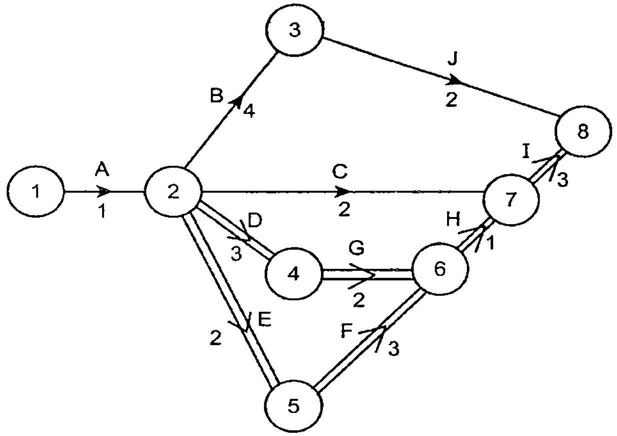


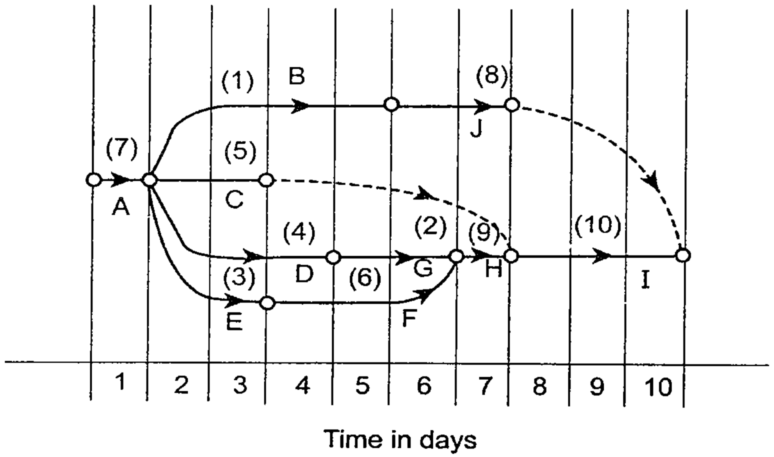
Fig. 31

(ii) There are two critical paths.

A – D – G – H – I and A – E – F – H – J.

Project duration = 10 days

(iii)



Time in days

Fig. 32

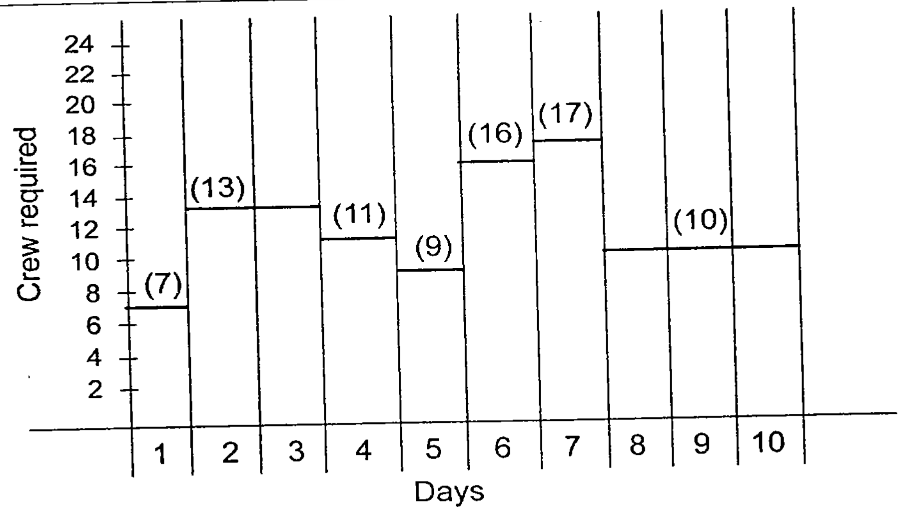


Fig. 33

Optimum requirement versus time

(ii)

	Paths	Duration
(a)	1 – 2 – 3 – 8	7
(b)	1 – 2 – 7 – 8	6
(c)	**1 – 2 – 4 – 6 – 7 – 8**	**10**
(d)	**1 – 2 – 5 – 6 – 7 – 8**	**10**

There are two critical paths. Project completion time = 10 days.

7.

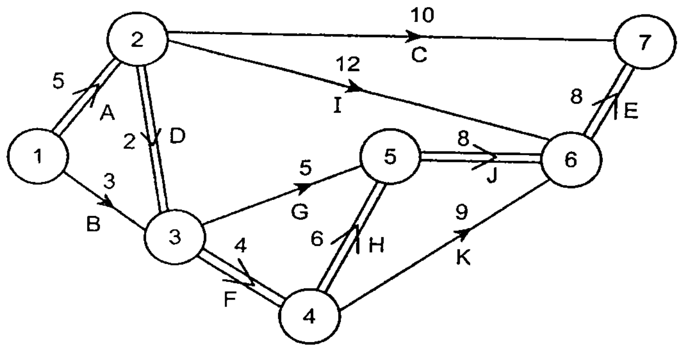


Fig. 33 *(a)*

Activity		Duration	Earliest		Latest		TF	FF
			Start	Finish	Start	Finish		
A	1 - 2	5	0	5	0	5	0	0
B	1 - 3	3	0	3	4	7	4	4
D	2 - 3	2	5	7	5	7	0	0
I	2 - 6	12	5	17	13	25	8	8
C	2 - 7	10	5	15	23	33	18	18
G	3 - 5	5	7	12	12	17	5	5
F	3 - 4	4	7	11	7	11	0	0
H	4 - 5	6	11	17	11	17	0	0
K	4 - 6	9	11	20	16	25	5	5
J	5 - 6	8	17	25	17	25	0	0
E	6 - 7	8	25	33	25	33	0	0

Critical path: 1 - 2 - 3 - 4 - 5 - 6 - 7

5.6 PROGRAMME EVALUATION REVIEW TECHNIQUE: (PERT)

This technique, unlike CPM, takes into account the uncertainty of project durations into account.

PERT calculations depend upon the following three time estimates.

***Optimistic (least) time estimate:* (t_o or a)** is the duration of any activity when everything goes on very well during the project. i.e., labourers are available and come in time, machines are working properly, money is available whenever needed, there is no scarcity of raw material needed etc.

***Pessimistic (greatest) time estimate:* (t_p or b)** is the duration of any activity when almost every thing goes against our will and a lot of difficulties is faced while doing a project.

***Most likely time estimate:* (t_m or m)** is the duration of any activity when sometimes things go on very well, sometimes things go on very bad while doing the project.

Two main assumptions made in PERT calculations are

(i) The activity durations are independent. i.e., the time required to complete an activity will have no bearing on the completion times of any other activity of the project.

(ii) The activity durations follow β – distribution.

β distribution is a probability distribution with density function $k(t - a)^{\alpha}\ (b - t)^{\beta}$ with mean $t_e = \frac{1}{3}\left[2t_m + \frac{1}{2}(t_o - t_p)\right]$ and the standard deviation $\sigma_t = \frac{t_p - t_o}{6}$.

PERT Procedure

(1) Draw the project net work

(2) Compute the expected duration of each activity $t_e = \frac{t_o + 4t_m + t_p}{6}$

(3) Compute the expected variance $\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$ of each activity.

(4) Compute the earliest start, earliest finish, latest start, latest finish and total float of each activity.

(5) Determine the critical path and identify critical activities.

(6) Compute the expected variance of the Project length (also called the variance of the critical path) σ_c^2 which is the sum of the variances of all the critical activities.

(7) Compute the expected standard deviation of the project length σ_c and calculate the standard normal deviate $\frac{T_S - T_E}{\sigma_c}$ where

T_S = Specified or Scheduled time to complete the project

T_E = Normal expected project duration

σ_c = Expected standard deviation of the project length.

(8) Using (7) one can estimate the probability of completing the project within a specified time, using the normal curve (Area) tables.

Note: (2), (3) are valid because of assumption (ii). (6) is valid because of assumption *(i)*.

5.7 BASIC DIFFERENCES BETWEEN PERT AND CPM

PERT

1. PERT was developed in a brand new R and D Project it had to consider and deal with the uncertainties associated with such projects. Thus the project duration is regarded as a random variable and therefore probabilities are calculated so as to characterise it.
2. Emphasis is given to important stages of completion of task rather than the activities required to be performed to reach a particular event or task in the analysis of network. i.e., PERT network is essentially an event – oriented network.
3. PERT is usually used for projects in which time estimates are uncertain. Example: R & D activities which are usually non–repetitive.
4. PERT helps in identifying critical areas in a project so that suitable necessary adjustments may be made to meet the scheduled completion date of the project.

CPM

1. CPM was developed for conventional projects like construction project which consists of well known routine tasks whose resource requirement and duration were known with certainty.
2. CPM is suited to establish a trade off for optimum balancing between schedule time and cost of the project.
3. CPM is used for projects involving well known activities of repetitive in nature.

   However the distinction between PERT and CPM is mostly historical.

Example 1 **Construct the network for the project whose activities and the three time estimates of these activities (in weeks) are given below. Compute**

(a) Expected duration of each activity

(b) Expected variance of each activity

(c) Expected variance of the project length

Activity	t_o	t_m	t_p
1 – 2	3	4	5
2 – 3	1	2	3
2 – 4	2	3	4
3 – 5	3	4	5
4 – 5	1	3	5
4 – 6	3	5	7
5 – 7	4	5	6
6 – 7	6	7	8
7 – 8	2	4	6
7 – 9	1	2	3
8 – 10	4	6	8
9 – 10	3	5	7

Solution: *(a) & (b)*

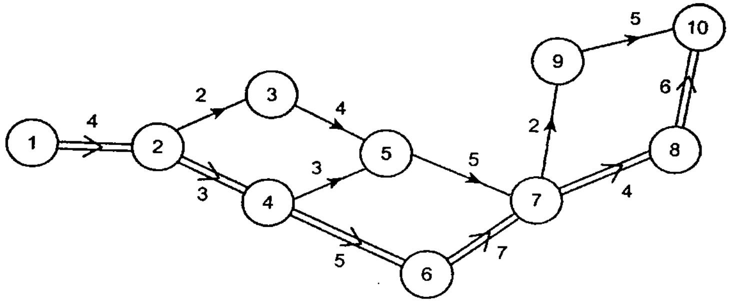


Fig. 34

Activity	t_o	t_m	t_p	Expected duration $t_e = \frac{t_o + 4t_m + t_p}{6}$	Expected Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1 – 2	3	4	5	4	**1/9 = 0.11** nearly
2 – 3	1	2	3	2	1/9 = 0.11
2 – 4	2	3	4	3	**1/9 = 0.11**
3 – 5	3	4	5	4	1/9 = 0.11
4 – 5	1	3	5	3	4/9 = 0.44
4 – 6	3	5	7	5	**4/9 = 0.44**
5 – 7	4	5	6	5	1/9 = 0.11
6 – 7	6	7	8	7	**1/9 = 0.11**
7 – 8	2	4	6	4	**4/9 = 0.44**
7 – 9	1	2	3	2	1/9 = 0.11
8 – 10	4	6	8	6	**4/9 = 0.44**
9 – 10	3	5	7	5	4/9 = 0.44

Critical path 1–2 –4 –6 –7–8–10. Expected Project duration = 29 weeks.

(c) Expected variance of the project length = Sum of the expected variances of all the critical activities

$$= \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = \frac{15}{9} = \frac{15}{9} = \frac{5}{3} = 1.67$$

or (0.11 + 0.11 + 0.44 + 0.11 + 0.44 + 0.44 = 1.32 + 0.33 = 1.65)

Example 2: The following table indicates the details of a project. The durations are in days. '*a*' refers to optimistic time, '*m*' refers to most likely time and '*b*' refers to pessimistic time duration.

Activity	**1–2**	**1–3**	**1–4**	**2–4**	**2–5**	**3–5**	**4–5**
a	**2**	**3**	**4**	**8**	**6**	**2**	**2**
m	**4**	**4**	**5**	**9**	**8**	**3**	**5**
b	**5**	**6**	**6**	**11**	**12**	**4**	**7**

(a) **Draw the network**

(b) **Find the critical path**

(c) **Determine the expected standard deviation of the completion time.**

Solution:

Activity	a	m	b	Expected duration t_e	Expected variance σ^2
1 – 2	2	4	5	3.83	**1/4**
1 – 3	3	4	6	4.17	$\frac{1}{4}$
1 – 4	4	5	6	5	$\frac{1}{9}$
2 – 4	8	9	11	9.17	$\mathbf{\frac{1}{4}}$
2 – 5	6	8	12	8.33	1
3 – 4	2	3	4	3	$\frac{1}{9}$
4 – 5	2	5	7	4.83	$\mathbf{\frac{25}{36}}$

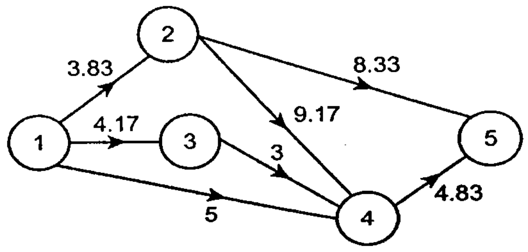


Fig 35

Critical path 1 – 2 – 4 – 5

Expected Project duration = 17.83 days

Expected variance of the completion time $= \frac{1}{4} + \frac{1}{4} + \frac{25}{36} = \frac{43}{36}$

Expected standard deviation of the completion time $= \sqrt{\frac{43}{36}} = 1.09$ nearly

Example 3 **A project consists of the following activities and time estimates:**

Activity	*Least time (days)*	*Greatest time (days)*	*Most likely time (days)*
1 – 2	**3**	**15**	**6**
2 – 3	**2**	**14**	**5**
1 – 4	**6**	**30**	**12**
2 – 5	**2**	**8**	**5**
2 – 6	**5**	**17**	**11**
3 – 6	**3**	**15**	**6**
4 – 7	**3**	**27**	**9**
5 – 7	**1**	**7**	**4**
6 – 7	**2**	**8**	**5**

(a) **Draw the network**

(b) **What is the probability that the project will be completed in 27 days ?**

Solution: Obviously Greatest time = Pessimistic time = t_p

Least time = Optimistic time = t_o

Most likely time = t_m

(a)

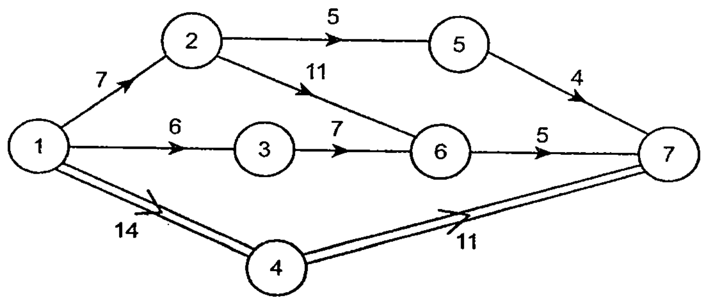


Fig. 36

Activity	t_o	t_p	t_m	$t_e = \frac{t_o+4t_m+t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1 – 2	3	15	6	7	4
1 – 3	2	14	5	6	4
1 – 4	6	30	12	14	**16**
2 – 5	2	8	5	5	1
2 – 6	5	17	11	11	4
3 – 6	3	15	6	7	4
4 – 7	3	27	9	11	**16**
5 – 7	1	7	4	4	1
6 – 7	2	8	5	5	1

Critical path 1 – 4 – 7.

Expected Project duration = 25 days

Expected variance of the project length = Sum of the expected variances of all the critical activities

critical activities

= 16 + 16 = 32.

σ_c = Standard deviation of the project length $= \sqrt{32} = 4\sqrt{2} = 5.656$

$$z = \frac{T_S - T_E}{\sigma_c} = \frac{27-25}{5.656} = \frac{2}{5.656} = 0.35$$

Probability that the project will be completed in 27 days

$= P(T_S \le 27) = P(Z \le 0.35)$

$= 0.6368 = 63.7\%$

Example 4: **Three time estimates (in months) of all activities of a project are as given below:**

Time in Months

Activity	*a*	*m*	*b*
1 – 2	0.8	1.0	1.2
2 – 3	3.7	5.6	9.9
2 – 4	6.2	6.6	15.4
3 – 4	2.1	2.7	6.1
4 – 5	0.8	3.4	3.6
5 – 6	0.9	1.0	1.1

(a) Find the expected duration and standard deviation of each activity

(b) Construct the project network

(c) Determine the critical path, expected project length and expected variance of the project length.

(d) What is the probability that the project will be completed (i) two months later than expected (ii) not more than 3 months earlier than expected (iii) What due date has about 90% chance of being met ?

Solution: *(a)*

Activity	*a*	*m*	*b*	t_e	σ
1 – 2	0.8	1.0	1.2	1	0.067
2 – 3	3.7	5.6	9.9	6	1.03
2 – 4	6.2	6.6	15.4	8	1.53
3 – 4	2.1	2.7	5.1	3	0.5
4 – 5	0.8	3.4	3.6	3	0.47
5 – 6	0.9	1.0	1.1	1	0.033

(b)

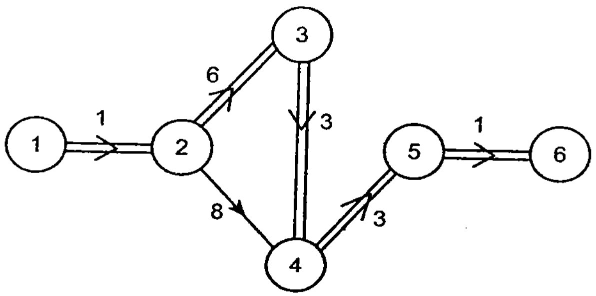


Fig. 37

(c) Critical path: 1 – 2 – 3 – 4 – 5 – 6

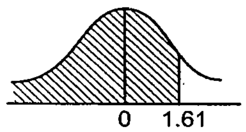
Expected Project length = 14 months

$$\text{Expected Variance} = (0.067)^2 + (1.03)^2 + (0.5)^2 + (0.47)^2 + (0.033)^2$$

$$= 1.5374$$

$$\therefore \sigma_c = \sqrt{1.5374} = 1.2399$$

(d). (i) $T_S = 16,\ T_E = 14,\ \sigma_c = 1.2399$



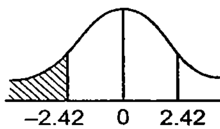
$$z = \frac{16-14}{\sigma_c} = \frac{2}{1.6023}$$

$$= 1.61$$

$$P(T_S \le 13) = 0.9463$$

$$P(T_S \le 13) = 94.63\ \% = \text{Required probability}$$

(d) (ii) $T_S = 11,\ T_E = 14,\ \sigma_c = 1.2399$



$$z = \frac{T_S - T_E}{\sigma_c} = \frac{11-14}{1.2399}$$

$$= -\frac{3}{1.2399} = -2.42$$

$$P(T_S \le 11) = 0.5 - 0.4922 = 0.0078$$

Required probability = 0.78 %

(iii) z value for 90% area in the table = 1.28 nearly

Let T_S be the required due date. Then $z = \dfrac{T_S - T_E}{\sigma_c}$

$$\text{i.e., } 1.28 = \frac{T_S - 14}{1.6023},$$

$$T_S = 14 + 1.28 \times 1.2399 = 15.59 \text{ months nearly.}$$

EXERCISE

I. (1) Explain the three time estimates used in PERT.

(2) What are the main assumptions under in PERT computations

(3) Distinguish between CPM and PERT.

[MU. MBA, Nov 95]

(4) Explain PERT procedure to determine (a) the expected critical path (b) expected variance of the project length and (c) the probability of completing the project with in a specified time.

II. 1. The data for a small PERT project is as given below where a represents optimistic time, *m* most likely time and *b* pessimistic time estimates (in days) of the activities A, B ...... J, K:

	A	B	C	D	E	F	G	H	I	J	K
a:	3	2	6	2	5	3	3	1	4	1	2
m:	6	5	12	5	11	6	9	4	19	2	4
b:	15	14	30	8	17	15	27	7	28	9	12

Precedence relationships A, B, C can start immediately.

A $\leq$ D, I ; B < G, F ; D < G, F; C < E ; E < H, K; F < H, K ; G, H < J.

Draw the arrow network of the project. Calculate earliest and latest expected times to reach each node and find critical path. What is the probability that the project will be completed 2 days later than expected ? *[MU. MBA. Nov 96]*

2. Consider a project consisting of 7 jobs A, B ...., G with the following precedence relations and time estimates:

Job	Predecessor	*a*	*m*	*b*
A	–	2	5	8
B	A	6	9	12
C	A	6	14	17
D	B	5	8	11
E	C, D	3	6	9
F	–	3	12	21
G	E, F	1	4	7

Draw the project network and find the probability of completing the project in 30 days. *[MU. BE. Apr 97]*

3. The operations of a PERT network along with their time estimates in weeks (*a* – optimistic, *m* – most likely, *b* – pessimistic) are given below. Draw the project network. Calculate the length and variance of the critical path. What is the probability that the project will be completed atleast two weeks earlier than expected ?

[MU. MBA. Apr 95]

Operation	*a*	*m*	*b*
1 – 2	4	7	16
1 – 3	7	9	14
2 – 3	2	4	9
2 – 4	7	12	18
3 – 4	0	0	0
3 – 5	8	14	17
4 - 6	3	4	5
5 – 7	8	13	15
5 – 8	7	8	12
6 – 8	5	9	13
7 – 9	4	7	13
8 – 9	6	9	12
9 – 10	7	13	16

4. The following table list the jobs of a network along with their time estimates

Jobs	1–2	1–3	2–4	3–4	4–5	3 – 5
Optimistic time	2	9	5	2	6	8
Most likely time	5	12	14	5	6	17
Pessimistic time	14	15	17	8	12	20

(*a*) Draw the network.

(*b*) Calculate the length and variance of the critical path

(*c*) Find the probability that the project will be completed within 30 days ? *[BE. BrU. Nov 96]*

5. The three estimates for the activities of a project are given below:

Activity	Estimated duration (days)		
	a	*m*	*b*
1 – 2	5	6	7
1 – 3	1	1	7
1 – 4	2	4	12
2 – 5	3	6	15
3 – 5	1	1	1
4 – 6	2	2	8
5 – 6	1	4	7

Draw the project network. Calculate the total slack and free slack of each activity. Find out the critical path of the project and project duration. What is the probability that the project will be completed atleast 5 days earlier than expected ? What is the probability of completing the project on or before 22 days ?

What is the probability that the project will be completed on 22 days ?

