22-09-23

# Fundamental Of Mathematics

## UNIT-1st

## Determinants

Evaluate the given determinants:

① $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$

$= 1\times3 - 2\times2$

$= 3-4$

$= -1$

Ans

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= (-1)^{1+1} a_{11}(a_{22}) + (-1)^{1+2} a_{12}(a_{21})$

$= \boxed{a_{11}a_{22} - a_{12}a_{21}}$

② $\begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} = (-1)(-1) - (-1)(-1) = 1-1 = 0$

③ $\begin{vmatrix} -2 & 50 \\ 40 & 30 \end{vmatrix}$

$|A| = -2\times30 - 50\times40$

$|A| = -60 - 2000$

$|A| = -2060$

④ $\begin{vmatrix} -1 & a \\ -1 & a \end{vmatrix}$

$|A| = (-1)(a) - (a)(-1)$

$|A| = -\cancel{a} + \cancel{a}$

$|A| = 0$

Evaluate the given determinants:

① $\begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 1(2-3) - 3(4-9) + 2(2-3) = 1\times(-1) - 3\times(-5) + 2\times(-1)$

$= -1 + 15 - 2$

$= 12$ Ans

(2) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-1) - 1(1-1) + 1(1-1) = 1\times0 - 1\times0 + 1\times0 = 0$

(3) $\begin{vmatrix} 2 & -3 & -2 \\ 1 & -2 & 3 \\ -2 & 1 & -3 \end{vmatrix} = 2(6-3) + 3(-3+6) - 2(1-4)$

$= 2(3) + 3(3) - 2(-3)$

$= 6 + 9 + 6 = 21$ Ans

(4) $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 2(5+2) + 1(20-2) + 3(-4-1)$

$= 2(7) + 1(18) + 3(-5)$

$= 14 + 18 - 15$

$= 17$ Ans

(5) $\begin{vmatrix} 1 & 0 & 2 \\ -1 & 3 & 2 \\ -2 & 1 & 0 \end{vmatrix}$

$= 1(0-2) - 0(0+4) + 2(-1+6)$

$= 1(-2) - 0 + 2(5)$

$= -2 - 0 + 10$

$= 8$ Ans

(6) $\begin{vmatrix} 2 & 0 & 2 \\ 1 & 3 & 2 \\ 3 & 1 & 0 \end{vmatrix}$

$= 2(0-2) - 0(0-6) + 2(1-9)$

$= 2(-2) - 0 + 2(-8)$

$= -4 - 0 - 16$

$= -20$ Ans

(7) $\begin{vmatrix} 2 & 1 & 2 \\ 1 & -1 & 2 \\ 3 & -2 & 0 \end{vmatrix}$

$= 2(0+4) - 1(0-6) + 2(-2+3)$

$= 2(4) - 1(-6) + 2(1)$

$= 8 + 6 + 2$

$= 16$ Ans

(8) $\begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 3 & -2 & 1 \end{vmatrix}$

$= 2(-1+6) - 1(1-9) + 0(2+3)$

$= 2(5) - 1(-8) + 0$

$= +10 + 8 = 18$ Ans

(9) $\begin{vmatrix} 1 & 2 & -1 & 5 \\ 2 & 1 & 0 & 2 \\ 1 & -1 & 3 & 2 \\ 3 & -2 & 1 & 0 \end{vmatrix}$

$= +1(8) - 2(-20) + (-1)(16) - 5(18)$

$= +8 + 40 - 16 - 90$

$= 48 - 106$

$= -58$ Ans

25-09-23

## Minors and Cofactors of determinants-

Minors- The minor $M_{ij}$ of the element $a_{ij}$ of matrix A of order n, is the determinant of square sum matrix of order (n-1) obtained by leaving $i^{th}$ row and $j^{th}$ column of A.

Ques- Find the minor matrix for the given matrix.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Sol^n If matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ then, minor matrix of $A = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$

~~Now, $M_{11} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$~~

Now, $M_{11} = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 4 - 2 = 2$

$M_{12} = -2 - 1 = -3$

$M_{13} = -2 - 2 = -4$

$M_{21} = -2 - 2 = -4$

$M_{22} = 2 - 1 = 1$

$M_{23} = 2 + 1 = 3$

$M_{31} = -1 - 2 = -3$

$M_{32} = 1 + 1 = 2$

$M_{33} = 2 - 1 = 1$

minor matrix of $A = \begin{bmatrix} 2 & -3 & -4 \\ -4 & 1 & 3 \\ -3 & 2 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 2 & -1 & 5 \\ 7 & 3 & 2 \\ 6 & -1 & 5 \end{bmatrix}$, the find the minor of A.

Sol$^n$- $M_{11} = 15+2 = 17$

$M_{12} = 35-12 = 23$

$M_{13} = -7-18 = -25$

$M_{21} = \begin{vmatrix} -1 & 5 \\ -1 & 5 \end{vmatrix} = -5+5 = 0$

$M_{22} = \begin{vmatrix} 2 & 5 \\ 6 & 5 \end{vmatrix} = 10-30 = -20$

$M_{23} = \begin{vmatrix} 2 & -1 \\ 6 & -1 \end{vmatrix} = -2+6 = 4$

$M_{31} = \begin{vmatrix} -1 & 5 \\ 3 & 2 \end{vmatrix} = -2-15 = -17$

$M_{32} = \begin{vmatrix} 2 & 5 \\ 7 & 2 \end{vmatrix} = 4-35 = -31$

$M_{33} = \begin{vmatrix} 2 & -1 \\ 7 & 3 \end{vmatrix} = 6+7 = 13$

If $A = \begin{bmatrix} 2 & -1 & 5 \\ 7 & 3 & 2 \\ 6 & -1 & 5 \end{bmatrix}$ then,

the minor of $A = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$

$A = \begin{bmatrix} 17 & 23 & -25 \\ 0 & -20 & 4 \\ -17 & -31 & 13 \end{bmatrix}$

26-09-23

Cofactor of matrix -

$C_{ij} = \begin{Bmatrix} M_{ij}, & \text{if } i+j \text{ is even} \\ -M_{ij}, & \text{if } i+j \text{ is odd} \end{Bmatrix}$ where $M_{ij}$ represents minor of matrix.

Q- Find the cofactor of the given matrix

(1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 4 \end{bmatrix}$

Sol$^n$- If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 4 \end{bmatrix}$, then cofactor of $A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 1(8-3) = 5$

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = -1(12-1) = -11$

$C_{13} = + \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = 9-2 = 7$

$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -(8-9) = 1$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = (4-3) = 1$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -(3-2) = -1$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = (2-6) = -4$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -(1-9) = 8$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = (2-6) = -4$

$\therefore$ Cofactor matrix of $A = \begin{bmatrix} 5 & -11 & 7 \\ 1 & 1 & -1 \\ -4 & 8 & -4 \end{bmatrix}$

② $A = \begin{bmatrix} 2 & 5 & -3 \\ 4 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix}$

If $A = \begin{bmatrix} 2 & 5 & -3 \\ 4 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix}$ then, cofactor matrix of $A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$C_{11} = +\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} = -1-10 = -11$

$C_{12} = -\begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix} = -(-4-6) = 10$

$C_{13} = (-1)^4 \begin{vmatrix} 4 & 1 \\ 3 & 5 \end{vmatrix} = (20-3) = 17$

$C_{21} = -\begin{vmatrix} 5 & -3 \\ 5 & -1 \end{vmatrix} = -(-5+15) = -10$

$C_{22} = \begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix} = -2+9 = 7$

$C_{23} = -\begin{vmatrix} 2 & 5 \\ 3 & 5 \end{vmatrix} = -(10-15) = 5$

$C_{31} = \begin{vmatrix} 5 & -3 \\ 1 & 2 \end{vmatrix} = (10+3) = 13$

$C_{32} = -\begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix} = -(4+12) = -16$

$C_{33} = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = 2-20 = -18$

Hence, cofactor matrix of A is

$A = \begin{bmatrix} -11 & 10 & 17 \\ -10 & 7 & 5 \\ 13 & -16 & -18 \end{bmatrix}$

27/09/23

## Properties of determinants –

Property 1 – If each entry in any row or column of a determinant is zero(0) then the value of the determinant is zero (0).

Ex-1 $A = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = 0(10-12) - 0(5-9) + 0(4-6) = 0+0+0 = 0$

**Property-2** ⇒ If two rows or column are interchange in a determinants, it written in its absolute value but changes its sign.

Example -

$$A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 1(2-3)-2(2-9)+1(1-3) = -1-2(-7)+(-2)$$

$$= -1+14-2$$

$$= \cancel{12}\ 11$$

$$A = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1(4-1)-1(2-3)+3(1-6) = 3+1-15$$

$$= -11$$

**Property 3** ⇒ If rows are changed into colums and columns into rows the determinants remain unchanged.

Example -

$$A' = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 1(2-3)-1(4-1)+3(6-1) = -1-3+15 = 11$$

**Property 4** ⇒ If two rows or columns are identical then value of determinants is zero.

Example -

$$A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1(4-1)-2(2-3)+1(1-6) = 3+2-5 = 0$$

**Property 5** ⇒ If any row or column is multiplied by a complex number k the determinant so obtained is k times of the original value.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example -

$$A = \begin{vmatrix} 1 & 2 & 1 \\ 10 & 15 & 20 \\ 3 & 1 & 2 \end{vmatrix} = 1(30-20)-2(20-60)+1(10-45) = 10+80-35$$

$$= 90-35$$

$$= 55$$

$$A' = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{vmatrix} = 1(6-4)-2(4-12)+1(2-9) = 2+16-7 = 11$$

Ques- Without expanding the determinants prove that -

$$\begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Soln A given determinants -

$$\begin{vmatrix} b+c & c+a & a+b \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 = R_1 + R_2$,

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$(a+b+c)$ is a factor of $R_1$, Hence,

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$R_1$ is identical with $R_3$, Hence, $= (a+b+c) \times 0$

$= 0 = RHS$ Hence proved.

Q- Find the value of the given determinants (without expanding)

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Soln- A given determinant - $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Interchanging row and column

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

$R_2 \leftarrow R_2 + R_3$

$$\begin{vmatrix} 1 & 1 & 1 \\ a+b+c & a+b+c & a+b+c \\ b+c & c+a & a+b \end{vmatrix}$$

$\therefore (a+b+c)$ is a factor of $R_2$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ b+c & c+a & a+b \end{vmatrix} \Rightarrow (a+b+c) \times 0 = 0$$

Q- $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Soln A given determinant $= \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$R_1 \to R_1 + R_2$

$\therefore \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$\therefore (x+y+z)$ is a factor of $R_1$

$\Rightarrow (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$\Rightarrow (x+y+z) \times 0$

$\Rightarrow 0$ Ans

04/10/23

Q- Evaluate a given determinants. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & 6 \end{vmatrix}$

Soln A given determinant $= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 3. & 6. \end{vmatrix}$

$\Rightarrow$ $R_2 = R_2 - C_1$ and $C_3 = C_3 - C_1$

$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 2 & 5 \end{vmatrix} = 1(5-4) = 1 \times 1 = 1$

- $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{vmatrix}$

$\therefore$ After interchanging row and column, we get $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & 6 \end{vmatrix} = 1$

Q- $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix}$

Sol$^n$- Applying $C_1 \to C_1 - C_2$, we get,

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{vmatrix}$$

Now, $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} 0 & -2 & -5 \\ 0 & -1 & -3 \\ 1 & 4 & 6 \end{vmatrix} = 1(+6-5) = 1$$

Q- Without expanding the determinant find the value of given determinant

$$\begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix}$$

Sol$^n$- $\underbrace{\begin{vmatrix} 3x & 2x & x \\ 4x & 3x & 3x \\ 5x & 4x & 6x \end{vmatrix}}_{A_1} + \underbrace{\begin{vmatrix} y & 2x & x \\ 3y & 3x & 3x \\ 6y & 4x & 6x \end{vmatrix}}_{A_2}$

$\Rightarrow$ Applying $C_1 \to C_1 - C_2$ in $A_1$

$$A_1 = \begin{vmatrix} x & 2x & x \\ x & 3x & 3x \\ x & 4x & 6x \end{vmatrix}$$

Now, $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$

$$A_1 = \begin{vmatrix} 0 & -2x & -5x \\ 0 & -x & -3x \\ x & 4x & 6x \end{vmatrix} = x(6x^2 - 5x^2) = x \times x^2 = x^3$$

Now, $A_2 = \begin{vmatrix} y & 2x & x \\ 3y & 3x & 3x \\ 6y & 4x & 6x \end{vmatrix} = 3\begin{vmatrix} y & 2x & x \\ y & x & x \\ 6y & 4x & 6x \end{vmatrix} = 3yx\begin{vmatrix} 1 & 2x & 1 \\ 1 & x & 1 \\ 6 & 4x & 6 \end{vmatrix}$

$= 3yx \times 0$

$= 0$

$$\begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = x^3 + 0 = x^3$$

5-10-23

Q- Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Soln $|A| = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$

$|A| = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$|A| = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Now, $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$|A| = (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$

$|A| = (a+b+c)(-b-c-a)(-c-a-b)$

$|A| = (a+b+c)(a+b+c)(a+b+c)$

$|A| = (a+b+c)^3$ H.P

Q- Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Soln Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$LHS = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ a & (b-a) & (c-a) \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & (b+a) & (c+a) \end{vmatrix}$

$= (b-a)(c-a)(c+a-b-a)$

$= (b-a)(c-a)(c-b)$

$= (a-b)(b-c)(c-a)$

$= RHS$ H.P

09-10-23

Q- Solve the given determinants.

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Soln- Applying $R_1 \to R_1 + R_2 + R_3$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Now, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 1(4-0) = 4 \quad \text{Ans}$$

Q- Prove that $\begin{vmatrix} -a^2 & ba & ca \\ ab & -b^2 & cb \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$.

