CSE 6140/ CX 4140
Computational Science and Engineering
ALGORITHMS

NP Completeness 4

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Based on slides by Prof. Ümit V. Çatalyürek

Genres of NP-complete problems

- Six basic genres of NP-complete problems and paradigmatic examples.
  - Packing problems: SET-PACKING, INDEPENDENT SET.
  - Covering problems: SET-COVER, VERTEX-COVER.
  - Constraint satisfaction problems: SAT, 3-SAT.
  - Sequencing problems: HAMILTONIAN-CYCLE, TSP.
  - Partitioning problems: 3-COLOR, 3D-MATCHING.
  - Numerical problems: 2-PARTITION, SUBSET-SUM, KNAPSACK.

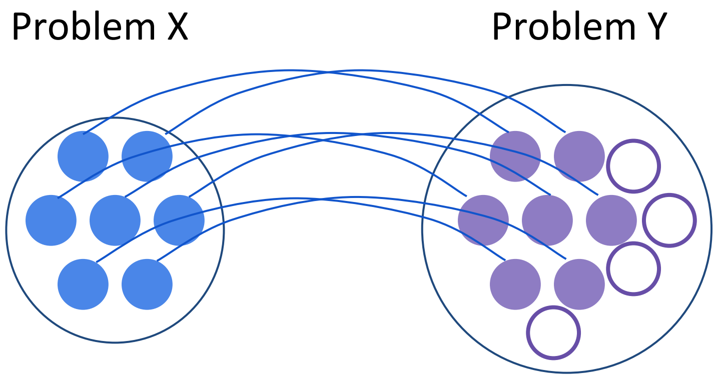
Basic reduction strategies.

- Reduction by simple equivalence.
  - INDEPENDENT-SET -- VERTEX-COVER
  - VERTEX COVER -- CLIQUE
- Reduction from special case to general case
  - VERTEX-COVER $\leq_P$ SET-COVER
- Reduction from general case to special case
  - SAT $\leq_P$ 3SAT
- Reduction by encoding with gadgets
  - 3-SAT $\leq_P$ INDEPENDENT-SET

Establishing NP-Completeness

- Recipe to establish NP-completeness of problem Y.
  - Step 1. Show that Y is in NP.
    - Describe how a potential **solution**/witness will be represented
    - Describe a **procedure to check** whether the potential witness is a correct solution to the problem instance, and argue that this procedure takes **polynomial time**
  - Step 2. Choose an NP-complete problem X.
  - Step 3. Prove that $X \leq_P Y$ (X is **poly-time reducible** to Y).
    - Describe a **procedure f that converts** the inputs i of X to inputs of Y in **polynomial time**
    - Show that the reduction is correct by showing that $X(i) = \text{YES} \Leftrightarrow Y(f(i)) = \text{YES}$ (**if and only if**, proof in both directions)

Prove that X $\leq_P$ Y



X is NP-Complete. Y is at least as difficult as X. Y is NP-complete.

Independent set

- Independent set (IS)
  - Given a graph G=(V,E), find the largest independent set: a set of vertices in the graph with no edges between them.
  - **Decision version**?
    - is there an independent set of at least K vertices?
- IS is NP (Step 1)
  - Certificate: set of vertices S
  - Certifier: Check size of S $\geq$ K, and no pair of vertices in S is connected by an edge, polynomial time

Independent set

- Independent set (IS)
  - Given a graph G=(V,E), find the largest independent set: a set of vertices in the graph with no edges between them.
  - Decision version: is there an independent set of at least K vertices?

  - Reduction by gadget (Step 2: choosing 3SAT)

3SAT → independent set of size K=m clauses in appropriately constructed graph G

3-Satisfiability Reduces to Independent Set

- **Claim**. 3-SAT $\leq_P$ INDEPENDENT-SET.
- **Pf**. Given an instance Φ of 3-SAT (I_1), we construct an instance (G, k) of INDEPENDENT-SET (I_2) that has an independent set of size k iff Φ is satisfiable.
- **Construction** (Step 3a)

