CSE 6140/ CX 4140

Computational Science and Engineering ALGORITHMS

NP Completeness 3

Instructor: Xiuwei Zhang
Assistant Professor
School of Computational Science and Engineering

Based on slides by Prof. Ümit V. Çatalyürek

Reductions

- Reduction from A to B is showing that we can solve A using the algorithm that solves B
- We say that <u>problem A is easier than problem B</u>, (i.e., we write **“A ≤ B”**)

Does it mean that the running time of A is less than B?

Not necessarily.

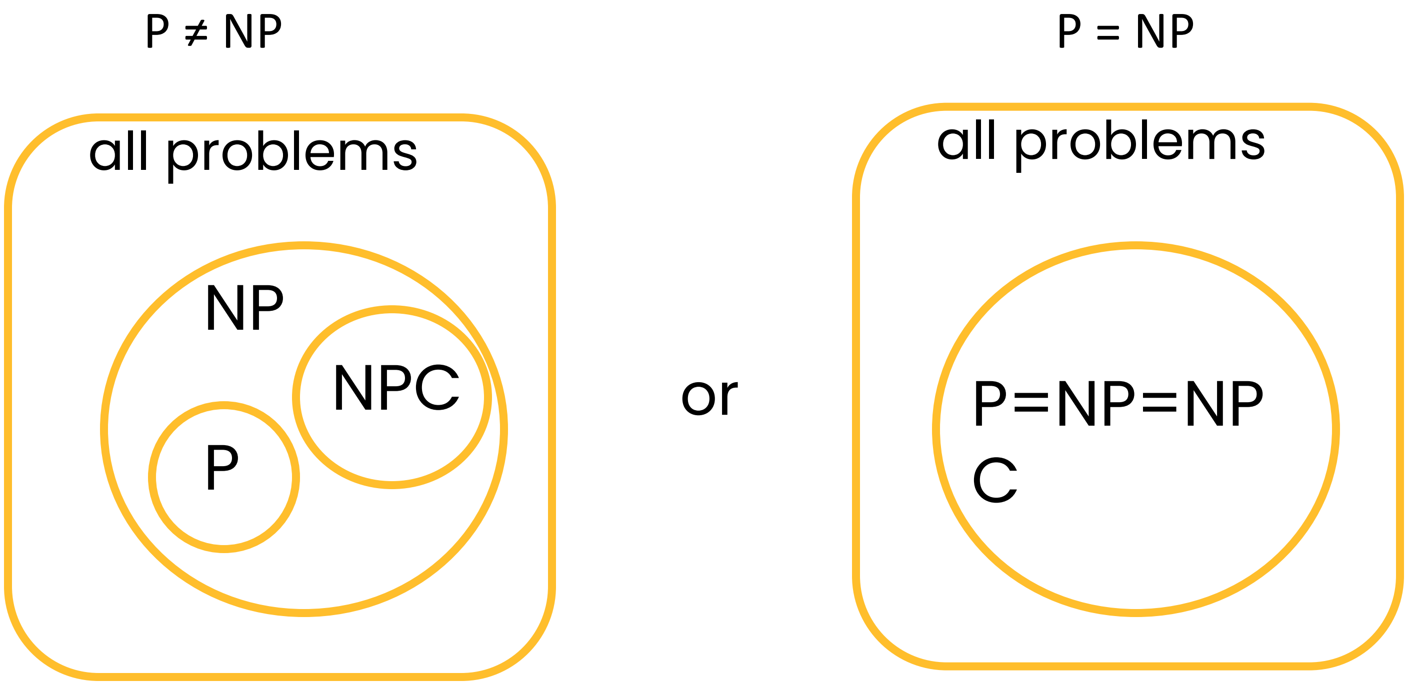
Polynomial Reductions

- Given two problems A, B, we say that A is polynomially **reducible** to B ($A \leq_p B$) if:
  1. There exists a function f that converts the input of A **to** inputs of B in polynomial time
  2. $A(i) = YES \Leftrightarrow B(f(i)) = YES$

Summary

- P
  - Decision problems that can be solved in polynomial time
  - Can be solved "efficiently"
- NP
  - Decision problems whose "YES" answer can be verified in polynomial time, if we already have the candidate solution
- NP-complete
  - The "hardest" problems in NP: a problem Y in NP with the property that for every problem X in NP, $X \leq_p Y$.

