CSE 6140/ CX 4140
Computational Science and Engineering
ALGORITHMS

NP Completeness 5

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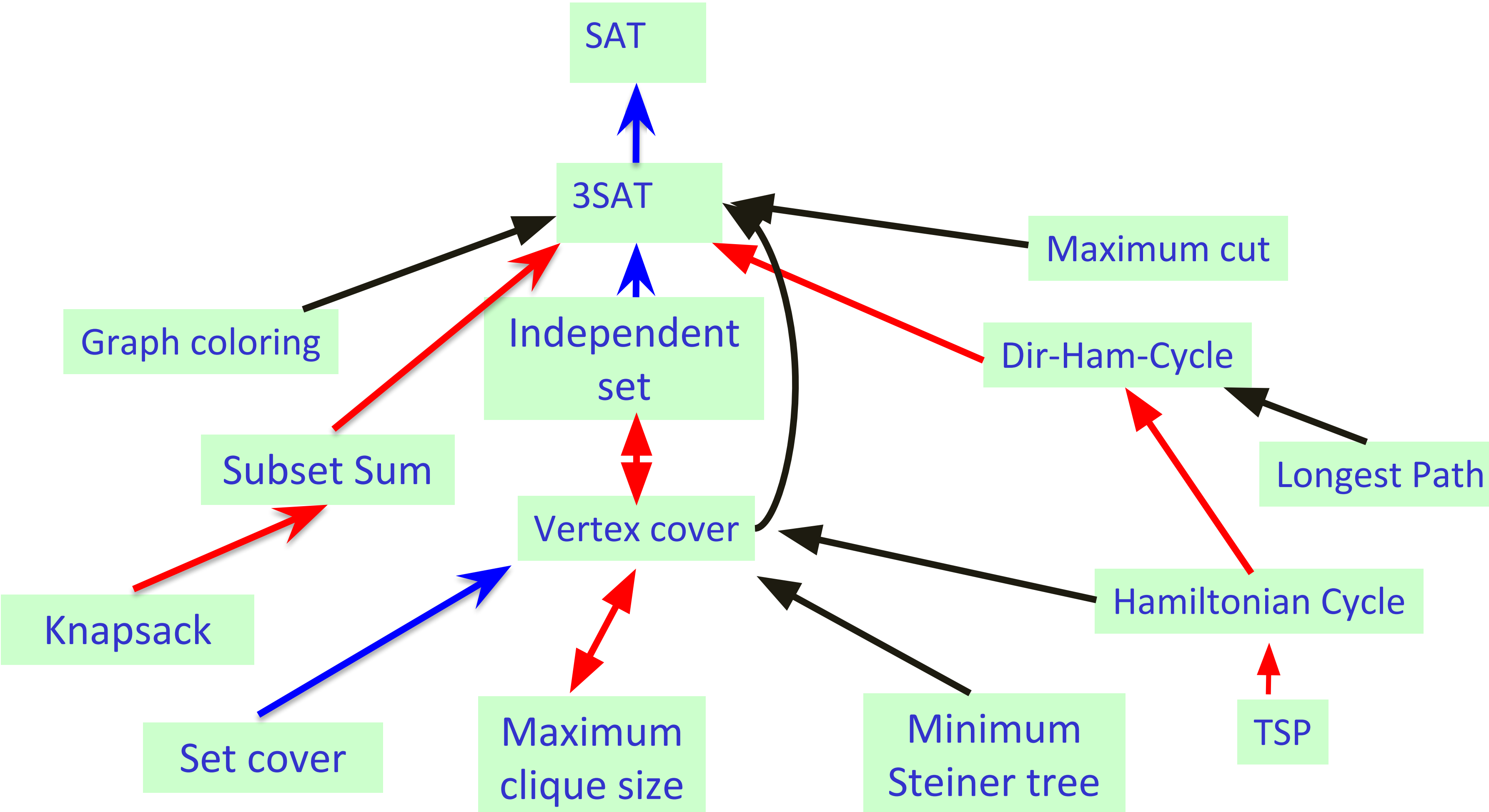
Based on slides by Prof. Ümit V. Çatalyürek

Course logistics

Test 2: Oct 28

Course project:
will be released and introduced in the lecture of next week.

Summary of some NPc problems



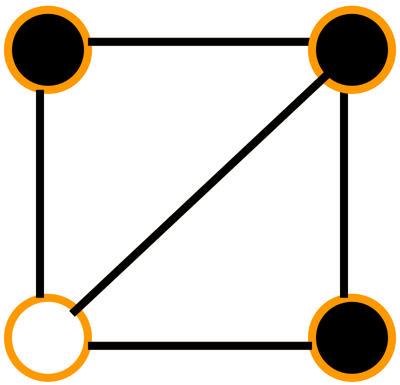
Basic reduction strategies.

- Reduction by simple equivalence.
  - INDEPENDENT-SET $\equiv_P$ VERTEX-COVER
  - VERTEX COVER $\equiv_P$ CLIQUE
- Reduction from special case to general case
  - VERTEX-COVER $\leq_P$ SET-COVER
- Reduction from general case to special case
  - SAT $\leq_P$ 3-SAT
- Reduction by encoding with gadgets
  - 3-SAT $\leq_P$ INDEPENDENT-SET

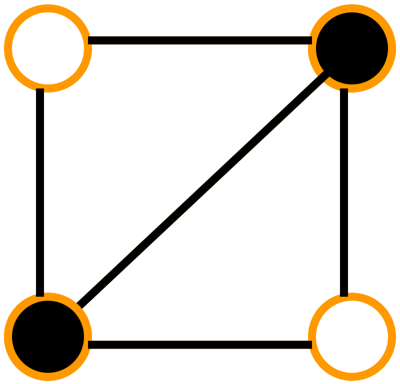
Vertex Cover

Vertex cover (VC)

- given a graph G=(V,E), find the *smallest* number of vertices that cover *each edge*
- Decision problem: is there a set of at most K vertices that cover *each edge*?



vertex cover of size 3



vertex cover of size 2

Vertex Cover Decision Problem

- VC(G,k): Given a graph G and an integer K, does G have a vertex cover of size at most K?

- **Theorem: VC is NP-complete.**
- Proof:
- 1) show VC is in NP:
  - Certificate: a subset V' of the vertices
  - Certifier: check in polynomial time O(n+m) if |V'| $\leq$ K and if every edge has at least one endpoint in V'.