6. The following table tests the jobs of a network with three time estimates.

Job		Duration	
i, j	Optimistic	Most likely	Pessimistic
1 – 2	3	6	15
1 – 6	2	5	14
2 – 3	6	12	30
2 – 4	2	5	8
3 – 5	5	11	17
4 – 5	3	6	15
6 – 7	3	9	27
5 – 8	1	4	7
7 – 8	4	10	28

***(i)* Draw the project network.**

***(ii)* What is the approximate probability that the jobs on the critical path will be completed by the date of 42 days ?**

7. Find the critical path and calculate the slack time for each event from the following PERT diagram.

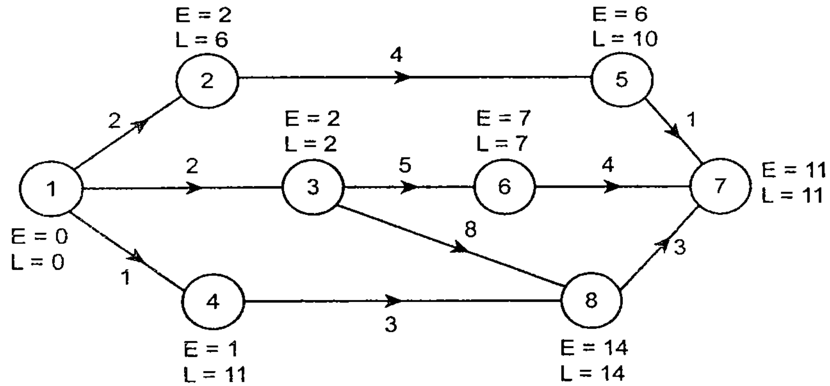


Fig. 38

8. A project consists of the following activities and time estimates:

Activity	Least time (days)	Greatest time (days)	Most likely (days)
1 – 2	3	15	6
1 – 3	2	14	5
1 – 4	6	30	12
2 – 5	2	8	5
2 – 6	5	17	11
3 – 6	3	15	6
4 – 7	3	27	9
5 – 7	1	7	4
6 – 7	2	8	5

***(a)* Draw the network.**

(b)* What is the probability that the project will be completed in 27 days ?** ***[MU. MCA. Nov 95]

***(c)* What due date has about 95% chance of being met ?**

9. A Small PERT Project consist of 12 activities whose time estimates in days are as below (a = of the time, b = most likely, b = Pessimistic)

Activity	*a*	*m*	*b*	ACTIVITY	A	M	B
1 - 2	7	10	12	2 - 3	26	33	40
2 - 4	5	8	10	3 - 4	7	10	12
3 - 5	4	5	8	4 - 6	5	7	10
5 - 8	8	11	12	6 - 7	2	3	7
6 - 10	6	9	13	7 - 9	5	6	9
8 - 9	5	11	17	9 - 10	2	5	14

Determine the critical path and its length and variable. Find the probability that the project will be completed that

(*a*) two days earlier than expected and

(*b*) one day later than expected *[MU. MBA. Apr 98]*

Critical path: 1 - 2 - 3 - 4 - 5 - 8 - 9 - 10

Length of the critical path = 75 days

variable of the critical path $= \frac{541}{36} = 15.03$

(*a*) 69.85 % (*b*) 60.26

10. The data for a small project is as given below:

Activity (*i* - *j*):	A	B	C	D	E	F
Optimistic time (days):	9	14	16	24	28	18
Most likely time (days):	10	20	20	30	36	20
Pessimistic time (days):	14	26	22	36	46	21

Precedence relationships: A, B can start immediately.

A < C, D; B < C, D; C < E; D, E < F.

Find the expected activity time, and their variances. Calculate the earliest and latest expected times to reach each node. What is the probability that the project will be computed w days earlier than expected ?

ANSWERS
EXERCISE

II. 1.

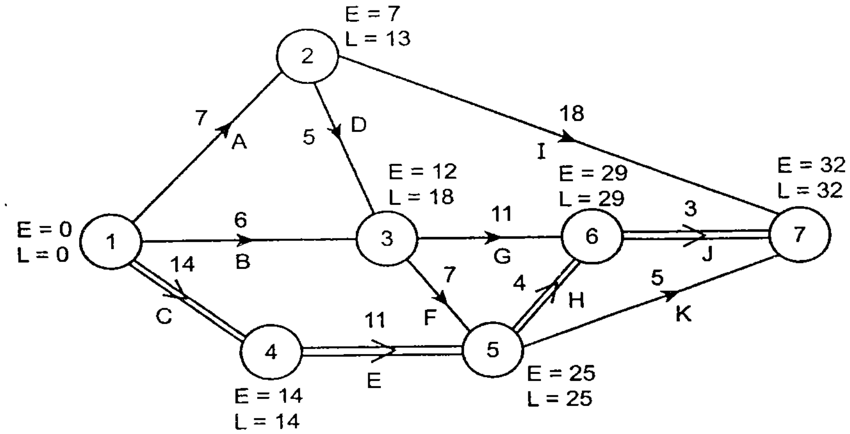


Fig. 39

Critical path C – E – H → J

Required probability 0.5628 or 56.28 %

2.

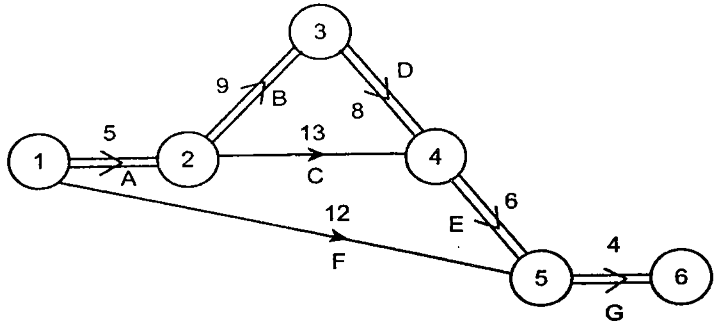


Fig. 40

Required probability = 18.67 %

3.

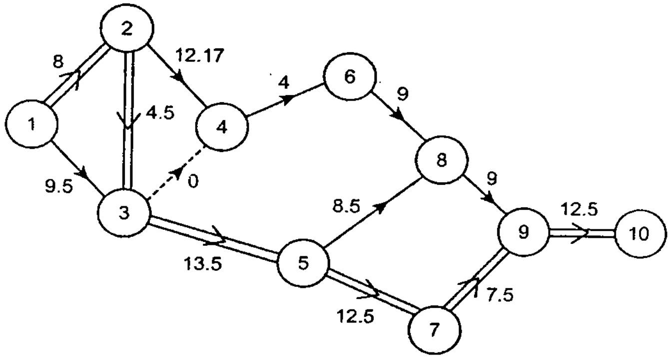


Fig. 41

Length of the critical path = 58.5

Variance of the critical path = $\frac{485}{36} = 13.47$

Required probability = 29.46 %

4. *(a)*

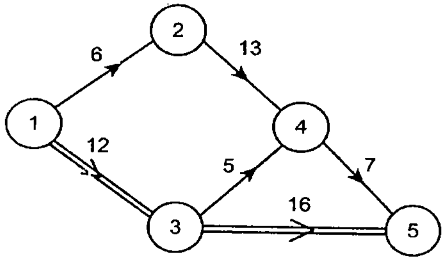


Fig. 42

(b) Length of the critical path = 28 days

Variance of the critical path = 5

(c) Required probability = 81.33 %

5.

Activity	**Total Float**	**Free Float**
1 – 2	0	0
1 – 3	10	0
1 – 4	9	0
2 – 5	0	0
3 – 5	10	10
4 – 6	9	9
5 – 6	0	0

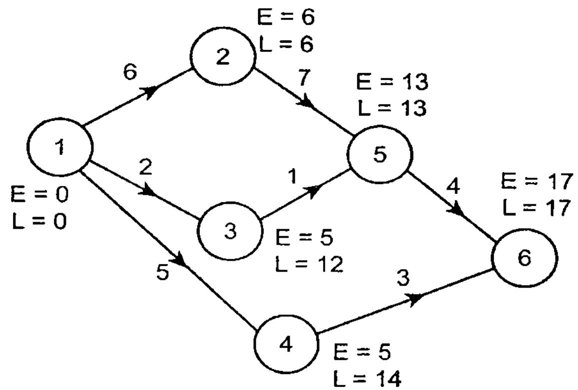


Fig. 43

Critical path 1 – 2 – 5 – 6.

Project duration = 17 days

Required probability = 0.0136.

Probability that the project will be completed on or before an 22 days = 98.64%.

Probability that the project will be completed exactly on 22 days = 0

6. *(i)*

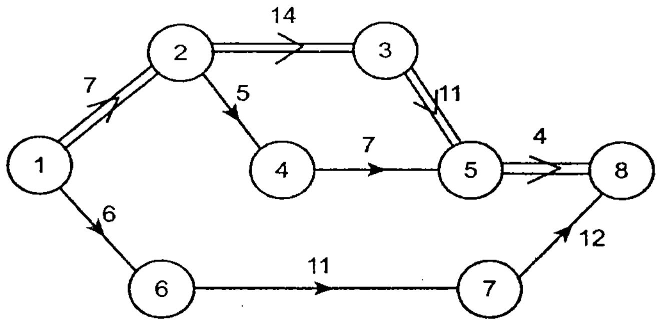


Fig. 44

(ii) Required probability 88.49%

7. Critical path 1 – 3 – 6 – 7 – 8.

Event	1	2	3	4	5	6	7	8
Slack	0	4	0	10	4	0	0	0

8. *(a)*

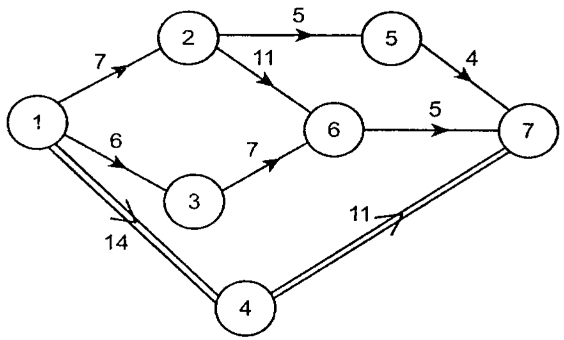


Fig 45

(b) Required probability = 92.07% *(c)* 27.33 days

9. Critical path: 1 - 2 - 3 - 5 - 8 - 9 - 10

Length of the critical paths = 75 days. Variance of the critical paths $= \frac{541}{36} = 15.03$

(a) **69.85 %,** *(b)* **60.26 %**

10.

Activity $i-j$	Optimistic time	Most likely time	Pessimistic time	t_p	σ^2
A	9	10	14	$\frac{63}{6}$	$\frac{25}{36}$
B	14	20	26	**20**	**4**
C	16	20	22	$\frac{118}{6}$	**1**
D	24	30	36	**30**	4
E	28	36	46	$\frac{218}{6}$	9
F	18	20	21	$\frac{119}{6}$	$\frac{1}{4}$

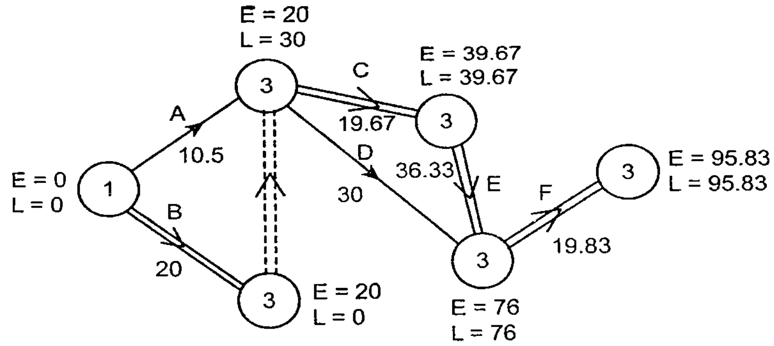


Fig. 45 *(a)*

$$\sigma_c^2 = \frac{57}{4}, \qquad \sigma_c = 3.77$$

$$T_E = 95.83, \qquad T_s = 93.83$$

$$Z = \frac{TS - TE}{\sigma c} = \frac{-2}{3.77} = 0.53$$

$$\therefore P(T \leq 93.83) = 0.5 - 0.2019 = 0.298)$$

$$= 29.81\% \text{ is the required probability.}$$

5.8 COST CONSIDERATIONS IN PERT AND CPM

Crashing

In all the earlier discussions on PERT and CPM it was assumed that the only constraint for an activity was the starting date and the completion date. Availability of resources and cost aspects were not considered.

There are two kinds of costs which can influence the project schedules. They are ***Direct costs*** and ***Indirect costs.***

Direct costs are the costs directly associated with each activity such as machine costs, labour costs etc for each activity.

Indirect costs are the costs due to management services, rentals, insurance including allocation of fixed expenses, cost of security etc.

Direct costs, increase when the duration of any activity is to be reduced since one has to use more machines, more labour, more money to shorten the duration. Indirect costs decrease when we shorten the duration of a project.

Therefore there is some optimum project duration – a balance between excessive direct costs for shortening the project and excessive indirect costs for lengthening the project.

There is a method for finding the optimum duration and the associated least cost called ***least cost schedule.*** It should be observed that there is no need to crash all jobs to get a project done faster. Only critical activities need be expedited. Least cost schedule will indicate which ***critical activities are to be crashed*** and by how much so as to get the optimum duration.

Step by Step procedure for least cost schedule

(1) Draw the network.

(2) Determine the critical path and the normal duration. Also identify the critical activities.

(3) Find the total Normal cost and the Normal duration of the project

(4) Compute the cost slope for each activity by using the formula

$$\text{cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal duration} - \text{Crash duration}}$$

(5) Crash the critical activity of least cost slope first to the maximum extent possible so that the project duration is really reduced.

(6) Calculate the new direct cost by cumulatively adding the cost of the crashing to the current direct cost.

Total cost = New direct cost + Current Indirect cost.

(7) When critical activities are crashed and the duration is reduced other paths may also become critical. Such critical paths are called ***parallel critical paths.*** When there is more than one critical path in a network, project duration can be reduced only when either the duration of a critical activity common to all critical paths is reduced or the durations of different suitable activities on different critical paths are simultaneously reduced.

(8) Stop when the total cost is minimum. This gives optimum (least cost) schedule called *optimum duration.*

Note: (1) ***Minimum (least duration) need not be the same as (least cost) optimum duration. (2) When no indirect cost is given the least cost optimum duration is the same as the normal duration where no crashing is done but least crashing cost duration may be found out.***

Example 1: The following data is pertaining to a project with normal time and crash time.

Jobs	Normal		Crash	
	Time	Cost	Time	cost
1 – 2	8	100	6	200
1 – 3	4	150	2	350
2 – 4	2	50	1	90
2 – 5	10	100	5	400
3 – 4	5	100	1	200
4 – 5	3	80	1	100

***(a)* If the indirect cost is Rs.100 per day find the least cost schedule (optimum duration).**

***(b)* What is the minimum duration ?**

Solution:

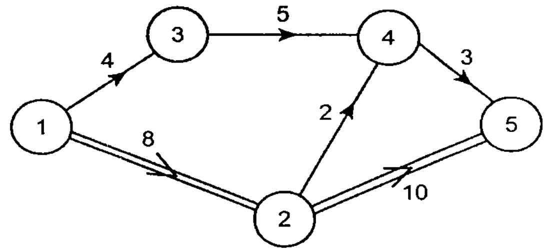


Fig 46

Critical path 1 – 2 – 5. Normal duration 18 days

Total cost = Indirect cost + direct cost

= 1800 + 580 = Rs. 2380

Cost – Slope table

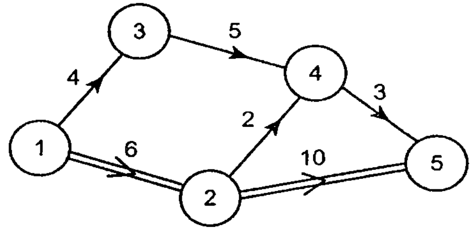
Activity	Cost slope	
1 – 2	50	(Cr = Critical)
1 – 3	100	
2 – 4	40	
2 – 5	60	(Cr)
3 – 4	25	
4 – 5	10	

Stage 1: 1 – 2 is the ***critical*** activity of least cost slope.

Crash 1 – 2 by 2 days. Current Critical path: 1 – 2 – 5

Current duration = 18 – 2 = 16 days

Current Total cost = 16 × 100 + 580 + 100 = 1600 + 680 = 2280

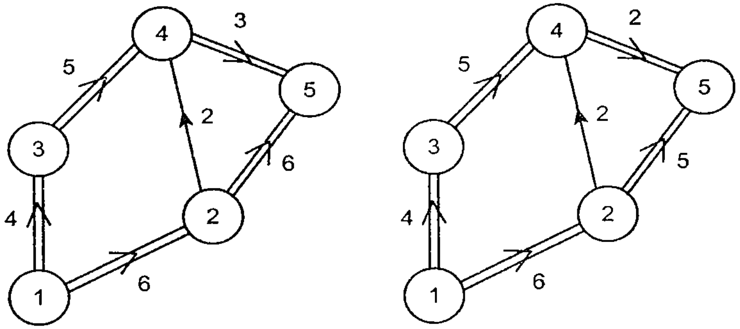


Stage 1

Fig 47

Stage 2: Critical activities 1 – 2 and 2 – 5. ***Crash 2 – 5 by 4 days*** since the duration of the path 1 – 3 – 4 – 5 is 12 days.

Current Critical paths: *(i)* 1 – 2 – 5 *(ii)* 1 – 3 – 4 – 5.
Current duration = 16 – 4 = 12 days
Current total cost = Rs. 12 × 100 + Rs. 680 + Rs. 240 = Rs. 2120.



Stage 2 Stage 3

Fig 48

Fig. 48 *(a)* Fig. 48*(b)*

Stage 3: Critical activities 1 – 2, 1 – 3, 2 – 5, 3 – 4 and 4 – 5.

Crash the critical activities 2 – 5 and 4 – 5 by 1 day each since the duration of the path 1 – 2 – 4 – 5 is 11 days and also the activity 2 – 5 can be crashed only by one day.

Current critical paths: *(i)* 1 – 2 – 5 *and* 1 – 3 – 4 – 5.

Current duration: 12 – 1 = 11 days.

Current total cost = Rs.11 × 100 + Rs. 920 + 1 × 60 + 1 × 10 = Rs.2090

No further crashing is possible since all the activities on the critical path 1 – 2 – 5 have been crashed to the maximum extent.

Hence the optimum duration = 11 days. Least cost = Rs. 2090

(b) Least or minimum duration is also 11 days.

Crashing schedule can be tabulated as follows:

Stage	Crash	Current duration	Direct Cost	Indirect Cost	Total Cost
(o)	0	18	580	1800	2380
(1)	1 - 2 by 2 days	16	680	1600	2280
(2)	2 - 5 by 4 days	12	920	1200	2120
(3)	2 - 5 and 4-5 by 1 days	11	990	1100	2090

Note: Optimum duration is the same as minimum duration ***in this particular problem.***

Example 2 **The following time–cost table (time in days, cost in rupees) applies to a project. Use it to arrive at the network associated with completing the project in minimum time at minimum cost.**
[MU. BE. Apr 94]

Activity	Normal		Crash	
	Time	Cost	Time	Cost
1 – 2	2	800	1	1400
1 – 3	5	1000	2	2000
1 – 4	5	1000	3	1800
2 – 4	1	500	1	500
2 – 5	5	1500	3	2100
3 – 4	4	2000	3	3000
3 – 5	6	1200	4	1600
4 – 5	3	900	2	1600

Solution:

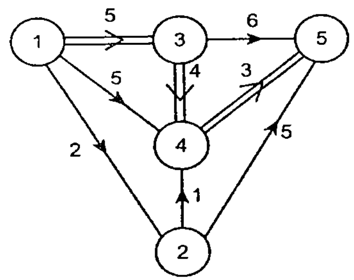


Fig. 49

Critical path 1 – 3 – 4 – 5.

Normal duration = 12 days
Total Normal cost = Rs. 8900

This cost is the minimum cost since no indirect cost is given in the problem. If we want the minimum duration with respect to crashing cost we proceed as follows: Critical activities are 1 – 3, 3 – 4 and 4 – 5.

Cost slope table

Activity Cost slope

1 – 2	600	
1 – 3	333.33	(Cr = Critical)
1 – 4	400	
2 – 4	–	cannot be crashed
2 – 5	300	
3 – 4	1000	(Cr)
3 – 5	200	
4 – 5	700	(Cr)

Stage 1: Crash the activity 1 – 3 by 3 days.

Since 1 - 3 is the critical activity of least cost slope.

Critical path continues to be the same.

No new critical path.

Now the duration = 9 days

Total cost = 8900 + 3 × 333.33 = 9900

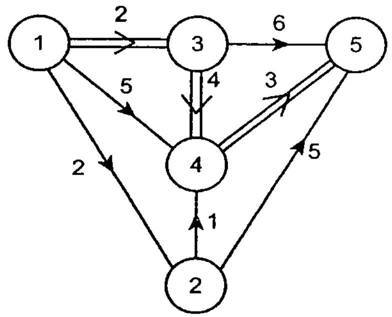


Fig. 50

Stage 2: Crash 4 – 5 by 1 day

Since 4 - 5 is the critical activity of least cost slope.

Duration = 8 days

There are 2 critical paths now

(i) 1 – 3 – 4 – 5

(ii) 1 – 3 – 5

Total cost = 9900 + 700 = 10600

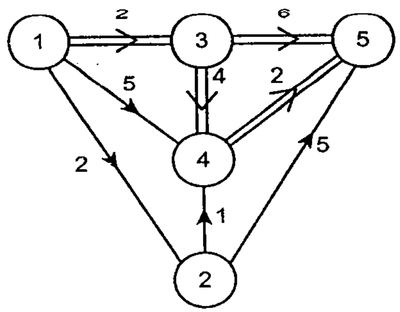


Fig. 51

Stage 3: ***Crash 3 – 4 by 1 day and 3 – 5 by 1 day***

Duration = 7 days

Critical paths

(i) 1 – 3 – 4 – 5 *(ii)* 1 – 3 – 5

(iii) 1 – 4 – 5 *(iv)* 1 – 2 – 5

Cost = 10600 + 1000 + 200 = 11800

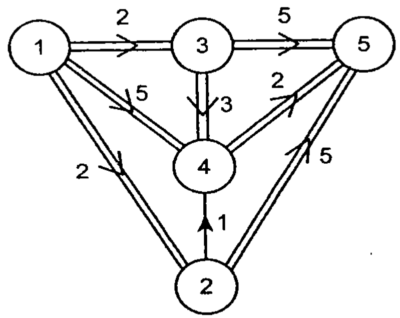


Fig. 52

Further crashing is not possible since the activities 1 – 3, 3 – 4 and 4 – 5 can never be crashed further and so the length of the critical path cannot be reduced below seven even though the durations of some other critical activities can be crashed.

∴ The least (Minimum) duration is 7 days with associated cost as Rs. 11800.

Note that the total normal cost corresponding to 12 days duration is Rs. 8900 is less than the cost. Rs. 11800 corresponding to 7 days duration.

Least cost crash schedule table.

Stage	Crash	Current duration	Direct Cost	Indirect Cost	Total Cost
(0)	0	12	8900	-	8900
(1)	1 - 3 by 3 days	9	8900 + 3 × 333.33	-	9900
(2)	4 - 5 by 1 day	8	9900 + 1 × 700	-	10600
(3)	3 - 4 by 1 day and 3 - 5 by 1 day	7	10600 + 1 × 1000 + 1 × 200	-	11800

Example 3: **A maintenance foreman has given the following estimate of times and cost of jobs in a maintenance project.**

Job	Predecessor	Normal Time hrs	Normal cost Rs.	Crash Time hrs	Crash cost Rs.
A	–	8	80	6	100
B	A	7	40	4	94
C	A	12	100	5	184
D	A	9	70	5	102
E	B,C,D	6	50	6	50

Overhead cost is Rs. 25/- per hour.