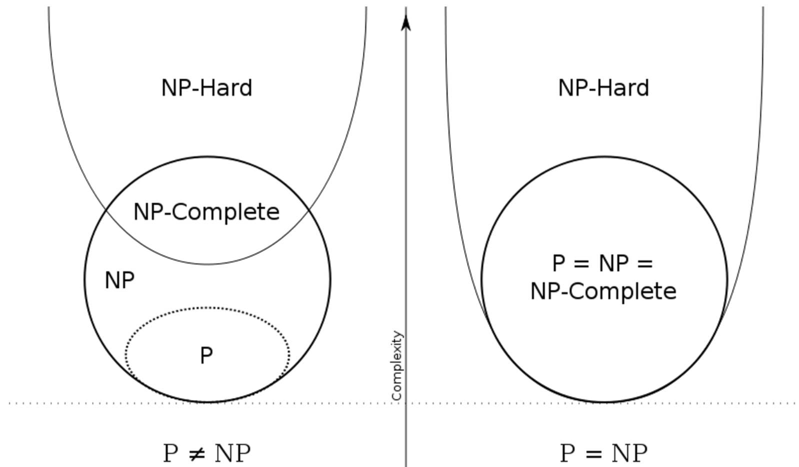
Possible Worlds



NPC: NP-complete

Revisit “Is P = NP?”

Georgia Tech



- **Theorem.** Suppose Y is an NP-complete problem. Y is solvable in poly-time if and only if P = NP.
- Pf. $\Leftarrow$ If P = NP then Y is in P. Hence Y can be solved in poly-time.
- Pf. $\Rightarrow$ Suppose Y can be solved in poly-time.
  - Let X be any problem in NP. Then, we know that $X \leq_p Y$ by definition of NP-complete and Y being NP-complete problem. Then we can solve X in poly-time by solving Y in poly-time. This implies any problem X in NP is also in P, i.e. NP $\subseteq$ P.
  - We already know P $\subseteq$ NP. Thus P = NP. ▪

Implications of Polynomial-Time Reductions

- Purpose. Classify problems according to relative difficulty.
- Design algorithms. If $X \leq_P Y$ and Y can be solved in polynomial-time, then X can also be solved in polynomial time.
- Establish intractability. If $X \leq_P Y$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial time.
- Establish equivalence. If $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$.

  up to cost of reduction
- Transitivity: if $X \leq_p Y$ and $Y \leq_p Z$, then $X \leq_p Z$

Reduction By Simple Equivalence

- Basic reduction strategies.
  - Reduction by simple equivalence.
  - Reduction from special case to general case.
  - Reduction by encoding with gadgets.

Establishing NP-Completeness

- Recipe to establish NP-completeness of problem Y.
  - Step 1. Show that Y is in NP.
    - Describe how a potential **solution**/witness will be represented
    - Describe a **procedure to check** whether the potential witness is a correct solution to the problem instance, and argue that this procedure takes **polynomial time**
  - Step 2. Choose an NP-complete problem X.
  - Step 3. Prove that $X \leq_P Y$ (X is **poly-time reducible** to Y).
    - Describe a **procedure f that converts** the inputs i of X to inputs of Y in **polynomial time**
    - Show that the reduction is correct by showing that $X(i) = \text{YES} \Leftrightarrow Y(f(i)) = \text{YES}$ (**if and only if**, proof in both directions)

P & NP-Complete Problems

- **Shortest simple path**
  - Given a graph G = (V, E) find a **shortest** path from a source to all other vertices
  - <u>Polynomial solution:</u> O(VE)
- **Longest simple path**
  - Given a graph G = (V, E) find a **longest** path from a source to all other vertices
  - <u>NP-complete</u>