Soln LHS $= \begin{vmatrix} -a^2 & ba & ca \\ ab & -b^2 & cb \\ ac & bc & -c^2 \end{vmatrix} = abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= a^2b^2c^2 \times 4$

$= 4a^2b^2c^2 = RHS$ H.P

Q- Prove that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

Solⁿ- LHS $= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1+C_2+C_3$

$$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2c+2a+2b & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Now, applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)\{1((a+b+c)(a+b+c)-0)\}$$

$$= 2(a+b+c)(a+b+c)^2$$

$$= 2(a+b+c)^3$$

$=$ RHS h.p

10-10-23

Q- Using properties of determinants show that

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$$

Soln- $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

$$LHS = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

multiplying by $a, b$ & $c$ in $R_1, R_2$ & $R_3$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1-1 & a-a & a^2-a^2 \\ 1-1 & b-b & b^2-b^2 \\ 1-1 & c-c & c^2-c^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0 = RHS$ H.P

11-10-23

Q- If $x, y$ and $z$ are different, then show that $1+xyz=0$, if $(1+xyz)(x-y)(y-z)(z-x)=0$

Q- Evaluate $\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+z & z+x \end{vmatrix}$

sol^n- $|A| = 1(\cancel{z}+x-y-\cancel{z}) = (x-y)$

Q- Evaluate $(1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$.

Sol$^n$ $LHS = (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow 1+xyz\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz)(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & (y+x) \\ 0 & 1 & (z+x) \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz)(y-x)(z-x)(z+\cancel{x}-y-\cancel{x}) = 0$$

$$\Rightarrow (1+xyz)(y-x)(z-x)(z-y) = 0$$

$$\Rightarrow (1+xyz)(x-y)(y-z)(z-x) = 0$$

But $(x-y) \neq 0$, $(y-z) \neq 0$, $(z-x) \neq 0$

So, $(1+xyz) = 0$ Ans

Q- $\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

~~Sol$^n$ $\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz)$~~

Sol$^n$- Interchange $C_2$ and $C_3$,

$$\Rightarrow -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Interchanging $C_1$ & $C_2$

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Rightarrow (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Now, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$

$$\Rightarrow (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{vmatrix}$$

$$\Rightarrow (1+xyz)(z-x)(y-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & (y+x) \\ 0 & 1 & (z+x) \end{vmatrix}$$

$$\Rightarrow (1+xyz)(z-x)(y-x)(z+\cancel{x}-y-\cancel{x})$$

$$\Rightarrow (1+xyz)(z-x)(y-x)(z-y)$$

$$\Rightarrow (1+xyz)(x-y)(y-z)(z-x) \quad \text{Ans}$$

Q- If $x, y, z$ are different $A = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

then prove that $1+xyz = 0$

Soln $A = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$A = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$\because x, y, z$ are the factor of $R_1, R_2$ and $R_3$, so take $x, y, z$ as common

$$A = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$\Rightarrow$ Interchanging $C_2$ and $C_3$, then $C_1$ & $C_2$

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

Now, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{vmatrix} = 0$$

$\because (y-x)$ is cofactor of $R_2$ and $(z-x)$ is cofactor of $R_3$, take $(y-x)$ & $(z-x)$ as common

$$\Rightarrow (1+xyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & (y+x) \\ 0 & 1 & (z+x) \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz)(y-x)(z-x)(z+x-y-x) = 0$$

$$\Rightarrow (1+xyz)(y-x)(z-x)(z-y) = 0$$

But above it is given that $x, y, z$ are different, so $(y-x) \neq 0$, $(z-x) \neq 0$ and $(z-y) \neq 0$

Hence, $(1+xyz) = 0$ H.P.

12-10-23

Q- using the properties of determinants show that

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Sol$^n$ LHS = $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

~~On adding~~ Applying $R_1 \to R_1 + R_2 + R_3$,

$$= \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

hence, in $R_1$ all the elements are zero, so by using property

$\Delta = 0$

i.e. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$ H.P

15/10/23

# Matrix

## Addition of matrix —

Q-1) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 5 \\ 6 & 7 & 8 \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 6 & 8 \\ 10 & 12 & 14 \\ 8 & 9 & 10 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} a & b & c \\ c & 2 & d \\ a & b & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1+a & 1+b & 2+c \\ 2+c & 5 & 4+d \\ 1+a & 1+b & 12 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$ Not possible

4) $\begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \\ a & b & d \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & d-c \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & d-c \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ a & b & d \\ 1 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ a & b & d \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \\ a & b & d \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

## Substraction

1) $\begin{bmatrix} -1 & -4 & -2 \\ -6 & -4 & -13 \\ -10 & -2 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 2 & -4 \\ -6 & -5 & -2 \\ -4 & -3 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1+1 & -4-2 & -2+4 \\ -6+6 & -4+5 & -13+2 \\ -10+4 & -2+3 & -1+2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & -6 & 2 \\ 0 & 1 & -11 \\ -6 & 1 & 1 \end{bmatrix}$

Multiplication of matrix -

1) $\cancel{\begin{bmatrix} 1 & 1 & 1 \\ 2 & & 2 \\ 3 & 1 & 0 \end{bmatrix}}$

1) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2+3 & 2+2 \\ 2+6 & 2+4 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 \\ 8 & 6 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1+1 & 1+2 \\ 1+2 & 1+4 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ = Not possible

~~4)~~ $\cancel{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1+0+2 & 1+0 \\ & \\ & \end{bmatrix}}$

4) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1+0+2 & 1+0+2 \\ 1+1+0 & 1+2+0 \\ 1+2+4 & 1+4+4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}$

$\Rightarrow \underset{(3\times 2)}{\begin{bmatrix} 3 & 3 \\ 2 & 3 \\ 7 & 9 \end{bmatrix}} \underset{(3\times 2)}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}}$ $\Rightarrow$ Not possible.

5) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times3} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{3\times1} \Rightarrow \begin{bmatrix} 1+1+1 \\ 1+1+1 \\ 1+1+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

6) $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$A^2 = A \times A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+2 \\ 1+1 & 2+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

7) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then show that $A^2 - 4A - 5I = 0$

Sol$^n$ $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$

$5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Now, $A^2 - 4A - 5I = 0$

$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$ H.P

Q- If $A = \begin{bmatrix} x & -3 \\ 4 & -y \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $A+B = 2I$.

Sol$^n$: $A+B = 2I$

$$\begin{bmatrix} x & -3 \\ 4 & -y \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x+3 & 1 \\ 5 & 2-y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\therefore x+3 = 2$

$\boxed{x = -1}$

$2-y = 2$

$-y = 0$

$\boxed{y = 0}$

Q- If $A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 1 & 6 \\ 0 & 2 & 5 \end{bmatrix}$, then find the value of $A^3$.

Sol$^n$: $A^3 = A \times A \times A$

$$A \times A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 1 & 6 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -4 & 0 \\ 2 & 1 & 6 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 9-8+0 & -12-4+0 & 0-24+0 \\ 6+2+0 & -8+1+12 & 0+6+30 \\ 0+4+0 & 0+2+10 & 0+12+25 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -16 & -24 \\ 8 & 5 & 36 \\ 4 & 12 & 37 \end{bmatrix}$$

Now, $$A^2 \cdot A = \begin{bmatrix} 1 & -16 & -24 \\ 8 & 5 & 36 \\ 4 & 12 & 37 \end{bmatrix} \begin{bmatrix} 3 & -4 & 0 \\ 2 & 1 & 6 \\ 0 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-32+0 & -4-16-48 & 0-96-120 \\ 24+10+0 & -32+5+72 & 0+30+180 \\ 12+24+0 & -16+12+74 & 0+72+185 \end{bmatrix}$$

$$= \begin{bmatrix} -29 & -68 & -216 \\ 34 & 45 & 210 \\ 36 & 70 & 257 \end{bmatrix}$$

# Types of Matrix -

1) Rectangular matrix - row ≠ column
2) Square matrix - row = column
3) Row matrix and column matrix -

matrix having only one row

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$$

matrix having only one column

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$$

4) Diagonal matrix -

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

5) Scalar matrix -

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{bmatrix}$$

6) ~~Scalar~~ Identity matrix - (unit)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7) Null matrix -

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

8) Triangular matrix -

(a) Upper Triangular Matrix -

$$\begin{bmatrix} a_1 & b_1 & b_2 & b_3 \\ 0 & a_2 & c_1 & c_2 \\ 0 & 0 & a_3 & d_1 \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$

(b) Lower Triangular matrix -

$$\begin{bmatrix} a_1 & 0 & 0 & 0 \\ d_1 & a_2 & 0 & 0 \\ c_1 & c_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & a_4 \end{bmatrix}$$

## 9) Transpose Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A' = A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

## 10) Symmetric Matrix

$$\boxed{A^T = A}$$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

i.e. $A = \begin{bmatrix} a & h & i \\ h & b & j \\ i & j & c \end{bmatrix}$

## 11) Skew symmetric matrix

$$\boxed{A^T = -A}$$

$$A = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & 6 \\ -5 & -6 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -4 & -5 \\ 4 & 0 & -6 \\ 5 & 6 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & 6 \\ -5 & -6 & 0 \end{bmatrix} = -A$$

i.e. $A = \begin{bmatrix} 0 & h & i \\ -h & 0 & g \\ -i & -g & 0 \end{bmatrix}$

Q- Prove that for $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$, $A + A^T$ is a symmetric matrix where $A^T$ is transpose of A.

Soln- $A^T = \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix}$

$$(A + A^T) = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 9 & 12 \end{bmatrix}$$

$$(A + A^T)^T = \begin{bmatrix} 4 & 9 \\ 9 & 12 \end{bmatrix}$$

$\therefore (A + A^T) = (A + A^T)^T$

So, $(A + A^T)$ is a symmetric matrix.

Q- If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, prove that $A - A^T$ is a skew symmetric matrix.

Soln. $A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

$$A - A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(A - A^T)^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -(A - A^T)$$

$\therefore (A - A^T)$ is a skew symmetric matrix.

16-10-23

Orthogonal matrix -

i) $\boxed{A \cdot A^T = A^T \cdot A = I}$

Q- Show that the matrix

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -\frac{1}{\sqrt{2}} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & \frac{1}{\sqrt{2}} \end{bmatrix}$$ is orthogonal matrix.

Sol$^n$ - $A^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

Now, $A \cdot A^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} 1/3 + 1/6 + 1/2 & 1/3 - 2/6 - 0 & 1/3 + 1/6 - 1/2 \\ 1/3 - 2/6 + 0 & 1/3 + 4/6 + 0 & 1/3 - 2/6 + 0 \\ 1/3 + 1/6 - 1/2 & 1/3 - 2/6 + 0 & 1/3 + 1/6 + 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2+1+3}{6} & 0 & \frac{2+1-3}{6} \\ 0 & \frac{3}{3} & 0 \\ \frac{2+1-3}{6} & 0 & \frac{2+1+3}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$ Hence, orthogonal matrix.

Q- Show that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal matrix.

Sol$^n$ - $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta.\sin\theta + \sin\theta.\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, it is an orthogonal matrix.

(Applicable only for square matrix)

- Idempotent matrix – $A^2 = A$
- Nilpotent matrix – $A^m = 0$, where m is even = 2 (and order of matrix).
- Involutary matrix – $A^2 = I$

Q- Show that matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent matrix.

Sol$^n$- $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} \cancel{4}+2-\cancel{4} & -4-6+8 & -8+8+12 \\ -2-\cancel{3}+4 & 2+9-8 & 4+\cancel{12}-\cancel{12} \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A \quad \text{Hence. proved.}$$

Q- Show that the matrix $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}_{2\times2}$ is nilpotent matrix.

Sol$^n$. $A^m = A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence, A is nilpotent matrix.

18/10/23

Conjugate matrix – $(\bar{A})$

If $A = \begin{bmatrix} 3+i & i+4 \\ 3-i & i+1 \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 3-i & -i+4 \\ 3+i & -i+1 \end{bmatrix}$

Ex- Find the conjugate matrix of $A = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

Soln $\bar{A} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Tranjugate matrix – $(A^\theta)$

$\therefore \boxed{A^\theta = (\bar{A})^T}$

Ex- Find the tranjugate of the given matrix $A = \begin{bmatrix} 1+i & i-1 & 0 \\ 3i & 4i & 1 \\ i & -i & i \end{bmatrix}$

Soln. $\bar{A} = \begin{bmatrix} 1-i & -i-1 & 0 \\ -3i & -4i & 1 \\ -i & i & -i \end{bmatrix}$

$A^\theta = (\bar{A})^T = \begin{bmatrix} 1-i & -3i & -i \\ -i-1 & -4i & i \\ 0 & 1 & -i \end{bmatrix}$

Unitary matrix = $\boxed{A^\theta . A = I}$

Hermition Matrix – $\boxed{A^\theta = A}$

Skew Hermition matrix – $\boxed{A^\theta = -A}$

Ex- If $A = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1+i \\ -1+i & 0 \end{bmatrix}$, then prove that $(A+B)^\theta = A^\theta + B^\theta$

Soln ⇒ $A+B = \begin{bmatrix} 3 & 2+3i \\ -i & 2 \end{bmatrix}$

Now, $\overline{(A+B)} = \begin{bmatrix} 3 & 2-3i \\ i & 2 \end{bmatrix}$

$(A+B)^\theta = (\overline{A+B})^T = \begin{bmatrix} 3 & i \\ 2-3i & 2 \end{bmatrix}$

Now, $\bar{A} = \begin{bmatrix} 3 & 1-2i \\ 1+2i & 2 \end{bmatrix}$ $\bar{B} = \begin{bmatrix} 0 & 1-i \\ -1-i & 0 \end{bmatrix}$

$A^\theta = (\bar{A})^T = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ $B^\theta = (\bar{B})^T = \begin{bmatrix} 0 & -1-i \\ 1-i & 0 \end{bmatrix}$

Now, $A^\theta + B^\theta = \begin{bmatrix} 3 & i \\ 2-3i & 2 \end{bmatrix}$

$\therefore$ $\boxed{(A+B)^\theta = A^\theta + B^\theta}$ Hence proved.

Q- Prove that $(AB)^\theta = B^\theta A^\theta$.

