CS100433
2D and 3D Transformation

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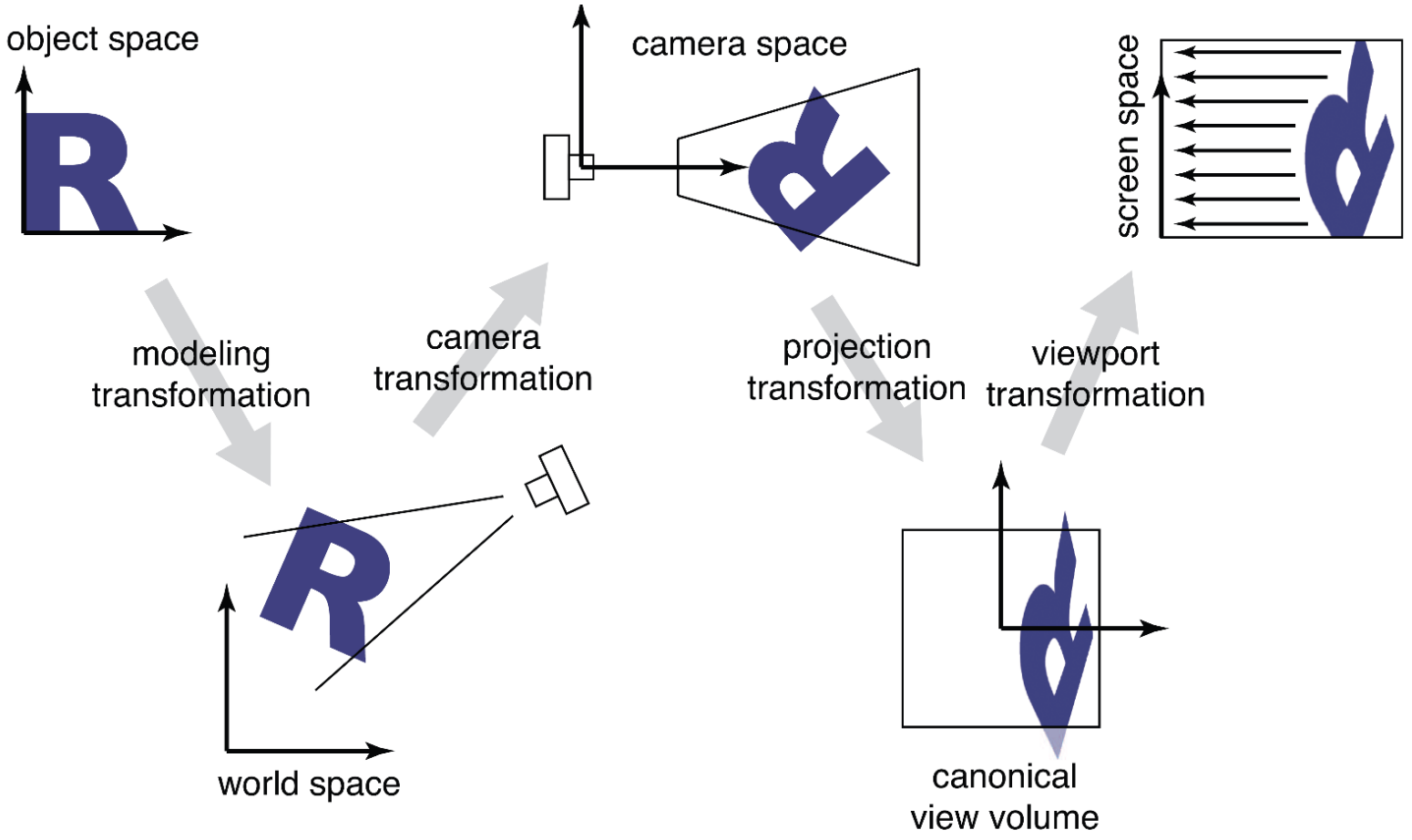
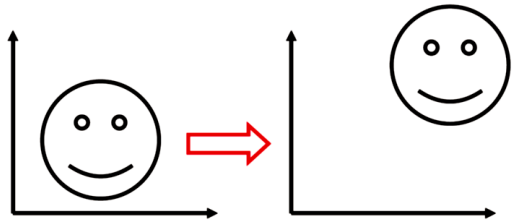
Why transformation?

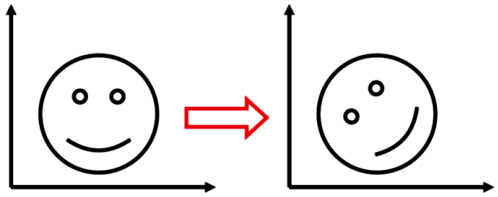


(2013 Steve Marschner)

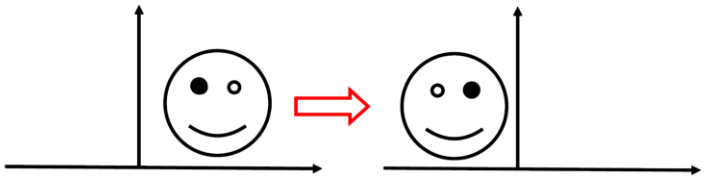
Transformations



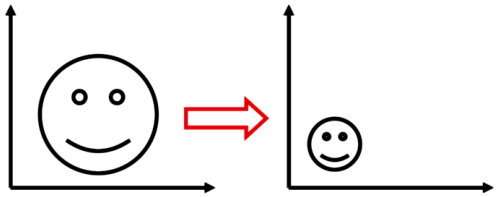
translation



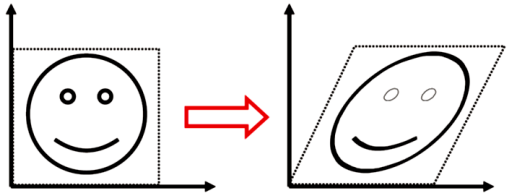
rotation



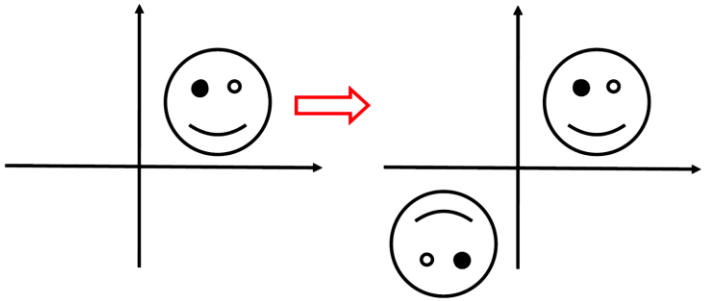
reflection with respect to the y-axis



scaling



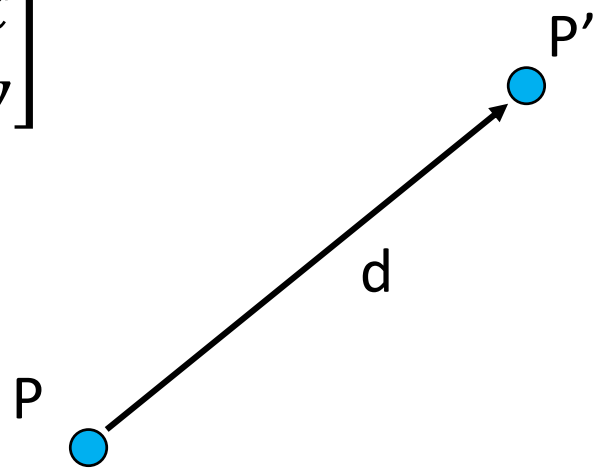
shear



reflection with respect to the origin

Translation

- Move (translate, displace) a point to a new location
- Displacement determined by a vector d
  - $P' = P + d$
  - $P = \begin{bmatrix} x \\ y \end{bmatrix},\ P' = \begin{bmatrix} x' \\ y' \end{bmatrix},\ d = \begin{bmatrix} dx \\ dy \end{bmatrix}$
- Simplest transformation:
  - $T(v) = v + u$
  - Inverse: $T^{-1}(v) = v - u$