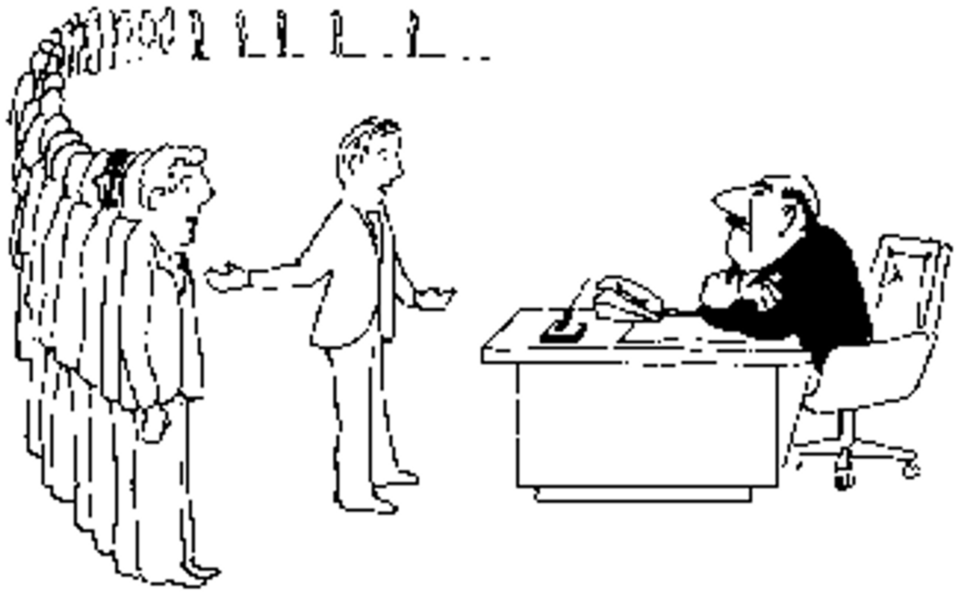
P & NP-Complete Problems

- **Euler tour**
  - G = (V, E) a connected, directed graph find a cycle that traverses **each edge** of G exactly once (may visit a vertex multiple times)
  - <u>Polynomial solution O(E)</u>
- **Hamiltonian cycle**
  - G = (V, E) a connected, directed graph find a cycle that visits **each vertex** of G exactly once
  - <u>NP-complete</u>

More Hard Computational Problems

- Aerospace engineering: optimal mesh partitioning for finite elements.
- Biology: protein folding.
- Chemical engineering: heat exchanger network synthesis.
- Civil engineering: equilibrium of urban traffic flow.
- Economics: computation of arbitrage in financial markets with friction.
- Electrical engineering: VLSI layout.
- Environmental engineering: optimal placement of contaminant sensors.
- Financial engineering: find minimum risk portfolio of given return.
- Game theory: find Nash equilibrium that maximizes social welfare.
- Genomics: phylogeny reconstruction.
- Mechanical engineering: structure of turbulence in sheared flows.
- Medicine: reconstructing 3-D shape from biplane angiocardiogram.
- Operations research: optimal resource allocation.
- Physics: partition function of 3-D Ising model in statistical mechanics.
- Politics: Shapley-Shubik voting power.
- Statistics: optimal experimental design.

Practical applications of NP-completeness



"I can't find an efficient algorithm, but neither can all these famous people."

[Garey & Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, 1979.]

Satisfiability Problem (SAT)

Satisfiability problem: given a logical expression Φ, find an assignment of values (F, T) to variables x_i that causes Φ to evaluate to T

$$\Phi = x_1 \vee \neg x_2 \wedge x_3 \vee \neg x_4$$

- boolean variables: take on values T or F
  - Ex: x, y
- literal: variable or negation of a variable
  - Ex: x, $\neg$ x (also denoted by $\bar{x}$)

Logical Operands

- $\mathrm{x} = \{0,1\}$ or $\{F, T\}$

Not

x	¬ x (negation)
0	1
1	0

And

x_1	x_2	$x_1 \wedge x_2$ (AND)
1	1	1
1	0	0
0	1	0
0	0	0

Or

x_1	x_2	$x_1 \vee x_2$ (OR)
1	1	1
1	0	1
0	1	1
0	0	0

Satisfiability Problem (SAT)