$A = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1+i \\ -1+i & 0 \end{bmatrix}$

Soln $AB = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix} \begin{bmatrix} 0 & 1+i \\ -1+i & 0 \end{bmatrix}$

$= \begin{bmatrix} 0+(1+2i)(-1+i) & 3+3i+0 \\ 0-2+2i \ \cancel{1-2i-2+2i} & (1-2i)(1+i)+0 \end{bmatrix}$

$= \begin{bmatrix} -1+i-2i+2i^2 & 3+3i \\ -2+2i & (1+i-2i-2i^2) \end{bmatrix}$

$= \begin{bmatrix} -1-i-2 & 3+3i \\ -2+2i & (1-i+2) \end{bmatrix}$

$AB = \begin{bmatrix} -3-i & 3+3i \\ -2+2i & 3-i \end{bmatrix}$

$\overline{AB} = \begin{bmatrix} -3+i & 3-3i \\ -2-2i & 3+i \end{bmatrix}$ $(AB)^\theta = (\overline{AB})' = \begin{bmatrix} -3+i & -2-2i \\ 3-3i & 3+i \end{bmatrix}$

Now, $A^\theta = (\bar{A})' = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ and $B^\theta = (\bar{B})' = \begin{bmatrix} 0 & -1-i \\ 1-i & 0 \end{bmatrix}$

$$B^{\theta}\cdot A^{\theta} = \begin{bmatrix} 0 & -1-i \\ 1-i & 0 \end{bmatrix}\begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0-(1+i)(1-2i) & 0-2-2i \\ 3-3i+0 & (1-i)(1+2i)+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0-(1-2i+i-2i^2) & -2-2i \\ 3-3i & (1+2i-i-2i^2) \end{bmatrix}$$

$$= \begin{bmatrix} -(1-i+2) & -2-2i \\ 3-3i & (1+i+2) \end{bmatrix}$$

$$B^{\theta}\cdot A^{\theta} = \begin{bmatrix} -3+i & -2-2i \\ 3-3i & 3+i \end{bmatrix}$$

$\therefore \boxed{LHS = RHS}$ H.P

19/10/23

Ques- If $A = \begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$, then find out the

i) $A + A^{\theta}$

ii) $A - A^{\theta}$

Sol$^n$ $\bar{A} = \begin{bmatrix} -2-3i & 1+i & 2-i \\ 3 & 4+5i & 5 \\ 1 & 1-i & -2-2i \end{bmatrix}$

$A^{\theta} = (\bar{A})' = \begin{bmatrix} -2-3i & 3 & 1 \\ 1+i & 4+5i & 1-i \\ 2-i & 5 & -2-2i \end{bmatrix}$

Now, $A + A^{\theta} = \begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix} + \begin{bmatrix} -2-3i & 3 & 1 \\ 1+i & 4+5i & 1-i \\ 2-i & 5 & -2-2i \end{bmatrix}$

$$= \begin{bmatrix} -4 & 4-i & 3+i \\ 4+i & 8 & 6-i \\ 3-i & 6+i & [illegible] \end{bmatrix}$$

ii) $A - A^{\theta} = \begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix} - \begin{bmatrix} -2-3i & 3 & 1 \\ 1+i & 4+5i & 1-i \\ 2-i & 5 & -2-2i \end{bmatrix}$

$= \begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & -10i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix}$ Ans

iii) $P = \frac{1}{2}(A + A^{\theta})$

Soln- $P = \frac{1}{2}(A+A^{\theta}) = \frac{1}{2}\begin{bmatrix} -4 & 4-i & 3+i \\ 4+i & 8 & 6-i \\ 3-i & 6+i & -4 \end{bmatrix}$

$P = \begin{bmatrix} -2 & 2-i/2 & 3/2+i/2 \\ 2+i/2 & 4 & 3-i/2 \\ 3/2-i/2 & 3+i/2 & -2 \end{bmatrix}$ Ans

iv) $Q = \frac{1}{2}(A - A^{\theta})$

Soln- $Q = \frac{1}{2}(A-A^{\theta}) = \frac{1}{2}\begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & -10i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix}$

$Q = \begin{bmatrix} 3i & -1-i/2 & 1/2+i/2 \\ 1-i/2 & -5i & 2+i/2 \\ -1/2+i/2 & -2+i/2 & 2i \end{bmatrix}$ Ans

Now, $P+Q = \begin{bmatrix} -2 & 2-i/2 & 3/2+i/2 \\ 2+i/2 & 4 & 3-i/2 \\ 3/2-i/2 & 3+i/2 & -2 \end{bmatrix} + \begin{bmatrix} 3i & -1-i/2 & 1/2+i/2 \\ 1-i/2 & -5i & 2+i/2 \\ -1/2+i/2 & -2+i/2 & 2i \end{bmatrix}$

$= \begin{bmatrix} -2+3i & 1-i & 2i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix} = A$ Ans

v) Find the tranjugate of P.

Sol$^n$ $\bar{P} = \begin{bmatrix} -2 & 2+i/2 & 3/2 - i/2 \\ 2-i/2 & 4 & 3+i/2 \\ 3/2+i/2 & 3-i/2 & -2 \end{bmatrix}$

$P^\theta = (\bar{P})' = \begin{bmatrix} -2 & 2-i/2 & 3/2+i/2 \\ 2+i/2 & 4 & 3-i/2 \\ 3/2-i/2 & 3+i/2 & -2 \end{bmatrix} = PAns.$

25/10/23

Q- Show that $\begin{bmatrix} 1/2(1+i) & -1/2(1-i) \\ 1/2(1+i) & 1/2(1-i) \end{bmatrix}$ is unitary matrix.

Sol$^n$ $\bar{A} = \begin{bmatrix} 1/2(1-i) & -1/2(1+i) \\ 1/2(1-i) & 1/2(1+i) \end{bmatrix}$

$A^\theta = (\bar{A})' = \begin{bmatrix} 1/2(1-i) & 1/2(1-i) \\ -1/2(1+i) & 1/2(1+i) \end{bmatrix}$

Now, $\boxed{A^\theta . A = I \text{ \{For unitary matrix\}}}$

$LHS = A^\theta . A = \begin{bmatrix} 1/2(1-i) & 1/2(1-i) \\ -1/2(1+i) & 1/2(1+i) \end{bmatrix} \begin{bmatrix} 1/2(1+i) & -1/2(1-i) \\ 1/2(1+i) & 1/2(1-i) \end{bmatrix}$

$= \begin{bmatrix} 1/4(1-i^2) + 1/4(1-i^2) & -1/4(1-i)^2 + 1/4(1-i)^2 \\ -1/4(1+i)^2 + 1/4(1+i)^2 & 1/4(1-i^2) + 1/4(1-i^2) \end{bmatrix}$

$= \begin{bmatrix} 1/2(1-(-1)) & 0 \\ 0 & 1/2(1-(-1)) \end{bmatrix}$

$= \begin{bmatrix} 2/2 & 0 \\ 0 & 2/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = RHS$ hp

## Adjoint of matrix- Adjoint of $A = [\text{cofactor of } A]^T$

Q- Find the cofactor matrix for given matrix

$$A = \begin{bmatrix} 6 & -5 \\ 7 & 3 \end{bmatrix}$$

Soln -
$a_{11} = 3$
$a_{12} = -7$
$a_{21} = +5$
$a_{22} = 6$

$$\text{Cofactor matrix of } A = \begin{bmatrix} 3 & -7 \\ 5 & 6 \end{bmatrix}$$

Q- Find adjoint of matrix $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

Soln
$C_{11} = (2-3) = -1$ $\quad C_{21} = -(-2-3) = 5$ $\quad C_{31} = (6+6) = 12$
$C_{12} = -(-1-0) = 1$ $\quad C_{22} = (-2-0) = -2$ $\quad C_{32} = -(6-3) = -3$
$C_{13} = (1-0) = 1$ $\quad C_{23} = -(2-0) = -2$ $\quad C_{33} = (-\cancel{4}-2) = -4$

$$\text{Cofactor matrix} = \begin{bmatrix} -1 & 1 & 1 \\ 5 & -2 & -2 \\ 12 & -3 & -\cancel{6} \end{bmatrix}$$

$$\text{Adjoint of } A = \begin{bmatrix} -1 & 5 & 12 \\ 1 & -2 & -3 \\ 1 & -2 & -\cancel{6} \end{bmatrix}$$

26-10-23

$$\text{Inverse of matrix } (A^{-1}) = \frac{\text{adj}(A)}{|A|}$$

Q- Find $A^{-1}$ where $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Soln $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$C_{11} = -1$ $\quad C_{21} = -(-1) = 1$ $\quad C_{31} = 1$

$C_{12} = -(-1) = 1$ $\quad C_{22} = -1$ $\quad C_{32} = -(-1) = 1$

$C_{13} = 1$ $\quad C_{33} = -(-1) = 1$ $\quad C_{33} = -1$

Cofactor matrix of $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Now, $\operatorname{adj}(A) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

~~$|A| =$~~ Now, $|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$

$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 1(1+1) = 2$

Now, $A^{-1} = \dfrac{\operatorname{adj} A}{|A|} = \dfrac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$

Q- Find $A^{-1}$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

Soln $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

$C_{11} = 36 - 25 = 11$

$C_{12} = -(12-5) = -7$

$C_{13} = 5 - 3 = 2$

$C_{21} = -(24-15) = -9$

$C_{22} = 12 - 3 = 9$

$C_{23} = -(5-2) = -3$

$C_{31} = 10 - 9 = 1$

$C_{32} = -(5-3) = -2$

$C_{33} = 3 - 2 = 1$

Cofactor matrix of $A = \begin{bmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{bmatrix}$ , $Adj(A) = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

Now, $|A| = 1 \times 11 + (-7) \times 2 + (2) \times 3$

$= 11 - 14 + 6$

$= 11 - 8 = 3$

$$A^{-1} = \frac{Adj(A)}{|A|} = \frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11/3 & -9/3 & 1/3 \\ -7/3 & 9/3 & -2/3 \\ 2/3 & -3/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 11/3 & -3 & 1/3 \\ -7/3 & 3 & -2/3 \\ 2/3 & -1 & 1/3 \end{bmatrix}$$

30/10/23

Q- If $A = \begin{bmatrix} 5 & 7 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix}$ then prove that

$Adj(AB) = Adj(B) \cdot Adj(A)$

Soln. $A \cdot B = \begin{bmatrix} 5 & 7 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 15-7 & 10+35 \\ 9+2 & 6-10 \end{bmatrix} = \begin{bmatrix} 8 & 45 \\ 11 & -4 \end{bmatrix}$

Now, Cofactor matrix of $AB = \begin{bmatrix} -4 & -11 \\ -45 & 8 \end{bmatrix}$

LHS ⇒ Now, $Adj(AB) = \begin{bmatrix} -4 & -45 \\ -11 & 8 \end{bmatrix}$

~~$|AB| = -32 - 495 = -527$~~

Cofactor matrix of $A = \begin{bmatrix} -2 & -3 \\ -7 & 5 \end{bmatrix}$

Cofactor matrix of $B = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$

$Adj(A) = \begin{bmatrix} -2 & -7 \\ -3 & 5 \end{bmatrix}$ and $Adj(B) = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

Now, $Adj(B) \cdot Adj(A)$.

$$\Rightarrow \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & -7 \\ -3 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -10+6 & -35-10 \\ -2+9 & -7+15 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -45 \\ -11 & 8 \end{bmatrix} \Rightarrow RHS$$

$\therefore$ $\boxed{LHS = RHS}$

i.e. $Adj(AB) = Adj(B) \cdot Adj(A)$

Q- If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ then find the following -

i) A-B

soln $A \cdot B = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 12+6 & 15+8 \\ 28+15 & 35+20 \end{bmatrix} = \begin{bmatrix} 18 & 23 \\ 43 & 55 \end{bmatrix}$

ii) $B^{-1}$

soln $Adj(B) = [\text{Cofactor of } B]'$

Cofactor of $B = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}$

$Adj(B) = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

$|B| = 16 - 15 = 1$

$B^{-1} = \frac{1}{1} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

$B^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

iii) $A^{-1}$

soln. Cofactor of $A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

Now, $Adj(A) = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$|A| = 15 - 14 = 1$

$A^{-1} = \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

(iv) $(AB)^{-1}$

Sol$^n$ $A \cdot B = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 12+6 & 15+8 \\ 28+15 & 35+20 \end{bmatrix} = \begin{bmatrix} 18 & 23 \\ 43 & 55 \end{bmatrix}$

Cofactor of AB $= \begin{bmatrix} 55 & -43 \\ -23 & 18 \end{bmatrix}$

$\text{Adj}(AB) = \begin{bmatrix} 55 & -23 \\ -43 & 18 \end{bmatrix}$

$|AB| = 18 \times 55 - 23 \times 43 = 990 - 989 = 1$

$(AB)^{-1} = \dfrac{\text{adj}(AB)}{|AB|} = \begin{bmatrix} 55 & -23 \\ -43 & 18 \end{bmatrix}$

v) $B^{-1} \cdot A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \begin{bmatrix} 20+35 & -8-15 \\ -15-28 & 6+12 \end{bmatrix} = \begin{bmatrix} 55 & -23 \\ -43 & 18 \end{bmatrix}$

$\therefore \boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$

Ques – If $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$, Verify that $[A^T]^{-1} = [A^{-1}]^T$

Sol$^n$. $A^T = \begin{bmatrix} 1 & 3 & -1 \\ 3 & -1 & 5 \\ 4 & 6 & 1 \end{bmatrix}$

Now, cofactor matrix of $A^T$,

$C_{11} = -1-30 = -31$ $\quad C_{21} = -(3+6) = -9$ $\quad C_{31} = 15-1 = 14$

$C_{12} = -(3-20) = 17$ $\quad C_{22} = 1+4 = 5$ $\quad C_{32} = -(5+3) = -8$

$C_{13} = 18+4 = 22$ $\quad C_{23} = -(6-12) = 6$ $\quad C_{33} = -1-9 = -10$

Now, Cofactor matrix of $A^T = \begin{bmatrix} -31 & 17 & 22 \\ -9 & 5 & 6 \\ 14 & -8 & -10 \end{bmatrix}$

$$Adj(A^T) = \begin{bmatrix} -31 & -9 & 14 \\ 17 & 5 & -8 \\ 22 & 6 & -10 \end{bmatrix}$$

$$|A^T| = \begin{vmatrix} 1 & 3 & -1 \\ 3 & -1 & 5 \\ 4 & 6 & 1 \end{vmatrix} = -31 + 51 - 22 \Rightarrow -53 + 51 = -2$$