Find

(a) **the normal duration of the project and the associated cost**

(b) **the minimum duration (optimum) of the project and associated cost.**

(c) **The least duration of the project and its cost**

(d) **If all the activities are crashed what will be the project duration and the corresponding cost.**

Solution:

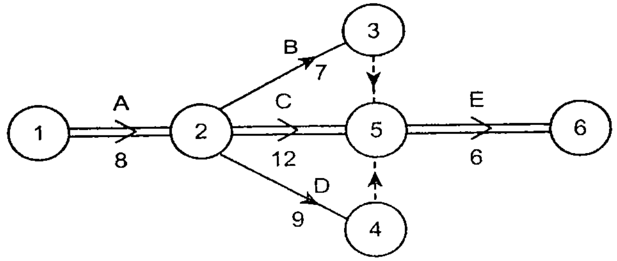


Fig. 53

(a) Critical path 1 – 2 – 5 – 6. Normal duration = 26 hrs

Total cost = Indirect cost + Direct cost = Rs.26×25 +Rs.340 = Rs. 990

(b)

	Activity	*Cost Slope*	
A	1 – 2	10	(Cr = Critical)
B	2 – 3	18	
C	2 – 5	12	
D	2 – 4	8	(Cr)
Dummy	3 – 5	–	
Dummy	4 – 5	–	
E	5 – 6	–	(Cr)

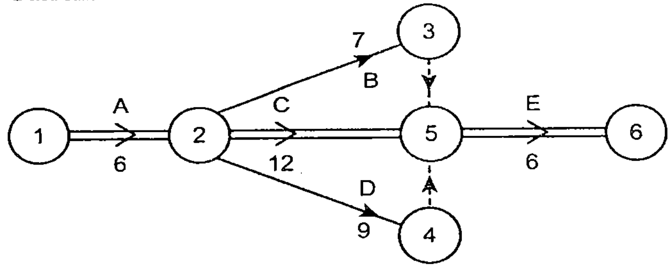
***Stage 1**: Crash the critical activity of least slope 1 – 2.*

Crash 1 – 2 by 2 hrs.

Current critical path = 1 – 2 – 5 – 6. continues to be the same.

Current duration = 26 – 2 = 24 hrs.

Current total cost = Rs. 24 × 25 + Rs. 340 + Rs. 2 × 10 = Rs. 960



Stage 1 Fig 54

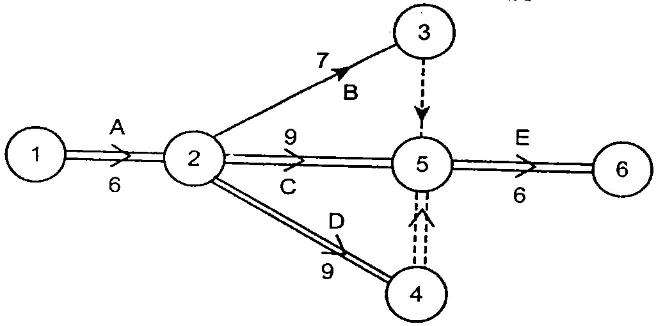
Stage 2: Critical activities 1 – 2, 2 – 5 and 5 – 6.

2 – 5 is the only critical activity which can be crashed. ***Even though the critical activity 2 – 5 can be crashed by 7 hrs. crash 2 – 5 by 3 hrs only since the path 1 – 2 – 4 – 5 - 6 has the duration 21 hrs.***

Current critical path: *(i)* 1 – 2 – 5 – 6 *(ii)* 1 – 2 – 4 – 5 – 6

Current duration = 24 – 3 = 21 hrs.

Current total cost = Rs. 21 × 25 + 360 + 36 = Rs. 921



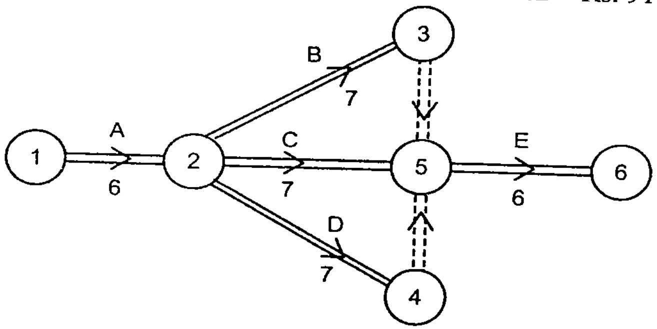
Stage 2 Fig 55

Stage 3: Critical activities: 1 – 2, 2 – 4, 2 – 5, 4 – 5 and 2 – 6.

Crash the critical activities 2 – 5 and 2 – 4, each by 2 hrs, since the duration of the path 1 – 2 – 3 – 5 – 6 is 19 hrs.

Current critical paths *(i)* 1 –2 –5 – 6 *(ii)* 1–2 –4–5– 6 *(ii)* 1–2–3–5– 6 *i.e.,* ***All paths are critical.*** Current duration = 21 – 2 = 19 hrs.

Current total cost = Rs. 19 × 25 + 396 + 2 × 8 + 2 × 12 = Rs. 911



Stage 3 Fig 56

Stage 4:

Critical activities: 1 – 2, 2 – 3, 2 – 4, 2 – 5, 3 – 5, 4 – 5 and 5 – 6.

Crash the critical activities 2 – 3, 2 – 4 and 2 – 5, each by 2 hrs

Current critical path: All paths

Current duration = 19 – 2 = 17 hrs.

Current total cost = Rs. 17 × 25 + Rs.436 +2 × 18 + 2 × 12 + 2 × 8 = Rs. 937.

No further crashing is possible since the duration of every activity along the critical path 1 – 2 – 5 – 6 has been crashed to the maximum extent. The

project duration cannot be reduced from 17 hrs even though the critical activity 2 – 3 can be crashed by 1 hr.

∴ Optimum (least cost) duration = 19 hrs

Least cost = Rs. 911

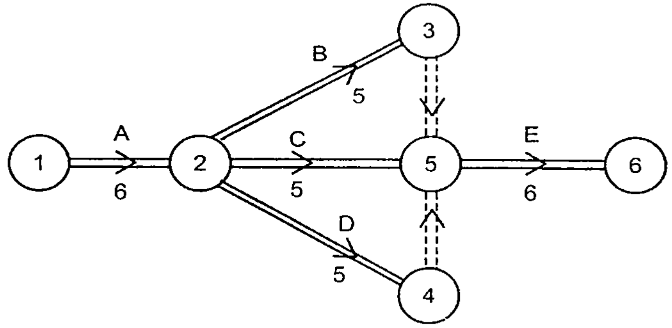


Fig. 57

(c) The minimum duration is 17 hrs as can be seen in stage 4. The associated cost is Rs. 937.

(d) If all activities are crashed (*i.e.,* the only critical activity 2 - 3 which can be crashed by 1 hr as seen in stage 4 which however cannot reduce the project duration). Even then the project duration is 17 hrs but the total cost is Rs. 937 + 18 = Rs. 955.

Least cost crash schedule table.

Stage	Crash	Current duration	Direct Cost (in Rs.)	Indirect Cost (in Rs.)	Total Cost (in Rs.)
(0)	0	26 hrs	340	650	990
(1)	1 - 2 by 2 days	24	360	600	960
(2)	2 - 5 by 4 days	21	396	525	921
(3)	2 - 5 and 4- 5 each by hrs	19	436	475	911
(4)	2-3, 2-4, and 2-5 each by 2 hrs.	17	512	425	937

5.9 PROJECT SCHEDULING WITH LIMITED RESOURCES

Introduction

Problems of resource scheduling vary in kind and complexity, depending upon the nature of the project and its organisational set up. It is but natural that activities are scheduled so that no two of them requiring the same facility occurs at the same time, wherever possible. The problem of scheduling activities so that none of the precedence relations are violated is an extremely difficult task even for projects of modest size. The problem of scheduling project with the limited resources is usually large, combinatorially. Even the powerful techniques aided by the fastest, sophisticated computer can solve only small projects having not more than about 100 activities. Analytical techniques are impractical for real world problems of this type usually. One turns to **Heuristic Programs** for such cases.

Lacking time or inclination to pursue more thorough problem solving procedures, one employs a rule of thumb arising out of experience, expertise and commonsense. In some cases rule of thumb is insufficient. It must be combined with other rules to take into additional factors or exceptional circumstances. A collection of such rules for solving a particular problem is called a ***heuristic program.*** Such a program may require a computer for its solution for complex problems.

Heuristic Programs for resource scheduling are

A. Resource levelling programs

B. Resource allocation programs

A. Resource Levelling Programs

These programs attempt to reduce peak resource requirements and smooth out period to period assignments without changing the constraint on project duration.

Using the resource requirements data of the early start schedule, the program attempts to reduce peak resource requirements by shifting jobs with slack to non peak periods. Resource limits are not specified but peak requirements are levelled as much as possible ***without delaying the specified due date.***

Steps

(1) Draw the early start schedule graph.

(2) Draw the corresponding manpower chart.

(3) Identify the activities with slack.

(4) Adjust the activities identified in step (3) and adjust them to level the peak resource requirements.

Example 1: The early start schedule graph of a project is given below. The manpower requirement for each activity is indicated in the parenthesis. Using resource levelling programming reduce the peak resource requirements. Alphabets denote activities.

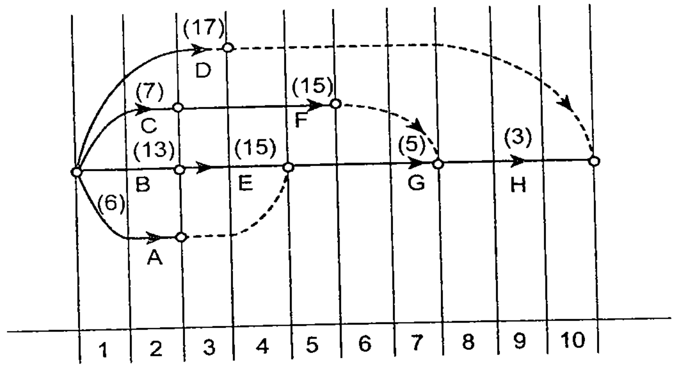
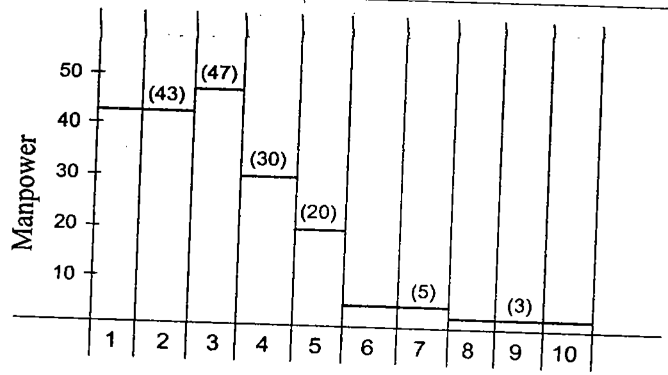


Fig.58

Solution: Draw the man power loading chart.

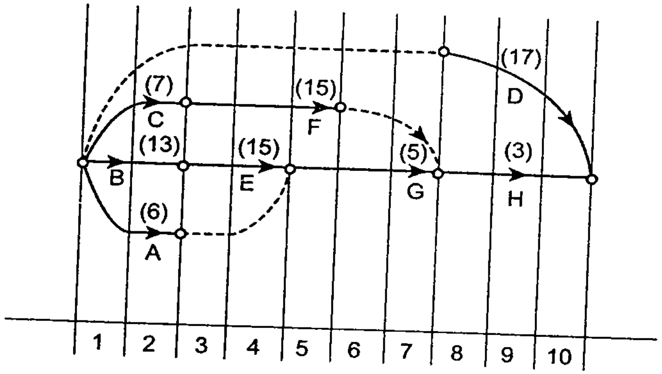


Manpower loading chart – Time in weeks

Fig. 59

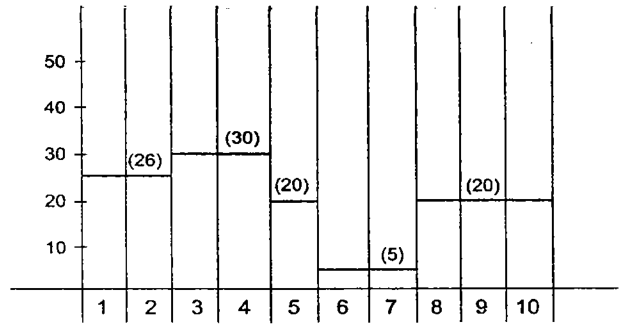
Peak manpower requirement is 47 men in the third week. A has two weeks, D and F have slack of 7 weeks and 2 weeks respectively. Therefore these jobs can be delayed using the slack without delaying the project completion date. B, E, G, H are on the critical path. Since D has maximum slack and peak resource requirement can be reduced using this slack, delay the start of the activity D by 7 weeks. The schedule graph and the man power loading chart will then be as follows:

Stage 1



Schedule graph

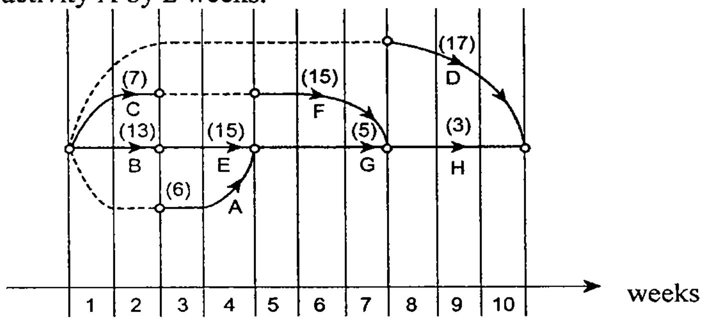
Fig. 60



Man power loading chart

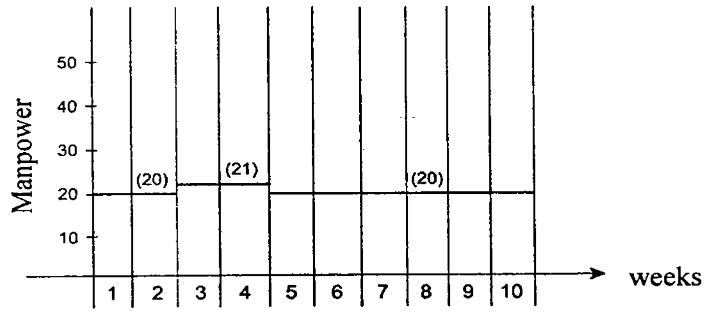
Fig.61

Stage 2 Shift the start of the activity F by 2 weeks and also delay the start of the activity A by 2 weeks.



Schedule graph

Fig. 62



Manpower loading chart

Fig. 63

Levelling further is impossible.

Example 2 **The manpower required for each activity of a project is given in the following table:**

Activity	Normal Time (days)	Manpower required per day
A 1 – 2	10	2
B 1 – 3	11	3
C 2 – 4	13	4
D 2 – 6	14	3
E 3 – 4	10	1
G 4 – 5	7	3
F 4 – 6	17	5
I 5 – 7	13	3
H 6 – 7	9	8
J 7 – 8	1	11

The contractor stipulates that the first 26 days, only 4 to 5 men and during the remaining days 8 to 11 men only are available. Find whether it is possible to rearrange the activity suitably for levelling the manpower resources satisfying the above condition.

Solution: Draw the network and the schedule graph.

Network:

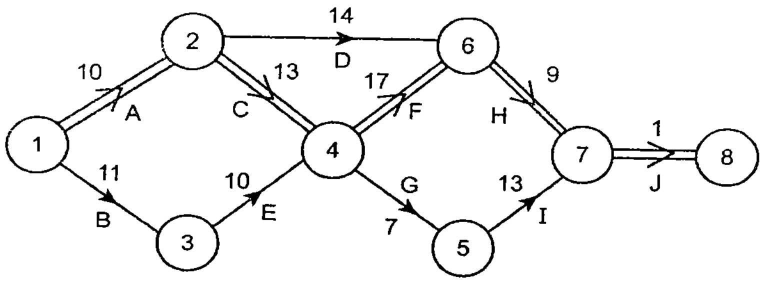
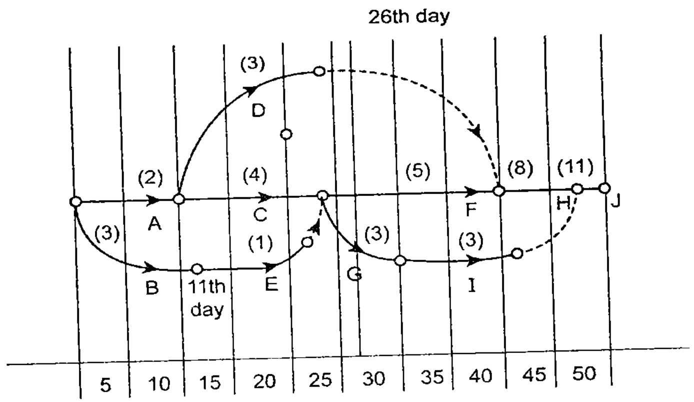


Fig. 64

Critical path 1 – 2 – 4 – 6 – 7 – 8

Project duration is 50 days



Schedule Graph

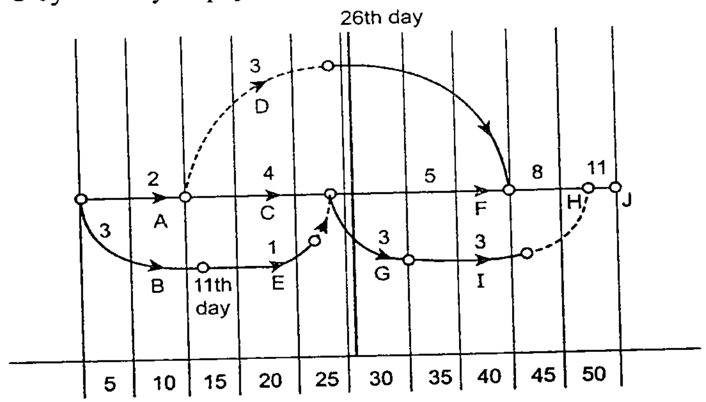
Fig. 65

Complete rearrangement satisfying the condition is not possible.

For,

From the schedule graph it is observed that by using the slack by D, I and E we have problem only on the 11th day where the manpower requirement is 7 whereas the availability is only 4 or 5. Unless the duration is lengthened or overtime arranged, 11th day cannot be managed. On all other days there is adequate manpower availability as indicated in the following figure.

Delay C by one day or pay overtime. Duration becomes 51 days.



Schedule Graph

Fig. 66

B. Resource Allocation Programme

Step 1: Allocate resources serially in time.

Step 2: If several jobs compete for the same resources, give preference to the jobs with the smallest slack.

Step 3: Reschedule non critical jobs, if possible, so as to make available the needed resources for rescheduling critical jobs.

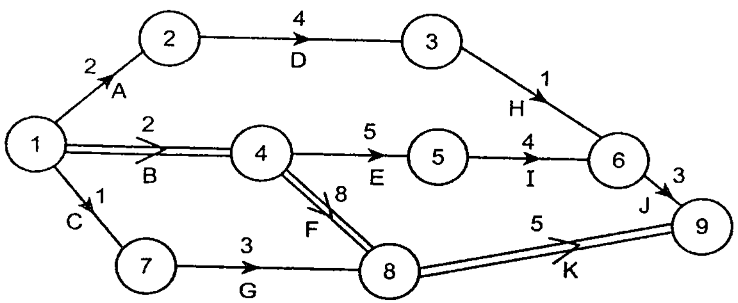
Example 3 **A project schedule has the following characteristics:**

Activity :	**1–2**	**1–4**	**1–7**	**2–3**	**3–6**	**4–5**
Duration:	**2**	**2**	**1**	**4**	**1**	**5**
Activity :	**4–8**	**5–6**	**6–9**	**7–8**	**8–9**	
Duration:	**8**	**4**	**3**	**3**	**5**	

(a) **Construct a PERT Network and find the critical path and the project duration.**

(b) **Activities 2–3, 4–5, 6–9 each requires one unit of the same key equipment to complete it. Do you think availability of one unit of the equipment in the organisation is sufficient for completing the project without delaying it ; if so what is the schedule of these activities ?**

Solution:



Schedule Graph

Fig. 67

Critical path 1 – 4 – 8 – 9 ; Project duration = 15 time units

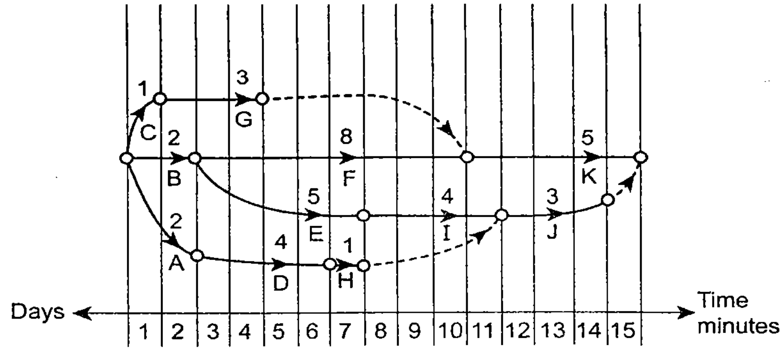


Fig. 68 - Schedule Graph on time scale

Start D after 7 time units and start J after 12 time units. No change in the start and other activities, as shown in the following diagram.

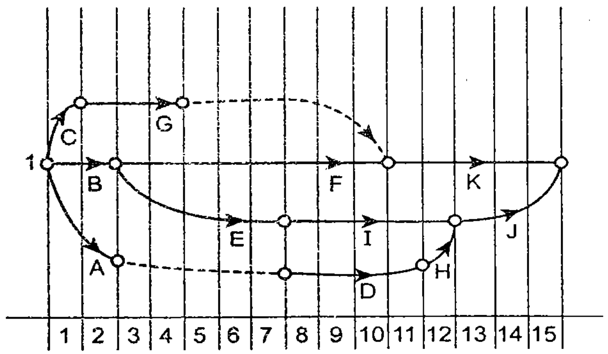


Fig.69

EXERCISE

1. What is resource levelling ?
2. What is resource scheduling ?
3. What is heuristic programming ?
4. What are the two main costs for a project ? Illustrate with examples.
5. Define (a) Indirect cost (b) Direct cost for a project.
6. The activities of a project with normal and crash time and cost are given below:

Activity	Normal Time hrs	Normal Cost Rs.	Crash Time hrs	Crash Cost Rs.
(1, 2)	20	2000	15	3000
(1, 3)	10	1500	7	2,400
(2, 5)	15	1000	10	1500
(3, 4)	16	3000	12	4000
(3, 5)	22	4500	16	5700
(4, 5)	14	1500	10	2100

Find the optimum scheduling of the project which minimises the total cost.