Linear Transformations

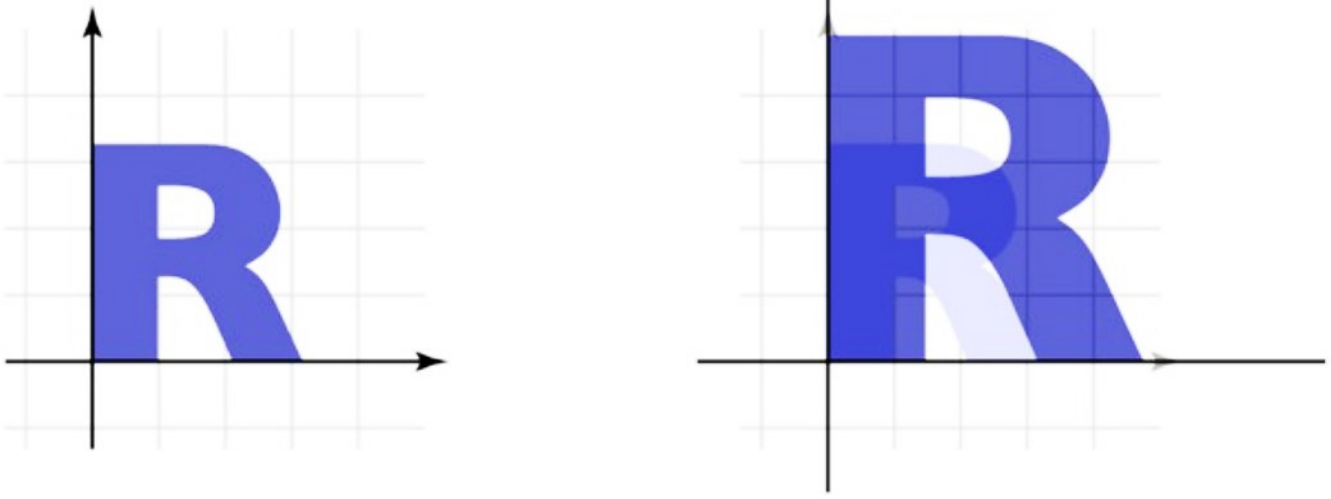
- Such transformations are *linear*:
  - $T(v+u) = T(v) + T(u)$
  - $T(\alpha v) = \alpha T(v)$
  - $T(\alpha v + \beta u) = \alpha T(v) + \beta T(u)$
- Linear Transformations can be represented in *matrix* form
- Rotation, scale, shear, reflection are all *linear*
- Is translation *linear*?

2D Linear Transformations

- $T(v) = Mv = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_v$
- *M* is *a* 2x2 transformation matrix
  - Uniform scale
  - Non-uniform scale
  - Rotation
  - Shear
  - Reflection

Uniform Scale

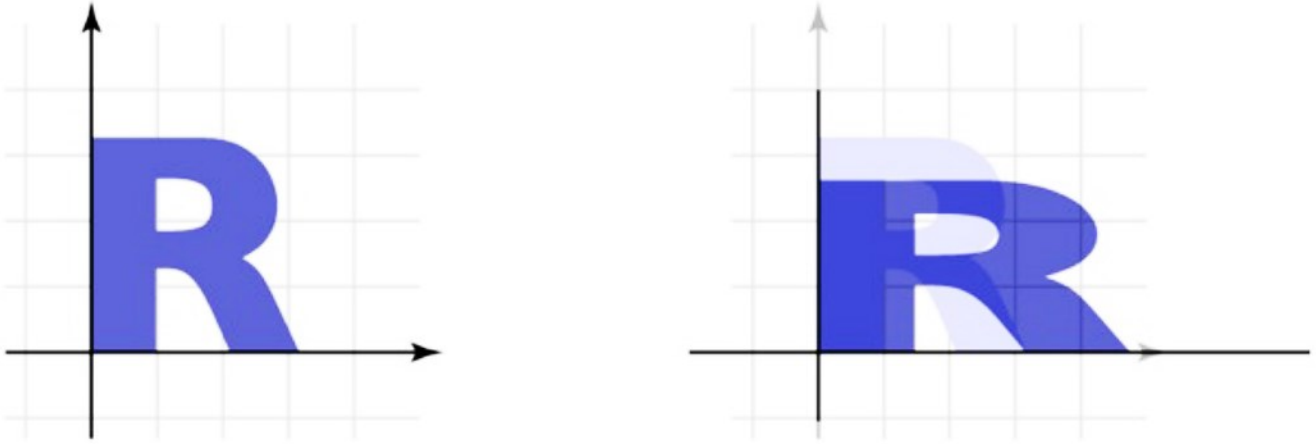
- $\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} sx \\ xy \end{bmatrix}$



Non-uniform Scale

- $\begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} sx \times x \\ sy \times y \end{bmatrix}$

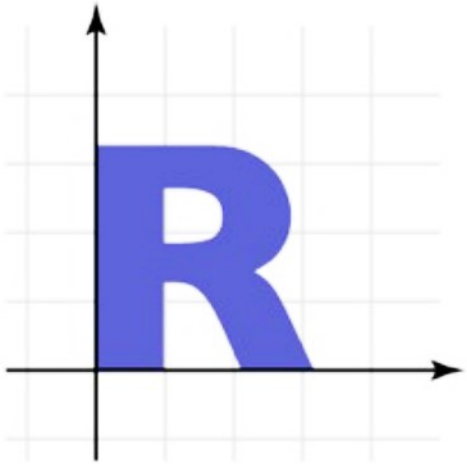
$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.8 \end{bmatrix}$$

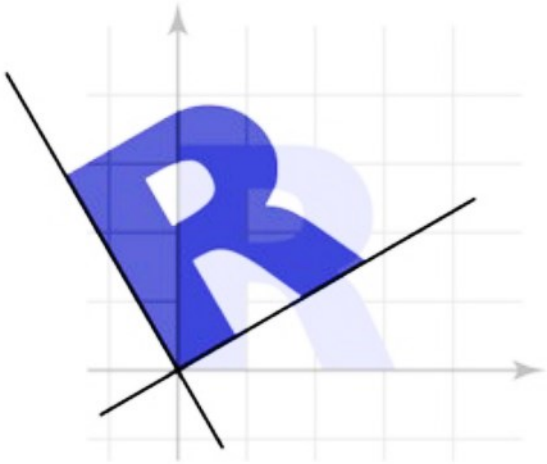


Rotation

- $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xcos\theta - ysin\theta \\ xsin\theta + ycos\theta \end{bmatrix}$

$$\begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$$



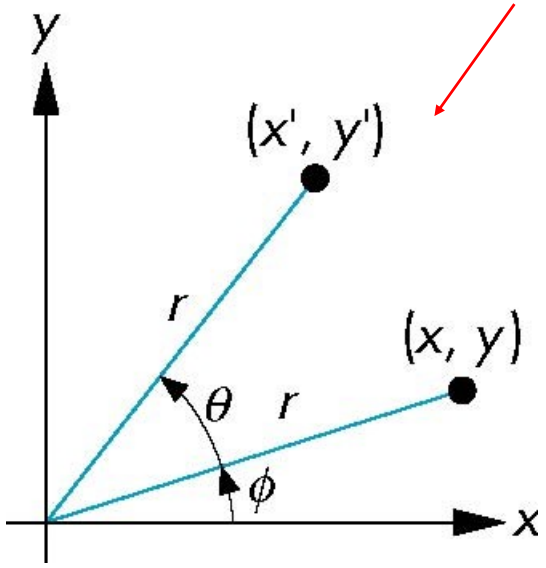


Rotation about the origin

Consider rotation about the origin by θ degrees

- radius stays the same, angle increases by θ

$$x' = r \cos(\phi + \theta)$$
$$y' = r \sin(\phi + \theta)$$



$$x = r \cos \phi$$
$$y = r \sin \phi$$

2D Rotation about the origin

$$x' = r.\cos(\theta + \phi) = r.\cos\phi.\cos\theta - r.\sin\phi.\sin\theta$$

$$y' = r.\sin(\theta + \phi) = r.\cos\phi.\sin\theta + r.\sin\phi.\cos\theta$$

Substituting for r :

$$x = r.\cos\phi$$

$$y = r.\sin\phi$$

Gives us :