Applying $C_1 \to C_1 + C_2$

$$|A^T| = \begin{vmatrix} 0 & 3 & -1 \\ 8 & -1 & 5 \\ 5 & 6 & 1 \end{vmatrix}$$

$$\text{Now, } |A^T| = -3(8-25) - 1(48+5) = -3(-23) - 1(53)$$
$$= 69 - 53 = 16$$

$$\text{Now, } [A^T]^{-1} = \frac{adj(A^T)}{|A^T|} = -\frac{1}{2} \begin{bmatrix} -31 & -9 & 14 \\ 17 & 5 & -8 \\ 22 & 6 & -10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj(A)}{|A|}$$

Cofactor matrix of $A \Rightarrow$

$C_{11} = -1 - 30 = -31$ $\quad C_{21} = -(3-20) = 17$ $\quad C_{31} = 22$

$C_{12} = -(3+6) = -9$ $\quad C_{22} = 1 + 4 = 5$ $\quad C_{32} = 6$

$C_{13} = 15 - 1 = 14$ $\quad C_{23} = -(5+3) = -8$ $\quad C_{33} = -10$

$$Adj(A) = \begin{bmatrix} -31 & -9 & 14 \\ 17 & 5 & -8 \\ 22 & 6 & -10 \end{bmatrix}^T = \begin{bmatrix} -31 & 17 & 22 \\ -9 & 5 & 6 \\ 14 & -8 & -10 \end{bmatrix}$$

$$\therefore |A| = -31 - 27 + 56 \Rightarrow 56 - 58 = -2$$

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} -31 & 17 & 22 \\ -9 & 5 & 6 \\ 14 & -8 & -10 \end{bmatrix} \qquad \therefore [A^{-1}]^T = -\frac{1}{2} \begin{bmatrix} -31 & -9 & 14 \\ 17 & 5 & -8 \\ 22 & 6 & -10 \end{bmatrix}$$

$$\therefore [A^T]^{-1} = [A^{-1}]^T$$

01/11/23

## Unit-2

# Limits And Continuity

Q-1) Evaluate $\lim_{x\to 3} \frac{x^2-1}{x}$.

Soln $\lim_{x\to 3} \frac{x^2-1}{x}$

$\Rightarrow \frac{3^2-1}{3} = \frac{9-1}{3} = \frac{8}{3}$ Ans.

$\lim_{x\to 3} \frac{x^2-1}{x} \Rightarrow \frac{\lim_{x\to 3} x^2-1}{\lim_{x\to 3} x}$

$\Rightarrow \frac{9-1}{3} = \frac{8}{3}$

2) Evaluate $\lim_{x\to 0} \frac{\sqrt{9+x}+\sqrt{9-x}}{3+x}$

Soln $\lim_{x\to 0} \frac{\sqrt{9+x}+\sqrt{9-x}}{3+x} \Rightarrow \frac{\lim_{x\to 0} \sqrt{9+x}+\sqrt{9-x}}{\lim_{x\to 0} 3+x}$

$\Rightarrow \frac{\sqrt{9+0}+\sqrt{9-0}}{3+0}$

$\Rightarrow \frac{\sqrt{9}+\sqrt{9}}{3} \Rightarrow \frac{3+3}{3} = \frac{6}{3} = 2$ Ans

## Factorisation Method-

* $a^2-b^2 = (a+b)(a-b)$
* $a^3-b^3 = (a-b)(a^2+ab+b^2)$
* $a^3+b^3 = (a+b)(a^2-ab+b^2)$ or $(a+b)^3-3ab(a+b)$
* $a^4-b^4 = (a^2)^2-(b^2)^2 = (a^2-b^2)(a^2+b^2)$
  $= (a+b)(a-b)(a^2+b^2)$

Q- $x^3-6x^2+11x-6=0$

~~Factor of 6 = ±2, ±3~~

$x = 0, \pm 1, \pm 2 \ldots$

Putting $x=1$

$x^3-6x^2+11x-6$

$\Rightarrow 1-6+11-6$

$\Rightarrow 12-12 = 0$

$(x-1)$ is a first factor of $x^3-6x^2+11x-6=0$

Now,

$x^3-6x^2+11x-6 = (x-1)(x^2-5x+6)$

factor

| | | | | |
|---|---|---|---|---|
| 1 | 1 | -6 | 11 | -6 |
| | | +1 | -5 | 6 |
| | 1 | -5 | 6 | 0 |
| | ↓ $x^2$ | ↓ $x$ | ↓ constant | |

i.e. $\boxed{x^2-5x+6}$

Q- Evaluate $\lim_{x\to1} \frac{x^3-1}{x-1}$

Sol$^n$ $\lim_{x\to1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$

$\Rightarrow \lim_{x\to1} x^2+x+1 \Rightarrow \lim_{x\to1} x^2 + \lim_{x\to1} x + \lim_{x\to1} 1$

$\Rightarrow 1^2+1+1$

$\Rightarrow 3$

Q- Evaluate $\lim_{x\to2} \frac{x^3-6x^2+11x-6}{x^2-6x+8}$

Sol$^n$: $x^3-6x^2+11x-6 = (x-1)(x^2-5x+6) = 0$

Now, $x^2-5x+6=0$

$x^2-2x-3x+6=0$

~~$x(x-2)+1($~~

$x^2-2x-3x+6=0$

$x(x-2)-3(x-2)=0$

$(x-3)(x-2)=0$

Now,

$x^2-6x+8=0$

$x^2-4x-2x+8=0$

$x(x-4)-2(x-4)=0$

$(x-4)(x-2)=0$

$\lim_{x\to2} \frac{(x-1)(x-3)\cancel{(x-2)}}{(x-4)\cancel{(x-2)}}$

$\Rightarrow \frac{(2-1)(2-3)}{(2-4)}$

$\Rightarrow \frac{1\times(-1)}{-2}$

$\Rightarrow \frac{1}{2}$ Ans

Q- Evaluate $\lim_{x\to 1/2} \frac{8x^3-1}{16x^4-1}$

Soln we have,

$$\lim_{x\to 1/2} \frac{8x^3-1}{16x^4-1} \Rightarrow \frac{8\times\frac{1}{8}-1}{16\times\frac{1}{16}-1} \Rightarrow \frac{1-1}{1-1} = \frac{0}{0}$$

i.e. Value is undefined at $x = 1/2$.

So, we will write.

Now, $\lim_{x\to 1/2} \frac{(2x)^3-1^3}{(2x)^4-1^4}$

$$\Rightarrow \lim_{x\to 1/2} \frac{\cancel{(2x-1)}(4x^2+1+2x)}{(2x+1)\cancel{(2x-1)}(4x^2+1)}$$

$$\Rightarrow \frac{\lim_{x\to 1/2} 4x^2+1+2x}{\lim_{x\to 1/2}(2x+1)(4x^2+1)}$$

$$\Rightarrow \frac{4\times\frac{1}{4}+1+2\times\frac{1}{2}}{\left(2\times\frac{1}{2}+1\right)\left(4\times\frac{1}{4}+1\right)}$$

$$\Rightarrow \frac{1+1+1}{2\times 2} \Rightarrow \frac{3}{4}$$

$\Rightarrow$ ~~8/4~~

Q- Evaluate $\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)$

Soln. We have,

$\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)$

After solving,

$$\Rightarrow \frac{2}{1-1^2}+\frac{1}{1-1}$$

$$\Rightarrow \frac{2}{0}+\frac{1}{0}$$

i.e. Value is undefined at $x=1$.

Now, $\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)$

$$\Rightarrow \lim_{x\to 1}\left(\frac{2}{(1+x)(1-x)}+\frac{1}{(x-1)}\right)$$

$$\Rightarrow \lim_{x\to 1}\left(\frac{2}{-(x-1)(-x-1)}+\frac{1}{(x-1)}\right)$$

$$\Rightarrow \lim_{x\to 1}\left[\frac{1}{(x-1)}\left(\frac{+2}{-x-1}+\frac{1}{\cancel{x}}\right)\right]$$

$$\Rightarrow \lim_{x\to 1}\left[\frac{1}{x-1}\left(\frac{+2-x-1}{-x-1}\right)\right]$$

$$\Rightarrow \lim_{x\to 1}\left[\frac{1}{\cancel{(x-1)}}\left(\frac{-\cancel{(x-1)}}{-(x+1)}\right)\right]$$

$$\Rightarrow \lim_{x\to 1}\frac{1}{(x+1)} \Rightarrow \frac{1}{1+1}\Rightarrow\frac{1}{2}$$

Q- Evaluate $\lim_{x\to 1}\left(\frac{1}{x^2+x-2}-\frac{x}{x^3-1}\right)$

Soln. $\lim_{x\to 1}\left(\frac{1}{x^2+x-2}-\frac{x}{x^3-1}\right)$

$$\Rightarrow \lim_{x\to 1}\left(\frac{1}{(x^2+x-2)}-\frac{x}{(x-1)(x^2+x+1)}\right)$$

$$\Rightarrow \lim_{x\to 1}\left(\frac{1}{(x-1)(x+2)}-\frac{x}{(x-1)(x^2+x+1)}\right)$$

$$\Rightarrow \lim_{x\to 1}\left[\frac{1}{(x-1)}\left(\frac{1}{x+2}-\frac{x}{(x^2+x+1)}\right)\right]$$

$$\Rightarrow \lim_{x\to 1}\left[\frac{1}{x-1}\left(\frac{\cancel{x^2}+x+1-\cancel{x^2}-2x}{(x+2)(x^2+x+1)}\right)\right]$$

$$\Rightarrow \lim_{x\to 1}\left[\frac{1}{\cancel{(x-1)}}\left(\frac{-\cancel{(x-1)}}{(x+2)(x^2+x+1)}\right)\right]$$

$$\Rightarrow \lim_{x\to 1}\frac{-1}{(x+2)(x^2+x+1)} \Rightarrow \frac{-1}{(1+2)(1+1+1)} = \frac{-1}{3\times 3} = \frac{-1}{9}$$

## Rationalisation method -

Q- Evaluate limit

$$\lim_{x\to 0} \frac{x}{\sqrt{a+x}-\sqrt{a-x}}$$

Soln. $\lim_{x\to 0} \frac{x(\sqrt{a+x}+\sqrt{a+x})}{(\sqrt{a+x}-\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}$

$\Rightarrow \lim_{x\to 0} \frac{x(\sqrt{a+x}+\sqrt{a-x})}{a+x-(a-x)}$

$\Rightarrow \lim_{x\to 0} \frac{x(\sqrt{a+x}+\sqrt{a-x})}{\cancel{a}+x-\cancel{a}+x}$

$\Rightarrow \lim_{x\to 0} \frac{\cancel{x}(\sqrt{a+x}+\sqrt{a-x})}{2\cancel{x}}$

$\Rightarrow \frac{\sqrt{a+0}+\sqrt{a-0}}{2}$

$\Rightarrow \frac{\sqrt{a}+\sqrt{a}}{2}$

$\Rightarrow \frac{\cancel{2}\sqrt{a}}{\cancel{2}}$

$\Rightarrow \sqrt{a}$

Q- $\lim_{x\to 4} \frac{x^2-16}{\sqrt{x^2+9}-5}$

Soln. $\lim_{x\to 4} \frac{(x-4)(x+4)\{\sqrt{x^2+9}+5\}}{(\sqrt{x^2+9}-5)(\sqrt{x^2+9}+5)}$

$\Rightarrow \lim_{x\to 4} \frac{(x-4)(x+4)(\sqrt{x^2+9}+5)}{x^2+9-25}$

$\Rightarrow \lim_{x\to 4} \frac{\cancel{(x-4)}\cancel{(x+4)}(\sqrt{x^2+9}+5)}{\cancel{(x-4)}\cancel{(x+4)}}$

$\Rightarrow \lim_{x\to 4} \sqrt{x^2+9}+5$

$\Rightarrow \sqrt{16+9}+5$

$\Rightarrow \sqrt{25}+5$

$\Rightarrow 5+5$

$\Rightarrow 10$

Q- $\lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x}$

Sol$^n$- $\lim_{x\to 2} \frac{(\sqrt{3-x}-1)(\sqrt{3-x}+1)}{(2-x)(\sqrt{3-x}+1)}$

$\Rightarrow \lim_{x\to 2} \frac{3-x-1}{(2-x)(\sqrt{3-x}+1)}$

$\Rightarrow \lim_{x\to 2} \frac{\cancel{2-x}}{\cancel{2-x}(\sqrt{3-x}+1)}$

$\Rightarrow \lim_{x\to 2} \frac{1}{(\sqrt{3-x}+1)}$

$\Rightarrow \frac{1}{\sqrt{3-2}+1}$

$\Rightarrow \frac{1}{\sqrt{1}+1} \Rightarrow 1/2$ Ans

## Evaluation of algebraic limit using some standard limits -

- $\boxed{\text{If } \lim_{x\to a} \frac{x^n-a^n}{x-a} \Rightarrow na^{n-1}}$

Q- Evaluate

$\lim_{x\to 2} \frac{x^{10}-1024}{x-2}$

$\Rightarrow \lim_{x\to 2} \frac{x^{10}-2^{10}}{x-2}$

$\Rightarrow 10\times 2^{10-1}$

$\Rightarrow 10\times 2^9$

$\Rightarrow 10\times 512$

$\Rightarrow 5120$

Q- Evaluate

$\lim_{x\to 2} \frac{x^5-32}{x-2}$

$\Rightarrow \lim_{x\to 2} \frac{x^5-2^5}{x-2}$

$\Rightarrow 5\times 2^{5-1}$

$\Rightarrow 5\times 2^4$

$\Rightarrow 5\times 16$

$\Rightarrow 80.$

Q- $\cancel{\lim_{x\to 2} \frac{x^{10}-1024}{x^5-32}} = 64$

Soln $\cancel{\lim_{x\to 2} \frac{(x^5)^2-(32)^2}{x^5-32}}$

$\cancel{\Rightarrow 2\times 2^{2-1}}$

$\cancel{\Rightarrow 2\times 2}$

$\cancel{\Rightarrow 4}$

Q- $\lim_{x\to a} \frac{x^m-a^m}{x^n-a^n}$

Soln $\lim_{x\to a} \frac{\frac{x^m-a^m}{x-a}}{\frac{x^n-a^n}{x-a}}$

$\Rightarrow \frac{\lim_{x\to a} \frac{x^m-a^m}{x-a}}{\lim_{x\to a} \frac{x^n-a^n}{x-a}}$