7. State the steps of Fulkerson's in giving node numbers
Explain: Total float, Free float and Independent Float.
State the difference between smoothing and levelling of resources.

8. Determine the optimum project duration and cost for the project given below:

Activity	Normal Time days	Normal Cost Rs.	Crash Time Days	Crash Cost Rs.
1 – 2	8	100	6	200
1 – 3	4	150	2	350
2 – 4	2	50	1	90
2 – 5	10	100	5	400
3 – 4	5	100	1	200
4 – 5	3	80	1	100

Overhead is Rs. 70/- per day.

9. The following table gives the activities in a construction project and other relevant information.

Activity $i-j$	Normal duration (days)	Crash duration (days)	Cost of Crashing (Rs per day)
1 – 2	9	6	20
1 – 3	8	5	25
1 – 4	15	10	30
2 – 4	5	3	10
3 – 4	10	6	15
4 – 5	2	1	40

(a) What is the normal project length and the minimum project length ?

(b) Determine the minimum crashing costs of schedules ranging from normal length down to and including the minimum length schedule.

(c) What is the optimal length schedule duration, of each job for your solution ?

Overhead cost for the project is Rs. 60 per day.

10. A small maintenance project consists of jobs given in the table below. With each job, normal time and crash time are given in days.

The cost of crashing a job in rupees per day is also given.

Job i	Job j	Normal duration days t_n	Crash duration days t_c	Cost of crashing Rs/day c
1	2	9	6	20
1	3	8	5	25
1	4	15	10	30
3	4	10	6	15
4	5	2	1	40

(i) **What is the normal length and the minimum project length ?**

(ii) **Determine the minimum crashing cost schedule for normal length, by each day, for 7 days.**

(iii) **If overhead costs total Rs 60 per day, what is the optimal length schedule in terms of both crashing and overhead costs.**

[BE. BNU. Nov 96]

Hint: Cost of crashing per day is nothing but the cost slope.

11. Draw the project network, based on information provided below:

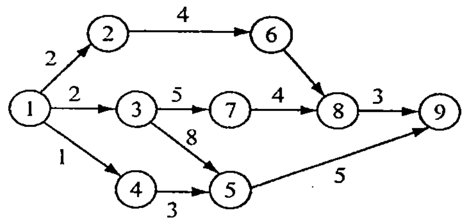
Activity	**1–2**	**1–3**	**1–4**	**1–6**	**2–3**	**2–5**	**3–4**
Duration (in weeks)	**5**	**3**	**2**	**8**	**2**	**7**	**1**
Cost (in Thousands)	**100**	**150**	**60**	**200**	**250**	**125**	**500**
Activity	**3–7**	**4–5**	**4 –7**	**5–6**	**5 –7**	**6 –7**	
Duration (in weeks)	**5**	**10**	**3**	**6**	**5**	**9**	
Cost (in Thousands)	**325**	**75**	**80**	**100**	**120**	**300**	

Identity the critical path, critical activities, the normal duration and normal cost for completing the project. If the project is to be completed one day earlier, what is the additional expenditure ? (Assume that an additional expenditure of 30% of normal cost is incurred, for reducing the duration of any one activity by one week)

12. What is the least cost schedule of a project ? How is it obtained ?

13. Write Short notes on the cost aspects of PERT.

14. Find the critical path and calculate the slack time for each event for the following PERT diagram.



15. Draw the project network, based on information provided below:

Activity	**(1-2)**	**(1-3)**	**(1-4)**	**(1-6)**	**(2-3)**	**(2-5)**	**(3-4)**
Duration (in weeks)	**5**	**3**	**2**	**8**	**2**	**7**	**1**
Cost (in thousands)	**100**	**150**	**80**	**200**	**250**	**125**	**500**
Activity	**(3-7)**	**(4-5)**	**(4-7)**	**(5-6)**	**(5-7)**	**(6-7)**	
Duration (in weeks)	**5**	**10**	**3**	**6**	**5**	**9**	
Cost (in thousands)	**325**	**75**	**80**	**100**	**120**	**300**	

Identify the critical path, critical activities, the normal duration and normal cost for completing the project. If the project is to be completed one day earlier, what is the additional expenditure ? (Assume that an additional expenditure of 30% of normal cost, is incurred, for reducing the duration of any one activity by one week.)

16. Time estimates for a project is given below:

Activity	Optimistic time	Most-likely time	Pessimistic time (days)
A (1, 2)	2	5	14
B (1, 3)	9	12	15
C (2, 4)	5	14	17
D (3, 4)	2	5	8
E (4, 5)	6	9	12
F (3, 5)	8	17	20

Find the critical path expected project completion time and the probability that the project will be completed in 30 days.

17. The following table lists the jobs of a network along with their time estimates:

Job	1-2	1-3	2-4	3-4	4-5	3-5
Optimistic time	2	9	5	2	6	8
Most likely time	5	12	14	5	6	17
Pessimistic time	14	15	17	8	12	20

(a) Draw the project network.

(b) Calculate the length and variance of the critical path.

(c) Find the probability that the project will be completed within 30 days.

18. The activities of a project are tabulated below with immediate predecessors and normal and crash time together with cost.

Activity	Immediate	Normal		Crash	
		Cost (Rs.)	Time (days)	Cost (Rs.)	Time (days)
A	–	200	3	400	2
B	–	250	8	700	5
C	–	320	5	380	4
D	A	410	9	800	4
E	C	600	2	670	1
F	B, E	400	6	950	1
G	B, E	550	12	1000	6
H	D	300	11	400	9

(a) Determine the critical path and the normal duration and cost of the project.

(b) Suitably crash the activities so that the normal duration may be reduced by 3 days at a minimum cost.

19. A project schedule has the following characteristics.

Activity	Time	Activity	Time
1-2	4	5-6	4
1-3	1	5-7	8
2-4	1	6-8	1
3-4	1	7-8	2
3-5	6	8-10	5
4-9	5	9-10	7

Construct PERT network and find the critical path.

20. How will you optimise a given dynamic programming problem.

(a) The following data pertain to a project:

Activity	1-2	1-4	1-5	2-3	2-5	2-6
t_0 (weeks)	5	1	2	7	8	5
t_m (weeks)	6	3	4	8	9	9
t_p (weeks)	8	4	5	10	13	10

Activity	3-4	3-6	4-6	4-7	5-6	5-7	6-7
t_0 (weeks)	4	3	4	5	9	4	3
t_m (weeks)	5	4	8	6	10	6	4
t_p (weeks)	6	5	10	8	15	8	5

t_0 = optimistic time estimate ; t_m = most likely time estimate

t_p = pessimistic time estimate. Draw the project network. Estimate the probability of completing the project in 33 weeks.

$$\text{Take } Z = \frac{T - T_{exp}}{\text{standard deviation}}$$

Z	1.0	1.5	2.0	2.5	3.0	0.5
Area under normal distribution curve	0.8413	0.9332	0.9772	0.9938	0.9987	0.6915

21. For the activities given below draw the project network and find *(a)* duration of the project, *(b)* the probability that the project will be finished within 30 weeks:

Activity	Preceding activity	Time Estimates (in weeks) Optimistic	Most likely	Pessimistic
A	None	2	4	12
B	None	10	12	26
C	A	8	9	10
D	A	10	15	20
E	A	7	7.5	11
F	B, C	9	9	9
G	D	3	3.5	7
H	E, F, G	5	5	5

22. The activities for the erection of a transformer and distribution lines for a new housing colony is given below:

Activity	Predecessor	Time (days) Normal	Crash	Direct cost (Rs.) Normal	Crash
A	–	4	3	60	40
B	–	6	4	150	250
C	–	2	1	38	60
D	A	5	3	150	250
E	C	2	2	100	100
F	A	7	5	115	175
G	D, B, E	4	2	100	240

Indirect costs vary as follows:

Days:	15	14	13	12	11	10	9	8	7	6
Cost (Rs.)	600	500	400	250	175	100	75	50	35	25

Determine the project duration which will return in minimum total project cost.

23. A small project consists of the jobs in the table given below: With each job is listed its normal time and crash time (in days). The cost of crashing each job per day (in Rs.) is also given:

Job	:	1-2	1-3	1-4	2-4	3-4	4-5
Normal time	:	9	8	15	5	10	2
Crash time	:	6	5	10	3	6	1
Cost of crashing	:	60	75	90	30	45	120

(a) What are the normal project length and the minimum project length ?

(b) Find the minimum crashing costs of schedules ranging from normal length down to minimum length assuming that the overhead cost per day is Rs. 200.

(c) What is the optimal length schedule duration of each job for your solution ?

24. The details of the activities of a project are given below:

Activity :	A	B	C	D	E	F	G	H	I	J	K	L
Dependence:	–	–	–	B, C	–	C	E	E	D,F,H	E	I, J	G
Duration (days) :	9	4	7	8	7	5	10	8	6	9	10	2

Draw the network and find earliest occurrence time, latest occurrence time, critical activities and project completion time.

25. The following table gives the activities in a construction project and other relevant information.

Activity $i-j$	Normal duration (days)	Crash duration (days)	Cost of crashing (Rs. per day)
1 – 2	9	6	20
1 – 3	8	5	25
1 – 4	15	10	30
2 – 4	5	3	10
3 – 4	10	6	15
4 – 5	2	1	40

(a) What is the normal project length and the minimum project length.

(b) Determine the minimum crashing costs of schedules ranging from normal length down to, and including the minimum length schedule.

(c) What is the optimal length schedule duration, of each job for your solution ?

Overhead cost for the project is Rs. 60 per day.

26. *(a)* **A certain project requires tasks A, B, C, ..., H, I to be completed. The notation M < N, means that task M must be completed before task N can begin. With this notation,**

A < D ; A < E ; B < F ; D < F ; C < G ; C < H ; F < I ; G < I

Draw the network graph to represent the sequence of tasks. Identify the critical path. Find the minimum time of completion of the project in days. The time for completion of each task in days, is as follows:

Task	A	B	C	D	E	F	G	H	I
Time	5	9	14	4	3	10	6	12	10

(b) **Making use of an example, discuss the algorithm for solving sequencing problem for the case of 5 jobs and three machines.**

27. **Consider the following data for the activities concerning a project:**

Activity	A	B	C	D	E	F
Immediate predecessor/s	–	A	A	B, C	–	E
Duration (days)	2	3	4	6	2	8

(a) Draw the network diagram.

(b) Find the minimum project completion time.

(c) Determine the critical path.

(d) Find 'slack' for each activity.

28. **A small maintenance project consists of the following 12 jobs. Draw arrow network of the project. Summarize CPM calculation and calculate three types of floats for the jobs. Determine the critical path:**

Activity	1–2	2–3	2–4	3–4	3–5	4–6	5–8	6–7	6–10	7–9	8–9	9–10
Duration (days)	2	7	3	3	5	3	5	8	4	4	1	7

29. **A project schedule has the following characteristics:**

Activity	Time	Activity	Time
1 – 2	4	5 – 6	4
1 – 3	1	5 – 7	8
2 – 4	1	6 – 8	1
3 – 4	1	7 – 8	2
3 – 5	6	8 – 10	5
4 – 9	5	9 – 10	7

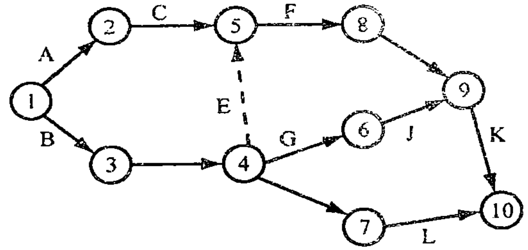
(a) Construct a network diagram.

(b) Compute earliest event time and latest event time.

(c) Determine critical path and total project duration.

(d) Compute total and free float for each activity.

30. **For the project represented by Fig. Q. 23 *(a)*, find the critical path, and the probability of completing the project within the schedule time of 34.67 units.**



Task	A	B	C	D	E	F	G	H	I	J	K	L
t_0	3	1	6	8	0	5	6	3	3	1	3	8
t_p	7	3	12	17	0	9	12	8	8	3	6	20
t_m	5	2	8	12	0	7	9	6	6	2	4	15

31.

Job	1 – 2	1 – 3	1 – 4	2 – 4
Normal time:	9	8	15	5
Crash time:	6	5	10	3
Cost of crashing in Rs.	60	75	90	30

Make use of area under normal distribution curve.

(a) What is the optimal length schedule duration of each job ?

(b) What are normal project length and minimum project length ?

32. The following tables shows the jobs of a network along with their time estimates.

Job	1–2	1–6	2–3	2–4	3–5	4–5	6–5	5–8	7–8
a (days)	1	2	2	2	7	5	5	3	8
m (days)	7	5	14	5	10	5	8	3	17
b (days)	13	14	26	8	19	17	29	9	32

Draw the project network and find the probability that the project is completed in 40 days.

33. The utility data for a network is given below. Determine the total free and independent floats and identify the critical path.

Activity	0–1	1–2	1–3	2–4	2–5	3–4	3–6	4–7	5–7	6–7
Duration	2	8	10	6	3	3	7	5	2	8

34. The data on normal time and normal cost and crash time and cost for a project are given in the table.

	NORMAL		CRASH	
Activity	Time (days)	Cost (Rs.)	Time (days)	Cost (Rs.)
1 – 2	8	100	6	200
1 – 3	4	150	2	350
2 – 4	2	50	1	90
2 – 5	10	100	5	400
3 – 4	5	100	1	200
4 – 5	3	80	1	100

Indirect cost = Rs. 70 / day.

Draw the network, crash the activities systematically and determine the optimum project completion time and cost.

35. (a) Distinguish between PERT and CPM.

(b) A project has the following time schedule.

Activity	Time (months)	Activity	Time (months)
1 – 2	2	4 – 6	3
1 – 3	2	5 – 8	1
1 – 4	1	6 – 9	5
2 – 5	4	7 – 8	4
3 – 6	8	8 – 9	3
3 – 7	5		

Construct a PERT Network and compute critical path.

36. A maintenance activity consists of the following jobs. Draw the network for the project and calculate the total float and free float for each activity.

Job	1–2	2–3	3–4	3–7	4–5	4–7	5–6	6–7
Duration (days)	3	4	4	4	2	2	3	2

37. Following are the manpower requirements for each activity in a project.

Activity	1–2	1–3	2–4	2–6	3–4
Normal time (days)	10	11	13	14	10
Manpower required/ day	2	3	4	3	1

Activity	4–5	4–6	5–7	6–7	7–8
Normal time (days)	7	17	13	9	1
Manpower required/ day	3	5	3	8	11

The contractor stipulates that during the first 26 days only 4 to 5 men and during the remaining days 8 to 11 men only can be made available. Rearrange the activities suitably by levelling the manpower resources satisfying the above condition.

38. **Tasks A, B, C, ..., H, I constitute a project. The notation X < Y means that the task X must be finished before Y can begin with this notation.**

A < D, A < E, B < F, D < F, C < G, C < H, F < I, G < I.

Draw a graph to represent the sequence of tasks and find the minimum time of completion of the project, when the time (in days) of completion of each task is as follows:

Task	A	B	C	D	E	F	G	H	I
Time	8	10	8	10	16	17	18	14	9

39. **Consider a project consisting of 7 jobs A, B, ..., G with the following precedence relations and time estimates.**

Job	Predecessor	a	m	b
A	–	2	5	8
B	A	6	9	12
C	A	5	14	17
D	B	5	8	11
E	C, D	3	6	9
F	–	3	12	21
G	E, F	1	4	7

Draw the project network and find the probability of completing the project in 30 days.

40. *(a)* **The jobs of a network with their time estimates are given in the following table:**

JobDuration (days)

$i-j$	Optimistic	Most likely	Pessimistic
1 – 2	3	6	15
1 – 6	2	5	14
2 – 3	6	12	30
2 – 4	2	5	8
3 – 5	5	11	17
4 – 5	3	6	15
6 – 7	3	9	27
5 – 8	1	4	7
7 – 8	4	19	28

(i) **Draw the project network.**

(ii) **Calculate the length and variance of the critical path.**

(iii) **What is the approximate probability that the jobs on the critical path will be completed by the due date of 42 days ?**

41. **The activities, activity durations and manpower requirements of a project are given in the table.**

Activity	1–2	1–3	1–4	2–5	2–6	3–7
Duration (days)	2	2	0	2	5	4
Manpower requirement	5	4	0	2	3	6

Activity	4–8	5–9	6–9	7–8	8–9
Duration (days)	5	6	3	4	6
Manpower requirement	2	8	7	4	3

There are eleven persons who can be employed for its project. Carryout the appropriate manpower levelling so that the fluctuations of work force requirement from day-to-day are as small as possible.

42. *(a)* **What is meant by updating the project in CPM ?**

(b) **A project has the following time schedule:**

Activity	Time in months
1 – 2	2
1 – 3	2
1 – 4	1
2 – 5	4
3 – 6	8
3 – 7	5
4 – 6	3
5 – 8	1
6 – 9	5
7 – 8	4
8 – 9	3

Draw the network and construct critical path.

43. The utility data for a project network is given below. If the indirect cost per day is Rs. 250, find the optimum project schedule:

	NORMAL		CRASH	
Activity	Time (days)	Cost (Rs.)	Time (days)	Cost (Rs.)
1 – 2	4	600	2	800
1 – 3	2	500	1	900
2 – 4	6	1000	3	1750
2 – 5	4	1200	4	1200
3 – 5	5	1000	3	1200
4 – 5	5	1300	5	1300

44. Draw the network for the following project consisting of 5 jobs (A, B, C, D and E) with the following job sequence. Job A precedes C and D. B precedes D. Jobs C and D precede E.

45. A small marketing project consists of the jobs in the table given below. With each job is listed its normal time and a minimum or crash time (in days). The cost (in Rs. per day) of crashing each job is given.

Job	Normal	Minimum (crash)	Cost of crashing (Rs. per day)
1 – 2	9	6	20
1 – 3	8	5	25
1 – 4	15	10	30
2 – 4	5	3	10
3 – 4	10	6	15
4 – 5	2	1	40

(i) What is the normal project length and minimum project length ?

(ii) Also determine the minimum crashing costs of schedules from normal length to minimum length schedule.

46. The following tables shows the jobs of a network along with their time estimates. The time estimates are in days.

Job	(1, 2)	(1, 6)	(2, 3)	(2, 4)	(3, 5)	(4, 5)	(5, 8)	(6, 7)	(7, 8)
a	3	2	6	2	5	3	1	3	4
m	6	5	12	5	11	6	4	9	19
b	15	14	30	8	17	15	7	27	28

Draw the project network. Find the critical path. Also find the probability that the project is completed in 31 days.

47. The following table indicates the details of the activities for a project. The duration of the activities are given in days.

Activity	A	B	C	D	E	F	G	H	I	J
Immediate Preceding activity	–	–	A	A	B,C	B,C	E	E	D,G	F,H,I
Duration in days	10	12	5	1	7	10	2	12	2	12

(i) Draw the network.

(ii) Find the total float and free float for each activity.

(iii) Indicate the critical path.

48. The total below shows jobs their normal time and cost and crash time and cost for a project.

	NORMAL		CRASH	
Activity	Time (days)	Cost (Rs.)	Time (days)	Cost (Rs.)
1 – 2	6	1400	4	1900
1 – 3	8	2000	5	2800
2 – 3	4	1100	2	1500
2 – 4	3	800	2	1400
3 – 4	Dummy	–	–	–
3 – 5	6	900	3	1600
4 – 6	10	2500	6	3500
5 – 6	3	500	2	800

(i) Draw the network of the project.

(ii) What is the normal duration and cost of the project ?

(iii) Find the optimum duration and minimum cost of the project. Indirect cost for the project is Rs. 300 per day.

49. **A project consists of a series of tasks labelled A, B, ... H, I with the following relationships. Construct the CPM network having the following constraints.**

A < D, E ; B, D < F ; C < G ; B < H ; F, G < I.

(X < Y, Z means Y & Z cannot start until X is completed. X, Y, Z cannot start until X & Y have been completed].

Tabulate the earliest start time, latest finish time and the block associated with each task, given the time (in days) of completion of each task as follows.

Task	A	B	C	D	E	F	G	H	I
Time	23	8	20	16	24	18	19	4	10

50. **Consider the optimistic, most likely, and pessimistic times for each activity that are given by the following table. Carry out a complete PERT analysis and determine the probability that the project will be completed three days earlier to the expected time.**

Activity	1–2	1–3	2–4	2–5	3–4	3–6	4–5	4–6	5–7	6–7
a (days)	1	2	5	6	3	4	9	2	9	3
m (days)	5	8	7	9	10	8	11	5	9	6
b (days)	9	20	15	24	17	18	13	14	15	9

51. **A maintenance project consists of the following jobs:**

Job	1–2	2–3	2–4	3–5	3–6
Duration (days)	2	3	5	4	1
Job	4–6	4–7	5–8	6–8	7–8
Duration (days)	6	2	8	7	4

(i) Draw the network.

(ii) Calculate the early start, early finish schedule, late start, late finish schedule.

(iii) Calculate the total float and free float for each activity.

52. **Determine the least cost schedule for the following project. Overhead cost is Rs. 60 per day.**

Activity	Normal duration (days)	Crash duration (days)	Cost of crashing per day (Rs.)
1 – 2	5	3	30
2 – 3	4	2	20
3 – 4	6	3	40
3 – 5	4	1	30
4 – 5	3	2	60

ANSWERS

6. Since indirect cost is not given the minimum total cost = normal cost. Normal duration is the minimum cost duration = 40 hrs with total cost 13500. If crashing is done using cost slope then the least duration is 29 hrs with the total cost as 16700. Further crashing is not possible.
8. 11 days, cost Rs. 1760
9. *(a)* 20 days, 12 days, *(b)* Let K be the initial direct cost.

Activity Crashed	No. of days crashed (days)	Current project duration (days)	Direct cost (Rs)	Indirect cost (Rs)	Total cost (Rs)
Nil	Nil	20	K	1200	1200 + K
4 – 5	1	19	K + 40	1140	1180 + K
3 – 4	3	16	K + 85	960	1045 +K
1–4&3 – 4	1 day	15	K + 130	900	1030 + K
2 – 4,1 – 3 & 1 – 4	2 days	13	K + 260	780	1040 +K
1 – 3,1 – 2 & 1 – 4	1 days	12	K + 335	720	1055 + K

10. *(i)* 20 days, 12 days, *(ii)* Let K be the total initial direct cost. Total cost = K + 1020. *(i)* to *(iv)* in the following table is the required minimum crashing cost schedule for 7 days.

Crash		Current duration	Direct cost	Indirect cost	Total cost
	–	20	K	1200	1200+K
(i)	3 – 4 by 3 days	17	K+45	1020	1065+K
(ii)	4–5 by 1 day	16	K+85	960	1045+K
(iii)	3–4 and 1–3 by 1 day each and 1–4 by 2 days	14	K+185	840	1025+K
(iv)	1 – 3,1–4 each by 1 day	13	K + 240	780	1020+K
(v)	1 – 3, 1–4 each by 1 day	12	K + 295	720	1015+K

(iii) optimum duration is 12 days as seen in the table above.

