Aug 24

(3) a, b, c int.

$a \mid b$, $a \nmid c$ $\Rightarrow a \nmid (b+c)$

Suppose $a \mid (b+c)$

$\hookrightarrow \exists d_1 \in \mathbb{N}$ s.t.

$b+c = a.d_1$

$a \mid b \Rightarrow \exists d_2 \in \mathbb{N}$ s.t. $b = a.d_2$

$c = (b+c) - b = a d_1 - a d_2$

$= a(\underbrace{d_1 - d_2}_{\in \mathbb{N}})$

$\hookrightarrow a \mid c$

Contradiction!

Therefore, $a \nmid (b+c)$

[redacted]

Aug. 24

4 $2n < 2^n - 1 \quad \forall n \geq 3$

- base - $n = 3 \Rightarrow 2.3 = 6 < 2^3 - 1 = 7$
- ind. hyp. $n = k$, $2k < 2^k - 1$
- ind. step. - Try to show that

$$2(k+1) < 2^{k+1} - 1$$

$$2(k+1) = \underbrace{2k} + 2 < (2^k - 1) + 2 \quad (\text{hyp})$$

$$< 2^k + 2^k - 1 \quad (2 < 2^k)$$

$$2^{k+1} - 1$$

[redacted]

8/29 ①

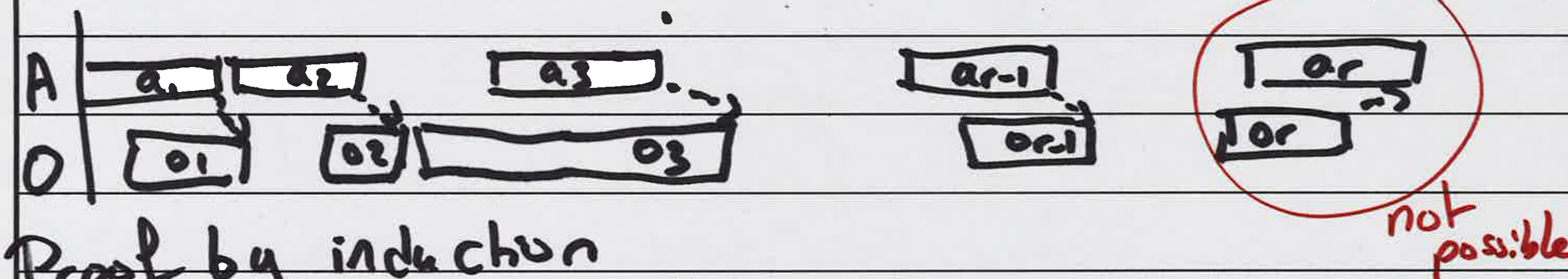
Greedy sol. A : $a_1, a_2 \dots a_k$ $f_{a_1} \leq f_{a_2} \leq \dots$

Opt. sol. O : $o_1, o_2, \dots o_m$ $m \geq k$

assume $f_{o_1} < f_{o_2} \dots$

Greedy stays ahead : claim : For all indices $r \leq k$

$f(a_r) \leq f(o_r)$



Proof by induction

base $r=1$ $f(a_1) \leq f(o_1)$ by greedy choice

(earliest finish time)

ind. hyp holds for $r-1$: $f(a_{r-1}) \leq f(o_{r-1})$

ind. step r

[redacted]

3/29 (2)

$f(o_{r-1}) \le s(o_r) \le f(o_r)$ $\boxed{o_{r-1}} \rightarrow \boxed{o_r}$

"feasibility of opt. sol."

$f(a_{r-1}) \le f(o_{r-1})$ by ind. hyp.