$\Rightarrow \frac{m a^{m-1}}{n a^{n-1}}$

$\Rightarrow \frac{m}{n} a^{m-1-n+1}$

$\Rightarrow \frac{m}{n} a^{m-n}$

Q- If $\lim_{x\to a} \frac{x^9+a^9}{x+a} = 9$, then find a

Soln $\lim_{x\to a} \frac{x^9-(-a)^9}{x-(-a)} = 9$

$\Rightarrow \cancel{\lim_{x\to a}} \; \cancel{9}$

$\Rightarrow 9(-a)^8 = 9$

$\Rightarrow a^8 = 9/9$

$\Rightarrow \boxed{a = \pm 1}$

$$\boxed{\begin{array}{c} \infty = \frac{1}{0} \\ \frac{1}{\infty} = 0 \end{array}}$$

03/11/23

method of evaluation of algebraic limits at infinity.-

Q- $\lim_{x\to\infty} \frac{5x-6}{\sqrt{4x^2+9}}$

soln. $\cancel{\lim_{x\to\infty} \frac{(5x-6)(\sqrt{4x^2+9})}{(\sqrt{4x^2+9})(\sqrt{4x^2+9})}}$

$\cancel{\Rightarrow \lim_{x\to\infty} \frac{(5x-6)(\sqrt{4x^2-9})}{4x^2+9}}$

$$\Rightarrow \lim_{x\to\infty} \frac{\cancel{\dfrac{(5x-6)(\sqrt{4x^2-9})}{x}}}{\cancel{4x^2+0}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\dfrac{5x-6}{x}}{\dfrac{\sqrt{4x^2+9}}{x}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{5-6/x}{\sqrt{\frac{4x^2}{x^2}+9/x^2}}$$

$$\Rightarrow \frac{5 - 6\lim_{x\to\infty}\frac{1}{x}}{\sqrt{4+9\lim_{x\to\infty}\frac{1}{x^2}}}$$

$$\Rightarrow \frac{5-6/\infty}{\sqrt{4+1/\infty}}$$

$$\Rightarrow \frac{5}{\sqrt{4}} \Rightarrow \frac{5}{2} \text{ Ans}$$

Q- $\lim_{x\to\infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)$

Sol$^n$ $\lim_{x\to\infty} \frac{\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}$

$$\Rightarrow \lim_{x\to\infty} \frac{\cancel{x^2}+x+\cancel{1}-\cancel{x^2}-\cancel{1}}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{x}{x\left(\dfrac{\sqrt{x^2+x+1}+\sqrt{x^2+1}}{x}\right)}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\left(\sqrt{1+1/x+1/x^2}+\sqrt{1+1/x^2}\right)}$$

$$\Rightarrow \lim_{x\to\infty} \frac{1}{\left(\sqrt{1+1/x+1/x^2}+\sqrt{1+1/x^2}\right)}$$

$$\Rightarrow \frac{1}{\left(\sqrt{1+1/\infty+1/\infty}+\sqrt{1+1/\infty}\right)}$$

$$\Rightarrow \frac{1}{\sqrt{1}+\sqrt{1}} \Rightarrow \frac{1}{1+1} = \frac{1}{2}$$

Q- $\lim_{a\to\infty} \left(\sqrt{a^2+a+1} - \sqrt{a^2+1}\right)$

Soln. $\lim_{a\to\infty} \frac{\left(\sqrt{a^2+a+1} - \sqrt{a^2+1}\right)\left(\sqrt{a^2+a+1} + \sqrt{a^2+1}\right)}{\left(\sqrt{a^2+a+1} + \sqrt{a^2+1}\right)}$

$\Rightarrow \lim_{a\to\infty} \frac{\cancel{a^2}+a+\cancel{1}-\cancel{a^2}-\cancel{1}}{\sqrt{a^2+a+1} + \sqrt{a^2+1}}$

$\Rightarrow \lim_{a\to\infty} \frac{1}{\left(\frac{\sqrt{a^2+a+1} + \sqrt{a^2+1}}{a}\right)}$

$\Rightarrow \lim_{a\to\infty} \frac{1}{\sqrt{1+1/a+1/a^2} + \sqrt{1+1/a^2}}$

$\Rightarrow \frac{1}{\sqrt{1+1/\infty+1/\infty} + \sqrt{1+1/\infty}}$

$\Rightarrow \frac{1}{\sqrt{1}+\sqrt{1}} \Rightarrow \frac{1}{1+1} \Rightarrow 1/2$ Ans

<u>05/11/23</u>

Q- Evaluate $\lim_{n\to\infty} \frac{1+2+3+\dots+n}{n^2}$

Soln. $\lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2}$

$\Rightarrow \lim_{n\to\infty} \frac{n^2}{2n^2} + \frac{1}{2n}$

$\Rightarrow \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n}$

$\Rightarrow \frac{1}{2} + \frac{1}{\infty}$

$\Rightarrow 1/2 + 0$

$\Rightarrow 1/2$

Q- Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$$

Sol^n $$\lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{x}}{\frac{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}{x}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}-\sqrt[3]{1+1/x^3}}{\sqrt{1+1/x^4}-\sqrt{1/x+1/x^5}}$$

$$\Rightarrow \frac{\sqrt{1+1/\infty}-\sqrt[3]{1+1/\infty}}{\sqrt[4]{1+\frac{1}{\infty}}-\sqrt[5]{1/\infty+1/\infty}}$$

$$\Rightarrow \frac{\sqrt{1+0}-\sqrt[3]{1+0}}{\sqrt[4]{1+0}-\sqrt[5]{0+0}}$$

$$\Rightarrow \frac{\sqrt{1}-\sqrt[3]{1}}{\sqrt[4]{1}-0}$$

$$\Rightarrow \frac{0}{\sqrt[4]{1}} \Rightarrow \frac{0}{1} \Rightarrow 0$$

Evaluation of Trigonometric function.-

- $\lim_{\theta\to 0} \cos\theta = 1$
- $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$
- $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$

$1-\cos\theta = 2\sin^2\theta/2$

$1-\sin\theta = 2\cos^2\theta/2$

- $\lim_{x\to 0} \frac{\sin^{-1}x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan^{-1}x}{x} = 1$

Q- Evaluate $\lim_{x \to 0} \frac{\sin 7x}{3x}$

Soln - $\lim_{x \to 0} \frac{7 \sin 7x}{3x \times 7}$

$\Rightarrow \frac{7}{3} \lim_{x \to 0} \frac{\sin 7x}{7x}$

$\Rightarrow \frac{7}{3} \times 1$

$\Rightarrow 7/3$

Q- $\lim_{x \to 0} \frac{\sin^2 3x}{x^2}$

$\Rightarrow \lim_{x \to 0} \frac{9 (\sin 3x)^2}{9 \times x \times x}$

$\Rightarrow 9 \lim_{x \to 0} \frac{\sin 3x}{3x} \times \frac{\sin 3x}{3x}$

$\Rightarrow 9 \left( \lim_{x \to 0} \frac{\sin 3x}{3x} \times \lim_{x \to 0} \frac{\sin 3x}{3x} \right)$

$\Rightarrow 9 (1 \times 1)$

$\Rightarrow 9$

Q- $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

Soln $\lim_{x \to 0} \frac{2 \sin^2 x/2}{x^2}$

$\Rightarrow \lim_{x \to 0} \frac{\frac{2}{4} \sin x/2 \sin x/2}{x/2 \times x/2}$

$\Rightarrow \frac{1}{2} \lim_{x \to 0} \frac{\sin x/2}{x/2} \times \frac{\sin x/2}{x/2}$

$\Rightarrow 1/2$

Q $\lim$

$\sin^2 x = \sin x \times \sin x$
$= (\sin x)^2$

Q. $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$

Soln $\lim_{x \to 0} \frac{\sin 5x}{\tan 3x}$

$\Rightarrow \lim_{x \to 0} \frac{5 \times \frac{\sin 5x}{5x}}{3 \times \frac{\tan 3x}{3x}}$

$\Rightarrow 5/3 \lim_{x \to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\tan 3x}{3x}}$

$\Rightarrow 5/3 \times \frac{1}{1}$

$\Rightarrow 5/3$

Q- $\lim_{x\to 0} \frac{\sin 6x}{x \cdot \cos x}$

Solⁿ $\lim_{x\to 0} \frac{6\sin 6x}{\frac{6x \cdot \cos x}{x}}$

$\Rightarrow 6 \lim_{x\to 0} \frac{\frac{\sin 6x}{6x}}{\cos x}$

$\Rightarrow \cancel{6 \lim_{x\to 6}} \Rightarrow 6 \times \frac{1}{1} \Rightarrow 6$

16/11/23

Q- $\lim_{x\to 0} \frac{1-\cos 2mx}{1-\cos 2nx}$ {Important}

Solⁿ $\lim_{x\to 0} \frac{2\sin^2 mx}{2\sin^2 nx}$

$\Rightarrow \lim_{x\to 0} \frac{\sin^2 mx}{\sin^2 nx} \Rightarrow \lim_{x\to 0} \frac{\sin mx \cdot \sin mx}{\sin nx \cdot \sin nx}$

$\Rightarrow \lim_{x\to 0} \frac{\sin mx \cdot \sin mx \times n^2x^2 \times m^2x^2}{\sin nx \cdot \sin nx \times n^2x^2 \times m^2x^2}$

$\Rightarrow \lim_{x\to 0} \left( \frac{\frac{\sin mx}{mx} \cdot \frac{\sin mx}{mx}}{\frac{\sin nx}{nx} \cdot \frac{\sin nx}{nx}} \right) \frac{m^2\cancel{x^2}}{n^2\cancel{x^2}}$

$\Rightarrow \frac{m^2}{n^2} \left( \frac{\lim_{x\to 0} \frac{\sin mx}{mx} \cdot \lim_{x\to 0} \frac{\sin mx}{mx}}{\lim_{x\to 0} \frac{\sin nx}{nx} \cdot \lim_{x\to 0} \frac{\sin nx}{nx}} \right)$

$\Rightarrow \frac{m^2}{n^2} \times \frac{1}{1} \times \frac{1}{1}$

$\Rightarrow \frac{m^2}{n^2}$

22/11/23 → (Important)

Q- $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ {Important}

Sol$^n$ $\lim_{x\to 0} \frac{\cancel{2\sin^2} 1-[1-2\sin^2 x/2]}{x^2}$

$\Rightarrow \lim_{x\to 0} \frac{1-1+2\sin^2 x/2}{x^2}$

$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2}$

$\Rightarrow \lim_{x\to 0} \frac{2(\sin x/2)^2}{x^2}$

$\Rightarrow \lim_{x\to 0} \frac{2 \sin x/2 \ \sin x/2}{4 \times x/2 \cdot x/2}$

$\Rightarrow \frac{2}{4} \lim_{x\to 0} \frac{\sin x/2}{x/2} \cdot \lim_{x\to 0} \frac{\sin x/2}{x/2}$

$\Rightarrow \frac{1}{2} \times 1 \times 1$

$\Rightarrow 1/2$

Q- $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$ {Imp}

Sol$^n$ $\lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$

$\Rightarrow \lim_{x\to 0} \frac{\sin x - \sin x \cos x}{\cos x \cdot x^3}$

$\Rightarrow \lim_{x\to 0} \frac{\sin x(1-\cos x)}{\cos x \cdot x^3}$

$\Rightarrow \lim_{x\to 0} \frac{\sin x\{1-(1-2\sin^2 x/2)\}}{\cos x \cdot x^3}$

$\Rightarrow \lim_{x\to 0} \frac{\sin x \cdot 2\sin^2 x/2}{\cos x \cdot x^3}$

$\Rightarrow 2\lim_{x\to 0} \frac{\sin x \cdot \sin x/2 \cdot \sin x/2}{4 \cdot \cos x \cdot x \cdot x/2 \cdot x/2}$

$\Rightarrow \frac{2}{4} \frac{\left\{\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x/2}{x/2} \cdot \lim_{x\to 0} \frac{\sin x/2}{x/2}\right\}}{\lim_{x\to 0} \cos x}$

$\Rightarrow \frac{1}{2} \times \frac{1 \times 1 \times 1}{1}$

$\Rightarrow 1/2$ Ans

Q- Evaluat $A^2\,2A-3I$, If $A=\begin{bmatrix}-1 & 0 & 1\\ -1 & 1 & 1\\ -3 & -2 & 0\end{bmatrix}$

Soln $A^2 = A\cdot A$

$$=\begin{bmatrix}-1 & 0 & 1\\ -1 & 1 & 1\\ -3 & -2 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ -1 & 1 & 1\\ -3 & -2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1+0-3 & 0+0-2 & -1+0+0\\ \cancel{1}-\cancel{1}-3 & 0+1-2 & \cancel{-1}+\cancel{1}+0\\ 3+2+0 & 0-2+0 & -3-2+0\end{bmatrix}=\begin{bmatrix}-2 & -2 & -1\\ -3 & -1 & 0\\ 5 & -2 & -5\end{bmatrix}$$

$$2A=\begin{bmatrix}-2 & 0 & 2\\ -2 & 2 & 2\\ -6 & -4 & 0\end{bmatrix} \qquad 3I=\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$$

Now, $A^2-2A-3I$

$$=\begin{bmatrix}-2 & -2 & -1\\ -3 & -1 & 0\\ 5 & -2 & -5\end{bmatrix}-\begin{bmatrix}-2 & 0 & 2\\ -2 & 2 & 2\\ -6 & -4 & 0\end{bmatrix}-\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-2 & -2 & -1\\ -3 & -1 & 0\\ 5 & -2 & -5\end{bmatrix}-\begin{bmatrix}1 & 0 & 2\\ -2 & 5 & 2\\ -6 & -4 & 3\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}-3 & -2 & -3\\ -1 & -6 & -2\\ 11 & 2 & -8\end{bmatrix}$$

Q- $\lim_{x\to a}\dfrac{x^9+a^9}{x+a}=9$

$\Rightarrow \lim_{x\to a}\dfrac{x^9-(-a)^9}{x-(-a)} \Rightarrow 9$ $\qquad$ { As we know that $\lim_{x\to a}\dfrac{x^n-a^n}{x-a}\Rightarrow na^{n-1}$ }

$9(a)^{9-1}=9$

$(a)^8=1$

$a=\sqrt{1} \Rightarrow \{a \pm \pm 1\}$

23/11/23

# Continuity

Q- If $\lim_{x \to 2} x^2$ then find the value of RHL and LHL.