11. Critical path: *(i)* 1 – 2 – 3 – 4 – 5 – 6 – 7

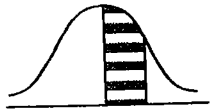
Normal Duration = 33 days ; Normal Cost = Rs.2385000

Additional Expenditure = Rs.3214 by reducing the activity 4 – 5 by 1 day.

APPENDIX

Table 1 : Area under a Standard Normal Curve

An entry in the table is the area under the entire curve which is between $z = 0$ and a positive value of z as shown in the figure. Area for negative values of z are obtained by symmetry.



Z	.00	.01	.02	.03	0.4	0.5	.06	0.7	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	0.019	0.239	0.279	0.319	0.359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2853
0.8	.2881	.2901	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4392	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	4515	.4525	.4535	4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Two Marks Q & A

UNIT – I

1. Operations Research (O.R.)

1. What is O.R? *[MU. BE. Oct 96]*
2. What is the scope of O.R?
3. What are the applications of O.R.?
4. List the uses of O.R.
5. Write a short note on the importance of operations research in production management *[MU. MBA April, 97]*
6. Write a short note on the role of operations research in marketing management. *[MU. MBA Nov.96, Nov. 97, April 98]*
7. Write a short note on the role of operations research in production planning. *[MU. MBA April, 96]*
8. What are the different phases of O.R.?
   [MKU. BE. Nov 97, BNU. Nov 96]
9. What are the characteristics of an O.R. problem?
   [MKU. BE. Nov 97, MS. BE. Nov 96]
10. What is a model?
11. Analogue models. *[MKU. BE. Apr 97, MU. BE. Apr 97]*
12. Iconic models. *[MKU. BE. Apr 97, MU. BE. Oct 96]*
13. What is a Mathematical Model? *[MU. BE. Oct 97]*
14. What are the different types of Models?
15. What are the characteristics of a good model?
16. What are the limitations of a mathematical model?
17. What are the limitations of an O.R. Model? *[MKU. BE. Nov 97]*
18. Explain the general methods of solving O.R. Models.
19. Explain the principles of modelling.

20. State the different types of Models used in O.R. [MU. BE. Oct 97]
21. What are the various phases in the study of operations research? [BRU. BE. Apr 97, Nov 97, Apr 98, MS. BE. Nov 96]
22. Answer the following questions with examples wherever necessary.
    (a) Necessities of OR in industry,
    (b) Fields of application of OR in industry
    (c) Deterministic models,
    (d) Mention atleast eight mathematical models. [MKU. BE. Nov 97]
23. What is an iconic model in the study of operations research? [MU. BE. Oct. '96]
24. Name some important fields in which OR has got wide scope. [BRU. Nov 96]
25. Write two major areas of applications of operations research. [MU. BE. Apr 97]
26. What are the parameters in the construction of an operations research problem? [MU. BE. Apr 97]
27. Differentiate between 'simulation model' and 'mathematical model' of a OR problem. [BRU. BE. Apr 97]
28. Give any two definitions of operations research. [MSU. BE. Apr 97]
29. State any four applications of operations research in industry. [MSU. BE. Apr 97]
30. As applied to OR, distinguish between a quantitative model and a qualitative model. [BRU. Apr 98]
31. List any three practical limitations of the OR techniques. [BRU. BE. Apr 98]
32. What is the underlying principle of operations research? [BRU. BE. Apr 98]
33. What is the meaning of models in operations research? [BRU. BE. Apr 98]

2. Linear Programming Formulation and Graphical Method

1. What do you mean by a general LPP? [MSU. BE. Nov. 97, BNU. BE. Nov 97, Nov 98, MKU. BE. Apr 97, BRU. BE. Apr 98]
2. Define feasible region. [MU. BE. Nov. 96, MKU. BE. Nov 96]
3. Define redundant constraint. [MU. BE. Nov. 96, MKU. BE. Nov 96, Apr 97]
4. When we say that an LPP have *(i)* unique solution *(ii)* an infinite number of solutions. *(iii)* an unbounded solution *(iv)* no solution
5. What do you understand by the term 'constraint'? Explain with examples. [BNU. BE. Nov. 98]
6. When can we use the graphical method for solving a LPP? [BRU. BE. Nov. 97]
7. Solve the following linear programming problem graphically:
   Max $Z = x + y$, subject to $2x + y \le 6$, $x + 2y \le 6$, $x, y \ge 0$.
   [*Ans:* Max $Z = 4$, $x_1 = 2$, $x_2 = 2$] [BRU. BE. Apr.97]
8. Give an example of an LPP having the feasible region as a square. [MU. BE. Oct. 98]

   Ans:
   $$\begin{aligned} \text{Max } Z &= C_1x_1 + C_2x_2 \\ \text{subject to } x_1 &\le b \\ x_2 &\le b \end{aligned}$$
   and $x_1, x_2 \ge 0$, $b > 0$.
9. State the limitations of the graphical method of solving a LPP. [BRU. BE. Apr. 97, Nov. 97, BNU. BE. Apr. 97, Nov. 97]
10. Linear programming is the most widely used method of problem solving. Why? [BRU. BE. Nov. 97]
11. What are the assumptions underlying Linear programming? [MU. BE. Nov.96, MKU. BE. Nov 96, Apr 97]
12. What does the non-negativity restriction mean? [MU. BE. Nov.96, MKU. BE. Nov. 96]

13. Define Iso-profit line. *[MU. BE. Nov. 96, MKU. BE. Nov. 96]*
14. What are the applications of linear programming? *[MKU.BE. Nov. 97, BNU.BE. Nov. 96, MU BE. Nov. 96,97] [BRU. BE. Nov 97, Apr 98]*
15. Name any two major requirements of a LPP. *[BRU BE. Nov. 96, Nov. 97, MKU. BE. Nov. 96, Apr. 97] [BNU. BE. Apr. 98]*
16. Distinguish between a resource and a constraint. *[BNU. BE. Nov. 96, Nov. 97]*
17. Solve the following LPP graphically:
    Max Z = $x + y$
    subject to $2x + y \leq 8$, $2x + 3y \leq 12$ and $x, y \geq 0$. *[BRU. BE. Nov. 96]*
    [*Ans:* Max Z = 5, $x = 3$, $y = 2$]
18. What are decision variables in the construction of operation research problems? *[MU. BE. Oct. 96, MKU. BE. Apr. 97]*
19. List the limitations of LPP. *[MSU. BE. Apr. 97, BNU. BE. Apr. 97, Nov. 97]*
20. What are the characteristics of linear programming? *[MU. BE. Apr. 98]*
21. State any three salient features of a L.P.P. *[BNU. BE. Apr. 98]*
22. What are decision variables in the construction of operations research problems? *[MU. BE. Oct. 96]*

Fill in the blanks.

1. Usage of raw materials is .... or .... to the availability of raw materials. [*Ans:* Less than, equal]
2. Production should always .... or .... to the requirement so as to meet the demand. [*Ans:* greater than, equal]
3. An LPP having more than one optimal solution is said to have .... or .... optimal solutions. [*Ans:* Multiple, alternative]
4. The set of feasible solutions to an LPP is a .... [*Ans:* convex set]
5. The feasible region of an LPP is always .... *[MU. BE: Oct.97]* [*Ans:* convex]

Say True or False.

1. Linear programming is a mathematical technique used to solve the problem of allocating limited resources among the competing activities. [*Ans:* True]
2. The main components of a decision model include the decision alternatives, the constraints and the objective function. [*Ans:* True]
3. The most important step towards solving a decision problem is the construction of an adequate model. [*Ans:* True]
4. A typical LPP must have atleast two decision alternatives. [*Ans:* True]
5. The number of alternatives in a LPP is typically finite. [*Ans:* False]
6. An LPP that has infinity of alternatives is usually unsolvable. [*Ans:* False]
7. A feasible alternative in a LPP must satisfy all the restrictions of the problem [*Ans:* True]
8. Linear programming models can be applied only in those situations where the constraints and the objective function are linear. [*Ans:* True]
9. Constraints specify the upper limit on the availability of a resource or a lower limit on necessary levels to achieve. [*Ans:* True]
10. LPP does not take into consideration the effect of time and uncertainty. [*Ans.* True]
11. LPP deals with problems involving only a single objective. [*Ans:* True]
12. Graphical method of linear programming is useful when the problem involves two variables. [*Ans:* True]
13. Optimal solution in the graphical method of linear programming always lies at one of the vertices in the feasible region. [*Ans:* True]

14. An LPP may have more than one optimal solution. [*Ans:* True]
15. Constraints appear as straight lines when plotted on a graph. [*Ans:* True]
16. An LPP is said to be infeasible if it has no solution that satisfies all constraints [*Ans:* True]
17. An LP solution when permitted to be infinitely large is called unbounded. [*Ans:* True]

3. General Linear Programming Problems – Simplex Methods

1. Define feasible solution.
2. Define optimal solution.
3. Define infeasible solution.
4. Define slack variables. *[MU. BE. Oct. 98]*
5. Define surplus variables. *[MKU. BE. Apr. 97, MSU. BE. Nov. 96, Apr. 97]*
6. Define unrestricted variables.
7. What do you mean by canonical form of a LPP? *[MU. B.Tech Oct 96]*
8. What do you mean by standard form of a LPP? *[MKU. BE. Nov. 96]*
9. Write the standard form of LPP in Matrix form. *[MU. BE. Oct. 97, MSU. BE. Nov. 96]*
10. Write the canonical form of LPP in Matrix form.
11. Is it meaningful to have a LPP for which the requirement vector $b = 0$?
12. State the characteristics of the canonical form.
13. State the characteristics of the standard form.
14. Express the following LPP in canonical form:

$$\begin{array}{lrcl} & \text{Min } Z & = & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 & \le & 1 \\ & 2x_1 + 3x_2 & = & 4 \\ & x_1,\ x_2 & \ge & 0. \end{array}$$

$$\begin{array}{lrcl} \textit{Ans:} & \text{Max } Z^* & = & -2x_1 - x_2 \\ \text{subject to} & x_1 + x_2 & \le & 1 \\ & 2x_1 + 3x & \le & 4 \\ & -2x_1 - 3x_2 & \le & -4 \\ & x_1,\ x_2 & \ge & 0. \end{array}$$

15. Express the following LPP in standard form:

$$\begin{array}{lrcl} & \text{Minimize } Z & = & x_1 + 3x_2 \\ \text{subject to} & 2x_1 + 3x_2 & \le & 2 \\ & x_1 + x_2 & \ge & -2 \\ & 3x_1 + x_2 & \le & -3 \\ & x_1, x_2 & \ge & 0. \end{array}$$

$$\begin{array}{lrcl} \textit{Ans:} & \text{Max } Z^* & = & -x_1 - 3x_2 \\ \text{subject to} & 2x_1 + 3x_2 + s_1 & = & 2 \\ & -x_1 - x_2 + s_2 & = & 2 \\ & -3x_1 - x_2 - s_3 & = & 3 \\ & x_1, x_2, s_1, s_2, s_3 & \ge & 0. \end{array}$$

16. Express the following LPP in standard form:

$$\begin{array}{lrcl} & \text{Minimize } Z & = & 2x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 & \le & 5 \\ & 3x_1 + x_2 & \ge & 3 \end{array}$$

$x_1 \ge 0$, x_2 is unrestricted.

$$\begin{array}{lrcl} \textit{Ans:} & \text{Max } Z^* & = & -2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} & x_1 + x_2' - x_2'' + s_1 & = & 2 \\ & 3x_1 + x_2' - x_2'' - s_2 & = & 3 \\ & x_1,\ x_2',\ x_2'',\ s_1,\ s_2 & \ge & 0. \end{array}$$

17. Define basic solution to an LPP. *[MU. BE. Oct.98]*
18. What do you mean by basic variables and non-basic variables?
19. Find all the basic solutions of the following LPP.

$$\begin{array}{lrcl} & \text{Maximize } Z & = & x_1 + x_2 + 2x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 & = & 4 \\ & 2x_1 + x_2 + 5x_3 & = & 5 \\ & x_1, x_2, x_3 & \ge & 0. \end{array}$$

Ans:	Basic	Non-basic
1.	$x_1 = 2,\ x_2 = 1$	$x_3 = 0$
2.	$x_1 = 5,\ x_3 = -1$	$x_2 = 0$
3.	$x_2 = \frac{5}{3},\ x_3 = \frac{2}{3}$	$x_1 = 0.$

20. Write the general mathematical model of LPP in matrix form. ***[MSU. BE. Nov. 96, MU. BE. Oct.97]***
21. What is meant by optimality test? ***[MU. BE. Nov.96]***
22. Define *(i)* feasible solution
    (ii) basic solution and
    (iii) basic feasible solution of a LPP.
    [MKU. BE. Nov. 97, MU. MBA. Apr. 95, Nov 96, BRU. BE. Apr 98, MSU. BE. Nov. 96, Apr. 97, Nov. 97]
23. Define *(i)* Non-degenerate basic solution
    (ii) Degenerate basic solution.
24. What is meant by degeneracy of a LPP?
    [MKU. BE. Nov. 96, Apr. 97, MU. BE., Oct. 97, Nov. 96 & BRU. BE. Nov. 97, BNU. BE. Nov 96, MU. MBA. Apr 97]
25. Explain the term feasible solution with an example. ***[BNU. BE. Nov. 96]***
26. What do you mean by an optimal solution of a LPP? ***[BRU. BE. Nov. 96]***
27. State the central problem of linear programming. ***[MU. MBA. Apr. 95]***
28. What are slack and surplus variables? ***[MU. MBA. Apr. 95, BE. Apr. 98, BRU. BE. Apr. 98]***
29. What is the Physical interpretation of a slack variable? ***[BNU. BE. Nov. 96]***
30. What is the difference between slack variable and surplus variable? ***[BNU. BE. Nov. 96, Nov. 97]***
31. Find the non-degenerate basic feasible solution of the LPP:

$$\text{Maximize } Z = x_1 + x_2 + x_3$$

$$\begin{aligned} \text{subject to} \quad x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 &= 0 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[*Ans:* No non-degenerate basic feasible solution]

32. Which of the following solutions are feasible, infeasible, degenerate and non-degenerate?
    (i) $x_1 = 2,\ x_2 = 5,$ *(ii)* $x_1 = 2,\ x_2 = -5$
    (iii) $x_1 = 0,\ x_2 = 3,$ *(iv)* $x_1 = 5,\ x_2 = 3$
    [*Ans:* *(i)* feasible *(ii)* infeasible
    (iii) degenerate *(iv)* non-degenerate]
33. Define a basic solution to a given system m simultaneous linear equations in n unknowns.
34. How many basic feasible solutions are there to a given system of 3 simultaneous linear equations in 4 unknowns? [*Ans:* $4C_3 = 4$]
35. Find all the basic feasible solutions of the simultaneous equations
    $2x_1 + 6x_2 + 2x_3 + x_4 = 3,\ 6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$
    [*Ans:* *(i)* $x_1 = 0,\ x_2 = \frac{1}{2},$ *(ii)* $x_2 = \frac{1}{2}, x_3 = 0$
    (iii) $x_2 = \frac{1}{2},\ x_4 = 0$
    Note that all the three solutions are degenerate.]
36. State the necessary and sufficient condition for a basic feasible solution of a LPP to be an optimum solution.
37. What is key column? How is it selected?
38. What is key row? How is it selected?
39. What is the test of optimality in the simplex method?
    And how is the solution read from simplex tableau?
    [MU. BE. Nov. 96, MKU. BE. Nov. 96, BRU. BE. Apr. 95]
40. What information can be derived from the optimum simplex tableau?
41. State the condition for an unbounded solution of a LPP? ***[MSU. BE. Nov. 96]***
42. When does the simplex arithmetic indicate that the LPP has unbounded solution? ***[BRU.BE. Apr. 97, MKU.BE. Nov.97]***
43. What is infeasible solution? How is it identified in the simplex tableau? ***[MU. BE. Apr 99]***

44. How can you find whether the solution to a LPP is unique or not?

[MKU. BE. Nov. 97]

45. How is the presence of more than one optimal solution found out from the optimal simplex table of a LPP?

46. How will you find whether a LPP has got an alternate optimal solution or not from the optimal simplex table?

[BRU. BE. Nov 96]

47. Why the artificial variables are called so? *[BRU. BE. Nov 96]*

48. What do you mean by Big–M Method? *[BRU B. Tech Oct 96]*

49. When does an LPP possess a pseudo optimal solution?

50. Define artificial variables.

[BRU. BE. Apr 97, Apr 98, MU. BE. Apr 99, MKU. BE. Nov. 96, Apr. 97, MSU. BE. Apr. 97]

51. What is the use of artificial variables?

[MU. BE. Oct. 97, MKU. BE. Apr. 97]

52. Write the solution of the maximization LPP from the following optimum simplex table. Also state the nature of the solution.

		C_j	(−1	2	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
2	x_2	2	$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	s_2	3	$\frac{-1}{2}$	0	1	1
$(Z_j - C_j)$		4	2	0	1	0

[*Ans:* max $Z = 4$, $x_1 = 0$, $x_2 = 2$ unique solution]

53. Write the nature of the solution of the maximization LPP from the following simplex table.

		C_j	(2	1	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
2	x_1	30	1	0	−1	1
1	x_2	20	0	1	−2	1
$(Z_j - C_j)$		80	0	0	−4	3

[*Ans:* Unbounded solution]

54. What are slack variables in the solution of LPP?

55. Point out basic approach used with two phase simplex method. *[BNU. BE. Nov 98]*

56. For the solution of any LPP by simplex method, existence of an initial feasible solution need not be assumed. Is this true? *[MU. BE. Oct. 96]*

57. While solving a LPP by simplex method, how will you find that the problem has got an unbounded solution? *[BRU. BE. Apr. 97, BNU. BE. Apr. 97]*

58. What is the difference between feasible solution and basic feasible solution? *[MKU. BE. Nov. 96, MU. BE. Apr. 98]*

59. What do you mean by two phase method for solving a given LPP? *[MKU. Nov. 96]*

60. What the disadvantages of Big-M method over two phase method? *[MRU. BE. Apr. 97]*

61. Define in the usual notation net salvation of the simplex procedure. *[MSU. BE. Apr. 97]*

Fill in the blanks.

1. A basic solution is said to be .... if none of the basic variable is zero [*Ans:* Non – degenerate]
2. A basic solution is said to be .... if one or more of the basic variable is zero [*Ans:* Degenerate]
3. A feasible solution which is also basic is called a .... [*Ans:* Basic feasible solution]
4. The leaving variable row is called the .... [*Ans:* Key row or pivot row or pivot equation]
5. The entering variable is the non-basic variable corresponding to the .... [*Ans:* Most negative value of $(Z_j - C_j)$]
6. The entering variables column is called the .... [*Ans:* Key column or pivot column]
7. The leaving variable is the basic variable corresponding to the .... [*Ans:* Minimum ratio θ, where $\theta = \frac{X_{Bk}}{a_{ir}}$, $a_{ir} > 0$]
8. The intersection of the pivot column and pivot row is called the .... [*Ans:* Pivot element or key element]

9. Constraints involving "equal to sign" do not require use of .... or .... variables. [*Ans:* slack, surplus]
10. Constraint involving '$\geq$' sign are reduced to equations by introducing .... [*Ans:* surplus variable]
11. The coefficients of artificial variables are –M in the objective function for .... [*Ans:* Maximization problems]
12. The coefficients of artificial variables are +M in the objective function for .... [*Ans:* Minimization problems]
13. Constraint involving '$\leq$' sign are reduced to equations by adding .... *[MU. BE. Apr 97]* [*Ans:* slack variable]
14. A feasible solution which optimizes the objective function is called .... [*Ans:* An optimal solution]
15. A constraint of the type $\geq$ can be converted to an equation by subtracting a .... variable from the left side of the constraint. *[MU. BE. Oct. 96]* [*Ans:* Surplus]
16. The number of basic feasible solutions to the system $2x_1 - x_2 + 4x_3 = 5,\ x_1 - 2x_2 + 2x_3 = 1$ is ...... *[MU. BE. Oct. 97]* [*Ans:* one]
17. The method designed to over come the difficulty arising in computer implementation of Big M – Method is known as .... *[MU. BE. Apr. 97]* [*Ans:* Two phase method]
18. In solving an LPP by the simplex method .... variable is associated with equality type constraint *[MU. BE. Apr. 98]* [*Ans:* Artificial]

State whether the following statements are 'True' or 'False'

1. Decision variables in a standard LPP should have either zero or positive values. [*Ans:* True]
2. Objective function specifies the dependent relationship between the decision objective and the decision variables. [*Ans:* True]
3. For the solution of any LPP by simplex method, existence of an initial feasible solution need not be assumed. Is this true? *[MU. BE. Oct 96]* [*Ans:* False]

4. Most mathematical OR models are solved by using iterative procedures. [*Ans:* True]
5. If a LPP has a feasible solution, then it also has a basic feasible solution [*Ans:* True]
6. There exists only a finite number of basic feasible solutions to a LPP. [*Ans:* True]
7. Every equality constraint can be replaced equivalently by two inequalities. [*Ans:* True]
8. The maximization of a function f subject to a set of constraints is exactly equivalent to the minimization of $g = -f$ subject to the same set of constraints, except that Min $g = -$ max f. [*Ans:* True]
9. In an LPP with m constraints, a simplex iteration may include more than m positive basic variables. [*Ans:* False]
10. A simplex iteration (basic solution) may not necessarily coincide with a feasible extreme point of the solution space. [*Ans:* False]
11. In the simplex method, all variables must be non-negative [*Ans:* True]
12. In the simplex method, the optimality conditions for the maximization and minimization problems are different. [*Ans:* True]
13. In the simplex method, the feasibility conditions for the maximization and minimization problems are different. [*Ans:* False]
14. If the leaving variable does not correspond to the minimum ratio, atleast one basic variable will definitely become negative in the next iteration. [*Ans:* True]
15. The selection of the entering variable from among the current non–basic variables as the one with the most negative value of $(z_j - c_j)$ guarantees the most increase in the objective value in the next iteration. [*Ans:* False]
16. The optimality condition always guarantees that the next solution will have a better objective value than in the immediately preceding iteration. [*Ans:* False]

17. In a simplex iteration, the pivot element can be zero or negative. [*Ans:* False]
18. An artificial variable column can be dropped all together from the simplex tablaue once the variable becomes non-basic. [*Ans:* True]
19. The two-phase method and M – technique require the same number of iterations for solving a LPP. [*Ans:* True]
20. Degeneracy can be avoided if redundant constraints can be deleted. [*Ans:* True]
21. The simplex method may not move to an adjacent extreme point if the current solution is degenerate. [*Ans:* True]
22. If a current solution is degenerate, then the solution in the next iteration will necessarily be degenerate. [*Ans:* False]
23. If the solution space is unbounded, the objective value always will always be unbounded. [*Ans:* False]
24. An LPP may have a feasible solution even though an artificial appears at a positive level in the optimal iteration. [*Ans:* False]

4. Revised Simplex Method and Bounded Variable Method

1. Write a brief note on revised Simplex Method.
2. What are the advantages of revised simplex method, when compared to simplex method? *[MU, MCA, Nov. '96, BRU BE Nov. '96]*
3. In what way do you consider the revised simplex method is superior to the usual simplex method?
4. Under what circumstance would you apply the bounded variable method to solve a LPP?
5. How will you select the leving variable in the upper bound algorithm?
6. What is the difference between simplex method and revised simplex method? *[MU., B.E., Oct. '97]*
7. What the disadvantages of ordinary simplex method over revised simplex method? *[MSU., BE., Nov. '96]*
8. List the two main merits of using revised simplex method for solving a LPP with the aid of a digital computer. *[BNU., B.E., Apr. '97]*

5. Sensitivity Analysis

1. What is sensitivity analysis? What does it signify?
2. What advantages can be realised from sensitivity analysis?
3. What is the purpose of sensitivity analysis?