$$x' = x.\cos\theta - y.\sin\theta$$

$$y' = x.\sin\theta + y.\cos\theta$$

Rewriting in Matrix form

$$x' = x.\cos\theta - y.\sin\theta$$

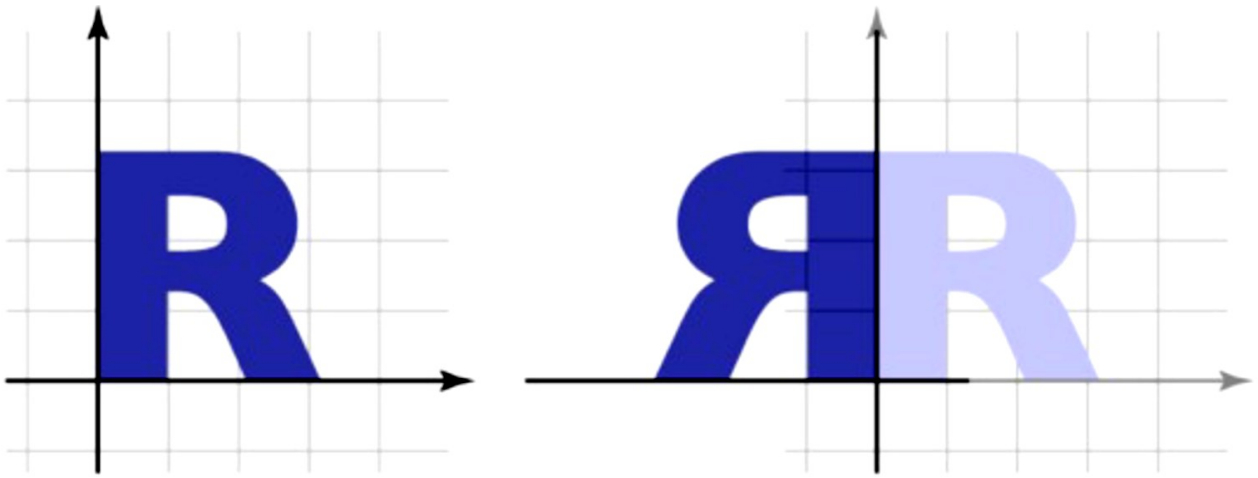
$$y' = x.\sin\theta + y.\cos\theta$$

Rewriting in matrix form gives us :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection

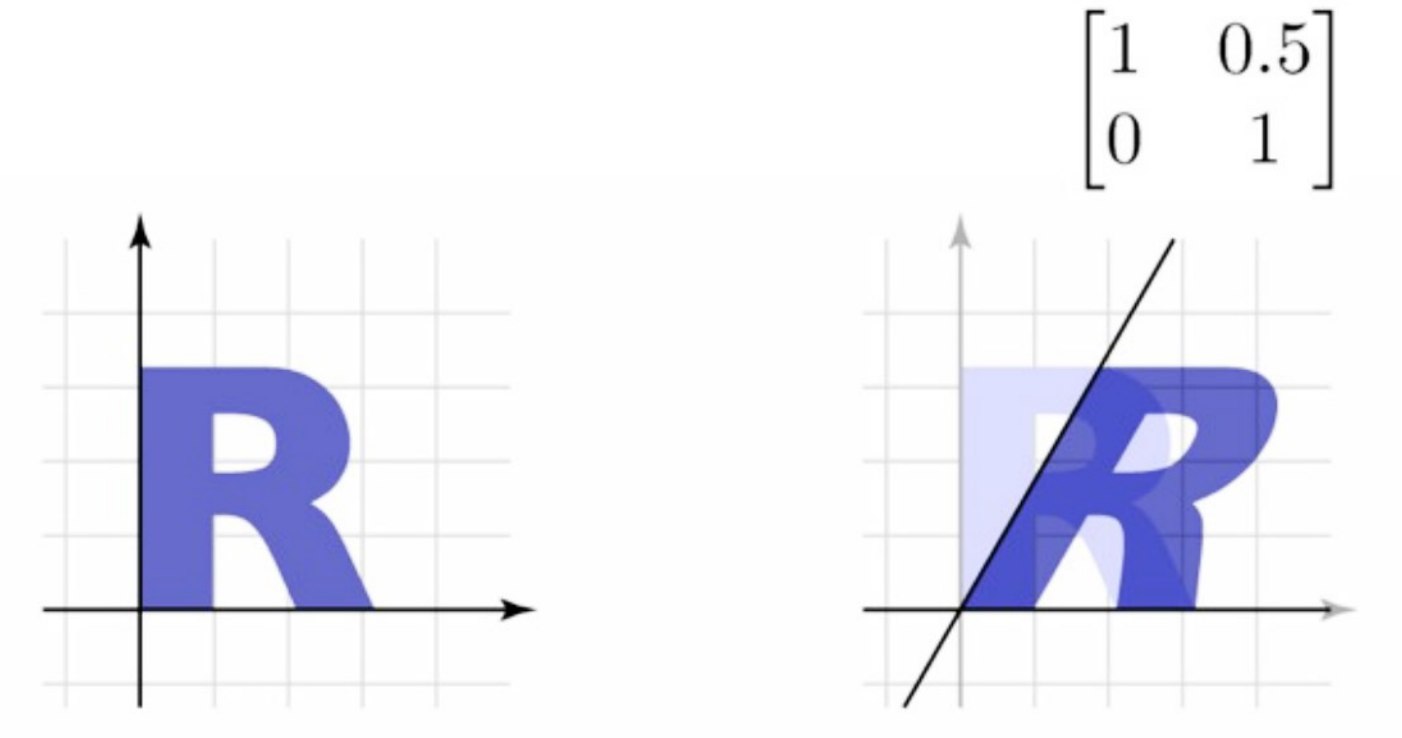
- Can be considered as a special case of non-uniform scale
- $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



Shear

- $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \alpha y \\ y \end{bmatrix}$

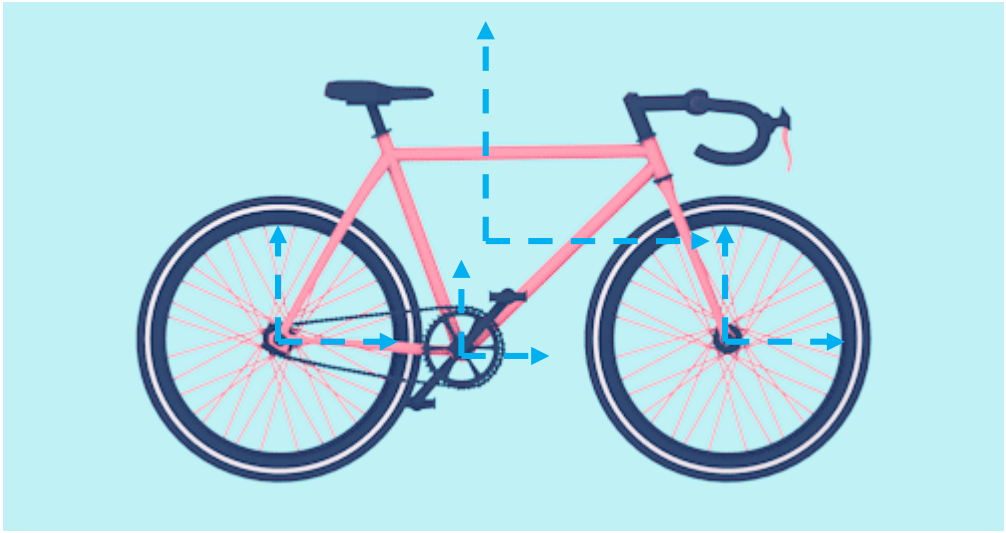
$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$



- Questions?

How to **animate** a bicycle?

How to **animate** a bicycle?



Composing transformations

- We “move” an object twice or more
  - $P \rightarrow T(P) = P'$
  - $P' \rightarrow S(P') = P''$
  - $P'' = S(P') = S\big(T(P)\big) = (S \circ T)(P)$
- $S \circ T$ is S compose T
- (**which one is firstly applied?**)
- Translations
  - $T(P) = P + d_T$
  - $S(P) = P + d_S$
  - $(S \circ T)(P) = P + (d_T + d_S)$
  - Therefore, translation by d_T and d_S is translation by $d_T + d_S$
  - Commutative!

Composing transformations

- Linear Transformations
  - $P \rightarrow T(P) = M_T P = P'$
  - $P' \rightarrow S(P') = M_S P' = P''$
  - $(S \circ T)(P) = M_S M_T P$
  - (**which one is firstly applied?**)
- *Be careful!*
  - *Only some are commutative*

Composing transformations

- Translation together with linear transformations - Affine transformation
- Applications require a single framework
  - $T(P) = M_T P + d_T$
  - $S(P) = M_S P + d_S$
  - $(S \circ T)(P) = M_S(M_T P + d_T) + d_S = M_S M_T P + M_S d_T + d_S$
  - Not elegant!

Homogeneous coordinates

- Extra component *w*
  - $V = [x, y, w]^T$
- If $w \neq 0$, divide by it to get Cartesian coordinates of point (*x*/*w*, *y*/*w*, 1)
  - $P = [x, y, 1]^T$
  - $P' = [x', y', 1]^T$
  - $d = [dx, dy, 0]^T$
- Hence
  - $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + \begin{bmatrix} dx \\ dy \\ 0 \end{bmatrix}$

note that this expression is in Three dimensions and expresses

point = point + vector

Point – point = ?

Can we translate vectors?