Soln. If $\lim_{x \to 2} x^2$

For RHL, $\lim_{x \to 2^+} x^2$

Let $x = 2+h$

If $\lim_{x \to 2}$ then $\lim_{h \to 0}$

Now, $\lim_{x \to 2^+} x^2$

$\Rightarrow \lim_{h \to 0} (2+h)^2$

$\Rightarrow \lim_{h \to 0} (4 + h^2 + 4h)$

$\Rightarrow \lim_{h \to 0} 4 + \lim_{h \to 0} h^2 + \lim_{h \to 0} 4h$

$\Rightarrow 4 + 0^2 + 4 \times 0$

$\Rightarrow 4 + 0 + 0$

$\Rightarrow 4$

For LHL, $\lim_{x \to 2^-} x^2$

Let $x = 2-h$

If $\lim_{x \to 2}$ then $\lim_{h \to 0}$

Now, $\lim_{x \to 2^-} x^2$

$\Rightarrow \lim_{h \to 0} (2-h)^2$

$\Rightarrow \lim_{h \to 0} (4 + h^2 - 4h)$

$\Rightarrow \lim_{h \to 0} 4 + \lim_{h \to 0} h^2 - \lim_{h \to 0} 4h$

$\Rightarrow 4 + 0^2 - 4 \times 0$

$\Rightarrow 4 + 0 - 0$

$\Rightarrow 4$

Q- If $f(x) = x \sin 1/x$ for $x \neq 0$ and $f(0) = 1$, then show that $f(x)$ is continuous at $x = 0$.

Soln. For continuous function,

$LHL = RHL = f(0)$

Now, RHL, $\lim_{x \to 0^+} x \sin 1/x$

Let $x = 0+h$,

If $\lim_{x \to 0^+}$ then $\lim_{h \to 0}$

Now, $\lim_{x \to 0^+} x \sin 1/x$

$\Rightarrow \lim_{h \to 0} (0+h) \sin \frac{1}{0+h} \Rightarrow \lim_{h \to 0} \frac{h \sin 1/h \times 1/h}{1/h} \Rightarrow \lim_{h \to 0} \frac{\sin 1/h}{1/h} \Rightarrow 1$

For LHL, $\lim_{x \to 0^-} x \sin 1/x$

Let $x = 0-h$,

If $\lim_{x \to 0^-}$ then $\lim_{h \to 0}$

Now, $\lim_{x \to 0^-} x \sin 1/x$

$\Rightarrow \lim_{h \to 0} (0-h) \sin \frac{1}{(0-h)}$

$\Rightarrow \lim_{h \to 0} -h \sin(-1/h)$

$\Rightarrow \lim_{h \to 0} h \sin 1/h$

$\Rightarrow \lim_{h \to 0} \frac{\sin 1/h}{1/h}$

$\Rightarrow 1$

Hence, LHL = RHL = $f(0) = 1$

So, $f(x)$ is a continuous function.

Q- $f(x) = \begin{cases} x+1 & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 < x \leq 2 \end{cases}$

Show that it is discontinuous at $x=0$ but it is continuous at $x=1$.

Soln To check continuity of $f(x)$ at $x=0$,

RHL at $x=0$,

$\lim_{x \to 0^+} x$

$\Rightarrow \lim_{h \to 0} 0+h$

$\Rightarrow 0$

LHL at $x=0$

$\lim_{x \to 0^-} x+1$

$\Rightarrow \lim_{h \to 0} 0-h+1$

$\Rightarrow 0-0+1$

$\Rightarrow 1$

$\therefore$ RHL $\neq$ LHL, Hence, $f(x)$ is discontinuous at $x=0$.

Now, To check continuity of $f(x)$ at $x=1$,

RHL at $x=1$,

$\lim_{x \to 1^+} 2-x \Rightarrow \lim_{h \to 0} 2-(1+h) \Rightarrow \lim_{h \to 0} 2-1-h \Rightarrow 1-0 \Rightarrow 1$

LHL at $x=1$,

$\lim_{x\to 1^-} x \Rightarrow \lim_{h\to 0} 1-h \Rightarrow 1-0 \Rightarrow 1$

Now, $f(x) = x$
$f(1) = 1$

$LHL = RHL = f(1) = 1$

Hence, $f(x)$ is continuous at $x=1$.

Q- Find the points of continuity and discontinuity of the function

$$f(x) = \begin{cases} 0 & \text{for } x=0 \\ \frac{1}{2}-x & \text{for } 0<x<\frac{1}{2} \\ \frac{1}{2} & \text{for } x=\frac{1}{2} \\ \frac{1}{2}-x & \text{for } \frac{1}{2}<x<1 \\ 1 & \text{for } x=1 \end{cases}$$

Solⁿ • To check continuity at $x=0$,

$RHL = \lim_{x\to 0^+} \frac{1}{2}-x \Rightarrow \lim_{h\to 0} \frac{1}{2}-(0+h) \Rightarrow \lim_{h\to 0} \frac{1}{2}-0-h$

$\Rightarrow \frac{1}{2}-0-0 \Rightarrow \frac{1}{2}$

$f(0) = 0$

$\therefore \quad RHL \neq f(0)$

Hence, $f(x)$ is discontinuous at $x=0$.

• To check continuity at $x=\frac{1}{2}$,

$RHL = \lim_{x\to \frac{1}{2}^+} \frac{1}{2}-x \Rightarrow \lim_{h\to 0} \frac{1}{2}-(\frac{1}{2}+h) = \lim_{h\to 0} \frac{1}{2}-\frac{1}{2}-h$

$\Rightarrow 0$

$LHL = \lim_{x\to \frac{1}{2}^-} \frac{1}{2}-x \Rightarrow \lim_{h\to 0} \frac{1}{2}-(\frac{1}{2}-h) \Rightarrow \lim_{h\to 0} \frac{1}{2}-\frac{1}{2}+h$

$\Rightarrow \lim_{h\to 0} h = 0$

$f(\frac{1}{2}) = \frac{1}{2}$

$\therefore$ LHL $\neq$ RHL $\neq f(1/2)$

Hence, $f(x)$ is discontinuous at $x = 1/2$.

- To check continuity at $x = 1$,

$$LHL = \lim_{x\to 1^-} \frac{1}{2} - x \Rightarrow \lim_{h\to 0} \frac{1}{2} - (1-h) \Rightarrow \lim_{h\to 0} \frac{1}{2} - 1 + h$$

$$\Rightarrow \lim_{h\to 0} -\frac{1}{2} + h$$

$$\Rightarrow -\frac{1}{2}$$

~~RHL = lim~~

$f(1) = 1$

$\therefore$ LHL $\neq f(1)$

Hence, $f(x)$ is discontinuous at $x = 1$.

24/11/

# Differentiation

$a \log b = \log b^a$

$e^{\log_e a} = a$

## Formula -

- $\frac{d(c)}{dx} = 0$
- $\frac{d[c \cdot f(x)]}{dx} = c\frac{d(f(x))}{dx}$
- $\frac{d}{dx}(x^n) = n x^{n-1}$
- $\frac{d(\log_a x)}{dx} = \frac{1}{\cancel{x}\log_e a} \Rightarrow \frac{1}{\log_a x}$
- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \log_e a$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$
- $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d(u.v)}{dx} = u\frac{dv}{dx} + v.\frac{du}{dx}$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\sinh x) = \cosh x$
- $\frac{d}{dx}(\cosh x) = \sinh x$

Q. Differentiate given function w.r.t x.

① $\frac{d}{dx}(ax^2+bx+c)$

$\Rightarrow \frac{d}{dx}ax^2 + \frac{d}{dx}bx + \frac{d}{dx}c$

$\Rightarrow 2ax + b + 0$

$\Rightarrow 2ax+b$

② $\frac{d}{dx}\left(\frac{ax^2+bx+c}{\sqrt{x}}\right)$

$$\Rightarrow \frac{\frac{d}{dx}(ax^2+bx+c)\cdot\sqrt{x} - (ax^2+bx+c)\cdot\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}$$

$$\Rightarrow \frac{(2ax+b)\sqrt{x} - \frac{1}{2\sqrt{x}}(ax^2+bx+c)}{x}$$

$$\Rightarrow \frac{(2ax+b)2x - (ax^2+bx+c)}{2x\cdot\sqrt{x}}$$

$$\Rightarrow \frac{4ax^2+2bx-ax^2-bx-c}{2x\cdot\sqrt{x}}$$

$$\Rightarrow \frac{3ax^2+bx-c}{2x\cdot\sqrt{x}}$$

$$\Rightarrow \frac{3ax^2}{2x\sqrt{x}} + \frac{bx}{2x\cdot\sqrt{x}} - \frac{c}{2x\cdot\sqrt{x}}$$

$$\Rightarrow \frac{3ax}{2x^{1/2}} + \frac{b}{2\sqrt{x}} - \frac{c}{2x\sqrt{x}}$$

$$\Rightarrow \frac{3ax^{1-1/2}}{2} + \frac{bx^{-1/2}}{2} - \frac{cx^{-3/2}}{2}$$

$$\Rightarrow \frac{3a}{2}x^{1/2} + \frac{b}{2}x^{-1/2} - \frac{c}{2}x^{-3/2}$$ Ans

③ $\frac{d}{dx} e^{x\log a}$

$\Rightarrow \frac{d}{dx} e^{\log a^x}$ we know that, $\{e^{\log_e a} \Rightarrow a\}$

$\Rightarrow \frac{d}{dx} a^x$

$\Rightarrow a^x \log_e a$

④ $\frac{d}{dx} e^{a\log x} \Rightarrow \frac{d}{dx} e^{\log x^a} \Rightarrow \frac{d}{dx} x^a \Rightarrow a x^{a-1}$

⑤ $\frac{d}{dx} e^{a\log a} \Rightarrow \frac{d}{dx} e^{\log a^a} \Rightarrow \cancel{a^a \log_e a} \frac{d}{dx} a^a \Rightarrow 0$

*⑥ $\frac{d}{dx} (e^{x\log a} + e^{a\log x} + e^{a\log a})$

$\Rightarrow \frac{d}{dx} e^{x\log a} + \frac{d}{dx} e^{a\log x} + \frac{d}{dx} e^{a\log a}$

$\Rightarrow \frac{d}{dx} e^{\log a^x} + \frac{d}{dx} e^{\log x^a} + \frac{d}{dx} e^{\log a^a}$

$\Rightarrow \frac{d}{dx} a^x + \frac{d}{dx} x^a + \frac{d}{dx} a^a$

$\Rightarrow a^x \log_e a + a x^{a-1} + 0$

$\Rightarrow a^x \log_e a + a x^{a-1}$ Ans

*⑦ $\frac{d}{dx} x^2 e^x \log x$

$\Rightarrow x^2 e^x \frac{d}{dx}\log x + \log x \frac{d}{dx} x^2 e^x$ $\{$ As $\frac{d\,uv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\}$

$\Rightarrow x^2 e^x \cdot \frac{1}{x} + \log x \left[x^2 \frac{de^x}{dx} + e^x \frac{dx^2}{dx}\right]$

$\Rightarrow x e^x + \log x \left[x^2 e^x + 2x e^x\right]$

$\Rightarrow x e^x \left[1 + \log x (x+2)\right] \Rightarrow x e^x \left[1 + x\log x + 2\log x\right]$

★ (8) $\frac{d}{dx}(x\sin x + \cos x)(e^x + x^2\log x)$

$\Rightarrow \cancel{\frac{d}{dx} x\sin x} + \cancel{\frac{d}{dx}\cos}$

$\Rightarrow (x\sin x + \cos x)\frac{d}{dx}(e^x + x^2\log x) + (e^x + x^2\log x)\cdot\frac{d}{dx}(x\sin x + \cos x)$

$\Rightarrow (x\sin x + \cos x)\left\{\frac{d}{dx}e^x + \frac{d}{dx}(x^2\log x)\right\} + (e^x + x^2\log x)\left\{\frac{d}{dx}x\sin x + \frac{d}{dx}\cos x\right\}$

$\Rightarrow (x\sin x + \cos x)\left\{e^x + \frac{x^2}{x} + 2x\log x\right\} + (e^x + x^2\log x)\left\{(x\cos x + \sin x) \cancel{- \sin x}\right\}$

$\Rightarrow (x\sin x + \cos x)\{e^x + x + 2x\log x\} + (e^x + x^2\log x)\{x\cos x\}$

$\Rightarrow (x\sin x + \cos x)(e^x + x + 2x\log x) + (e^x + x^2\log x)(x\cos x)$

★ (9) $\frac{d}{dx}\cancel{\log\sin x}$ Differentiate $\log\sin x$ w.r.t $\sqrt{\cos x}$ $\left\{\frac{d\log\sin x}{d\sqrt{\cos x}}\right.$

Sol$^n$ let $u = \log\sin x$, $v = \sqrt{\cos x}$

Now, $\frac{du}{dx} = \frac{d\log\sin x}{dx}$ and $\frac{dv}{dx} = \frac{d\sqrt{\cos x}}{dx}$

$\frac{du}{dx} = \frac{d\log\sin x}{dx} \times \frac{d\sin x}{d\sin x}$

$= \frac{d\log\sin x}{d\sin x} \times \frac{d\sin x}{dx}$

$\Rightarrow \frac{1}{\sin x} \times \cos x$

$\frac{du}{dx} \Rightarrow \frac{\cos x}{\sin x}$

$\frac{dv}{dx} = \frac{d(\cos x)^{1/2}}{dx} \times \frac{d\cos x}{d\cos x}$

$= \frac{d(\cos x)^{1/2}}{d\cos x} \times \frac{d\cos x}{dx}$

$= \frac{d\sqrt{\cos x}}{dx} \times -\sin x$

$= \frac{1}{2\sqrt{\cos x}} \times -\sin x$

$\frac{dv}{dx} = \frac{-\sin x}{2\sqrt{\cos x}}$

As $\frac{d\log\sin x}{d\sqrt{\cos x}} = \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{\cos x}{\sin x}}{\frac{-\sin x}{2\sqrt{\cos x}}} = \frac{-2\cos x\sqrt{\cos x}}{\sin^2 x}$

$= \frac{-2\cos^{3/2}x}{\sin^2 x}$

06/12/23

# Maxima And Minima

Q- Find the maximum and minimum points and value for the given function in the interval [1,3].