Fill in the blanks.

1. By introducing one more constraint in the maximization type of LPP, the optimal value is .... [*Ans:* unchanged]
2. Addition of a new constraint may affect only the .... of the current solution. [*Ans:* Feasibility]
3. By introducing a new variable in the maximizing type of LPP, the optimal value is .... [*Ans:* Improved]
4. Deletion of a basic variable may affect the ...... of the current solution. [*Ans:* optimality]
5. The deletion of an unbinding constraint will not affect the ...... [*Ans:* optimal solution]
6. Deletion of a basic variable ...... the feasibility and optimality of the current solution. [*Ans:* does not affect]
7. Generally the addition of a new constraint will never ...... the value of the objective function. [*Ans:* improve]

State True or False.

1. Sensitivity analysis determines how sensitive the optimum solution is to making changes in the original model. [*Ans:* True]
2. Sensitivity analysis forms an integral part of construction a LPP. [*Ans:* True]
3. Sensitivity analysis is normally carried out after the optimal solution is obtained. [*Ans:* True]
4. Optimum solution to a LPP is not very sensitive to the changes in right hand side values of the constraints. [*Ans:* True]
5. Changes in the coefficients of the variables in the objective function can affect optimality of the solution. [*Ans:* True]

6. The variation in the right hand side of constraints will affect only the feasibility of the solution. [*Ans:* True]
7. Addition of new constraints may affect only the feasibility of the current solution. [*Ans:* True]
8. The addition of a new constraint can improve the objective value. [*Ans:* False]
9. The addition of a new constraint cannot improve the objective value. [*Ans:* True]
10. Any change in the coefficients of the non-basic variables in the objective function will affect only its net evaluation coefficients and not others. [*Ans:* True]
11. Any change in the coefficients of the basic variables in the objective function will affect only its net evaluation coefficients and the value of the objective function. [*Ans:* True]
12. Any change in the coefficients a_{ij} of the non-basic variables in the constraints will not affect the feasibility of the solution, but may affect the optimality of the current solution. [*Ans:* True]
13. Any change in the coefficients a_{ij} of basic variables in the constraints may affect both the feasibility and optimality of the current solution. [*Ans:* True]
14. The addition of a new variable can improve the objective value. [*Ans:* True]
15. The addition of a new variable can never worsen the value of the objective function. [*Ans:* True]
16. If we change both the right side of the constraints and the coefficients of the objective function, we can destroy both the optimality and feasibility of the solution. [*Ans:* True]
17. Deletion of a non-basic variable does not affect the feasibility and optimality the current solution. [*Ans:* True]
18. Deletion of a basic variable may affect the optimality of the current solution. [*Ans:* True]

UNIT - II

6. Duality in LPP and Networks

1. What is a dual problem in linear programming? *[MSU. BE. Nov. 97, MU. BE. Apr. 98]*
2. What do you mean by primal – dual problems?
3. What is duality in LPP? Explain its application. *[MU. MCA. May 92]*
4. Give economic interpretation of dual. *[MU. BE. Nov 96, MKU. B.E. Nov. 96, Nov. 97]*
5. State the importance of the duality concept in LPP.
6. What are the two important forms of primal–dual pairs?
7. State the fundamental theorem of duality. *[MU. BE. Oct 96, MSU. BE. Nov. 97, BNU. BE. Apr. 98]*
8. Explain the importance of duality in LPP.
9. State the necessary and sufficient condition for any LPP and its dual to have an optimal solution.
10. State the existence theorem of duality.
11. State the complementary slackness theorem of duality.
12. Explain the primal-dual relationship. *[MKU. B.E. Apr. 97, Nov. 97]*
13. How do you convert the dual into primal? *[MKU. B.E. Nov. 97]*
14. Write down the dual of the following LPP.

$$\text{Maximize } Z = 5x_1 + 12x_2 + 4x_3$$

$$\text{subject to} \quad \begin{aligned} x_1 + 2x_2 + x_3 &\le 5 \\ 2x_1 - x_2 + 3x_3 &= 2 \\ x_i &\ge 0. \end{aligned}$$

[BRU. BE. Nov 96]

Ans: Min $W = 5y_1 + 2y_2$

$$\text{subject to} \quad \begin{aligned} y_1 + 2y_2 &\ge 5 \\ 2y_1 - y_2 &\ge 12 \\ y_1 + 3y_2 &\ge 4 \\ y_1 &\ge 0, \; y_2 \text{ unrestricted.} \end{aligned}$$

15. Write the dual of Max $Z = 3x_1 + 17x_2 + 9x_3$

subject to
$$\begin{aligned} x_1 - x_2 + x_3 &\geq 3 \\ -3x_1 + 2x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$
[MU. B.Tech Oct 96]

Ans: Min $W = -3y_1 + y_2$

subject to
$$\begin{aligned} -y_1 - 3y_2 &\geq 3 \\ y_1 &\geq 17 \\ -y_1 + 2y_2 &\geq 9 \\ y_1, y_2 &\geq 0. \end{aligned}$$

16. Obtain the dual of the LPP

Minimize $Z = 4x_1 + x_2$

subject to
$$\begin{aligned} x_1 + x_2 &\geq 10 \\ 2x_1 + 3x_2 &\geq 24 \\ x_1, x_2 &\geq 0. \end{aligned}$$
[BNU. BE. Nov 96]

Ans: Max $W = 10y_1 + 24y_2$

subject to
$$\begin{aligned} y_1 + 2y_2 &\leq 4 \\ y_1 + 3y_2 &\leq 1 \\ y_1, y_2 &\geq 0. \end{aligned}$$

17. Write the dual of the problem

Maximize $Z = 2x_1 + 4x_2 + x_3$

subject to
$$\begin{aligned} x_1 + 2x_2 - x_3 &\leq 5 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ -x_1 + 2x_2 + 2x_3 &\geq 1 \\ x_1, x_2 &\geq 0, \ x_3 \text{ unrestricted.} \end{aligned}$$
[BRU. BE. Nov 96]

Ans: Min $W = 5y_1 + 2y_2 - y_3$

subject to
$$\begin{aligned} y_1 + 2y_2 + y_3 &\geq 2 \\ 2y_1 - y_2 - 2y_3 &\geq 4 \\ -y_1 + 2y_2 - 2y_3 &= 1 \\ y_1, y_3 &\geq 0, \ y_2 \text{ unrestricted.} \end{aligned}$$

18. When do you use dual simplex method? *[BRU. BE. Apr. 97, BNU. BE. Nov. 98]*

19. What is the difference between regular simplex method and dual simplex method? *[BRU. M.Sc. 83]*

20. What do you mean by shadow prices? *[MU. BE. Nov. 97, MKU. BE. Nov. 96]*

21. State the optimality condition in dual simplex method.

22. State the feasibility condition in dual simplex method.

23. Obtain the dual of the following linear programming problem:

Maximize $Z = 3x_1 + x_2 + x_3 - x_4$

subject to the constraints
$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &\leq 5 \\ x_1 + x_2 &= 1 \\ x_3 - x_4 &\leq -5 \\ \text{and } x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$
[MU. BE. Apr 97]

Ans: Min $W = 5y_1 + y_2 - 5y_3$

subject to
$$\begin{aligned} y_1 + y_2 &\geq 3 \\ 2y_1 - y_2 &\geq 1 \\ -3y_1 + y_3 &\geq 1 \\ 4y_1 - y_3 &\geq -1 \\ y_1, y_3 &\geq 0, \ y_2 \text{ unrestricted.} \end{aligned}$$

24. Write the dual of: Max $Z = 2x_1 + 3x_2 + x_3$

subject to
$$\begin{aligned} 4x_1 + 3x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 5x_3 &= 4 \\ \text{and } x_1, x_2, x_3 &\geq 0. \end{aligned}$$
[MU. BE. Oct 97]

Ans: Min $W = 6y_1 + 4y_2$

subject to
$$\begin{aligned} 4y_1 + y_2 &\geq 2 \\ 3y_1 + 2y_2 &\geq 3 \\ y_1 + 5y_2 &\geq 1 \end{aligned}$$
and y_1, y_2 are unrestricted.

25. Write the dual of:

$$\text{Max Z} = x_1 + 2x_2$$

subject to

$$2x_1 + 3x_2 \le 8$$
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

[MU. BE. Oct 98]

Ans.

$$\text{Min W} = 8y_1 + 4y_2$$

subject to

$$2y_1 + y_2 \ge 1$$
$$3y_1 + y_2 \ge 2$$
$$y_1, y_2 \ge 0$$

26. Obtain the dual of the given L.P.P.

$2x_1 + x_3 \le 4$; $x_1 + x_2 + 2x_3 \le 10$; $x_1, x_2, x_3 \ge 0$

Maximize $Z = x_1 + 2x_2 + 3x_3$ *[BNU. BE. Nov. 98]*

Ans. Min $W = 4y_1 + 10y_2$

subject to

$$2y_1 + y_2 \ge 1$$
$$y_2 \ge 2$$
$$y_1 + 2y_2 \ge 3$$
$$y_1, y_2 \ge 0$$

27. Write the dual of Max $Z = x_1 + x_2 + x_3$

subject to

$$x_1 - 3x_2 + 4x_3 = 5$$
$$x_1 - 2x_2 \le 3$$
$$2x_2 - x_3 \ge 4$$

$x_1, x_2 \ge 0$, x_3 unrestricted in sign. *[MU. BE. Apr. 99]*

$2x_1 + x_3 \le 4$; $x_1 + x_2 + 2x_3 \le 10$; $x_1, x_2, x_3 \ge 0$

[***Ans.*** Min $W = 5y_1 + 3y_2 - 4y_3$

subject to $y_1 + y_2 \ge 1$, $-3y_1 - 2y_2 - 2y_3 \ge 1$, $4y_1 + y_3 = 1$

and $y_2, y_3 \ge 0$, y_1 unrestricted.]

28. Compare the starting solution used with conventional simplex method with that of dual simplex method. *[BNU. BE. Apr. 97]*

29. Differentiate between primal problem and dual problem. *[BNU. BE. Apr. 98]*

Fill in the blanks.

1. The Maximization problem in the primal becomes the ...... problem in the dual. [***Ans:*** Minimization]
2. The two important forms of primal-dual pairs are ...... and .... [***Ans:*** Symmetric form, Unsymmetric form]
3. If the primal contains m constraints and n variables, then the dual will contain ......and ...... [***Ans:*** n constraints, m variables]
4. The constants $c_1, c_2 \dots c_n$ in the objective function of the primal appear in the ...... of the dual. [***Ans:*** Constraints]
5. The constants $b_1, b_2 \dots b_m$ in the constraints of the primal appear in the ...... of the dual. [***Ans:*** Objective function]
6. If the k^{th} constraint of the primal problem is an equality, then the corresponding dual variable y_k is ...... [***Ans:*** Unrestricted in sign]
7. If the k^{th} variable of the primal problem is unrestricted in sign, the corresponding k^{th} constraint in the dual problem will be ...... [***Ans:*** an equality]
8. The dual of the dual is .... [***Ans:*** primal] *[MU. BE. Apr. 98]*
9. The method used to solve LPP without the use of artificial variables is called the ...... method. *[MU. BE. Oct.96]* [***Ans:*** dual simplex]
10. If either the primal or the dual problem has an unbounded solution, then the other problem has ...... [***Ans:*** no feasible solution]
11. If the dual has no feasible solution, then the primal ...... [***Ans:*** has no feasible solution]
12. If a primal variable is positive, then the corresponding dual constraint is ...... at the optimum. [***Ans:*** an equation]
13. If a dual variable is positive, then the corresponding primal constraint is ...... at the optimum [***Ans:*** an equation]
14. If a primal constraint is a strict inequality, then the corresponding dual variable is ...... at the optimum. [***Ans:*** zero]
15. If a dual constraint is a strict inequality, then the corresponding primal variable is ...... at the optimum. [***Ans:*** zero]

Say 'True or False'.

1. In duality, the values of the dual variable are called shadow prices. [*Ans:* True]
2. If either the primal or the dual problem has an unbounded solution, then the other problem has no feasible solution. [*Ans:* True]
3. If dual has no feasible solution, then primal also admits no feasible solution. [*Ans:* True]
4. If i^{th} dual constraint is multiplied by –1, then ith primal variable computed from the $(Z_j - C_j)$ row of the dual problem must be multiplied by –1. [*Ans:* True]
5. The value of the objective function z for any feasible solution of the primal is less than or equal to the value of the objective function w for any feasible solution of the dual. [*Ans:* True]
6. If a primal variable is positive, then the corresponding dual constraint is an equation at the optimum and vice versa. [*Ans:* True]
7. If a primal constraint is a strict inequality, then the corresponding dual variable is zero at the optimum and vice versa. [*Ans:* True]
8. The dual simplex method starts with a basic infeasible but optimal solution and works towards feasibility. [*Ans:* True]
9. In dual simplex method, we first determine the leaving variable and then the entering variable. [*Ans:* True]
10. There is no difference between the regular simplex method and the dual simplex method. [*Ans:* False]

7. Transportation Models

1. What do you mean by transportation problem?
2. Define feasible and basic feasible solution of a transportation problem. ***[BRU. BE. Nov 97]***
3. Write down the basic steps involved in solving a transportation problem. ***[BRU. BE. Nov 97]***
4. Describe a situation where we encounter transportation problem. ***[MU. BE. Oct 96]***
5. How the problem of degeneracy arise in a transportation problem? ***[MU. BE. Oct 97]***
6. List any three approaches used with transportation problem, for determining the starting solution. ***[BNU. BE. Nov 96]***
7. What is the use of MODI Method? ***[BNU. BE. Nov 97]***
8. What is the purpose of MODI Method? ***[BNU. BE. Nov 96, MSU. BE. Nov 97]***
9. Explain MODI method of revising a non-optimal solution of a transportation problem. ***[BNU. BE. Nov 96, BRU. BE. Nov. 96, Nov. 97]***
10. Define degenerate and non-degenerate basic feasible solution of a transportation problem. ***[BRU. BE. Nov. 97, MU. BE. Apr 98]***
11. Define the optimal solution to a transportation problem.
12. State the necessary and sufficient condition for the existence of a feasible solution to a transportation problem.
13. When does a transportation problem has a unique solution?
14. What do you understand by degeneracy in a transportation problem? ***[BNU. BE. Nov 96, Apr 98, BRU. BE. Nov 96, Apr 98, MSU. BE. Nov 96, Apr 97, MKU. BE. Nov 97, MU. BE. Apr 99]***
15. Explain how degeneracy in a transportation problem may be resolved?

16. What is the difference between LPP and transportation problem?
17. What is unbalanced transportation problem? How to solve it? ***[MSU. BE. Nov 97, MKU. BE. Apr 97]***
18. Give mathematical formulation of a transportation problem.
19. When does a transportation problem has an alternate optimal solution? ***[BRU. BE. Apr 97, Apr 98]***
20. How do you convert an unbalanced transportation problem into a balanced one?
21. How many basic variables will be there for a balanced transportation problem with 3 rows and 3 columns? ***[MU. BE. Oct. 97]*** [*Ans:* 3 + 3 – 1 = 5]
22. Find a basic feasible solution to the following transportation problem using north west corner rule.

		To E	F	G	H	Availability
	A	4	8	10	16	100
From	B	7	2	3	1	200
	C	5	9	11	2	300
Requirement		160	240	105	95	

[MU. BE. Apr 99]

[*Ans:* $x_{11} = 100$, $x_{21} = 60$, $x_{22} = 140$, $x_{31} = 100$, $x_{32} = 105$, $x_{33} = 95$]

23. Determine an initial basic feasible solution to the following transportation problem by least cost method.

		To 1	2	3	Availability
	A	10	13	6	10
From	B	16	7	13	12
	C	8	22	2	8
Requirement		6	11	13	

[MU. BE. Apr 99]

[*Ans:* $x_{11} = 5$, $x_{13} = 5$, $x_{21} = 1$, $x_{22} = 11$, $x_{33} = 8$]

24. What are the merits and limitations of using the north west corner method for obtaining the initial basic feasible solution of a transportation problem? ***[BNU. BE. Nov. 98]***
25. What is the use of Vogel's Approximation Method? ***[BNU. BE. Nov. 98]***
26. Explain North-West Corner rule. ***[BNU. BE. Apr 97]***
27. Name the rules of writing initial basic feasible solutions of a transportation problem. ***[MSU. BE. Nov. 96, Apr. 97, BNU. BE. Nov. 97]***
28. State the objective of the transportation problem. ***[BNU. BE. Apr. 98]***
29. Vogeles approximation method results in the most economical initial basic feasible solution. Is this true? ***[MU. BE. Apr. 97]***
30. Give three reasons why LPP solution techniques, is not made use for solving a transportation problem. ***[BNU. BE. Apr. 97, Nov. 97]***
31. Give step by step procedure of VAM. ***[BRU. BE. Apr. 97]***
32. What are the merits of VAM over north west corner method. ***[BNU. BE. Nov. 97]***

State "True or False".

1. A transportation problem is special case of linear programming problem. [*Ans:* True]
2. Only those problems where total demand equals the total supply can be solved by the technique of transportation model. [*Ans:* True]
3. A balanced transportation model will always have a feasible solution. [*Ans:* True]
4. A transportation model that is initially unbalanced may require the addition of both a dummy source and a dummy destination to effect balancing. [*Ans:* False]
5. In a transportation problem, north west corner rule gives a better starting solution than VAM method. [*Ans:* False]
6. In a transportation model, VAM method gives a better starting solution than Least Cost method. [*Ans:* True]
7. In a transportation model, north west corner rule starting solution is recommended because it ensures that there will be $(m + n - 1)$ allocations. [*Ans:* False]

8. A transportation problem can always be represented by balanced model. [*Ans:* True]
9. In the solution of the transportation model, the amounts shipped from a dummy source to the destinations actually represent shortage at the destinations. [*Ans:* True]
10. The transportation model is restricted to dealing with a single commodity only. [*Ans:* False]
11. The transportation technique essentially uses the same steps of the simplex method. [*Ans:* True]
12. If a constant value is added to every cost element c_{ij} in the transportation table, the optimal values of the variables x_{ij} will change. [*Ans:* False]
13. A balanced transportation model may not have any feasible solution. [*Ans:* False]
14. Degeneracy in a transportation model occurs when the sum of sources and the destinations equals the number of occupied cells. [*Ans:* False]
15. Alternate + and – signs are assigned on the closed loop by moving counter clockwise and starting with a + sign in the selected unoccupied cell for improving the current solution. [*Ans:* True]
16. Allocations to the dummy destinations represent the surplus at the supply points. [*Ans:* True]
17. Allocations to the dummy sources represent the number of units to be produced by use of over time. [*Ans:* True]
18. An alternate optimal solution to a transportation problem is said to exist when one or more of the unoccupied cells have zero value for the net cost change in the optimal solution. [*Ans:* True]

Fill in the blanks.

1. A transportation problem is said to be balanced if ....
   [*Ans:* $\Sigma a_i = \Sigma b_j$]
2. A transportation problem is said to be unbalanced if ....
   [*Ans:* $\Sigma a_i \neq \Sigma b_j$]
3. For any transportation problem, the coefficients of all x_{ij} in the constraints are .... [*Ans:* Unity]
4. Degeneracy in a $m \times n$ transportation problem occurs when the number of occupied cells is .... than ....
   [*Ans:* Less, $m+n-1$]

5. A solution that satisfies all conditions of supply and demand but it may or may not be optimal is called ....
   [*Ans:* Initial feasible solution]
6. To solve degeneracy, an unoccupied cell with .... cost is converted into occupied cell by assigning .... to it.
   [*Ans:* lowest, ∈ infinitely small amount]
7. In a northwest corner rule, if the demand in the column is satisfied, one must move to the .... cell in the next ....
   [*Ans:* right, column]
8. Row wise and column wise difference between two minimum costs is calculated under .... method. [*Ans:* VAM]
9. An optimum solution results when net cost change values of all unoccupied cells are .... [*Ans:* non–negative]
10. The number of non-basic variables in the balanced transportation problem with m rows and n columns is ....
    [MU. BE. Apr. 98] [*Ans:* $mn-(m+n-1)$]
11. The number of non-basic variables in the balanced transportation problem with 4 rows and 5 columns is ....
    [MU. BE. Oct 96] [*Ans:* 12]
12. MODI method associated with transportation problem, MODI stands for .... *[MU.BE.Oct 97]* [*Ans:* Modified distribution]
13. The number of basic variables in the balanced transportation problem with 3 rows and 5 columns is ....
    [MU.BE.Apr 97] [*Ans:* 7]

8. Assignment

1. What are assignment problems? Describe Mathematical formulation of an assignment problem?
   [MU. BE. Apr 93, Oct 96, BRU. BE. Nov 96, MU. MBA. Nov 96,]
2. Distinguish between transportation model and assignment model.
   [MU. BE. Nov 94, MKU. BE. Nov. 96, BNU. BE. Apr. 98]
3. Explain how the assignment problem can be treated as a particular case of transportation problem? Why this method is not preferred?
   [MU. MBA Nov. 95, Apr. 97, MKU. BE. Apr. 97]
4. Explain the steps in the Hungarian Method used for solving assignment problems.
   [MU. MBA. Apr 95, BRU. BE. Nov 96, MU. MCA. Nov 95, BNU. BE. Apr. 98]

5. Define an unbalanced assignment problem and describe the steps involved in solving it.
6. Explain how maximization problems are solved using assignment model technique?
7. What do you understand by restricted assignments? Explain how should one overcome it?
8. Enumerate the steps to solve an unbalanced profit maximization problem containing one or more restricted assignments?
9. Is it possible to have more than one optimal solution to an assignment problem? How is the presence of an alternate solution established?
10. What is the difference between assignment problem and travelling salesman problem?
11. What is the objective of the travelling salesman problem? *[MU. BE. Oct 96]*
12. What is travelling salesman problem? *[BRU. BE. Nov 96, MKU. BE. Nov. 96, BNU. BE. Apr. 97, Nov. 97]*
13. Define an assignment problem. *[MKU. BE. Nov 96, BNU. BE. Nov 96, Nov 97, BRU BE Nov 96, Nov 97, MU. BE. Oct 96, Apr 98, Apr 99, MSU. BE. Nov 96, Nov 97]*
14. What do you mean by a balanced assignment problem?
15. Give the Linear Programming form of the assignment problem.
16. Give a mathematical formulation of assignment problem. *[MKU. BE. Nov 97]*
17. State the difference between transportation problem and assignment problem. *[MU. BE. Nov 94, MSU. BE. Nov 97, BNU. BE. Apr 97]*
18. What do you understand by impossible assignments?
19. How do you convert the maximization assignment problem into a minimization one?
20. How will you revise the opportunity cost matrix of an assignment problem, if it does not give the optimal solution? *[BRU. BE. Nov 96, Apr 97]*
21. If each entry is increased by 3 in a 4×4 assignment problem, what is the effect in the optimal value? *[MU. BE. Oct 98]*
[*Ans:* The optimal value will be increased by 12]
22. Give two areas of operation of assignment problem. *[MKU. BE. Apr 97, MSU. BE. Apr 97, Nov 97, MU. BE. Apr 98]*

23. State the objective of an assignment problem. *[BNU. BE. Apr 98, Nov 98]*
24. Indicate the algorithm to solve an assignment problem. *[BNU. BE. Nov. 97]*
25. Where assignment problems are encountered? *[MU. BE. Apr. 97]*

Fill in the blanks.