Translation Matrix

We can now express translation using a

3 x 3 matrix $\mathbf{T_d}$ in homogeneous coordinates

$\mathbf{p'} = \mathbf{T_d p}$, where

$$\mathrm{T_d} = \mathrm{T_d}(dx, dy) = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + dx \\ y + dy \\ 1 \end{bmatrix}$$

Properties of translations

- $T(0,0) = I$
- $T(s_x, s_y) \cdot T(t_x, t_y) = T(s_x + t_x, s_y + t_y)$
- $T(s_x, s_y) \cdot T(t_x, t_y) = T(t_x, t_y) \cdot T(s_x, s_y)$
- $T^{-1}(s_x, s_y) = T(-s_x, -s_y)$

- Note : translation matrices are *commutative.*

Rotation Matrix

- With Cartesian coordinate frame
  - $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$
- With homogenous coordinate frame

$$\mathrm{R} = \mathrm{R}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} xcos\theta - ysin\theta \\ xsin\theta + ycos\theta \\ 1 \end{bmatrix}$$

Properties of rotation

- Rotation matrices are orthogonal, i.e.:
  - $R^{-1}(\theta) = R^{T}(\theta)$
- $R^{-1}(\theta) = R(-\theta)$
- $R(0) = I$
- $\mathrm{Det(R)} = 1$
- $R(\theta) \cdot R(\phi) = R(\theta + \phi)$
- And
- $R(\theta) \cdot R(\phi) = R(\phi) \cdot R(\theta)$
  - But this is only valid when the axis of rotation is the same
  - Not the case in 3D!

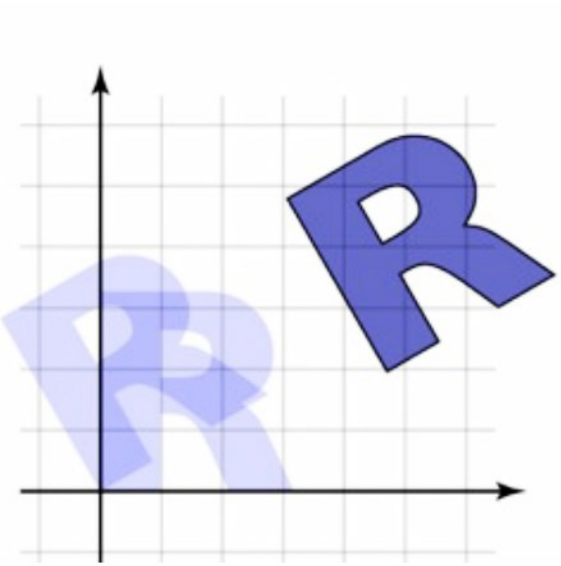
Composing transformations

- Compose translation and linear transformations using homogenous coordinates
- $$\begin{bmatrix} M_S & d_S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_T & d_T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix} = \begin{bmatrix} M_S M_T P + M_S d_T + d_S \\ 1 \end{bmatrix}$$
- This form is better for implementation because all **Affine Transformations** can be expressed this way and multiple transformations can be concatenated together

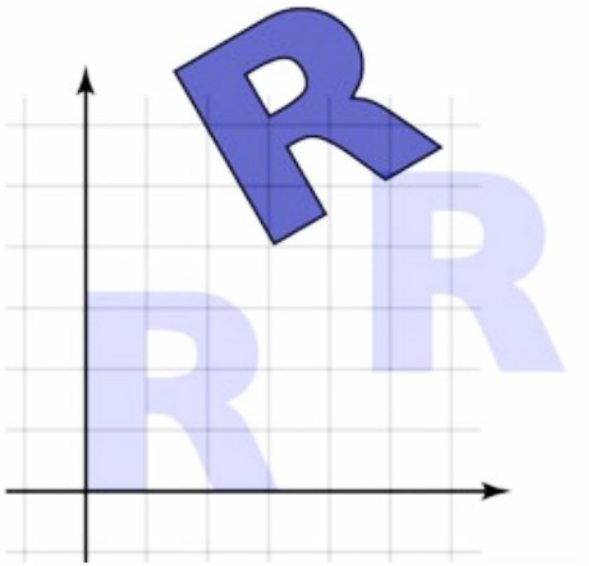
- Questions?

Composite Transformations

- In general NOT commutative

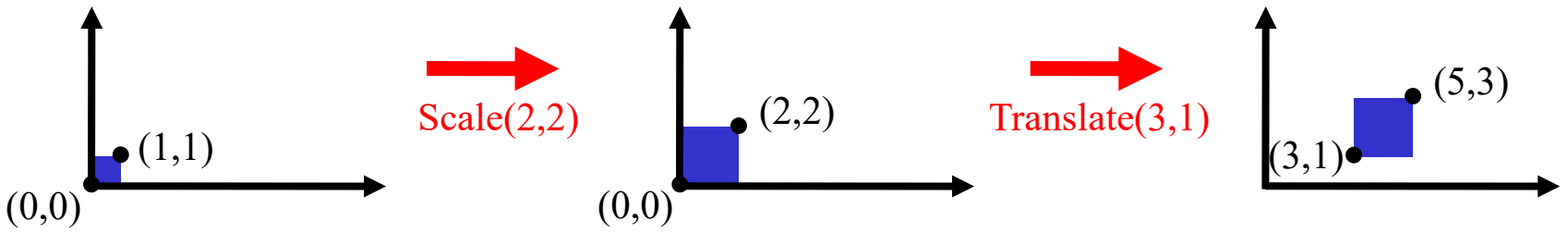


rotate, then translate

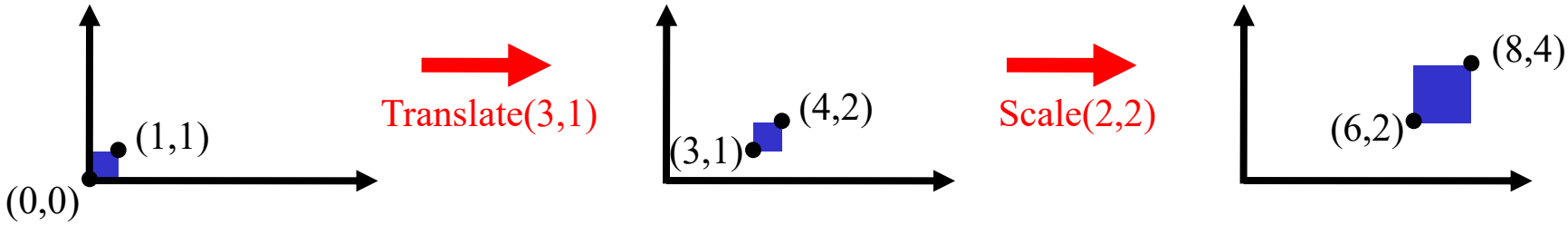


translate, then rotate

Scale then Translate: $p' = T(S\,p) = TS\,p$



Translate then Scale: $p' = S(T\,p) = ST\,p$



Scale then Translate: $p' = T(S\,p) = TS\,p$

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Translate then Scale: $p' = S(T\,p) = ST\,p$

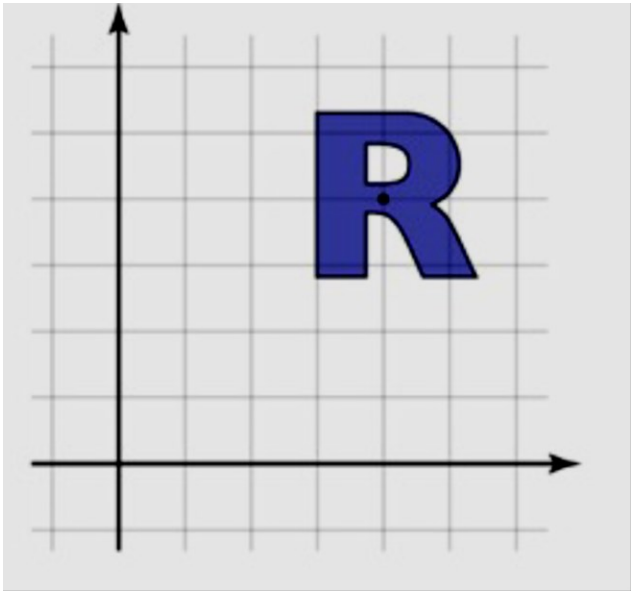
$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Properties of Affine transformations

- Rotation and translation
  - Angles and distances are preserved
  - Unit cube is always unit cube
  - ***Rigid-Body* transformations or *rigid motion***
- Rotation, translation and scale.
  - Angles & distances not preserved.
  - But parallel lines are.