How?

vertex cover in G of size k

$\Longleftrightarrow$

independent set in G'=G of size k'=|V| - k

Vertex Cover and Independent Set [KT 8.1]

- **Claim**. We show a set of vertices S is an independent set of G iff V − S is a vertex cover of G.
- $\Rightarrow$
  - Let S be any independent set.
  - Consider an arbitrary edge (u, v).
  - S independent $\Rightarrow$ u $\notin$ S or v $\notin$ S $\Rightarrow$ u $\in$ V − S or v $\in$ V − S.
  - Thus, V − S covers any edge (u, v) . V-S is a vertex cover.
- $\Leftarrow$
  - Let V − S be any vertex cover.
  - Consider any two nodes u $\in$ S and v $\in$ S.
  - Observe that (u, v) $\notin$ E since V − S is a vertex cover and would have needed to cover edge (u, v) by including one of its endpoints.
  - Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set. ▪

Claim. Solving IS(G, k) is equivalent to solving VC(G, n-k) and hence
VC $\leq_P$ IS and IS $\leq_P$ VC.

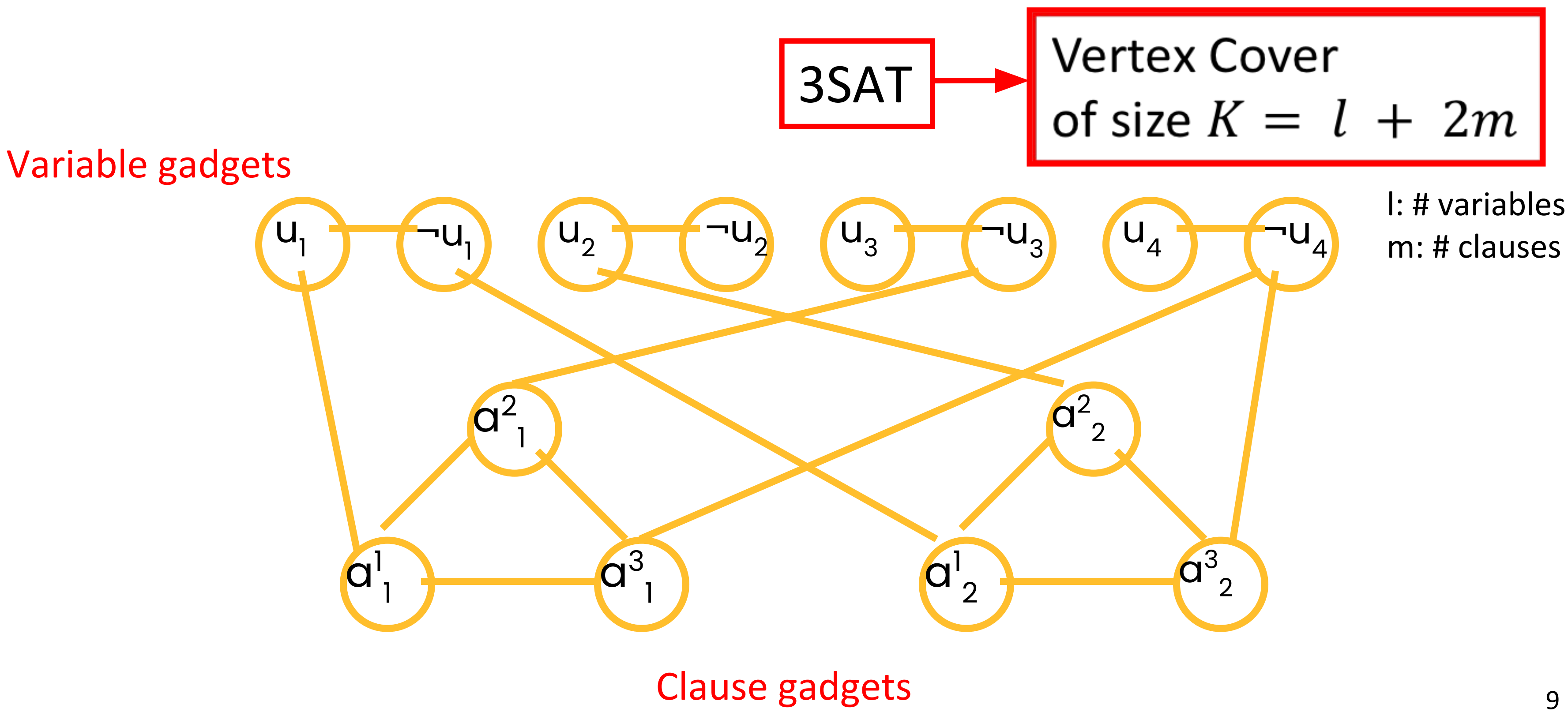
Vertex Cover is NP-complete

- Independent set: IS(G,k)
  - given a graph G=(V,E), find the largest independent set: a set of vertices in the graph with no edges between them.
  - Decision version: is there an independent set of at least k vertices?
- Vertex Cover: VC(G,k)
  - Given a graph G and an integer k, does G have a vertex cover of size at most K?

- VC is NP-complete because we showed:
  - VC is NP
  - IS is NP-complete and IS $\leq_P$ VC, hence VC is NP-complete
  - (Given IS(G,k), reduce it solving VC(G'=G, k'=|V|-k), correctness by proof on previous slide)