$f(a_{r-1}) \le s(o_r)$

$\rightarrow$ o_r is compatible with $a_1 \dots a_{r-1}$

$\rightarrow$ it was an option for greedy

$\rightarrow$ job a_r was chosen by greedy

$\Rightarrow f(a_r) \le f(o_r)$

[redacted]

Th Greedy is optimal ($k = m$)

$A: a_1 \dots a_k$ $\qquad$ $m \geq k$ (by opt. of O)

$O: o_1 \dots \; - - \; o_m$

Assume that $k < m$

k: our claim says that $f(a_k) \leq f(o_k)$

opt. must have o_{k+1} ($k < m$)

opt. O is feasible

$\rightarrow f(o_k) \leq s(o_{k+1})$

$\rightarrow o_{k+1}$ is an option for greedy.

after iter. k

$\rightarrow$ contradicts with greedy stopping at iter. k !

$\Rightarrow k = m$

[redacted]

Aug. 31 (1)

Dijkstra's Algo correctness: $(x \to y)$

Greedy stays ahead.

<u>Invariant</u>: For each $u \in S$, $d(u)$ is the length of the shortest $s \to u$ path

Proof by induction (size of S).

<u>base</u>: $|S| = 1$ $S = \{s\}$ $d(s) = 0$ ✓

<u>ind. hyp.</u> when $|S| = k \geq 1$, the inv. holds

$\forall u \in S$ $d(u)$ is shortest path length.

<u>ind. step.</u> Consider next node added

to S by Dijkstra.

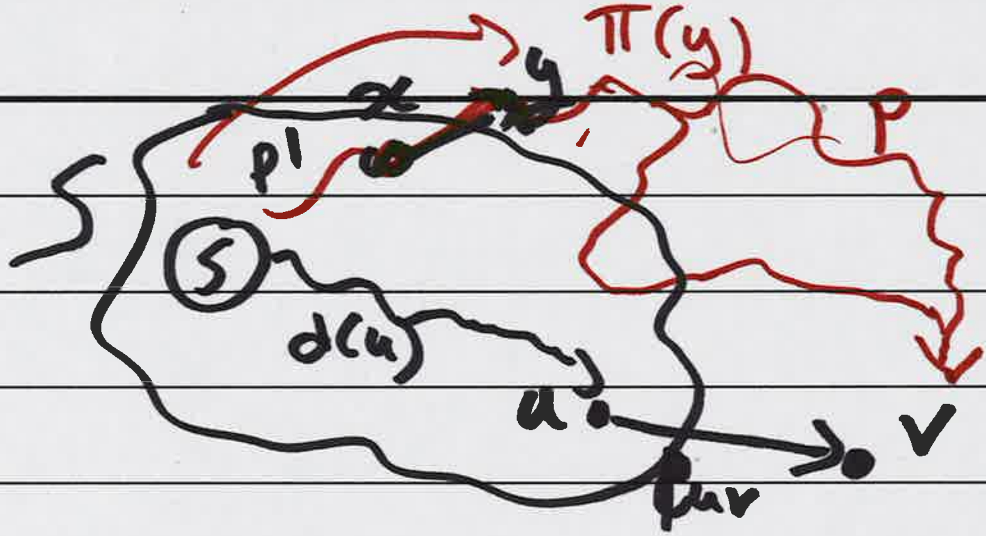
We call it v.

$S \leftarrow S \cup \{v\}$

size $k+1$

[redacted]

Aug. 31 ②



v must have edge (u,v) s.t. $u \in S$

$\pi(v) = d(u) + e_{uv}$

and $\pi(v)$ is best choice (greedy).

Show that $d(v) = \pi(v) = d(u) + e_{uv}$

is shortest path length

[redacted]

Consider any $s \rightsquigarrow v$ path P

P must leave S

$\Rightarrow (x, y)$ with $x \in S$, $y \notin S$

Aug. 31 (3)

$$\begin{aligned} \ell(p) &\geq \ell(p') + e_{xy} \\ &\geq d(x) + e_{xy} && \text{(because } x \in S\text{)} \\ &\geq \pi(y) && \text{(by def).} \\ &\geq \pi(v) && \text{(by greedy choice)} \end{aligned}$$

For any path, $\ell(P) \geq \pi(v) = d(v)$

$\Rightarrow d(v)$ is shortest!

[redacted]