Want to rotate about a particular point

- We know how to rotate about the origin
  - So translate that point to the origin

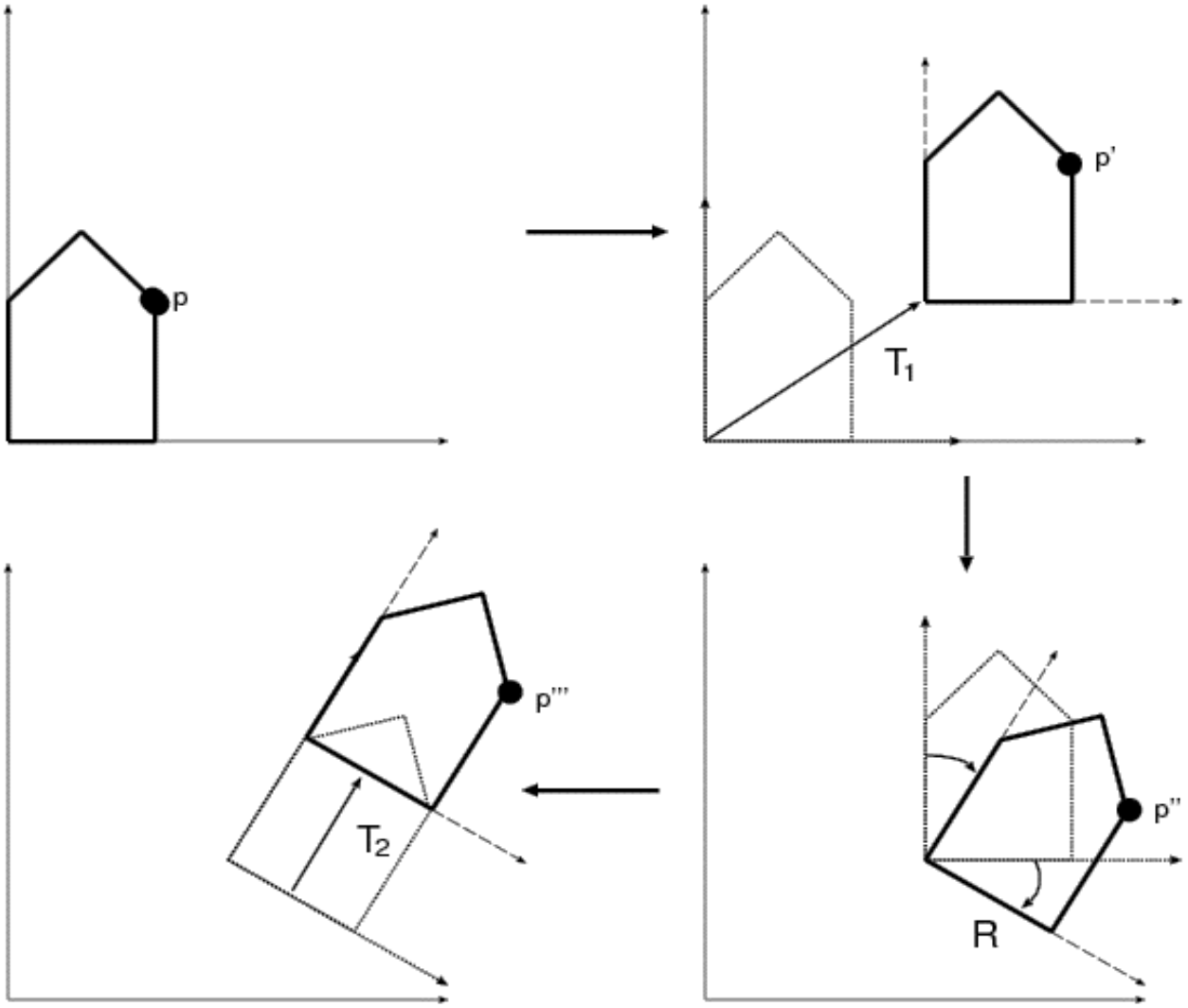


$$M = T^{-1}RT$$

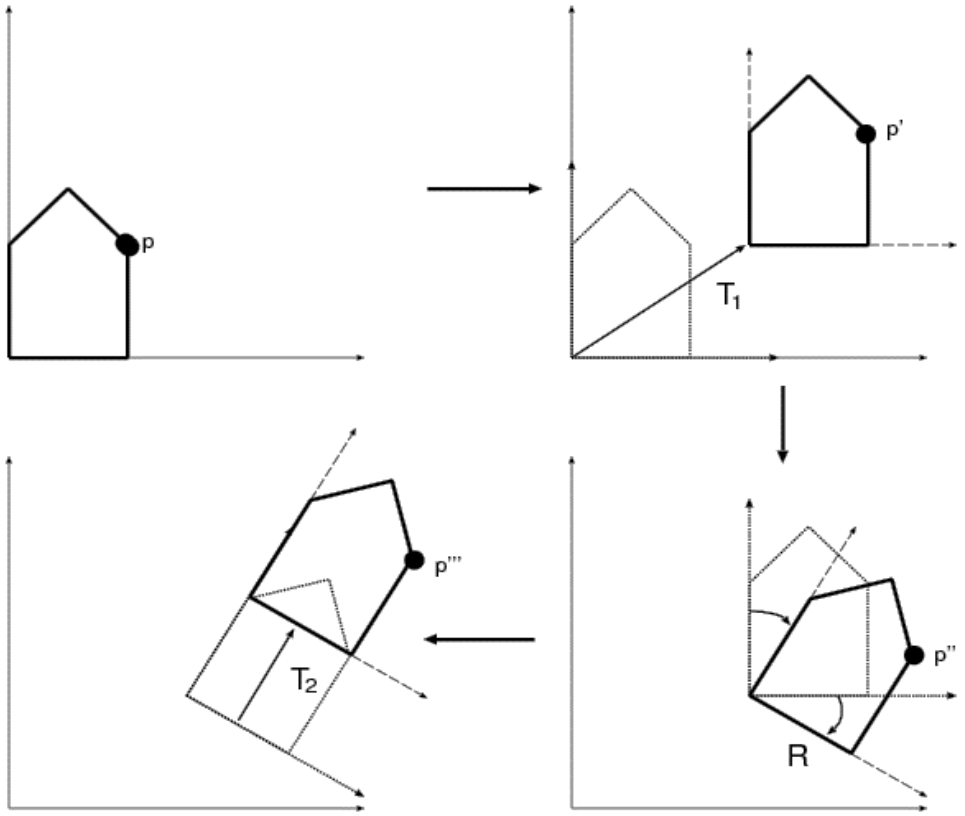
Transformations of local coordinate systems

- Have been discussing transformations as transforming points
- Always need to think the transformation in the world coordinate system
- Sometimes this might not be convenient
  - i.e. rotating objects at its location

Transformations of local coordinate systems - Example



Transformations of local coordinate systems - Example



$p' = T_1\, p$

$p'' = T_1\, R\, p$

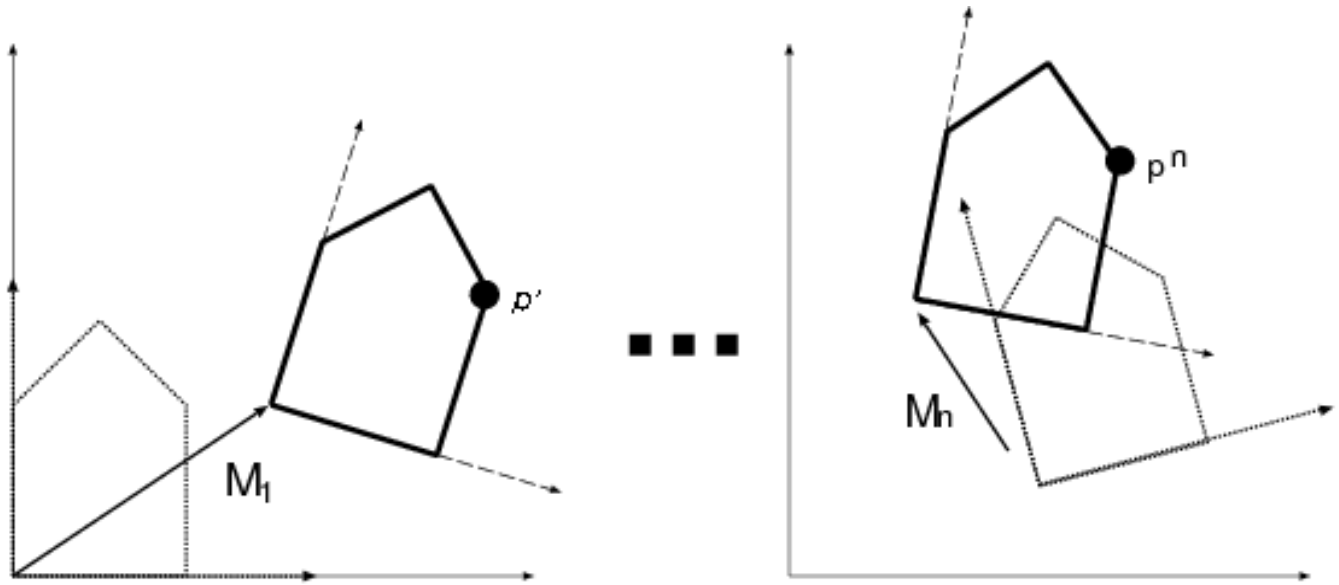
$p''' = T_1\, R\, T_2\, p$

- **Concatenate local transformation matrices from left to right**
- Can obtain the local – world transformation matrix
- p', p'', p''' are the world coordinates of p after each transformation