$f(x) = x^5 - 5x^4 + 5x^3 - 10$

Sol$^n$- For maxima and minima point,

$f'(x) = 0$

So, $f'(x) = \frac{d}{dx}(x^5 - 5x^4 + 5x^3 - 10)$

$= \frac{dx^5}{dx} - \frac{d5x^4}{dx} + \frac{d5x^3}{dx} - \frac{d10}{dx}$

$= 5x^4 - 20x^3 + 15x^2 - 0$

$= 5x^2(x^2 - 4x + 3)$

$\therefore f'(x) = 0$

$5x^2(x^2 - 4x + 3) = 0$

$5x^2(x^2 - 3x - x + 3) = 0$

$5x^2(x-3)(x-1) = 0$

$\therefore 5x^2 = 0$ $[x = 0]$ | $x - 3 = 0$ $[x = 3]$ | $x - 1 = 0$ $[x = 1]$

Now,
For maxima and minima,
find $f''(x)$

$f''(x) = \frac{d}{dx}(5x^4 - 20x^3 + 15x^2)$

$= 20x^3 - 60x^2 + 30x$

Now,

Putting $x = 1$

$f''(1) = 20 - 60 + 30 = -10$

$\therefore f''(x) < 0$

Maximum value of function at $x=1$,

$f(1) \Rightarrow x^5 - 5x^4 + 5x^3 - 10$

$\Rightarrow 1 - \cancel{5} + \cancel{5} - 10$

$\Rightarrow -9$

Putting $x = 3$,

$f''(3) = \cancel{20}\,10(2x^3 - 6x^2 + 3x)$

$= 10(2\times27 - 6\times9 + 3\times3)$

$= 10(54 - 54 + 9)$

$= 90$

$\therefore f''(3) > 0$

Minimum value of function is at $x = 3$.

Minimum value $[f(3)]$

$\Rightarrow x^5 - 5x^4 + 5x^3 - 10$

$\Rightarrow$ 243 $\cancel{188 - 860 + 135} - 10$

$\Rightarrow -37$

(Important) 07/12/23

Q- Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$

Sol$^n$ we know that $F(x) = \left(\frac{1}{x}\right)^x$

let $y = \left(\frac{1}{x}\right)^x$

Taking log both the sides,

$\log y = \log (1/x)^x$

$\log y = x \log 1/x$ $\{$ As $\log m^n = n \log m \}$

$\log y = x \log (x^{-1})$

$\log y = -x \log x$

Now differentiate w.r.t to x. both sides,

$\frac{d \log y}{dx} = \frac{d(-x \log x)}{dx}$

$\frac{d \log y}{dy} * \frac{dy}{dx} = -\left[ x \frac{d \log x}{dx} + \log x \frac{dx}{dx} \right]$

$\frac{1}{y} * \frac{dy}{dx} = -\left[ x \times \frac{1}{x} + \log x \right]$

$\frac{1}{y} * \frac{dy}{dx} = -[1 + \log x]$

$\frac{dy}{dx} = -y[1 + \log x]$

For maxima and minima point,

$\frac{dy}{dx} = 0$

$-y[1 + \log x] = 0$ $\{\log_e e = 1\}$

$\therefore -y \neq 0$

So, $1 + \log x = 0$

$\log x = -1 \Rightarrow \log x = -1 \log_e e \Rightarrow \log x = \log_e e^{-1}$

$[x = e^{-1}]$

$\left[x = \frac{1}{e}\right]$

For maxima ~~po fing~~ find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\{y(1+\log x)\}$$

$$\frac{d^2y}{dx^2} = -\left[y\frac{d}{dx}(\log x+1) + (1+\log x)\frac{dy}{dx}\right]$$

$$\frac{d^2y}{dx^2} = -\left[y \times \frac{1}{x} + (1+\log x)[-y(1+\log x)]\right]$$

$$\frac{d^2y}{dx^2} = -\left[\frac{y}{x} - y(1+\log x)^2\right]$$

$$\frac{d^2y}{dx^2} = y\left[(1+\log x)^2 - \frac{1}{x}\right]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{x}\right)^x\left[(1+\log x)^2 - \frac{1}{x}\right]$$

Now, Putting $x = 1/e$ in double differentiation,

$$\frac{d^2y}{dx^2} = \left(\frac{1}{1/e}\right)^{1/e}\left[(1+\log_e e^{-1}) - e\right]$$

$$\frac{d^2y}{dx^2} = e^{1/e}\left[(1-\log_e e) - e\right]$$

$$\frac{d^2y}{dx^2} = e^{1/e}\left[(1-1) - e\right]$$

$$\frac{d^2y}{dx^2} = e^{1/e} \times -e \Rightarrow -e \cdot e^{1/e}$$

i.e. $\frac{d^2y}{dx^2} < 0$

Hence, the maximum value of given function at $x = 1/e$.

$$\left(\frac{1}{x}\right)^x = \left(\frac{1}{1/e}\right)^{1/e} = (e)^{1/e}$$

H.P

12/12/23

Q. Find the maximum (largest) and minimum (smallest) value of given ~~given~~ equation $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$, in the given range $[0, 2]$

Sol$^n$- For maxima and minima,

$$f'(x) = \frac{d}{dx}[3x^4 - 2x^3 - 6x^2 + 6x + 1]$$

$$f'(x) = \frac{d}{dx}3x^4 - \frac{d}{dx}2x^3 - \frac{d}{dx}6x^2 + \frac{d}{dx}6x + \frac{d}{dx}1$$

$$f'(x) = 12x^3 - 6x^2 - 12x + 6 + 0$$

$$f'(x) = 12x^3 - 6x^2 - 12x + 6$$

For maxima and minima point,

$$f'(x) = 0$$

$$12x^3 - 6x^2 - 12x + 6 = 0$$

$$6(2x^3 - x^2 - 2x + 1) = 0$$

$$6(x+1)(x-1)(2x-1) = 0$$

$\therefore x + 1 = 0$ | $x - 1 = 0$ | $2x - 1 = 0$

$[x = -1]$ | $[x = 1]$ | $[x = 1/2]$

Not considered

Now, For maxima and minima.

$$f''(x) = 36x^2 - 12x - 12$$

Now, put $x = 1$

$$f''(1) = 36 - 12 - 12$$
$$= 36 - 24$$
$$= 12$$

$\therefore f''(x)$ is positive

$\therefore f''(x) > 0$

minimum value at $x = 1$.

$$f(1) = 3(1)^4 - 2(1)^3 - 6(1)^2 + 6(1) + 1$$
$$= 3 - 2 - \cancel{6} + \cancel{6} + 1$$
$$= 2$$

Put $x = 1/2$

$$f''(1/2) = \overset{9}{\cancel{36}} \times \frac{1}{\cancel{4}} - \overset{6}{\cancel{12}} \times \frac{1}{\cancel{2}} - 12$$
$$= 9 - 18$$
$$= -9$$

$$f''(x) < 0$$

Maximum value at $x = 1/2$.

$$f(1/2) = 3 \times \frac{1}{16} - 2 \times \frac{1}{8} - 6 \times \frac{1}{4} + 6 \times \frac{1}{2} + 1$$

$$= \frac{3 - 4 - 24 + 48 + 16}{16}$$

$$= \frac{39}{16}$$

Q. Find the maximum and minimum value of $f(x) = x^3 - 18x^2 + 96x$

Soln For maxima and minima,

$f'(x) = \frac{d}{dx}(x^3 - 18x^2 + 96x)$

$= 3x^2 - 36x + 96$

$\therefore$ For maxima and minima point.

$f'(x) = 0$

$3x^2 - 36x + 96 = 0$

$3(x^2 - 12x + 32) = 0$

$3(x^2 - 8x - 4x + 32) = 0$

$x(x-8) - 4(x-8) = 0$

$\therefore (x-4)(x-8) = 0$

| | |
|---|---|
| $x - 4 = 0$ | $x - 8 = 0$ |
| $[x = 4]$ | $[x = 8]$ |

Now, for maxima and minima

~~Now, Put x=4,~~ $f''(x) = 6x - 36$

Now,

| | |
|---|---|
| Put $x = 4$ | Put $x = 8$ |
| $f''(4) = 24 - 36$ | $f''(8) = 6 \times 8 - 36$ |
| $= -12$ | $= 48 - 36$ |
| $\therefore f''(4) < 0$ | $= 12$ |
| maxima value, | $\therefore f''(8) > 0$ |
| $f(4) = (4)^3 - 18 \times 4^2 + 96 \times 4$ | minima value, |
| $= 64 - 288 + 384$ | $f(8) = (8)^3 - 18 \times 8^2 + 96 \times 8$ |
| $= 160$ | $= 512 - 1152 + 768$ |
| | $= 1280 - 1152$ |
| | $= 128$ |

18/12/23

# UNIT-4

# INTEGRATION

$\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq 1) + c$

$\int \frac{1}{x} dx = \log_e x + c$

$\int e^x dx = e^x + c$

$\int a^x dx = \frac{a^x}{\log_e a} + c$

$\int \sin x\, dx = -\cos x + c$

$\int \cos x\, dx = \sin x + c$

$\int \tan x\, dx = -\log \cos x + c$

$\int \cot x\, dx = \log \sin x + c$

$\int \sec x\, dx = \log(\sec x + \tan x) + c$

$\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x) + c$

$\int \sec^2 x\, dx = \tan x + c$

$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\frac{x}{a} + c$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2} \log \frac{x-a}{x+a} + c$

$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a} + c$

$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + c$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + c$

$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}(\sqrt{a^2-x^2}) + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

$\int dx = x + c$

$\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}(\sqrt{a^2+x^2}) + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$

$\int a^a dx = x \cdot a^a + c$

$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}(\sqrt{x^2-a^2}) - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c$

$\int \sec x \tan x\, dx = \sec x + c$

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$

$\int \sinh x\, dx = \cosh x + c$

$\int \coth x\, dx = \log \sinh x + c$

$\int \cosh x\, dx = \sinh x + c$

$\int \operatorname{sech}^2 x\, dx = \tanh x + c$

$\int \tanh x\, dx = \log \cosh x + c$

$\int \operatorname{cosech}^2 x\, dx = -\coth x + c$

Q- $\int (x^5+x^4+x+1)dx = \frac{x^6}{6} + \frac{x^5}{5} + \frac{x^2}{2} + x + C$

Q. $\int \frac{x^5+x^4+x^3+x^2+x+1}{x^2} dx$

we know that

$\int (u+v+w)\,dx = \int u\,dx + \int v\,dx + \int w\,dx$

Now,

$\int \frac{x^5+x^4+x^3+x^2+x+1}{x^2} dx = \int \frac{x^5}{x^2}dx + \int \frac{x^4}{x^2}dx + \int \frac{x^3}{x^2}dx +$

$\int \frac{x^2}{x^2}dx + \int \frac{x}{x^2}dx + \int \frac{1}{x^2}dx$

$= \int x^3 dx + \int x^2 dx + \int x\,dx + \int dx +$

$\int \frac{1}{x}dx + \int \frac{1}{x^2}dx$

$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \log x - \frac{1}{x} + C$

Q- $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

$\Rightarrow \int \left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 + 2\right) dx$

$\Rightarrow \int \left(x + \frac{1}{x} + 2\right) dx = \frac{x^2}{2} + \log x + 2x + C$

Q- $\int \frac{1}{2}\sec^2 x\,dx = \frac{1}{2}\tan x + C$

Imp Q- $\int \left(e^{a\log x} + e^{x\log a} + e^{a\log a}\right) dx$

$\Rightarrow \int \left(e^{\log x^a} + e^{\log a^x} + e^{\log a^a}\right) dx$

$\Rightarrow \int (a^x + x^a + a^a)\,dx$

$\Rightarrow \int a^x dx + \int x^a dx + \int a^a dx$

$\Rightarrow \frac{a^x}{\log_e a} + \frac{x^{a+1}}{a+1} + x.a^a + C$

19/12/23

Q- $\int (4\sin x - 3\cos x + 5\sec^2 x - 6\operatorname{cosec}^2 x)dx$

$\Rightarrow \int 4\sin x\,dx - \int 3\cos x\,dx + \int 5\sec^2 x\,dx - \int 6\operatorname{cosec}^2 x\,dx$

$\Rightarrow -4\cos x - 3\sin x + 5\tan x + 6\cot x + C$

Q- $\int \frac{1}{\sin^2 x \cdot \cos^2 x} dx$

$\Rightarrow \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx$

$\Rightarrow \int \left( \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x} + \frac{\cos^2 x}{\cos^2 x \sin^2 x} \right) dx$