1. The assignment problem can be stated in the form of a $n \times n$ matrix (c_{ij}) called the .... [*Ans*: cost matrix (or) Effective matrix]
2. An assignment problem represents a transportation problem with all demands and supplies equal to .... [*Ans:* 1]
3. An assignment problem is a completely .... form of a transportation problem. [*Ans:* Degenerate]
4. An assignment problem is said to be balanced if .... [*Ans:* No.of rows = No. of columns]
5. An assignment problem is said to be unbalanced if .... [*Ans:* No.of rows ≠ No. of columns]
6. The transportation technique or simplex method cannot be used to solve the assignment problem because of ... [*Ans:* Degeneracy]

State true or false.

1. Assignment problem is a particular case of LPP. [*Ans:* True]
2. Assignment problem is a particular case of transportation problem. [*Ans:* True]
3. Assignment problem is a completely degenerate form of a transportation problem. [*Ans:* True]
4. Assignment problem represents a transportation problem with all demands and supplies equal to 1. [*Ans:* True]
5. A major constraint in the use of assignment technique is that number of tasks must equal the number of facilities. [*Ans:* True]
6. Assignment technique is of little use to a firm whose facilities are perfect substitute for each other. [*Ans:* True]
7. Assignment technique is essentially a minimization technique. [*Ans:* True]

8. The transportation technique or simplex method can not be used to solve the assignment problem because of degeneracy.
[*Ans:* True]

9. The optimum assignment schedule remains unaltered if we add or subtract a constant from all the elements of a row or column of the assignment cost matrix. [*Ans:* True]

10. If for an assignment problem all $c_{ij} \geq 0$, then an assignment schedule (x_{ij}) which satisfies $\Sigma\Sigma\ c_{ij}\, x_{ij} = 0$, must be optimal.
[*Ans:* True]

11. If the final cost matrix of an assignment problem contains more than one zero at independent positions, then the problem will have multiple optimal solutions. [*Ans:* True]

12. In a travelling salesman problem, assignment can be made along the diagonal line of the cost matrix. [*Ans:* False]

13. In a travelling salesman problem, the salesman should not visit a city twice except the starting city. [*Ans:* True]

14. Not every basic solution in the assignment problem is degenerate.
[*Ans:* False]

15. The assignment problem cannot be solved by the transportation technique. [*Ans:* False]

UNIT-III

9. Integer Programming

1. What is integer programming?
[MU. MCA. May 89, BNU. BE. Nov 96]

2. Define Pure integer programming problem.

3. Define Mixed integer programming problem.

4. Differentiate between pure and mixed integer programming problems. ***[MU.BE. Oct 96, BNU. BE. Nov 97]***

5. Explain the importance of the integer programming problems.
[MU. BE. 80]

6. Give any two applications of integer programming.
[MU. BE. 79,80, MSU. BE. Nov 96]

7. Why not round off the optimum values instead of resorting to integer programming? Explain. ***[MU. MCA. May 89]***

8. Write down the Gomory's fractional cut corresponding to the equation $x_1 + \frac{2}{3}x_3 - \frac{1}{3}x_4 = \frac{2}{3}$ that appears in the non–integer optimal simplex table of an integer programming problem.
[BRU. BE. Nov 96] $\left[\textit{Ans}: \frac{-2}{3}x_3 - \frac{2}{3}x_4 + s_1 = \frac{-2}{3} \right]$

9. Can we apply the Branch and Bound method for both pure as well as mixed integer programming problems.

10. What is the fractional part of $\frac{-2}{3}$? ***[MU. BE. Oct 98]***
$\left[\textit{Ans.}\ \frac{1}{3} \right]$

11. What is Gomory's fractional algorithm? ***[MU. BE. Nov 96]***

12. The fractional part of the negative number $\frac{-7}{3}$ is ....
$\left[\textit{Ans}: \frac{2}{3} \right]$ ***[MU. BE. Apr 97]***

13. The greatest integer less than or equal to $\frac{-22}{7}$ is ....
[*Ans* : -4] ***[MU. BE. Oct 97]***

14. What do you mean by integer linear programming?
[BRU. BE. Apr 97]

15. Write down the Gomory's fractional cut corresponding to the equation $x_2 - \frac{1}{7}x_3 + \frac{5}{14}x_4 = \frac{12}{7}$, that occurs in the non-integer optimal simplex table of an integer programming problem.
$\left[\textit{Ans}: \frac{-6}{7}x_3 - \frac{5}{14}x_4 + s_1 = \frac{-5}{7} \right]$ ***[BRU. BE. Apr 97]***

16. In the optimal solution of an I.P.P. by simplex method, the basic variable x_1 is not an integer. The corresponding row in the table is

X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$x_1 = 3\frac{1}{4}$	1	$\frac{3}{2}$	$\frac{-5}{3}$	0	2	0	$\frac{-11}{4}$

Construct a Gomary's constraint for this.

$\left[Ans: \frac{-1}{2}x_2 - \frac{1}{3}x_3 - \frac{1}{4}x_7 + s_1 = \frac{-1}{4} \right]$ *[MU. BE. Apr. 99]*

17. Where Branch and bound method is used? *[MU. BE. Apr. 97]*
18. What is Gomory constant that is inclined in the simplex table is otherwise known? *[MU. BE. Apr. 97]*

    [*Ans.* Gomorian slack]
19. What are the methods of solving an integer programming problem? *[MSU. BE. Apr. 97]*
20. What is the use of Gomory's constraint in integer programming problem? *[BNU. BE. Apr. 98]*

State True or False

1. Every integer program, mixed or pure, can be expressed in terms of zero–one variables. *[MU. BE. Oct 97]* [*Ans:* True]
2. It is impossible to obtain a feasible integer solution by rounding the continuous optimum of a linear programming problem that originally contains strict equality constraints. [*Ans:* True]
3. The integer programming cuts modify the continuous solution space in a manner that will produce a continuous optimum that satisfies the integer conditions on the variables. [*Ans:* True]
4. The optimum integer solution of a problem can produce a better objective value than its associated continuous optimum.

   [*Ans:* False]
5. The construction of the fractional cut does not require the slack variables to be integer. [*Ans:* False]

6. In the application of the cutting methods, it is necessary to retain all the generated cuts in the simplex table until the optimum integer is reached. [*Ans:* False]
7. A cut may eliminate an integer feasible point as long as it is not the optimum integer. [*Ans:* False]
8. If immediately following an application of a cut, the dual simplex method does not produce a feasible (continuous or integer) solution, the problem has no integer feasible solution.

   [*Ans:* True]
9. The mixed cut can eventually produce the optimum integer solution of a problem in which all variables are integers.

   [*Ans:* True]
10. If a dual cutting plane method is stopped prematurely, the last available solution can be considered a good feasible solution to the integer problem. [*Ans:* False]
11. In Branch and Bound method, the bounding step sets a lower (upper) limit on the objective value in the case of maximization (minimization) provided a feasible integer solution is encountered.

    [*Ans:* True]
12. In the Branch and Bound method, the branching step effectively removes continuous parts of the feasible space. [*Ans:* True]
13. The number of sub-problems created by the branching step can be reduced drastically if a good bound is discovered at the early stages of computations. [*Ans:* True]
14. A bound obtained by the branch and bound method may not necessarily be associated with a feasible point of the integer problem [*Ans:* False]
15. In the branch and bound method, the available rules for selecting the branching variables at a node guarantee encountering a good bound rapidly. [*Ans:* False]

16. The basic disadvantage of the branch and bound method is that the number of sub problems created is more. [*Ans:* True]

17. The effect of round–off error in the branch and bound method is just as bad as in the cutting plane method. [*Ans:* False]

18. In the branch and bound method, the branching variable at a current node cannot be branched again at a subsequent node that is generated from the current one. [*Ans:* False]

10. Dynamic Programming

1. What is dynamic Programming? ***[MU. BE. Oct 96, 97, MSU. BE. Nov 96, BNU. BE. Apr 97]***
2. State Bellman's Principle of optimality. ***[MU. BE. Oct 98, Apr 99, BNU. BE. Nov. 97, Nov 98]***
3. What are the essential characteristics of dynamic programming problems? ***[MU. BE. Nov 97]***
4. Write the advantage of dynamic programming. ***[MU. BE. Apr 97]***
5. State the application of dynamic programming.
6. State the application of the principle of optimality in dynamic programming ***[MU. BE. Apr 97, MSU. BE. Apr 97]***
7. What difficulties you overcome when dynamic programming is designed?
8. A problem which does not satisfy the Principle of optimality cannot be solved by dynamic programming. Is it true or not? ***[Ans : True]***
9. Write down the recursive equation for the following problem, treating it as a dynamic programming problem:

   Maximize : $f(x) = x_1 \; x_2 \; \ldots x_n$.

   Subject to $x_1 + x_2 + \ldots\ldots + x_n = k$

   $x_i \geq 0,$ ***[BRU. BE. Nov. 96]***

   [Ans: $f_n(k) = \underset{0 \leq y \leq k}{\text{Max}} \{y \; f_{n-1}(k - y)\}$]
10. Where can we apply dynamic programming ? ***[BRU. BE. Nov. 97]***

11. State the Bellman's principle of optimality used to solve dynamic programming problems. ***[BRU. BE. Apr. 97, MU. BE. Apr. 98]***
12. When is the tabulation method chosen while solving a problem by dynamic programming technique? ***[BNU. BE. Apr. 98, Nov. 98]***
13. State any two merits of dynamic programming techniques. ***[BNU. BE. Nov 96]***
14. In dynamic programming, it is usually more difficult to define stages rather than states. Is it true? ***[MU. BE. Apr 98]***
15. What is the objective in dynamic programming? ***[BRU. BE. Apr 98]***

UNIT-IV

11. Classical Optimisation Theory

1. Investigate $f(x) = x^3 + x^2$ for extrema

   Ans: x = 0 minimises and $x = -\frac{2}{3}$ maximises

   Min f = 0, Max $f = \frac{4}{27}$
2. Define a stationary point of $Z = f(x, y)$
3. What is a saddle point of $Z = f(x, y)$?
4. Write down the conditions for $Z = f(x, y)$ to have a maximum at (a, b).
5. Write down the conditions for $Z = f(x, y)$ to have a minimum at (a, b).
6. Determine the extreme points of $Z = x^2 - x + y^2 - y$

   Ans: $x = \frac{1}{2}, y = \frac{1}{2}$ minimises Z Min $Z = -\frac{1}{2}$, extreme print is $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

7. Investigate $Z = 3x - x^2 - y^2$ for extrema.

Ans: $Z = \frac{9}{4}$

8. Lab (x_0, λ_0) be the stationary point of the Lagrangean function $L(X, \lambda)$. Let H_0^B be the bordered Hessian matrix computed at (X_0, λ_0). State the sufficient conditions for the stationary point to be a maximum point.

9. For the Lagrangean in question no. 8 state the sufficient condition for the stationary point to be a minimum point.

10. Write down the Kuhn – Tucker conditions for the problem:

$$\text{Max} f\ (x_1, x_2, x_3) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$$

subject to

$$2x_1 + 3x_2 \leq 12$$
$$x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

11. Write down the Kuhn - Tucker conditions for the problem:

$$\text{Min} f\ (x_1, x_2) = 2x_1^2 + 12\, x_1\, x_2 - 7x_2^2$$

subject to

$$2x_1 + 5x_2 \leq 98$$
$$x_1, x_2 \geq 0$$

12. Write down the Kuhn - Tucker conditions for the problem:

$$\text{Min } Z = x_1^2 + x_2^2 + x_3^2$$

subject to

$$2x_1 + x_2 \leq 5$$
$$x_1 + x_2 \leq 2$$
$$x_1 \geq 1$$
$$x_2 \geq 2$$
$$x_2 \geq 0$$

13. Write down the Kuhn Tucker conditions for the problem:

$$\text{Min } f\ (x_1, x_2) = \log x_1 - \log x_2$$

subject to

$$x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

14. What are the Kuhn - Tucker conditions for the problem:

$$\text{Max } Z = x_1^2 - x_2^2 + x_1\, x_3^2$$

subject to

$$x_1 + x_2^2 + x_3 = 5$$
$$5x_1^2 - x_2^2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0\ z$$

15. Write down the Kuhn – Tucker conditions for the problem:

$$\text{Min } Z = x_1^2 + x_2^4 + 6x_1\, x_2\, x_3$$

subject to

$$x_1^2 - x_2^2 + x_3^2 \leq 15$$
$$x_1^3 + x_2^2 + 4x_3^2 \geq 25$$

UNIT – V

12. Project Scheduling by PERT and CPM

1. What do you mean by a Project ? *[BRU. B.E. Nov96]*
2. What are the three main phases of Project ? *[BRU. B.E. Nov96]*
3. What do you mean by reviewing a Project ?
4. What do you mean by an activity of a Project ?
5. What is the basic relationship which helps us to represent a Project as a network ?
6. What is do you mean by a network ? ***[BRU. Apr 97, MU. B.E.Oct 96]***
7. What are the rules for constructing a Project network ? ***[MU. B.E. Nov 97]***

8. Explain the method of constructing a network diagram. *[MKU. B.E. Nov 97]*

9. Explain the process of numbering the events of a network.

10. Can a loop be present in a Project network ?

11. What is dangling in a network ?

12. What is dummy activity ?

13. Define total float of an activity.

14. When do you call an activity critical ? *[BRU. B.E. Nov 96]*

15. Earliest start of an activity can be calculated by the formula......

16. Define critical path of a Project network. *[BRU. Apr 97, Nov 97, MKU. B.E. Nov 96, MS. B.E. Nov 96]*

17. What is critical path in PERT/CPM ? Explain its importance. *[MU. B.E. Nov 96]*

18. Total float of an activity is positive implies abundant resources for the activity. True or False ? *[Ans :* True*]*

19. Indicate Dummy activities in a network ? *[MKU. B.E. Nov 97]*

20. What is a redundant activity ? *[MKU. B.E. Nov 97]*

21. The path of least float in a Project is called .... *[Ans :* Critical path*]*

22. Can the total float be negative ? *[Ans :* Yes]

23. Distinguish between float and slack.

24. Explain how you compute ES, EF, LS and LF times of any job. *[BRU. B.E. Apr 97]*

25. What do you mean by slack of a job ? *[BRU. B.E. Apr 97, BRU. B.E. Apr 98]*

26. Can dummy activity appear on the critical path of a Project network ? *[Ans :* Yes*]*

27. The Project duration is affected if the duration of any activity is changed. True or False ? *[Ans :* False*]*

28. A critical activity must have its total and free floats equal to zero. – Is it true ? *[Ans :* Not necessary*] [MU. B.E. Oct. 97]*

29. Describe the total float and free float of an activity. *[MU. M.C.A. Nov 97]*

30. What do you mean by independent float of an activity ?

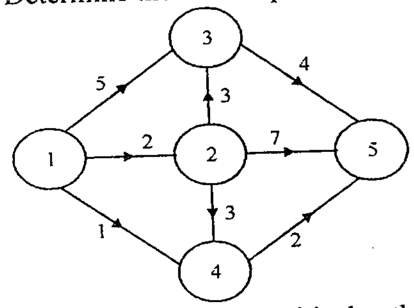
31. Write the relationship between total float, free float and independent float.

32. What is meant by interfering float ? How do you compute it ?

33. Define three types of floats in CPM. *[MKU. B.E. Nov. 97]*

34. Latest finish of an activity can be calculated by the formula ....

35. Determine the critical path for the following network.



[Ans : There are two critical paths 1 – 2 – 5, 1 – 3 – 5*]*

36. If the total float of an activity 3 – 4 is 18 and the latest and earliest occurrences of the events 3 and 4 are 15, 12 and 22 10 respectively, what is the free float of 3 – 4 ? *[Ans :* 6*]*

37. What is the independent float of the activity 3 – 4 in question (23) ? *[Ans :* 3*]*

38. What is the difference between PERT and CPM ? *[MKU. Nov 97, BRU. B.E. Nov 96, 97, MU. B.E. Oct 96, BRU. Apr 97]*

39. "PERT takes care of uncertain duration ?" How far is this statement correct ? *[MU.B.E. Oct 97]*

40. Define the expected variance of a Project length.

41. The expected standard deviations of Project length is equal to the sum of the expected standard deviations of all the critical activities of the Project. True or False ? *[Ans :* False*]*

42. Express the expected duration of an activity of a Project in terms of optimistic, pessimistic and most likely time estimates. *[MS BE. Apr 97]*

43. Express the expected variance of an activity interms of the optimistic and the pessimistic time estimates. *[MS BE. Apr 97]*

44. The name of the probability distribution (used in PERT) which estimates the expected duration and the expected variance of an activity is............... *[Ans :* β Distribution*]*

45. Write down atleast two main assumptions in PERT network calculations.

46. In PERT analysis, the variance of a job having optimistic time 5, most likely time 8 and pessimistic time 17 is...... *[Ans : 9]*

[MU. BE. Oct 97]

47. If the critical path of a network is 1 – 2 – 4 – 6 – 7 and the variance of 1 – 2, 2 – 3, 2 – 4, 4 – 5, 4 – 6, 2 – 6, 4 – 7 and 6 – 7 are 3,6,5,7,6,10,1,2 what is the variance of the Project length ?

[Ans : 16]

48. If the standard deviations of all the 5 critical activities of a project network are 1, 4 8, 9, 3 then the standard deviation of the project length is (in times units) (a) 25 (b) 5 (c) 17 (d) 13.08

[Ans : d*]*

49. If the expected project duration is 15 months and the specified due date is 13 months find the probability that the project may be completed before 13 months given that the variance of the project length is 4. *[Ans :* 0.1587*]*

50. If the expected project duration is 15 months and the specified due data is 17 months find the probability that the project will be completed before 17 months given that the variance of the project length is 4. *[Ans :* 0.8413*]*

51. For an activity in PERT analysis, the three times a,m and b are in arithmetic progression with common difference 5. What is the variance of this activity ? *[MU. B.E. Oct 98]*

$$\left[Ans: \frac{3a+15}{6}\right]$$

52. For a standard normal variate Z, P (O ≤ Z ≤ 1) = 0.3413. If the expected duration of a project is 40 days and the standard deviation of the critical path in 5 days, what is the probability of completing the project in 35 days ? *[MU. B.E. Oct 98]*
[Ans : 0.1587*]*

53. Discuss what is meant by critical path. *[BNU. B.E. Nov 98]*

54. Justify the use of three time estimates in PERT.
[BNU. B.E.Nov 98]

55. What is meant by crashing ? *[MU. B.E.Nov. 96, Apr 98]*

56. Explain the stepwise method of the least cost schedule.

57. What is resource leveling in network analysis ?
[MS. MU. B.E. Nov 97,BNU. BE. Apr 97, MU. BE. Nov 96, MKU Nov 96, MU. BE. Apr 98, BRU. BE. Apr 98, Nov 97]

58. What is resource scheduling ?

59. What is heuristic programming ?

60. Define cost-time slope. *[MU. B.E. Oct 98, BNU. Apr 97]*

61. Define optimum duration of a project.

62. Define the least duration of a project.

63. Reducing the activity of least cost slop, the project duration can be reduced, True or False ? *[Ans :* False*]*

64. The Duration corresponding to the least cost schedule is the minimum duration of a project. True or False ? *[Ans :* False*]*

65. If all paths of a network are critical paths then the project duration cannot be reduced further. True or False ? *[Ans :* False*]*

66. What are the two main costs for a project ?

67. Define (a) Indirect cost (b) Direct cost for a project.

68. When the duration of an activity is decreased the direct cost increases and the indirect cost decreases in general for the activity. True or False ? *[Ans :* True*]*

69. Project duration is fixed in resource leveling programme – True or False ? *[Ans :* True*]*

70. What do you mean by parallel activities ?

71. Briefly mention the area of application of network techniques.
[MU. B.E. Oct 97]

72. The cost-time slope of an activity having normal time 4 hours, crash time 2 hours, normal cost Rs. 150 and crash cost Rs. 350 is................ *[Ans :* 100*] [MU. B.E. Oct 97]*

73. What is the purpose of project scheduling ? *[MS. B.E. Nov 97]*

74. What is the purpose of cost analysis in project management ?
[BRU. B.E. Nov 97]

75. Why is squared network superior to conventional network ?
[MS. B.E. Nov 97]

76. Define Makespan. *[BRU. B.E. Nov 97]*

77. Explain resource allocation problem with an example.
[BRU. B.E. Apr 97]

78. What is the need of resource leveling ? Make use of an example.
[BNU. B.E. Nov 98]

79. The number of time estimates involved in PERT problems is ....
[Ans : 3*]*
[MU. BE. Oct 96]

80. For a non-critical activity, the total float is always.........
[Ans : Non-Zero*]*
[MU. BE. Oct 96]

81. The probability to complete a project in the expected time is ......
[Ans : 0.5*]*
[MU. BE. Oct 96]

82. What is mean by crash time ? *[BNU. BE. Nov 96]*

83. In what 3 major ways does a CPM network differ from a PERT network ? *[BNU. BE. Nov 96]*

84. In PERT analysis, the critical path is obtained by joining event having ....
[Ans : +ve slack/zero slack*]*
[BNU. BE. Nov 96]

85. Define total float with reference to PERT chart.
[MKU. BE. Nov 96]

86. Explain the normal time. *[MKU. BE. Nov 96]*

87. Explain with suitable example, the term CPM.
[MKU. BE. Nov 96]

88. What is the purpose of crashing of networks ?
[BRU. BE. Nov 96]

89. Name the different phases of project scheduling.
[BRU. BE. Nov 96]

90. When do you call an activity critical. *[BRU. BE. Nov 96]*

91. What is free float in CPM calculations. *[MS. BE. Nov 96]*

92. The first two phases of project scheduling by PERT-CPM are .... and .... *[MU. BE. Apr 97]*

93. The total float for a critical activity is ....
[Ans : Either negative or zero*]*

94. For an activity, the normal time is 8 days, normal cost is 100, crash time is 6 days and crash cost is Rs. 200. The cost-time slope for this activity is .... *[Ans :* 50*]*
[MU. BE. Apr 97]