Transformations of local coordinate systems - example

- p^n is the world coordinate of point p after n transformations

$$p^n = M_1 M_2 \cdots M_n p$$

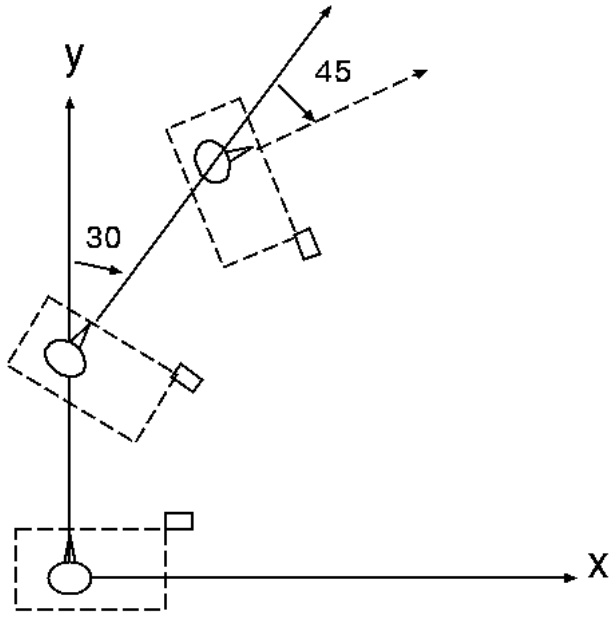


Quiz

- I sat in the car, and find the side mirror is 0.4m on my right and 0.3m in my front
- I started my car and drove 5m forward, turned 30 degrees to right, moved 5m forward again, and turned 45 degrees to the right, and stopped
- What is the position of the side mirror now, relative to where I was sitting in the beginning?

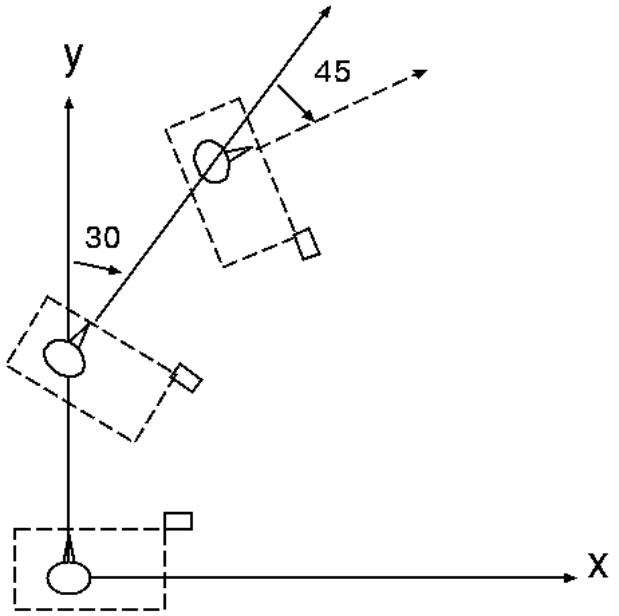
Solution

- The side mirror position is locally (0.4, 0.3)
- The matrix of first driving forward 5m is
- $T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$



Solution

- The matrix to turn to the right 30 and 45 degrees (rotating -30 and -45 degrees around the origin) are
- $R_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $R_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

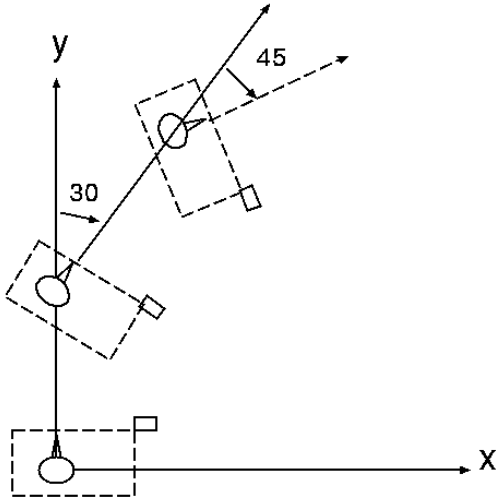


Solution

- The local-to-global transformation matrix at the last configuration of the car is

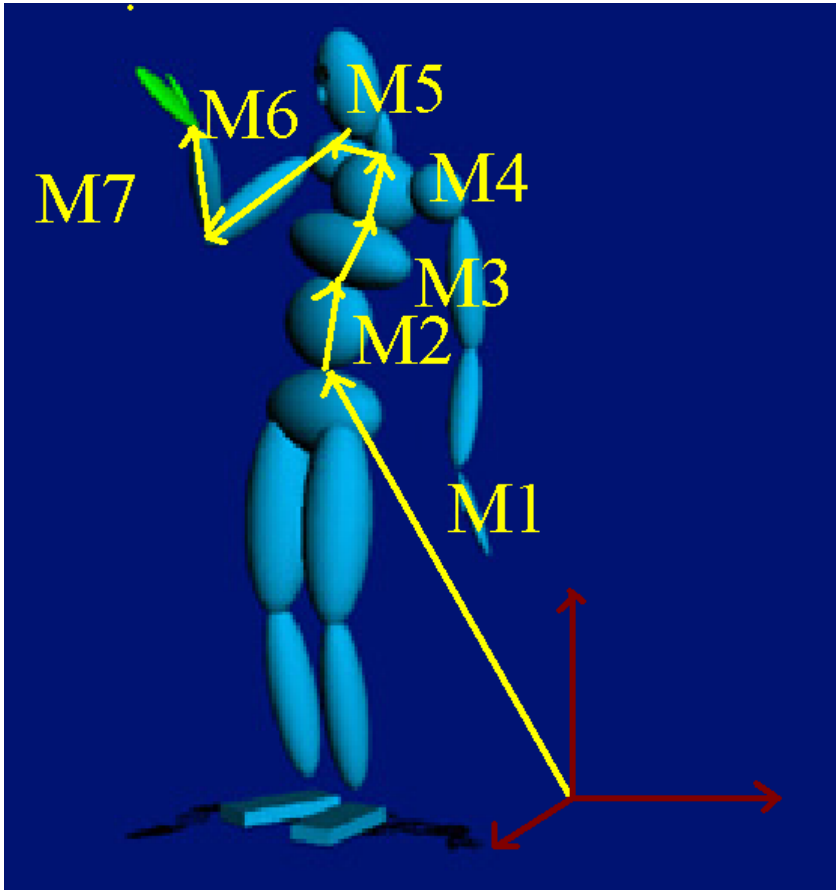
- $$TR_1TR_2P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.3 \\ 1 \end{bmatrix}$$

The final position of the side mirror can be computed by TR_1TR_2P which is around (2.89331, 9.0214)



This is convenient for character animation / robotics

- In robotics / animation, we often want to know what is the current 3D location of the end effectors (like the hand)
- Can concatenate matrices from the origin of the body towards the end effecter



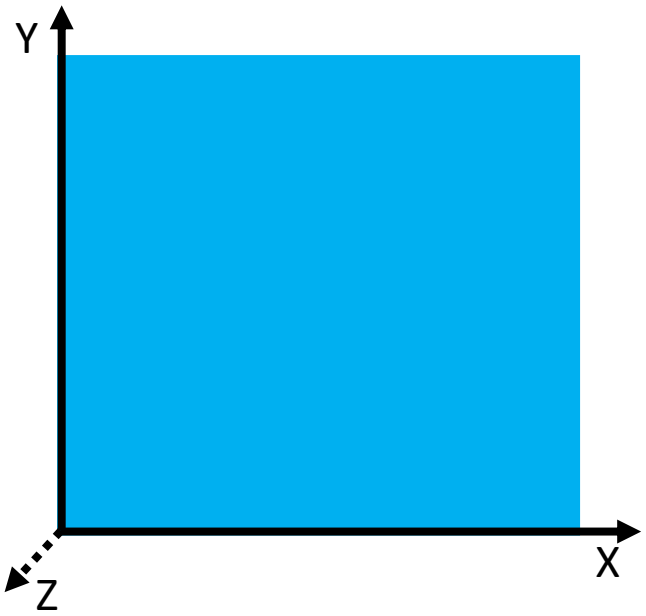
How to implement?