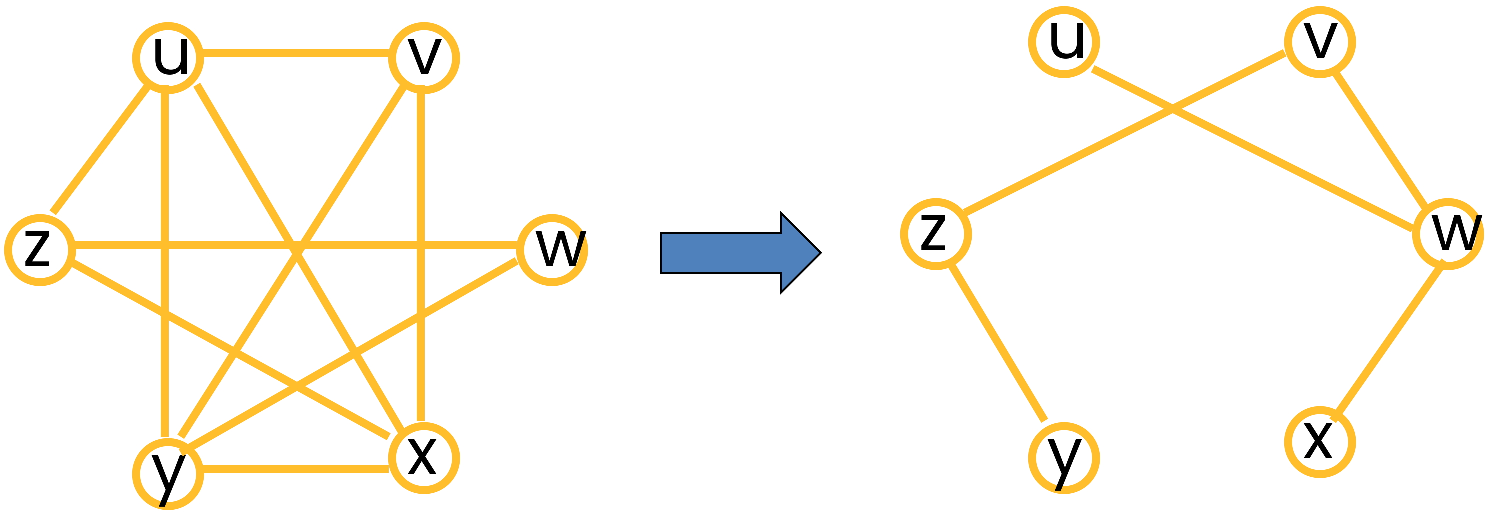
3SAT reduces to Vertex Cover

- Vertex cover (VC)
  - Given a graph G=(V,E) and an integer K, is there a set of at most K vertices that cover *each edge*?
    - By gadget (similar to 3SAT to IS)



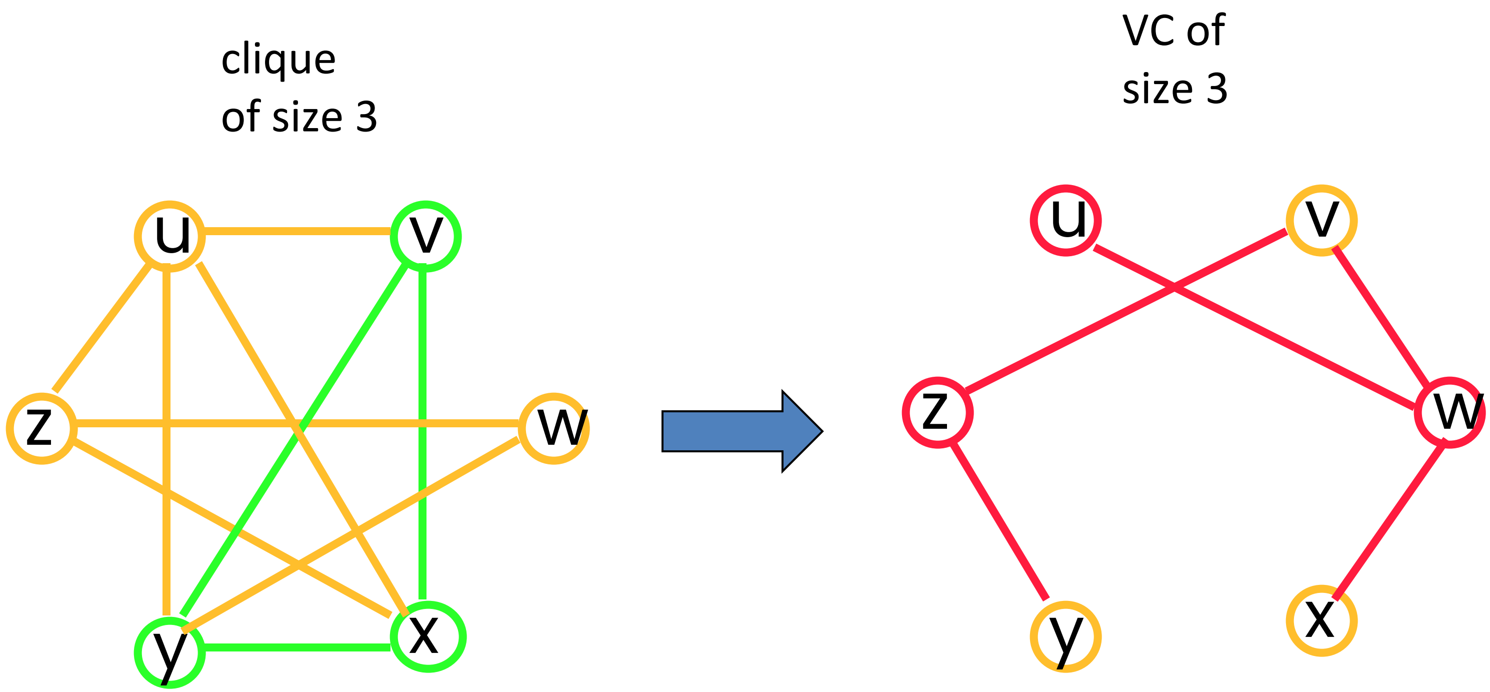
CLIQUE vs. VC (simple equivalence)

- CLIQUE (G, k): does G contain a completely connected subgraph of size at least K?
- The complement of graph G = (V, E) is the graph G_c = (V, E_c), where E_c consists of all the edges that are missing in G.
- CLIQUE(G, k) equivalent to VC(G_c, n-k)



CLIQUE vs. VC

- **Theorem**: V' is a clique of G if and only if V – V' is a vertex cover of G_c.