k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

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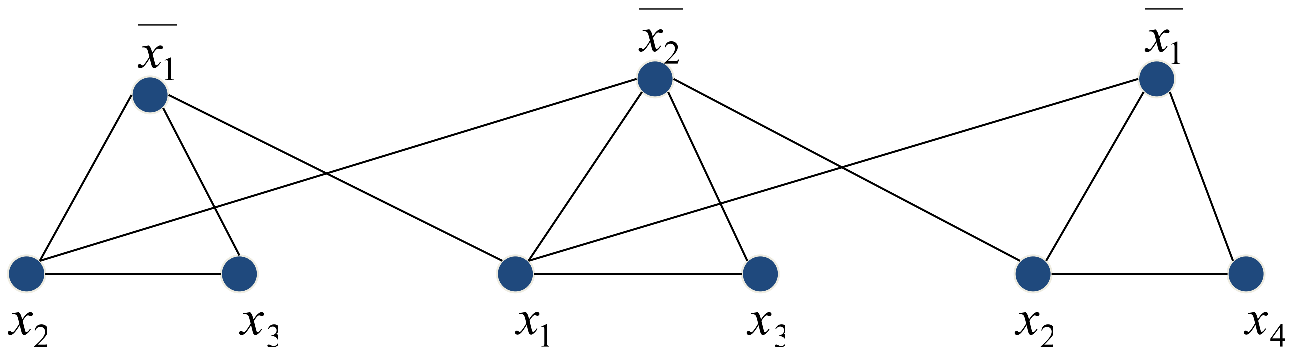
- **Construction** (Step 3a)
  - G contains 3 vertices for each clause, one for each literal.
  - Connect 3 literals in a clause in a triangle.
  - Connect literal to each of its negations.
  - *The size of I_2 is polynomial in the size of I_1*

G



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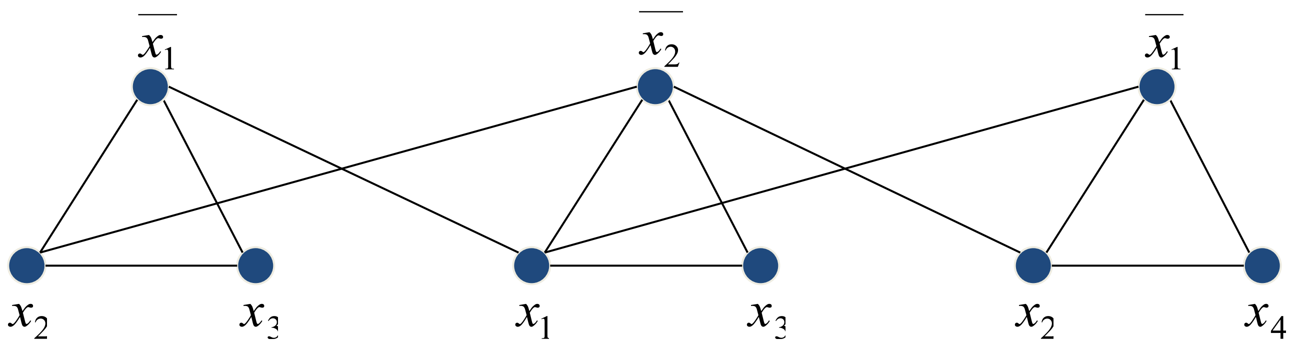
3 Satisfiability Reduces to Independent Set

- Claim. Φ is satisfiable iff G contains independent set of size k = $|\Phi|$
- (I_1 has a solution ⇔ I_2 has a solution)

Step 3b

- => Given satisfying assignment (sol to I_1), select one true literal from each triangle. Show the corresponding set of vertices is an independent set of size k.
  - Call this set of vertices S. $|S|=k$
  - Show S is an independent set by contradiction:
    - Assume S is not an IS.
    - There are two vertices sharing an edge. Say y_i and y_j, $y_i \in S$, $y_j \in S$.
    - One vertex from each triangle. $y_i = \neg y_j$, contradicts with that both y_i and y_j are True.

G



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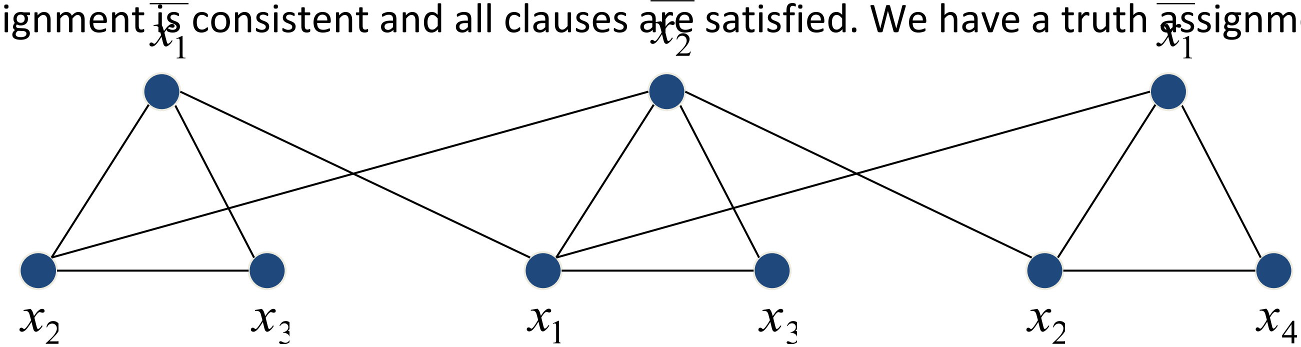
3 Satisfiability Reduces to Independent Set