95. PERT is the .... oriented technique. *[BNU. BE. Apr 97]*

96. What is meant by 'total float of an activity' ?*[BNU. BE. Apr 97]*

97. Explain Fulkerson's rule in numbering the events in a network.
[BNU. BE. Apr 97]

98. What is the use of crash time in network analysis ?
[BNU. BE. Apr 97]

99. Mention the cases where PERT/CPM techniques are used.
[MKU. BE. Apr 97]

100. Define free float with reference to a PERT chart.
[MKU. BE. Apr 97]

101. Define an activity in a PERT network. *[MKU. BE. Apr 97]*

102. What is critical path analysis ? *[MKU. BE. Apr 97]*

103. What do you mean by project duration ? *[BRU. BE. APR 97]*

104. Write the formula for cost slope for a project in PERT/CPM.
[MS. BE. Apr 97]

105. Mention the applications of PERT/CPM techniques to industrial problems. *[BRU. BE. Nov 97]*

106. Explain independent float. *[BRU. BE. Nov 97]*

107. Explain resource smoothing.
[MKU. BE.Apr 97, BRU. BE. Nov 97, MS. BE. Nov 97]

108. What is crash point ? *[MSU. BE. Nov 97]*

109. What is meant by cost slope content ? *[MSU. BE. Nov 97]*

110. The expansion of the abbreviation of PERT is ......
[MU. BE. Apr 98]

111. If the standard deviation of the critical activities in a project are 2, 3 and 6 then the standard deviation of the critical path is ......
[Ans : 7*]*
[MU. BE. Apr 98]

112. What is the purpose of cost analysis with reference to project management ? *[BNU. BE. Apr 98]*

113. Define network. *[BNU. BE. Apr 98]*

114. What are the three time estimates used in PERT network ? *[BRU. BE. Apr 98]*

115. What is the use of the Gantt bar chart ? *[BNU. BE. Nov 96]*

116. What is dummy activity, and when is this needed ? *[BNU. BE. Nov 96]*

117. What is meant by crashing of an activity, and when it is justified? *[BNU. BE. Nov 96]*

118. Distinguish briefly between the approaches made use of in PERT and CPM. *[BNU. BE. Nov 96]*

119. What is crashing ? *[MKU. BE. Nov 96]*

120. Can a project have multiple critical paths ? *[MKU. BE. Nov 96]*

121. What is the significance of a dummy activity ? *[MKU. BE. Nov 96]*

122. What are the rules for constructing a project network ? *[MKU. BE. Nov 96]*

123. Define the three time estimates used in PERT and write down the formulas for expected job time and S.D. of job time. *[BRU. BE. Nov 96]*

124. Explain the terms : normal and crash durations and cost of crashing for a job. *[BRU. BE. Nov 96]*

125. Define optimistic and pessimistic time in PERT. *[MS. BE. Nov 96]*

126. What are the advantages of network techniques ? *[MS. BE. Nov 96]*

127. Define optimistic time and normal time in PERT. *[MU. BE. Apr 97]*

128. What are the three constituent basic phases, as related to project scheduling by PERT/CPM ? *[BNU. BE. Apr 97]*

129. Distinguish between 'critical activity' and 'non-critical activity'. *[BNU. BE. Apr 97]*

130. Distinguish between 'total float' and 'free float' associated with non-critical activity. *[BNU. BE. Apr 97]*

131. Indicate total float by a diagram. *[MKU. BE. Apr 97]*

132. Define free float and independent float with reference to a PERT chart. *[MSU. BE. Apr 97]*

133. Distinguish between 'sequencing' and 'scheduling'. *[BNU. BE. Nov 97, BRU. BE. Nov 97]*

134. Discuss briefly the total float and free float of an activity. *[MU. BE. Apr 98]*

135. Explain reasons for incorporating dummy activities in a network diagram. *[MU. BE. Apr 98]*

136. What is the significance of 'float' in CPM ? *[BNU. BE. Apr 98]*

137. Distinguish between 'earliest expected time' and 'latest expected time'. *[BNU. BE. Apr 98]*

138. Why do we use three estimates of time in PERT ? *[BNU. BE. Apr 98]*

139. What is the use of Gantt chart ? *[BNU. BE. Apr 98]*

140. What is the objective of resource smoothing ? *[BNU. BE. Apr 98]*

141. Explain resource allocation with an example. *[BNU. BE. Apr 98]*

MODEL QUESTION PAPER - I

LATEST ANNA UNIVERSITY SYLLABUS

RESOURCE MANAGEMENT TECHNIQUES

Time : Three hours **Maximum :100 marks**

Answer ALL Questions

PART – A (10 × 2 = 20 marks)

1. What are slack and surplus variables, used in a linear programming problems?
2. What is sensitivity Analysis?
3. Obtain the dual of the following Linear programming problem:

   Maximise $Z = 3x_1 + x_2 + x_3 - x_4$?

   subject to

$$x_1 + 2x_2 - 3x_3 + 4x_4 \leq 5$$

$$x_1 + x_2 = 1$$

$$x_3 + x_4 \leq -5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

4. Distinguish between Dijkstra's algorithm and Floyd's algorithm used to determine shortest route in a distance net work.
5. What is Gomory's fractional algorithm?
6. State Bellman's principle of optimality.
7. Give a set of conditions for a point (a, b) to be a saddle print of the function $Z = f(x, y)$.
8. State kuhn-Tucker conditions.
9. Define critical path for a project network.
10. What are the main steps followed is Resource leveling programmer?

PART – B (5 × 16 = 80 marks)

11(a) Solve the following linear programming problem graphically:

Minimise $Z = 6000\, x_1 + 4000\, x_2$

subject to

$$3x_1 + x_2 \geq 40$$
$$x_1 + 2.5\, x_2 \geq 22$$
$$3x_1 + 3x_2 \geq 40$$

and $x_1\, x_2 \geq 0$

OR

11(b) Use simplex method to solve the following linear programming problem:

Maximise $Z = 20\, x_1 + 6\, x_2 + 8x_3$

subject to

$$8x_1 + 2x_2 + 3x_3 \leq 250$$
$$4x_1 + 3\, x_2 \leq 150$$
$$2x_1 + x_2 \leq 50$$
$$x_1, x_2, x_3 \geq 0$$

12(a) Write down the dual of the following linear programming problem and hence solve it.

Max $Z = 4\, x_1 + 2\, x_2$

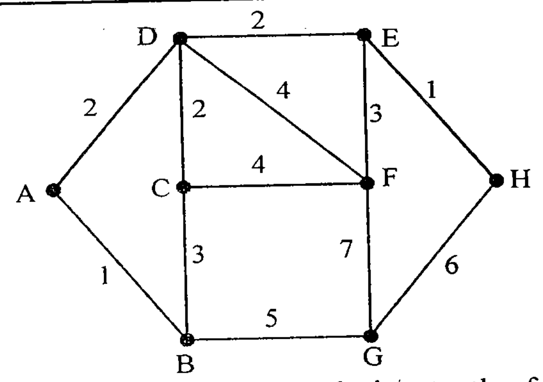
subject to

$$-x_1 - x_2 \leq -3$$
$$-x_1 + x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

Hence or otherwise write down the solution of the primal

(OR)

12 (b) Find the shortest route and its length from the vertex 1 to vertex 8 using Dijkstra's algorithm.



13 (a) Find the optimal integer solution to the following linear programming problem using cutting plane method

Max $Z = x_1 + x_2$

subject to

$$3x_1 + 2x_2 \leq 5$$
$$x_2 \leq 2$$
$$x_1 \geq 0, x_2 \geq 0, x_1, x_2 \text{ are integers}$$

(OR)

13 (b) Use dynamic programming, to solve

Max $Z = y_1\, y_2\, y_3$

subject to

$$y_1 + y_2 + y_3 = 5$$
$$y_1, y_2, y_3 \geq 0$$

14(a) Investigate the function $Z = x^4 + y^4 - x^2 - y^2 + 1$ for maxima, minima and saddle points.

(OR)

14(b) Optimise $Z = x_1^2 + x_2^2 + x_3^2$

subject to

$$x_1 + x_2 + 3x_3 = 2$$
$$5x_1 + 2x_2 + x_3 = 5$$
$$x_1, x_2, x_3 \geq 0$$

using Lagrange's multipliers.

15(a) Construct the network for the project whose activities are given below and computer the total, free and independent float of each activity and hence determine the cirtical path and the project duration.

Activity	0 – 1	1 – 2	1 – 3	2 – 4	2 – 5
Druation in weeks	3	8	12	6	3
Activity	3 – 4	3 – 6	4 – 7	5 – 7	6 – 7
Druation in weeks	3	8	5	3	8

(OR)

(b) Three time estimates (in months) of all activities of a project are given below:

	Time in months		
Activity	a	m	b
1 – 2	0.8	1.0	1.2
2 – 3	3.7	5.6	9.9
2 – 4	6.2	6.6	15.4
3 – 4	2.1	2.7	6.1
4 – 5	0.8	3.4	3.6
5 – 5	0.9	1.0	1.1

(i) Find the expected duration and standard deviation of each activity.

(ii) Construct the project network

(iii) Determine the critical path, expected project length and expeted variance of the project length

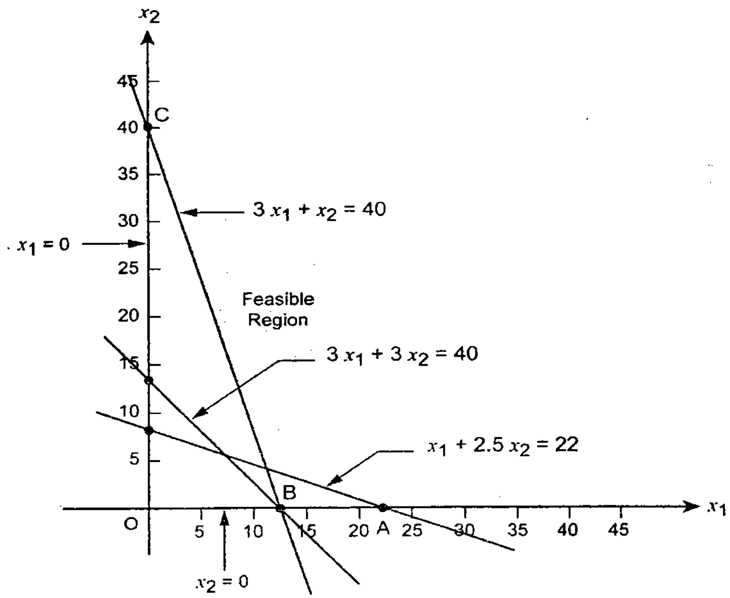
(iv) What is the probability that the project will be completed

(a) two months later than expected?

(b) not more than 3 months earlier than expected?

ANSWERS TO PART B OF MODEL QUESTION PAPER 1

11(a)



Ans : $x_1 = 12$ days, $x_2 = 4$ days

Max Z = Rs 88, 000

11(b) $x_1 = 0$ $x_2 = 50, x_3 = 50$

Max Z = 700

12(a) No optimal basic feasible solution no finite optimal to the primal

12(b) $A \rightarrow D \rightarrow E \rightarrow H$, Length 5 units

13(a) Max Z = 2, $x_1 = 0$ $x_2 = 2$

13(b) Optimal poilcy is $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$

Max $Z = \frac{125}{27}$

14(a) Min Z $=$ 0, at x = 0, y = 0

$$\text{Max } Z = \frac{1}{2}, \text{ at } \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$$

$\left(0, \pm\frac{1}{\sqrt{2}}\right)$ and $\left(\pm\frac{1}{2}, 0\right)$ are saddle points.

14(b) Min Z $=$ 0.857

$$x_1 = \frac{37}{46}, x_2 = \frac{8}{23}, x_3 = \frac{13}{46}$$

15(a)

Activity	Total Fhat	Fred Float	Independent Float
0 – 1	0	0	0
1 – 2	9	0	0
1 – 3	0	0	0
2 – 4	9	1	– 8
2 – 5	14	0	– 9
3 – 4	8	0	0
3 – 6	0	0	0
4 – 7	8	8	0
5 – 7	14	14	0
6 – 7	0	0	0

Crited path 0 – 1 – 3 – 6 – 7

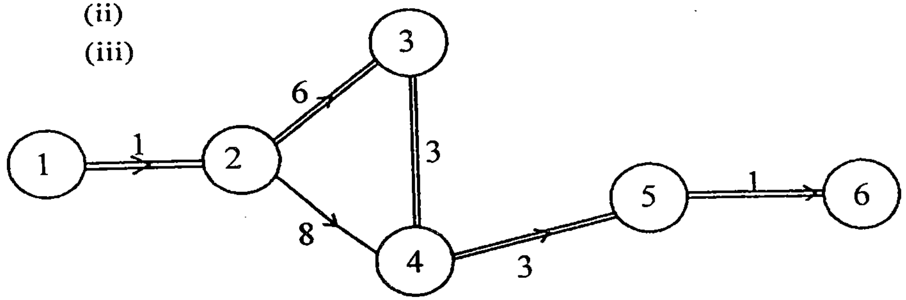
Project ducation $=$ 31 weeks

15(a)(i)

Activity	Expected duration	Expected Standard deviation
1 – 2	1	0.067
2 – 3	6	1.03
2 – 4	8	1.53
3 – 4	3	0.5
4 – 5	3	0.47
5 – 6	1	0.033

(ii)

(iii)



Critical path 1 – 2 – 3 – 4 – 5 – 6

Expected project length 14 months

Expected variance of the project length 1.537

(iv) (a) 0.9463

(b) 0.0078

MODEL QUESTION PAPER -II
LATEST ANNA UNIVERSITY SYLLABUS
RESOURCE MANAGEMENT TECHNIQUES

Time : Three hours **Maximum :100 marks**

Answer ALL Questions

PART – A (10 × 2 = 20 marks)

1. How will you find whether a linear programming problem has got an alternate optimal solution or not from the optimal simplex table?
2. What advantages can be realised from sensitivity analysis?
3. Distinguish between Transportation model and Assignment model.
4. Set up Dijkstra's algorithm to find the shortest path in a two terminal network.
5. Give any two applications of integer programming.
6. What are the essential characteristics of Dynamic programming problems?
7. Determine the nature of the extreme point (5, 11) of the function $Z = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.
8. Find the necessary conditions for the function $xy + 2xz + 2yz$ with the constraint $xyz = V$ (a constant) is to have an extremum using Lagrangean Multiplier.
9. Define total float, free float and Independent float in CPM.
10. If the critical path of a project network is 1 – 2 – 4 – 6 – 7 and the variances of all the activities namely 1 – 2, 2 – 3, 2 – 4, 4 – 5, 4 – 6, 2 – 6, 4 – 7 and 6 – 7 are 3, 6, 5, 7, 6, 10,1 2 respectively then find the standard deviation of the project length.

PART – B (5 × 16 = 80 marks)

11(a) Use penalty method to

Maximise $Z = 2x_1 + x_2 + x_3$

subject to

$$4x_1 + 6x_2 + 3x_3 \le 8$$
$$3x_1 - 6x_2 - 4x_3 \le 1$$
$$2x_1 + 3x_2 - 5x_3 \ge 4$$
$$x_1, x_2, x_3 \ge 0$$

(OR)

11 (b) Use two-phase simplex method to solve

Maximise $Z = 5x_1 + 8x_2$

subject to

$$3x_1 + 2x_2 \ge 3$$
$$x_1 + 4x_2 \ge 4$$
$$x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

12(a) We dual simplex method to move the following linear programming problem:

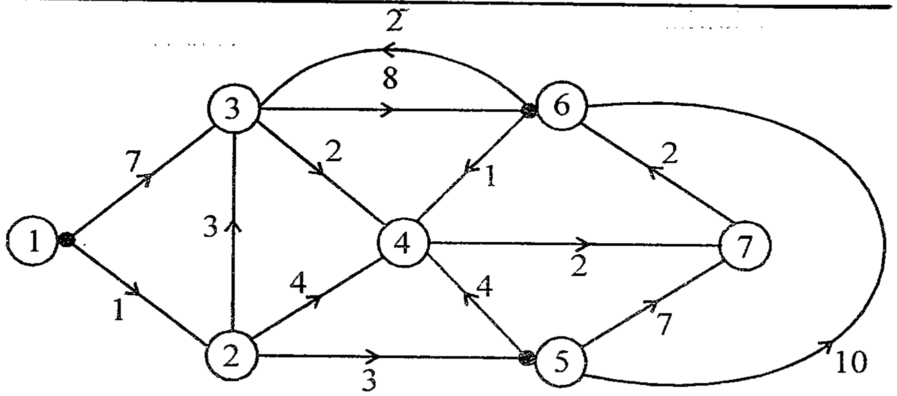
Maximise $Z = -3x_1 - 2x_2$

subject to

$$x_1 + x_2 \ge 1$$
$$x_1 + x_2 \le 7$$
$$x_1 + 2x_2 \ge 10$$
$$x_2 \le 3$$
$$x_1, x_2 \ge 0$$

(OR)

12 (b) We Dijkstra's algorithm to determine the shortest route between the node 1 to node 7 in the following distance network.



13(a) Use Branch and Bound technique to solve

Maximise $Z = x_1 + 4x_2$

subject to

$$2x_1 + 4x_2 \leq 7$$
$$5x_1 + 3x_2 \leq 15$$
$$x_1, x_2 \geq 0 ; x_1, x_2 \text{ are integers}$$

(OR)

13(b) Using dynamic programming technique, solve

Manimise $Z = x_1^2 + x_2^2 + x_3^2$

subject to

$$x_1 + x_2 + x_3 \geq 15$$
$$x_1, x_2, x_3 \geq 0$$

14 (a) An aquarium with rectangular sides and bottom (and no top) is to hold 32 litres. Find its dimensions so that it will use the least amount of material

(OR)

14 (b) Using kuhn – Tuckes conditions solve the problem:

Maximise $Z = 4x_1 - x_1^3 + 2x_2$

subject to

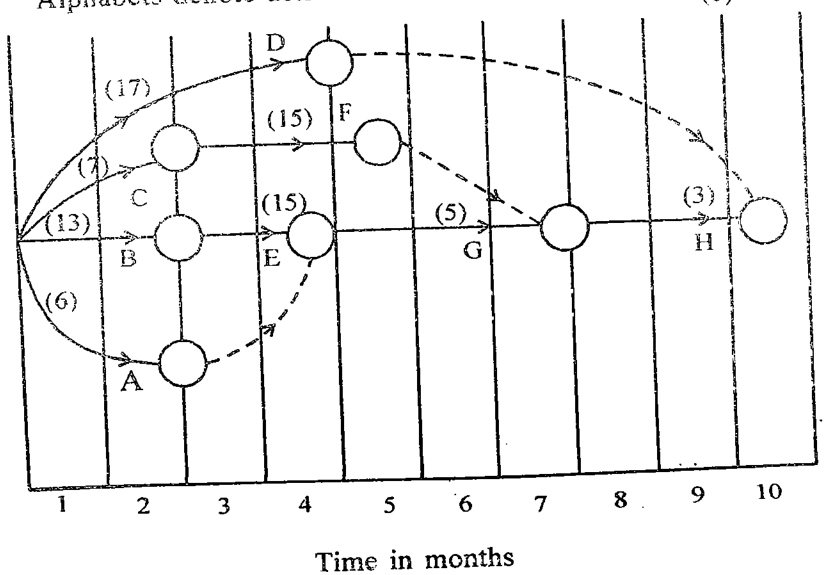
$$x_1 + x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

15 (a) The following time - cost table (time in days, cost in rupees) applies to a project. Use it to arrive at the network associated with completing the project in minimum time at minimum cost.

	Normal		Crash	
Activity	Time	Cost	Time	Cost
1 – 2	2	800	1	1400
1 – 3	5	1000	2	2000
1 – 4	5	1000	3	1800
2 – 4	1	500	1	500
2 – 5	5	1500	3	2100
3 – 4	4	2000	3	3000
3 – 5	6	1200	4	1600
4 – 5	3	900	2	1600

(OR)

15 (b) The early start schedule graph of a project in given below. The manpower requirement for each activity is indicated in the parenthesis. Using resource leveling programming, reduce the peak resource requirements.
Alphabets denote activities. (6)



ANSWERS TO PART B OF MODEL REVSTION PAPER 2

11(a) Max Z $= \frac{64}{21}, x_1 = \frac{9}{7}, x_2 = \frac{10}{22}, x_3 = 0$

11(b) Max Z = 40, $x_1 = 0, x_2 = 5$

12(a) Max Z $= -18, x_1 = 4, x_2 = 3$

12(b) Length of the shortest path = units

Shortest path 1 – 2 – 4 – 7

13(a) Max Z = 5, $x_1 = 1, x_2 = 1$

13(b) Optimal policy is (5, 5, 5)

Min Z = 75

14(a) 4, 4, 3 units

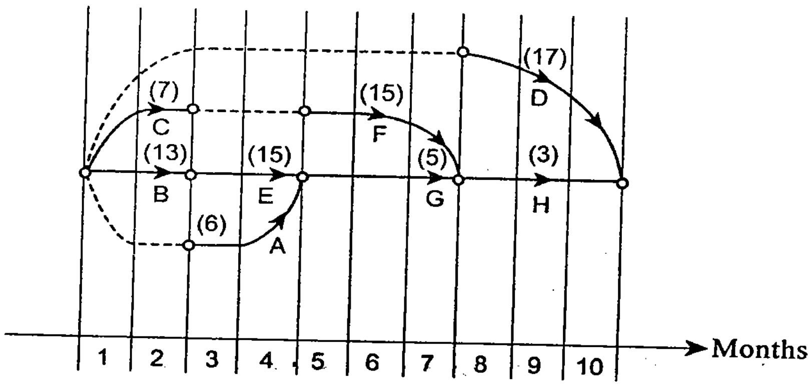
14(b) $x_1 = 0.8165,\ x_2 = 0.1835$

Max Z = 3.6323

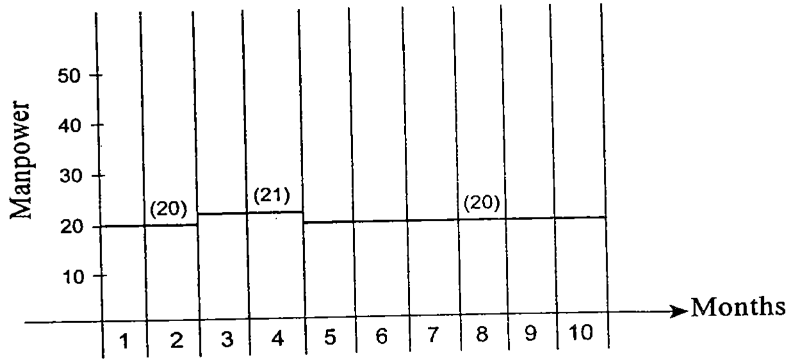
15(a) Miximum druration = 7 days

Associated volt = Rs. 11800.

Manpower loading chart



Schedule graph



Leveling further is not possible

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NOTES