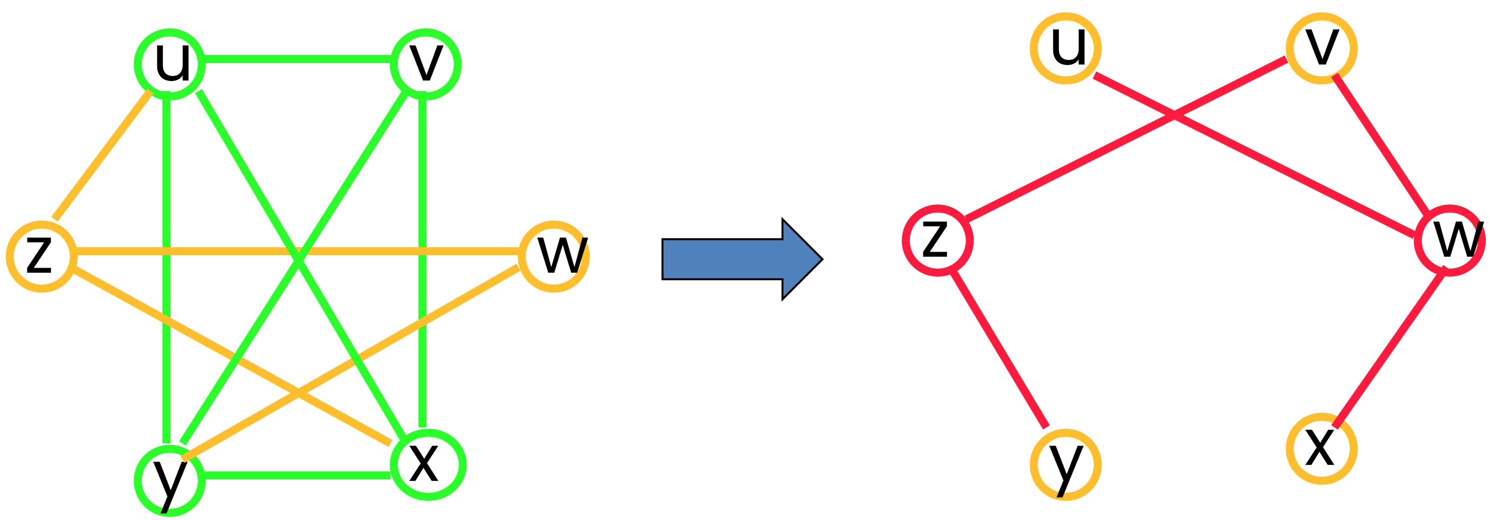
the vertices in V' would only "cover" missing edges and thus are not needed in G_c

CLIQUE vs. VC

- **Theorem**: V' is a clique of G if and only if V – V' is a vertex cover of G_c.

clique
of size 4

VC of
size 2



the vertices in V' would only "cover" missing edges and thus are not needed in G_c

CLIQUE vs. VC

Theorem: S is a clique of G if and only if V–S is a vertex cover of G_c.

(=>) G has a clique S. To show: V-S is a VC for G_c

Let's assume for the sake of contradiction V-S is not VC for G_c

Let e'=(u, v) be any edge in E_c that is not covered by V-S (such edge must exist if V-S is not VC)

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Let e'=(u, v) be any edge in E_c that is not covered by V-S (such edge must exist if V-S is not VC)

⇒ Both u not in V-S, and v not in V-S (by definition of not "covered" for edge (u, v))

⇒ u in S and v in S, but we also know that S is a clique in G

⇒ there must be an edge (u, v) in G, and hence e'=(u, v) is in E

If e' is in E, it cannot be in E_c, contradiction.

So V-S must be a VC for G_c

CLIQUE vs. VC

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CLIQUE vs. VC

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Let's assume for the sake of contradiction that S is not a clique in G (i.e., there must be at least one edge missing among S)

⇒ Exist 2 nodes u, v in S such that edge (u, v) is not in E

⇒ edge (u, v) must be in E_c (be definition of complement)

⇒ but neither u not v are in V-S, so edge (u, v) is not covered by V-S, contradiction with V-S being a VC for G_c

So S must be a clique in G. ▪

VC and CLIQUE

- Can use previous observation to show that
  - VC $\leq$p CLIQUE (given VC(G, k), solve CLIQUE(G'=G_c, k'=|V|-k)
- and also to show that
  - CLIQUE $\leq$p VC (given CLIQUE(G, k), solve VC'(G'=G_c, k'=|V|-k)
- How about IS $\leq$p CLIQUE and CLIQUE $\leq$p IS?
  - simple equivalence as well

Genres of NP-complete problems

- Six basic genres of NP-complete problems and paradigmatic examples.
  - Packing problems: SET-PACKING, INDEPENDENT SET.
  - Covering problems: SET-COVER, VERTEX-COVER.
  - Constraint satisfaction problems: SAT, 3-SAT.
  - Sequencing problems: HAMILTONIAN-CYCLE, TSP.
  - Partitioning problems: 3-COLOR, 3D-MATCHING.
  - Numerical problems: 2-PARTITION, SUBSET-SUM, KNAPSACK.

Subset Sum

SUBSET-SUM. Given natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Ex: { 1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344 }, W = 3754.

Yes. 1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754.

Claim. 3-SAT $\leq_P$ SUBSET-SUM.

Subset Sum

Construction. Given 3-SAT instance Φ with n variables and k clauses, form $2n + 2k$ decimal integers, each of $n+k$ digits, as illustrated below.

Claim. Φ is satisfiable iff there exists a subset that sums to W.

Pf. No carries possible.

$$C_1 = \bar{x} \vee y \vee z$$
$$C_2 = x \vee \bar{y} \vee z$$
$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

	x	y	z	C_1	C_2	C_3	
x	1	0	0	0	1	0	100,010
¬ x	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
¬ y	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
¬ z	0	0	1	0	0	1	1,001
2k dummies to get clause columns to sum to 4	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i has value $v_i > 0$ and weighs $w_i > 0$. ← we'll assume $w_i \leq W$
- Knapsack can carry weight up to W.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack is NP-Complete

(Decision) KNAPSACK: Given a finite set X, nonnegative weights w_i, nonnegative values v_i, a weight limit W, and a target value V, is there a subset S ⊆ X such that:

$$\sum_{i \in S} w_i \leq W$$
$$\sum_{i \in S} v_i \geq V$$

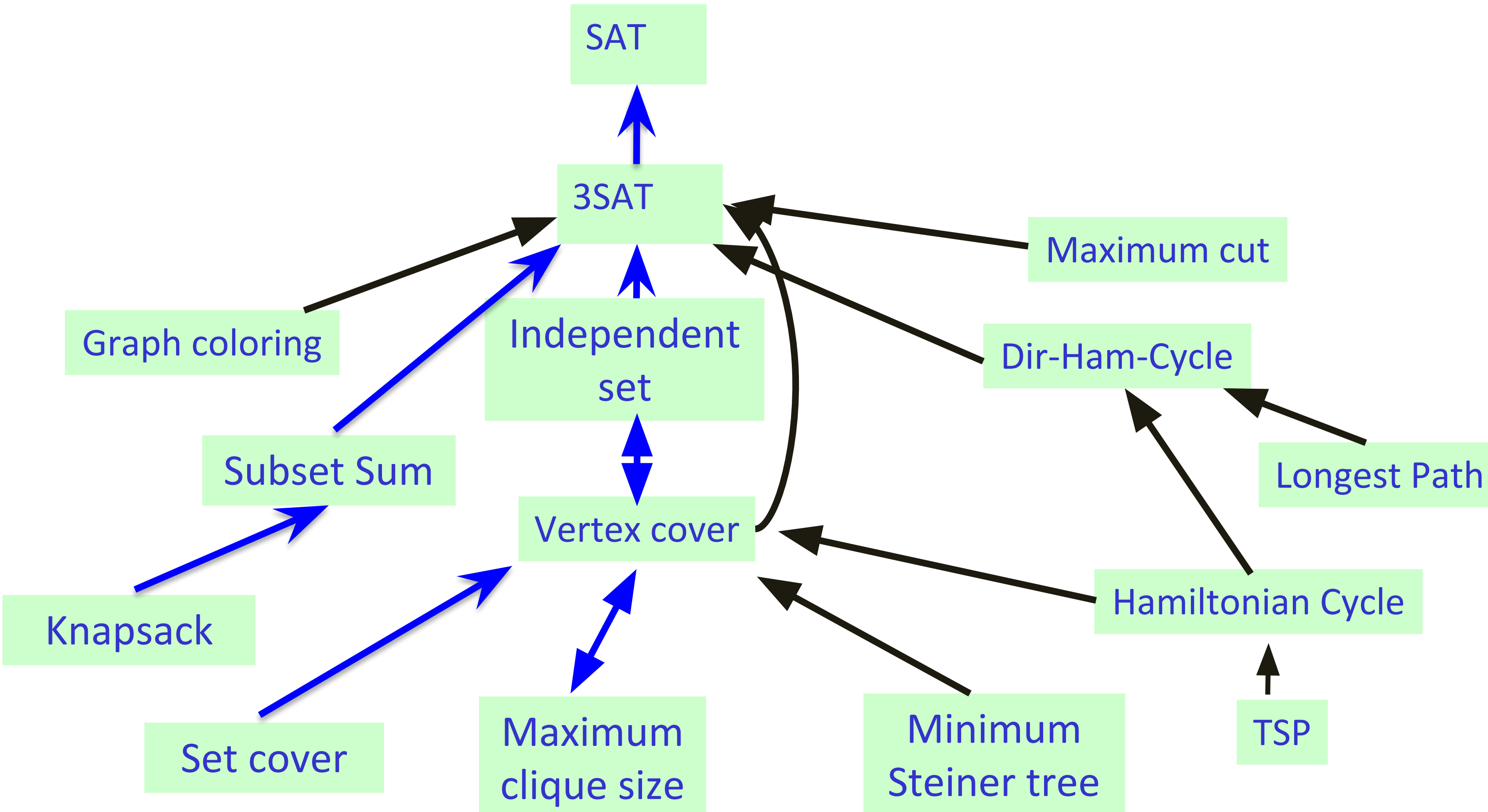
SUBSET-SUM: Given a finite set Y, nonnegative values u_i, and an integer U, is there a subset S' ⊆ Y whose elements sum to exactly U?

Claim. SUBSET-SUM $\leq_P$ KNAPSACK.

Reduction. Given instance (u_1, …, u_n, U) of SUBSET-SUM, create KNAPSACK instance:

$$v_i = w_i = u_i \qquad \sum_{i \in S} u_i \leq U$$
$$V = W = U \qquad \sum_{i \in S} u_i \geq U$$

Summary of some NPc problems



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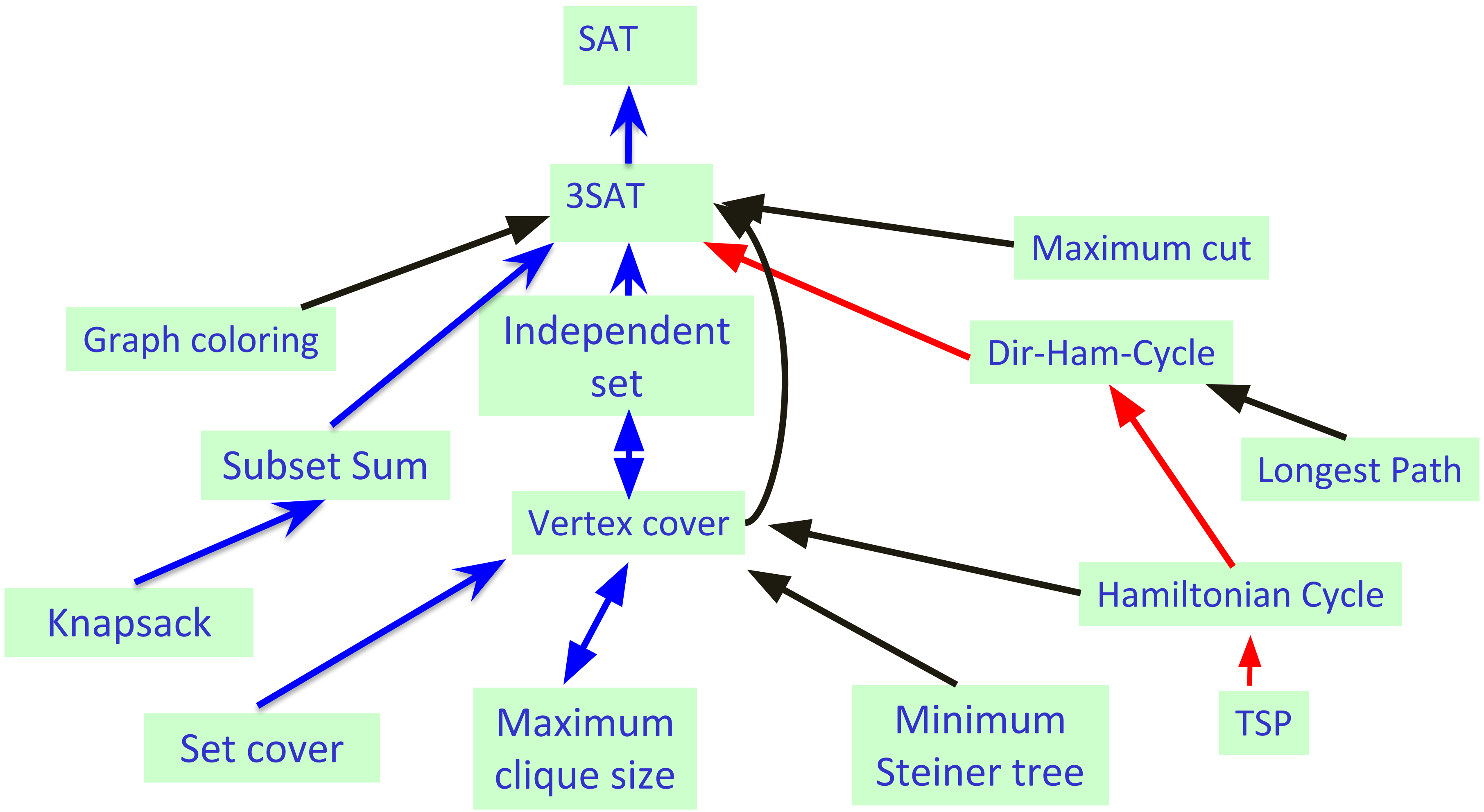