- **Satisfiability problem**: given a logical expression Φ, find an assignment of values (F, T) to variables x_i that causes Φ to evaluate to T

$$\Phi = x_1 \vee \neg x_2 \wedge x_3 \vee \neg x_4$$

- SAT is in NP: given a value assignment, check the Boolean logic of Φ evaluates to True (linear time)

- SAT was the first problem shown to be NP-complete! **(Cook–Levin theorem)**

CNF Satisfiability

- CNF is a special case of SAT
- Φ is in "Conjunctive Normal Form" (CNF)
  - "AND" of expressions (i.e., clauses)
  - Each clause contains only "OR"s of the variables and their negations

E.g.: $\Phi = (x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_2)$

clauses

SAT-CNF is NP-Complete

In the following, SAT means SAT-CNF

Definition of 3SAT / 3CNF

- A subcase of CNF problem:
  - Contains three literals per clause
- E.g.:
  - $\Phi = (x1 \vee \neg x1 \vee \neg x2) \wedge (x3 \vee x2 \vee x4) \wedge (\neg x1 \vee \neg x3 \vee \neg x4)$
- Is 3SAT in NP?
  - Yes, because SAT is in NP. Also easy to prove it directly.
- Is 3SAT NP-complete?
  - Not obvious. It has a more regular structure, which can perhaps be exploited to get an efficient algorithm
  - In fact, 2SAT does have a polynomial time algorithm

Showing 3SAT is NP-Complete by Reduction

- (1) To show 3SAT is in NP:
  - A certificate is a truth (0/1) assignment to the variables
  - Certifier: check that each clause has at least one literal set to true according to the certificate
- (2) Choose SAT as known NP-complete problem
- (3) Describe a reduction from SAT inputs to 3SAT inputs
  - Computable in poly time
  - SAT input is satisfiable iff constructed 3SAT input is satisfiable

  - (3a) Transform I_1 (instance of SAT) into I_2 (instance of 3SAT) in polynomial time
  - (3b,3c) Prove that I_1 has a solution $\Leftrightarrow$ I_2 has a solution

(3a) Reduction from SAT to 3SAT: I_1 -> I_2

- We are given an arbitrary CNF formula $C = c_1 \wedge c_2 \wedge \ldots \wedge c_m$ over set of variables U, this is instance I_1
  - each c_i is a clause (disjunction of literals)

- We will replace each clause c_i with a set of clauses C_i', and may use some extra variables U_i' just for this clause
- Each clause in C_i' will have exactly 3 literals
- Transformed input will be conjunction of all the clauses in all the C_i', this is an instance I_2 of 3SAT
- New clauses are carefully chosen…

(3a) Reduction from SAT to 3SAT: I_1 -> I_2

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ (the z's are literals)

- Case 1: $k = 1$.
  - E.g. $c_i = z_1$
  - Use extra variables y_i^1 and y_i^2.
  - Replace clause c_i with 4 clauses:

$$(z_1 \vee \overline{y_i^1} \vee y_i^2)$$
$$(z_1 \vee y_i^1 \vee \overline{y_i^2})$$
$$(z_1 \vee \overline{y_i^1} \vee \overline{y_i^2})$$
$$(z_1 \vee y_i^1 \vee y_i^2)$$

z_1 can be replaced by the intersection of these 4 clauses

  - Note that no matter what values we give the y variables, in one of the 4 clauses we will be forced to use z_1 to satisfy it

(3a) Reduction from SAT to 3SAT: I_1 -> I_2

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 2: $k = 2$.
  - E.g. $c_i = z_1 \vee z_2$
  - Use extra variable y_i^1.
  - Replace c_i with 2 clauses:

  $(z_1 \vee z_2 \vee \overline{y_i^1})$

  $(z_1 \vee z_2 \vee y_i^1)$

(3a) Reduction from SAT to 3SAT: I_1 -> I_2

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 3: $k = 3$.
  - No extra variables are needed.
  - Keep c_i:

    $(z_1 \vee z_2 \vee z_3)$

(3a) Reduction from SAT to 3SAT: I_1 -> I_2

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 4: k > 3.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

$$(z_1 \vee z_2 \vee y_i^1)$$

$$(\overline{y_i^1} \vee z_3 \vee y_i^2)$$

$$(\overline{y_i^2} \vee z_4 \vee y_i^3)$$

$$\ldots$$

$$(\overline{y_i^{k-5}} \vee z_{k-3} \vee y_i^{k-4})$$

$$(\overline{y_i^{k-4}} \vee z_{k-2} \vee y_i^{k-3})$$

$$(\overline{y_i^{k-3}} \vee z_{k-1} \vee z_k)$$

(3a) Why is Reduction Poly Time?

- The running time of the reduction (the algorithm to compute the 3SAT formula C', given the SAT formula C) is proportional to the size of C'
- Rules for constructing C' are simple to calculate

(3a) Size of New Formula

- **Original clause with 1 literal** becomes 4 clauses with 3 literals each (1 to 12 literals conversion)
- **Original clause with 2 literals** becomes 2 clauses with 3 literals each (2 to 6 literals conversion)
- **Original clause with 3 literals** becomes 1 clause with 3 literals
- **Original clause with k > 3 literals** becomes k-2 clauses with 3 literals each (k to 3(k-2) literals conversion)

- So new formula C' is only a constant factor larger than the original formula
  - total *L* literals in formula C to *cL* literals in C', where c is a constant

(3bc) Correctness of Reduction

- Show that CNF formula C is satisfiable iff the 3-CNF formula C' constructed is satisfiable, i.e., $sol(I_1) \Leftrightarrow sol(I_2)$
- Step 3b (=>) Suppose original CNF formula C is satisfiable, i.e., I_1 has a solution. That means it has a truth assignment A to the variables that make the formula C evaluate to true.
- Come up with a satisfying truth assignment for the reduced 3SAT formula C', i.e., a solution to instance I_2.
  - For variables in U, use same truth assignments as for C.
  - How to assign T/F to the new variables in C'?

(3b) Truth Assignment for New Var.: sol(I_1) => sol(I_2)

Let $c_i = z_1$

- Case 1: $k = 1$.
- Use extra variables y_i^1 and y_i^2.
  - Replace c_i with 4 clauses:

$(z_1 \vee \overline{y_i^1} \vee y_i^2)$

$(z_1 \vee y_i^1 \vee \overline{y_i^2})$

$(z_1 \vee \overline{y_i^1} \vee \overline{y_i^2})$

$(z_1 \vee y_i^1 \vee y_i^2)$

Since z_1 is true, it does not matter how we assign y_i^1 and y_i^2

(3b) Truth Assignment for New Var.: $sol(I_1) \Rightarrow sol(I_2)$

Let $c_i = (z_1 \vee z_2)$

- Case 2: $k = 2$.
  - Use extra variable y_i^1.
  - Replace c_i with 2 clauses:

    $(z_1 \vee z_2 \vee \overline{y_i^1})$

    $(z_1 \vee z_2 \vee y_i^1)$

Since either z_1 or z_2 is true, it does not matter how we assign y_i^1

(3b) Truth Assignment for New Var.: sol(I_1) => sol(I_2)

Let $c_i = z_1 \vee z_2 \vee z_3$

- Case 3: $k = 3$.
  - No extra variables are needed.
  - Keep c_i:

    $(z_1 \vee z_2 \vee z_3)$

No new variables.