- Queue?
- Stack?
- Tree?

- Questions?

3D Transformations

- Use homogeneous coordinates, just as in 2D case
- Transformations are now 4x4 matrices
- We will use a right-handed (world) coordinate system - (z out of the page)



Translation

- $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scale

- $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Reflection

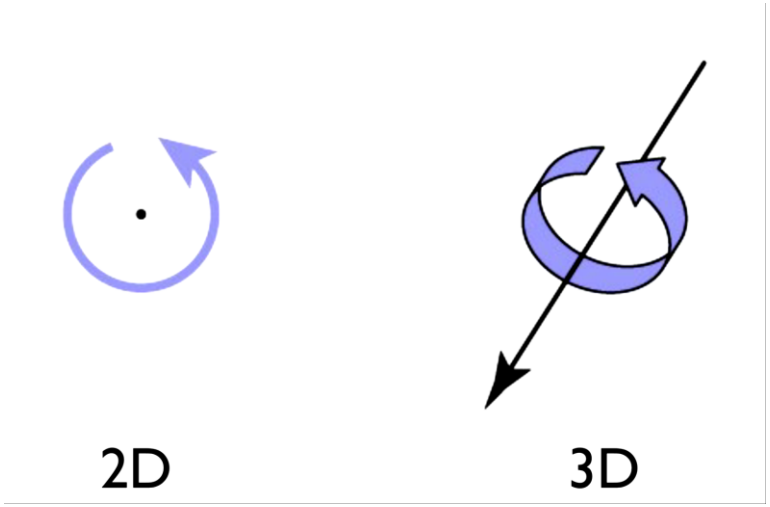
- $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Shear

- $$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a_1 & a_2 & 0 \\ a_3 & 1 & a_4 & 0 \\ a_5 & a_6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

General Rotation Matrices

- Rotation in 2D?
  - Around an arbitrary point
- Rotation in 3D?
  - Around an arbitrary axis
  - There are many more 3D rotations than 2D

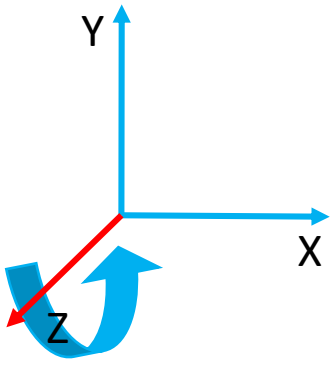


Specifying Rotations

- In 2D, just an angle θ
- In 3D, is more complex
  - Basic rotation about origin: Axis and angle
  - Convention: positive rotation is CCW (right handed)
  - Many ways
    - Directly through
      - Euler angles: 3 angles about 3 axes
    - Indirectly through frame transformations
    - Quaternions

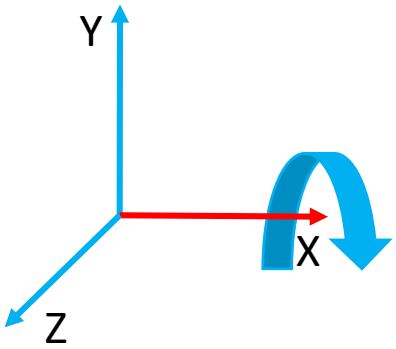
Rotation

- $R_z(\theta) = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$



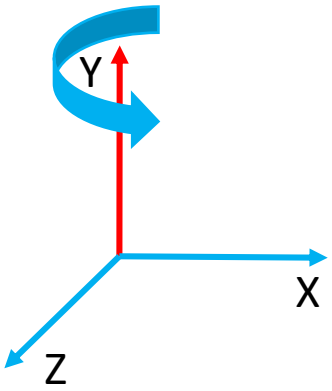
Rotation

- $R_x(\theta) = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$



Rotation

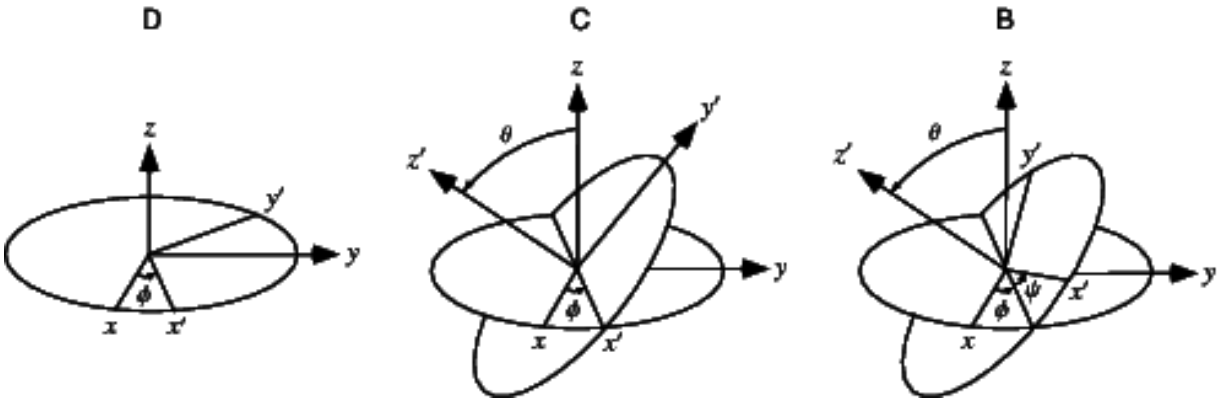
- $R_y(\theta) = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta & 0 & sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$



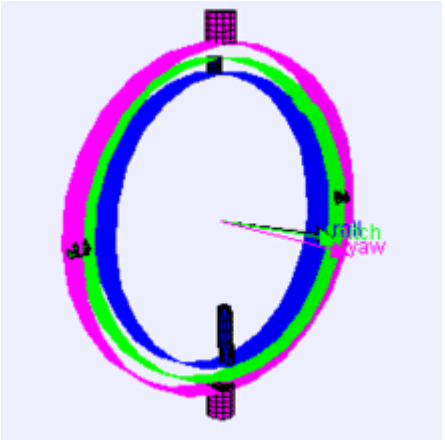
Euler Angles

- Any rotation may be described using three angles. If the rotations are written in terms of rotation matrices D, C, and B, then a general rotation A can be written as:

$$A = BCD$$

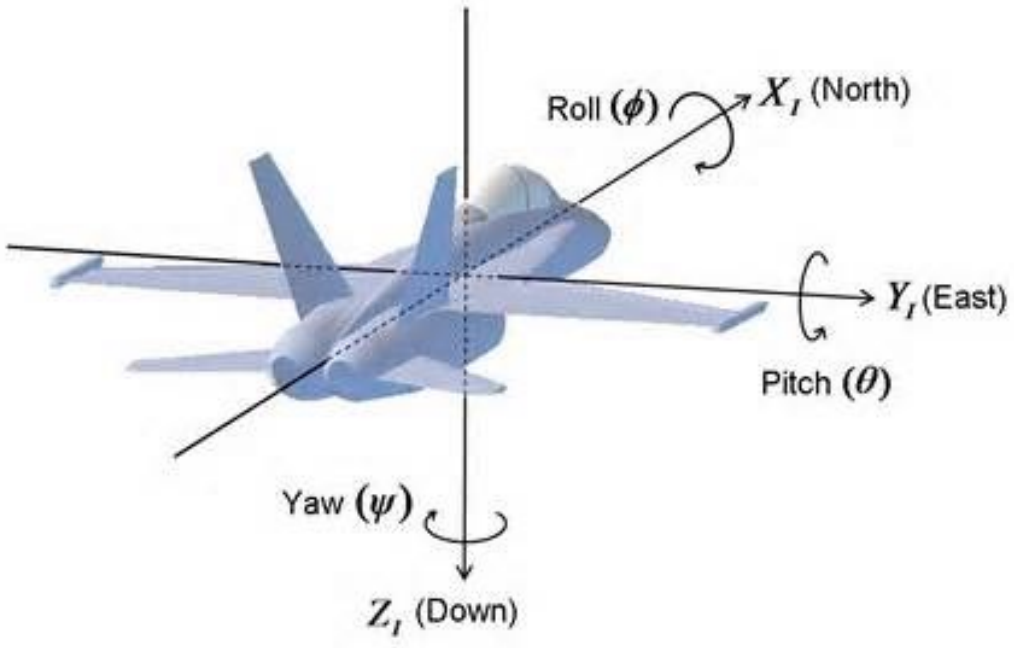


Gimbal



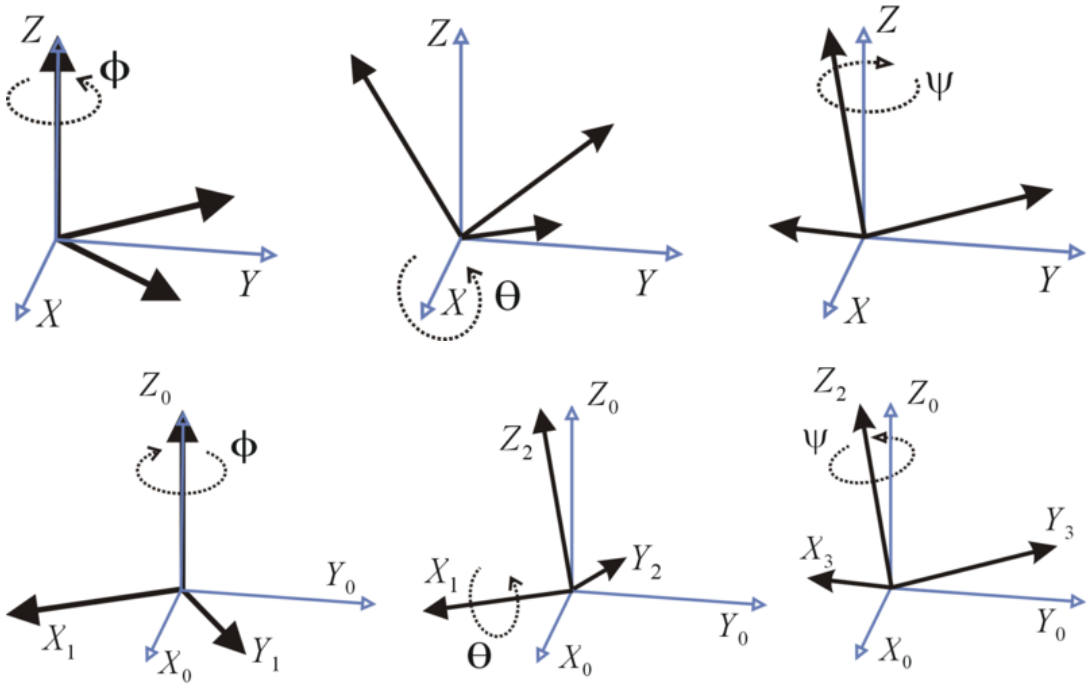
Euler Angles

- Roll(Φ) around X
- Pitch(Θ) around Y
- Yaw(Ψ) around Z



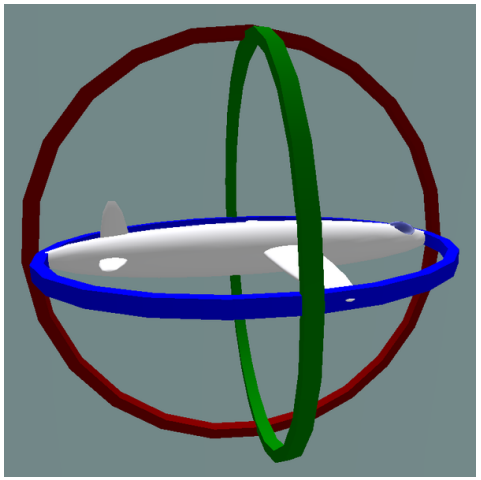
Extrinsic vs Intrinsic Rotations

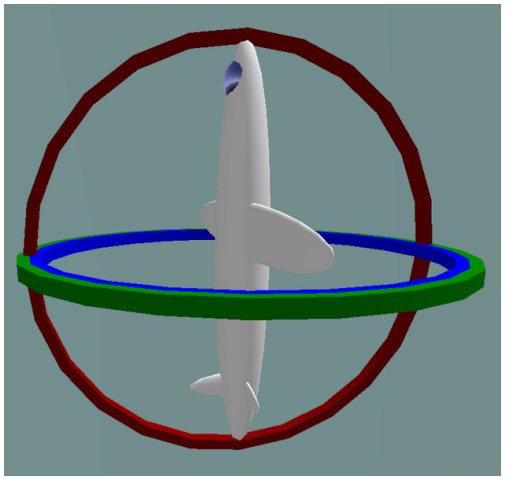
- Any extrinsic rotation is equivalent to an intrinsic rotation by the same angles but with inverted order of elemental rotations

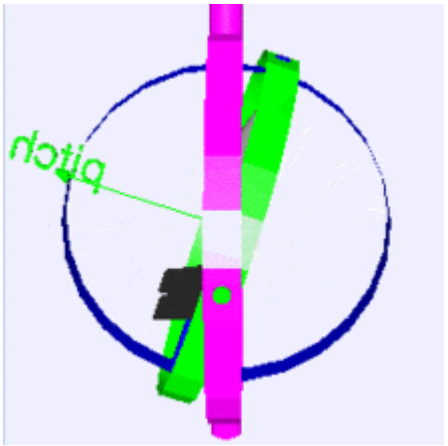


Gimbal Lock

- Euler basic motions are never expressed in a frame, but a **mixed axes of rotation**
- $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha \\ 0 & sin\alpha & cos\alpha \end{bmatrix} \begin{bmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{bmatrix} \begin{bmatrix} cos\gamma & -sin\gamma & 0 \\ sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Set $\beta = \pi/2$?





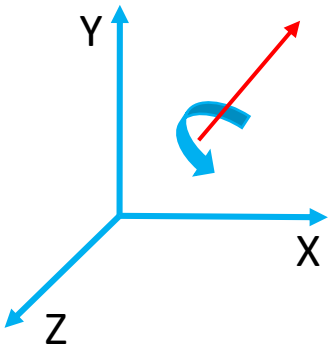


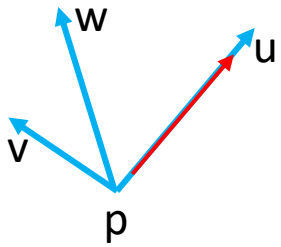
Matrices for Axis-angle Rotations

- Ruler angles are for coordinates axes
  - What if we want rotation about some random axis?
- Compute by composing elementary transforms
  - Transform rotation axis to align with x axis
  - Apply rotation
  - Inverse transform back into position

Building General Rotations

- Using elementary transforms
  - Translate axis to pass through origin
  - Rotate about y to get into x-y plane
  - Rotate about z to get into x axis
- Alternative: construct a frame and change coordinates
  - Choose p, u, v, w
  - Apply transform $T = FR_x(\theta)F^{-1}$
  - $F = \begin{bmatrix} u & v & w & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$





Quaternions

- Compare to Euler angles and Rotation matrices
  - Can avoid the gimbal lock
  - Can smoothly interpolate over a sphere
- Representation
  - A quaternion is composed of one real element and three complex elements

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij = k, ji = -k \\ jk = i, kj = -i \\ ki = j, ik = -j \end{cases}$$

Quaternions

- A rotation through an angle of θ around the axis defined by a unit vector $n = \left[n_{x,}\, n_y, n_z\right]^T$ can be represented by a quaternion:

$$q = \left[cos\frac{\theta}{2}, n_x\, sin\frac{\theta}{2}, n_y\, sin\frac{\theta}{2}, n_z\, sin\frac{\theta}{2}\right]^T$$

- Rotation： $p' = qpq^{-1}$
- Rotation matrix, Rotation Axis, Euler Angle and Quaternions are exchangeable.

Rodriguez formula

$$R = cos(\theta)\ \cdot I + \big(1 - cos(\theta)\big)\phi\phi^T + sin(\theta)\phi\times$$

- *I* identity matrix, ϕ axis-angle rotation
- x is skew-symmetric matrix of a vector:

$$(a\times) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

Transformations with GLM

```
#include <glm/glm.hpp>
#include <glm/gtc/matrix_transform.hpp>
#include <glm/gtc/type_ptr.hpp>
…
glm::vec4 vec(1.0f, 0.0f, 0.0f, 1.0f);
glm::mat4 trans = glm::mat4(1.0f);
trans = glm::translate(trans, glm::vec3(1.0f, 1.0f, 0.0f));
trans = glm::rotate(trans, glm::radians(90.0f),
glm::vec3(0.0, 0.0, 1.0));
vec = trans * vec;
```

- Questions?

References

- Ed Angel, CS/EECE 433 Computer Graphics, University of New Mexico
- Durand and Cuter, MIT EECS 6.837
- Steve Marschner, CS4620/5620 Computer Graphics, Cornell
- Tom Thorne, COMPUTER GRAPHICS, The University of Edinburgh
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