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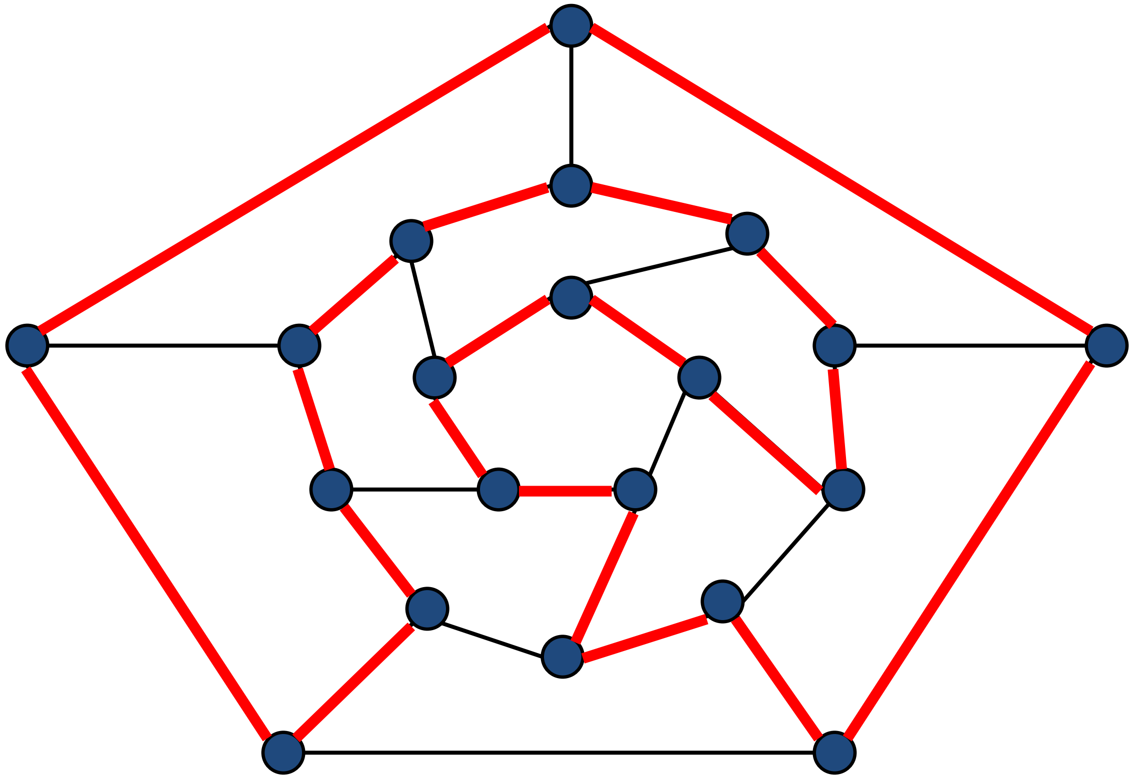
Hamiltonian Cycle

- HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle Γ that contains every node in V.



YES: vertices and faces of a dodecahedron.

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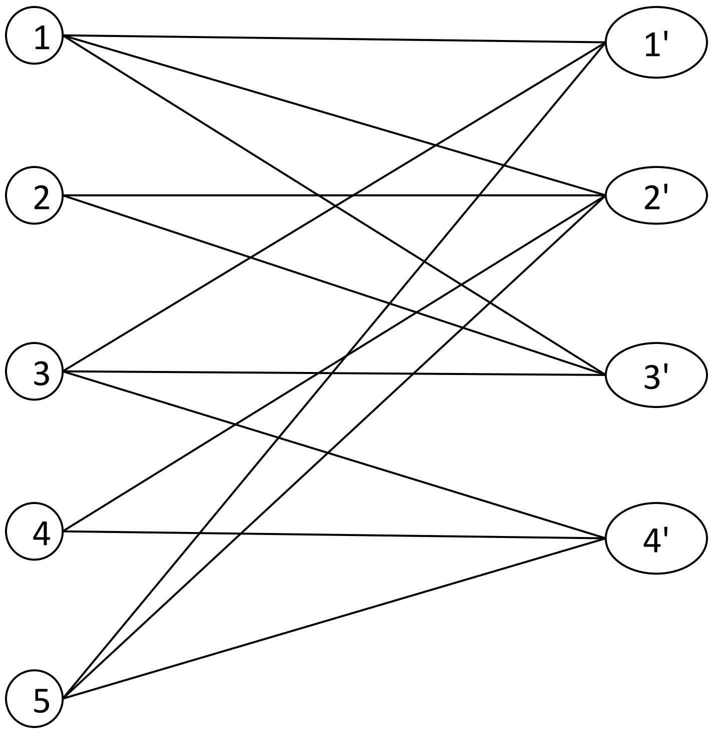


NO: bipartite graph with odd number of nodes.

Directed Hamiltonian Cycle

- DIR-HAM-CYCLE: given a digraph G = (V, E), does there exists a simple directed cycle Γ that contains every node in V?

- HAM-CYCLE: given an undirected graph G = (V, E), does there exists a simple directed cycle Γ that contains every node in V?

- DIR-HAM-CYCLE (HAM-CYCLE) is in NP
  - Certificate: Sequence of vertices
  - Certifier: Check if every vertex (except the first) appears exactly once, and that consecutive vertices are connected by a directed (undirected) edge

3-SAT Reduces to Dir. Hamiltonian Cycle

- Claim. 3-SAT $\leq_P$ DIR-HAM-CYCLE.

- Pf. Given an instance Φ of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff Φ is satisfiable.