(3b) Truth Assignment for New Var.: sol(I_1) => sol(I_2)

Georgia Tech

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 4: $k > 3$.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

$$(z_1 \vee z_2 \vee y_i^1)$$
$$(\overline{y_i^1} \vee z_3 \vee y_i^2)$$
$$(\overline{y_i^2} \vee z_4 \vee y_i^3)$$
$$\ldots$$
$$(\overline{y_i^{k-5}} \vee z_{k-3} \vee y_i^{k-4})$$
$$(\overline{y_i^{k-4}} \vee z_{k-2} \vee y_i^{k-3})$$
$$(\overline{y_i^{k-3}} \vee z_{k-1} \vee z_k)$$

If first true literal is z_1 or z_2, set all y_i's to false: then all later clauses have a true literal

(3b) Truth Assignment for New Var.: $sol(I_1) \Rightarrow sol(I_2)$

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 4: $k > 3$.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

$$(z_1 \vee z_2 \vee y_i^1)$$
$$(\overline{y_i^1} \vee z_3 \vee y_i^2)$$
$$(\overline{y_i^2} \vee z_4 \vee y_i^3)$$
$$\ldots$$
$$\ldots$$
$$(\overline{y_i^{k-5}} \vee z_{k-3} \vee y_i^{k-4})$$
$$(\overline{y_i^{k-4}} \vee z_{k-2} \vee y_i^{k-3})$$
$$(\overline{y_i^{k-3}} \vee z_{k-1} \vee z_k)$$

If first true literal is z_{k-1} or z_k, set all y_i's to true: then all earlier clauses have a true literal

(3b) Truth Assignment for New Var.: $sol(I_1) \Rightarrow sol(I_2)$

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 4: $k > 3$.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

$(z_1 \vee z_2 \vee y_i^1)$

$(\overline{y_i^1} \vee z_3 \vee y_i^2)$

$(\overline{y_i^2} \vee z_4 \vee y_i^3)$

$\ldots$

$\ldots$

$(\overline{y_i^{k-5}} \vee z_{k-3} \vee y_i^{k-4})$

$(\overline{y_i^{k-4}} \vee z_{k-2} \vee y_i^{k-3})$

$(\overline{y_i^{k-3}} \vee z_{k-1} \vee z_k)$

If first true literal is in between, set all earlier y_i's to true and set all later y_i's to false

(3c) Correctness of Reduction: sol(I_2) => sol(I_1)

- (<=) Suppose the newly constructed 3SAT formula C' is satisfiable, i.e., I_2 has a solution. We must show that the original SAT formula C is also satisfiable, i.e., I_1 has a solution.
- Use the same satisfying truth assignment for C as for C' (ignoring new variables).
- Show each original clause has at least one true literal in it.

(3c) $sol(I_2) \Rightarrow sol(I_1)$

Let $c_i = z_1$

- Case 1: $k = 1$.
- Use extra variables y_i^1 and y_i^2.
  - Replace c_i with 4 clauses:

$$(z_1 \lor \overline{y_i^1} \lor y_i^2)$$
$$(z_1 \lor y_i^1 \lor \overline{y_i^2})$$
$$(z_1 \lor \overline{y_i^1} \lor \overline{y_i^2})$$
$$(z_1 \lor y_i^1 \lor y_i^2)$$

For every assignment of y_i^1 and y_i^2 , in order for all 4 clauses to have a true literal, z_1 must be true.

(3c) $sol(I_2) \Rightarrow sol(I_1)$

Let $c_i = (z_1 \vee z_2)$

- Case 2: $k = 2$.
  - Use extra variable y_i^1.
  - Replace c_i with 2 clauses:

    $(z_1 \vee z_2 \vee \overline{y_i^1})$

    $(z_1 \vee z_2 \vee y_i^1)$

For either assignment of y_i^1, in order for both clauses to have a true literal, z_1 or z_2 must be true.

(3c) $sol(I_2) \Rightarrow sol(I_1)$

Let $c_i = z_1 \vee z_2 \vee z_3$

- Case 3: $k = 3$.
  - No extra variables are needed.
  - Keep c_i:

    $(z_1 \vee z_2 \vee z_3)$

No new variables.