- Claim. Φ is satisfiable iff G contains independent set of size $k = |\Phi|$
- (I_1 has a solution ⇔ I_2 has a solution)

Step 3c

- <= Let S be independent set of size k (sol. to I_2). Show that Φ has a truth assignment.
  - S must contain exactly one vertex in each triangle.
  - Set the literals corresponding to the vertices in S to True. (and any other variables in a consistent way)
    - Is this a "valid" assignment (can a variable to assigned both True and False?)?
    - Assume there exists such a variable x. If both x and ¬x are True, then both are in S. Contradicts with that S is an independent set.
  - Truth assignment is consistent and all clauses are satisfied. We have a truth assignment for Φ

G



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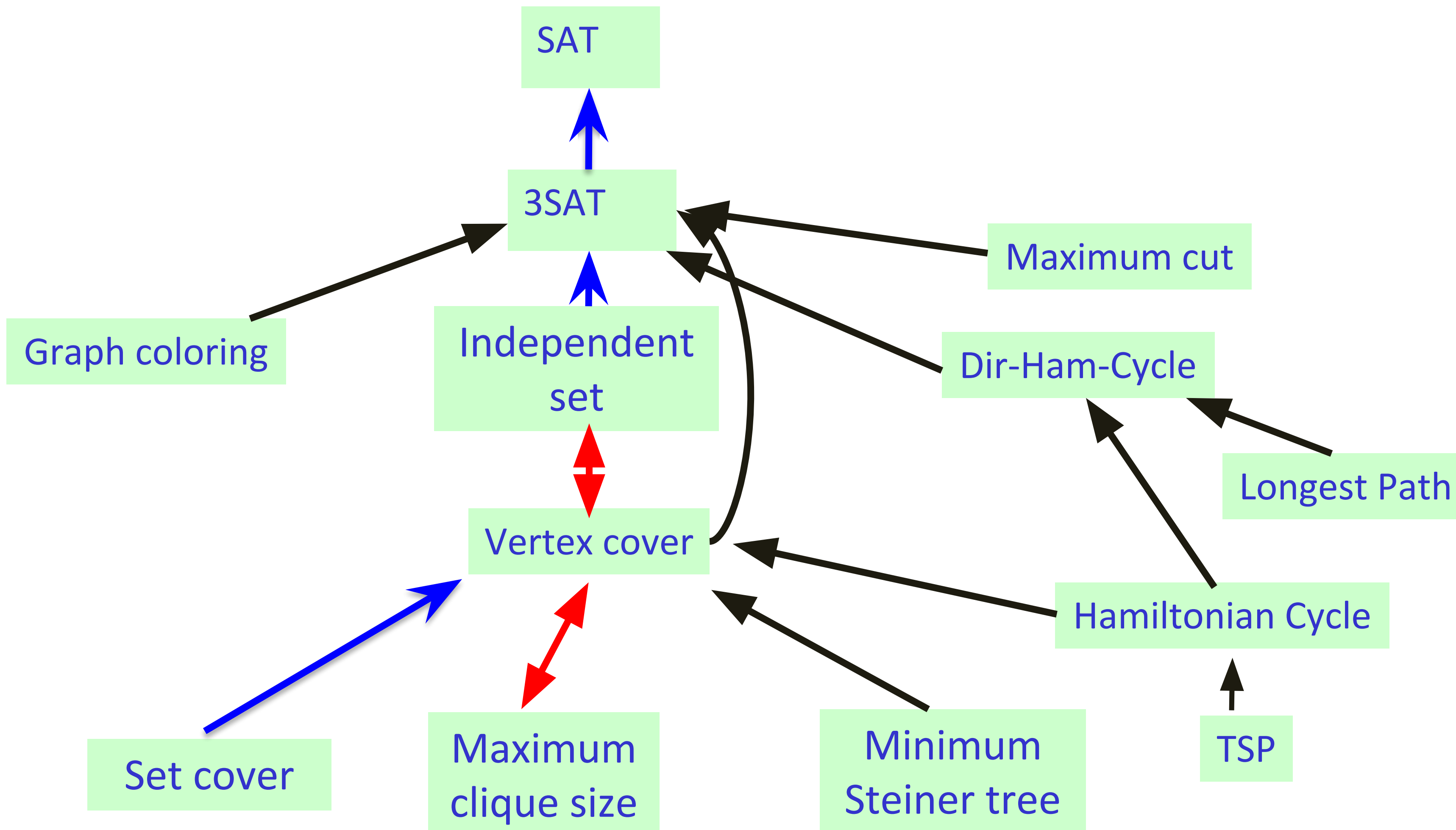
$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Examples of NP-complete problems

Summary of some NPc problems



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