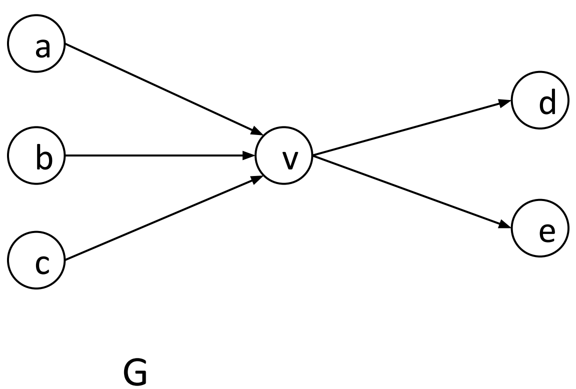
- See [KT 8.5].

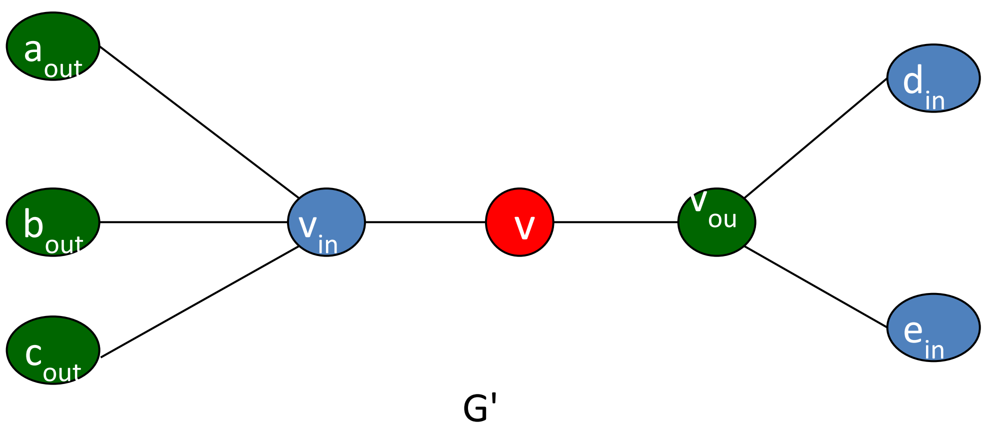
Directed Hamiltonian Cycle

- DIR-HAM-CYCLE: given a digraph G = (V, E), does there exists a simple directed cycle Γ that contains every node in V?

- Claim. DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE.

- Pf. Given a directed graph G = (V, E), construct an undirected graph G' with 3n nodes.



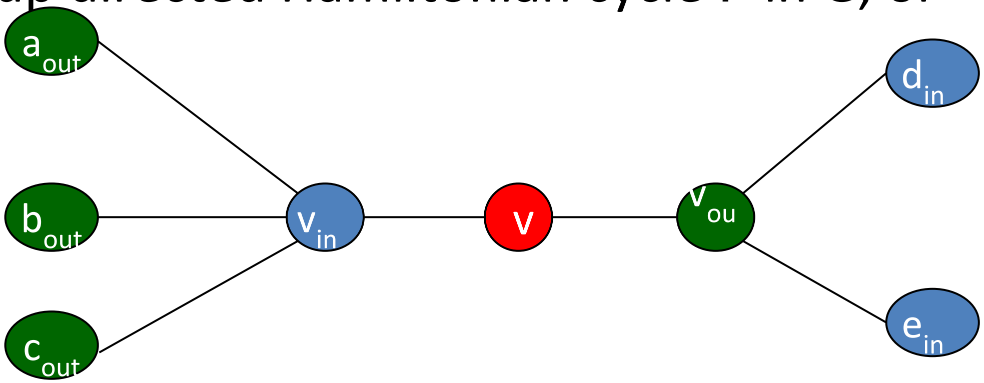


Directed Hamiltonian Cycle

- Claim. G has a Hamiltonian cycle iff G' does.
- Pf. ⇒
  - Suppose G has a directed Hamiltonian cycle Γ.
  - Then G' has an undirected Hamiltonian cycle (same order).
- Pf. ⇐
  - Suppose G' has an undirected Hamiltonian cycle Γ'.
  - Γ' must visit nodes in G' using one of following two orders:

    …, B, G, R, B, G, R, B, G, R, B, …

    …, B, R, G, B, R, G, B, R, G, B, …
  - Blue nodes in Γ' make up directed Hamiltonian cycle Γ in G, or reverse of one. ▪



G'

Traveling Salesperson Problem

- TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?

All 13,509 cities in US with a population of at least 500
Reference: http://www.math.uwaterloo.ca/tsp/