(3c) $sol(I_2) \Rightarrow sol(I_1)$

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

- Case 4: $k > 3$.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

Suppose in contradiction all z_i's are false.

$$(z_1 \lor z_2 \lor y_i^1)$$
$$(\overline{y_i^1} \lor z_3 \lor y_i^2)$$
$$(\overline{y_i^2} \lor z_4 \lor y_i^3)$$
$$\ldots$$
$$(\overline{y_i^{k-5}} \lor z_{k-3} \lor y_i^{k-4})$$
$$(\overline{y_i^{k-4}} \lor z_{k-2} \lor y_i^{k-3})$$
$$(\overline{y_i^{k-3}} \lor z_{k-1} \lor z_k)$$

(3c) $sol(I_2) \Rightarrow sol(I_1)$

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 4: $k > 3$.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

Suppose in contradiction all z_i's are false.
Then y_i^1 must be true.

$(z_1 \vee z_2 \vee y_i^1)$

$(\overline{y_i^1} \vee z_3 \vee y_i^2)$

$(\overline{y_i^2} \vee z_4 \vee y_i^3)$

$\ldots$

$\ldots$

$(\overline{y_i^{k-5}} \vee z_{k-3} \vee y_i^{k-4})$

$(\overline{y_i^{k-4}} \vee z_{k-2} \vee y_i^{k-3})$

$(\overline{y_i^{k-3}} \vee z_{k-1} \vee z_k)$

(3c) $sol(I_2) \Rightarrow sol(I_1)$

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 4: $k > 3$.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

Suppose in contradiction all z_i's are false.
Then y_i^1 must be true.
Then y_i^2 must be true...

$(z_1 \vee z_2 \vee y_i^1)$

$(\overline{y_i^1} \vee z_3 \vee y_i^2)$

$(\overline{y_i^2} \vee z_4 \vee y_i^3)$

$\ldots$

$\ldots$

$(\overline{y_i^{k-5}} \vee z_{k-3} \vee y_i^{k-4})$

$(\overline{y_i^{k-4}} \vee z_{k-2} \vee y_i^{k-3})$

$(\overline{y_i^{k-3}} \vee z_{k-1} \vee z_k)$

(3c) $sol(I_2) \Rightarrow sol(I_1)$

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 4: $k > 3$.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

Suppose in contradiction all z_i's are false.
Then y_i^1 must be true.
Then y_i^2 must be true...

$(z_1 \vee z_2 \vee y_i^1)$

$(\overline{y_i^1} \vee z_3 \vee y_i^2)$

$(\overline{y_i^2} \vee z_4 \vee y_i^3)$

$\ldots$

$\ldots$

$(\overline{y_i^{k-5}} \vee z_{k-3} \vee y_i^{k-4})$

$(\overline{y_i^{k-4}} \vee z_{k-2} \vee y_i^{k-3})$

$(\overline{y_i^{k-3}} \vee z_{k-1} \vee z_k)$

(3c) sol(I_2) => sol(I_1)

Let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$

- Case 4: k > 3.
  - Use extra variables $y_i^1, \ldots, y_i^{k-3}$.
  - Replace c_i with k-2 clauses:

$(z_1 \vee z_2 \vee y_i^1)$

$(\overline{y_i^1} \vee z_3 \vee y_i^2)$

$(\overline{y_i^2} \vee z_4 \vee y_i^3)$

$\ldots$

$\ldots$

$(\overline{y_i^{k-5}} \vee z_{k-3} \vee y_i^{k-4})$

$(\overline{y_i^{k-4}} \vee z_{k-2} \vee y_i^{k-3})$

$(\overline{y_i^{k-3}} \vee z_{k-1} \vee z_k)$

Suppose in contradiction all z_i's are false.
Then y_i^1 must be true.
Then y_i^2 must be true…
…
So the last clause is False