$\Rightarrow \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$

$\Rightarrow \tan x - \cot x + C$

Q) $\int \sqrt{1+\sin 2x}\; dx.$

$\Rightarrow \int \sqrt{1 + 2\sin x \cos x}\, dx$

$\Rightarrow \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\, dx$

$\Rightarrow \int \sqrt{(\sin x + \cos x)^2}\, dx$

$\Rightarrow \int (\sin x + \cos x)\, dx$

$\Rightarrow -\cos x + \sin x + C$

Q- $\int \sqrt{1+\cos 2x}\; dx$

$\Rightarrow \int \sqrt{\cancel{1} + 2\cos^2 x - \cancel{1}}\; dx$

$\Rightarrow \int \sqrt{2\cos^2 x}\; dx$

$\Rightarrow \int \sqrt{2}\cos x\, dx.$

$\Rightarrow \sqrt{2}\int \cos x\, dx$

$\Rightarrow \sqrt{2}\sin x + C$

~~Q- $\int \sqrt{x^2}$~~

Q- $\int \left(x^2 + \frac{1}{x^2}\right)^3 dx$

$\Rightarrow \int (x^2)^3 + \left(\frac{1}{x^2}\right)^3 + 3x^2 \cdot \frac{1}{x^2}\left(x^2 + \frac{1}{x^2}\right) dx$

$\Rightarrow \int x^6 + \frac{1}{x^6} + 3x^2 + \frac{3}{x^2}$

$\Rightarrow \frac{x^7}{7} - \frac{1}{5x^5} + \frac{3x^3}{3} - \frac{3}{x} + C$

$\Rightarrow \frac{x^7}{7} - \frac{1}{5x^5} + x^3 - \frac{3}{x} + C$

20/12/23 (Important)

Q - $\int \frac{\cos 2x - \cos 2a}{\cos x - \cos a} dx$

Sol^n - $\int \frac{2\cos^2 x - \cancel{1} - 2\cos^2 a + \cancel{1}}{\cos x - \cos a} dx$

$\Rightarrow 2\int \frac{(\cos x + \cos a)\cancel{(\cos x - \cos a)}}{\cancel{\cos x - \cos a}} dx$

$\Rightarrow 2[\sin x + x\cos a] + C$

Q - $\int \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} dx$

$\Rightarrow \int \tan^{-1}\sqrt{\frac{\frac{1-\cos 2x}{2}}{\frac{1+\cos 2x}{2}}} dx$

$\Rightarrow \int \tan^{-1}\sqrt{\frac{\sin^2 x}{\cos^2 x}} dx$

$\Rightarrow \int \tan^{-1}\sqrt{\left(\frac{\sin x}{\cos x}\right)^2} dx$

$\Rightarrow \int \tan^{-1}\sqrt{(\tan x)^2} dx$

$\Rightarrow \int \tan^{-1}(\tan x) dx$

$\Rightarrow \int x\, dx$

$\Rightarrow \frac{x^2}{2} + C$

~~Q - $\int \sec x \cdot \tan$~~

Q - $\int \frac{\sin x}{1+\sin x} dx$

$= \int \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$

$= \int \frac{\sin x(1-\sin x)}{1-\sin^2 x} dx$

$= \int \frac{\sin x(1-\sin x)}{\cos^2 x} dx$

$= \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$

$= \int \frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} dx$

$= \int \tan x \sec x - \tan^2 x\, dx$

$= \int \tan x \sec x - (\sec^2 x - 1) dx$

$= \int \tan x \sec x\, dx - \int \sec^2 x\, dx + \int dx$

$= \sec x - \tan x + x + C$

21/12/23

# Integration by Parts

$$\int \underset{I}{u} \cdot \underset{II}{v}\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx$$

ILATE → Exponential

I → Inverse

L → Logarithmic

A → Algebraic

T → Trigonometric

Q. $\int \underset{I}{x} \cdot \underset{II}{e^x}\,dx$

⇒ We have ~~$\int x \cdot e^x$~~ $\int u \cdot v\,dx = \int x \cdot e^x\,dx$,

Let $u = x$ and $v = e^x$

We know that,

$\int u \cdot v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$

so,

$\int x \cdot e^x\,dx = x\int e^x\,dx - \int\left[\frac{dx}{dx}\int e^x\,dx\right]dx$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x + c$

$= e^x(x-1) + c$

Q $\int x^2 \cdot e^x\,dx$

Let $u = x^2$ and $v = e^x$

Now,

$\int x^2 \cdot e^x\,dx = x^2\int e^x\,dx - \int\left[\frac{dx^2}{dx} \cdot \int e^x\,dx\right]dx$

$= x^2e^x - \int 2x \cdot e^x\,dx$

$= x^2e^x - 2\int(x \cdot e^x)\,dx$

$= x^2e^x - 2(xe^x - e^x) + c$ → from above question.

$= x^2e^x - 2xe^x + 2e^x + c$

$= e^x(x^2 - 2x + 2) + c$

Q- $\int x . \log x \, dx$
  II  I

let $u = \log x$ and $v = x$,

Now,

$\int \log x . x \, dx$

$\Rightarrow \log x \int x \, dx - \int \left[\frac{d}{dx} \log x . \int x\right] dx$

~~$\Rightarrow \frac{1}{2} \log x^2 - \int \frac{1}{x}$.~~

$\Rightarrow \log x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx$

$\Rightarrow \frac{1}{2} \log x . x^2 - \frac{1}{2} \int x \, dx$

$\Rightarrow \frac{1}{2} x^2 \log x - \frac{1}{2} \times \frac{x^2}{2} + C$

$\Rightarrow \frac{1}{2} x^2 \left(\log x - \frac{1}{2}\right) + C$

Q $\int x^2 . \sin x \, dx$
  I  II

let $u = x^2$ and $v = \sin x$

Now,

$\int x^2 \sin x \, dx$

$\Rightarrow x^2 \int \sin x \, dx - \int \left[\frac{d x^2}{x} , \int \sin x\right] dx$

$\Rightarrow -x^2 \cos x + \int 2x \cos x \, dx$ —(i)

$\Rightarrow -x^2 \cos x + I_1$

Now,

$I_1 = \int 2x \cos x \, dx$

$= 2x \int \cos x \, dx - \int \left[\frac{d 2x}{dx} . \int \cos x\right] dx$

$= 2x \sin x - \int 2 \sin x \, dx$

$= 2x \sin x + 2 \cos x + C_1$

From equ^n (i)

$\int x^2 . \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$

24/12/23

Q- $\int \log x \, dx$

$\Rightarrow \int 1 . \log x \, dx$

$\Rightarrow \int \log x . 1 \, dx$
  I  II

$\Rightarrow \log x \int dx - \int \left[\frac{d}{dx} \log x . \int dx\right] dx$

$\Rightarrow \log x . x - \int \frac{1}{x} . x \, dx$

$\Rightarrow x \log x - \int 1 \, dx \Rightarrow x \log x - x + C \Rightarrow x(\log x - 1) + C$

$$\frac{d}{dx}(\log x)^2 = \frac{2\log x}{x}$$

Q $\int (\log x)^2 dx$

$\Rightarrow \int \underset{I}{(\log x)^2} \cdot \underset{II}{1}\, dx$

$\Rightarrow (\log x)^2 \int dx - \int \left[ \frac{d(\log x)^2}{dx} \cdot \int dx \right] dx$

$\Rightarrow x(\log x)^2 - \int \left( \frac{2\log x}{x} \cdot x \right) dx$

$\Rightarrow x(\log x)^2 - 2\int \log x\, dx$

$\Rightarrow x(\log x)^2 - 2(x\log x - x) + C$

$\Rightarrow x(\log x)^2 - 2x\log x + 2x + C$

$\Rightarrow x[(\log x)^2 - 2\log x + 2] + C$

Q. $\int \frac{1}{x^2} \cdot \log x\, dx$

$\Rightarrow \int \log x \cdot x^{-2} dx$

$\Rightarrow \log x \int x^{-2} dx - \int \left[ \frac{d}{dx} \log x \cdot \int x^{-2} \right] dx$

$\Rightarrow \log x \cdot \frac{x^{-1}}{-1} - \int \frac{1}{x} \cdot \frac{x^{-1}}{-1}\, dx$

$\Rightarrow \frac{-1}{x} \log x + \int \frac{1}{x^2} dx$

$\Rightarrow -1/x \cdot \log x + \frac{1}{-x} + C$

$\Rightarrow -1/x \cdot \log x - \frac{1}{x} + C$

$\Rightarrow \frac{1}{x}(-\log x - 1) + C$

$\Rightarrow \frac{1}{x}(-\log x - 1) + C$

Q- $\int x^3 \log 2x \, dx$

$\Rightarrow \int \log 2x . x^3 \, dx$

$\Rightarrow \log 2x \int x^3 dx - \int \left[ \frac{d}{dx} \log 2x . \int x^3 dx \right] dx$

$\Rightarrow \log 2x . \frac{x^4}{4} - \int \left( \frac{1}{\cancel{2}x} . \cancel{2} . \frac{x^{\cancel{4}3}}{4} \right) dx$

$\Rightarrow \frac{x^4}{4} \log 2x - \frac{1}{4} \int x^3 dx$

$\Rightarrow \frac{x^4}{4} \log 2x - \frac{1}{4} \times \frac{x^4}{4} + C \Rightarrow \frac{x^4}{4} \log 2x - \frac{x^4}{16} + C$

$\Rightarrow \cancel{\frac{x^4}{4} \log 2x - \frac{x^5}{40} + C}$

Q- $\int \frac{\cos 2x - \cos 2a}{\cos x - \cos a} dx$ (Important)

$\Rightarrow \int \frac{2\cos^2 x - \cancel{1} - 2\cos^2 a + \cancel{1}}{\cos x - \cos a} dx$

$\Rightarrow 2 \int \frac{\cos^2 x - \cos^2 a}{\cos x - \cos a} dx$

$\Rightarrow 2 \int \frac{(\cos x + \cos a)\cancel{(\cos x - \cos a)}}{\cancel{(\cos x - \cos a)}} dx$

$\Rightarrow 2 \int (\cos x + \cos a) dx$

$\Rightarrow 2(\sin x + x \cos a) + C$

$\cos 2x = 2\cos^2 x - 1$
or
$1 - 2\sin^2 x$

$\cos x = 2\cos^2 x/2 - 1$
or
$1 - 2\sin^2 x/2$

$\sin 2x = 2\sin x . \cos x$

$\sin x = 2\sin x/2 \cos x/2$

$\cos(\pi/2 + x) = -\sin x$

$\sin(\pi/2 + x) = \cos x$

Q- $\int \tan^{-1}(\sec x + \tan x)\, dx$ (Important)

Sol^n We have, $\int \tan^{-1}(\sec x + \tan x) dx$

Now,

$\int \tan^{-1}(\sec x + \tan x) dx$

$\Rightarrow \int \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) dx$

$\Rightarrow \int \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) dx$

$\Rightarrow \int \tan^{-1}\left(\frac{1 - \cos(\pi/2 + x)}{\sin(\pi/2 + x)}\right) dx$

$\left\{ \begin{array}{l} \cos(\pi/2 + x) = -\sin x \\ \sin x = -\cos(\pi/2 + x) \end{array} \right\}$

$\Rightarrow \int \tan^{-1}\left(\frac{1 - [1 - 2\sin^2(\frac{\pi/2 + x}{2})]}{2\sin(\frac{\pi/2 + x}{2}) . \cos(\frac{\pi/2 + x}{2})}\right) dx$

$\Rightarrow \int \tan^{-1}\left(\frac{\cancel{2}\sin^2(\pi/4 + x/2)}{\cancel{2}\sin(\pi/4 + x/2) . \cos(\pi/4 + x/2)}\right) dx$

$\Rightarrow \int \tan^{-1}\left(\frac{\sin(\pi/4 + x/2)}{\cos(\pi/4 + x/2)}\right) dx$

$\Rightarrow \int \tan^{-1}(\tan(\pi/4 + x/2)) dx$

$\Rightarrow \int (\frac{\pi}{4} + x/2) dx \Rightarrow \frac{\pi}{4}x + \frac{x^2}{4} + c$

Q- $\int \frac{x^4+x^2+1}{x^2-x+1}\,dx$ (Important)

Sol^n $\int \frac{x^4+x^2+1+x^2-x^2}{x^2-x+1}\,dx$

$\Rightarrow \int \frac{x^4+2x^2+1-x^2}{x^2-x+1}\,dx$

$\Rightarrow \int \frac{(x^2)^2+2(x^2)\cdot 1^2+1^2-x^2}{x^2-x+1}\,dx$

$\Rightarrow \int \frac{(x^2+1)^2-x^2}{x^2-x+1}\,dx$

$\Rightarrow \int \frac{\cancel{(x^2+1-x)}(x^2+1+x)}{\cancel{(x^2-x+1)}}\,dx$

$\Rightarrow \int (x^2+1+x)\,dx$

$\Rightarrow \int x^2dx+\int x\,dx+\int dx$

$\Rightarrow \frac{x^3}{3}+\frac{x^2}{2}+x+C$

$a^3+b^3=(a+b)^3-3ab(a+b)$

Q- $\int \frac{\sin^6x+\cos^6x}{\sin^2x * \cos^2x}\,dx$

$\Rightarrow \int \frac{(\sin^2x)^3+(\cos^2x)^3}{\sin^2x\cdot\cos^2x}\,dx$

$\Rightarrow \int \frac{(\sin^2x+\cos^2x)^3-3\sin^2x(\cos^2x(\sin^2x+\cos^2x))}{\sin^2x\cdot\cos^2x}\,dx$

$\Rightarrow \int \frac{1-3\sin^2x\cos^2x}{\sin^2x\cos^2x}\,dx \Rightarrow \int\frac{1}{\sin^2x\cos^2x}\,dx - 3\int dx$

$\Rightarrow \int \frac{\sin^2x+\cos^2x}{\sin^2x\cos^2x}\,dx - 3\int dx \Rightarrow \int\frac{1}{\cos^2x}\,dx+\int\frac{1}{\sin^2x}\,dx - 3\int dx$

$\Rightarrow \int \sec^2x\,dx+\int \operatorname{cosec}^2x\,dx-3\int dx$

$\Rightarrow \tan x-\cot x-3x+C$