Traveling Salesperson Problem

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Optimal TSP tour
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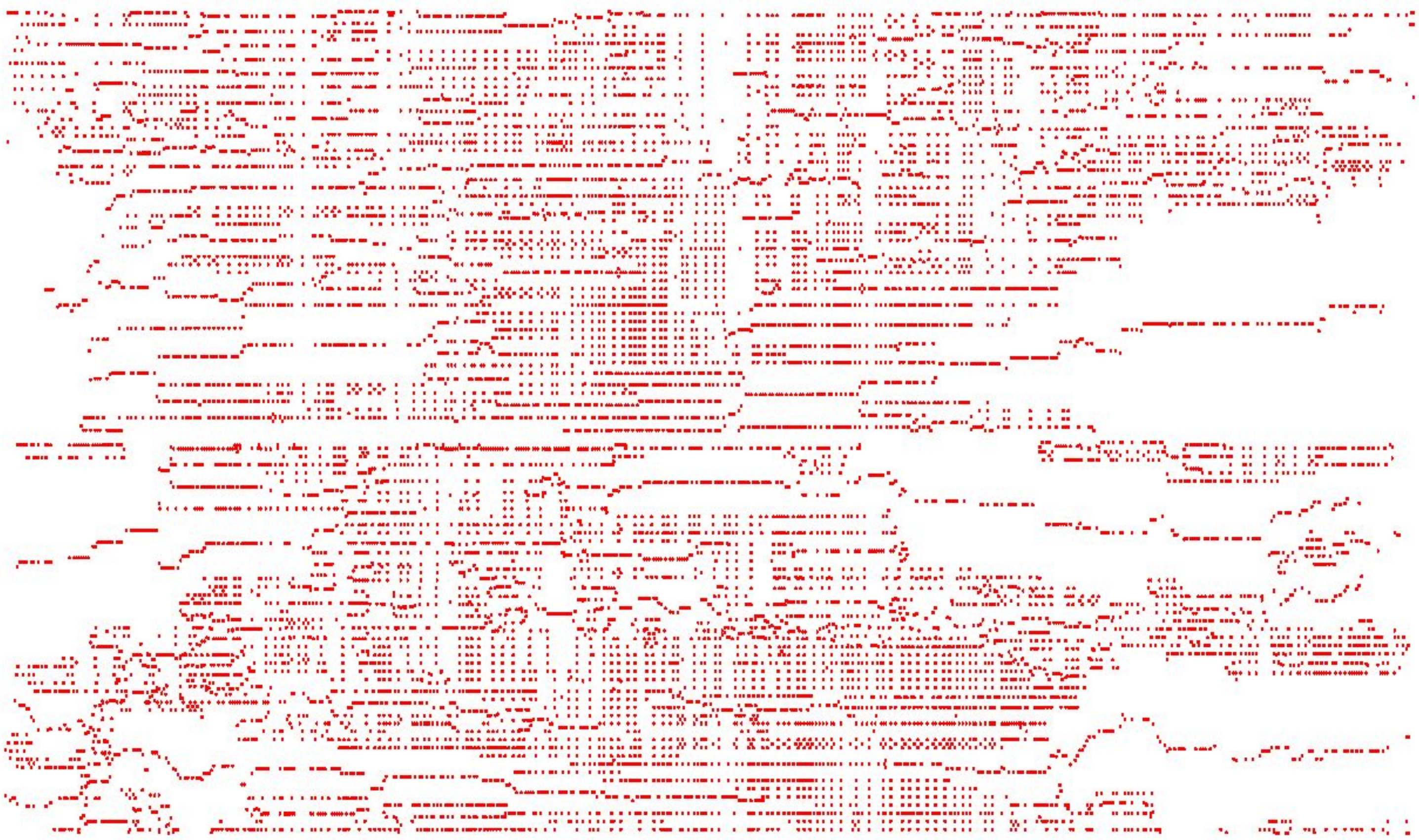

11,849 holes to drill in a programmed logic array
Reference: http://www.math.uwaterloo.ca/tsp/

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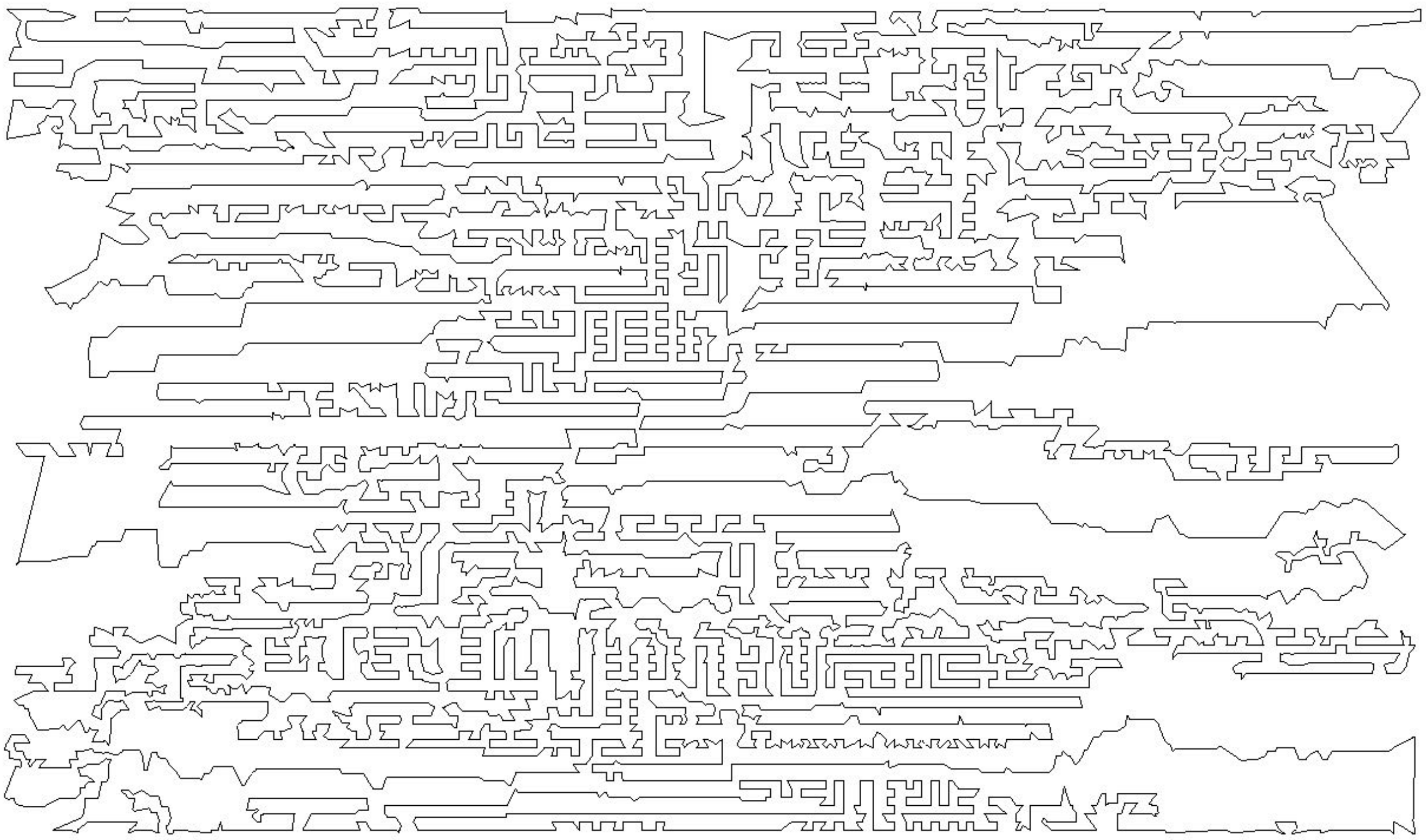

Optimal TSP tour
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Traveling Salesman Problem

- TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?
- HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V.

- Claim. HAM-CYCLE $\leq_P$ TSP.
- Pf.
  - Given instance G = (V, E) of HAM-CYCLE, create n cities with distance function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

  - G is Hamiltonian iff TSP instance has tour of length ≤ D=n. (proof is omitted)▪
- Remark. TSP instance in reduction satisfies Δ-inequality.