Conclusion

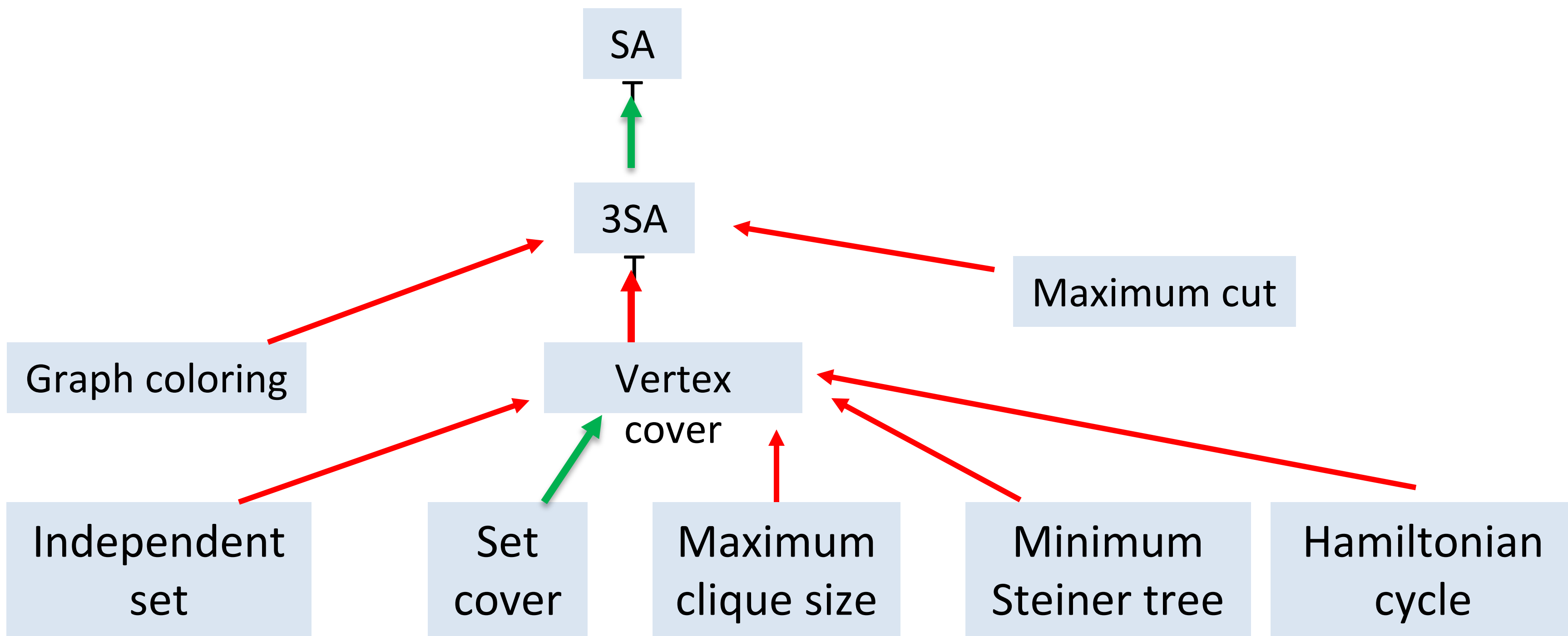
- (1) 3SAT is in NP
- (2) We know that SAT is NPC, we want to prove that 3SAT is more difficult than SAT, hence SAT $\leq_P$ 3SAT
- (3a) Take an instance I_1 of SAT, transform it in polynomial time into an instance I_2 of 3SAT
- (3b) Show that if I_1 has a solution, then I_2 has a solution
- (3c) Show that if I_2 has a solution, then I_1 has a solution

- 3SAT is NP-complete! This is your very first NP-completeness proof. Now you can do reductions from 3SAT.

- (All pbs in NP) $\leq_P$ SAT $\leq_P$ 3SAT

Examples of NP-complete problems

Summary of some NPC problems



find more NP-complete problems in

- http://en.wikipedia.org/wiki/List_of_NP-complete_problems
- **Garey-Johnson: Computers and Intractability: A Guide to the Theory of NP-Completeness**

Genres of NP-complete problems

- Six basic genres of NP-complete problems and paradigmatic examples.
  - Packing problems: SET-PACKING, INDEPENDENT SET.
  - Covering problems: SET-COVER, VERTEX-COVER.
  - Constraint satisfaction problems: SAT, 3-SAT.
  - Sequencing problems: HAMILTONIAN-CYCLE, TSP.
  - Partitioning problems: 3-COLOR, 3D-MATCHING.
  - Numerical problems: 2-PARTITION, SUBSET-SUM, KNAPSACK.